ASPECTS OF THE RICCI FLOW

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This thesis contains several projects investigating aspects of the Ricci flow (RF), from preserved curvature conditions, Harnack estimates, long-time existence results, to gradient Ricci solitons.

Recently, Wilking [98] proved a theorem giving a simple criterion to check if a curvature condition is preserved along the RF. Using his approach, we show another criterion with slightly different flavor (interpolations of cone conditions). The abstract formulation also recovers a known preserved condition.

Another project was initially concerned with the Ricci flow on a manifold with a warped product structure. Interestingly, that led to a dual problem of studying more abstract flows. Using the monotone framework, we derive several estimates for the adapted heat conjugate fundamental solution which include an analog of G. Perelman's differential Harnack inequality as in [81].

The behavior of the curvature towards the first finite singular time is also a topic of great interest. Here we provide a systematic approach to the mean value inequality method, suggested by N. Le [63] and F. He [59], and display a close connection to the time slice analysis as in [97]. Applications are obtained for a Ricci flow with nonnegative isotropic curvature assumption.

Finally, we investigate the Weyl tensor within a gradient Ricci soliton structure. First, we prove a Bochner-Weitzenböck type formula for the norm of the self-dual Weyl tensor and discuss its applications. We are also concerned with the interplay of curvature components and the potential function.

BIOGRAPHICAL SKETCH

Hung Tran was born on August 22, 1986 in Xuan Thuy, Nam Ha, Viet Nam. He graduated from Hanoi-Amsterdam high school before moving to Berea College, Kentucky, US . Hung received a BA degree in Mathematics and Economics in the summer of 2009 and, that same year, he joined the Mathematics Department at Cornell University to pursue doctoral studies. In addition to mathematics, he likes tennis, chess, soccer, poetry, daffodil, and programming.

To my parents,

Tran Quoc Luc and Nguyen Thi Ky

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CHAPTER 1

INTRODUCTION

This thesis is devoted to studying several aspects of the Ricci flow introduced by R. Hamilton [51], from preserved curvature conditions, Harnack estimates, long-time existence results, to gradient Ricci solitons.

Definition 1.0.1. *$(M, g(t))$, $0 \leq t \leq T \leq \infty$, a manifold equipped a one-parameter family of Riemannian metrics, is a solution to the Ricci flow if,*

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Rc}(t). \tag{1.1}$$

It is a powerful tool to prove the existence of canonical metrics on a manifold with suitable initial data. Even though the equation is a weakly-parabolic system, using DeTurck's trick [41], we can transform it to a strictly parabolic flow. Uniqueness and short-time existence follows but the flow generally develops singularities in finite time. The theory, hence, depends largely on understanding the formulation of singularity models, as limits in an appropriate sense. The recent breakthrough was obtained by G. Perelman, whose non-collapsing result makes it possible to take a limit in a general setting [81]. For dimension three, building on Hamilton's work, Perelman's surgery essentially completed the arguments for the Poincaré conjecture [82] . Since then, the Ricci flow played a key role in the proofs of the Space Form theorem for manifolds with 2-positive curvature operators by C. Böhm and B. Wilking [10] and the Differentiable Sphere theorem by S. Brendle and R. Schoen ([13, 15]) for point-wise 1/4-pinched manifolds.

Nevertheless, several aspects of the field remain elusive and intriguing. A preserved curvature condition is a restriction on the curvature tensor that would be passed on to the limit. The Harnack estimate developed by G. Perelman [81] plays a role in proving that it is possible to take a limit. Long-time existence results concern with conditions on the curvature approaching the first finite singular time. Finally, a gradient Ricci soliton is a self-similar solution to the Ricci flow and, thus, a special singularity model but it arises frequently in practice.

Now we describe the organization of the thesis. For preparation, Chapter 2 and 3 collect well-known facts about Riemannian geometry and the Ricci flow. There is little original research in those chapters but the narrative can be speculative occasionally, possibly reflecting the author's naive perspective.

In Chapter 4, we investigate preserved conditions along the Ricci flow. Since such a condition could be passed on to the limit, it is a key ingredient in applications of the Ricci flow (such as in celebrated works of [51, 81, 10, 13]).In a recent development, Wilking [98] proved a theorem giving a simple criterion in the Lie Algebra language. Using that approach, we show another criterion with slightly different flavor (interpolations of cone conditions). The abstract formulation also recovers some known preserved condition developed in [13].

Chapter 5 is initially concerned with the Ricci flow on a manifold with a warped product structure. That leads to a dual problem of studying more abstract geometric flows. Using the framework of monotone formulas, we derive several estimates for the adapted heat conjugate fundamental solution which include an analog of G. Perelman's differential Harnack inequality in [81]. The proof here is inspired by [78].

In Chapter 6 we study the behavior of the curvature towards the first finite singular time. This topic has been intensively investigated but simple questions, such as whether the scalar curvature blows up, persistently remain open. Here we provide a systematic approach to the mean value inequality method, suggested by N. Le [63] and F. He [59]. We also display a close connection between this method and time slice analysis as in [97]. Applications are derived for a Ricci flow with the nonnegative isotropic curvature assumption.

Chapter 7 is about the Weyl tensor within a gradient Ricci soliton structure (GRS). The Ricci flow in low dimension is relatively well understood thanks to classification results of gradient Ricci solitons. In higher dimension, $n > 3$, the situation is subtler mainly because of the non-triviality of the Weyl tensor. Thus, it is interesting to investigate that setting, particularly in dimension four, by combining different techniques including flow equations and a normal form used to study Einstein manifolds. First, we prove a Bochner-Weitzenböck type formula for the norm of the self-dual Weyl tensor and discuss its applications, including connections between geometry and topology. We are also concerned with the interaction of different components of Riemannian curvature and (gradient and Hessian of) the soliton potential function. The Weyl tensor arises naturally in these investigations. Applications here are rigidity results.

CHAPTER 2

PRELIMINARIES

In this chapter we first fix our notation and then review some basics of Riemannian geometry which will be used throughout the document.

2.1 Notations and Conventions

Let (M^n, g) be an n-dimensional manifold with Riemannian metric g. The Levi-Civita connection is defined by,

$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle_g =& X\langle Y, Z\rangle_g + Y\langle X, Z\rangle_g - Z\langle X, Y\rangle_g \\ &+ \langle [X, Y], Z\rangle_g - \langle [X, Z], Y\rangle_g - \langle [Y, Z], X\rangle_g . \end{aligned}$$

Also, we denote $\nabla^2_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z$ and $\{e_i\}_{i=1}^n$ a local coordinate. Consequently, the Christoffel symbol can be calculated explicitly,

$$\left\langle \nabla_{e_i} e_j, e_k \right\rangle \doteq \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial}{\partial e_i} g_{jl} + \frac{\partial}{\partial e_j} g_{il} - \frac{\partial}{\partial e_l} g_{ij}\right).$$

2.1.1 Operators

Given 1-forms $\omega^i \in T^*M$, we define,

$$(\omega^1 \wedge ... \wedge \omega^p)(X_1, ...X_p) \doteq \det[\omega^i(X_j)].$$

The wedge product $\wedge$ can be extended for all forms using linearity and associativity.

The volume form $d\mu$ is, for a positively oriented basis $\{\omega^i\}_{i=1}^n \in T^*M$,

$$d\mu \doteqdot \sqrt{\det(g_{ij})}\omega^1 \wedge ... \wedge \omega^n.$$

The exterior derivative d and interior product ι are defined as follows:

$$d(f\omega^1 \wedge ... \wedge \omega^p) = (df) \wedge \omega^1 \wedge .... \wedge \omega^p,$$

$$(d\omega)(X_0, ...X_p) = \Sigma_0^p(-1)^j(\nabla_{X_j}\beta)(X_0, ..., \hat{X}_j, ..., X_p),$$

$$d(\omega \wedge \psi) = (d\omega) \wedge \psi + (-1)^p\omega \wedge (d\psi),$$

$$(\iota_X\omega)(X_1, ..., X_p) = \omega(X, X_1, ...X_p),$$

$$\iota_X(\omega \wedge \psi) = (\iota_X\omega) \wedge \psi + (-1)^p\omega \wedge (\iota_X\psi).$$

Remark 2.1.1. *Our convention for d and ι follows [83] and differs from [38] by scaling.*

For differential forms γ, η of the same type p, the inner product is agreed to be, for $i_1 < .... < i_p, j_1 < ....j_p$,

$$\langle\gamma, \eta\rangle = g^{i_1j_1}....g^{i_pj_p}\gamma_{i_1...i_p}\eta_{j_1...j_p}.$$

In particular,

$$\left\langle\omega^{i_1} \wedge ... \wedge \omega^{i_p}, \omega^{j_1} \wedge ... \wedge \omega^{j_p}\right\rangle \doteqdot \det(\delta^{i_kj_l}).$$

The Hodge $*$ operator $\Lambda^pT^*M \to \Lambda^{n-p}T^*M$ is defined via the volume form $d\mu$:

$$(*\gamma) \wedge \eta \doteqdot \langle\gamma, \eta\rangle\, d\mu.$$

$$*(\omega^1 \wedge ... \wedge \omega^p) = \omega^{p+1} \wedge ... \wedge \omega^n.$$

The Lie derivative is defined though diffeomorphisms. Let X be a vector field and ϕ_t the corresponding (locally-defined) flow. The Lie derivative of a tensor

$\mathcal{D}$ in the direction of X is just the first order term in a suitable Taylor expansion of that tensor moved by the flow ϕ_t. That is,

$$L_X\mathcal{D} = \lim_{t\to 0}\frac{1}{t}(\mathcal{D} - (\phi_t)_*\mathcal{D}).$$

In particular, for a function f, vector fields X, Y, and tensors ω, ψ,

$$L_X f = Xf,$$

$$L_X Y = [X, Y],$$

$$L_X(\omega \wedge \psi) = (L_X\omega) \wedge \psi + \omega \wedge (L_X\psi).$$

Also, we have the H. Cartan's magic formula,

$$L_X = d \circ \iota_X + \iota_X \circ d.$$

Finally, the divergence δ (or div) and Laplacian Δ are defined as, with an orthonormal coordinate,

$$(\delta T)(X_1, ...X_m) = \mathrm{tr}(w \to (\nabla_w)(X_1, ...X_m)) = \sum_i (\nabla_{e_i} T)(e_i, X_1, ...X_m);$$

$$\Delta T = \mathrm{tr}(\nabla^2)T = \sum_i \nabla^2_{e_i,e_i} S.$$

We also take the chance here to introduce the heat operator,

$$\Box = \frac{\partial}{\partial t} - \Delta.$$

Remark 2.1.2. *In an appropriate context, the divergence can be identified with the co-differential (adjoint of d) with an opposite sign [8].*

2.1.2 Curvature Notions

The Riemannian curvature is defined by,

$$\mathrm{R}(X, Y, Z) = -\nabla^2_{X,Y} Z + \nabla^2_{Y,X} Z$$
$$\mathrm{R}(X, Y, Z, W) = -\left\langle \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, W \right\rangle_g .$$

Remark 2.1.3. *Our (3,1) curvature sign agrees with [1, 8] and opposite to [11, 38, 51, 83]. Our (4,0) curvature convention, however, is the same as [1, 8, 11, 51] and opposite to [38, 83]. Consequently,*

$$\mathrm{R}^l_{ijk} = \frac{\partial}{\partial e_j}\Gamma^l_{ik} - \frac{\partial}{\partial e_i}\Gamma^l_{jk} + \Gamma^m_{ik}\Gamma^l_{im} - \Gamma^m_{jk}\Gamma^l_{im}.$$

If $P \subset T_x M$ is a 2-plane with an orthonormal basis $\{e_1, e_2\}$, the sectional curvature of P is defined by

$$K(P) = \mathrm{R}(e_1, e_2, e_1, e_2) = \mathrm{R}_{1212}.$$

The Ricci and scalar curvature are defined by, respectively,

$$\mathrm{R}_{ij} = g^{pq}\mathrm{R}_{ipjq},$$
$$\mathrm{S} = g^{ij}\mathrm{R}_{ij}.$$

We take the chance to define the conjugate heat operator, along a Ricci flow,

$$\Box^* = -\frac{\partial}{\partial t} - \Delta + \mathrm{S}.$$

In order to define the Weyl tensor, we first need to recall the Kulkarni-Nomizu product for $(2, 0)$ symmetric tensors A and B,

$$(A \circ B)_{ijkl} = A_{ik}B_{jl} + A_{jl}B_{ik} - A_{il}B_{jk} - A_{jk}B_{il}.$$

Then we have the following decomposition of curvature, for $\mathrm{E} = \mathrm{Rc} - \frac{\mathrm{S}g}{4}$, W the Weyl tensor,

$$\mathrm{R} = \mathrm{W} + \frac{S\,g \circ g}{2n(n-1)} + \frac{\mathrm{E} \circ g}{n-2} = \mathrm{W} - \frac{S\,g \circ g}{2(n-2)(n-1)} + \frac{\mathrm{Rc} \circ g}{n-2}. \tag{2.1}$$

It can be seen from the equation that W inherits most of the symmetry from R, see Section 2.3.

2.1.3 Identification between tensors and operators

Using the point-wise induced inner product, any anti-symmetric (2,0) tensor α (a two-form) can be seen as an operator on the tangent space by,

$$\alpha(X, Y) = \langle -\alpha(X), Y \rangle = \langle X, \alpha(Y) \rangle = \langle \alpha, X \wedge Y \rangle .$$

In particular, a bi-vector acts on a vector X as follows

$$(U \wedge V)X = \langle V, X \rangle\, U - \langle U, X \rangle\, V.$$

For instance, in dimension four, for $e_{ij} = e_i \wedge e_j$:

	$e_{12} + e_{34}$	$e_{13} - e_{24}$	$e_{14} + e_{23}$	$e_{12} - e_{34}$	$e_{13} + e_{24}$	$e_{14} - e_{23}$
e_1	$-e_2$	$-e_3$	$-e_4$	$-e_2$	$-e_3$	$-e_4$
e_2	e_1	e_4	$-e_3$	e_1	$-e_4$	e_3
e_3	$-e_4$	e_1	e_2	e_4	e_1	$-e_2$
e_4	e_3	$-e_2$	e_1	$-e_3$	e_2	e_1

(2.2)

In a similar manner, any symmetric $(2, 0)$ tensor b can be seen as an operator on the tangent space,

$$b(X, Y) = \langle b(X), Y \rangle = \langle X, b(Y) \rangle = \langle b, X \wedge Y \rangle .$$

Consequently, when b is viewed as a 1-form valued 1-form, $d_\nabla b$ denotes the exterior derivative (a 1-form valued 2-form). That is,

$$(d_\nabla b)(X, Y, Z) = (\nabla b)(X, Y, Z) + (-1)^1(\nabla b)(Y, X, Z) = \nabla_X b(Y, Z) - \nabla_Y b(X, Z).$$

Similarly, a $(4, 0)$ tensor such as R, W can be interpreted as an operator on two-forms, that is, a map from $\Lambda_2(TM) \to \Lambda_2(TM)$. Then, we normally take the operator norm (sum of squares of eigenvalues) (this agrees with the tensor norm defined in [38] for $(2, 0)$ tensors but differs by $1/4$-factor for $(4, 0)$ tensors). More precisely, for an orthonormal frame or coordinate,

$$|\mathrm{W}|^2 = \sum_{i<j;k<l} \mathrm{W}_{ijkl}^2.$$

In addition, the norm of covariant derivative and divergence on these tensors can be defined accordingly,

$$|\nabla \mathrm{W}|^2 = \sum_i \sum_{a<b;c<d} (\nabla_i \mathrm{W}_{abcd})^2,$$
$$|\delta \mathrm{W}|^2 = \sum_i \sum_{a<b} ((\delta \mathrm{W})_{iab})^2.$$

For a tensor $T : \Lambda_2(TM) \otimes (TM) \to \mathbb{R}$, we define

$$\langle T, \delta \mathrm{W}\rangle = \sum_{i<j;k} T_{ijk}(\delta \mathrm{W})_{kij}, \tag{2.3}$$
$$\langle T, i_X \mathrm{W}\rangle = \sum_{i<j;k} T_{ijk}(i_X \mathrm{W})_{kij}. \tag{2.4}$$

Also, the Einstein summation convention is used when dealing with indices. Finally, when the context is clear, we will omit the measure when integrating.

2.1.4 Coordinate versus Frame

In order to study the geometry of a smooth manifold, it is essential to be able to carry out various computation (such as calculating the curvature given its metric). The two most popular tools are a local coordinate and a local frame. Because of the dominance of these two concepts, let's distinguish them first.

Let p be a point in a smooth manifold and U an open neighborhood of p.

A local coordinate $\{\frac{\partial}{\partial x_i}\}_{i=1}^n$ is associated with a local coordinate chart $\{x_i\}_{i=1}^n$ which is a diffeomorphic map between U and a open subset of the Euclidean space $\mathbb{R}^n$. The shorthand notation for $\frac{\partial}{\partial x_i}$ is just ∂_i when the context is clear.

A local frame $\{E_i\}_{i=1}^n$ is a collection of vector fields on V such that they are linearly independent and span the tangent space at each point in U. A local frame is orthonormal if $\left\langle E_i, E_j \right\rangle = \delta_i^j$.

In practice, it is often convenient to work with a normal coordinate (that is, $\nabla\partial_i \mid_p= 0$) or a normal orthonormal frame ($\nabla E_i \mid_p= 0$). It can be shown that, given an orthonormal basis $\{e_i\}_{i=1}^n$ of the tangent space at p, there exist a normal orthonormal frame and a normal coordinate around p such that their restrictions to that tangent space are exactly the given basis [83, Chapter 2].

Indeed, a local coordinate is usually constructed via the exponential map while a local frame can be built via parallel translations. To illustrate the difference in calculation involved with each method, we'll provide both perspectives on certain calculation such as Section 2.2 or Lemma 2.5.1.

2.2 The Riemannian Curvature

The purpose of this section is to show how to compute the Riemannian curvature given its metric and review some of its properties.

2.2.1 Coordinate Calculation

In a local coordinate, the curvature can be calculated from the Christoffel symbols Γ^k_{ij} as discussed earlier:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \tag{2.5}$$

$$\mathrm{R}^l_{ijk} = -\partial_i \Gamma^l_{jk} + \partial_j \Gamma^l_{ik} - \Gamma^r_{jk}\Gamma^l_{ip} + \Gamma^p_{ik}\Gamma^l_{jp} \tag{2.6}$$

$$\mathrm{R}_{ijkl} = g_{lm}\mathrm{R}^m_{ijk}.$$

Remark 2.2.1. *The formulae make clear that curvature components are essentially 2nd derivatives of the metric. In that sense, the Bianchi identities (2.8), (2.9) essentially expose the symmetry of the metric at the 2nd and 3rd orders.*

2.2.2 Frame Calculation

The curvature can also be calculated by using a frame via Cartan's structure equations. Our treatment here follows [38, Chapter 1]. Let $\{e_i\}$ be an orthonormal frame and $\{\omega^i\}$ its dual, i.e. $\omega^i(e_j) = \delta^i_j$. The connection 1-form ω^j_i is defined as,

$$\left\langle \nabla_X e_i, e_j \right\rangle = \omega^j_i(X).$$

Furthermore, it satisfies the following properties,

$$\omega^i_j = -\omega^j_i,$$

$$\nabla_X \omega^i = -\omega^i_j(X)\omega^j,$$

$$\nabla e_i = \omega^j_i \otimes e_j.$$

Remark 2.2.2. $\omega^k_j(e_i) \sim \Gamma^k_{ij}$ *but one is defined by a local frame while the other by a local coordinate.*

Define $\mathrm{Rm}^j_i(X, Y) = \frac{1}{2}\left\langle \mathrm{R}(X, Y)e_j, e_i \right\rangle$ then we have Cartan's equations:

$$d\omega^i = \omega^j \wedge \omega^i_j,$$

$$\mathrm{Rm}^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k.$$

Also, for computation convenience,

$$\omega^k_i(e_j) = d\omega^i(e_j, e_k) + d\omega^j(e_i, e_k) - d\omega^k(e_j, e_i).$$

2.2.3 Properties

Recall that,

$$\mathrm{R}(X, Y, Z, W) = -\left\langle \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z, W \right\rangle.$$

So it is easy to see the symmetry,

$$\mathrm{R}(X, Y, Z, W) = -\mathrm{R}(Y, X, Z, W) = \mathrm{R}(Z, W, X, Y).$$

Also, the $(4, 0)$ curvature tensor R satisfies the following Bianchi first and second identities,

(2.7)

$$\mathrm{R}_{ijkl} + \mathrm{R}_{jkil} + \mathrm{R}_{kijl} = 0, \tag{2.8}$$

$$\nabla_i \mathrm{R}_{jklm} + \nabla_j \mathrm{R}_{kilm} + \nabla_k \mathrm{R}_{ijlm} = 0. \tag{2.9}$$

As a consequence, we have the following contracted 2nd Bianchi identity in terms of the divergence:

$$\delta(\mathrm{Rc} - \frac{1}{2}\mathrm{S}g) = 0 \tag{2.10}$$

An immediate application is the flowing well-known fact.

Lemma 2.2.1. *On a closed Riemannian manifold, for any smooth function f,*

$$\int_M \Big(2\left\langle \mathrm{Rc}, \nabla^2 f \right\rangle_g - \mathrm{S}\Delta f\Big) d\mu = 0$$

Proof. We have,

$$\begin{aligned} \delta(\mathrm{Rc}\nabla f) &= (\delta \mathrm{Rc})\nabla f + \left\langle \mathrm{Rc}, \nabla^2 f \right\rangle_g \\ &= \frac{1}{2}\nabla \mathrm{S} \nabla f + \left\langle \mathrm{Rc}, \nabla^2 f \right\rangle_g . \end{aligned}$$

Applying the divergence theorem yields,

$$\int_M \Big(2\left\langle \mathrm{Rc}, \nabla^2 f \right\rangle_g - \mathrm{S}\Delta f\Big) d\mu = \int_M \Big(- \nabla \mathrm{S} \nabla f - \mathrm{S}\Delta f\Big) d\mu = 0.$$

□

2.3 The Weyl Tensor in Dimension Four

In this section, we give a brief review of the Weyl tensor on an oriented four-manifold (M, g).

2.3.1 Decomposition of the Curvature

Recall the curvature decomposition (2.1):

$$\mathrm{R} = \mathrm{W} + \frac{\mathrm{S}g \circ g}{2n(n-1)} + \frac{\mathrm{E} \circ g}{n-2} = \mathrm{W} - \frac{\mathrm{S}g \circ g}{2(n-2)(n-1)} + \frac{\mathrm{Rc} \circ g}{n-2}.$$

We note that, as $(4,0)$ tensors, W, $\mathrm{E} \circ g$, $g \circ g$ are orthogonal. Consequently, the Weyl tensor inherits algebraic properties of the curvature tensor and is also traceless. Then it is easy to see the followings, for an orthonormal frame,

$$W_{1212} = \sum_{2<i<j} W_{ijij}.$$

More generally, if the tangent space is decomposed into orthogonal subspaces N_1, N_2 then,

$$\mathrm{W}_{N_1} \doteqdot \sum_{i<j, i,j \in N_1} W_{ijij} = \sum_{k<l, k,l \in N_2} W_{klkl}.$$

That is, the Weyl "sectional curvature"s of complementing subspaces are relatively comparable and then well-defined. [1] Also, it is noted that if the co-dimension of N_1 is 0 or 1 then $\mathrm{W}_{N_1} = 0$.

In dimension four the decomposition becomes,

$$\begin{aligned}
\mathrm{R} &= \mathrm{W} + \frac{\mathrm{S}}{24} g \circ g + \frac{1}{2} \mathrm{E} \circ g \doteqdot \mathrm{W} + U + V, \\
|\mathrm{R}|^2 &= |\mathrm{W}|^2 + |U|^2 + |V|^2, \\
|U|^2 &= \frac{1}{2n(n-1)} \mathrm{S}^2 = \frac{1}{24} \mathrm{S}^2, \\
|V|^2 &= \frac{1}{n-2} |\mathrm{E}|^2 = \frac{1}{2} |\mathrm{E}|^2.
\end{aligned}$$

A special feature of dimension four is that the Hodge $*$ operator decomposes the space of two-forms (Λ_2) orthogonally according to eigenvalues ± 1.

[1] Berger's inequalities (Lemma 6.4.2) compare sectional curvatures of the curvature tensor.

Let $\operatorname{sign}(i, j, k)$ be the sign-um of the permutation of $\{1, 2, 3\}$ and $\{\alpha_i\}_{i=1}^3$ a positive-oriented orthogonal basis of Λ_2^+ with $|\alpha_i| = \sqrt{2}$ and $\operatorname{sign}(i, j, k) = 1$, then, according to [2],

$$\alpha_i^2 = -\text{Identity},$$
$$\alpha_i \alpha_j = \alpha_k = -\alpha_j \alpha_i,$$
$$\left\langle \alpha_i(X), \alpha_j(X) \right\rangle = \left\langle X, -\alpha_i \alpha_j X \right\rangle = \langle X, \alpha_k X \rangle = 0.$$

An example of such a basis is given by multiplying $\sqrt{2}$ the basis given in (2.11). Consequently, we have the following result.

Lemma 2.3.1. *Suppose (M, g) is a four-dimensional Riemannian manifold and X is a vector field on M. At any point p such that $X_p \neq 0$,*

$$T_p M = X_p \oplus \Lambda_2^+(X_p),$$

in which $\Lambda_2^+(X) = \{\alpha(X_p), \alpha \in \Lambda_2^+\}$.

Proof. Pick an orthogonal basis of Λ_2^+ as above then it follows that $\{\alpha_i(X_p)\}_{i=1}^3$ are three orthogonal vectors and each is perpendicular to X_p. So the statement follows. □

Let $\{e_i\}_{i=1}^4$ be a positively oriented orthonormal basis of $T_p M$, then a pair of orthonormal bases of $\Lambda_2^\pm$ is given by,

$$\begin{aligned} &\{\frac{1}{\sqrt{2}}(e_{12} + e_{34}), \frac{1}{\sqrt{2}}(e_{13} - e_{24}), \frac{1}{\sqrt{2}}(e_{14} + e_{23})\} \text{ for } \Lambda_2^+, \\ &\{\frac{1}{\sqrt{2}}(e_{12} - e_{34}), \frac{1}{\sqrt{2}}(e_{13} + e_{24}), \frac{1}{\sqrt{2}}(e_{14} - e_{23})\} \text{ for } \Lambda_2^-. \end{aligned} \tag{2.11}$$

Accordingly, the curvature is,

$$\mathrm{R} = \begin{pmatrix} A^+ & C \\ C^T & A^- \end{pmatrix}, \tag{2.12}$$

for C essentially the traceless Ricci. In addition,

$$A^{\pm} = \mathrm{W}^{\pm} + \frac{S}{12}\mathrm{Id}^{\pm},$$
$$|A^{\pm}|^2 = |\mathrm{W}^{\pm}|^2 + \frac{\mathrm{S}^2}{48},$$
$$|\mathrm{Rc}|^2 - \frac{\mathrm{S}^2}{4} = |\mathrm{E}|^2 = 4|C|^2 = 4\mathrm{tr}(CC^T).$$

Also, we observe that $\mathrm{W}(\Lambda_2^{\pm}) \in \Lambda_2^{\pm}$, so it is unambiguous to define $\mathrm{W}^{\pm} \doteqdot \mathrm{W}^{|\Lambda_{\pm}}$. In particular, with $\alpha^{\pm}$ and $\beta^{\pm}$ the projection of α, β onto $\Lambda_2^{\pm}$,

$$\mathrm{W}^{\pm}(\alpha, \beta) = \mathrm{W}(\alpha^{\pm}, \beta^{\pm}). \tag{2.13}$$

2.3.2 Normal form of the Weyl Tensor

As W is traceless and satisfies the first Bianchi identity, there is a normal form developed by M. Berger [7, 93] (it first came to our attention through the works of [25, 80]). That is, there exists an orthonormal basis $\{e_i\}_{i=1}^4$ of T_pM, consequently $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$ being a basis of Λ_2, such that, for $A = \mathrm{diag}(a_1, a_2, a_3)$, $B = \mathrm{diag}(b_1, b_2, b_3)$, and $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 0$,

$$\mathrm{W} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{2.14}$$

Then, by (2.13),

$$\mathrm{W}^{\pm} = \begin{pmatrix} \frac{A \pm B}{2} & \frac{B \pm A}{2} \\ \frac{B \pm A}{2} & \frac{A \pm B}{2} \end{pmatrix}.$$

With respect to the basis given in (2.11),

$$\mathrm{W} = \begin{pmatrix} A + B & 0 \\ 0 & A - B \end{pmatrix}.$$

Hence we obtain the following well-known identities [39, 2.31].

Lemma 2.3.2. *Let* (M^4, g) *be a four-dimensional Riemannian manifold, then the following tensorial equations hold,*

$$(\mathrm{W}^{\pm})_{ikpq}(\mathrm{W}^{\pm})_j{}^{kpq} = |\mathrm{W}^{\pm}|^2 g_{ij}, \tag{2.15}$$
$$(\mathrm{W}^{\pm})_{ikpq}(\mathrm{W}^{\pm})^{kpq}{}_j = \frac{1}{2}|\mathrm{W}^{\pm}|^2 g_{ij}.$$

Proof. We observe that these tensorial identities only depend on the structure of these tensors. In particular, it suffices to prove for the Weyl tensor. Using the normal form discussed above,

$$\begin{aligned}\mathrm{W}_{1kpq}\mathrm{W}^{kpq}{}_1 &= \sum_{i=1}^{3} a_i^2 - 2(b_1b_2 + b_2b_3 + b_3b_1)\\ &= \sum_{i=1}^{3}(a_i^2 + b_i^2).\end{aligned}$$

Calculation can be done for other pairs of indexes to verify the statements. □

2.3.3 Some Geometry of the Weyl Tensor

If the manifold is closed, then the Gauss-Bonnet-Chern formula for the Euler characteristic and Hirzebruch formulas for the signature [8] are given by,

$$\begin{aligned}8\pi^2\chi(M) &= \int_M (|\mathrm{W}|^2 - |V|^2 + |U|^2) = \int_M (|\mathrm{W}|^2 - \frac{1}{2}|\mathrm{E}|^2 + \frac{S^2}{24})\\ &= \int_M (|\mathrm{R}|^2 - |\mathrm{E}|^2),\end{aligned} \tag{2.16}$$

$$12\pi^2\tau(M) = \int_M (|\mathrm{W}^+|^2 - |\mathrm{W}^-|^2). \tag{2.17}$$

Remark 2.3.1. *It follows immediately that if M admits an Einsterin metric* $\mathrm{E} = 0$*, then we have the Hitchin-Thorpe inequality*

$$|\tau(M)| \leq \frac{2}{3}\chi(M).$$

Furthermore, the Weyl tensor is involved in the definition of the Weitzenböck operator, the curvature term that arises in the classical Weitzenböck formula [80, Section 2].

Definition 2.3.3. *Acting on two-forms, the Weitzenböck operator is,*

$$P = \frac{S}{6} Id - \mathrm{W}.$$

Using (2.12), $P \geq 0$ is equivalent to $\frac{S}{4}\mathrm{Id}_{\pm} - A_{\pm} = \mathrm{tr}(A_{\pm})\mathrm{Id}_{\pm} - A_{\pm} = \frac{S}{6}\mathrm{Id}_{\pm} - \mathrm{W}_{\pm} \geq 0$. A necessary condition is that $|\mathrm{W}_{\pm}|^2 \leq \frac{S^2}{24}$ [80, Lemma 3.2]. Note that the converse is not true.

Lemma 2.3.4. *If* $\frac{S}{6}Id_{\pm} - \mathrm{W}_{\pm} \geq 0$ *then* $|\mathrm{W}_{\pm}|^2 \leq \frac{S^2}{6}$.

Proof. Let $\lambda_1, \lambda_2, \lambda_3$ be eigenvalues of W_+ then we have:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \text{ and } -\frac{S}{3} \leq \lambda_i \leq \frac{S}{6}.$$

Consider the function $f(a, b, c) = a^2 + b^2 + c^2$ then we want to maximize f on the plane $a + b + c = 0$ bounded by the the tube $-\frac{S}{3} \leq a, b, c \leq \frac{S}{6}$. Since the region is compact, the function attains its maximum.

Suppose (a, b, c) maximizes the function then we can assume $a \leq b \leq c$. Then $a \leq 0 \leq c$. If $a > -\frac{S}{3}$ then we can always increase the function by decreasing a and increasing either b or c. Thus $a = -\frac{S}{3}$ and the result follows. □

Using the elementary technique above, we also obtain the following estimate.

Lemma 2.3.5. *In dimension 4, for* $\mathrm{E} = \mathrm{Rc} - \frac{Sg}{4}$, $|\mathrm{W}(\mathrm{E}, \mathrm{E})| \leq \frac{1}{2\sqrt{3}} |\mathrm{W}||\mathrm{E}|^2$.

Proof. Since E is symmetric, we can choose an orthonormal basis that diagonalizes both E and g. Then,

$$W(E,E) = \sum_{i<j} W_{ijij}\lambda_i\lambda_j = W_{1212}(\lambda_1\lambda_2+\lambda_3\lambda_4)+W_{1313}(\lambda_1\lambda_3+\lambda_2\lambda_4)+W_{1414}(\lambda_1\lambda_4+\lambda_3\lambda_2).$$

Algebraically, W and E are independent so we can think of W as fixed and try to maximize $f(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ given the constraint $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i^2 = |E|^2$ Towards that end, we repeatedly apply the Lagrange-Euler equation to obtain:

$$W_{1414}(-2\lambda_1-\lambda_2-\lambda_3)+W_{1313}(\lambda_3-\lambda_2)+W_{1212}(\lambda_2-\lambda_3) = 2\mu(2\lambda_1+\lambda_2+\lambda_3),$$

$$W_{1313}(-\lambda_1-2\lambda_2-\lambda_3)+W_{1414}(\lambda_3-\lambda_1)+W_{1212}(\lambda_1-\lambda_3) = 2\mu(\lambda_1+2\lambda_2+\lambda_3),$$

$$W_{1212}(-\lambda_1-\lambda_2-2\lambda_3)+W_{1313}(\lambda_1-\lambda_2)+W_{1414}(\lambda_2-\lambda_1) = 2\mu(\lambda_1+\lambda_2+2\lambda_3).$$

Using $-W_{1212} = W_{1313} + W_{1414}$ we can rewrite these equations as

$$W_{1212}(\lambda_1+\lambda_2)+W_{1313}(\lambda_1+\lambda_3) = \mu(2\lambda_1+\lambda_2+\lambda_3),$$

$$W_{1212}(\lambda_1+\lambda_2)+W_{1414}(\lambda_2+\lambda_3) = \mu(\lambda_1+2\lambda_2+\lambda_3),$$

$$W_{1313}(\lambda_1+\lambda_3)+W_{1414}(\lambda_2+\lambda_3) = \mu(\lambda_1+\lambda_2+2\lambda_3).$$

It is obvious that the system reduces further to

$$W_{1212}(\lambda_1+\lambda_2) = \mu(\lambda_1+\lambda_2),$$

$$W_{1313}(\lambda_1+\lambda_3) = \mu(\lambda_1+\lambda_3),$$

$$W_{1414}(\lambda_2+\lambda_3) = \mu(\lambda_2+\lambda_3).$$

Case 0. μ is different from all W_{ijij}. Then the system can only be satisfied if $\lambda_1+\lambda_2 = \lambda_1+\lambda_3 = \lambda_2+\lambda_3 = 0$. So $\lambda_i = 0$ and $f = 0$.

Case 1. $W_{1212} = W_{1313} = W_{1414} = 0$ then $f = 0$.

Case 2. $W_{1212} = W_{1313} = \frac{-1}{2}W_{1414} \neq 0$ then the above system can be satisfied in two sub-cases:

Subcase 21: $\mu = W_{1212} = W_{1313}$ then $\lambda_2 + \lambda_3 = 0$ and consequently $\lambda_1 + \lambda_4 = 0$. Direct calculation yields that $|f| = \frac{|E|^2}{2} \frac{|W|}{2\sqrt{3}} = \frac{|W||E|^2}{4\sqrt{3}}$.

Subcase 22: $\mu = W_{1414}$ then $\lambda_1 + \lambda_3 = \lambda_1 + \lambda_2 = 0$. Thus direct calculation yields $|f| = \frac{|W||E|^2}{2\sqrt{3}}$.

Case 3. $W_{1212}, W_{1313}, W_{1414}$ are distinct and $W_{1212} = \mu$ then, similar to sub-case 22, $\lambda_1 = \lambda_2 = -\lambda_3 = -\lambda_4$ and $|f| = |E|^2|W_{1313} + W_{1414}| < \frac{|W||E|^2}{2\sqrt{3}}$.

Summarizing these cases we have $|f| \leq \frac{|W||E|^2}{2\sqrt{3}}$. □

Remark 2.3.2. *For a general dimension n and $E = Rc - \frac{S}{n}g$, it was proved that $|W(E, E)| \leq \sqrt{\frac{n-2}{2(n-1)}}|W||E|^2$ [60, Lemma 3.4]. If $n = 4$, the constant is $\frac{1}{\sqrt{3}}$.*

2.4 Variational Formulas

In this section, we collect several variational formulas (as δ is reserved to denote variation here, the divergence goes by div). Let (M, g) be a Riemannian manifold and v a symmetric $(2, 0)$ tensor. We consider the variation,

$$\delta(g) = v.$$

Then we have, for $V = \mathrm{tr}_g(v)$, [8, Theorem 1.174]:

$$2\langle(\delta\nabla)_X Y, Z\rangle_g = \nabla_X v(Y,Z) + \nabla_Y v(X,Z) - \nabla_Z v(X,Y), \tag{2.18}$$

$$\delta\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}) \tag{2.19}$$

$$(\delta \mathrm{R})(X,Y)Z = (\nabla_Y \delta(\nabla))(X,Z) - (\nabla_X \delta(\nabla))(Y,Z), \tag{2.20}$$

$$\begin{aligned} 2\delta(\mathrm{R})(X,Y,Z,U) &= \nabla^2_{Y,Z} v(X,U) + \nabla^2_{X,U} v(Y,Z) - \nabla^2_{X,Z} v(Y,U) - \nabla^2_{Y,U} v(X,Z) \\ &\quad + v(\mathrm{R}(X,Y)Z,U) - v(\mathrm{R}(X,Y)U,Z), \end{aligned} \tag{2.21}$$

$$\begin{aligned} \delta(\mathrm{R}_{ij}) &= \frac{1}{2}\nabla^l(\nabla_i v_{jl} + \nabla_j v_{il} - \nabla_l v_{ij}) - \frac{1}{2}\nabla_i\nabla_j V \\ &= -\frac{1}{2}\Delta_L v_{ij} - \mathrm{div}^*(\mathrm{div} v)_{ij} - \frac{1}{2}\nabla_i V_j, \end{aligned} \tag{2.22}$$

$$\delta(\mathrm{S}) = -\Delta V + \mathrm{div}(\mathrm{div}(v)) - \langle v, \mathrm{Rc}\rangle \tag{2.23}$$

$$\delta(d\mu) = \frac{V}{2}d\mu. \tag{2.24}$$

Here,

$$\Delta_L v_{ij} = \Delta v_{ij} + 2\mathrm{R}_{ikjl}v^{kl} - \mathrm{R}_{ik}v_{jk} - \mathrm{R}_{jk}v_{ik},$$

$$-2\mathrm{div}^*(\mathrm{div} v)_{ij} = \nabla_i(\mathrm{div} v)_j + \nabla_j(\mathrm{div} v)_i.$$

Now suppose M has boundary Σ with second fundamental form A_{ij}, mean curvature H, and the inward normal vector e_0. Then by [68, Section 3], with $\overline{\nabla}$ the induced connection on Σ and $i, j, k \neq 0$,

$$A_{ij} = \left\langle e_0, \nabla_{e_i} e_j \right\rangle = -\frac{1}{2}\partial_0 g_{ij},$$

$$\delta(e_0) = -\frac{1}{2}v_{00}e_0 - v_0^i e_i, \tag{2.25}$$

$$\begin{aligned} \delta(A_{ij}) &= \frac{1}{2}(\nabla_i v_{0j} + \nabla_j v_{0i} - \nabla_0 v_{ij} - A_{ij}v_{00}) \\ &= \frac{1}{2}(\overline{\nabla}_i v_{0j} + A_{ki}v_j^k + \overline{\nabla}_j v_{0i} + A_{kj}v_i^k - \nabla_0 v_{ij} - A_{ij}v_{00}), \end{aligned} \tag{2.26}$$

$$\delta(H) = -v^{ij}A_{ij} + g^{ij}\delta(A_{ij}) = \overline{\nabla}_i v_0^i - \frac{1}{2}(g^{ij}\nabla_0 v_{ij} + Hv_{00}), \tag{2.27}$$

$$\delta(d\mu_\Sigma) = \frac{1}{2}v_i^i d\mu_\Sigma. \tag{2.28}$$

With above formulae, it is easy to calculate variations of well-known functionals. For example, here is Perelman's energy [81],

$$\mathcal{F}(g, f) = \int_M (|\nabla f|^2 + \mathrm{S})e^{-f} dV.$$

Lemma 2.4.1. *Let $\delta g = v$ and $\delta f = \ell$ then,*

$$\delta\mathcal{F} = \int_M \Big[-v^{ij}(R_{ij} + \nabla_i\nabla_j f) + (\frac{1}{2}V - \ell)(2\Delta f - |\nabla f|^2 + \mathrm{S})\Big]e^{-f} dV.$$

Another example is the Einstein-Hilbert functional:

$$\mathcal{E}(g) = \mathrm{Vol}(M)^{\frac{2-n}{n}} \int_M \mathrm{S} d\mu.$$

Lemma 2.4.2. *If $\delta g = v$ and* S *is constant then*

$$\delta\mathcal{E} = Vol(M)^{\frac{2-n}{n}} \int_M \left\langle -\mathrm{Rc} + \frac{\mathrm{S}}{n} g, v \right\rangle d\mu. \tag{2.29}$$

If g is Einstein and the variation is volume-preserving then, the second variation,

$$\begin{aligned} \delta^2\mathcal{E} &= \frac{1}{2}\int_M \langle v, \Delta v + 2div^*(divv) + 2\mathrm{R}{*}v\rangle\, d\mu \\ &+ \frac{1}{2}\int_M \Big(2div^2 v - \Delta V - \frac{\mathrm{S}}{n}V\Big) V d\mu, \end{aligned} \tag{2.30}$$

where $(\mathrm{R} * v)_{ij} = \mathrm{R}_{iljp} v^{lp}$.

Furthermore, if the variation is conformal, i.e. $v = fg$, then

$$\delta^2\mathcal{E} = \frac{n-2}{2} \int_M \langle (1-n)\Delta f - \mathrm{S}f, f\rangle\, d\mu. \tag{2.31}$$

Definition 2.4.3. *A variation v is called transverse-traceless if $divv = 0 = V$.*

Remark 2.4.1. *A transverse-traceless variation can not be conformal.*

2.5 Conformal Transformation

Since conformal transformation is of general interest, we devote this section to collect its related formulas. Most of the computation are readily adjusted from the previous section. If (M, g) is a smooth Riemannian manifold and $u = e^f$ a smooth function, a conformal change is given by:

$$\tilde{g} = e^{2f} g.$$

Then, for any quantity $\mathfrak{D}$ with respect to g, the corresponding for $\tilde{g}$ will be $\widetilde{\mathfrak{D}}$. Adjusting formula (2.18), we have:

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(f)Y + Y(f)X - (X, Y)\nabla f. \tag{2.32}$$

Consequently, for $a = \mathrm{Hess} f - df \otimes df + \frac{1}{2}|\nabla f|^2 g$,

$$\widetilde{\mathrm{R}} = e^{2f}\mathrm{R} - e^{2f} a \circ g. \tag{2.33}$$

Also,

$$\begin{aligned}
d\widetilde{\mu} &= e^{nf} d\mu, \\
\widetilde{\Delta} h &= e^{-2f}\Big(\Delta h + (n-2)\nabla^k f \nabla_k h\Big), \\
\widetilde{\mathrm{W}} &= e^{2f}\mathrm{W}, \\
\widetilde{\mathrm{Rc}} &= \mathrm{Rc} - (n-2)a - \Big(\triangle f + \frac{n-2}{2}|\nabla f|^2\Big) g, \\
\widetilde{\mathrm{S}} &= e^{-2f}\Big(\mathrm{S} - 2(n-1)\triangle f - (n-2)(n-1)|\nabla f|^2\Big) \\
&= e^{-2f}\Big(\mathrm{S} - \frac{4(n-1)}{n-2} e^{-\frac{n-2}{2} f} \triangle (e^{\frac{n-2}{2} f})\Big).
\end{aligned}$$

When the covariant derivative is involved, the transformation is nontrivial.

Lemma 2.5.1. *Under the conformal change $\tilde{g} = u^2 g$, the divergence of the Weyl tensor is given by,*

$$\delta\widetilde{\mathrm{W}}(X, Y, Z) = \delta \mathrm{W}(X, Y, Z) + (n-3)\mathrm{W}(\frac{\nabla u}{u}, X, Y, Z).$$

Proof. We provide two ways of doing calculation: by a coordinate and a frame. First, let $\{e_i\}_{i=1}^n$ be a normal coordinate (note that it does not stay normal under a conformal transformation). In fact, by (2.32),

$$
\begin{aligned}
\widetilde{\nabla}_{e_i} e_j =& \nabla_i j + \frac{u_i}{u} e_j + \frac{u_j}{u} e_i - \frac{\nabla u \delta_{ij}}{u} \\
=& \frac{u_i}{u} e_j + \frac{u_j}{u} e_i - \frac{\nabla u \delta_{ij}}{u}.
\end{aligned}
$$

Then, we compute,

$$
\begin{aligned}
\delta\widetilde{\mathrm{W}}(jkl) =& \mathrm{trace}(w \to (\widetilde{\nabla}_w \widetilde{\mathrm{W}})(jkl)) = \tilde{g}^{ij} \left\langle (\widetilde{\nabla}_i \widetilde{\mathrm{W}})(jkl), e_j \right\rangle = \tilde{g}^{ii} \left\langle (\widetilde{\nabla}_i \widetilde{\mathrm{W}})(jkl), e_i \right\rangle \\
=& u^{-2} \widetilde{\nabla}_i (\widetilde{\mathrm{W}}_{ijkl}) - u^{-2} \widetilde{\mathrm{W}}(\widetilde{\nabla}_i i, j, k, l) - u^{-2} \widetilde{\mathrm{W}}(i, \widetilde{\nabla}_i j, k, l) \\
& - u^{-2} \widetilde{\mathrm{W}}(i, j, \widetilde{\nabla}_i k, l) - u^{-2} \widetilde{\mathrm{W}}(i, j, k, \widetilde{\nabla}_i l) \\
=& u^{-2}(u^2 \mathrm{W}_{ijkl}) - \mathrm{W}(2\frac{u_i}{u} e_i - \frac{\nabla u}{u}, j, k, l) - \mathrm{W}(i, \frac{u_i}{u} j + \frac{u_j}{u} i - \delta_{ij} \frac{\nabla u}{u}, k, l) \\
& - \mathrm{W}(i, j, \frac{u_k}{u} i + \frac{u_i}{u} k - \delta_{ik} \frac{\nabla u}{u}, l) - \mathrm{W}(i, j, k, \frac{u_l}{u} i + \frac{u_i}{u} l - \delta_{il} \frac{\nabla u}{u}) \\
=& \delta \mathrm{W}(j, k, l) + 2\mathrm{W}(\frac{\nabla u}{u}, j, k, l) + n\mathrm{W}(\frac{\nabla u}{u}, j, k, l) - 2\mathrm{W}(\frac{\nabla u}{u}, j, k, l) \\
& - 3\mathrm{W}(\frac{\nabla u}{u}, j, k, l) + \mathrm{W}(j, \frac{\nabla u}{u}, k, l) + \mathrm{W}(k, j, \frac{\nabla u}{u}, l) + \mathrm{W}(l, j, k, \frac{\nabla u}{u}) \\
=& \delta \mathrm{W}(j, k, l) + (n-3)\mathrm{W}(\frac{\nabla u}{u}, j, k, l).
\end{aligned}
$$

Calculation using the frame: Let $\{e_i\}_{i=1}^n$ be a normal orthonormal frame. Then, correspondingly, $\{\tilde{e}_i = \frac{e_i}{u}\}_{i=1}^n$ is an orthonormal frame with respect to $\tilde{g}$. Then,

$$
\begin{aligned}
\delta\widetilde{\mathrm{W}}(X, Y, Z) =& (\widetilde{\nabla}_{\tilde{e}_i} \widetilde{\mathrm{W}})(\tilde{e}_i, X, Y, Z) \\
=& \widetilde{\nabla}_{\tilde{e}_i} (\widetilde{\mathrm{W}}(\tilde{e}_i, X, Y, Z)) - \widetilde{\mathrm{W}}(\widetilde{\nabla}_{\tilde{e}_i} \tilde{e}_i, X, Y, Z) \\
& - \widetilde{\mathrm{W}}(\tilde{e}_i, \widetilde{\nabla}_{\tilde{e}_i} X, Y, Z) - \widetilde{\mathrm{W}}(\tilde{e}_i, X, \widetilde{\nabla}_{\tilde{e}_i} Y, Z) - \widetilde{\mathrm{W}}(\tilde{e}_i, X, Y, \widetilde{\nabla}_{\tilde{e}_i} Z).
\end{aligned}
$$

By equation (2.32),

$$
\begin{aligned}
\widetilde{\nabla}_{\tilde{e}_i} \tilde{e}_i =& \frac{1}{u} \nabla_i (\frac{e_i}{u}) + 2u^{-3} u_i e_i - u^{-2} \frac{\nabla u}{u} \\
=& u^{-2} \nabla_i e_i + u^{-3} u_i e_i - u^{-2} \frac{\nabla u}{u}, \\
\widetilde{\nabla}_{\tilde{e}_i} X =& u^{-1} \nabla_i X + u^{-2} u_i X + \nabla_X f \tilde{e}_i - \langle X, \tilde{e}_i \rangle \frac{\nabla u}{u}.
\end{aligned}
$$

Therefore, since $\{e_i\}_{i=1}^n$ is normal,

$$\begin{aligned}
\widetilde{\nabla}_{\tilde{e}_i}(\widetilde{W}(\tilde{e}_i, X, Y, Z)) =& \delta W(X, Y, Z) + W(\frac{\nabla u}{u}, X, Y, Z),\\
\widetilde{W}(\widetilde{\nabla}_{\tilde{e}_i}\tilde{e}_i, X, Y, Z) =& W(\frac{\nabla u}{u}, X, Y, Z) - nW(\frac{\nabla u}{u}, X, Y, Z)\\
\widetilde{W}(\tilde{e}_i, \widetilde{\nabla}_{\tilde{e}_i}X, Y, Z) =& 2W(\frac{\nabla u}{u}, X, Y, Z),\\
\widetilde{W}(\tilde{e}_i, X, \widetilde{\nabla}_{\tilde{e}_i}Y, Z) =& W(\frac{\nabla u}{u}, X, Y, Z) - W(Y, X, \frac{\nabla u}{u}, Z),\\
\widetilde{W}(\tilde{e}_i, X, Y, \widetilde{\nabla}_{\tilde{e}_i}Z) =& W(\frac{\nabla u}{u}, X, Y, Z) - W(Z, X, Y, \frac{\nabla u}{u}).
\end{aligned}$$

The result then follows immediately. □

Now we restrict to $n = 4$ and notice that,

$$\widetilde{S} = u^3(-6\Delta + S)u,$$

$$\widetilde{W}_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = u^{-4}\widetilde{W}_{abcd} = u^{-2}W_{abcd},$$

$$\det\widetilde{W_+} = u^{-6}\det W_+.$$

Then we have the following formula for a conformal change for the covariant derivative of the Weyl tensor.

Lemma 2.5.2. *Let (M^4, g) be a Riemmanian manifolds and $\tilde{g} = u^2 g$. Then,*

$$|\widetilde{\nabla}\widetilde{W}|^2 = u^{-6}|\nabla W|^2 + 18u^{-8}|\nabla u|^2|W|^2 - 10u^{-7}\nabla u\nabla|W|^2 + 16\,\langle \delta W, \iota_{\nabla u}W\rangle\,.$$

Proof. We calculate:

$$
\begin{aligned}
|\widetilde{\nabla}\widetilde{W}|^2 =& u^{-10}\Big((\widetilde{\nabla}_{e_i}\widetilde{W})_{abcd}\Big)^2,\\
(\widetilde{\nabla}_{e_i}\widetilde{W})_{abcd} =& \nabla_i(u^2 W_{abcd}) - u^2\Big(W(\widetilde{\nabla}_{e_i}a, b, c, d) + W(a, \widetilde{\nabla}_{e_i}b, c, d)\\
&+ W(a, b, \widetilde{\nabla}_{e_i}c, d) + W(a, b, c, \widetilde{\nabla}_{e_i}d)\Big)\\
=& u^2\nabla_i W_{abcd} - 2uu_i W_{abcd} + u\delta_{ia}W_{\nabla u bcd} - uW_{ibcd}u_a\\
&+ u\delta_{ib}W_{a\nabla u cd} - uW_{aicd}u_b + u\delta_{ic}W_{ab\nabla u d} - uW_{abid}u_c\\
&+ u\delta_{id}W_{abc\nabla u} - uW_{abci}u_d.
\end{aligned}
$$

Now by summation over all indices using Lemma 2.3.2, we have:

$$
\begin{aligned}
(\nabla_i W_{abcd})^2 &= |\nabla W|^2, & (u_i W_{abcd})^2 &= |\nabla u|^2|W|^2,\\
(\delta_{ia}W_{\nabla u bcd})^2 &= 4(W_{\nabla u bcd})^2 = 4|\nabla u|^2|W|^2, & (W_{ibcd}u_a)^2 &= |\nabla u|^2|W|^2,\\
2\nabla_i W_{abcd}u_i W_{abcd} &= \Big\langle \nabla|W|^2, \nabla u\Big\rangle, & \nabla_i W_{abcd}\delta_{ia}W_{\nabla u bcd} &= \langle \delta W, i_{\nabla u}W\rangle,\\
\nabla_i W_{abcd}W_{ibcd}u_a &= \Big\langle \nabla|W|^2, \nabla u\Big\rangle - \langle \delta W, i_{\nabla u}W\rangle, & u_i W_{abcd}\delta_{ia}W_{\nabla u bcd} &= |\nabla u|^2|W|^2,\\
u_i W_{abcd}W_{ibcd}u_a &= |\nabla u|^2|W|^2, & \delta_{ia}W_{\nabla u bcd}W_{ibcd}u_a &= |\nabla u|^2|W|^2,\\
\delta_{ia}W_{\nabla u bcd}\delta_{ib}W_{a\nabla u cd} &= -|\nabla u|^2|W|^2, & W_{ibcd}u_a W_{aicd}u_b &= -|\nabla u|^2|W|^2.
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\delta_{ia}W_{\nabla u bcd}W_{aicd}u_b = \delta_{ia}W_{\nabla u bcd}W_{abid}u_c = \delta_{ia}W_{\nabla u bcd}W_{abci}u_d = 0,\\
&\delta_{ia}W_{\nabla u bcd}\delta_{ic}W_{ab\nabla u d} = W_{\nabla u bid}W_{bid\nabla u} = \frac{1}{2}|\nabla u|^2|W|^2,\\
&W_{ibcd}u_a W_{abid}u_c = W_{ib\nabla u d}W_{\nabla u bid} = \frac{1}{2}|\nabla u|^2|W|^2.
\end{aligned}
$$

The result then follows. □

Now we calculate the conformal change of a quantity related to a Bochner-Weitzenbock's formula.

Lemma 2.5.3. *Let (M^4, g) be a Riemmanian manifolds and*

$$h = \Delta|\mathrm{W}_+|^2 - 2|\nabla \mathrm{W}_+|^2 - \mathrm{S}|\mathrm{W}_+|^2 + 36 det \mathrm{W}_+. \tag{2.34}$$

Then under the conformal change $\tilde{g} = u^2 g$ for any positive C^2-function u,

$$u^6 \tilde{h} = h - 20(\frac{|\nabla u|}{u})^2 |\mathrm{W}_+|^2 + 2\frac{\Delta u}{u}|\mathrm{W}_+|^2 + 10\frac{\nabla u}{u}\nabla|\mathrm{W}_+|^2 - 32u^{-1}\langle \delta \mathrm{W}_+, \iota_{\nabla u}\mathrm{W}_+\rangle. \tag{2.35}$$

Proof. We abuse notation here to let $\mathrm{W} = \mathrm{W}_+$ and calculate,

$$\begin{aligned}
\widetilde{\Delta}|\widetilde{\mathrm{W}}|^2 =& \widetilde{\Delta}(u^{-4}|\mathrm{W}|^2) = u^{-2}(\Delta(u^{-4}|\mathrm{W}|^2) - 2\frac{\nabla u}{u}\nabla(u^{-4}|\mathrm{W}|^2),\\
=& u^{-2}\Big(u^{-4}\Delta|\mathrm{W}|^2 + |\mathrm{W}|^2\Delta u^{-4} + 2\nabla u^{-4}\nabla|\mathrm{W}|^2\\
& 2|\mathrm{W}|^2\frac{\nabla u}{u}\nabla u^{-4} - 2u^{-4}\frac{\nabla u}{u}\nabla|\mathrm{W}|^2\Big),\\
=& u^{-6}\Delta|\mathrm{W}|^2 + 20u^{-8}|\mathrm{W}|^2|\nabla u|^2 - 4u^{-7}|\mathrm{W}|^2\Delta u\\
& - 10u^{-7}\nabla u\nabla|\mathrm{W}|^2 + 8u^{-8}|\nabla u|^2|\mathrm{W}|^2,\\
=& u^{-6}\Delta|\mathrm{W}|^2 + 28u^{-8}|\mathrm{W}|^2|\nabla u|^2 - 4u^{-7}|\mathrm{W}|^2\Delta u - 10u^{-7}\nabla u\nabla|\mathrm{W}|^2.\\
\widetilde{\mathrm{S}}|\widetilde{\mathrm{W}}|^2 =& u^{-6}\mathrm{S}|\mathrm{W}|^2 - 6u^{-7}|\mathrm{W}|^2\Delta u.
\end{aligned}$$

The result then follows by combining these equations with Lemma 2.5.2 which is also valid for $\mathrm{W}_\pm$. □

2.6 Hyper-surfaces and Warped Products

In this section, we state a few calculation tools involved with hyper-surfaces and warped products.

2.6.1 Coordinate Perspective

First, we start with a general computation which is similar to [35]. Consider $(N^n, g(s))$, $s \in (a,b)$, a manifold with an one-parameter family of metrics such that $\frac{d}{ds}g = 2v$. Let M be the manifold $N \times (a,b)$ induced with the metric $\overline{g} = ds^2 + g(s)$.

Remark 2.6.1. *The choices of v, $-\mathrm{Rc}(g(s))$ and $\frac{\partial_s f}{f}g$, correspond to the space-time construction for the Ricci flow and the warped product $\overline{g} = ds^2 + f^2(s)g$ respectively. For a general hyper-surface, it can be understood that $v = -A$, the second fundamental form.*

Lemma 2.6.1. *Let $\{e_i\}_{i=1}^n$ be a local coordinate on $g(s)$ and $\partial_s = e_0$ then*

$$\begin{aligned}
&\overline{\Gamma}_{ij}^0 = -v_{ij},\\
&\overline{\Gamma}_{ij}^k = \Gamma_{ij}^k,\\
&\overline{\Gamma}_{i0}^k = \overline{\Gamma}_{0i}^k = v_i^k,\\
&\overline{\Gamma}_{00}^k = \overline{\Gamma}_{i0}^0 = \overline{\Gamma}_{00}^0 = 0,\\
&\overline{\mathrm{R}}_{ijk}^l = \mathrm{R}_{ijk}^l + v_{jk}v_i^l - v_{ik}v_j^l,\\
&\overline{\mathrm{R}}_{i00}^l = \partial_s v_i^l + v_i^p v_p^l,\\
&\overline{\nabla}_{i,j}^2 f = \nabla_{i,j}^2 f - \Gamma_{ij}^0 \partial_0 f = \nabla_{i,j}^2 f + v_{ij} f_s,\\
&\overline{\nabla}_{i,0}^2 f = \partial_i \partial_0 f - v_i^k \partial_k f.
\end{aligned}$$

If the coordinate is chosen to be normal then

$$\begin{aligned}
&\overline{\mathrm{R}}_{0jk}^l = \overline{\mathrm{R}}_{0jkl} = -\nabla_k v_{jl} + \nabla_l v_{jk},\\
&\overline{\mathrm{Rc}}_{00} = -\partial_s V - |v|^2 \; (V = tr_{g(s)} v),\\
&\overline{\mathrm{Rc}}_{i0} = -\nabla_i V + \nabla^j v_{ij},\\
&\overline{\mathrm{Rc}}_{ij} = \mathrm{Rc}_{ij} - \partial_s v_{ij} - V v_{ij}.
\end{aligned}$$

Proof. First using (2.5) we calculate the Christoffel symbols. Then the curvature is computed by (2.6), and the variation of the Christoffel symbol by (2.19). Finally, the Hessian is,

$$\overline{\nabla}^2_{ij} = \frac{\partial^2}{\partial_i \partial_j} - \overline{\Gamma}^k_{ij}\partial_k.$$

□

2.6.2 Frame Perspective

Now, we change the perspective and consider $M = N^n \times_f F^p$ with the metric $g_M = \overline{g} = g_N + f^2 g_F$. The calculation here is comparable to [8, Section 9.J] or [83, Section 3.2].

Remark 2.6.2. *Depending on the choice of N, F, and f, the manifold can be considered as structurally different warped products as discussed later.*

The computation below makes use of Cartan's structure equations. Let $\{e_i\}$ ($\{\omega^i\}$) be a normal orthonormal (co)frame on N while $\{e_\alpha\}$ ($\{\omega^\alpha\}$) on F. Then

$$\overline{e}_i = e_i \text{ and } \overline{e}_\alpha = \frac{1}{f} e_\alpha \ (\ \overline{\omega}^i = \omega^i \text{ and } \overline{\omega}^\alpha = f\omega^\alpha \ )$$

are the orthonormal (co)frame for $\overline{g}$.

Thus, we have:

$$
\begin{aligned}
\overline{\omega}_i^k &= \omega_i^k, \\
\overline{\omega}_i^\alpha &= \frac{f_i}{f}\overline{\omega}^\alpha = f_i\omega^\alpha, \\
\overline{\omega}_\beta^\alpha &= \omega_\beta^\alpha, \\
\overline{\mathrm{R}}_i^j &= \mathrm{R}_i^j, \\
\overline{\mathrm{R}}_i^\alpha &= \frac{f_{ij}}{f}\overline{\omega}^j \wedge \overline{\omega}^\alpha - \frac{f_j}{f}\overline{\omega}_i^j \wedge \overline{\omega}^\alpha, \\
\overline{\mathrm{R}}_\alpha^\beta &= \mathrm{R}_\alpha^\beta + \frac{|\nabla_N f|^2}{f^2}\overline{\omega}^\alpha \wedge \overline{\omega}^\beta.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\overline{\mathrm{R}}(\overline{e_i}, \overline{e_j}, \overline{e_k}, \overline{e_l}) &= \mathrm{R}(e_i, e_j, e_k, e_l), \\
\overline{\mathrm{R}}(\overline{e}_\alpha, \overline{e}_i, \overline{e}_\beta, \overline{e}_j) &= \frac{-f_{ij}}{f}\delta_{\alpha\beta}, \\
\overline{\mathrm{R}}(\overline{e}_\alpha, \overline{e}_\gamma, \overline{e}_\beta, \overline{e}_\gamma) &= \frac{1}{f^2}\mathrm{R}(e_\alpha, e_\gamma, e_\beta, e_\gamma) - \frac{|\nabla_N f|^2}{f^2}\delta_{\alpha\beta}, \\
\overline{\mathrm{R}}(e_i, e_\alpha, e_i, e_j) &= \overline{\mathrm{R}}(e_\alpha, e_i, e_\alpha, e_\beta) = 0, \\
\overline{\mathrm{Rc}}(\overline{e}_\alpha, \overline{e}_\beta) &= -\frac{\triangle f}{f}\delta_{\alpha\beta} + \frac{1}{f^2}\mathrm{Rc}(e_\alpha, e_\beta) - (p-1)\frac{|\nabla_N f|^2}{f^2}\delta_{\alpha\beta}, \\
\overline{\mathrm{Rc}}(\overline{e}_i, \overline{e}_j) &= \mathrm{Rc}(e_i, e_j) - p\frac{f_{ij}}{f}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\nabla^2_{X,Y}\alpha &= \nabla_X\nabla_Y\alpha - \nabla_{\nabla_X Y}\alpha, \\
\nabla^2_{i,j}\alpha &= \nabla_i(\nabla_j\alpha) - \omega_j^k(e_i)\nabla_k\alpha.
\end{aligned}
$$

Thus, for a function Φ,

$$\begin{aligned}
\overline{\nabla}^2_{\bar{\alpha}\bar{\beta}}\Phi &= \frac{1}{f^2}\nabla^2_{\alpha\beta}\Phi + \frac{\nabla_N\Phi\nabla_N f}{f}\delta_{\alpha\beta}, \\
\overline{\nabla}^2_{\bar{\alpha}\bar{i}}\Phi &= \frac{1}{f}\partial_\alpha\partial_i\Phi, \\
\overline{\nabla}^2_{\bar{i},\bar{j}}\Phi &= \nabla^2_{i,j}\Phi, \\
\overline{\Delta}\Phi &= \triangle_N\Phi + \frac{1}{f^2}\Delta_F\Phi + p\frac{\nabla_N\Phi\nabla_N f}{f}.
\end{aligned}$$

Next, we'll show how the computation simplify for warped products.

2.6.3 Warped Product with an Interval Base

Given an interval $I = (a, b)$, and $(N, g(x))$, $x \in I$, let

$$M = N \times I, \overline{g} = h^2(x)dx^2 + f^2(x)g(x) = ds^2 + f^2(s)g(s).$$

Adapted to this setting, we have:

$$
\begin{aligned}
\overline{\omega}_i^k &= \omega_i^k, \\
\overline{\omega}_0^k &= f_s \omega^k, \\
\overline{\mathrm{R}}_i^j &= \mathrm{R}_i^j - (\frac{f_s}{f})^2 \omega^j \wedge \omega^i, \\
\overline{\mathrm{R}}_0^j &= \frac{f_{ss}}{f} \omega^0 \wedge \omega^j, \\
\overline{\mathrm{R}}(e_i, e_j, e_i, e_j) &= \frac{1}{f^2} \mathrm{R}(\overline{e_i}, \overline{e_j}, \overline{e_i}, \overline{e_j}) - (\frac{f_s}{f})^2, \\
\overline{\mathrm{R}}(e_i, \partial_s, e_j, \partial_s) &= \frac{-f_{ss}}{f} \delta_{ij}, \\
\overline{\mathrm{R}}(e_i, \partial_s, e_j, e_k) &= 0, \\
\overline{\mathrm{R}}(e_i, e_j, e_k, e_l) &= \frac{1}{f^2} \mathrm{R}(\overline{e_i}, \overline{e_j}, \overline{e_k}, \overline{e_l}), \\
\overline{\mathrm{Rc}}_{00} &= -(n-1) \frac{f_{ss}}{f} = -\frac{n-1}{h^3} (f'' h - h' f'), \\
\overline{\mathrm{Rc}}_{ii} - \frac{1}{f^2} \mathrm{Rc}_{ii} &= -(n-2) (\frac{f_s}{f})^2 - \frac{f_{ss}}{f} = -(n-2) (\frac{f'}{f h})^2 - \frac{1}{h^3} (f'' h - h' f'), \\
\overline{\mathrm{Rc}}_{ij} &= \frac{1}{f^2} \mathrm{Rc}_{ij}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
\overline{\nabla}^2_{00} \Phi &= \Phi_{ss}, \\
\overline{\nabla}^2_{0,i} \Phi &= \partial_s e_i \Phi - \frac{f_s}{f} (e_i \Phi), \\
\overline{\nabla}^2_{i,i} \Phi &= e_i e_i \Phi - \Sigma_{k=1}^n \omega_i^k (e_i)(e_k \Phi) + \frac{f_s}{f} (\Phi_s) = \frac{1}{f^2} \nabla_{i,i} \Phi + \frac{f_s}{f} \Phi_s, \\
\overline{\nabla}^2_{i,j} \Phi &= e_i e_j \Phi - \Sigma_{k=1}^n \omega_i^k (e_j)(e_k \Phi) = \frac{1}{f^2} \nabla_{i,j} \Phi, \\
\overline{\Delta} - \frac{1}{f^2} \Delta &= \partial_s^2 + (n-1) \frac{f_s}{f} \partial_s = \frac{1}{h^2} \partial_x^2 + \frac{1}{h^2} \Big((n-1) \frac{f'}{f} + \frac{h'}{h} \Big) \partial_x.
\end{aligned}
$$

2.6.4 Warped Product with a Manifold Base

Let $\overline{g} = g + f^2 dx^2$ be a warped product metric on $M = N \times I$. Then,

$$\begin{aligned}
\overline{\omega}_i^k &= \omega_i^k, \\
\overline{\omega}_i^0 &= \frac{f_i}{f}\omega^0, \\
\overline{\mathrm{R}}_i^j &= \mathrm{R}_i^j, \\
\overline{\mathrm{R}}_i^0 &= \frac{f_{ij}}{f}\omega^j \wedge \omega^0 - \frac{f_j}{f}\omega_i^j \wedge \omega^n, \\
\overline{\mathrm{R}}(\overline{e_i}, \overline{e_j}, \overline{e_k}, \overline{e_l}) &= \mathrm{R}(e_i, e_j, e_k, e_l), \\
\overline{\mathrm{R}}(e_i, \partial_s, e_j, \partial_s) &= \frac{-f_{ij}}{f}, \\
\overline{\mathrm{R}}(e_i, \partial_s, e_j, e_k) &= 0, \\
\overline{\mathrm{Rc}}(\partial_s, \partial_s) &= -\frac{\triangle f}{f}, \\
\overline{\mathrm{Rc}}_{ij} &= -\frac{f_{ij}}{f} + \overline{\mathrm{R}}_{ii}.
\end{aligned}$$

As before, for a function Φ,

$$\begin{aligned}
\overline{\nabla}^2_{00}\Phi &= \Phi_{00} + \frac{\nabla\Phi\nabla f}{f}, \\
\overline{\nabla}^2_{0,i}\Phi &= \Phi_{0i}, \\
\overline{\nabla}^2_{i,j}\Phi &= \nabla_{i,i}\Phi, \\
\overline{\Delta}\Phi &= \Delta\Phi + \Phi_{00} + \frac{\nabla\Phi\nabla f}{f}.
\end{aligned}$$

CHAPTER 3

FUNDAMENTALS OF THE RICCI FLOW

3.1 Existence

As mentioned earlier, the uniqueness and short-time existence of a Ricci flow follows immediately from DeTurck's trick . However, it generally develops finite-time singularities. We say that $(M, g(t))$, $t \in [0, T)$, is a maximal solution if it becomes singular at time T. In his first paper on this topic, Hamilton described a characterization of the curvature approaching the singular time:

Theorem 3.1.1. *[51, Theorem 14.1] Let $(M, g(t))$ $0 \leq t < T < \infty$ be a solution of the Ricci flow on a closed manifold. Then the solution can be extended past time T or*

$$\lim_{t \to T} \max_{M} |\mathrm{R}(x,t)| = \infty.$$

However, qualitatively the solution does not blow up too fast:

Lemma 3.1.2. (Doubling-time estimate) *If $(M, g(t))$ is a Ricci flow on a closed manifold and $Q_0 = \max_M |\mathrm{R}(x,0)|$ then for all $t \in [0, \frac{1}{16Q_0})$,*

$$\mathrm{R}(x,t) \leq 2Q_0.$$

For a proof, see [38, Lemma 6.1].

3.2 Evolution Equations

The Ricci flow is a deformation of the metric along the Ricci direction. The first step in understanding the flow is to observe how the geometry evolves.

3.2.1 Curvature

Here we collect evolution equations related to various notions of curvature. First, the curvature tensor satisfies the following equation [51, Theorem 7.1],

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{R}_{ijkl} &= \Delta \mathrm{R}_{ijkl} + 2(\mathrm{R}^2 + \mathrm{R}^\sharp)_{ijkl} && (3.1)\\
&\quad - g^{pq}(\mathrm{R}_{ip}\mathrm{R}_{pjkl} + \mathrm{R}_{jp}\mathrm{R}_{ipkl} + \mathrm{R}_{kp}\mathrm{R}_{ijpl} + \mathrm{R}_{lp}\mathrm{R}_{ijkp}),\\
\mathrm{R}^2(X,Y,Z,W) &= \frac{1}{2}\mathrm{R}(X,Y,e_p,e_q)\mathrm{R}(Z,W,e_p,e_q) && (3.2)\\
\mathrm{R}^\sharp(X,Y,Z,W) &= \mathrm{R}(X,e_p,Z,e_q)\mathrm{R}(Ye_p,W,e_q) - \mathrm{R}(X,e_p,W,e_q)\mathrm{R}(Y,e_p,Z,e_q). && (3.3)
\end{aligned}$$

It is possible to simplify the equation using the Uhlenbeck's trick. The main idea is to evolve the frame in calculation. To be precise, first we pick a vector bundle $V \to M$ isomorphic to the tangent bundle $TM \to M$ and a bundle isomorphism $\iota_0 : V \to TM$. By pulling back the metric on TM at a fixed initial time, we obtain a metric on the fiber of V. We let the isometry evolve by the equation

$$\frac{\partial}{\partial t}\iota(t) = \mathrm{Rc}(t) \circ \iota(t).$$

Here Rc is a bundle map $TM \to TM$. Then it can be shown that $\iota(t)$ pullbacks varying metric $g(t)$ on TM to the fixed metric on V. Consequently, the evolution equation of the pullback of the curvature tensor is,

$$\frac{\partial}{\partial t}\mathrm{R} = \Delta\mathrm{R} + 2(\mathrm{R}^2 + \mathrm{R}^\sharp) = \Delta\mathrm{R} + 2Q(R). \tag{3.4}$$

It is easy to see that $Q(\mathrm{R})$ can be seen as an algebraic curvature tensor and,

$$\mathrm{Rc}(Q(\mathrm{R}))_{ik} = \sum_{p,q}\mathrm{R}_{ipkq}\mathrm{Rc}_{pq} \tag{3.5}$$

$$\mathrm{S}(Q(\mathrm{R})) = |\mathrm{Rc}|^2. \tag{3.6}$$

Then we can write down the evolution equation for the Ricci curvature and the scalar curvature,

$$\frac{\partial}{\partial t}\mathrm{Rc}(X,Y) = \Delta\mathrm{Rc}(X,Y) + 2\sum_{p,q}\mathrm{R}(X,e_p,Y,e_q)\mathrm{Rc}(e_p,e_q) \tag{3.7}$$

$$\frac{\partial}{\partial t}\mathrm{S} = \Delta\mathrm{S} + 2|\mathrm{Rc}|^2. \tag{3.8}$$

Finally, the equation of the Weyl tensor can be deduced from (3.1) as in [32, Prop 1.1],

$$\begin{aligned}
\frac{\partial}{\partial t}\mathrm{W}(t)_{ijkl} =&\Delta(\mathrm{W}_{ijkl}) + 2(C_{ijkl} - C_{ijlk} + C_{ikjl} - C_{iljk})\\
&- g^{pq}(\mathrm{Rc}_{ip}\mathrm{W}_{qjkl} + \mathrm{Rc}_{jp}\mathrm{W}_{iqkl} + \mathrm{Rc}_{kp}\mathrm{W}_{ijql} + \mathrm{Rc}_{ip}\mathrm{W}_{qjkl})\\
&+ \frac{2}{(n-2)^2}g^{pq}(\mathrm{Rc}_{ip}\mathrm{Rc}_{qk}g_{jl} - \mathrm{Rc}_{ip}\mathrm{Rc}_{ql}g_{jk} + \mathrm{Rc}_{jp}\mathrm{Rc}_{ql}g_{ik} - \mathrm{Rc}_{jp}\mathrm{Rc}_{qk}g_{il})\\
&+ \frac{2\mathrm{S}}{(n-2)^2}(\mathrm{Rc}_{ik}g_{jl} - \mathrm{Rc}_{il}g_{jk} + \mathrm{Rc}_{jl}g_{ik} - \mathrm{Rc}_{jk}g_{il}) \qquad (3.9)\\
&+ \frac{2}{n-2}(\mathrm{R}_{ik}\mathrm{R}_{jl} - \mathrm{R}_{jk}\mathrm{R}_{il}) + \frac{2(S^2 - |\mathrm{Rc}|^2)}{(n-1)(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk}),\\
C_{ijkl} =&g^{pq}g^{rs}\mathrm{W}_{pijr}\mathrm{W}_{slkq}.
\end{aligned}$$

Remark 3.2.1. *Since (3.9) is calculated from (3.1), it does not use the Uhlenbeck's trick.*

3.2.2 Geometric Quantities

It is also of great interest is to study how geometric operators like the Laplacian and quantities such as distance and volume evolve along the flow.

The Laplacian on functions and volume form evolve by,

$$\frac{\partial}{\partial t}(\Delta(t)) = 2\mathrm{R}_{ij}\cdot\nabla_i\nabla_j,$$
$$\frac{\partial}{\partial t}d\mu(t) = -\mathrm{S}d\mu(t).$$

If $\gamma : [a,b] \mapsto M$ is a fixed path then its length at time t is given by

$$L(t) = \int_a^b |\frac{d\gamma}{du}(u)|_{g(t)} du = \int_\gamma ds.$$

Differentiating yields

$$\frac{\partial L}{\partial t} = \frac{1}{2}\int_a^b |\frac{d\gamma}{du}|^{-1} \frac{\partial g}{\partial t}(\frac{d\gamma}{du}, \frac{d\gamma}{du}) du = -\int_\gamma \mathrm{Rc}(\dot\gamma, \dot\gamma) ds.$$

Thus,

$$\min_\gamma \Big(-\int_\gamma \mathrm{Rc}(\dot\gamma,\dot\gamma)ds \Big) \dotplus \frac{\partial^-}{\partial t}|_{t=t_0}\, d(x,y)$$
$$\leq \max_\gamma \Big(-\int_\gamma \mathrm{Rc}(\dot\gamma,\dot\gamma)ds \Big) \dotplus \frac{\partial_-}{\partial t}|_{t=t_0}\, d_t(x,y)$$

where the extrema are taken over all minimal geodesics, with respect to $g(t_0)$, γ joining x to y.

Remark 3.2.2. *The distance function might not be smooth in t for fixed x, y but at least Lipschitz continuous. Thus, the inequalities are understood in the sense of limsup and liminf of forward (superindex) or backwark (lowerindex) quotients. For a more detailed discussion, see [56, Lemma 17.3] and [36, Section 18.1]. If* $P(t) = \sup_M |\mathrm{Rc}(t)|$ *then it follows that,*

$$\frac{\partial^+ d(x,y)}{\partial t} \leq P(t) d(x,y),$$
$$|\ln \frac{d_{t_2}(x,y)}{d_{t_1}(x,y)}| \leq \int_{t_1}^{t_2} P(t).$$

3.2.3 In Dimension Four

Here we collect some evolution equations for quantities in dimension four.

Lemma 3.2.1. *Let* $(M^4, g(t))$, $0 \leq t < T \leq \infty$, *be a solution to Ricci flow, and the curvature operator is decomposed as in (2.12). Then, for a moving frame,*

$$\frac{\partial}{\partial t}\mathrm{W}^+ = \Delta \mathrm{W}^+ + 2(\mathrm{W}^+)^2 + 4(\mathrm{W}^+)^\sharp + 2(CC^T - \frac{1}{3}|C|^2 I^+). \tag{3.10}$$

Proof. By [71, Prop. 4.6], for a moving (in time) frame (Uhlenbeck's trick), we have the following equations,

$$\frac{\partial}{\partial t}A^+ = \Delta A^+ + 2(A^+)^2 + 4(A^+)^\sharp + 2CC^T, \tag{3.11}$$

$$A^+ = \mathrm{W}^+ + \frac{\mathrm{S}}{12}I^+, \tag{3.12}$$

$$\mathrm{S} = 4\mathrm{tr}(A^+). \tag{3.13}$$

Furthermore, if A^+ is diagonalized with eigenvalues a_1, a_2, a_3 then

$$(A^+)^2 = \mathrm{diag}(a_1^2\ a_2^2\ a_3^2), \tag{3.14}$$

$$(A^+)^\sharp = \mathrm{diag}(a_2a_3\ a_1a_3\ a_1a_2). \tag{3.15}$$

From (3.11), (3.13), (3.14), and (3.15), we arrive at,

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{\mathrm{S}}{4}) =& \frac{\partial}{\partial t}\mathrm{tr}(A^+) = \Delta\mathrm{tr}(A^+) + 2\mathrm{tr}((A^+)^2) + 4\mathrm{tr}(A^\sharp) + 2\mathrm{tr}(CC^T)\\ =& \Delta\mathrm{tr}(A^+) + 2(\mathrm{tr}A^+)^2 + 2|C|^2.\end{aligned}$$

Thus, by (3.12), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathrm{W}^+ =& \frac{\partial}{\partial t}A^+ - \frac{1}{3}(\frac{\partial}{\partial t}\frac{\mathrm{S}}{4})I^+,\\ =& \Delta A^+ + 2(A^+)^2 + 4(A^+)^\sharp + 2CC^T - \frac{1}{3}(\Delta\mathrm{tr}(A^+) + 2(\mathrm{tr}A^+)^2 + 2|C|^2)I^+\\ =& \Delta\mathrm{W}^+ + 2[(A^+)^2 - \frac{(\mathrm{tr}A^+)^2}{3}I^+] + 4(A^+)^\sharp + 2(CC^T - \frac{1}{3}|C|^2I^+).\end{aligned} \tag{3.16}$$

If we denote $\lambda_i = a_i - \frac{\mathrm{S}}{12}$, for $i = 1, 2, 3$, then they are eigenvalues of W^+ and we calculate each term in the diagonal of $(A^+)^2 - \frac{(\mathrm{tr}A^+)^2}{3}I^+$ to be,

$$(\lambda_i + \frac{\mathrm{S}}{12})^2 - \frac{1}{3}\frac{\mathrm{S}^2}{16} = \lambda_i^2 + \frac{\lambda_i\mathrm{S}}{6} - \frac{\mathrm{S}^2}{72}.$$

In addition, each term in the diagonal of $2(A^+)^\sharp$ is exactly,

$$2(\lambda_j + \frac{\mathrm{S}}{12})(\lambda_k + \frac{\mathrm{S}}{12}) = 2\lambda_j\lambda_k - \frac{\mathrm{S}\lambda_i}{6} + \frac{\mathrm{S}^2}{72}.$$

Therefore, (3.16) reduces to,

$$\frac{\partial}{\partial t}\mathrm{W}^+ = \Delta \mathrm{W}^+ + 2(\mathrm{W}^+)^2 + 4(\mathrm{W}^+)^\sharp + 2(CC^T - \frac{1}{3}|C|^2 I^+).$$

□

Remark 3.2.3. *Our convention agrees with [71] but differs from [52].*

The following result will come in handy later.

Lemma 3.2.2. *For a four-dimensional Riemmanian manifold (M, g), if the curvature is represented as in (2.12), then,*

$$\left\langle \mathrm{W}^+, CC^T \right\rangle = \frac{1}{4}\left\langle \mathrm{W}^+, \mathrm{Rc} \circ \mathrm{Rc} \right\rangle. \tag{3.17}$$

Proof. Since the equation is certainly coordinate free, it suffices to show it for a particular basis, namely one constructed by eigenvectors of Rc. With that basis, let $\alpha_i^\pm$, $i = 1, 2, 3$ as in (2.11) be a basis of $\Lambda_2^\pm$. Then C is diagonalized and,

$$\begin{aligned} C(\alpha_1^+, \alpha_1^-) =& \frac{1}{2}\mathrm{R}(12+34, 12-34) = \frac{1}{2}(\mathrm{R}_{1212} - \mathrm{R}_{3434}) \\ =& \frac{1}{4}(\mathrm{Rc}_{11} + \mathrm{Rc}_{22} - \mathrm{Rc}_{33} - \mathrm{Rc}_{44}). \end{aligned}$$

Therefore,

$$\begin{aligned} 4(CC^T)(\alpha_1^+, \alpha_1^+) =& \frac{1}{4}(\mathrm{S}^2 - 4(\mathrm{Rc}_{11} + \mathrm{Rc}_{22})(\mathrm{Rc}_{33} + \mathrm{Rc}_{44})) \\ =& \frac{\mathrm{S}^2}{4} - (\mathrm{Rc}_{11} + \mathrm{Rc}_{22})(\mathrm{Rc}_{33} + \mathrm{Rc}_{44}) - \mathrm{Rc}_{11}\mathrm{Rc}_{22} - \mathrm{Rc}_{33}\mathrm{Rc}_{44} \\ & + \mathrm{Rc}_{11}\mathrm{Rc}_{22} + \mathrm{Rc}_{33}\mathrm{Rc}_{44} \\ =& \frac{\mathrm{S}^2}{4} - \frac{1}{2}(\mathrm{S}^2 - |\mathrm{Rc}|^2) + \mathrm{Rc}_{11}\mathrm{Rc}_{22} + \mathrm{Rc}_{33}\mathrm{Rc}_{44} \\ =& \frac{\mathrm{S}^2}{4} - \frac{1}{2}(\mathrm{S}^2 - |\mathrm{Rc}|^2) + (\mathrm{Rc} \circ \mathrm{Rc})(\alpha_1, \alpha_1). \end{aligned}$$

Similar calculation holds for α_2^+ and α_3^+. As W^+ is traceless, we obtain,

$$\left\langle \mathrm{W}^+, CC^T \right\rangle = \frac{1}{4} \left\langle \mathrm{W}^+, \mathrm{Rc} \circ \mathrm{Rc} \right\rangle.$$

□

Theorem 3.2.3. *Let $(M^4, g(t))$, $0 \le t < T \le \infty$, be a solution to the Ricci flow then we have following evolution equation,*

$$(\frac{\partial}{\partial t} - \Delta)|\mathrm{W}^+|^2 = -2|\nabla \mathrm{W}^+|^2 + 36 det_{\Lambda_+^2} \mathrm{W}^+ + \left\langle \mathrm{Rc} \circ \mathrm{Rc}, \mathrm{W}^+ \right\rangle. \tag{3.18}$$

Proof. The calculation below is done for a local moving (in time) normal orthonormal (in space) frame (using the Uhlenberk's trick). First, since the pull-back metric is fixed, we observe,

$$\begin{aligned} \frac{\partial}{\partial t}|\mathrm{W}^+|^2 =& \frac{\partial}{\partial t} \sum_{i,j,k,l} (\mathrm{W}^+_{ijkl})^2 \\ =& \left\langle \mathrm{W}^+, \frac{\partial}{\partial t}\mathrm{W}^+ \right\rangle. \\ \Delta|\mathrm{W}^+|^2 =& 2|\nabla \mathrm{W}^+|^2 + 2\left\langle \mathrm{W}^+, \Delta \mathrm{W}^+ \right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} (\frac{\partial}{\partial t} - \Delta)|\mathrm{W}^+|^2 =& -2|\nabla \mathrm{W}^+|^2 + 2\left\langle \mathrm{W}^+, (\frac{\partial}{\partial t} - \Delta)\mathrm{W}^+ \right\rangle \\ =& -2|\nabla \mathrm{W}^+|^2 + 2\left\langle \mathrm{W}^+, 2(\mathrm{W}^+)^2 + 4(\mathrm{W}^+)^\sharp + 2(CC^T - \frac{1}{3}|C|^2 I^+) \right\rangle. \end{aligned}$$

We use Lemma 3.2.1 in the second step. By (3.14) and (3.15) and that W^+ is traceless, we have,

$$\begin{aligned} 2\left\langle \mathrm{W}^+, 2(\mathrm{W}^+)^2 \right\rangle =& 12 \det \mathrm{W}^+, \\ 2\left\langle \mathrm{W}^+, 4(\mathrm{W}^+)^\sharp \right\rangle =& 24 \det \mathrm{W}^+, \\ \left\langle \mathrm{W}^+, \mathrm{tr}|C|^2 I^+ \right\rangle =& 0. \end{aligned}$$

Moreover, applying (3.17) yields,

$$2\left\langle W^+, 2CC^T\right\rangle = \langle W^+, \mathrm{Rc}\circ\mathrm{Rc}\rangle.$$

The result then follows. □

Remark 3.2.4. *The Weyl tensor is considered as the traceless part of the curvature operator (module out the Ricci and scalar components). Thus, it is interesting to compare the above calculation with the evolution equation for the traceless part of the Ricci curvature $f = |E|^2$,*

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)f^2 =& -2|\nabla \mathrm{Rc}|^2 + \frac{|\nabla S|^2}{2} + \frac{2}{3}Sf - 4E^3 + 4W(E,E)\\ =& -2\nabla f\nabla(\ln S) - \frac{2}{S^2}|S\nabla\mathrm{Rc} - \mathrm{Rc}\nabla S|^2\\ &+ 2f^2(2|\nabla(\ln S)|^2 + \frac{S}{3}) - 4E^3 + 4W(E,E).\end{aligned}$$

This follows from, see [24],

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)|\mathrm{Rc}|^2 =& -|\nabla\mathrm{Rc}|^2 + 4R(\mathrm{Rc},\mathrm{Rc}),\\ R(\mathrm{Rc},\mathrm{Rc}) =& \frac{1}{n-2}(\frac{2n-1}{n-1}S|\mathrm{Rc}|^2 - 2\mathrm{Rc}^3 - \frac{S^3}{n-1}) + W(\mathrm{Rc},\mathrm{Rc}),\\ \mathrm{Rc}^3 =& E^3 + \frac{3}{n}SE^2 + \frac{S^3}{n^2}.\end{aligned}$$

Corollary 3.2.4. *Let $(M^4, g(t))$, $0 \le t < T \le \infty$, be a Ricci flow solution, then*

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)(\frac{|W^+|^2}{S^2}) =& -\frac{2}{S^4}|S\nabla W^+ - W^+\nabla S|^2 + \left\langle \nabla(\frac{|W^+|^2}{S^2}), \nabla \ln S^2\right\rangle\\ &+ 36\frac{det_{\Lambda^2_+}W^+}{S^2} + \frac{\langle \mathrm{Rc}\circ\mathrm{Rc}, W^+\rangle}{S^2} - 4\frac{|W^+|^2|\mathrm{Rc}|^2}{S^3}.\end{aligned} \tag{3.19}$$

Proof. Notice that

$$\begin{aligned}\frac{\partial}{\partial t}(\frac{A}{B}) =& \frac{B\frac{\partial}{\partial t}A - A\frac{\partial}{\partial t}B}{B^2},\\ \Delta(\frac{A}{B}) =& \frac{B\Delta A - A\Delta B}{B^2} - \left\langle \nabla(\frac{A}{B}), \nabla(\ln B^2)\right\rangle.\end{aligned}$$

Then applying the evolution equation $\frac{\partial}{\partial t}\mathrm{S}^2 = \Delta \mathrm{S}^2 - 2|\nabla \mathrm{S}|^2 + 4\mathrm{S}|\mathrm{Rc}|^2$ and Theorem 3.2.3 yields the statement. □

Remark 3.2.5. *On a GRS, the equation becomes*

$$-\Delta_f\left(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\right) = -\frac{2}{\mathrm{S}^4}|\mathrm{S}\nabla \mathrm{W}^+ - \mathrm{W}^+\nabla \mathrm{S}|^2 + \left\langle \nabla\left(\frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}\right), \nabla \ln \mathrm{S}^2 \right\rangle + 36\frac{det_{\Lambda^2_+}\mathrm{W}^+}{\mathrm{S}^2} + \frac{\langle \mathrm{Rc}\circ \mathrm{Rc}, \mathrm{W}^+\rangle}{\mathrm{S}^2} - 4\frac{|\mathrm{W}^+|^2|\mathrm{Rc}|^2}{\mathrm{S}^3}.$$

Proposition 3.2.5. *Let $(M^4, g(t))$, $0 \le t < T \le \infty$, be a Ricci flow solution on a closed manifold M. If $det_{\Lambda^2_+}\mathrm{W}^+$ is nonpositive along the flow then there exists a constant $C = C(g(0))$ such that $\frac{|\mathrm{W}^+|}{\mathrm{S}} < C$ is preserved along the flow.*

Proof. Denote $f = \frac{|\mathrm{W}^+|^2}{\mathrm{S}^2}$ and $h = \frac{|\mathrm{Rc}|^2}{\mathrm{S}^2}$. At a given point, we pick an orthonormal basis which diagonalizes the metric g and the Ricci tensor Rc simultaneously. Then by direct calculation,

$$\begin{aligned}\langle \mathrm{Rc}\circ \mathrm{Rc}, \mathrm{W}^+\rangle =& (\mathrm{W}_{1234} + \mathrm{W}_{1212})(\mathrm{R}_{11}\mathrm{R}_{22} + \mathrm{R}_{33}\mathrm{R}_{44}) \\ &+ (\mathrm{W}_{1342} + \mathrm{W}_{1313})(\mathrm{R}_{11}\mathrm{R}_{33} + \mathrm{R}_{22}\mathrm{R}_{44}) \\ &+ (\mathrm{W}_{1423} + \mathrm{W}_{1414})(\mathrm{R}_{11}\mathrm{R}_{44} + \mathrm{R}_{33}\mathrm{R}_{22}).\end{aligned}$$

Therefore, by elementary inequalities,

$$\langle \mathrm{Rc}\circ \mathrm{Rc}, \mathrm{W}^+\rangle \le \frac{\sqrt{3}}{2}|\mathrm{W}^+||\mathrm{Rc}|^2.$$

At the maximum of f, since $\det_{\Lambda^2_+}\mathrm{W}^+ \le 0$,

$$\frac{\partial}{\partial t}f_{max} \le \mathrm{S}fh\left(\frac{\sqrt{3}}{2\sqrt{f}} - 4\right).$$

Consequently, if $f_{max} > \frac{3}{64}$ then $\frac{\partial}{\partial t}f_{max} < 0$. The result follows by the maximum principle. □

Remark 3.2.6. *Without the assumption on $det_{\Lambda^2_+}\mathrm{W}^+$ then we have the following inequality:*

$$\frac{\partial}{\partial t}f_{max} \le \mathrm{S}\sqrt{f}\left(2\sqrt{6}f - 4h\sqrt{f} + \frac{\sqrt{3}h}{2}\right).$$

3.3 Convergence

A key step in the theory of the Ricci flow is to obtain a limit in an appropriate sense. In this section, we describe that process. Our main references are [83, Chapter 10], [46] and [1, Chapter 8].

3.3.1 Gromov-Hausdorff Distance

First, in order to talk about the convergence of manifolds, we need to develop a notion about how to compare manifolds with different geometries. The appropriate concept is the Gromov-Hausdorff distance. $B(P, r)$ denotes a ball of radius r around P.

Definition 3.3.1. *Suppose Z is a metric space, A_1, A_2 two subsets of Z, then the Hausdorff distance between them is defined as:*

$$d_H(A_1, A_2) = \inf\{\, r \,|A_2 \in B(A_1, r) \ \textit{and}\ A_1 \in B(A_2, r)\}.$$

Suppose X, Y are two metric spaces, the Gromov-Hausdorff distance is defined as,

$$d_{GH}(X, Y) = \inf_{i,j,Z}\{\, d_H(i(X), j(Y)) \mid i : X \mapsto Z, j : Y \mapsto Z \ \textit{are isometric embeddings}\}$$

Suppose (X, x),(Y, y) are pointed metric spaces, the pointed GH distance is defined as,

$$d_{GH}((X, x), (Y, y)) = \inf\{d_{GH}(X, Y) + d(x, y)\}.$$

Here $d(x, y)$ is calculated according to the isometric embedding.

In practice, the following notion is useful.

Definition 3.3.2. *Let (X, d_X, x) and (Y, d_Y, y) be pointed metric spaces. A map f : $(X, x) \mapsto (Y, y)$ is called an ϵ-pointed GH approximation if*

$$(f(x), y) < \epsilon,$$
$$B(y, \frac{1}{\epsilon}) \subset B(f[B(x, \frac{1}{\epsilon})], \epsilon),$$
$$|d(x_1, x_2) - d(f(x_1), f(x_2))| < \epsilon \text{ for all } x_1, x_2 \in B(x, \frac{1}{\epsilon}).$$

Remark 3.3.1. *The 2nd condition says that f maps X to almost all of Y while the 3rd condition essential implies f is almost an isometry. Then the pointed GH distance is equivalent to the infimum of ϵ such that there exist ϵ-pointed GH approximations f : $(X, x) \mapsto (Y, y)$ and $g : (Y, y) \mapsto (X, x)$.*

It is not difficult to show that these notions of distance satisfy the traditional axioms including the triangle inequality. Moreover, using this formulation, Gromov proved in the 80s the following result:

Definition 3.3.3. *A family (X_i, x_i) of path metric spaces is precompact if for each $r > 0$, the family of balls $B(x_i, r) \in X_i$ is precompact with respect to the GH distance.*

Theorem 3.3.4. [46, Theorem 5.3] *The set of n-dimensional pointed Riemannian manifolds with Ricci curvature uniformly bounded below is precompact with respect to the pointed GH topology*

Remark 3.3.2. *The limit is actually a length space with curvature bounded from below in the sense of an Alexandrov space [16].*

A closer look at the proof of that theorem reveals that the Ricci curvature bound is mostly used to obtain volume estimates. To be precise, let's explain the key lemma of that theorem.

Definition 3.3.5. *For each $\epsilon > 0, r > 0$ let $N(\epsilon, r, X)$ be the maximum number of disjoint balls of radius ϵ that fits within the ball of radius r centered at an $x \in X$.*

The following result relates the precompactness with the boundedness of $N(\epsilon, r, M)$.

Lemma 3.3.6. *[46, Proposition 5.2] A family (X_i, x_i) of pointed path metric space is precompact iff each function $N(\epsilon, r, \cdot)$ is bounded on X_i. In this case, the family is relatively compact, i.e, each sequence in the X_i admits a subsequence that converges to a complete, locally compact path metric space in the GH topology.*

The lemma clearly shows that estimates on the volume of small balls are the key to prove compactness. Along a Ricci flow, there are certain situations when we obtain volume estimates without using the bound on Ricci curvature. An example is the following GH convergence for gradient shrinking ricci solitons.

Definition 3.3.7. *A normalized gradient shrinking Ricci soliton is a triple (M, g, f) such that,*

$$\mathrm{Rc} + Hess f = \frac{1}{2} g.$$

The entropy is given by,

$$\mu(g) = \int_M (2\Delta f - |\nabla f|^2 + \mathrm{S} + f - n)(4\pi)^{-n/2} e^{-f}.$$

To see why the formula is well-defined, consult [58, Section 2].

Theorem 3.3.8. *[58, Theorem 2.3] Let (M_i, g_i, f_i) be a sequence of normalized gradient shrinking Ricci solitons with entropy uniformly bounded below $\mu(g_i) \geq \mu > -\infty$ then the sequence is volume non-collapsed at finite distances from the base points (minimum of f_i) and a subsequence converges to a complete metric space in the pointed GH topology.*

Remark 3.3.3. *The volume estimates in this theorem nevertheless come from the volume comparison theorem for the Bakry-Emery Ricci tensor.*

3.3.2 Smooth Convergence

When there is control over the curvature, it is possible to obtain smooth convergence.

Definition 3.3.9. *(**Smooth Cheeger-Gromov convergence**) A sequence (M_i, g_i, p_i) of complete pointed Riemannian manifolds converges to a pointed Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ if there exists:*

1. *An exhaustion $U_i \subset M_\infty$ with $p_\infty \in U_i$.*
2. *A sequence of diffeomorphisms $\Phi_i : U_i \mapsto V_i \subset M_i$ with $\Phi(p_\infty) = p_i$ such that $(\Phi_i^* g_i)$ converges in C^∞-topology to g_∞ on compact subsets in M_∞.*

The following theorem gives necessary criteria, curvature bound at each order and lower injectivity radius, to obtain a smooth Cheeger-Gromov convergence.

Theorem 3.3.10. *(**Cheeger-Gromov Compactness Theorem**) Let (M_i, g_i, p_i) be a sequence of complete pointed Riemannian manifolds satisfying*

1. *$|\nabla^p Rm_{g_i}| < C_p$ on M_i for each $p \geq 0$*
2. *$inj_{g_i}(p_i) \geq \kappa$ for some uniform $\kappa > 0$*

Then there exists a subsequence that converges in the smooth Cheeger-Gromov sense to a complete pointed manifold $(M_\infty, g_\infty, p_\infty)$.

In the theory of the Ricci flow, due to Shi's estimates [92](see also [56]), Hamilton proved the following version.

Theorem 3.3.11. *[55, Theorem 1.2] Suppose* $(M_i, g_i(t), x_i)_{i\in N}$, $t \in (\alpha,\beta) \ni 0$, *is a sequence of complete pointed Ricci flow solutions satisfying:*

1. $|\mathrm{R}(g_i(t))|_{g_i(t)} \leq C$ *on* $M_i \times (\alpha,\beta)$,
2. $inj_{g_i(0)}(x_i) \geq \delta > 0$.

Then the sequence sub-converges to a pointed complete solution of the Ricci flow $(M_\infty, g_\infty(t), x_\infty)$, $t \in (\alpha, \omega)$.

Remark 3.3.4. *The curvature bound can be replaced by various local uniform bounds at the expense of the completeness, see [95].*

The lower bound on injectivity radius is intrinsically related to the lower bound on the volume ratio.

Theorem 3.3.12. *(**Cheeger-Gromov-Taylor**) For any constant* $c > 0, s > 0, n \in N$. *there exists a constant* $\delta_0 > 0$ *such that the following holds. Suppose* (M, g) *is a complete Riemannian manifold with* $|\mathrm{R}| < 1$ *and* p *is a point such that, for all* $r \in (0, s]$,

$$\frac{Vol(B(p,r))}{r^n} \geq c.$$

Then we have,

$$inj(p) \geq \delta_0.$$

Remark 3.3.5. *For a proof, see [38, 5.42].*

For the Ricci flow, the lower bound on the volume ratio then follows from Perelman's non-collapsing result, Theorem 3.4.7. Therefore, we obtain the following convergence result.

Definition 3.3.13. *Given a constant* $1 \leq C < \infty$ *let*

$$M_C = \{(x,t) : |\mathrm{R}(x,t)| \geq \frac{1}{C} \max_M |\mathrm{R}(.,t)|\}. \tag{3.20}$$

From a sequence (x_i, t_i) *of a solution* $(M, g(t))$*, the parabolic dilation is defined as, for* $K_i = |\mathrm{R}(x_i, t_i)|$,

$$g_i(t) = K_i g(t_i + \frac{t}{K_i}). \tag{3.21}$$

Theorem 3.3.14. *[38, Theorem 8.4] Let* $(M, g(t))$, $0 \leq t < T < \infty$*, be a maximal solution to the Ricci flow on a closed manifold. If* (x_i, t_i) *is a sequence satisfying (3.20), then* $(M, g_i(t), x_i)$ *as defined by (3.21) sub-converges uniformly in every* C^k*-norm on compact sets to a complete solution* $(M_\infty, g_\infty, x_\infty)$ *of the Ricci flow.*

3.4 Entropy functionals

In this section, we recall the definition and basic properties of Perelman's functionals along a Ricci flow.

3.4.1 Motivation and Definition

This subsection follows the discussion in [77]. On a closed manifold the heat equation $\partial_t u = \Delta u$ is the L^2-gradient flow of the Dirichlet functional,

$$D(u) = \int_M \frac{1}{2} |\nabla u|^2 dV,$$
$$\partial_t D = \int_M -(\triangle u)^2 dV \leq 0.$$

The Nash entropy is, for $u = e^{-f}$,

$$N(u) = \int_M u \ln u dV,$$
$$\partial_t N = \int_M -|\nabla f|^2 e^{-f} dV \leq 0.$$

Taking the 2nd derivative yields,

$$\partial_t^2 N = \int_M 2u(|\mathrm{Hess}(f)|^2 + \mathrm{Rc}(\nabla f, \nabla f))dV.$$

Now we write a positive function u in the normalized form $u = (4\pi t)^{-n/2} e^{-f}$, $\int_M u = 1$ and define the following functional (like $N + \partial_t N$):

$$\Psi(u, t) = \int_M (t|\nabla f|^2 + f - n)udV. \tag{3.22}$$

Also, we denote,

$$W(u, t) = t(2\Delta f - |\nabla f|^2) + f - n \tag{3.23}$$

For u satisfying the heat equation, then, because of $(\Delta f - |\nabla f|^2)u = -\Delta u$ and integration by parts,

$$\Psi(u, t) = \int_M W u dV.$$

Furthermore,

$$(\partial_t - \Delta)(Wu) = -2ut\Big(|\mathrm{Hess} - \frac{g}{2t}|^2 + \mathrm{Rc}(\nabla f, \nabla f)\Big).$$

Thus, we obtain,

$$\partial_t \Psi(u, t) = -\int_M 2ut\Big(|\mathrm{Hess}(f) - \frac{g}{2t}|^2 + \mathrm{Rc}(\nabla f, \nabla f)\Big)dV$$

That motivates the definitions below.

Remark 3.4.1. *For the discussion below, along the Ricci flow, it is convenient to let* $\tau = T - t > 0$ *and then* $\Box^* = \partial_\tau - \Delta - \mathrm{S}$.

Definition 3.4.1. *On a closed manifold, the $\mathcal{F}$ functional is defined as, for $\int e^{-f} = 1$:*

$$\mathcal{F}(g, f) = \int_M (|\nabla f|^2 + \mathrm{S})e^{-f} dV \tag{3.24}$$

Remark 3.4.2. *In particular, if $\square^*(e^{-}f) = 0$ then $(\partial_t + \Delta)f = |\nabla f|^2 - \mathrm{S}$. Thus, by Lemma 2.4.1, along the Ricci flow, $\partial_t \mathcal{F} = \int_M 2u|Hess(f) + \mathrm{Rc}|^2 dV \geq 0$. Also we note that* S *appears when calculating the evolution of the volume form.*

Definition 3.4.2. *On a closed manifold, for $u = (4\pi\tau)^{-n/2}e^{-f}$, $\int u = 1$, define:*

$$\begin{aligned} W &= \tau(2\Delta f - |\nabla f|^2 + \mathrm{S}) + f - n, \\ \Psi(g, u, \tau) &= \int_M W u dV = \int_M \Big[\tau(|\nabla f|^2 + R) + (f - n)\Big] u dV \end{aligned} \tag{3.25}$$

Remark 3.4.3. *If $\square^* u = 0$ then $(\partial_t + \triangle)f = |\nabla f|^2 - \mathrm{S} + \frac{n}{2\tau}$. Furthermore, along the Ricci flow,*

$$\square^*(Wu) = -2\tau|\mathrm{Rc} + Hess f - \frac{g}{2\tau}|^2 u.$$

Then,

$$\begin{aligned} \frac{d}{dt}\Psi(g, u, \tau) &= \partial_t \int_M W u dV = -\int_M \square^*(Wu) dV \\ &= \int_M 2\tau \Big|\mathrm{Rc} + Hess f - \frac{g}{2\tau}\Big|^2 u dV \geq 0 \end{aligned}$$

3.4.2 Applications of Functionals

Here we collect some results on the fundamental solution of the conjugate heat equation and applications of Perelman's functional for a Ricci flow [81]. First the theorem below describes a Harnack inequality along a Ricci solution.

Theorem 3.4.3. *Let u be a positive solution to $\square^* u = 0$ and u tends to a δ-function as $\tau \to 0$. Then $W \leq 0$ for all $\tau > 0$. Furthermore, the maximum value of W is non-decreasing in t.*

The following corollary is immediate.

Corollary 3.4.4. *Under the assumptions as above, for any smooth curve $\gamma(t)$in M holds*

$$-\frac{\partial}{\partial t}f(\gamma(t),t) \leq \frac{1}{2}(|\dot{\gamma}(t)|^2 + \mathrm{S}(\gamma(t),t)) - \frac{f(\gamma(t),t)}{2(T-t)},$$
$$-\frac{\partial}{\partial t}(2\sqrt{\tau}f) \leq \sqrt{\tau}(\mathrm{S} + |\dot{\gamma}|^2).$$

Then it is natural to define the backwards reduced geometry as follows.

Definition 3.4.5. *Fix a point p and let $\Gamma(q,\tau) = \{\gamma : [0,\overline{\tau}] \mapsto M, \gamma(0) = p, \gamma(\overline{\tau}) = q\}$. The reduced distance is defined as*

$$\ell(q,\overline{\tau}) = \inf_{\gamma\in\Gamma}\Big\{\frac{1}{2\sqrt{\overline{\tau}}}\int_0^{\overline{\tau}} \sqrt{\tau}(R + |\dot{\gamma}|^2)d\tau\Big\}. \tag{3.26}$$

The backwards reduced volume is,

$$V(\overline{\tau}) = \int_M (4\pi\overline{\tau})^{-n/2}e^{-\ell(q,\overline{\tau})}dV(q). \tag{3.27}$$

Using this machinery, Perelman was able to prove non-collapsing results.

Definition 3.4.6. *A Riemannian manifold (M,g), is κ-non-collapsed at the scale r if any metric ball B of radius r, with $|\mathrm{R}|(x) \leq r^{-2}$ $\forall x \in B$, has volume at least κr^n. It is κ-non-collapsed if it is κ-non-collapsed at every scale.*

Then the following statement holds..

Theorem 3.4.7. *For a Ricci flow solution $(M^n, g(t))$, $0 \leq t < T < \infty$ and $\rho \in (0,\infty)$, there exists a constant $\kappa = \kappa(n, g(0), T, \rho)$ such that $(M, g(t)$ is κ-non-collapsed below the scale ρ. In that case, the solution is κ-non-collapsed.*

Remark 3.4.4. *There is an improved version, also due to Perelman, where only an upper bound on the scalar curvature is needed. If the scalar curvature is uniformly bounded by a constant C then we can pick $\rho = 1/\sqrt{C} > 0$. For any $p \in M$, $r < \rho$ holds $VolB(p,r) \geq \kappa r^n$.*

3.5 Singularity Model: Gradient Ricci Soliton

As a weakly parabolic system, the Ricci flow can develop finite-time singularities and, consequently, the study of singularity models becomes essentially crucial. In this section, we introduce some essential facts about gradient Ricci solitons (GRS), which are self-similar solutions of the Ricci flow and arise naturally in the analysis of singularities.

A GRS (M, g, f, λ) is a Riemannian manifold endowed with a special structure given by a (soliton) potential function f, a constant λ, and the equation:

$$\mathrm{Rc} + \nabla\nabla f = \lambda g. \tag{3.28}$$

Depending on the sign of λ, a GRS is called shrinking (positive), steady (zero), or expanding (negative). In particular an Einstein manifold N can be considered as a special case of a GRS where f is a constant and λ becomes the Einstein constant. A less trivial example is a Gaussian soliton $(\mathbb{R}^k, g_{\mathrm{sd}}, \lambda\frac{|x|^2}{2}, \lambda)$ with g_{sd} being the standard metric on Euclidean space. It is interesting to note that λ can be an arbitrary real number and that the Gaussian soliton can be either shrinking, steady or expanding. Furthermore, a combination of those two above, by the notation of P. Petersen and W. Wylie [84], is called a *rank k rigid GRS*, namely a quotient of $N \times \mathbb{R}^k$. Other nontrivial examples of GRS are rare and mostly Kähler, see [19, 43].

In recent years, following the interest in the Ricci flow, there have been various efforts to study the geometry and classification of GRS's; for example, see [20] and the citations therein. In particular, the low-dimensional cases ($n = 2, 3$) are relatively well-understood. For $n = 2$, Hamilton [53] completely classified shrinking gradient solitons with bounded curvature and showed that they must be either the round sphere, projective space, or Euclidean space with standard metric. For $n = 3$, utilizing the Hamilton-Ivey estimate, Perelman [82] proved an analogous theorem. Other significant results include recent development of Brendle [12] showing that a non-collapsed steady GRS must be rotationally symmetric and is, therefore, isometric to the Bryant soliton.

In higher dimensions, the situation is more subtle mainly due to the non-triviality of the Weyl tensor (W) which is vacuously zero for dimension less than four. One general approach to the classification problem so far has been imposing certain restrictions on the curvature operator. An analogue of Hamilton-Perelman results was obtained by A. Naber proving that a four dimensional complete non-compact GRS with bounded nonnegative curvature operator must be a finite quotient of R^4, $S^2 \times \mathrm{R}^2$ or $S^3 \times \mathrm{R}$ [75]. In [62], B. Kotschwar classified all rotationally symmetric GRS's with given diffeomorphic types on R^n, $S^{n-1} \times \mathrm{R}$ or S^n. Note that any rotationally symmetric Riemannian manifold has vanishing Weyl tensor.

Thus, a natural development is to impose certain conditions on that Weyl tensor. If the dimension is at least four, then a complete shrinking GRS with vanishing Weyl tensor must be a finite quotient of R^n, or $S^{n-1} \times \mathrm{R}$ or S^n following the works of [79, 103, 28, 85]; a steady GRS is flat or rotationally symmetric (that is, a Bryant Soliton) by [21]. The assumption $\mathrm{W} \equiv 0$ can be weakened to $\delta \mathrm{W} \equiv 0$, a closed or non-compact shrinking GRS must be rigid [28, 44, 74]; or in dimension four, to the vanishing of self-dual Weyl tensor only, a shrinking GRS with bounded curvature must be a finite quotient of R^4, $S^3 \times \mathrm{R}$, S^n, or CP^2, and steady GRS must be a Bryant soliton or flat [34]. There are some other classifications based on, for instance, Bach flatness [18] or assumptions on the radial sectional curvature [85].

Next, we collect important identities associated with a GRS. Algebraic manipulation of (3.28) and application of the Bianchi identities lead to following formulas (for a proof see [38]),

$$\mathrm{S} + \Delta f = n\lambda, \tag{3.29}$$

$$\frac{1}{2}\nabla_i \mathrm{S} = \nabla^j \mathrm{R}_{ij} = \mathrm{R}_{ij}\nabla^j f, \tag{3.30}$$

$$\mathrm{Rc}(\nabla f) = \frac{1}{2}\nabla \mathrm{S}, \tag{3.31}$$

$$\mathrm{S} + |\nabla f|^2 - 2\lambda f = \text{constant}, \tag{3.32}$$

$$\triangle \mathrm{S} + 2|\mathrm{Rc}|^2 = \langle \nabla f, \nabla \mathrm{S}\rangle + 2\lambda \mathrm{S}. \tag{3.33}$$

Remark 3.5.1. *If $\lambda \geq 0$, then $\mathrm{S} \geq 0$ by the maximum principle and equation* (3.33). *Moreover, a complete GRS has positive scalar curvature unless it is isometric to the flat Euclidean space [86].*

One motivation of the study to GRS's is that they arise naturally as self-similar solutions to the Ricci flow. For a fixed GRS given by (3.28) with $g(0) = g$ and $f(0) = f$, we define $\rho(t) := 1 - 2\lambda t > 0$, and let $\phi(t) : M^n \to M^n$ be a one-parameter family of diffeomorphisms generated by $X(t) := \frac{1}{\rho(t)}\nabla_{g(0)} f$. By pulling back,

$$\begin{aligned} g(t) &= \rho(t)\phi(t)^* g(0), \\ \mathrm{Rc}(t) &= \phi^* \mathrm{Rc}(0) = \frac{\lambda}{\rho(t)} g(t) - \mathrm{Hess}_{g(t)} f(t). \end{aligned}$$

Then $(M, g(t))$, $0 \leq t < T$, is a solution to the Ricci flow, where $T = \frac{1}{2\lambda}$ $(= \infty)$ if

$\lambda > 0$ ($\lambda \leq 0$). Other important quantities along the flow are given below,

$$\begin{aligned}
f(t) &= f(0) \circ \phi(t) = \phi(t)^* f, \\
\mathrm{S}(t) &= \mathrm{trace}(\mathrm{Rc}(t)) = \frac{n\lambda}{\rho(t)} - \Delta_{g(t)} f(t), \\
f_t &= |\nabla f|^2_{g(t)}, \\
\tau(t) &= T - t = \frac{\rho(t)}{2\lambda}, \\
u &= (4\pi\tau)^{-n/2} e^{-f}, \\
\Psi(g, \tau, f) &= \int_M \Big(\tau(|\nabla f|^2 + \mathrm{S}) + f - n\Big) u d\mu \\
&= -\tau C(t) \int_M u d\mu.
\end{aligned}$$

3.5.1 New Sectional Curvature

In this subsection, we prove some results in dimension four to illustrate that classical techniques for Einstein 4-manifolds can be adapted to study GRS's. For a four-dimensional GRS (M, g, f, λ), we define

$$H = \mathrm{Hess} f \circ g. \tag{3.34}$$

Then, with respect to bases given by (2.11), we have

$$H = \begin{pmatrix} A & B \\ B^T & A \end{pmatrix}, \tag{3.35}$$

with

$$\begin{aligned}
A &= \frac{\Delta f}{2} \mathrm{Id}, \\
B &= \begin{pmatrix} \frac{f_{11}+f_{22}-f_{33}-f_{44}}{2} & f_{23} - f_{14} & f_{24} + f_{13} \\ f_{23} + f_{14} & \frac{f_{11}+f_{33}-f_{22}-f_{44}}{2} & f_{34} - f_{12} \\ f_{24} - f_{13} & f_{34} + f_{12} & \frac{f_{11}+f_{44}-f_{22}-f_{33}}{2} \end{pmatrix}.
\end{aligned}$$

Remark 3.5.2. *In particular* $\langle H, \mathrm{W}\rangle = 0$.

We further define a new "curvature" tensor $\overline{\mathrm{R}}$ by

$$\begin{aligned}\overline{\mathrm{R}} &= \mathrm{R} + \frac{1}{2}H \\ &= \mathrm{W} + \frac{S}{24}g \circ g + \frac{1}{2}(\mathrm{Rc} - \frac{S}{4}g) \circ g + \frac{1}{2}H \\ &= \mathrm{W} - \frac{S}{12}g \circ g + \frac{1}{2}\lambda g \circ g = \mathrm{W} + (\frac{\lambda}{2} - \frac{S}{12})g \circ g.\end{aligned} \tag{3.36}$$

Thus, it follows immediately that, with respect to (2.11),

$$\overline{\mathrm{R}} = \begin{pmatrix} \overline{A}^{+} & 0 \\ 0 & \overline{A}^{-} \end{pmatrix},$$

with $\overline{A}^{\pm} = \mathrm{W}^{\pm} + (\lambda - \frac{S}{6})\mathrm{Id} = \mathrm{W}^{\pm} + (\frac{\Delta f}{4} + \frac{S}{12})\mathrm{Id}$. Furthermore, following the argument in [7], we obtain,

Proposition 3.5.1. *There exists a normal form for* $\overline{\mathrm{R}}$. *More precisely, at each point, there exits an orthonormal base* $\{e_i\}_{i=1}^4$, *such that with respect to the corresponding base* $\{e_{12}, e_{13}, e_{14}, e_{34}, e_{42}, e_{23}\}$ *for* Λ^2 *and as an operator on 2-forms,*

$$\overline{\mathrm{R}} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

with $A = diag(a_1, a_2, a_3)$ *and* $B = diag(b_1, b_2, b_3)$. *Moreover,* $a_1 = \min \overline{K}$, $a_3 = \max \overline{K}$ *and* $|b_i - b_j| \leq |a_i - a_j|$, *where* $\overline{K}$ *is the "sectional curvature" of* $\overline{\mathrm{R}}$, *i.e.,* $\overline{K}(e_1, e_2) = \overline{\mathrm{R}}_{1212}$ *for any orthonormal vectors* e_1 *and* e_2.

Remark 3.5.3. *Can a GRS be characterized by the existence of such a function f with* $\overline{\mathrm{R}}$ *constructed as above having the normal form?*

Next, we investigate the assumption of having a lower bound on this new sectional curvature similar to [49]. For $\epsilon < 1/3$, suppose that

$$\overline{K} \geq \epsilon\lambda. \tag{3.37}$$

Equivalently, for any orthonormal pairs e_i and e_j, that is

$$\overline{\mathrm{R}}_{ijij} \geq \epsilon\lambda \Leftrightarrow \mathrm{R}_{ijij} + \frac{f_{ii} + f_{jj}}{2} \geq \epsilon\lambda. \tag{3.38}$$

Then we have the following lemma.

Lemma 3.5.2. *Let (M, g, f, λ) be a GRS, then assumption (3.37) implies the following:*

$$\mathrm{S} + 3\Delta f \geq 12\epsilon\lambda,$$
$$\mathrm{S} \leq 6(1-\epsilon)\lambda,$$
$$\Delta f \geq 2(3\epsilon - 1)\lambda,$$
$$\frac{1}{\sqrt{6}}(|\mathrm{W}^+| + |\mathrm{W}^-|) \leq 2(1-\epsilon)\lambda - \frac{S}{3}.$$

The equality happens in the last formula if and only if $\mathrm{W}^\pm$ has the form $a^\pm diag(-1,-1,2)$, with $a^\pm \geq 0$ and

$$a^+ + a^- = 2(1-\epsilon)\lambda - \frac{\mathrm{S}}{3}.$$

Proof. All inequalities follow from tracing equation (3.38) and the soliton equation $S + \Delta f = 4\lambda$ except the last one.

For the last inequality, first note that any two form ϕ can be written as a simple wedge product of 1-forms iff $\phi \wedge \phi = 0$. In dimension four, with respect to (2.11), that is equivalent to $\phi = \phi^+ + \phi^-$ and $|\phi^+| = |\phi^-$. Therefore, in light of Proposition 2.14, assumption (3.37) is equivalent to

$$a^+ + a^- + 2\lambda - \frac{S}{3} \geq 2\epsilon\lambda \tag{3.39}$$

with a^+, a^- are the smallest eigenvalues of $\mathrm{W}^\pm$. Using the algebraic inequalities

$$-a^+ \geq \frac{1}{\sqrt{6}}|\mathrm{W}^+|, \tag{3.40}$$

$$-a^- \geq \frac{1}{\sqrt{6}}|\mathrm{W}^-|, \tag{3.41}$$

we obtain:

$$2(1-\epsilon)\lambda - \frac{S}{3} \geq \frac{1}{\sqrt{6}}(|W^+| + |W^-|).$$

Equality happens if and only if the equality happens in (3.39) and (3.40) (or (3.41)). The result then follows immediately.

□

Lemma 3.5.3. *Let (M, g, f, λ) be a closed GRS with assumption (3.37), then*

$$\int_M (|W^+| + |W^-|)^2 \leq \int_M \frac{2S^2}{3} d\mu - 8(1-\epsilon)(1+3\epsilon)\lambda^2 V(M).$$

Again equality holds if $W^\pm$ *has the form* $a^\pm diag(-1,-1,2)$ *with* $a^\pm \geq 0$ *and*

$$a^+ + a^- = 2(1-\epsilon)\lambda - \frac{S}{3}.$$

Proof. Applying Lemma 3.5.2, we compute

$$\begin{aligned}\int_M (2(1-\epsilon)\lambda - \frac{S}{3})^2 &= 4(1-\epsilon)^2\lambda^2 V(M) - \frac{4(1-\epsilon)\lambda}{3}\int_M S + \int_M \frac{S^2}{9} \\ &= 4(1-\epsilon)^2\lambda^2 V(M) - \frac{4(1-\epsilon)\lambda}{3} 4\lambda V(M) + \int_M \frac{S^2}{9} \\ &= 4(1-\epsilon)\lambda^2 V(M)(-\epsilon - \frac{1}{3}) + \int_M \frac{S^2}{9}.\end{aligned}$$

□

Remark 3.5.4. *If we use $S \leq 6(1-\epsilon)\lambda$, then*

$$\int_M (|W^+| + |W^-|)^2 \leq (\int_M S^2 d\mu)(\frac{2}{3} - \frac{2(1+3\epsilon)}{9(1-\epsilon)}) = \frac{4(1-3\epsilon)}{9(1-\epsilon)} \int_M S^2.$$

Lemma 3.5.4. *Let (M, g, f, λ) be a closed GRS, then*

$$\int_M |Rc|^2 = \int_M \frac{S^2}{2} - 4\lambda^2 V(M).$$

Proof. Using equation (3.33), we compute:

$$\begin{aligned}2\int_M |\mathrm{Rc}|^2 d\mu &= \int_M (2\lambda \mathrm{S} + \langle \nabla f, \nabla \mathrm{S}\rangle) d\mu \\ &= 2\lambda 4\lambda V(M) - \int_M \Delta f \mathrm{S} d\mu d\mu \\ &= 8\lambda^2 V(M) - \int_M (4\lambda - \mathrm{S})\mathrm{S} d\mu \\ &= -8\lambda^2 V(M) + \int_M \mathrm{S}^2 d\mu.\end{aligned}$$

□

The above results lead to the following estimate on the Euler characteristic.

Proposition 3.5.5. *Let* (M, g, f, λ) *be a closed non-flat GRS with unit volume, satisfying assumption (3.37), then*

$$8\pi^2\chi(M) < \frac{7}{12}\int_M \mathrm{S}^2 d\mu + 2\lambda^2(12\epsilon^2 - 8\epsilon - 3).$$

Proof. By the Gauss-Bonnet-Chern formula,

$$\begin{aligned}8\pi^2\chi(M) &= \int_M (|\mathrm{W}|^2 - \frac{1}{2}|\mathrm{E}|^2 + \frac{\mathrm{S}^2}{24}) d\mu \\ &\leq \int_M (|\mathrm{W}^+| + |\mathrm{W}^-|)^2 d\mu + \frac{1}{2}\int_M |\mathrm{Rc}|^2 d\mu - \int_M \frac{\mathrm{S}^2}{12} d\mu.\end{aligned}$$

Applying Lemmas 3.5.3 and 3.5.4 yields the inequality.

We now claim that the equality case can not happen. Suppose otherwise then $|\mathrm{W}^+||\mathrm{W}^-| = 0$ and equality also happens in Lemma 3.5.3. By the regularity theory for solitons [4], we can choose an orientation such that $|\mathrm{W}^-| \equiv 0$. Hence $W^+ = \mathrm{diag}(-a^+, -a^+, 2a^+)$ with $a^+ = 2(1-\epsilon)\lambda - \frac{S}{3}$, then by [34, Theorem 1.1], we have $W^+ = 0$ or $\mathrm{Rc} = 0$.

In the first case, by the classification of locally conformally flat four-dimensional closed GRS's as discussed in Introduction, (M, g) is flat, this is a contradiction.

In the second case, $\mathrm{Rc} = 0$ implies $\mathrm{S} = 0 = \lambda$, and since equality happens in Lemma 3.5.3, $\mathrm{W}^+ = 0$. Hence the above argument applies.

□

Remark 3.5.5. *The Euler characteristic of a closed Ricci soliton has been studied by [40]. If the manifold is Einstein and $\epsilon = 0$, we recover some results of [49].*

CHAPTER 4

PRESERVED CONDITIONS

Here, we investigate preserved conditions along the Ricci flow. Since such a condition could be passed on to the limit, it is a key ingredient in applications of the Ricci flow, such as in celebrated works of [51, 81, 10, 13].

Hamilton first observed that a closed manifold with positive Ricci curvature in dimension three and, more generally, a closed manifold with positive curvature operator remains so along the flow [51, 52]. Then other important conditions are shown to be preserved along the flow such as two-positive curvature [56, 33] and positive isotropic curvature and its variants [13, 76]. In a recent development, Wilking [98] proved a theorem giving a simple criterion to check whether a curvature condition is preserved.

Using his Lie algebra approach, we show another criterion with slightly different flavor (interpolations of cone conditions). The abstract formulation also recovers some known preserved condition developed in [13].

The organization of this chapter is as follows. Section 4.1 discusses the basic setting and techniques of the Lie algebra approach by Wilking [98]. In Section 4.2, we prove our main results.

4.1 The Lie Algebra Approach

In this section, we discuss the notation and basic setting of the Lie algebra approach developed by Wiling [98] and collect some preliminary results.

4.1.1 Identification of Vector Spaces and Complexification

A two-form can be seen as an operator on the associated tangent space (Chapter 2). Therefore, the space of two forms $\Lambda^2(V)$ can be identified with the orthogonal Lie algebra of skew-symmetric real matrices $\mathfrak{so}(n,\mathbb{R})$. The inner products on those spaces are, correspondingly,

$$\begin{aligned}\langle X \wedge Y, U \wedge V\rangle &= \langle X, U\rangle \langle Y, V\rangle - \langle X, V\rangle \langle Y, U\rangle\,,\\ \langle u, v\rangle &= \frac{1}{2}\mathrm{tr}(u^T v) = -\frac{1}{2}\mathrm{tr}(uv).\end{aligned}$$

Furthermore, given $A, B \in SO(n,\mathbb{R})$, $u, v \in \mathfrak{so}(n,\mathbb{R})$, the adjoint representation of the Lie group (algebra) is given by conjugation (commutator),

$$\begin{aligned}\mathrm{Ad}_A v &= AvA^{-1},\\ \mathrm{ad}_u v &= [u, v] = uv - vu.\end{aligned}$$

Remark 4.1.1. *For more background on Lie algbera, see [45].*

Next, we complexify the real vector space, $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. That is, $Z \in V_{\mathbb{C}}$ if and only if $Z = X + iY$ for some $X, Y \in V$. Then $\Lambda^2(V_{\mathbb{C}}) \leftrightarrow \mathfrak{so}(n,\mathbb{C})$ accordingly.

The inner product on $\mathfrak{so}(n,\mathbb{R})$ extends to a bilinear form $\langle .,.\rangle$ on $\mathfrak{so}(n,\mathbb{C})$. The inner product $(.,.)$ on $\mathfrak{so}(n,\mathbb{C})$ is defined as

$$(u, v) = \langle u, \overline{v}\rangle = \frac{1}{2}tr(u^T, \overline{v}).$$

Noted that the bilinear form (but not the inner product) is adjoint-invariant,

$$\begin{aligned}\langle u, v\rangle &= \langle \mathrm{Ad}_A u, \mathrm{Ad}_A v\rangle\,,\\ (v, v) = \langle v, \overline{v}\rangle &= \langle Ad_A v, Ad_A \overline{v}\rangle \neq \langle Ad_A v, Ad_{\overline{A}} \overline{v}\rangle = (Ad_A v, Ad_A v).\end{aligned}$$

4.1.2 Space of Algebraic Curvature Operators

An algebraic curvature operator R can be seen as a symmetric operator satisfying the first Bianchi identity on $\Lambda^2(V)$ and so a map: $\mathfrak{so}(n,\mathbb{R}) \to \mathfrak{so}(n,\mathbb{R})$. We denote the space of all these maps by $S_B^2(\Lambda^2(V)) \leftrightarrow S_B^2(\mathfrak{so}(n,\mathbb{R}))$.

Under the complexification procedure, the curvature is an operator on $\mathfrak{so}(n,\mathbb{C})$ by linear extension. It is noted that R is a Hermitian operator,

$$(\mathrm{R}A, B) = \left\langle \mathrm{R}A, \overline{B} \right\rangle = \left\langle A, \mathrm{R}\overline{B} \right\rangle = (A, \mathrm{R}B) = R(A, \overline{B}).$$

Immediately we obtain the following results.

Corollary 4.1.1. *The operator $-ad_v \mathrm{R} ad_{\overline{v}}$ is Hermitian.*

Lemma 4.1.2. *If A is a nonnegative Hermitian operator on a finite dimensional vector space V and B is nonnegative Hermitian operator on the image of A then $tr(AB) \geq 0$*

Proof. Since A is a nonnegative Hermitian operator, V has an orthonormal basis consisting of eigenvectors of A. Call them $\{x_i\}_{i=1}^n$ with eigenvalues $\lambda_i \geq 0$. Then we have $\mathrm{tr}(BA) = (BAx_i, x_i) = \lambda_i(Bx_i, x_i) \geq 0$ because, when $\lambda_i > 0$, x_i is in the image of A. □

We also observe the following result.

Lemma 4.1.3. *The followings are equivalent:*

a. R *is a* $\frac{n(n-1)}{2} - 1$ *nonnegative operator on* $\mathfrak{so}(n,\mathbb{R})$

b. $\mathrm{R}(v,v) \leq tr(\mathrm{R})$ *for any unit vector v in* $\mathfrak{so}(n,\mathbb{R})$

c. $\mathrm{R}(v,\overline{v}) = \langle \mathrm{R}v, \overline{v} \rangle = (\mathrm{R}v, v) \leq tr(\mathrm{R})$ *for any unit vector v in* $\mathfrak{so}(n,\mathbb{C})$

d. R *is a* $\frac{n(n-1)}{2} - 1$ *nonnegative operator on* $\mathfrak{so}(n,\mathbb{C})$

Proof. ($a \Leftrightarrow b$) and ($c \Leftrightarrow d$) are obvious. The only nontriviality part is $b \Leftrightarrow c$.

b $\Rightarrow$ **c**. If v is any unit vector in $\mathfrak{so}(n, C)$ then there exist $x, y \in \mathfrak{so}(n, \mathbb{R})$

$$
\begin{aligned}
v =& x + iy, \\
1 =& |x|^2 + |y|^2.
\end{aligned}
$$

Then, $\mathrm{R}(v, \bar{v}) = \mathrm{R}(x + iy, x - iy) = \mathrm{R}(x, x) + \mathrm{R}(y, y) \leq |x|^2 tr(\mathrm{R}) + |y|^2 \mathrm{tr}(\mathrm{R}) = \mathrm{tr}(\mathrm{R})$.

c $\Rightarrow$ **b**. Let v be any unit vector in $\mathfrak{so}(n, R)$ then v is also a unit vector in $\mathfrak{so}(n, \mathbb{C})$ and the result follows. □

Let F be a closed-convex set in $S^2_B(\mathfrak{so}(n, \mathbb{R}))$ which is invariant under the natural action of $O(n)$. The following theorem is essential in the study of preserved conditions along a Ricci flow. For $Q(\mathrm{R})$, see (3.4).

Theorem 4.1.4. *[51] Suppose F is invariant under the Hamilton ODE,*

$$
\frac{\partial}{\partial t}\mathrm{R} = Q(\mathrm{R}).
$$

If $(M, g(t))$, $t \in [0, T)$, is a solution to the Ricci flow such that $\mathrm{R}_{(p,0)} \in F$ for all points $p \in M$ then $\mathrm{R}_{(p,t)} \in F$ for all $t \in [0, T)$.

The theorem effectively reduces the study of the PDE system to the study of the corresponding ODE. Then to check a set is invariant under the ODE, it suffices to show that if $\mathrm{R} \in \partial F$ then $Q(\mathrm{R}) \in F$.

4.1.3 Basics of $Q(\mathrm{R})$ and $\mathrm{R}^\sharp$

Let $\{\phi^\alpha\}$ be an orthonormal basis of $\Lambda^2(V)$ or, equivalently, $\mathfrak{so}(n, \mathbb{R})$ and the structure constants are defined as, $c^{\gamma\eta}_\alpha = ([\phi^\gamma, \phi^\eta], \phi^\alpha)$. Notice that the structure constants are fully skew-symmetric. Equations (3.2), (3.3) become,

$$\begin{aligned}
\mathrm{R}^2_{\alpha\beta} =& \mathrm{R}_{\alpha\gamma}\mathrm{R}_{\beta\gamma},\\
\mathrm{R}^\sharp_{\alpha\beta} =& \frac{1}{2}c_\alpha^{\gamma\eta}c_\beta^{\delta\theta}\mathrm{R}_{\gamma\delta}\mathrm{R}_{\eta\theta}.
\end{aligned}$$

It follows that R^2 is just the matrix multiplication. The main difficulty when studying $Q(\mathrm{R})$, hence, is to understand $\mathrm{R}^\sharp$. One important observation is that $\mathrm{R}^\sharp$ can be realized as trace of an operator.

Lemma 4.1.5. $\left\langle \mathrm{R}^\sharp u, v\right\rangle = -\frac{1}{2}tr(ad_u \mathrm{R} ad_v \mathrm{R})$.

Proof. Since every operator involved is linear it suffices to show the statement for $u = \phi^1$ and $v = \phi^2$. Towards that end, we calculate $\mathrm{tr}(\mathrm{ad}_{\phi^1}\mathrm{R}\mathrm{ad}_{\phi^2}\mathrm{R})$. Let $M = \mathrm{ad}_{\phi^1}\mathrm{R}$ and $N = \mathrm{ad}_{\phi^2}\mathrm{R}$. The matrix of ad_{ϕ^i} is given by $(\mathrm{ad}_{\phi^i})_{jk} = ([\phi^i,\phi^j],\phi^k) = c_k^{ij}$; therefore,

$$\begin{aligned}
M_{ij} &= (\mathrm{ad}_{\phi^1})_{ik}\mathrm{R}_{kj} = c_k^{1i}\mathrm{R}_{kj},\\
N_{ji} &= (\mathrm{ad}_{\phi^2})_{jl}\mathrm{R}_{li} = c_l^{2j}\mathrm{R}_{li}.
\end{aligned}$$

Then,

$$\mathrm{tr}(MN) = \sum_{i,j} M_{ij}N_{ji} = c_k^{1i}c_l^{2j}\mathrm{R}_{kj}\mathrm{R}_{li} = -c_1^{ki}c_2^{jl}\mathrm{R}_{kj}\mathrm{R}_{li} = -2\mathrm{R}^\sharp_{12}.$$

□

Lemma 4.1.6. *Let* $\{\phi^\alpha\}$ *be an orthonormal basis diagonalizing* R *with eigenvalues* $\{\lambda^\alpha\}$*,*

$$\begin{aligned}
\varphi_{\alpha\beta} =& [\phi^\alpha,\phi^\beta],\\
d^i_{\alpha\beta} =& \left\langle [v_i,\phi^\alpha],\phi^\beta\right\rangle,
\end{aligned}$$

for any vector v_i*. Then we have:*

a. $\mathrm{R}^\sharp(v_i,v_i) = \frac{1}{2}(d^i_{\alpha\beta})^2\lambda^\alpha\lambda^\beta$.

b. $tr(\mathrm{R}^\sharp) = \frac{1}{2}|\varphi_{\alpha\beta}|^2\lambda^\alpha\lambda^\beta$.

Proof. **a.** By Lemma 3.3, $2\mathrm{R}^\sharp(v_i, v_i) = -\mathrm{tr}(\mathrm{ad}_{v_i}\mathrm{R}\mathrm{ad}_{v_i}R)$. Matrix A of ad_{v_i} with respect to the base $\{\phi^\alpha\}$, is given by $M_{\alpha\beta} = \left\langle [v_i, \phi^\alpha], \phi^\beta \right\rangle = d^i_{\alpha\beta}$. Also, since $d^i_{jk} = -d^i_{kj}$, the result follows immediately.

b. Now let $\{v_i\}$ be an orthonormal basis of $\Lambda^2(V)$ then by part a,

$$\mathrm{tr}(\mathrm{R}^\sharp) = \frac{1}{2}(\sum_i (d^i_{\alpha\beta})^2)\lambda^\alpha\lambda^\beta.$$

We also observe,

$$\sum_i (d^i_{\alpha\beta})^2 = \sum_i \left\langle \mathrm{ad}_{v_i}\phi^\alpha, \phi^\beta \right\rangle^2 = \sum_i \left\langle [\phi_k, \phi_j], v_i \right\rangle^2.$$

As $\left\langle [\phi^\alpha, \phi^\beta], v_i \right\rangle$ is the magnitude of the projection of $\varphi_{\alpha\beta}$ on v_i and$\{v_i\}$ is an orthonormal basis, the right hand side is exactly $|\varphi_{\alpha\beta}|^2$.

$\square$

Remark 4.1.2. *By (3.4), tr*$(\mathrm{R}^2 + \mathrm{R}^\sharp) = \frac{1}{2}|\mathrm{Rc}|^2$*, thus*

$$|\varphi_{\alpha\beta}|^2\lambda^\alpha\lambda^\beta = |\mathrm{Rc}|^2 - \frac{1}{2}|\mathrm{R}|^2 = \frac{1}{(n-1)(n-2)}\mathrm{S}^2 + \frac{n-4}{n-2}|\mathrm{Rc}|^2 - \frac{1}{2}|\mathrm{W}|^2.$$

If $n = 3$*, structure constants are 1,* $|\mathrm{R}|^2 = 4|\mathrm{Rc}|^2 - \mathrm{S}^2$*. If* $n = 4$*, it becomes* $\frac{1}{6}\mathrm{S}^2 - \frac{1}{2}|W|^2$.

If R is pure, $\mathrm{R}^\sharp$ can be calculated explicitly.

Lemma 4.1.7. *If the curvature operator is pure, then the* $\mathrm{R}^\sharp$ *is diagonalized by the same basis and*

$$\mathrm{R}^\sharp(e_{ij}, e_{ij}) = \mathrm{R}(e_{ik}, e_{ik})\mathrm{R}(e_{jk}, e_{jk})$$

Proof. Let e_i be a basis that diagonalizes the curvature operator. Note that for distinct indices i, j, k, l,

$$[e_{ij}, e_{kl}] = 0,$$

$$[e_{ij}, e_{ik}] = e_{jk},$$

$$\left\langle e_{ij}, e_{ik} \right\rangle = 0.$$

So the only nonzero structure constants are $\left\langle [e_{ij}, e_{ik}], e_{jk} \right\rangle = 1$. Therefore,

$$\begin{aligned}
&\mathrm{R}^\sharp(e_{ij}, e_{kl}) = \mathrm{R}^\sharp(e_{ij}, e_{lk}) = 0 \\
&\mathrm{R}^\sharp(e_{ij}, e_{ij}) = \frac{1}{2}\left\langle [e_{kl}, e_{mn}], e_{ij} \right\rangle^2 \mathrm{R}(e_{kl}, e_{kl})\mathrm{R}(e_{mn}, e_{mn}) = \mathrm{R}(e_{ik}, e_{ik})\mathrm{R}(e_{jk}, e_{jk}).
\end{aligned}$$

□

When $\mathrm{W} = 0$, the curvature is pure and we obtain the following result.

Corollary 4.1.8. *If, along the Ricci flow,* $\mathrm{W} = 0$ *then positive Ricci curvature is preserved.*

Proof. Let $\lambda_i = \mathrm{R}_{ii}$ and $\lambda = \frac{1}{n-1}\Sigma_i \lambda_i$. By the curvature decomposition, since $W = 0$,

$$\mathrm{R}_{ijij} = \frac{1}{n-2}(\lambda_i + \lambda_j - \lambda).$$

By the above lemma and equation (3.5),

$$\begin{aligned}
\frac{d}{dt}\lambda_1 &= \Sigma_{k\neq 1}\mathrm{R}_{1k1k}\lambda_k \\
&= \frac{1}{n-2}\Sigma_{k\neq 1}\lambda_k(\lambda_1 + \lambda_k - \lambda) \\
&= \frac{1}{n-2}\Big(\Sigma_{k\neq 1}\lambda_k^2 - (\lambda - \lambda_1)\Sigma_{k\neq 1}\lambda_k\Big) \\
&= \frac{1}{n-2}\Big(\Sigma_{k\neq 1}\lambda_k^2 - \frac{1}{n-1}(\Sigma_{k\neq 1}\lambda_k)^2 + \frac{n-2}{n-1}\lambda_1\Sigma_{k\neq 1}\lambda_k\Big)
\end{aligned}$$

Now since $\Sigma_{k\neq 1}\lambda_k^2 - \frac{1}{n-1}(\Sigma_{k\neq 1}\lambda_k)^2 \geq 0$, the result follows. □

Before we proceed further, let's summarize the set up.

$\Lambda^2(V)$	$\Lambda^2(V_{\mathbb{C}})$	$\mathfrak{so}(n,\mathbb{R})$	$\mathfrak{so}(n,\mathbb{C})$
$X \wedge Y$	linear ext	$u : Z \mapsto \langle Y, Z\rangle Z - \langle Z, Z\rangle Y$	linear ext
$\langle X \wedge Y, Z \wedge W\rangle$	linear ext	$\langle u, v\rangle = \frac{1}{2}tr(u^T v) = -\frac{1}{2}tr(uv)$	conjugation
$S^2_B(\Lambda^2(V))$	$S^2_B(\Lambda^2(V_{\mathbb{C}}))$	$S^2_B(\mathfrak{so}(n,\mathbb{R}))$	$S^2_B(\mathfrak{so}(n,\mathbb{C}))$
		$\left\langle \mathrm{R}^\sharp u, v\right\rangle = -\frac{1}{2}tr(\mathrm{ad}_u \mathrm{R} \mathrm{ad}_v \mathrm{R})$	linear ext
		$\left\langle \mathrm{R}^2 u, v\right\rangle = \mathrm{R}(u, \phi^\alpha)\mathrm{R}(v, \phi^\alpha)$	linear ext

4.2 Main Results

First, we recall Wilking's result and its consequences.

Definition 4.2.1. *Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{so}(n,\mathbb{R})$ or $\mathfrak{u}(n,\mathbb{R})$, and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, $\mathfrak{so}(n,\mathbb{C})$ or $\mathfrak{u}(n,\mathbb{C})$. For a set* S *in $\mathfrak{g}_{\mathbb{C}}$, and a real number h, we define,*

$$C(\mathrm{S}, h) = \{\mathrm{R} \in \mathrm{S}^2_B(\mathfrak{g}_{\mathbb{C}}) \mid \mathrm{R}(v, \overline{v}) \geq h, \forall v \in \mathrm{S}\}.$$

Also the Lie group associated with that Lie algebra is denoted $G_{\mathbb{C}}$.

Wilking's theorem asserts that if S is invariant under the adjoint representation of $G_{\mathbb{C}}$ then $C(\mathrm{S}, h)$ is invariant under the ODE $\mathrm{R}' = Q(\mathrm{R})$. That statement along with Hamilton's ODE-PDE theorem 4.1.4 capture several preserved conditions along the Ricci flow. For example, setting $h = 0$ and choosing appropriate set S's, we recover some well-known results summarized below.

Conditions	Choice of Set
NC	$\mathrm{S} = \mathfrak{so}(n, \mathbb{C})$
2NC	$\mathrm{S} = \mathrm{S}_{2+} = \{v \in \mathfrak{so}(n, \mathbb{C}), \mathrm{tr}(v^2) = 0\}$
NIC	$\mathrm{S} = \mathrm{S}_0 = \{v \in \mathfrak{so}(n, \mathbb{C}), rank(v) = 2, v^2 = 0\}$
NIC1	$\mathrm{S} = \mathrm{S}_1 = \{v \in \mathfrak{so}(n, \mathbb{C}), rank(v) = 2, v^3 = 0\}$
NIC2	$\mathrm{S} = \mathrm{S}_2 = \{v \in \mathfrak{so}(n, \mathbb{C}), rank(v) = 2\}$

Explanation of these conditions, $\phi, \psi \in \mathfrak{so}(n, \mathbb{R})$, $\eta, \zeta \in \mathfrak{so}(n, \mathbb{C})$, $\{e_i\}_{i=1}^4$ orthonormal:

- NC: Nonnegative curvature, $\mathrm{R}(\phi, \phi) \geq 0$.
- 2NC: Two-nonnegative curvature, $\mathrm{R}(\phi, \phi) + \mathrm{R}(\psi, \psi) \geq 0$, $\forall |\phi| = |\psi|, \langle \phi, \psi \rangle = 0$.
- NIC: Nonnegative isotropic curvature, $\mathrm{R}(\eta, \zeta, \overline{\eta}, \overline{\zeta}) \geq 0$, for all $\langle \eta, \eta \rangle = \langle \eta, \zeta \rangle = \langle \eta, \eta \rangle = 0$, or,

$$\mathrm{R}_{1313} + \mathrm{R}_{1414} + \mathrm{R}_{2323} + \mathrm{R}_{2424} - 2\mathrm{R}_{1234} \geq 0.$$

- NIC1: $\mathrm{R}(\eta, \zeta, \overline{\eta}, \overline{\zeta}) \geq 0$ for all $\langle \eta, \eta \rangle \langle \zeta, \zeta \rangle = \langle \eta, \zeta \rangle^2$ or

$$\mathrm{R}_{1313} + \lambda^2 \mathrm{R}_{1414} + \mathrm{R}_{2323} + \lambda^2 \mathrm{R}_{2424} - 2\lambda \mathrm{R}_{1234} \geq 0 \text{ for all } \lambda \in [0, 1].$$

- NIC2: $\mathrm{R}(\eta, \zeta, \overline{\eta}, \overline{\zeta}) \geq 0$ or

$$\mathrm{R}_{1313} + \lambda^2 \mathrm{R}_{1414} + \mu^2 \mathrm{R}_{2323} + \lambda^2 \mu^2 \mathrm{R}_{2424} - 2\lambda\mu \mathrm{R}_{1234} \geq 0 \text{ for all } \lambda, \mu \in [0, 1].$$

Remark 4.2.1. *By Theorem 4.1.4, a priori requirement for $C(S, h)$ to be invariant under the Ricci flow is $O(n, \mathbb{R})$-invariant. But that is equivalent to say that S is invariant under the adjoint representation of $O(n, \mathbb{R})$. Thus, Wilking's theorem is a partial converse statement.*

Lemma 4.2.2. $S_1 = S_2 \cap S_{2+}$.

Proof. First, let $u \in S_2 \cap S_{2+}$. Since $rank(u) = 2$ there exist $X, Y \in V_C$ such that $X \wedge Y \leftrightarrow u$ by the correspondence above.

Claim: For any skew-symmetric matrix of rank 2, $u^3 = \frac{1}{2}tr(u^2)u$.

To see the claim, we observe,

$$
\begin{aligned}
u(Z) &= \langle Y, Z\rangle X - \langle X, Z\rangle Y, \\
u^2 Z &= (\langle Y, Z\rangle \langle Y, X\rangle - \langle X, Z\rangle \langle Y, Y\rangle)X - (\langle Y, Z\rangle \langle X, X\rangle - \langle X, Z\rangle \langle X, Y\rangle)y \\
u^3 Z &= -\rho u Z \text{ with } \rho = \langle X, X\rangle \langle Y, Y\rangle - \langle X, Y\rangle^2 = \langle X \wedge Y, X \wedge Y\rangle = -\frac{1}{2}tr(u^2).
\end{aligned}
$$

Thus, if $u \in S_2 \cap S_{2+}$, then $u^3 = 0$ and so $u \in S_1$. The converse is similar. □

Remark 4.2.2. $C(S_\alpha \cup S_\beta, 0) = C(S_\alpha, 0) \cap C(S_\beta, 0)$ *but* $C(S_\alpha \cap S_\beta, 0) \supsetneq C(S_\alpha, 0) \cup C(S_\beta, 0)$.

Remark 4.2.3. *There have been subsequent works based on Wilking's criterion to study the convergence of a Ricci solution [50, 87].*

Inspired by Wilking's theorem, we come up with the following theorem.

Definition 4.2.3. *For a set* $\mathrm{S} \subset \mathfrak{g}_\mathbb{C}$ *and* $h \in \mathbb{R}$ *we define,*

$$C_{tr}(S, h) = \{\mathrm{R} \in \mathrm{S}^2_B(\mathfrak{g}_\mathbb{C}), \forall v \in S \mid \mathrm{R}(v, \overline{v}) + h|tr(v^2)| \geq 0\}.$$

Remark 4.2.4. *Note that* $C_{tr}(S, h)$ *is in general not a cone for* $h \neq 0$.

Theorem 4.2.4. *If* S *is invariant under the adjoint representation of* $G_\mathbb{C}$ *then* $C_{tr}(S, h)$ *is invariant under the ODE,* $\mathrm{R}' = Q(\mathrm{R})$

The following lemma is essential in the proof of the theorem.

Lemma 4.2.5. *Let* S *be an invariant set under the adjoint representation of* $G_\mathbb{C}$. *If* $\mathrm{R}(v, \overline{v}) + h|tr(v^2)| \geq 0, \forall v \in \mathrm{S}$ *and* $\mathrm{R}(u, \overline{u}) + h|tr(u^2)| = 0$ *for some* $u \in \mathrm{S}$ *then* $\mathrm{R}^\sharp(u, \overline{u}) \geq 0$.

Proof. We pick an arbitrary $v \in \mathfrak{g}_{\mathbb{C}}$ and define

$$f_u(t) = \mathrm{R}(\mathrm{Ad}_{\exp tv}u, \mathrm{Ad}_{\exp t\overline{v}}\overline{u}) + h|\mathrm{tr}((\mathrm{Ad}_{\exp tv}u)^2)|.$$

Since S is invariant under the adjoint representation of $G_{\mathbb{C}}$, $\mathrm{Ad}_{\exp tv}u \in \mathrm{S}$ and thus the function f_u is nonnegative for all t and zero at $t = 0$.
Since the adjoint representation is given by $\mathrm{Ad}_A(u) = AYA^{-1}$,

$$|\mathrm{tr}((\mathrm{Ad}_{\exp tv}u)^2)| = |tr(u^2)|.$$

Therefore, differentiating twice with respect to t and evaluating at $t = 0$ yield,

$$0 \leq 2\mathrm{R}(\mathrm{ad}_v u, \mathrm{ad}_{\overline{v}}\overline{u}) + \mathrm{R}(\mathrm{ad}_v\mathrm{ad}_v u, \overline{u}) + \mathrm{R}(u, \mathrm{ad}_{\overline{v}}\mathrm{ad}_{\overline{v}}\overline{u}).$$

Replacing v by iv and summing the 2 inequalities yields, for all $X \in \mathfrak{so}(n, C)$

$$0 \leq \mathrm{R}(\mathrm{ad}_v u, \mathrm{ad}_{\overline{v}}\overline{u}) = \mathrm{R}(\mathrm{ad}_u v, \mathrm{ad}_{\overline{u}}\overline{v}).$$

The last equation implies that $-\mathrm{ad}_{\overline{u}}\mathrm{R}\mathrm{ad}_u$ and its conjugate $-\mathrm{ad}_u\mathrm{R}\mathrm{ad}_{\overline{u}}$ are nonnegative on $\mathfrak{g}$ and R induces a nonnegative operator in the image of ad_u. By Lemma 4.1.5,

$$\left\langle \mathrm{R}^{\sharp}u, \overline{u} \right\rangle = -\frac{1}{2}\mathrm{tr}(\mathrm{ad}_u\mathrm{R}\mathrm{ad}_{\overline{u}}\mathrm{R}).$$

Thus, the statement follows from Lemma 4.1.2. □

Proof. (**Theorem 4.2.4**). Since trace is invariant under the adjoint representation, $C_{tr}(S, h)$ is convex and $O(n, , \mathbb{R})$-invariant. Furthermore as $|c^2| = |c|^2$ the set is scaling-invariant. Then, by Theorem 4.1.4 and the fact that R^2 is weakly positive definite, the statement follows from Lemma 4.2.5. □

Remark 4.2.5. *The complex set up allows the interchange of v and iv. That manipulation is powerful because we can compare algebraically the curvature operator acting on perpendicular elements.*

It is interesting to observe some relations between new invariant sets and Wilking's original sets.

Proposition 4.2.6. *If* S *be invariant under the adjoint representation of* $G_{\mathbb{C}}$ *then*

a. $C(\mathrm{S}, 0) = \cap_{h>0} C_{tr}(\mathrm{S}, h)$.

b. $C_{tr}(\mathrm{S}, \frac{1}{2}) \subset \{\mathrm{R} \in \mathrm{S}^2_B(\mathfrak{so}(n, \mathbb{C})), \mathrm{R} + Id \in C(\mathrm{S}, 0)\}$.

c. $C(\mathrm{S} \cap \mathrm{S}_{2+}, 0) = \cup_{h>0} C_{tr}(\mathrm{S}, h)$.

To prove Prop 4.2.6, first, we need the following lemma.

Lemma 4.2.7. *For* $v \in \mathfrak{g}_{\mathbb{C}}$, $\frac{1}{2}|tr(v^2)| \leq |v|^2$.

Proof. If $v \in \mathfrak{g}_{\mathbb{C}}$, it can be written as $v = \phi + i\psi$, $\phi, \psi \in \mathfrak{g}$. Then we have,

$$|v|^2 = \langle v, \overline{v} \rangle = |\phi|^2 + |\psi|^2.$$

Furthermore, using the Cauchy-Schwarz inequality for the inner product on $\mathfrak{g}$,

$$\begin{aligned} |\frac{1}{2}\mathrm{tr}(v^2)|^2 &= |\langle v, v\rangle|^2 = ||\phi|^2 - |\psi|^2 + 2i\langle \phi, \psi\rangle|^2 \\ &= |\phi|^4 + |\psi|^4 - 2|\phi|^2|\psi|^2 + 4\langle \phi, \psi\rangle^2 \\ &\leq |\phi|^4 + |\psi|^4 + 2|\phi|^2|\psi|^2 = (|\phi|^2 + |\psi|^2)^2 = |v|^4 \end{aligned}$$

Thus, $\frac{1}{2}|\mathrm{tr}(v^2)| \leq |v|^2$. □

Proof. (**Prop. 4.2.6**) Without loss of generality, we can assume that S is closed and scaling invariant.

a. Obviously if, $\forall v \in \mathrm{S}$, $\mathrm{R}(v, \overline{v}) \geq 0$, then for $h > 0$, $\mathrm{R}(v, \overline{v}) + h|\mathrm{tr}(v^2)| \geq 0$. Thus, $C(\mathrm{S}) \subset C_{\mathrm{tr}}(\mathrm{S}, h)$ for each $h > 0$.

For the other direction, we observe that if $\mathrm{R} \in \cap_{h>0} C_{\mathrm{tr}}(\mathrm{S}, h)$ then for each $v \in \mathrm{S}$, $\mathrm{R}(v, \overline{v}) \geq -h|tr(v^2)|$ for all $h > 0$. Letting $h \to 0^+$ we have $\mathrm{R}(v, \overline{v}) \geq 0$. Then the result follows.

b. If $R \in C_{tr}(S, \frac{1}{2})$ then for all $v \in S$, by Lemma 4.2.7,

$$0 \leq R(v, \overline{v}) + \frac{1}{2}|tr(v^2)| \leq R(v, \overline{v}) + |v|^2 = R(v, \overline{v}) + Id(v, \overline{v}).$$

Therefore, $R + Id \in C(S, 0)$.

c. Let $R \in C_{tr}(S, h)$ then, for any $v \in S \cap S_{2+}$ $tr(v^2) = 0$, $R(v, \overline{v}) \geq 0$. Thus, $C_{tr}(S, h) \subset C(S \cap S_{2+})$.

For the other direction, we proceed by contradiction. Let $R \in C(S \cap S_{2+})$ and suppose that $R \notin C_{tr}(S, h)$ for any $h > 0$. That is, we can find sequences $h_j \to +\infty$, $v_j \in S$ such that $R(v_j, \overline{v_j}) + h_j|tr(v_j^2)| < 0$. Since the inequality is scaling invariant we can assume that $|v_j| = 1$. Then, by compactness, we can obtain a subsequence $h_i \to +\infty$ and $v_i \to v$ such that

$$R(v_i, \overline{v_i}) + h_i|tr(v_i^2)| < 0. \tag{4.1}$$

If $|tr(v^2)| \neq 0$ then the second term of (4.1) approaches positive infinity and, thus, we obtain a contradiction as $R(v_i, \overline{v_i}) \to R(v, \overline{v}) < \infty$. If $|tr(v^2)| = 0$ then $v \in S \cap S_{2+}$. Since $R \in C(S \cap S_{2+})$, $R(v, \overline{v}) \geq 0$. But that is also a contradiction with (4.1).Therefore, $R \in C_{tr}(S, h)$ for some $h > 0$. □

Also, by choosing $S = S_2$, $h = \frac{1}{2}$ we recover the following result which plays a role in the proof of the differentiable sphere theorem [13].

Corollary 4.2.8. *Let C be the set of algebraic curvatures such that, for $\{e_i\}_{i=1}^4$ orthonormal, $\lambda, \mu \in [0, 1]$,*

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} + (1 - \lambda^2)(1 - \mu^2) \geq 0.$$

Then C is invariant under the Hamilton ODE.

Proof. We observe that if $z = e_1 + i\mu e_2$ and $w = e_3 + i\lambda e_4$ then,

$$
\begin{aligned}
\mathrm{R}(z, w, \bar{z}, \overline{w}) =& \mathrm{R}_{1313} + \lambda^2 \mathrm{R}_{1414} + \mu^2 \mathrm{R}_{2323} + \lambda^2\mu^2 \mathrm{R}_{2424} - 2\lambda\mu \mathrm{R}_{1234},\\
-\frac{1}{2}\mathrm{tr}((z \wedge w)^2) =& \langle z \wedge w, z \wedge w \rangle = 1 + \lambda^2\mu^2 - \lambda^2 - \mu^2.
\end{aligned}
$$

Then by the correspondence between $\mathfrak{so}(n, \mathbb{C})$ and $\Lambda^2(V \otimes_{\mathbb{R}} \mathbb{C})$ (see [11, Appendix B]), the statement follows. □

CHAPTER 5

HARNACK ESTIMATES

In this chapter, we show Harnack estimates on a closed manifold M^{n+p} with warped product symmetry along the Ricci flow. Given (F^p, g_F) Ricci flat and (N^n, g_N) a closed Riemannian manifold, let $M^{n+p} = N^n \times F^p$ with the warped product metric [1]:

$$g_M = g_N + f^2 g_F = g_N + e^{2u} g_F, \tag{5.1}$$

which evolves under the Ricci flow

$$\frac{\partial}{\partial t} g_M = -2\mathrm{Rc}_M. \tag{5.2}$$

The Ricci flow on a warped product has been investigated by several authors such as Cao [22], Lott-Sesum [69]. Harnack inequalities have a long history with fundamental contribution by, for example, Li-Yau [66] on parabolic equations. For the Ricci flow, key results were proved by Hamilton [54] and Perelman [81]. Our main theorem gives estimates for a fundamental solution to the adapted conjugate heat equation. The inequality is structurally similar to Perelman's but for a slightly more general setting.

Before proceeding further, let's fix the notation. We will use A_X to denote a quantity with respect to metric g_X on manifold X. We'll also omit the subscript when it is clear that the calculation is carried on N. Also if the flow exists for $0 \leq t \leq T$, it is convenient to define $\tau = T - t$. The conjugate heat operator with respect to the Ricci flow on M is $\square^*_M = \partial_\tau - \Delta_M + \mathrm{S}_M$. For the warped product setting, the adapted operator on N is given by, for $\mathrm{S}_w = \mathrm{S}_N - p|\nabla u|^2$, $\square^*_w = \partial_\tau - \triangle_N + \mathrm{S}_w$.

[1] The warped structure can also be defined more generally as in Section 2.6 but that one is not preserved by the Ricci flow in general.

The rest of this chapter is organized as follows. In section 2, we discuss the adaptation of the Ricci flow for a warped product and an equivalent system obtained via diffeomorphisms. In section 3, we derive modified monotonicity formulas and functionals. In section 4, we prove several gradient estimates with respect to the equivalent system. Section 5 collects some applications.

5.1 Basics of Ricci Flow on Warped Products

Let (M, g_M) be a warped product as in (5.1) then, by Section 2.6, for a function h,

$$\begin{aligned}
\mathrm{Hess}_M h &= [\mathrm{Hess} h]_N \oplus [f \langle \nabla h, \nabla f \rangle_N \, g_F]_F \qquad (5.3)\\
&= [\mathrm{Hess} h]_N \oplus [e^{2u} \langle \nabla h, \nabla u \rangle_N \, g_F]_F,\\
\triangle_M h &= \Delta_N h + p \langle \nabla u, \nabla h \rangle_N = \Delta_N h + \frac{p}{f} \langle \nabla f, \nabla h \rangle_N\,, \qquad (5.4)\\
d\mu_M &= d\mu_N f^p d\mu_F = d\mu_N e^{pu} d\mu_F, \qquad (5.5)\\
\mathrm{Rc}_M &= [\mathrm{Rc} - \frac{p}{f}\mathrm{Hess}(f)]_N \oplus [-(f \triangle f + (p-1)|\nabla f|^2) g_F]_F \qquad (5.6)\\
&= [\mathrm{Rc} - p\mathrm{Hess}(u) - p du \otimes du]_N \oplus [-e^{2u}(\Delta u + p|\nabla u|^2) g_F]_F,\\
\mathrm{S}_M &= \mathrm{S} - 2p\frac{\triangle f}{f} - p(p-1)\frac{|\nabla f|^2}{f^2} \qquad (5.7)\\
&= \mathrm{S} - 2p\Delta u - p(p+1)|\nabla u|^2.
\end{aligned}$$

Lemma 5.1.1. *Let $(M, g_M(t))$, $0 \le t \le T$, be a solution to the Ricci flow and $g_M(0)$ is a warped product metric as in (5.1). The flow preserves that warped structure and can be considered as a flow on $(N, g(t))$:*

$$\begin{aligned}
\frac{\partial}{\partial t} g &= -2Rc + 2p\frac{Hess f}{f} = -2\mathrm{Rc} + 2pHess u + 2p du \otimes du, \qquad (5.8)\\
\frac{\partial}{\partial t} f &= \triangle f + (p-1)\frac{|\nabla f|^2}{f}\\
\frac{\partial}{\partial t} u &= \triangle u + p|\nabla u|^2 = \Delta_M u.
\end{aligned}$$

Proof. Suppose (g_N, f) evolves as above then we can check that g_M evolves by the Ricci flow. By the uniqueness theorem for Ricci flow [51, Section 5], the result follows. □

Since u satisfies the heat equation, the maximum principle applies that if $u(.,0) \leq C$ then $u(.,t) \leq C$ as long as the flow exists. Furthermore, extensive use of the maximum principle yields interior estimates.

Lemma 5.1.2. *Let $(M, g_M(t))$, $0 \leq t \leq T$, be a solution to the Ricci flow and $g_M(0)$ is a warped product metric as in (5.1). Then for each $\alpha > 0$, there exists a constant $C(m, n, \alpha)$ such that if*

$$|Rm|_M(.,t) < k \text{ for all } t \in [0, \frac{\alpha}{k}]$$

then

$$|\nabla^m u|_{\overline{g}(t)} \leq \frac{C|u(.,0)|_{L^\infty}}{t^{m/2}}$$

for all $t \in [0, \frac{\alpha}{k}]$.

Proof. Since $\frac{\partial}{\partial t}u = \triangle_{g_M} u$ and $|u(.,0)|_{L^\infty}$ is preserved, the method of Shi's estimates applies. For a detailed calculation, see lemma 3.6 of [11]. □

Remark 5.1.1. *The essence of this lemma is that the constant only depends on degree and dimension. Therefore, under suitable dilation limit analysis, it holds for any small compact interval under a uniform curvature bound.*

5.1.1 Transform by Diffeomorphisms

Here, we discuss the procedure of transforming the flow system on N by a family of diffeomorphisms and collect some useful evolution equations. Most of the calculation here are similar to that of [67] or [73].

We consider diffeomorphisms generated by $-p\nabla u$, $\frac{\partial}{\partial t}\varphi(t)(x) = (-p\nabla u)(\varphi(t)(x))$. Pullbacks $\tilde{g}(t) = \varphi^*(t)(g(t))$, $\tilde{u}(t) = \sqrt{p}\varphi^*(t)(u(t)) = \sqrt{p}u(t) \circ \varphi(t)$ yield

$$\begin{aligned}
\frac{\partial}{\partial t}\tilde{g}(t) &= L_{-p\nabla u}(\varphi^*(t)(g(t))) + \varphi^*(t)(\frac{\partial}{\partial t}g(t)) \\
&= \varphi^*(t)(\frac{\partial}{\partial t}g(t) + L_{-p\nabla u}g(t)) = \varphi^*(t)(-2\mathrm{Rc} + 2pdu \otimes du) \\
&= -2\widetilde{\mathrm{Rc}} + 2d\tilde{u} \otimes \tilde{u}, \\
\frac{\partial}{\partial t}\tilde{u}(t) &= \sqrt{p}L_{-p\nabla u}(\varphi^*(t)(u(t))) + \sqrt{p}\varphi^*(t)(\frac{\partial}{\partial t}u(t)) \\
&= \sqrt{p}\varphi^*(t)(\frac{\partial}{\partial t}u(t) + L_{-p\nabla u}u(t)) = \sqrt{p}\varphi^*(t)(\triangle u) = \triangle \tilde{u}.
\end{aligned}$$

So (5.8) is transformed into the following system on N (we abuse notation here as tildes are removed):

$$\begin{aligned}
\mathcal{S} &= \mathrm{Rc} - du \otimes du \\
\frac{\partial g}{\partial t} &= -2\mathrm{Rc} + 2du \otimes du = -2\mathcal{S} \\
\frac{\partial u}{\partial t} &= \triangle u.
\end{aligned} \tag{5.9}$$

Remark 5.1.2. *Thus, results in [69] extend to a slightly more general setting: the fiber can be any Ricci flat manifold instead of S^1.*

Then the Christoffel symbols evolve by

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma^k_{ij} &= -g^{kl}(\nabla_i \mathcal{S}_{jl} + \nabla_j \mathcal{S}_{il} - \nabla_l \mathcal{S}_{ij}) \\
&= g^{kl}(-\nabla_i \mathrm{Rc}_{jl} - \nabla_j \mathrm{Rc}_{il} + \nabla_l \mathrm{Rc}_{ij} + 2\nabla_i \nabla_j u \partial_l u).
\end{aligned}$$

Lemma 5.1.3. *If $(N, u(., t), g(t))$ is a solution to (5.9) then the Laplacian acting on function evolves by*

$$\frac{\partial}{\partial t}\Delta = 2\mathcal{S}_{ij} \cdot \nabla_i \nabla_j - 2\Delta u \langle \nabla u, \nabla(.) \rangle \tag{5.10}$$

Proof. We compute

$$\frac{\partial}{\partial t}\Delta = \frac{\partial}{\partial t}(g^{ij}\nabla_i\nabla_j) = \frac{\partial}{\partial t}(g^{ij}(\partial_i\partial_j - \Gamma_{ij}^k\partial_k))$$
$$= (-\frac{\partial}{\partial t}g_{ij})\nabla_i\nabla_j - g^{ij}(\frac{\partial}{\partial t}\Gamma_{ij}^k)\partial_k.$$

Using the evolution equation for Γ_{ij}^k yields

$$g^{ij}(\frac{\partial}{\partial t}\Gamma_{ij}^k)\partial_k = g^{kl}(-2g^{ij}\nabla_i\mathrm{Rc}_{jl} + \nabla_l\mathrm{R}) + 2g^{kl}\Delta u\partial_l u\partial_k$$
$$= 2\Delta u\,\langle\nabla u, \nabla(.)\rangle\,,$$

where we use the contracted 2nd Bianchi identity. The result follows. □

Now we derive evolution equations for some geometrical quantities. Recall $\mathrm{S}_w = \mathrm{tr}(\mathcal{S}) = \mathrm{R} - |\nabla u|^2$ and we compute:

$$\frac{\partial}{\partial t}|\nabla u|^2 = \frac{\partial}{\partial t}(g^{ij}\nabla_i u\nabla_j u) = 2\mathcal{S}(\nabla u, \nabla u) + 2\left\langle\nabla u, \nabla\frac{\partial}{\partial t}u\right\rangle,$$
$$= 2\mathrm{Rc}(\nabla u, \nabla u) - 2|\nabla u|^4 + 2\,\langle\nabla u, \nabla\triangle u\rangle\,,$$
$$\Delta|\nabla u|^2 = 2\left\langle\nabla u, \nabla\frac{\partial}{\partial t}u\right\rangle + 2\mathrm{Rc}(\nabla u, \nabla u) + 2|\mathrm{Hess}u|^2 \text{ (Bochner's formula).}$$

Combining equations above yields

$$\Box|\nabla u|^2 = -2|\mathrm{Hess}u|^2 - 2|\nabla u|^4. \tag{5.11}$$

By Section 2.4 $\frac{\partial}{\partial t}g = v$ then $\frac{\partial}{\partial t}R = -\triangle\mathrm{trace}(v) + \mathrm{div}(\mathrm{div}v) - (v, \mathrm{Rc})$. Here,

$$\mathrm{div}(\mathrm{div}2\mathrm{Rc}) = \nabla_i 2\nabla_j\mathrm{Rc}_{ij} = \nabla_i\nabla_i\mathrm{S} = \Delta\mathrm{S},$$
$$\mathrm{div}(\mathrm{div}du\otimes du) = \nabla_i(\nabla_j(\nabla_i u\nabla_j u)) = \frac{1}{2}\triangle|\nabla u|^2 + \langle\nabla u, \nabla\triangle u\rangle + |\Delta u|^2,$$
$$\frac{\partial}{\partial t}\mathrm{S} = -\Delta(-2\mathrm{S}_w) - \Delta\mathrm{S} + \Delta|\nabla u|^2 + 2\,\langle\nabla u, \nabla\Delta u\rangle$$
$$+ 2|\Delta u|^2 + 2|\mathrm{Rc}|^2 - 2\mathrm{Rc}(\nabla u, \nabla u)$$
$$= \Delta\mathrm{S}_w + 2\,\langle\nabla u, \nabla\Delta u\rangle + 2|\Delta u|^2 + 2|\mathrm{Rc}|^2 - 2\mathrm{Rc}(\nabla u, \nabla u).$$

Combining equations above yields

$$\frac{\partial}{\partial t}\mathrm{S}_w = \Delta S + 2|\Delta u|^2 + 2|\mathcal{S}_{ij}|^2. \tag{5.12}$$

Remark 5.1.3. *[67] considers a similar system with a constant α_n associated with the term $du \otimes du$. However, in case $\alpha_n \geq 0$ letting $\tilde{u} = \sqrt{\alpha_n}u$ recovers (5.9). So every result in section 4 holds for $\alpha_n \geq 0$ as well.*

Remark 5.1.4. *A generalization of that system is so-called the Ricci-Harmonic flow first introduced by R. Muller in [73] and it is interesting to extend the result here for that setting (see, [9]).*

5.2 Monotonicity Formulae

We shall derive the adapted and modified forms of monotonicity formulas and associated functionals to the warped product setting. First, to adapt these formulas (see Section 3.4) to our setting, we observe the following relations.

Lemma 5.2.1. **a.** *If $\overline{H} = e^{-\overline{h}}$ and $H = \overline{H}e^{pu} = e^{-h}$ then*

$$\begin{aligned} h &= \overline{h} - pu, \\ \overline{h}_t &= |\nabla\overline{h}|^2 - \triangle_M\overline{h} - \mathrm{R}_M \text{ iff} \\ h_t &= -S - \triangle_N h + \nabla h(\nabla h + p\nabla u). \end{aligned}$$

b. *If $\overline{H} = (4\pi\tau)^{-(n+p)/2}e^{-\overline{h}}$ and $H = \overline{H}e^{pu} = (4\pi\tau)^{-n/2}e^{-h}$ then*

$$\begin{aligned} h &= \overline{h} - pu + \frac{p}{2}\ln(4\pi\tau) \text{ and} \\ \overline{h}_t &= |\nabla\overline{h}|^2 - \triangle_M\overline{h} - \mathrm{R}_M + \frac{n+p}{2\tau} \text{ iff} \\ h_t &= -S - \triangle_N h + \nabla h(\nabla h + p\nabla u) + \frac{n}{2\tau}. \end{aligned}$$

Proof. **a.** Using (5.4) and (5.7), we compute

$$\begin{aligned}
\overline{h}_t &= |\nabla\overline{h}|^2 - \triangle_M\overline{h} - \mathrm{R}_M \\
&= |\nabla\overline{h}|^2 - \triangle_N\overline{h} - p\nabla\overline{h}\nabla u - \mathrm{R}_N + 2p\triangle_N u + p(p+1)|\nabla u|^2, \\
h_t &= \overline{h}_t - pu_t \\
&= |\nabla\overline{h}|^2 - \triangle_N\overline{h} - p\nabla\overline{h}\nabla u - \mathrm{R}_N + 2p\triangle_N u + p(p+1)|\nabla u|^2 \\
&- p\triangle_N u - p^2|\nabla u|^2 \\
&= -\triangle_N h - \mathrm{R}_N + p|\nabla u|^2 + \nabla(h+pu)\nabla h.
\end{aligned}$$

b. This follows from a similar computation. □

Lemma 5.2.2. *Adapted to the Ricci flow on warped product metric given in (5.1) , the monotonicity formulas on $(N, g(t))$ are given by:*

a. $\mathcal{F}(g,u,h) = \int_N (\mathrm{S}_w + |\nabla h|^2)e^{-h}d\mu$ *restricted to* $\int_N e^{-h}d\mu = \frac{1}{V(F)}$.

Furthermore if $h_t = -\mathrm{S}_w - \Delta h + \nabla h(\nabla h + p\nabla u)$ *then*

$$\frac{d}{dt}\mathcal{F} = 2\int_N \Big(|\mathcal{S} + Hess(h)|^2 + p|\Delta u - \nabla u\nabla h|^2\Big)e^{-h}d\mu.$$

a′. *W.r.t system (5.9),* $h_t = -\mathrm{S}_w - \Delta h + |\nabla h|^2$.

b. *Restricted to* $\int_N Hd\mu = \int_N (4\pi\tau)^{-n/2}e^{-h}d\mu = \frac{1}{V(F)}$,

$$\Psi(g,u,\tau,h) = \int_N \Big[\tau(|\nabla h|^2 + \mathrm{S}_w) + (h + pu - n - p) - \frac{p}{2}\ln(4\pi\tau)\Big]Hd\mu_N.$$

And if $h_t = -\mathrm{S}_w - \Delta h + \nabla h(\nabla h + p\nabla u) + \frac{n}{2\tau}$ *then*

$$\frac{d}{dt}\Psi = 2\tau\int_N (|\mathcal{S} + Hessh - \frac{g}{2\tau}|^2 + p|\Delta u - \nabla u\nabla h + \frac{1}{2\tau}|^2)(4\pi\tau)^{-n/2}e^{-h}d\mu.$$

b′. *W.r.t system (5.9),* $h_t = -S - \triangle h + |\nabla h|^2 + \frac{n}{2\tau}$.

Proof. **a.** We will use formulas (5.7), (5.4), (5.5) to compute:

$$\begin{aligned}\mathcal{F}(g_M,\overline{h}) &= \int_M (\mathrm{S}_M + |\nabla\overline{h}|^2)e^{-\overline{h}}d\mu_M\\ &= \int_N\int_F (\mathrm{S}_N - 2p\Delta u - p(p+1)|\nabla u|^2 + |\nabla\overline{h}|^2)e^{-\overline{h}}e^{pu}d\mu_N d\mu_F\\ &= V(F)\int_N (\mathrm{S} - p|\nabla u|^2 + |\nabla h|^2)e^{-h}d\mu,\end{aligned}$$

where we use integration by parts (IBP) to simplify

$$\int_N 2p\Delta u e^{-h}d\mu = \int_N 2p\nabla h\nabla u e^{-h}d\mu.$$

Furthermore, using (5.3) and (5.6), we calculate

$$\begin{aligned}\overline{h}_t &= |\nabla\overline{h}|^2 - \Delta_M\overline{h} - \mathrm{S}_M, \text{ then}\\ \frac{d}{dt}\mathcal{F} &= 2\int_M (|\mathrm{Rc}_M + \mathrm{Hess}_M\overline{h}|^2 d\mu_M\\ &= 2V(F)\int_N \Big(\Big|\mathrm{Rc} - p du\otimes du - p\mathrm{Hess}(u) + \mathrm{Hess}(h+pu)\Big|^2\\ &+ p\Big| - \triangle_N u - p|\nabla u|^2 + \nabla u\nabla(h+pu)\Big|^2\Big)d\mu\\ &= 2V(F)\int_N (|\mathrm{Rc} - p du\otimes du + \mathrm{Hess}h|^2 + p|\triangle u - \nabla u\nabla h|^2)e^{-h}d\mu.\end{aligned}$$

The result then follows from lemma 5.2.1.

a′. It follows from $L_{-p\nabla u}h = -p\nabla u\nabla h$.

b. and **b′.** are similar using part b) of lemma 5.2.1. □

Corollary 5.2.3. *For* $(N, g(t))$ *along (5.9), if*

$$\Psi_w(g,u,\tau,h) = \int_N \Big(\tau(|\nabla h|^2 + \mathrm{S}_w) + (h-n)\Big)(4\pi\tau)^{-n/2}e^{-h}d\mu_N$$
$$h_t = -\mathrm{S}_w - \Delta h + \nabla h(\nabla h + p\nabla u) + \frac{n}{2\tau},$$

then,

$$\frac{d}{dt}\Psi_w = 2\tau\int_N (|\mathcal{S} + Hessh - \frac{g}{2\tau}|^2 + p|\triangle u - \nabla u\nabla h|^2)(4\pi\tau)^{-n/2}e^{-h}d\mu_N.$$

Proof. We have

$$\Psi_w = \Psi + \int_N (p - u + \frac{p}{2}\ln(4\pi\tau))\overline{H}e^{pu}d\mu_N.$$

Since u satisfies the heat equation on M and $\overline{H}$ the conjugate, $\frac{d}{dt}\int_M u\overline{H}d\mu_M = 0$. Thus,

$$\begin{aligned}\frac{d}{dt}\Psi_w &= \frac{d}{dt}\Psi + \frac{p}{2}\Big(\frac{d}{dt}\ln(4\pi\tau)\Big)\int_N \overline{H}e^{pu}d\mu_N \\ &= \frac{d}{dt}\Psi - \frac{p}{2\tau}\int_N \overline{H}e^{pu}d\mu_N.\end{aligned}$$

On the other hand,

$$\begin{aligned}|\Delta u - \nabla u\nabla h + \frac{1}{2\tau}|^2 &= |\Delta u - \nabla u\nabla h|^2 + \frac{1}{4\tau^2} + \frac{1}{\tau}(\Delta u - \nabla u\nabla h), \\ \int_N \triangle u e^{-h}d\mu_N &= \int_N \nabla u\nabla h e^{-h}d\mu_N \text{ by Stoke's theorem.}\end{aligned}$$

The result follows. □

An immediate application from the above calculation is the following result.

Proposition 5.2.4. *Let (M, g_M) be a closed warped product given as in (5.1). If M is a gradient soliton and the soliton function is constant on each fiber then (N, g_N) is Ricci flat and f is a constant function.*

Proof. Suppose (M, g) is a gradient soliton satisfying

$$\mathrm{Rc}_M + \mathrm{Hess}_M\overline{h} = \lambda g_M,$$

with $\overline{h}$ constant on each fiber. Let $h = \overline{h} + pu$ and follow the calculation from previous lemmas, we obtain:

$$\begin{aligned}0 = \int_M |\mathrm{Rc}_M + \mathrm{Hess}_M\overline{h} - \lambda g_M|^2 e^{-\overline{h}}d\mu_M &= \int_M |\mathcal{S} + \mathrm{Hess}_N h - \lambda g_N|^2 e^{-\overline{h}}d\mu_M \\ &\quad + V(F)\int_N p|\Delta_N u - \nabla u\nabla h + \lambda|^2 d\mu_N.\end{aligned}$$

On the other hand,

$$|\Delta_N u - \nabla u \nabla h + \lambda|^2 = |\Delta_N u - \nabla u \nabla h|^2 + \lambda^2 + 2\lambda(\Delta_N u - \nabla u \nabla h),$$
$$\int_N \Delta_N u e^{-h} d\mu_N = \int_N \nabla u \nabla h e^{-h} d\mu_N \text{ by Stoke's theorem.}$$

Thus $\lambda = 0$ and $(N \times F, g_N + f^2 g_F)$ is a gradient steady soliton. As $N \times F$ is closed, by either theorem 2.4 of [81] or 20.1 of [56], the manifold is Ricci flat. That is

$$0 = f\Delta_N f + (p-1)|\nabla f|^2 = \Delta_N u + p|\nabla u|^2,$$
$$0 = \mathrm{Rc}_N - p\frac{\mathrm{Hess}_N(f)}{f}.$$

However, as $\int_N \Delta_N u d\mu_N = 0$, the first equality implies that $\nabla u = 0$ and so f must be constant. Plugging into the 2nd equality yields the result. □

Remark 5.2.1. *Also computation above shows that monotone functionals in [67] are just suitable modification of ones developed by Perelman for warped products. For completeness, we'll repeat the definition here.*

Definition 5.2.5. *Along the flow given by (5.8) or (5.9), restricted to $\int_N e^{-h} d\mu_N = 1$,*

$$\mathcal{F}_w(g, u, h) = \int_N (\mathrm{S}_w + |\nabla h|^2) e^{-h} d\mu_N. \tag{5.13}$$

Restricted to $\int_N (4\pi\tau)^{-n/2} e^{-h} d\mu_N = 1$,

$$\Psi_w(g, u, \tau, h) = \int_N \Big(\tau(|\nabla h|^2 + \mathrm{S}_w) + (h - n)\Big)(4\pi\tau)^{-n/2} e^{-h} d\mu_N. \tag{5.14}$$

Furthermore, associated functionals can be defined similarly as follows:

$$\mu_w(g, u, \tau) = \inf_h \Psi_w(g, u, h, \tau), \tag{5.15}$$

$$\upsilon_w(g, u) = \inf_{\tau > 0} \mu_w(g, u, \tau), \tag{5.16}$$

$$\lambda_w(g, u) = \inf_h \mathcal{F}_w(g, u, h) \geq \lambda(g_M). \tag{5.17}$$

Remark 5.2.2. *These functionals satisfy diffeomorphism and scaling invariance:*

$$\Psi_w(g, u, \tau, h) = \Psi_w(cg, u, c\tau, h),$$

$$\mu_w(g, u, \tau) = \mu_w(cg, u, c\tau),$$

$$\upsilon_w(g, u) = \upsilon_w(cg, u).$$

Also the reduced geometry can be motivationally defined in an analogous manner (see also [72]).

Definition 5.2.6. *We define the $\mathcal{L}_w$-length of a curve $\gamma : [\tau_0, \tau_1] \mapsto N$, $[\tau_0, \tau_1] \subset [0, T]$ by*

$$\mathcal{L}_w(\gamma) := \int_{\tau_0}^{\tau_1} \sqrt{\tau}(\mathrm{S}_w(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2)d\tau.$$

For a fixed point $y \in N$ and $\tau_0 = 0$, the backward reduced distance is defined as

$$\ell_w(x, \tau_1) := \inf_{\gamma\in\Gamma}\{\frac{1}{2\tau_1}\mathcal{L}_w(\gamma)\}, \tag{5.18}$$

where $\Gamma = \{\gamma : [0, \tau_1] \mapsto M, \gamma(0) = y, \gamma(\tau_1) = x\}$.

The backward reduced volume is

$$V_w(\tau) := \int_M (4\pi\tau)^{-n/2} e^{-\ell_w(y,\tau)} d\mu_\tau(y).$$

Remark 5.2.3. *The functionals here differ from ones for the Ricci flow by replacing* S *with* S_w*. So it is natural that these new quantities behave similarly. First, we collect some lemmas.*

Lemma 5.2.7. *For any metric g, smooth function u on closed N anf $\tau > 0$,*

a. *There exists a smooth minimizer f_τ for $\Psi_w(g, u, ., \tau)$ which satisfies*

$$\tau(2\Delta f_\tau - |\nabla f_\tau|^2 + \mathrm{S}_w) + f_\tau - n = \mu_w(g, u, \tau).$$

b. *$\mu_w(g, u, \tau)$ is finite.*

c. *Along the flow, $0 \le t_1 \le t_2 \le T$ and $\tau(t) > 0$, $\frac{d}{dt}\tau = -1$ then*

$$\mu_w(g(t_2), \tau(t_2)) \ge \mu_w(g(t_1), \tau(t_1)).$$

d. $\lim_{\tau\to 0^+}\mu_w(g,u,\tau)=0.$

Proof. The arguments are identical to the counterpart for the Ricci flow, such as [37, Chapter 6] and [36, chapter 17]. Also details for the Ricci-Harmonic map flow which our setting is a special case of are given in [73, Section 7]. The proof of part d is verbatim to that of [91, Prop 3.2], replacing S by S_w.

□

Remark 5.2.4. *It is interesting to note that the functional μ_w can be defined without the flow context but the proof of* **d** *use some monotonicity formula of the flow.*

Lemma 5.2.8. *Assume as above and $\lambda_w(g,u)>0$ then $\lim_{\tau\to\infty}\mu_w(g,u,\tau)=+\infty$. Thus, $\upsilon(g,u)$ is well-defined and finite.*

Proof. The argument is verbatim to [37, Lemma 6.30] replacing S by S_w. □

An immediate application of the monotone framework is the theorem below which resembles a result of P. Topping in [94] using scalar curvature to control diameter for a compact manifold along the Ricci flow. The proof is verbatim by replacing monotonicity formulas μ and υ by μ_w and υ_w with required features described in Lemmas 5.2.7 and 5.2.8.

Theorem 5.2.9. *Let $n\geq 3$ and $(N^n, g(t), u(.,t))$ be a solution to (5.8) with $\upsilon_w(g,u)\geq -\infty$ then there exists a C depending on n, $\upsilon_w(g,u)$ such that*

$$diam(N,g)\leq C\int_N (S_w)_+^{(n-1)/2}d\mu_N = C\int_N (S-p|\nabla u|^2)_+^{(n-1)/2}d\mu_N.$$

Remark 5.2.5. *The + subscript denotes the positive part and $C=\max\{\frac{12}{\omega_n}, 6e^{3^n 37-\upsilon_w(g,u)}\}$.*

Corollary 5.2.10. *Let $n \geq 3$ and Let $(M, g_M(t))$, $0 \leq t \leq T$, be a solution to the Ricci flow and $g_M(0)$ is a warped product metric as in (5.1). Furthermore assume that $\lambda_w(g(0)) > 0$ then there exists C_1, C_2 depending on the initial conditions such that*

$$diam(M, g) \leq C_1 + C_2 \int_N (\mathrm{S} - p|\nabla u|^2)_+^{(n-1)/2} d\mu_N.$$

Proof. Since the flow preserves the warped product setting, (F, g_F) is closed, $|u(., t)|_{L^\infty} \leq |u(., 0)|_{L^\infty}$, the result follows from triangle inequalities and theorem 5.2.9. □

Remark 5.2.6. *Applying Topping result directly yields the bound $C \int_N (\mathrm{S} - 2p\Delta u - p(p+1)|\nabla u|^2)^{(n+p-1)/2} e^{pu} d\mu_N$. Thus, the above corollary gives a better estimate.*

5.3 Gradient Estimates and Harnack Inequality

For this section, we restric ourselves to system (5.9) and prove gradient estimates and a differential Harnack inequality for solutions to the conjugate heat equation. This section might be of independent interest and some arguments here are similar to those in [78].

Recall $\square_w^* = -\frac{\partial}{\partial t} - \Delta + \mathrm{S}_w$ is the adapted conjugate operator. Following standard theory on heat equations, for example [36, Chapter 23, 24], we denote:

$$H(x, t; y, T) = (4\pi(T - t))^{-n/2} e^{-h} = (4\pi\tau)^{-n/2} e^{-h},$$

for $\tau = T - t > 0$, to be the heat kernel. That is, for fixed (x, t), H is the fundamental solution of equation $\square H = 0$ based at (x, t), and similarly for fixed (y, T) and equation $\square_w^* H = 0$. The ultimate goal is to prove the following theorem.

Theorem 5.3.1. *Let $(N, u(., t), g(t))$, $0 \leq t \leq T$, be a solution to (5.9). Fix (y, T), let $H = (4\pi\tau)^{-n/2}e^{-h}$ be the fundamental solution of $\Box_w^* H = 0$, and*

$$v = \Big((T-t)(2\Delta h - |\nabla h|^2 + \mathrm{S}_w) + h - n\Big)H,$$

then for all $t < T$, we have

$$v \leq 0.$$

First let us recall the asymptotic behavior of the heat kernel as $t \to T$.

Theorem 5.3.2. *[36, Theorem 24.21] For $\tau = T - t$,*

$$H(x, t; y, T) \sim \frac{e^{-\frac{d_T^2(x,y)}{4\tau}}}{(4\pi\tau)^{n/2}} \Sigma_{j=0}^{\infty} \tau^j u_j(x, y, \tau).$$

More precisely, there exist $t_0 > 0$ and a sequence $u_j \in C^\infty(M \times M \times [0, t_0])$ such that,

$$H(x, t; y, T) - \frac{e^{-\frac{d_T^2(x,y)}{4\tau}}}{(4\pi\tau)^{n/2}} \Sigma_{j=0}^{k} \tau^j u_j(x, y, T - l) = w_k(x, y, \tau),$$

with

$$u_0(x, x, 0) = 1,$$

and

$$w_k(x, y, \tau) = O(\tau^{k+1-\frac{n}{2}})$$

as $\tau \to 0$ uniformly for all $x, y \in M$.

Next we derive a general estimate on the kernel. The proof is inspired by [29].

Lemma 5.3.3. *Let $B = -\inf_{0<\tau\leq T} \mu_w(g(0), \tau)$(B is well-defined due to Lemma 5.2.8) and $D = \min\{0, \inf_{N\times\{0\}} \mathrm{S}_w\}$, then we have*

$$H(x, t, y, T) \leq e^{B-(T-t)D/3}(4\pi(T-t))^{-n/2}.$$

Proof. Without loss of generality, we may assume that $t = 0$. Let $\Phi(y,t)$ be any positive solution to the heat equation along the flow. First, we obtain an upper bound for the L^∞-norm of $\Phi(.,T)$ in terms of L^1-norm of $\Phi(.,0)$.

Set $p(l) = \frac{T}{T-l} = \frac{T}{\tau}$ then $p(0) = 1$ and $\lim_{l\to T} p(l) = \infty$. For $A = \sqrt{\int_N \Phi^p d\mu}$, $v = A^{-1}\Phi^{p/2}$ and $\nabla\Phi\nabla(v^2\Phi^{-1}) = (p-1)p^{-2}4|\nabla v|^2$, integration by parts (IBP) yields

$$\begin{aligned}
\partial_t(\ln \|\Phi\|_{L^p}) &= -p'p^{-2}\ln(\int_N \Phi^p d\mu) + (p\int_N \Phi^p d\mu)^{-1}\partial_t(\int_N \Phi^p d\mu)\\
&= -p'p^{-2}\ln(\int_N \Phi^p d\mu) + (p\int_N \Phi^p d\mu)^{-1}\Big(\int_N \Phi^p(p\Phi^{-1}\Phi' + p'\ln\Phi - \mathrm{S}_w)d\mu\Big)\\
&= -p'p^{-2}\ln(A^2) + p^{-1}A^{-2}\Big(\int_N A^2v^2(p\Phi^{-1}\Phi' + p'\frac{2}{p}\ln(Av) - \mathrm{S}_w)d\mu\Big)\\
&= \int_N v^2\Phi^{-1}\triangle\Phi d\mu + p'p^{-2}\int v^2\ln v^2 - p^{-1}\int_N \mathrm{S}_w v^2 d\mu\\
&= p'p^{-2}\int_N v^2\ln v^2 d\mu - (p-1)p^{-2}\int_N 4|\nabla v|^2 d\mu - p^{-1}\int_N \mathrm{S}_w v^2 d\mu\\
&= p'p^{-2}\Big(\int_N v^2\ln v^2 d\mu - \frac{p-1}{p'}\int_N 4|\nabla v|^2 d\mu - \frac{p-1}{p'}\int_N \mathrm{S}_w v^2 dv\Big)\\
&+ ((p-1)p^{-2} - p^{-1})\int_N \mathrm{S}_w v^2 d\mu.
\end{aligned}$$

Note that if we set $v^2 = (4\pi\tau)^{-n/2}e^{-h}$ then the first term becomes,

$$-p'p^{-2}\Psi_w(g,u,\frac{p-1}{p'},h) - n - \frac{n}{2}\ln(4\pi\frac{p-1}{p'}).$$

We have

$$p'p^{-2} = \frac{1}{T}, \frac{p-1}{p'} = \frac{l(T-l)}{T}, \text{ and } (p-1)p^{-2} - p^{-1} = -\frac{(T-l)^2}{T^2}.$$

For $0 < t_0 < T$, $\tau(t_0) = \frac{t_0(T-t_0)}{T}$ and $\frac{d}{dt}\tau = -1$ then $0 < \tau(0) = \frac{t_0(2T-t_0)}{T} < T$. By Lemma 5.2.7, we arrive at

$$-p'p^{-2}\Psi_w(g(l),u,\frac{p-1}{p'},h) \le -\frac{1}{T}\Psi_w(g(0),u,\tau(0),h) \le -\frac{1}{T}\inf_{0<\tau\le T}\mu_w(g(0),\tau) = \frac{B}{T}.$$

Thus

$$T\partial_t(\ln|\Phi\|_{L^p}) \le B - n - \frac{n}{2}\ln(4\pi\frac{t(T-t)}{T}) - \frac{(T-t)^2}{T}D,$$

since, by (5.12), the minimum of S_w is nondecreasing along the flow. Integrating the above inequality yields

$$T \ln \frac{\|\Phi(.,T)\|_{L^\infty}}{\|\Phi(.,0)\|_{L^1}} \leq T(B - n - \frac{n}{2}(\ln(4\pi T) - 2)) - \frac{T^2}{3}D.$$

Then

$$\|\Phi(.,T)\|_{L^\infty} \leq e^{B-TD/3}(4\pi T)^{-n/2}\|\Phi(.,0)\|_{L^1}.$$

Since

$$\Phi(y,T) = \int_N H(x,0,y,T)\Phi(x,0)d\mu_{g(0)}(x), \tag{5.19}$$

and the above inequality holds for any arbitrary positive heat equation, we obtain

$$H(x,0,y,T) \leq e^{B-TD/3}(4\pi T)^{-n/2}.$$

□

Lemma 5.3.4. *Assume there exist* $k_1, k_2, k_3 \geq 0$ *such that, on* $N \times [0,T]$*,*

$$\mathrm{Rc}(g(t)) \geq -k_1 g(t),$$
$$\max\{S_w, |\nabla S_w|^2\} \leq k_2,$$
$$|\nabla u|^2 \leq k_3.$$

Let q *be any positive solution to the equation* $\Box_w^* q = 0$ *on* $N\times[0,T]$ *and* $\tau = T-t$*. If* $q < A$ *hen there exist* C_1, C_2 *depending on* k_1, k_2, k_3 *and* n *such that for* $0 < \tau \leq \min\{1, T, \frac{1}{2k_2}\}$*, we have*

$$\tau\frac{|\nabla q|^2}{q^2} \leq (1 + C_1\tau)(\ln\frac{A}{q} + C_2\tau). \tag{5.20}$$

Proof. We compute

$$
\begin{aligned}
(-\partial_t - \triangle)\frac{|\nabla q|^2}{q} &= S\frac{|\nabla q|^2}{q} + \frac{1}{q}(-\partial_t - \triangle)|\nabla q|^2 + 2|\nabla q|^2\nabla\frac{1}{q}\nabla \ln q - 2\nabla|\nabla q|^2\nabla\frac{1}{q},\\
\frac{1}{q}(-\partial_t - \triangle)|\nabla q|^2 &= \frac{1}{q}\Big[-2\mathcal{S}(\nabla q,\nabla q) - 2\mathrm{Rc}(\nabla q,\nabla q) - 2\nabla q\nabla(S q) - 2|\nabla^2 q|^2\Big],\\
2|\nabla q|^2\nabla\frac{1}{q}\nabla \ln q &= -2\frac{|\nabla q|^4}{q^3},\\
-2\nabla|\nabla q|^2\nabla\frac{1}{q} &= 4\frac{\nabla^2 q(\nabla q,\nabla q)}{q^2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(-\partial_t - \triangle)\frac{|\nabla q|^2}{q} &= \frac{-2}{q}|\nabla^2 q - \frac{dq\otimes dq}{q}|^2 + \frac{-4\mathrm{Rc}(\nabla q,\nabla q) + 2(\nabla u\nabla q)^2 - 2\nabla q\nabla(\mathrm{S}_w q)}{q} + \mathrm{S}_w\frac{|\nabla q|^2}{q}\\
&\leq [(4+n)k_1 + 3k_3 + 1]\frac{|\nabla q|^2}{q} + k_2 q.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
(-\partial_t - \triangle)(q\ln\frac{A}{q}) &= -\mathrm{S}_w q\ln\frac{A}{q} + \mathrm{S}_w q + \frac{|\nabla q|^2}{q}\\
&\geq \frac{|\nabla q|^2}{q} - (nk_1 + k_3)q - k_2 q\ln\frac{A}{q}.
\end{aligned}
$$

Let $\Phi = a(\tau)\frac{|\nabla q|^2}{q} - b(\tau)q\ln\frac{A}{q} - cq$, and we can choose a,b,c appropriately such that $(-\partial_t - \triangle)\Phi \leq 0$. For example,

$$
\begin{aligned}
a &= \frac{\tau}{1 + [(4+n)k_1 + 3k_3 + 1]\tau},\\
b &= e^{k_2\tau},\\
c &= (e^{k_2\tau}(nk_1 + k_3) + k_2)\tau.
\end{aligned}
$$

Then by the maximum principle, noticing that $\Phi \leq 0$ at $\tau = 0$,

$$
a\frac{|\nabla q|^2}{q} \leq b(\tau)q\ln\frac{A}{q} + cq.
$$

The result then follows from simple algebra.

□

The next result, mainly from [72], relates the reduced distance defined in (5.18) with the regular distance at time T.

Lemma 5.3.5. *Let $L_w(x,\tau) = 4\tau\ell_w(x,\tau)$ then we have.*

a. *Assume that there exists $k_1, k_2 \geq 0$ such that $-k_1 g(t) \leq \mathcal{S}(t) \leq k_2 g(t)$ for $t \in [0,T]$ then L_w is smooth amost everywhere and a local Lipschitz function on $N \times [0,T]$. Furthermore,*

$$e^{-2k_1\tau} d_T^2(x,y) - \frac{4k_1 n}{3}\tau^2 \leq L_w(x,\tau) \leq e^{2k_2\tau} d_T^2(x,y) + \frac{4k_2 n}{3}\tau^2.$$

b. $\Box_w^*\left(\frac{e^{-\frac{L_w(x,\tau)}{4\tau}}}{(4\pi\tau)^{n/2}}\right) \leq 0.$

c. $H(x,t;y,T) = (4\pi\tau)^{-n/2}e^{-h}$ *then* $h(x,t;y,T) \leq \ell_w(x,T-t)$.

Proof. **a.** This follows from the result [72, Lemma 4.1] for general flows.

b. This follows from [72, Lemma 5.15]. The key assumption is the non-negativity of the quantity,

$$\mathcal{D}(\mathcal{S},X) = \partial_t \mathrm{S}_w - \Delta \mathrm{S}_w - 2|\mathcal{S}|^2 + 4(\nabla_i \mathcal{S}_{ij})X_j - 2(\nabla_j S)X_j + 2(\mathrm{Rc} - \mathcal{S})(X,X).$$

In our case, applying (5.12) and the second Bianchi identity yields

$$\begin{aligned}\mathcal{D}(\mathcal{S},X) &= 2(\Delta u)^2 + 4\nabla^i(\mathrm{R}_{ij} - u_i u_j)X^j - 2\nabla_j(\mathrm{S} - |\nabla u|^2)X^j + 2du \otimes du(X,X)\\ &= 2(\Delta u)^2 - 4\Delta u \langle \nabla u, X\rangle + 2\langle \nabla u, X\rangle^2 = 2(\Delta u - \langle \nabla u, X\rangle)^2 \geq 0.\end{aligned}$$

c. We first observe that part a) implies $\lim_{\tau\to 0} L_w(x,\tau) = d_T^2(y,x)$ and, hence,

$$\lim_{\tau\to 0} \frac{e^{-\frac{L_w(x,\tau)}{4\tau}}}{(4\pi\tau)^{n/2}} = \delta_y(x),$$

since locally Riemannian manifolds look like Euclidean. It follows immediately from part b) and the maximum principle that,

$$H(x,t;y,T) \geq \frac{e^{-\frac{L_w(x,\tau)}{4\tau}}}{(4\pi\tau)^{n/2}} = \frac{e^{-\frac{L_w(x,T-t)}{4\tau}}}{(4\pi(T-t))^{n/2}}.$$

Hence we have,

$$h(x,t;y,T) \leq \frac{L_w(x,\tau)}{4\tau} = \ell_w(x,\tau) = \ell_w(x,T-l).$$

$\square$

A direct consequence is the following estimate on the heat kernel.

Lemma 5.3.6. *We have* $\int_N hH\Phi d\mu_N \leq \frac{n}{2}\Phi(y,T)$, *i.e,* $\int_N (h-\frac{n}{2})H\Phi d\mu_N \leq 0$.

Proof. By lemma 5.3.5 we have

$$\begin{aligned}\limsup_{\tau\to 0}\int_N hH\Phi d\mu_N &\leq \limsup_{\tau\to 0}\int_N \ell_w(x,\tau)H\Phi d\mu_N(x)\\ &\leq \limsup_{\tau\to 0}\int_N \frac{d_T^2(x,y)}{4\tau}H\Phi d\mu_N(x).\end{aligned}$$

Using Lemma 5.3.2,

$$\lim_{\tau\to 0}\int_N \frac{d_T^2(x,y)}{4\tau}H\Phi d\mu_N(x) = \lim_{\tau\to 0}\int_N \frac{d_T^2(x,y)}{4\tau}\frac{e^{-\frac{d_T^2(x,y)}{4\tau}}}{(4\pi\tau)^{n/2}}\Phi d\mu_N(x).$$

Either by differentiating twice under the integral sign or using these following identities on Euclidean spaces

$$\int_{-\infty}^{\infty} e^{-a\mathbf{x}^2}d\mathbf{x} = \sqrt{\frac{\pi}{a}} \text{ and } \int_{-\infty}^{\infty}\mathbf{x}^2 e^{-a\mathbf{x}^2}d\mathbf{x} = \frac{1}{2a}\sqrt{\frac{\pi}{a}},$$

we obtain

$$\int_{\mathbb{R}^n}|x|^2 e^{-a|x|^2}dx = n\Big(\int_{-\infty}^{\infty}\mathbf{x}^2 e^{-a\mathbf{x}^2}d\mathbf{x}\Big)\Big(\int_{-\infty}^{\infty} e^{-a\mathbf{x}^2}d\mathbf{x}\Big)^{n-1} = \frac{n}{2a}(\frac{\pi}{a})^{n/2}.$$

Therefore,

$$\lim_{\tau\to 0}\frac{d_T^2(x,y)}{4\tau}\frac{e^{-\frac{d_T^2(x,y)}{4\tau}}}{(4\pi\tau)^{n/2}} = \frac{n}{2}\delta_y(x)$$

and so

$$\lim_{\tau\to 0}\int_N \frac{d_T^2(x,y)}{4\tau}\frac{e^{-\frac{d_T^2(x,y)}{4\tau}}}{(4\pi\tau)^{n/2}}\Phi d\mu_N(x) = \frac{n}{2}\Phi(y,T).$$

Thus the result follows. $\square$

Remark 5.3.1. *In fact, the equality actually holds (See the proof of Theorem 5.4.2).*

Proposition 5.3.7. *Let* $v = \Big((T-t)(2\Delta h - |\nabla h|^2 + \mathrm{S}_w) + h - n\Big)H$ *then*

a. $\Box_w^* v = -2(T-t)\Big(|\mathcal{S} + Hessh - \frac{g}{2\tau}|^2 + |\Delta u - \nabla u \nabla h|^2\Big)H \leq 0;$

b. *If* $\rho_\Phi(t) = \int_N v\Phi d\mu_N$, *then* $\lim_{t\to T}\rho_\Phi(t) = 0$.

Proof. **a.** Let $q = 2\Delta h - |\nabla h|^2 + \mathrm{S}_w$ then

$$\begin{aligned} H^{-1}\Box_w^* v &= -(\partial_t + \Delta)(\tau q + h) - 2\left\langle \nabla(\tau q + h), H^{-1}\nabla H\right\rangle \\ &= q - \tau(\partial_t + \Delta)q - (\partial_t + \Delta)h + 2\tau\langle\nabla q, \nabla h\rangle + 2|\nabla h|^2. \end{aligned}$$

As H satisfies $\Box_w^* H = 0$, $(\partial_t + \Delta)h = -\mathrm{S}_w + |\nabla h|^2 + \frac{n}{2\tau}$. We compute

$$\begin{aligned} (\partial_t + \Delta)\Delta h &= \Delta\frac{\partial h}{\partial t} + 2\langle\mathcal{S}, \mathrm{Hess}(h)\rangle - 2\Delta u\langle\nabla u, \nabla h\rangle + \Delta(\Delta h) \\ &= \Delta(-\Delta h + |\nabla h|^2 - \mathrm{S}_w + \frac{n}{2\tau} + \Delta(\Delta h) \\ &\quad + 2\langle\mathcal{S}, \mathrm{Hess}(h)\rangle - 2\Delta u\langle\nabla u, \nabla h\rangle \\ &= \Delta(|\nabla h|^2 - \mathrm{S}_w) + 2\langle\mathcal{S}, \mathrm{Hess}(h)\rangle - 2\Delta u\langle\nabla u, \nabla h\rangle, \end{aligned}$$

where we use Lemma 5.1.3.

$$\begin{aligned} (\partial_t + \Delta)|\nabla h|^2 =& 2\mathcal{S}(\nabla h, \nabla h) + 2\left\langle\nabla h, \nabla\frac{\partial h}{\partial t}\right\rangle + \Delta|\nabla h|^2 \\ =& 2\left\langle\nabla h, \nabla(-\triangle h + |\nabla h|^2 - \mathrm{S}_w)\right\rangle \\ &+ 2\mathcal{S}(\nabla h, \nabla h) + \Delta|\nabla h|^2. \end{aligned}$$

Recall from (5.12), $(\partial_t + \Delta)\mathrm{S}_w = 2\Delta\mathrm{S}_w + 2|\mathcal{S}|^2 + 2|\Delta u|^2$, and

$$2\mathcal{S}(\nabla h, \nabla h) = 2\mathrm{Rc}(\nabla h, \nabla h) - 2du\otimes du(\nabla h, \nabla h) = 2\mathrm{Rc}(\nabla h, \nabla h) - 2\langle\nabla u, \nabla h\rangle^2$$

$$\Delta|\nabla h|^2 = 2\mathrm{Hess}(h)^2 + 2\langle\nabla h, \nabla\Delta h\rangle + 2\mathrm{Rc}(\nabla h, \nabla h),$$

where the second equation is by Bochner's identity. Combining those above yields,

$$\begin{aligned}(\partial_t + \Delta)q =& 4\langle \mathcal{S}, \mathrm{Hess}(h)\rangle - 4\Delta u \langle \nabla u, \nabla h\rangle + \Delta|\nabla h|^2 \\ & - 2\mathcal{S}(\nabla h, \nabla h) - 2\left\langle \nabla h, \nabla(-\triangle h + |\nabla h|^2 - \mathrm{S}_w)\right\rangle + 2|\mathcal{S}|^2 + 2|\Delta u|^2 \\ =& 4\langle \mathcal{S}, \mathrm{Hess}(h)\rangle - 4\Delta u \langle \nabla u, \nabla h\rangle + 2\langle \nabla h, \nabla q\rangle + 2\mathrm{Hess}(h)^2 \\ & + 2|\mathcal{S}|^2 + 2|\Delta u|^2 + 2\langle \nabla u, \nabla h\rangle^2 \\ =& 2|\mathcal{S} + \mathrm{Hess}(h)|^2 + 2|\Delta u - \langle \nabla u, \nabla h\rangle|^2 + 2\langle \nabla h, \nabla q\rangle .\end{aligned}$$

Thus,

$$\begin{aligned}H^{-1}\square_w^* v &= q + \mathrm{S}_w - |\nabla h|^2 - \frac{n}{2\tau} + 2|\nabla h|^2 \\ &\quad - 2\tau(|\mathcal{S} + \mathrm{Hess}(h)|^2 + 2|\Delta u - \langle \nabla u, \nabla h\rangle|^2) \\ &= -2\tau\Big(|\mathcal{S} + \mathrm{Hess}(h) - \frac{g}{2\tau}|^2 + |\triangle u - \nabla u \nabla h|^2\Big).\end{aligned}$$

The result follows.

b. IBP yields

$$\begin{aligned}\rho_\Phi(t) &= \int_N \Big(\tau(2\Delta h - |\nabla h|^2 + \mathrm{S}_w) + h - n\Big)H\Phi d\mu_N \\ &= -\int_N 2\tau\nabla h\nabla(H\Phi)d\mu_N - \int_N \tau|\nabla h|^2 H\Phi d\mu_N + \int_N (\tau\mathrm{S}_w + h - n)H\Phi d\mu_N \\ &= \int_N \tau|\nabla h|^2 H\Phi d\mu_N - 2\tau\int_N \nabla\Phi\nabla h H d\mu_N + \int_N (\tau\mathrm{S}_w + h - n)H\Phi d\mu_N \\ &= \int_N \tau|\nabla h|^2 H\Phi d\mu_N - 2\tau\int_N H\triangle\Phi d\mu_N + \int_N (\tau\mathrm{S}_w + h - n)H\Phi d\mu_N \\ &= \int_N \tau|\nabla h|^2 H\Phi d\mu_N + \int_N hH\Phi d\mu_N - 2\tau\int_N H\triangle\Phi d\mu_N + \int_N (\tau\mathrm{S}_w - n)H\Phi d\mu_N.\end{aligned}$$

For the first term, using Lemmas 5.3.3 and 5.3.4 for $N \times [\frac{\tau}{2}, \tau]$ to arrive at

$$\begin{aligned}\tau\int_N |\nabla h|^2 H\Phi d\mu_N &\le (2 + C_1\tau)\int_N (\ln(\frac{C_3 e^{-D\tau/3}}{H(4\pi\tau)^{n/2}}) + C_2\tau)H\Phi d\mu_N \\ &\le (2 + C_1\tau)\int_N (\ln C_3 - \frac{D\tau}{3} + h + C_2\tau)H\Phi d\mu_N,\end{aligned}$$

with C_1, C_2 as in Lemma 5.3.4 while $C_3 = \frac{e^B}{2^{n/2}}$.

Therefore, applying Lemma 5.3.6,

$$\lim_{\tau\to 0}\Big(\int_N \tau|\nabla h|^2 d\mu_N + \int_N hH\Phi d\mu_N\Big) \leq 3\int_N hH\Phi d\mu_N + 2\ln C_3\Phi(x,T)$$
$$\leq \Big(\frac{3n}{2} + 2\ln C_3\Big)\Phi(x,T).$$

Now we observe that expect for the first 2 terms, the rest approaches $-n\Phi(y,T)$ as $\tau \to 0$. Thus

$$\lim_{t\to T}\rho_\Phi(t) \leq C_4\Phi(x,T).$$

Furthermore, since Φ is a positive test function satisfying the heat equation $\partial_t\Phi = \triangle\Phi$, hence,

$$\partial_t\rho_\Phi(t) = \partial_t\int_N v\Phi d\mu_N = \int_N(\Box\Phi v - \Phi\Box_w^* v)d\mu_N \geq 0. \tag{5.21}$$

The above conditions imply that there exists α such that

$$\lim_{t\to T}\rho_\Phi(t) = \alpha.$$

Hence $\lim_{\tau\to 0}(\rho_\Phi(T-\tau) - \rho_\Phi(T-\frac{\tau}{2})) = 0$. By equation (5.21), part a), and the mean-value theorem, there exists a sequence $\tau_i \to 0$ such that

$$\lim_{\tau_i\to 0}\tau_i^2\int_N\Big(|\mathcal{S} + \mathrm{Hess}h - \frac{g}{2\tau}|^2 + |\Delta u - \nabla u\nabla h|^2\Big)H\Phi d\mu_N = 0.$$

Now using standard inequalitites yield,

$$\Big(\int_N \tau_i(\mathrm{S}_w + \triangle h - \frac{n}{2\tau_i})H\Phi d\mu_N\Big)^2$$
$$\leq \Big(\int_N \tau_i^2(\mathrm{S}_w + \triangle h - \frac{n}{2\tau_i})^2 H\Phi d\mu_N\Big)\Big(\int_N H\Phi d\mu_N\Big)$$
$$\leq \Big(\int_N \tau_i^2|\mathcal{S} + \mathrm{Hess}h - \frac{g}{2\tau}|^2 H\Phi d\mu_N\Big)\Big(\int_N H\Phi d\mu_N\Big).$$

Since $\lim_{\tau_i\to 0}\int_N H\Phi d\mu_N = \Phi(y,T) < \infty$ and $|\triangle u - \nabla u\nabla h|^2 \geq 0$,

$$\lim_{\tau_i\to 0}\int_N \tau_i(\mathrm{S}_w + \triangle h - \frac{n}{2\tau_i})H\Phi d\mu_N = 0.$$

Therefore, by Lemma 5.3.6,

$$\begin{aligned}\lim_{t\to T}\rho_\Phi(t) &= \lim_{t\to T}\int_N (\tau_i(2\triangle h - |\nabla h|^2 + \mathrm{S}_w) + h - n)H\Phi d\mu_N \\ &= \lim_{t\to T}\int_N (\tau_i(\triangle h - |\nabla h|^2) + h - \frac{n}{2})H\Phi d\mu_N \\ &= \lim_{t\to T}(\int_N (-\tau_i H\triangle\Phi d\mu_N + \int_N (h - \frac{n}{2})H\Phi d\mu_N) \\ &= \int_N (h - \frac{n}{2})H\Phi d\mu_N \le 0.\end{aligned}$$

So $\alpha \le 0$. To show that equality holds, we proceed by contradiction. Without loss of generality, we may assume $\Phi(y, T) = 1$. Let $H\Phi = (4\pi\tau)^{-n/2}e^{\tilde{h}}$ (that is, $\tilde{h} = h - \ln\Phi$), then IBP yields,

$$\rho_\Phi(t) = \Psi_w(g, u, \tau, \tilde{h}) + \int_N \Big(\tau(\frac{|\nabla\Phi|^2}{\Phi}) - \Phi\ln\Phi\Big)Hd\mu_N. \tag{5.22}$$

By the choice of Φ the last term converges to 0 as $\tau \to 0$. So if $\lim_{t\to T}\rho_\Phi(t) = \alpha < 0$ then $\lim_{\tau\to 0}\mu_w(g, u, \tau) < 0$ and, thus, contradictss Lemma 5.2.8. Therefore $\alpha = 0$. The result then follows. $\square$

Now Theorem 5.3.1 follows immediately.

Proof. (Theorem 5.3.1) Recall from inequality (5.21)

$$\partial_t \int_N v\Phi d\mu_N = \int_N (v\Box\Phi - \Phi\Box_w^* v)d\mu_N \ge 0.$$

By Proposition 5.3.7, $\lim_{t\to T}\int_N v\Phi d\mu_N = 0$. Since Φ is arbitrary, $v \le 0$. $\square$

5.4 Applications

In this section, we collect some applications of the estimates proved in the previous sections.

An immediate consequence is the following LYH-type Harnack estimate.

Corollary 5.4.1. *Let $(N, g(t))$, $0 \le t \le T$, $\tau = T - t$, be a solution to (5.9), along any smooth curve $\gamma(t)$ in N, we have*

$$-\partial_t h(\gamma(t), t) \le \frac{1}{2}(\mathrm{S}_w(\gamma(t), t) + |\dot\gamma(t)|^2) - \frac{1}{2(T-t)} h(\gamma(t), t),$$
$$\partial_\tau (2\sqrt{\tau} h) \le \sqrt{\tau}(\mathrm{S}_w + |\dot\gamma(t)|^2).$$

Proof. As H satisfies $\Box_w^* H = 0$,

$$h_t = -\mathrm{S}_w - \Delta h + |\nabla h|^2 + \frac{n}{2\tau}.$$

Substituting that into $\partial_t h(\gamma(t), t) = \nabla h \dot\gamma(t) + h_t \ge h_t - \frac{1}{2}(|\nabla h|^2 + |\dot\gamma(t)|^2)$ and applying $v \le 0$ prove the result. □

The next theorem exposes relations between fundamental solutions and the reduced distance defined with respect to the same reference point (y, T).

Theorem 5.4.2. *Let $(N, g(t))$, $0 \le t \le T$, $\tau = T - t$, be a solution to (5.9). Let $H = (4\pi\tau)^{-n/2} e^{-h}$ be a positive fundamental solution of $\Box_w^* H = 0$ centered at (y, T). If Φ is a positive solution to the heat equation $\partial_t \Phi = \triangle_M \Phi$, then the following hold:*

a. $h(x, l; y, t) \le \ell_w(x, T - l)$,

b. $\lim_{\tau \to 0} 4\tau \ell_w(x, \tau) = d_T^2(y, x)$,

c. $\lim_{\tau \to 0} \int_N hH\Phi d\mu_N = \lim_{\tau \to 0} \int_N \ell_w(x, \tau) H \Phi d\mu_N = \frac{n}{2}\Phi(y, T)$.

Proof. Part a) and b) are proved in Lemma 5.3.5. Part c) follows from Lemma 5.3.6 and the proof of Lemma 5.3.7, where it is shown that equality must hold. □

Remark 5.4.1. *If H satisfies (5.25), then $\overline{H} = He^{-u}$ satisfies the conjugate heat equation on (M, g_M). However, $\overline{H}$ is not fundamental because it blows up on the whole fiber over*

(y, T). That partially explains the following result which is interesting because if $\tilde{H}$ ***were*** *a fundamental solution then the limit would be zero.*

Corollary 5.4.3. *Let Ψ be Perelamn's Ψ-functional (3.25) and $\overline{H}$ as above. Let*

$$\widetilde{H} = \frac{1}{V(F)}\overline{H} = (4\pi\tau)^{-(n+p)/2}e^{-\widetilde{h}}, \tag{5.23}$$

for $V(F)$ denotes the volume of (F, g_F). Then $\lim_{\tau\to 0}\Psi(g_M, \tau, \widetilde{h}) = \infty$.

Proof. We abuse notation here by writing,

$$\Psi(g_M, \tau, \overline{h}) = \int_M \Big(\tau(|\nabla\overline{h}|^2 + \mathrm{S}_M) + \overline{h} - n - p\Big)(4\pi\tau)^{-(n+1)/2}e^{-\overline{h}}d\mu_M,$$

for $\int_M \overline{H}d\mu_M = V(F)$.

Let $\Phi = 1$ be the constant function in Prop. 5.3.7 then $\rho_1(t) = \Psi_w(g, u, \tau, h)$.

By Lemmas 5.2.1 and 5.2.2,

$$\begin{aligned}\Psi(g_M, \tau, \overline{h}) &= V(F)\int_N \Big(\tau(\mathrm{S}_w + |\nabla h|^2) + h - n - p + pu - \frac{p}{2}\ln(4\pi\tau))Hd\mu_N \\ &= V(F)\Psi_w(g, u, \tau, h) + pV(F)\int_N (u - 1 - \frac{1}{2}\ln(4\pi\tau))Hd\mu_N.\end{aligned}$$

Since $\lim_{\tau\to 0}\ln(4\pi\tau) = -\infty$, by Lemma 5.3.7, $\lim_{\tau\to 0}\Psi(g_M, \tau, \overline{h}) = +\infty$. A direct calculation yields that,

$$\Psi(g_M, \tau, \tilde{h}) = \frac{1}{V(F)}\Psi(g_M, \tau, \overline{h}) + \ln(V(F)). \tag{5.24}$$

Thus the result follows. □

Finally, we state the Harnack inequality translated to the warped product.

Theorem 5.4.4. *Let $(M, g_M(t))$, $0 \le t \le T$, $\tau = T - t$, be a solution to the Ricci flow and $g_M(0)$ a warped product metric as in (5.1). Let $\overline{H}$ be a positive, fiber-constant function on M such that, on N, $H = \overline{H}e^u = (4\pi\tau)^{-n/2}e^{-h}$ is the fundamental solution of*

$$\Box_w^* H + p\nabla u\nabla H = 0 \tag{5.25}$$

centered at (y, T)*. If* $v = \Big(\tau(2\Delta h - |\nabla h|^2 + \mathrm{S}_w) + h - n\Big)H$*, then, for all* $0 < \tau \leq T$, $v \leq 0$.

Proof. By the diffeomorphism discussed in Section 2, the result follow from Theorem 5.3.1. Note that if, with respect to (5.9), Φ (H) is a positive function satisfying the equation $\partial_t \Phi = \triangle_N \Phi$ $(\Box_w^* H = 0)$ then pulling back by the diffeomorphism, with respect to (5.8),

$$\partial_t \Phi = \Delta_N \Phi + p \nabla u \nabla \Phi = \triangle_{g_M} \Phi,$$

$$\partial_t H = -\triangle_N H + \mathrm{S} - wH + p \nabla u \nabla H.$$

□

CHAPTER 6

CONDITIONS TO EXTEND THE RICCI FLOW

This chapter describes a joint work with X. Cao [27] examining conditions related to the first finite singularity time. In particular, we provide a systematic approach to the mean value inequality method, suggested by N. Le [63] and F. He [59]. We also display a close connection between this method and time slice analysis as in [97].

It was first shown by Hamilton that $|R|$ must blow up approaching the first finite singular time (Theorem 3.1.1). More recently, by using an application of the non-collapsing result of Perelman (Section 3.4), Sesum was able to prove that if $|Rc|$ is bounded then the flow can be extended [90]. Since then, the obvious generalized question of whether the scalar curvature must behave similarly has received extensive attention. It is still open but considerable progress has been made: the Type I case is resolved by J. Ender, R. Muller and P. Topping [42], also independently by Q. Zhang and the X. Cao in [30, 24], while the Kähler case is solved by Z. Zhang [102]. There are various other relevant results such as estimates relating the scalar curvature and the Weyl tensor [24], comparable growth rates of different components of the curvature tensor [97], [96], and integral conditions by Le and Sesum [64].

It is interesting that elementary but clever analytical techniques proved fruitful to study this problem. Following the mean value inequality trick of Le [63] for the mean curvature flow, F. He developed a logarithmic-improvement condition for the Ricci flow [59]. Our contribution is to provide a more systematic treatment of the mean value inequality method and to find a close connection to the time slice analysis method suggested by B. Wang [97]. Then we apply our analysis to a particular context of Ricci flow with a uniform-growth condition defined below.

For the rest of this chapter, we will use the following notation:

$$Q(t) = \sup_{M\times\{t\}} |\mathrm{R}|,\ P(t) = \sup_{M\times\{t\}} |\mathrm{Rc}|,\ O(t) = \sup_{M\times\{t\}} |\mathrm{S}|.$$

Our first theorem gives a logarithmic-improvement condition relating the Ricci curvature and the Riemannian curvature tensor (in comparison, the logarithmic result in [59] involves a double integral of just the Riemannian curvature).

Theorem 6.0.5. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution on M. If for some $0 \le p \le 1$, we have*

$$\int_0^T \frac{P(t)}{(\ln(1 + Q(t)))^p} dt < \infty,$$

then the solution can be extended past time T.

Since we are interested in the behavior of the scalar curvature at a singular time, this motivates the following definition.

Definition 6.0.6. *A Ricci flow solution on a closed manifold is said to satisfy the* ***uniform-growth*** *condition if it develops a singularity in finite time, and any singularity model obtained by parabolic rescaling at the scale of the maximum curvature tensor must has non-flat scalar curvature.*

Under the Ricci flow, the uniform-growth condition generalizes both Type I and (non-flat) nonnegative isotropic curvature (NIC) conditions. Combining the above mean value inequality method with the uniform-growth condition yields the following logarithmic-improvement result.

Theorem 6.0.7. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition on M. If for some $0 \le p \le 1$, we have*

$$\int_0^T \int_M \frac{|\mathrm{S}|^{n/2+1}}{(\ln(1+|\mathrm{S}|))^p} d\mu dt < \infty, \tag{6.1}$$

then the solution can be extended past time T.

The organization is as follows. In Section 2, we recover a result of [59] by elementary continuity analysis. Section 3 discuss mean value inequalities and provide the proof of Theorem 6.0.5. Section 4 displays a close connection to the time-slice analysis and thus gives another proof of the above result as well as some independent estimates. In Section 5 we apply our method to the context of nonnegative isotropic curvature and its generalization.

6.1 Continuity Analysis

This section is to prove the following result:

Theorem 6.1.1. *Let $(M, g(t))$, $0 \le t < T < \infty$ be a solution to the Ricci flow. If $F(x) = \int_0^T |Rc|(x,t)dt$ is continuous on M then the solution can be extended past time T.*

Remark 6.1.1. *The result is also proved in [59] using the Sobolev machinery.*

Let $H_x(t_1, t_2) = \int_{t_1}^{t_2} |\mathrm{Rc}|(x,t)dt$. H is uniformly continuous if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|t_2 - t_1| < \delta$ then $H_x(t_1, t_2) < \epsilon,\ \forall x \in M$.

Lemma 6.1.2. *H is uniformly continuous under one of those assumptions:*

a. $\int_0^T P(t)dt < C$.

b. $F(x) = \int_0^T |\mathrm{Rc}|(x,t)dt$ *is continuous on M*

Proof. **a.** Since $\int_0^T \mathrm{P}(t)dt$ is finite, we can choose η such that $H(t,T) < \epsilon$ for all $\eta \leq t$. If $\eta \leq t_1 \leq t_2 \leq T$ then obviously, $H(t_1,t_2) < \epsilon$. Let $c = \max_{[0,\frac{T+\eta}{2}]} |P(t)|$ and choose $\delta < \min\{\frac{\eta}{2}, \frac{\epsilon}{c}\}$ then the result follows.

b. Let $\mathcal{F}(x,t) = \int_0^t |\mathrm{Rc}|(x,t)dt$. By the assumption and M is closed and T finite, $\mathcal{F}$ is uniformly continuous on $M \times [0,T]$. The argument carries over. □

Remark 6.1.2. *Is it possible to replace $\int_0^T \mathrm{P}(t)dt$ by $\int_0^T |\mathrm{Rc}(t)|dt$ at any point in M?*

Lemma 6.1.3. *Let $(M, g(t))$, $0 \leq t < T < \infty$ be a solution to the Ricci flow. If H is uniformly continuous then $g(t)$ is uniformly continuous.*

Proof. For any $x \in M$ and any $V \in T_xM$ we have:

$$|\ln \frac{g(x,t_2)(V,V)}{g(x,t_1)(V,V)}| = |\int_{t_1}^{t_2} \frac{\partial_t g(x,t)(V,V)}{g(x,t)(V,V)}| \leq 2\int_{t_1}^{t_2} |\mathrm{Rc}|(x,t) = H_x(t_1,t_2).$$

□

We are ready to prove the main theorem.

Proof. (of Theorem 6.1.1) The proof is modeled after that of [38, Theorem 6.40]. By Lemmas 6.1.2 and 6.1.3, the metric is uniformly continuous. Thus the same argument as in the aforementioned reference would apply if we can show that the singularity model is Ricci flat.

If T is the singular time then by Theorem 3.1.1, there exist a sequence $t_j \to T$, $Q_j = \max_M |\mathrm{R}(x,t_j)| \to \infty$. We dilate the solution by $g_j(t) = Q_j g(t_j + \frac{t}{Q_j})$. Then $|\mathrm{Rc}|_j(x,t) = \frac{1}{Q_j}|\mathrm{Rc}|(x, t_j + \frac{t}{Q_j})$ and therefore,

$$\int_{-1}^{0} |\mathrm{Rc}_j|(x,t)dt = \int_{t_j-\frac{1}{Q_j}}^{t_j} \frac{|\mathrm{Rc}|(x,s)}{Q_j} Q_j ds$$

$$= \int_{t_j-\frac{1}{Q_j}}^{t_j} |\mathrm{Rc}|(x,s)ds$$

Since $Q_j \to \infty$, $t_j - \frac{1}{Q_j} \to T$. As in Lemma 6.1.2, $\mathcal{F}(x,t) = \int_0^t |\mathrm{Rc}|(x,t)dt$ is uniformly continuous on $M \times [0,T]$. Therefore, the last integral above is approaching zero as $j \to \infty$. By the convergence theory (see 3.3.14), $\int_{-1}^0 |\mathrm{Rc}|_\infty(x,t)dt = 0$ and the solution is Ricci flat. The result then follows. □

Remark 6.1.3. *Let (S^n, g_0) be the space form of constant sectional curvature 1. The Ricci flow has the solution $g(t) = (1 - 2(n-1)t)g(0)$ with $T = \frac{1}{2(n-1)}$ is the first singular time. The family $g(t)$ is not uniformly continuous because*

$$|g(t_1) - g(t_0)|_{g(t_0)} = 2(n-1)|t_1 - t_0||g(0)|_{g(t_0)} = \frac{2(n-1)|t_1 - t_0|}{1 - 2(n-1)t_0}|g(0)|_{g(0)}.$$

6.2 Mean Value Inequalities

In this section, we describe the method of mean value inequalities to study conditions to extend the Ricci flow. The key idea is to generalize a simple but clever trick from [63] which involves an integral with a carefully chosen weight function. The conclusion is that, regarding the blow-up behavior, the weight function does not really matter.

Lemma 6.2.1. *Let $f, G : [0,T) \to [0,\infty)$ be continuous functions and $\psi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that*

$$\int_1^\infty \frac{1}{\psi(s)} ds = \infty. \tag{6.2}$$

If there is a mean value inequality of the form

$$f(t) \le C_1 \int_0^t \psi(f(s))G(s)ds + C_2 = h(t) \tag{6.3}$$

and $\int_0^T G(t)dt < \infty$, then $\limsup_{t\to T} f(t) < \infty$.

Proof. For any $T_0 < T$,

$$\begin{aligned}\int_0^{T_0} C_1 G(t)dt &= \int_0^{T_0} \frac{1}{\psi(f(t))} C_1 \psi(f(t)) G(t) dt \\ &= \int_{h(0)}^{h(T_0)} \frac{1}{\psi(f(h^{-1}(s)))} ds \text{ (let } s = h(t), ds = h'(t)dt) \\ &\geq \int_{h(0)}^{h(T_0)} \frac{1}{\psi(s)} ds.\end{aligned}$$

The last inequality is because of $f(t) \leq h(t)$. If $\int_0^T C_1 G(t) dt < \infty$, then by the choice of ψ, $h(T_0) \leq C < \infty$. Now by the mean value inequality, $f(T_0) \leq h(T_0) \leq C$. Since T_0 is arbitrary, $\sup_{[0,T)} f \leq C < \infty$. □

Next, we will establish a mean value inequality connecting $Q(t)$ and $P(t)$.

Lemma 6.2.2. *Let $\Sigma(M, \kappa, C_0) = \{g(t) | t \in [0,1], g(t)$ is κ-noncollapsed, $Q(0) \leq C_0\}$ be a set of complete Ricci flow solutions on M^n. Then there exists a constant $C = C(n, \kappa, C_0)$ such that for any $g(t) \in \Sigma$,*

$$\sup_{[0,1]} Q(t) \leq C \int_0^1 Q(t) P(t) dt + 32 C_0. \tag{6.4}$$

Proof. The proof is by contradiction. Suppose that the statement is false then there exists a sequence of $g_i(t) \in \Sigma$ and $a_i \to \infty$ such that

$$\sup_{[0,1]} Q_i(t) \geq a_i \int_0^1 Q_i(t) P_i(t) dt + 32 C_0.$$

Let $Q_i = \sup_{[0,1]} Q_i(t)$ then we can find (x_i, t_i) such that Q_i is attained. Since $Q_i > 32C_0$ there exists t_{i0} being the first time backward such that $Q_i(t_{i0}) = \frac{1}{2} Q_i$. Consequently, for $t \in [t_{i0}, t_i]$, $32C_0 < Q_i < 2Q_i(t)$, $Q_i(t_{i0}) > 16C_0$ and by Lemma 3.1.2, $t_{i0} > \frac{1}{16C_0}$.

Claim: There exists a constant $\epsilon_0 = \epsilon_0(n, \kappa)$ such that the following holds: for any $t_0 > 0$, $D \geq \max\{1/t_0, \max_{[0,t_0]} Q\}$, let $t_1 > t_0$ be the first time, if exists, such that

$Q(t_1) = D$, and $t_2 > t_1$ be the first time, if exists, such that $|\ln(Q(t_2)/Q(t_1))| = \ln 2$, then

$$\int_{t_1}^{t_2} P(t)dt > \epsilon_0.$$

Proof of claim: This is essentially just a restatement of [97, Lemma 3.2]. If there are no such t_1, t_2, the statement is vacuously true. If they exist then we dilate the solution by $\tilde{g}(t) = Dg(t_1 + t/D)$ then $\tilde{g}(t)$ satisfies the condition of the aforementioned result and the claim follows after rescaling back.

Applying the claim above yields

$$\int_{t_{i0}}^{t_i} P_i(t)dt > \epsilon_0. \tag{6.5}$$

Thus,

$$Q_i \geq 32C_0 + a_i \int_{t_{i0}}^{t_i} Q_i(t)P_i(t)dt \geq 32C_0 + a_i 16C_0\epsilon_0. \tag{6.6}$$

On the other hand,

$$\begin{aligned} Q_i \int_{t_{i0}}^{t_i} P_i(t)dt &\leq 2 \int_{t_{i0}}^{t_i} Q_i(t)P_i(t)dt \\ &\leq 2 \int_0^1 Q_i(t)P_i(t)dt \\ &\leq 2\frac{Q_i - 32C_0}{a_i}, \end{aligned}$$

hence

$$\int_{t_{i0}}^{t_i} P_i(t)dt \leq \frac{2}{a_i}\frac{Q_i - 32C_0}{Q_i} \to 0,$$

the last limit follows from (6.6) and $a_i \to \infty$. This is in contradiction with (6.5), so the lemma follows. □

We are now in the position to state our mean value inequality.

Proposition 6.2.3. *Let* $(M, g(t))$, $0 \leq t < T$, *be a Ricci flow solution. There exist:*

$$C_0 = C_0(n, \kappa, Q(0)),$$
$$C_1 = 32Q(0),$$

such that,

$$\sup_{[0,t]} Q \leq C_0 \int_0^t Q(u)P(u)du + C_1. \tag{6.7}$$

Proof. For $t \in [0, \frac{1}{16Q(0)})$ the statement is true by Lemma 3.1.2. For any $t \in [\frac{1}{16Q(0)}, T)$ define

$$\tilde{g}(s) = \frac{1}{t} g(ts), \ \ s \in [0, 1],$$
$$\tilde{Q}(s) = tQ(s).$$

Since the non-collapsing constant is a scaling invariant, Lemma 6.2.2 yields

$$\sup_{[0,1]} \tilde{Q} \leq C_0 \int_0^1 \tilde{Q}(s)\tilde{P}(s)ds + 32\tilde{Q}(0),$$
$$\sup_{[0,t]} tQ \leq C_0 t \int_0^t Q(u)P(u)du + 32tQ(0) \ (u = ts),$$
$$\sup_{[0,t]} Q \leq C_0 \int_0^t Q(u)P(u)du + 32Q(0).$$

□

Now we can finish the proof of Theorem 6.0.5.

Proof. (**Theorem 6.0.5**) First observe that if T is the first singular time then

$$\lim_{t \to T} Q(t) = \infty$$

by Theorem 3.1.1. Now applying Lemma 6.2.1 with the function $\psi(s) = s \ln(1 + s)^p$, $0 \leq p \leq 1$ (it is easy to check that it is nondecreasing and $\int_1^\infty \frac{1}{\psi(s)} ds = \infty$) and Proposition 6.2.3 yields the result. □

6.3 Time Slice Approach

In the last section, the essential ingredient to obtain the mean value inequality relating $Q(t)$ and $P(t)$ is the estimate in Lemma 6.2.2. That result points out that, when the curvature doubles, the integral of the maximum of the Ricci tensor norm is bounded below by a universal constant. It turns out that using the time slice analysis, we can deduce similar results in a slightly different manner. To be more precise, the logarithmic quantity and $ln(\int_0^T P(t)dt)$ blow up together at the first singular time. We shall also derive some other results which might be of independent interest.

For a Ricci flow solution developing a finite time singularity, let s_i be the first time such that $Q(s_i) = 2^{i+4}Q(0)$. Then by Lemma 3.1.2,

$$s_{i+1} \geq s_i + \frac{1}{16Q(s_i)} = s_i + \frac{1}{8Q(s_{i+1})}. \tag{6.8}$$

Lemma 6.3.1. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Then*

$$\sup_{[0,t]} Q(s) \leq 2^{\frac{1}{\epsilon_0}\int_0^t P(s)ds+1} 16Q(0), \tag{6.9}$$

where ϵ_0 is the constant from the claim of Lemma 6.2.2.

Proof. The result can be deduced directly from [97, Theorem 3.1]. For completeness, we provide a proof here. From the claim in Lemma 6.2.2, we have

$$\int_{s_i}^{s_{i+1}} P(t)dt \geq \epsilon_0.$$

Let N be the largest interger such that $s_N \leq t$ then

$$N\epsilon_0 \leq \int_{s_0}^{s_N} P(s)ds \leq \int_0^t P(s)ds,$$

hence

$$N \leq \frac{1}{\epsilon_0} \int_0^t P(s) ds.$$

Thus it follows that

$$\sup_{[0,t]} Q(s) \leq 2^{N+1} 16 Q(0) \leq 2^{\frac{1}{\epsilon_0} \int_0^t P(s) ds + 1} 16 Q(0).$$

□

Next we derive a mean value type inequality using the time slice argument.

Theorem 6.3.2. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Furthermore, let*

$$G(u) = \ln(16Q(0)) + 2\ln 2 + \frac{\ln 2}{\epsilon_0} \int_0^u P(s) ds.$$

Then for $0 \leq p \leq 1$, we have

$$\ln(G(t)) \leq C_1 \int_0^t \frac{P(s)}{(\ln(1 + Q(s)))^p} ds + C_2, \tag{6.10}$$

where $C_1 > 0$ only depends on ϵ_0, $C_2 > 0$ depends on ϵ_0 and $Q(0)$.

Proof. First, without loss of generality, let $Q = \sup_{[0,t]} Q(s) > 2$ and observe that for $0 \leq p \leq 1$,

$$(\ln(1 + Q(s)))^p \leq \ln(1 + Q(s)) \leq \ln(1 + Q).$$

Applying Lemma 6.3.1,

$$1 + Q \leq 2^{\frac{1}{\epsilon_0} \int_0^t P(s) ds + 2} 16 Q(0),$$

$$\ln(1 + Q) \leq \ln(16Q(0)) + 2\ln 2 + \frac{\ln 2}{\epsilon_0} \int_0^t P(s) ds.$$

Since $G(u) = \ln(16Q(0)) + 2\ln 2 + \ln 2 \int_0^u P(s) ds$, we have

$$G'(s) = \frac{\ln 2}{\epsilon_0} P(s) > 0,$$

and

$$G(s) \geq (\ln(1 + Q(s)))^p.$$

Therefore,

$$\begin{aligned}\frac{\ln 2}{\epsilon_0}\int_0^t \frac{P(s)}{(\ln(1+Q(s)))^p}ds &\geq \int_0^t \frac{G'(s)}{G(s)}ds \\ &= \ln G(t) - \ln G(0).\end{aligned}$$

The statement now follows immediately. □

Remark 6.3.1. *Theorem 6.0.5 now follows from Theorem 6.3.2 and the fact that* $\int_0^T P(s)ds$ *needs to blow up at the first singular time* T *[97].*

Next we apply the same method to a slightly different setting.

Lemma 6.3.3. *Let* $(M, g(t))$*,* $t \in [0, T)$*, be a maximal* κ*-noncollapsed Ricci flow solution on* M*. Then there exists a constant* $C = C(Q(0), \kappa)$*, such that*

$$Q(s_{i+1}) \leq C \int_{s_i}^{s_{i+1}} \int_M |\mathrm{R}|^{\frac{n}{2}+2} d\mu_{g(s)} ds, \tag{6.11}$$

and thus

$$\frac{1}{C} \leq \int_{s_i}^{s_{i+1}} \int_M |\mathrm{R}|^{\frac{n}{2}+1} d\mu_{g(s)} ds. \tag{6.12}$$

Proof. Suppose that the statement is false then as $j \to \infty$, there exist $s_{i_j} \to T$ and $a_j \to \infty$, such that

$$a_j \int_{s_{i_j}}^{s_{i_j+1}} \int_M |\mathrm{R}|^{n/2+2} d\mu_{g(s)} ds \leq Q(s_{i_j+1}).$$

Therefore, we can choose a blow-up sequence (x_j, s_{i_j+1}) and rescale (see Section 3.3 to obtain a singularity model $(M_\infty, g_\infty(s), x_\infty)$ with $|\mathrm{R}_\infty(x_\infty, 0)| = 1$.

On the other hand, due to (6.8),

$$\begin{aligned}\int_{-1/8}^{0}\int_M |\mathrm{R}(g_j(t))|^{\frac{n}{2}+2}d\mu_{g_j(t)}dt &= \frac{1}{Q(s_{i_j+1})}\int_{s_{i_j+1}-\frac{1}{8Q(s_{i_j+1})}}^{s_{i_j+1}}\int_M |\mathrm{R}(g(s)|^{\frac{n}{2}+2}d\mu_{g(s)}ds \\ &\leq \frac{1}{Q(s_{i_j+1})}\int_{s_{i_j}}^{s_{i_j+1}}\int_M |\mathrm{R}(g(s)|^{\frac{n}{2}+2}d\mu_{g(s)}ds \\ &\leq \frac{1}{a_j} \to 0.\end{aligned}$$

Hence, the limit solution is flat, a contradiction. The second statement follows from the first immediately. □

Note that Lemma 6.3.3 involves a time slice estimate similar in the spirit of the claim in Lemma 6.2.2 and, thus, applying the same method as before yields the following results. The proofs are omitted as they are almost identical to those of Lemma 6.3.1 and Theorem 6.3.2.

Proposition 6.3.4. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Then*

$$\sup_{[0,t]} Q(s) \leq 2^{C\int_0^t\int_M |\mathrm{R}|^{\frac{n}{2}+1}d\mu_{g(s)}ds+1}16Q(0). \tag{6.13}$$

Theorem 6.3.5. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal κ-noncollapsed Ricci flow solution on M. Let*

$$G(u) = \ln(16Q(0)) + 2\ln 2 + C\ln 2\int_0^u\int_M |\mathrm{R}|^{\frac{n}{2}+1}d\mu_{g(s)}ds.$$

Then for $0 \leq p \leq 1$, we have

$$\ln(G(t)) \leq C_1\int_0^t\int_M \frac{|\mathrm{R}|^{\frac{n}{2}+1}}{(\ln(1+\mathrm{R}))^p}d\mu_{g(s)}ds + C_2, \tag{6.14}$$

where $C_1 > 0$ and C_2 only depend on κ and $Q(0)$.

Remark 6.3.2. *It is shown in [96] that the function $G(t)$ must blow up as t approaches the first singular time. Therefore, Theorem 6.3.5 implies [59, Theorem 1.6].*

6.4 Nonnegative Isotropic Curvature Condition

The notion of nonnegative isotropic curvature (NIC) was first introduced by M. Micallef and J. D. Moore in [70]. A Riemannian manifold M of dimension $n \geq 4$ is said to have nonnegative isotropic curvature if for every orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$, that

$$\mathrm{R}_{1313} + \mathrm{R}_{1414} + \mathrm{R}_{2323} + \mathrm{R}_{2424} - 2\mathrm{R}_{1234} \geq 0.$$

The positive condition is defined similarly by replacing the above with a strict inequality. The isotropic curvature is also related to complex sectional curvatures described as follows. For each $p \in M$, let $T_p^C M = T_p M \otimes_{\mathbb{R}} \mathbb{C}$, then the Riemannian metric g extends naturally to a complex bilinear form

$$g : T_p^C M \times T_p^C M \to \mathbb{C},$$

and so is the Riemannian curvature tensor R to a complex multilinear form

$$\mathrm{R} : T_p^C M \times T_p^C M \times T_p^C M \times T_p^C M \to \mathbb{C}.$$

Then M has NIC if and only if,

$$\mathrm{R}(\theta, \eta, \overline{\theta}, \overline{\eta}) \geq 0$$

for all (complex) vectors θ, η satisfying $g(\theta, \theta) = g(\eta, \eta) = g(\theta, \eta) = 0$ (such a plane spanned by θ and η is called an isotropic plane, for more details, see [11]). Furthermore, this NIC condition is implied by several other commonly used curvature conditions, such as nonnegative curvature operator or point-wise $\frac{1}{4}$-pinched sectional curvature conditions, and it implies nonnegative scalar curvature. For more details, see [70] or [11].

Another interesting and relevant fact is that this condition is preserved along the Ricci flow. In dimension 4, it was proved by Hamilton [57]; higher dimension analog was extended by S. Brendle and R. Schoen [13] and also by H. Nguyen [76] independently. Using minimal surface technique, Micallef and Moore [70] showed that any compact, simply connected manifold with positive isotropic curvature is homeomorphic to S^n. By utilizing the Ricci flow and the aforementioned perseverance, Brendle and Schoen further proved the Differentiable Sphere theorem, which has been a long time conjecture since the (topological) $\frac{1}{4}$-pinched Sphere theorem was proved by M. Berger [5] and W. Klingenberg [61] around 60's. More precisely, Brendle and Schoen showed that any compact Riemannian manifold with point-wise $\frac{1}{4}$-pinched sectional curvature is diffeomorphic to a spherical space form [13].

In this section, we apply our analysis to the context of non-flat manifolds with NIC or, slightly more generally, satisfying the uniform-growth assumption as in Definition 6.0.6. Let's first recall the definition of flag curvature and Berger's Lemma.

Definition 6.4.1. *Given a unit vector e, the flag curvature on the direction e is a symmetric bilinear form on $V_e = e^{\perp}$ (the perpendicular compliment of e in $V = \mathbb{R}^n$) given by* $\mathrm{R}_e(X, X) = \mathrm{R}(e, X, e, X)$ *for any $X \in V_e$.*

We further define $\rho_e = \sup_{|X|=|Y|=1, <X,Y>=0} (\mathrm{R}_e(X, X) - \mathrm{R}_e(Y, Y))$ and $\rho = \sup_e \rho_e$.

Remark 6.4.1. *It is clear that ρ is no more than the difference between the maximum and minimum of sectional curvatures at each point.*

Lemma 6.4.2 (Berger [6]). *For orthonormal vectors U, V, X, W in T_pM, we have*

a) $|\mathrm{R}(U, V, U, W)| \leq \frac{1}{2}\rho_U$,

b) $|\mathrm{R}(U, V, X, W)| \leq \frac{1}{6}\rho_{U+X} + \frac{1}{6}\rho_{U-X} + \frac{1}{6}\rho_{U+W} + \frac{1}{6}\rho_{U-W} \leq \frac{2}{3}\rho$.

It is well-known [70, 71, 80] that in dimension four, NIC is equivalent to the non-negativity of the Weitzenböck operator as in Subsection 2.3.3. The following result is well-known, for example, see [89] or [11, Prop. 7.3]. We'll provide a proof for completeness.

Lemma 6.4.3. *Let (M^n, g), $n > 4$, be a Riemannian manifold with NIC then* $|\mathrm{R}| \le c(n)\mathrm{S}$.

Proof. We have

$$\begin{aligned}
&\mathrm{R}_{ikik} + \mathrm{R}_{ilil} + \mathrm{R}_{jkjk} + \mathrm{R}_{jljl} \ge 0,\\
&\mathrm{R}_{ii} + \mathrm{R}_{jj} \ge 2\mathrm{R}_{ijij},\\
&(n-4)\mathrm{R}_{ii} + \mathrm{S} \ge 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathrm{R}_{ii} \ge -\frac{\mathrm{S}}{n-4},\\
&\mathrm{R}_{ii} = \mathrm{S} - \Sigma_{j\neq i}\mathrm{R}_{jj} \le \mathrm{S} + (n-1)\frac{\mathrm{S}}{n-4} = c_1\mathrm{S},\\
&\mathrm{R}_{ijij} \le \frac{1}{2}(\mathrm{R}_{ii} + \mathrm{R}_{jj}) \le c_1\mathrm{S},\\
&\mathrm{R}_{ijij} \ge -3c_1\mathrm{S},
\end{aligned}$$

Now by Lemma 6.4.2,

$$|\mathrm{R}_{ijik}| \le 2c_1\mathrm{S}, \tag{6.15}$$

$$|\mathrm{R}_{ijkl}| \le \frac{8}{3}c_1\mathrm{S}. \tag{6.16}$$

Thus, there exists a constant $c(n)$ such that

$$|\mathrm{R}| \le c(n)|\mathrm{S}|.$$

□

A direct consequence of the above lemma is the following proposition.

Proposition 6.4.4. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal Ricci flow solution with NIC, then there exists $c = c(n, g(0))$ such that $|\mathrm{R}| \leq c\mathrm{S}$ along the flow.*

Proof. If $n > 4$ then the result follows from part b) of Lemma 6.4.3.
If $n = 4$, then by the pinching estimate of [24] and Lemma 2.3.4,

$$\frac{|\mathring{\mathrm{Rc}}|}{\mathrm{S}} \leq c_1(n, g(0)) + c_2(n) \sup_{M\times[0,T)} \sqrt{\frac{|\mathrm{W}|}{\mathrm{S}}} \leq c_1 + c_2\sqrt{\frac{1}{\sqrt{6}}}.$$

Furthermore, $|\mathrm{R}|^2 = |\mathrm{W}|^2 + \frac{\mathrm{S}^2}{6} + 2|\mathring{\mathrm{Rc}}|^2$, the result follows. □

Remark 6.4.2. *One easy consequence is that a non-flat Ricci flow solution on a closed manifold with NIC satisfies the uniform-growth condition as in Definition 6.0.6.*

Theorem 6.4.5. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition. If either*

$$\int_M |\mathrm{S}|^\alpha d\mu_{g(t)} < \infty, \text{ for some } \alpha > n/2,$$

or

$$\int_0^T \int_M |\mathrm{S}|^\alpha d\mu_{g(t)} dt < \infty \text{ for some } \alpha \geq \frac{n}{2} + 1,$$

then the solution can be extended past time T.

Proof. First we observe that, by Holder inequality, for the second condition, it suffices to prove the case $\alpha = \frac{n}{2} + 1$.

The proof is by a contradiction argument. Suppose the flow develops a singularity at time T then we carry a point-picking and rescaling procedure described in Section 3.3 to obtain a singularity model $(M_\infty, g_\infty(s), x_\infty)$ with

$$|\mathrm{R}_\infty(x_\infty, 0)| = 1. \tag{6.17}$$

Recalling the scaling property of S, we calculate:

$$\begin{aligned}\int_M |\mathrm{S}(g_i(.))|^\alpha d\mu_{g_i(.)} &= \int_M Q_i^{-\alpha}|\mathrm{S}(g(.)|^\alpha Q_i^{n/2} d\mu_{g(.)}\\ &= Q_i^{\frac{n}{2}-\alpha}\int_M |\mathrm{S}|^\alpha d\mu_{g(.)} \to 0 \text{ as } i \to \infty.\end{aligned}$$

In the second case, we have:

$$\begin{aligned}\int_{-1}^0\int_M |\mathrm{S}(g_i(s))|^{\frac{n}{2}+1} d\mu_{g_i(s)} ds &= \int_{t_i-\frac{1}{Q_i}}^{t_i}\int_M Q_i^{-\frac{n}{2}-1}|\mathrm{S}(g(t)|^{\frac{n}{2}+1} Q_i^{n/2} d\mu_{g(t)} Q_i dt\\ &= \int_{t_i-\frac{1}{Q_i}}^{t_i}\int_M |\mathrm{S}(g(t)|^{\frac{n}{2}+1} d\mu_{g(t)} dt \to 0 \text{ as } i \to \infty.\end{aligned}$$

By the dominating convergence theorem, the singularity model $(M_\infty, g_\infty(s), x_\infty)$ is scalar flat, which is a contradiction to our uniform-growth condition. □

Applying Lemma 6.2.1 in this context, we obtain the following lemma.

Lemma 6.4.6. *Let $(M, g(t))$, $t \in [0, T)$, be a Ricci flow solution satisfying the uniform-growth condition. Suppose $\psi : (0, \infty) \to (0, \infty)$ is a nondecreasing function such that*

$$\int_1^\infty \frac{1}{\psi(s)} ds = \infty. \tag{6.18}$$

If there is a mean value inequality of the form

$$O(t) \leq \int_0^t C_1\psi(O(s))G(s)ds + C_2 = h(t), \tag{6.19}$$

and $\int_0^T G(t)dt < \infty$, then the solution can be extended past time T.

Proof. If T is a first singular time then, by Theorem 3.1.1, $\lim_{t\to T} Q(t) = \infty$. The uniform-growth condition implies that the curvature tensor and the scalar curvature blow up together. Applying Lemma 6.2.1 we obtain a contradiction, hence the result holds. □

We are ready to state a mean value inequality.

Lemma 6.4.7. *Let $(M, g(t))$, $t \in [0, T)$, be a maximal Ricci flow solution satisfying the uniform-growth condition. Then the following mean value inequality holds: there exists $C_1 = C_1(n, g(0))$ and C_0 such that,*

$$\sup_{[0,t]} O(t) \leq C_0 \int_0^t \int_M |\mathrm{S}(g(t))|^{n/2+2} d\mu_{g(t)} dt + C_1 \tag{6.20}$$

for all $t < T$.

Proof. First we observe that there is a constant $c_0(n)$ such that $|\mathrm{S}|(x,t) \leq c_0 |\mathrm{R}|(x,t)$. Also by Lemma 3.1.2, if $t \leq \frac{1}{16Q_0}$ then

$$O(t) \leq c_0 Q(t) \leq 2c_0 Q(0). \tag{6.21}$$

Let $C_1 = 2c_0 Q(0)$. Suppose the statement is false then there exist sequences $t_i \to T$ and $a_i \to \infty$ such that

$$a_i \int_0^{t_i} \int_M |\mathrm{S}|^{n/2+2} d\mu_{g(s)} ds + 2c_0 Q(0) \leq \sup_{[0,t_i]} O(t) \leq c_0 \sup_{[0,t_i]} Q(t).$$

Let $Q_i = \sup_{[0,t_i]} Q(t)$ then there exist x_i, $\tilde{t}_i \to T$ such that $Q_i = |\mathrm{R}(x_i, \tilde{t}_i)|$. Now we can invoke a convergence process again to obtain a singularity model $(M_\infty, g_\infty(t), x_\infty)$, $t \in [-\infty, 0]$, with $|\mathrm{R}_\infty(x_\infty, 0)| = 1$.

On the other hand, we have

$$\begin{aligned} \int_{-1}^0 \int_M |\mathrm{S}(g_i(s))|^{n/2+2} d\mu_{g_i(s)} ds &= \frac{1}{Q_i} \int_{\tilde{t}_i - \frac{1}{Q_i}}^{\tilde{t}_i} \int_M |\mathrm{S}(g(t)|^{n/2+2} d\mu_{g(t)} dt \\ &\leq \frac{c_0 Q_i - 2c_0 Q(0)}{a_i Q_i} \to 0. \end{aligned}$$

Thus, by the dominating convergence theorem, the limit solution is scalar flat, which is a contradiction to the uniform-growth condition. □

Proof. (**Theorem 6.0.7**) Applying Lemma 6.4.6 with the function $\psi(s) = s \ln(1 + s)^p$, $0 \leq p \leq 1$ (it is easy to check that it is nondecreasing and $\int_1^\infty \frac{1}{\psi(s)} ds = \infty$) and Lemma 6.4.7 yields the result. □

CHAPTER 7

THE WEYL TENSOR OF A GRADIENT RICCI SOLITON

As the major obstruction to understand GRS in higher dimensions is the non-triviality of the Weyl tensor, this chapter is devoted to studying the delicate role of the Weyl tensor within a gradient soliton structure. This is joint work with X. Cao [26].

In particular, we derive several new identities on the Weyl tensor of GRS in dimension four. In the first part, we prove the following Bochner-Weitzenböck type formula for the norm of the self-dual Weyl tensor using flow equations and some ideas related to Einstein manifolds.

Theorem 7.0.8. *Let (M, g, f, λ) be a four-dimensional GRS. Then we have the following Bochner-Weitzenböck formula:*

$$\begin{aligned}\Delta_f|W^+|^2 =& 2|\nabla W^+|^2 + 4\lambda|W^+|^2 - 36 det W^+ - \langle Rc \circ Rc, W^+\rangle \\ =& 2|\nabla W^+|^2 + 4\lambda|W^+|^2 - 36 det W^+ - \langle Hess f \circ Hess f, W^+\rangle \,. \end{aligned} \tag{7.1}$$

It potentially has several applications and we will present a couple of them in Section 7.2 including a gap theorem. More precisely, if the GRS is not locally conformally flat and the divergence of the Weyl tensor is relatively small, then the L_2-norm of the Weyl tensor is bounded below by a topological constant (cf. Theorem 7.2.1). The proof, in a similar manner to that of [48], uses some ideas from the solution to the Yamabe problem.

In the second part, we are mostly concerned with the interaction of different curvature components, gradient and Hessian of the potential function. In particular, an interesting connection is illustrated by the following integration by parts formula.

Theorem 7.0.9. *Let (M, g, f, λ) be a closed GRS. Then we have the following identity:*

$$\int_M \langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = \int_M \langle \mathrm{W}, Hess f \circ Hess f \rangle = \int_M \mathrm{W}(Hess f, Hess f) = \int_M \mathrm{W}_{ijkl} f_{ik} f_{jl}$$
$$= \frac{1}{n-3} \int_M \langle \delta \mathrm{W}, (n-4)M + (n-2)P \rangle \,. \tag{7.2}$$

In particular, in dimension four, the identity becomes

$$\int_M \langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = 4 \int_M |\delta \mathrm{W}|^2. \tag{7.3}$$

Remark 7.0.3. *For definitions of M and P, see Section 7.3. In dimension four, the statement also holds if replacing* W *by* $\mathrm{W}^{\pm}$*, see Corollary 7.3.8. This result exposes the intriguing interaction between the Weyl tensor and the potential function f on a GRS. It will be interesting to extend those identities to a (possibly non-compact) smooth metric measure space or generalized Einstein manifold.*

The interactions of various curvature components and the soliton potential function can be applied to study the classification problem. For example, Theorem 7.4.1 asserts rigidity of the Ricci curvature tensor in dimension four. More precisely, if the Ricci tensor at each point has at most two eigenvalues with multiplicity one and three, then any such closed GRS must be rigid. It is interesting to compare this result with classical classification results of the Codazzi tensor, which requires both distribution of eigenvalues and information on the first derivative (see [8, Chapter 16, Section C]).

This rest of the chapter is organized as follows. Section 7.1 provides a proof of Theorem 7.0.8 and Section 7.2 gives immediate applications of the new Bochner-Weitzenböck type formula. In Section 7.3, we first discuss a general framework to study the interaction of different components of the curvature with the potential function, and then prove Theorem 7.0.9. Then, in the last Section, we apply our framework to obtain various rigidity results.

7.1 Bochner-Weitzenböck Formula

In this section, we prove Theorem 7.0.8, a new Bochner-Weitzenböck formula for the Weyl tensor of GRS's, which generalizes the one for Einstein manifolds. Bochner-Weitzenböck formulas have been proven a powerful tool to find connections between topology and geometry with certain curvature conditions (for example, see [47, 83, 99]).

Particularly, in dimension four, if $\delta W^+ = 0$ (this contains all Einstein manifolds), we have the following well-known formula (see [8, 16.73]),

$$\Delta |W^+|^2 = 2|\nabla W^+|^2 + S|W^+|^2 - 36\det W^+. \tag{7.4}$$

This equation plays a crucial role to obtain a L_2-gap theorem of the Weyl tensor and to study the classification problem of Einstein manifolds (cf. [48, 49, 101]).

Our first technical lemma gives a formula of $\Delta_f W$ in a local frame. Also it is noticed that the Einstein summation convention is used repeatedly here.

Lemma 7.1.1. *Let (M, g, f, λ) be a GRS and $\{e_i\}_{i=1}^n$ be a local normal frame, then the following holds,*

$$\begin{aligned}
\Delta_f W_{ijkl} =& 2\lambda W_{ijkl} - 2(C_{ijkl} - C_{ijlk} + C_{ikjl} - C_{iljk}) \\
&- \frac{2}{(n-2)^2} g^{pq}(\mathrm{Rc}_{ip}\mathrm{Rc}_{qk}g_{jl} - \mathrm{Rc}_{ip}\mathrm{Rc}_{ql}g_{jk} + \mathrm{Rc}_{jp}\mathrm{Rc}_{ql}g_{ik} - \mathrm{Rc}_{jp}\mathrm{Rc}_{qk}g_{il}) \\
&+ \frac{2S}{(n-2)^2}(\mathrm{Rc}_{ik}g_{jl} - \mathrm{Rc}_{il}g_{jk} + \mathrm{Rc}_{jl}g_{ik} - \mathrm{Rc}_{jk}g_{il}) \\
&- \frac{2}{n-2}(R_{ik}R_{jl} - R_{jk}R_{il}) - \frac{2(S^2 - |\mathrm{Rc}|^2)}{(n-1)(n-2)^2}(g_{ik}g_{jl} - g_{il}g_{jk}),
\end{aligned} \tag{7.5}$$

here $C_{ijkl} = g^{pq}g^{rs}W_{pijr}W_{slkq}$.

Proof. First, as in Section 3.5, a GRS can be realized as a self-similar solution to the Ricci flow via $\phi(x,t)$, a family of diffeomorphisms generated by, for $\tau(t) = 1 - 2\lambda t$, $X = \frac{1}{\tau}\nabla f$. In particular, $\mathrm{W}(t) = \tau\phi^*\mathrm{W}$. Let p be a point in M and $\{e_i\}_{i=1}^n$ be a basis of T_pM, and we obtain a local normal frame via extending e_i to a neighborhood by parallel translation along geodesics with respect to $g(0)$. We observe, at that chosen point,

$$\frac{d}{dt}\mathrm{W}(t)_{ijkl}\mid_{t=0} = (\frac{d}{dt}\tau\phi^*\mathrm{W})_{ijkl}\mid_{t=0} = -\frac{2\lambda}{\tau}\mathrm{W}_{ijkl} + (L_{\nabla f}\mathrm{W})_{ijkl}. \tag{7.6}$$

Furthermore,

$$\begin{aligned} L_{\nabla f}\mathrm{W}_{ijkl} =& \nabla f(\mathrm{W}_{ijkl}) - \mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) - \mathrm{W}(e_i, [\nabla f, e_j], , e_k, e_l) \\ & - \mathrm{W}(e_i, e_j, [\nabla f, e_k], e_l) - \mathrm{W}(e_i, e_j, e_k, [\nabla f, e_l]). \end{aligned} \tag{7.7}$$

We calculate that

$$\mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) = \mathrm{W}(\nabla_{\nabla f} e_i - \nabla_{e_i}\nabla f, e_j, e_k, e_l) = -\mathrm{W}(\nabla_{e_i}\nabla f, e_j, e_k, e_l).$$

By the soliton structure, $\nabla_{e_i}\nabla_. f = -\mathrm{Rc}(e_i, .) + \lambda g(e_i, .)$. Thus,

$$\begin{aligned} \mathrm{W}([\nabla f, e_i], e_j, e_k, e_l) &= -\mathrm{W}(\lambda e_i - \mathrm{Rc}(e_i), e_j, e_k, e_l) \\ &= -\lambda \mathrm{W}_{ijkl} + g^{pq}\mathrm{Rc}_{ip}\mathrm{W}_{qjkl}. \end{aligned} \tag{7.8}$$

Combining (7.6),(7.7), and (7.8) we obtain,

$$\begin{aligned} \frac{d}{dt}\mathrm{W}(t)_{ijkl}\mid_{t=0} =& \nabla f(\mathrm{W}_{ijkl}) + 2\lambda \mathrm{W}_{ijkl} \\ & - g^{pq}(\mathrm{Rc}_{ip}\mathrm{W}_{qjkl} + \mathrm{Rc}_{jp}\mathrm{W}_{iqkl} + \mathrm{Rc}_{kp}\mathrm{W}_{ijql} + \mathrm{Rc}_{ip}\mathrm{W}_{qjkl}). \end{aligned}$$

Now, in combination with (3.9), the result follows. □

In dimension four, we obtain simplification due to the special structure given by the Hodge operator. That gives the proof of the first main theorem.

Proof. **(Theorem 7.0.8)** We observe that,

$$\left\langle W^+, \Delta_f W^+ \right\rangle = \langle W^+, \Delta W^+ \rangle - \left\langle W^+, \nabla_{\nabla f} W^+ \right\rangle$$
$$= \langle W^+, \Delta W^+ \rangle - \frac{1}{2}\nabla_{\nabla f}|W^+|^2.$$

Therefore,

$$\Delta_f|W^+|^2 = \Delta|W^+|^2 - \nabla_{\nabla f}|W^+|^2 = 2\left\langle W^+, \Delta_f W^+ \right\rangle + 2|\nabla W^+|^2.$$

To calculate the first term of the right hand side, we use the normal form of the Weyl tensor (2.14). As usual, a local normal frame is obtained by parallel translation along geodesic lines. Then (2.11) gives a basis of eigenvectors $\{\alpha_i\}_{i=1}^3$ of W^+ with corresponding eigenvalues $\lambda_i = a_i + b_i$. Consequently,

$$\left\langle W^+, \Delta_f W^+ \right\rangle = \sum_i \lambda_i \Delta_f W^+(\alpha_i, \alpha_i). \tag{7.9}$$

In order to use Lemma 7.1.1, it is necessary to calculate the C_{ijkl} terms. By the normal form, we have

$$\begin{aligned}
C_{1212} &= a_1^2 + b_2^2 + b_3^2, & C_{1234} &= -2a_1b_3,\\
C_{1221} &= -2b_2b_3, & C_{1243} &= 2a_1b_2,\\
C_{1122} &= 2a_2a_3, & C_{1324} &= 2a_2b_3,\\
C_{1221} &= -2b_2b_3, & C_{1423} &= -2a_3b_2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta_f W_{1212} =& 2\lambda a_1 - 2(a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3)\\
&- \frac{1}{2}\sum_p (\mathrm{Rc}_{1p}^2 + \mathrm{Rc}_{2p}^2) + \frac{S}{2}(\mathrm{Rc}_{11} + \mathrm{Rc}_{12})\\
&- (\mathrm{Rc}_{11}\mathrm{R}_{22} - \mathrm{Rc}_{12}^2) - \frac{1}{6}(S^2 - |\mathrm{Rc}|^2),\\
\Delta_f W_{1234} =& 2\lambda b_1 - 4(a_1b_1 + a_2b_3 + a_3b_2) + (\mathrm{Rc}_{13}\mathrm{Rc}_{24} - \mathrm{Rc}_{23}\mathrm{Rc}_{14}).
\end{aligned}$$

Therefore,

$$\Delta_f W^+(\alpha_1, \alpha_1) = 2\lambda\lambda_1 - 2\lambda_1^2 - 4\lambda_2\lambda_3 - \frac{1}{12}(|Rc|^2 - S^2) - T_1, \tag{7.10}$$

in which,

$$\begin{aligned} 2T_1 =& Rc_{11}Rc_{22} + Rc_{33}Rc_{44} + 2Rc_{13}Rc_{24} - Rc_{12}^2 - 2Rc_{23}Rc_{14} - Rc_{34}^2 \\ =& (Rc \circ Rc)(\alpha_1, \alpha_1). \end{aligned}$$

Similar calculations hold when replacing α_1 by α_2, α_3,

$$\Delta_f W^+(\alpha_2, \alpha_2) = 2\lambda\lambda_2 - 2\lambda_2^2 - 4\lambda_1\lambda_3 - \frac{1}{12}(|Rc|^2 - S^2) - \frac{1}{2}Rc \circ Rc(\alpha_2, \alpha_2), \tag{7.11}$$

$$\Delta_f W^+(\alpha_3, \alpha_3) = 2\lambda\lambda_3 - 2\lambda_3^2 - 4\lambda_1\lambda_2 - \frac{1}{12}(|Rc|^2 - S^2) - \frac{1}{2}Rc \circ Rc(\alpha_3, \alpha_3). \tag{7.12}$$

Combining (7.9), (7.10), (7.11), (7.12) yields,

$$\begin{aligned} \left\langle W^+, \Delta_f W^+ \right\rangle =& 2\lambda|W^+|^2 - 18\det W^+ - \sum_i T_i\lambda_i \\ =& 2\lambda|W^+|^2 - 18\det W^+ - \frac{1}{2}\left\langle Rc \circ Rc, W^+ \right\rangle. \end{aligned}$$

The first equality then follows. The second equality comes from the soliton equation, the property that W^+ is trace-free and Remark 3.5.2. □

7.2 Applications of the Bochner-Weitzenböck Formula

This section presents some applications of the Bochner-Weitzenböck formula.

7.2.1 A Gap Theorem for the Weyl Tensor

In [48], under the assumptions $W^+ \neq 0$, $\delta W^+ = 0$, and the positivity of the Yamabe constant, M. Gursky proves the following inequality, relating $\|W^+\|_{L_2}$ with topological invariants of a closed four-manifold,

$$\int_M |W^+|^2 d\mu \geq \frac{4}{3}\pi^2(2\chi(M) + 3\tau(M)). \tag{7.13}$$

Our main result here is to prove an analog for GRS's. It is noted that the particular structure of GRS allows us to relax the harmonic self-dual condition at the expense of a worse coefficient due to the lack of an improved Kato's inequality.

Theorem 7.2.1. *Let* (M, g, f, λ) *be a closed four-dimensional shrinking GRS with*

$$\int_M \langle W^+, Hess f \circ Hess f \rangle \leq \frac{2}{3}\int S|W^+|^2, \tag{7.14}$$

then, unless $W^+ \equiv 0$*,*

$$\int_M |W^+|^2 d\mu > \frac{4}{11}\pi^2(2\chi(M) + 3\tau(M)). \tag{7.15}$$

Remark 7.2.1. *By Corollary 7.3.8, assumption (7.14) is equivalent to*

$$\int |\delta W^+|^2 \leq \int \frac{S}{6}|W^+|^2.$$

To prove Theorem 7.2.1, we follow an idea of [48] and introduce a Yamabe-type conformal invariant. First, the conformal Laplacian is given by,

$$L = -6\Delta + S.$$

Furthermore, we define that

$$\begin{aligned} F_{a,b} =& aS - b|W^+|, \\ L_{a,b} =& -6a\Delta_g + F_{a,b} = aL - bW^+, \end{aligned}$$

where a and b are constants to be determined later. Under a conformal transformation as described in Section 2.5, for any function Φ, we have

$$\begin{aligned}
\tilde{L}(\Phi) &= u^{-3} L(\Phi u), \\
\widetilde{L_{a,b}}\Phi &= u^{-3} L_{a,b}(\Phi u), \\
\widetilde{F_{a,b}} &= u^{-3}(-6a\Delta_g + F_{a,b})u, \\
\int_M \widetilde{F_{a,b}} d\widetilde{\mu} &= \int_M u(-6a\Delta_g + F_{a,b}) u d\mu \\
&= \int_M (F_{a,b}u^2 + 6a|\nabla u|^2) d\mu.
\end{aligned}$$

The Yamabe problem is, for a given Riemannian manifold (M, g), to find a constant scalar curvature metric in its conformal class $[g]$. That is equivalent to find a critical point of the following functional, for any C^2 positive function u, let $\tilde{g} = u^2 g$, define

$$Y_g[u] = \frac{\langle u, Lu\rangle_{L_2}}{\|u\|^2_{L_4}} = \frac{\int_M \widetilde{S} d\widetilde{\mu}}{\sqrt{\int_M d\widetilde{\mu}}}.$$

Then the conformal invariant Y is defined as

$$Y(M, [g]) = \inf\{Y_g[u] \colon \text{u is a positive } C^2 \text{ function on M}\}.$$

For an expository account on the Yamabe problem, see [65].

As $F_{a,b}$ conformally transforms like the scalar curvature, in analogy with the discussion above, we can define the following conformal invariant.

Definition 7.2.2. *Given a Riemannian manifold (M, g), define*

$$\hat{Y}_{a,b}(M, [g]) = \inf\{(\hat{Y}_{a,b})_g[u] \colon u \text{ is a positive } C^2 \text{ function on } M\},$$

where

$$(\hat{Y}_{a,b})_g[u] = \frac{\langle u, L_{a,b}u\rangle_{L_2}}{\|u\|^2_{L_4}} = \frac{\int_M \widetilde{F_{a,b}} d\widetilde{\mu}}{\sqrt{\int_M d\widetilde{\mu}}}.$$

For the case of interest, we shall denote

$$F = F_{1,6\sqrt{6}} = \mathrm{S} - 6\sqrt{6}|\mathrm{W}^+|,$$
$$\hat{Y}(M) = \hat{Y}_{1,6\sqrt{6}}(M, [g]),$$

when the context is clear. First we observe the following simple inequality.

Lemma 7.2.3. *Let (M^n, g) be a closed n-dimensional Riemannian manifold which is not locally conformally flat, and (S^n, g_{sd}) be the sphere with standard metric. Then*

$$\hat{Y}(M, [g]) \leq Y(M, [g]) < Y(S^n, [g_{sd}]) = \hat{Y}(S^n, [g_{sd}]). \tag{7.16}$$

Proof. The first inequality follows from the definition and the following observation. Given a metric g, a positive function u and $b \geq 0$, then

$$\langle u, Lu \rangle_{L_2} - \langle u, L_{1,b} u \rangle_{L_2} = \int_M b|\mathrm{W}^+|u^2 d\mu \geq 0.$$

The second inequality is a result of T. Aubin [3] and R. Schoen [88]. The last inequality is an immediate consequence of the fact that the standard metric on S^n is locally conformally flat (W = 0). □

On a complete gradient shrinking soliton, the scalar curvature is positive unless the soliton is isometric to the flat Euclidean space [86]. Therefore, if the GRS is not flat then the existence of a solution to the Yamabe problem [65] implies that $Y_g > 0$. This observation is essential because of the following result.

Proposition 7.2.4. *Let (M, g) be a closed four-dimensional Riemannian manifold. If $Y(M) > 0$ and $\hat{Y}(M) \leq 0$, then there is a smooth metric $\tilde{g} = u^2 g$ such that*

$$\int_M \widetilde{\mathrm{S}}^2 d\widetilde{\mu} \leq 216 \int_M |\widetilde{\mathrm{W}}^+|^2 d\widetilde{\mu}. \tag{7.17}$$

Furthermore, the equality holds only if $\hat{Y}(M) = 0$ and $\widetilde{\mathrm{S}} = 6\sqrt{6}|\widetilde{\mathrm{W}}|$.

Proof. The proof is almost identical to [48, Prop 3.5]. Thus, we provide a brief argument here. Through a conformal transformation, the Yamabe problem can be solved via variational approach for an appropriate eigenvalue PDE problem. In particular, the existence of solution under the assumption $Y(M) < Y(S^n)$ depends solely on the analysis of regularity of the Laplacian operator (but not on the reaction term) [65, Theorem 4.5].

In our case, F conformally transforms as scalar curvature and Lemma 7.2.3 holds, then there exists a minimizer v for $\hat{Y}_g[.]$, such that under normalization $\|v\|_{L_4} = 1$, the metric $\tilde{g} = v^2 g$ satisfies $\widetilde{F} = \widetilde{\mathrm{S}} - 6\sqrt{6}|\widetilde{\mathrm{W}}^+| = \hat{Y}(M)$. Applying $Y(M) > 0$ and $\hat{Y}(M) \leq 0$ we obtain,

$$\begin{aligned}
\int_M \widetilde{\mathrm{S}}^2 d\tilde{\mu} &= \int_M 6\sqrt{6}|\widetilde{\mathrm{W}}^+|\widetilde{\mathrm{S}} d\tilde{\mu} + \hat{Y}(M)\int_M \widetilde{\mathrm{S}} d\tilde{\mu} \\
&\leq \int_M 6\sqrt{6}|\widetilde{\mathrm{W}}^+|\widetilde{\mathrm{S}} d\tilde{\mu} \\
&\leq 6\sqrt{6}\Big(\int_M |\widetilde{\mathrm{W}}^+|^2 d\tilde{\mu}\Big)^{1/2}\Big(\int_M |\widetilde{\mathrm{S}}|^2 d\tilde{\mu}\Big)^{1/2}.
\end{aligned}$$

Therefore, $\int_M \widetilde{\mathrm{S}}^2 d\tilde{\mu} \leq 216 \int_M |\widetilde{\mathrm{W}}^+|^2 d\tilde{\mu}$. The equality case is attained if only if $\tilde{g}$ attains the infimum, $\hat{Y}(M) = 0$ and $\widetilde{\mathrm{S}} = 6\sqrt{6}|\widetilde{\mathrm{W}}|$. □

Proposition 7.2.5. *Let (M, g, f, λ) be a closed four-dimensional shrinking GRS satisfying (7.14) and $\mathrm{W}^+ \neq 0$, then $\hat{Y}(M) \leq 0$. Moreover, equality holds only if W^+ has the form $\omega diag(-1, -1, 2)$ for some $\omega \geq 0$ at each point.*

Proof. By Theorem 7.0.8, we have

$$\Delta_f |\mathrm{W}^+|^2 = 2|\nabla \mathrm{W}^+|^2 + 4\lambda|\mathrm{W}^+|^2 - 36\det\nolimits_{\Lambda^2_+}\mathrm{W}^+ - \langle \mathrm{Rc}\circ\mathrm{Rc}, \mathrm{W}^+\rangle .$$

Integrating both sides and applying (7.14) yield

$$\int_M \Delta_f|\mathrm{W}^+|^2 d\mu \geq \int_M \Big[2|\nabla \mathrm{W}^+|^2 + (\frac{\mathrm{S}}{3} + \Delta f)|\mathrm{W}^+|^2 - 36\det\nolimits_{\Lambda^2_+}\mathrm{W}^+\Big].$$

Via integration by parts, we have

$$\int_M \nabla f(|W^+|^2)d\mu = \int_M \left\langle \nabla f, \nabla |W^+|^2 \right\rangle d\mu = -\int_M \Delta f |W^+|^2 d\mu.$$

Therefore, we arrive at

$$0 \geq \int_M \Big(2|\nabla W^+|^2 + \frac{S}{3}|W^+|^2 - 36\det_{\Lambda^2_+} W^+\Big).$$

We also have the following pointwise estimates,

$$|\nabla W^+|^2 \geq |\nabla |W^+||^2,$$
$$-18\det W^+ \geq -\sqrt{6}|W^+|^3.$$

The first one is the classical Kato's inequality while the second one is purely algebraic. Thus, for $u = |W^+|$,

$$\int_M (\frac{1}{3}Fu^2 + 2|\nabla u|^2)d\mu \leq 0.$$

Hence, if $|\widetilde{W^+}| > 0$ everywhere then the statement follows. If $|\widetilde{W^+}| = 0$ somewhere, let M_ϵ be the set of points at which $|\widetilde{W^+}| < \epsilon$. By the analyticity of a closed GRS [4], $\mathrm{Vol}(M_\epsilon) \to 0$ as $\epsilon \to 0$. Let $\eta_\epsilon : [0,\infty) \to [0,\infty)$ be a C^2 positive function which is $\epsilon/2$ on $[0,\epsilon/2]$, identity on $[\epsilon,\infty)$ and $0 \leq \eta'_\epsilon \leq 10$. If $u_\epsilon = \eta_\epsilon \circ u$, then u_ϵ is C^2 and positive. In addition, we have,

$$\int_M Fu_\epsilon^2 d\mu \leq \int_{M-M_\epsilon} Fu^2 d\mu + C\epsilon^2 \mathrm{Vol}(M_\epsilon),$$
$$\int_M |\nabla u_\epsilon|^2 d\mu = \int_M |\eta'_\epsilon \nabla u|^2 d\mu \leq \int_{M-M_\epsilon} |\nabla u|^2 d\mu + C\mathrm{Vol}(M_\epsilon),$$

where C is a constant depending on the metric. Therefore, we have,

$$\inf_{\epsilon>0}\{\int_M (Fu_\epsilon^2 + 6|\nabla u_\epsilon|^2)d\mu\} \leq 0.$$

Consequently, $\hat{Y}(M) \leq 0$.

Now, equality holds only if $\int_M (\frac{1}{3}Fu^2 + 2|\nabla u|^2) d\mu = 0$ and the equality happens in each point-wise estimate above. The result then follows.

□

We are now ready to prove the main result of this subsection.

Proof. (**Theorem 7.2.1**)

By Proposition 7.2.5, we have $\hat{Y}(g) \leq 0$ and $Y(M) > 0$. Otherwise $\mathrm{S} = 0$ and the GRS is flat by [86], which is a contradiction to $\mathrm{W}^+ \neq 0$. Therefore, following Proposition 7.2.4, there is a conformal transformation $\tilde{g} = u^2 g$ with

$$\int_M \widetilde{\mathrm{S}}^2 d\widetilde{\mu} \leq 216 \int_M |\widetilde{\mathrm{W}}^+|^2 d\widetilde{\mu}. \tag{7.18}$$

According to (2.16) and (2.17),

$$\begin{aligned} 2\pi^2(2\chi(M) + 3\tau(M)) &= \int_M |\widetilde{\mathrm{W}}^+|^2 d\widetilde{\mu} - \frac{1}{4}\int_M |\widetilde{\mathrm{E}}|^2 d\widetilde{\mu} + \frac{1}{48}\int_M \widetilde{\mathrm{S}}^2 d\widetilde{\mu} \\ &\leq \int_M |\widetilde{\mathrm{W}}^+|^2 d\widetilde{\mu} + \frac{1}{48}\int_M \widetilde{\mathrm{S}}^2 d\widetilde{\mu} \\ &\leq (1 + \frac{9}{2})\int_M |\widetilde{\mathrm{W}}^+|^2 d\widetilde{\mu}. \end{aligned} \tag{7.19}$$

Here we used (7.18) in the last step. Since $\|\mathrm{W}^+\|_{L_2}$ is conformally invariant, (7.15) then follows.

Now the equality holds only if all equalities hold in (7.19), (7.18) and (7.14). The first one implies that $\tilde{g}$ is Einstein. Therefore, by [49, Theorem 1], inequality (7.18) is strict unless $\mathrm{S} \equiv 0$. But this is a contradiction to $Y(M) > 0$. Thus the inequality is strict.

□

7.2.2 Isotropic Curvature

Another application is the following inequality which is an improvement of [100, Prop 2.6].

Proposition 7.2.6. *Let (M, g, f, λ) be a four-dimensional GRS, then we have*

$$\Delta_f u \leq (2\lambda + \frac{3}{2}u - \mathrm{S})u - \frac{1}{4}|\mathrm{Rc}|^2 \tag{7.20}$$

in the distribution sense where $u(x)$ is the smallest eigenvalue of $\frac{S}{3} - 2\mathrm{W}_{\pm}$.

Proof. Let $X_{1234} = \frac{S}{3} - 2\mathrm{W}(e_{12} + e_{34}, e_{12} + e_{34})$ for any 4–orthonormal basis. We use the normal form discussed in (2.14) and obtain a local frame by parallel translation along geodesic lines. We denote $\{\alpha_i\}_{i=1}^3$ the basis of Λ_2^+ as in (2.11) with corresponding eigenvalues $\lambda_i = a_i + b_i$. Without loss of generality, we can assume $a_1 + b_1 \geq a_2 + b_2 \geq a_3 + b_3$ and thus $u(x) = X_{1234}(x)$. Using Lemma 7.1.1, we compute

$$\begin{aligned}\Delta_f \mathrm{W}_{1212} =& 2\lambda a_1 - 2(a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3)\\ &- \frac{1}{2}\sum_p(\mathrm{Rc}_{1p}^2 + \mathrm{Rc}_{2p}^2) + \frac{S}{2}(\mathrm{Rc}_{11} + \mathrm{Rc}_{12})\\ &- (\mathrm{Rc}_{11}\mathrm{R}_{22} - \mathrm{Rc}_{12}^2) - \frac{1}{6}(S^2 - |\mathrm{Rc}|^2),\\ \Delta_f \mathrm{W}_{1234} =& 2\lambda b_1 - 4(a_1b_1 + a_2b_3 + a_3b_2) + (\mathrm{Rc}_{13}\mathrm{Rc}_{24} - \mathrm{Rc}_{23}\mathrm{Rc}_{14}).\end{aligned}$$

Let us recall that, $\Delta_f \mathrm{S} = 2\lambda\mathrm{S} - 2|\mathrm{Rc}|^2$. Thus, for $2T_1 = (\mathrm{Rc} \circ \mathrm{Rc})(\alpha_1, \alpha_1)$, we have

$$\begin{aligned}\Delta_f(X_{1234}) =& 2\lambda\frac{\mathrm{S}}{3} - \frac{2}{3}|\mathrm{Rc}|^2 - 4\lambda(a_1 + b_1) + 4\lambda_1^2 + 8\lambda_2\lambda_3 + \frac{1}{6}(|\mathrm{Rc}|^2 - \mathrm{S}^2) + T_1\\ =& 2\lambda X_{1234} - \frac{1}{2}|\mathrm{Rc}|^2 + 4\lambda_1^2 + 8\lambda_2\lambda_3 - \frac{1}{6}\mathrm{S}^2 + T_1.\end{aligned}$$

Next we observe that $\lambda_2 + \lambda_3 = -\lambda_1$ and $8\lambda_2\lambda_3 \leq 2\lambda_1^2$. By Cauchy-Schwartz inequality, $T_1 \leq \frac{1}{4}|\mathrm{Rc}|^2$. Therefore,

$$\begin{aligned}\Delta_f(X_{1234}) \leq & 2\lambda X_{1234} - \frac{1}{4}|\mathrm{Rc}|^2 + 6(\frac{\frac{\mathrm{S}}{3} - X_{1234}}{2})^2 - \frac{1}{6}\mathrm{S}^2 \\ \leq & 2\lambda X_{1234} + \frac{3}{2}X_{1234}^2 - \mathrm{S}X_{1234} - \frac{1}{4}|\mathrm{Rc}|^2 = u(2\lambda + \frac{3}{2}u - \mathrm{S}) - \frac{1}{4}|\mathrm{Rc}|^2.\end{aligned}$$

Since $\Delta_f u \leq \Delta_f(X_{1234})$ in the barrier sense of E. Calabi (see[17]), the result then follows. □

7.3 A Framework Approach

In this section, we shall propose a framework to study interactions between components of curvature operator and the potential function on a GRS (M, g, f, λ). In particular, we represent the divergence and the interior product $i_{\nabla f}$ on each curvature component as linear combinations of four operators P, Q, M, N. The geometry of these operators, in turn, gives us information about the original objects. It should be noted that some identities here have already appeared elsewhere.

Now we define the elements of the framework, first via a local frame and then provides a coordinate-free version. Let $\alpha \in \Lambda_2$, $X, Y, Z \in TM$, and $\{e_i\}_{i=1}^n$ be a local normal orthonormal frame on a GRS (M^n, g, f, λ).

Definition 7.3.1. *The tensors* $P, Q, M, N : \Lambda_2 TM \otimes TM \to \mathbb{R}$ *are defined as:*

$$P_{ijk} = \nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik} = \nabla_j f_{ik} - \nabla_i f_{jk} = \mathrm{R}_{jikp} \nabla^p f, \tag{7.21}$$

$$P(X \wedge Y, Z) = -\mathrm{R}(X, Y, Z, \nabla f) = (d_\nabla \mathrm{Rc})(X, Y, Z) = \delta \mathrm{R}(Z, X, Y),$$

$$P(\alpha, Z) = \mathrm{R}(\alpha, \nabla f \wedge Z) = \delta \mathrm{R}(Z, \alpha);$$

$$Q_{ijk} = g_{ki} \nabla_j \mathrm{S} - g_{kj} \nabla_i \mathrm{S} = 2(g_{ki} \mathrm{R}_{jp} - g_{kj} \mathrm{R}_{ip}) \nabla^p f, \tag{7.22}$$

$$Q(X \wedge Y, Z) = 2(X, Z)\mathrm{Rc}(Y, \nabla f) - 2(Y, Z)\mathrm{Rc}(X, \nabla f),$$

$$Q(\alpha, Z) = -2\mathrm{Rc}(\alpha(Z), \nabla f) = -2 \langle \alpha Z, \mathrm{Rc}(\nabla f) \rangle ;$$

$$M_{ijk} = \mathrm{R}_{kj} \nabla_i f - \mathrm{R}_{ki} \nabla_j f, \tag{7.23}$$

$$M(X \wedge Y, Z) = \mathrm{Rc}(Y, Z)\nabla_X f - \mathrm{Rc}(X, Z)\nabla_Y f = -\mathrm{Rc}((X \wedge Y)\nabla f, Z),$$

$$M(\alpha, Z) = -\mathrm{Rc}(\alpha(\nabla f), Z) = -\langle \alpha \nabla f, \mathrm{Rc}(Z) \rangle ;$$

$$N_{ijk} = g_{kj} \nabla_i f - g_{ki} \nabla_j f, \tag{7.24}$$

$$N(X \wedge Y, Z) = \langle Y, Z \rangle \nabla_X f - \langle X, Z \rangle \nabla_Y f = \langle (X \wedge Y)Z, \nabla f \rangle ,$$

$$N(\alpha, Z) = \langle \alpha Z, \nabla f \rangle = -\alpha(Z, \nabla f).$$

Remark 7.3.1. *The tensors* $P^\pm, Q^\pm, M^\pm, N^\pm : \Lambda_2^\pm TM \otimes TM \to \mathbb{R}$ *are defined by restricting* $\alpha \in \Lambda_2^\pm TM$*. They can be seen as operators on* Λ_2 *by standard projection.*

Remark 7.3.2. *Before proceeding further, let us remark on the essence of these tensors.* $P \equiv 0$ *if and only if the curvature is harmonic;* $Q \equiv 0$ *if and only if the scalar curvature is constant;* $N \equiv 0$ *if and only if the potential function* f *is constant; finally,* $M \equiv 0$ *if and only if either* $\nabla f = 0$ *or* Rc *vanishes on the orthogonal complement of* ∇f.

7.3.1 Decomposition Lemmas

Using the framework above, we now can represent the interior product $i_{\nabla f}$ on components of the curvature tensor as follows. Again the Einstein summation convention is used here.

Lemma 7.3.2. *Let (M, g, f, λ) be a GRS, for P, Q, M, N as in Definition 7.3.1, in a local normal orthonormal frame, we have*

$$\mathrm{R}_{ijkp}\nabla^p f = \mathrm{R}(e_i, e_j, e_k, \nabla f) = -P_{ijk} = \nabla^p \mathrm{R}_{ijkp} = -\delta \mathrm{R}(e_k, e_i, e_j), \tag{7.25}$$

$$(g \circ g)_{ijkp}\nabla^p f = (g \circ g)(e_i, e_j, e_k, \nabla f) = -2N_{ijk}, \tag{7.26}$$

$$(\mathrm{Rc} \circ g)_{ijkp}\nabla^p f = (\mathrm{Rc} \circ g)(e_i, e_j, e_k, \nabla f) = \frac{1}{2}Q_{ijk} - M_{ijk}, \tag{7.27}$$

$$\mathrm{H}_{ijkp}\nabla^p f = H(e_i, e_j, e_k, \nabla f) = M_{ijk} - \frac{1}{2}Q_{ijk} - 2\lambda N_{ijk}, \tag{7.28}$$

$$\begin{aligned} \mathrm{W}_{ijkp}\nabla^p f &= \mathrm{W}(e_i, e_j, e_k, \nabla f) \\ &= -P_{ijk} - \frac{Q_{ijk}}{2(n-2)} + \frac{M_{ijk}}{(n-2)} - \frac{\mathrm{S}N_{ijk}}{(n-1)(n-2)}. \end{aligned} \tag{7.29}$$

Proof. The first formula is well-known (cf. [23]), following from the soliton equation and Bianchi identities. For the second, we compute,

$$\begin{aligned} (g \circ g)_{ijkp}\nabla^p f =& 2(g_{ik}g_{jp} - g_{ip}g_{jk})\nabla^p f \\ =& 2g_{ik}\nabla_j f - 2g_{jk}\nabla_i f = -2N_{ijk}. \end{aligned}$$

For the third, we use (3.31) to calculate

$$\begin{aligned} (\mathrm{Rc} \circ g)_{ijkp}\nabla^p f =& (\mathrm{Rc}_{ik}g_{jp} + \mathrm{Rc}_{jp}g_{ik} - \mathrm{Rc}_{ip}g_{jk} - \mathrm{Rc}_{jk}g_{ip})\nabla^p f \\ =& \mathrm{Rc}_{ik}\nabla_j f + \frac{1}{2}(g_{ik}\nabla_j \mathrm{S} - g_{jk}\nabla_i \mathrm{S}) - \mathrm{Rc}_{jk}\nabla_i f \\ =& \frac{1}{2}Q_{ijk} - M_{ijk}. \end{aligned}$$

The next formula is a consequence of the above formulas, definition of H (3.34) and the soliton equation (3.28). Finally, the last one comes from decomposition of the curvature operator (2.1) and previous formulas; it appeared, for example, in [34]. □

In addition, the divergence on these components can be written as linear combinations of P, Q, M, N.

Lemma 7.3.3. *Let (M, g, f, λ) be a GRS, for P, Q, M, N as in Definition 7.3.1, in a local normal orthonormal frame, we have*

$$\nabla^p \mathrm{R}_{ijkp} = -P_{ijk}, \tag{7.30}$$

$$\nabla^p (\mathrm{S}g \circ g)_{ijkp} = 2Q_{ijk}, \tag{7.31}$$

$$\nabla^p (\mathrm{Rc} \circ g)_{ijkp} = -\nabla^p \mathrm{H}_{ijkp} = -P_{ijk} + \frac{1}{2} Q_{ijk}, \tag{7.32}$$

$$\nabla^p \mathrm{W}_{ijkp} = -\frac{n-3}{n-2} P_{ijk} - \frac{n-3}{2(n-1)(n-2)} Q_{ijk} := -\frac{n-3}{n-2} C_{ijk}. \tag{7.33}$$

Proof. The first formula is well-known and comes from the second Bianchi identity [23]. For the second, we compute,

$$\begin{aligned}
\nabla^p (\mathrm{S}g \circ g)_{ijkp} &= 2\nabla^p (\mathrm{S}g_{ik} g_{jp} - \mathrm{S}g_{ip} g_{jk}) \\
&= 2g_{ik} g_{jp} \nabla^p \mathrm{S} - g_{jk} g_{ip} \nabla^p \mathrm{S} \\
&= 2g_{ik} \nabla_j \mathrm{S} - g_{jk} \nabla_i \mathrm{S} = 2Q_{ijk}.
\end{aligned}$$

For the next one, we use (3.30) to calculate,

$$\begin{aligned}
\nabla^p (\mathrm{Rc} \circ g)_{ijkp} &= \nabla^p (\mathrm{Rc}_{ik} g_{jp} + \mathrm{Rc}_{jp} g_{ik} - \mathrm{Rc}_{ip} g_{jk} - \mathrm{Rc}_{jk} g_{ip}) \\
&= g_{jp} \nabla^p \mathrm{Rc}_{ik} + g_{ik} \nabla^p \mathrm{Rc}_{jp} - g_{jk} \nabla^p \mathrm{Rc}_{ip} - g_{ip} \nabla^p \mathrm{Rc}_{jk} \\
&= \nabla_j \mathrm{Rc}_{ik} + \frac{1}{2}(g_{ik} \nabla_j \mathrm{S} - g_{jk} \nabla_i \mathrm{S}) - \nabla_i \mathrm{Rc}_{jk} \\
&= \frac{1}{2} Q_{ijk} - P_{ijk}.
\end{aligned}$$

Finally, the last one comes from decomposition of curvature (2.1) and previous formulas; it also appeared in, for example, [39, Eq. (9)]. □

Remark 7.3.3. *C defined in (7.33) is also called the Cotton tensor in literature.*

Remark 7.3.4. *By the standard projection, and*

$$(\delta \mathrm{W})^{\pm} = \delta(\mathrm{W}^{\pm}),$$
$$(i_{\nabla f}\mathrm{W})^{\pm} = i_{\nabla f}\mathrm{W}^{\pm},$$

the analogous identities hold if replacing W, P, Q, M, N *in Lemmas 7.3.2 and 7.3.3 by* $\mathrm{W}^{\pm}, P^{\pm}, Q^{\pm}, M^{\pm}, N^{\pm}$*, respectively.*

The following observation is an immediate consequence of Lemma 7.3.3.

Proposition 7.3.4. *Let* (M^n, g, f, λ)*,* $n > 2$*, be a GRS and* H *given by (3.34). Then the tensor*

$$F = \mathrm{W} + \frac{n-3}{n-2}\mathrm{H} + \frac{n(n-3)\mathrm{S}}{4(n-1)(n-2)} g \circ g$$

is divergence free.

Remark 7.3.5. *The result can be viewed as a generalization of the harmonicity of the Weyl tensor on an Einstein manifold.*

Lastly, we introduce the following tensor D which plays a crucial role in the classification problem (cf. [18], [21], [34]),

$$\begin{aligned} D_{ijk} &= -\frac{Q_{ijk}}{2(n-1)(n-2)} + \frac{M_{ijk}}{n-2} - \frac{\mathrm{S}N_{ijk}}{(n-1)(n-2)} \\ &= C_{ijk} + \mathrm{W}_{ijkp}\nabla^p f. \end{aligned} \tag{7.34}$$

7.3.2 Norm Calculations

Lemma 7.3.5. *Let* (M, g, f, λ) *be a GRS, then the following identities hold:*

$$
\begin{aligned}
2\langle P, Q\rangle &= -|\nabla \mathrm{S}|^2, \\
2\langle P, N\rangle &= \langle \nabla f, \nabla \mathrm{S}\rangle, \\
2\langle Q, Q\rangle &= 2(n-1)|\nabla \mathrm{S}|^2, \\
2\langle M, M\rangle &= 2|\mathrm{Rc}|^2|\nabla f|^2 - \frac{1}{2}|\nabla \mathrm{S}|^2, \\
2\langle N, N\rangle &= 2(n-1)|\nabla f|^2, \\
2\langle Q, M\rangle &= |\nabla \mathrm{S}|^2 - 2\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle, \\
2\langle Q, N\rangle &= -2(n-1)\langle \nabla f, \nabla \mathrm{S}\rangle, \\
2\langle M, N\rangle &= 2S|\nabla f|^2 - \langle \nabla f, \nabla \mathrm{S}\rangle.
\end{aligned}
$$

Furthermore, if M is closed, then

$$
\begin{aligned}
\int_M 2\langle P, P\rangle e^{-f} &= \int_M |\nabla \mathrm{Rc}|^2 e^{-f}, \\
\int_M 2\langle P, M\rangle &= 2\int_M (\lambda|\mathrm{Rc}|^2 - \mathrm{Rc}^3) + \int_M \left\langle \nabla f, \nabla|\mathrm{Rc}|^2\right\rangle + \frac{1}{2}\int_M |\nabla \mathrm{S}|^2.
\end{aligned}
$$

Proof. The main technique is to compute under a normal orthonormal local frame. For example,

$$
\begin{aligned}
2\langle P, Q\rangle &= P_{ijk}Q_{ijk} \\
&= (\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik})(g_{ki}\nabla_j \mathrm{S} - g_{kj}\nabla_i \mathrm{S}) \\
&= 2(\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik})g_{ki}\nabla_j \mathrm{S} \\
&= 2\nabla_j \mathrm{S}(\nabla_k \mathrm{Rc}_{kj} - \nabla_j \mathrm{Rc}_{kk}) \\
&= |\nabla \mathrm{S}|^2 - 2|\nabla \mathrm{S}|^2 = -|\nabla \mathrm{S}|^2.
\end{aligned}
$$

Other equations follow from similar calculation.

When M is closed, we can integrate by parts. In particular, the first equation was first derived in [23]. For the second, we compute that

$$
\begin{aligned}
\int_M 2\langle P, M\rangle &= 2\int_M (\nabla_i \mathrm{Rc}_{jk} - \nabla_j \mathrm{Rc}_{ik})\mathrm{Rc}_{kj}\nabla_i f \\
&= \int_M \nabla_i f \nabla_i \mathrm{Rc}^2_{jk} - 2\int_M \nabla_j \mathrm{Rc}_{ik}\mathrm{Rc}_{kj}\nabla_i f, \\
\int_M \nabla_j \mathrm{Rc}_{ik}\mathrm{Rc}_{kj}\nabla_i f &= -\int_M \mathrm{Rc}_{ik}\mathrm{Rc}_{kj} f_{ij} - \int_M \mathrm{Rc}_{ik} f_i \nabla_j \mathrm{Rc}_{kj} \\
&= -\int_M (\lambda|\mathrm{Rc}|^2 - \mathrm{Rc}^3) - \frac{1}{4}\int_M |\nabla \mathrm{S}|^2.
\end{aligned}
$$

Hence, the statement follows. □

Remark 7.3.6. *The factor of 2 is due to our convention of calculating norm. Some special cases of dimension four also appeared in [14, Proposition 4].*

An interesting consequence of the above calculation is the following corollary, which exposes the orthogonality of Q, N versus $i_{\nabla f}\mathrm{W}$, $\delta\mathrm{W}$.

Corollary 7.3.6. *Let (M, g, f, λ) be a GRS.*

a. *At each point, we have*

$$0 = \left\langle Q, i_{\nabla f}\mathrm{W}\right\rangle = \left\langle N, i_{\nabla f}\mathrm{W}\right\rangle = \langle Q, \delta\mathrm{W}\rangle = \langle N, \delta\mathrm{W}\rangle .$$

b. *If M is closed, then,*

$$\int_M 2|\delta\mathrm{W}|^2 e^{-f} = (\frac{n-3}{n-2})^2 \int_M (|\nabla \mathrm{Rc}|^2 - \frac{1}{(n-1)}|\nabla \mathrm{S}|^2)e^{-f}. \tag{7.35}$$

Proof. Part a) follows immediately from Lemmas 7.3.2, 7.3.3, 7.3.5, and our convention (2.3). For example,

$$
\begin{aligned}
\left\langle Q, i_{\nabla f}\mathrm{W}\right\rangle =& \sum_{i<j} Q_{ijk}(i_{\nabla f}\mathrm{W})_{kij} \\
=& \sum_{i<j} Q_{ijk}\nabla^p f \mathrm{W}_{pkij} = -\sum_{i<j} Q_{ijk}\mathrm{W}_{ijkp}\nabla^p f \\
=& \left\langle Q, P + \frac{Q}{2(n-2)} - \frac{M}{n-2} + \frac{\mathrm{S}N}{(n-1)(n-2)}\right\rangle \\
=& -\frac{|\nabla \mathrm{S}|^2}{2} + \frac{(n-1)|\nabla \mathrm{S}|^2}{2(n-2)} - \frac{|\nabla \mathrm{S}|^2}{2(n-2)} + \frac{\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle}{n-2} - \frac{(n-1)\mathrm{S}\langle \nabla f, \nabla \mathrm{S}\rangle}{(n-1)(n-2)} \\
=& 0.
\end{aligned}
$$

Other formulas follow from similar calculations.

For part b) we observe that,

$$
\begin{aligned}
|\delta \mathrm{W}|^2 =& (\frac{n-3}{n-2})^2 \left\langle P + \frac{Q}{2(n-1)}, P + \frac{Q}{2(n-1)}\right\rangle \\
=& (\frac{n-3}{n-2})^2 \left\langle P + \frac{Q}{2(n-1)}, P\right\rangle.
\end{aligned}
$$

Notice that we apply part a) in the last step. Consequently, applying Lemma 7.3.5 again yields

$$
\begin{aligned}
2\int_M |\delta \mathrm{W}|^2 e^{-f} &= (\frac{n-3}{n-2})^2 \int_M 2\left\langle P + \frac{Q}{2(n-1)}, P\right\rangle e^{-f} \\
&= (\frac{n-3}{n-2})^2 \int_M (|\nabla \mathrm{Rc}|^2 - \frac{|\nabla \mathrm{S}|^2}{2(n-1)})e^{-f}.
\end{aligned}
$$

□

Remark 7.3.7. *Part b) recovers the well-known fact that harmonic curvature implies harmonic Weyl tensor and constant scalar curvature.*

Now we are ready to prove Theorem 7.0.9.

Proof. **(Theorem 7.0.9)** First, we observe,

$$\begin{aligned}
\langle \mathrm{W}, \mathrm{Hess} f \circ \mathrm{Hess} f \rangle &= \sum_{i<j,k<l} \mathrm{W}_{ijkl} (\mathrm{Hess} f \circ \mathrm{Hess} f)_{ijkl} \\
&= \frac{1}{2} \sum_{k<l;i,j} \mathrm{W}_{ijkl} (\mathrm{Hess} f \circ \mathrm{Hess} f)_{ijkl} \\
&= \sum_{k<l;i,j} \mathrm{W}_{ijkl} (f_{ik} f_{jl} - f_{il} f_{jk}) \\
&= \sum_{i,j,k,l} \mathrm{W}_{ijkl} f_{ik} f_{jl}.
\end{aligned}$$

Next, subduing the summation notation, we integrate by parts,

$$\int_M \mathrm{W}_{ijkl} f_{ik} f_{jl} = -\int_M \nabla_i \mathrm{W}_{ijkl} f_k f_{jl} - \int_M \mathrm{W}_{ijkl} f_k \nabla_i f_{jl}.$$

The first term can be written as

$$\begin{aligned}
\int_M \nabla_i \mathrm{W}_{ijkl} f_k f_{jl} &= \int_M \nabla_i \mathrm{W}_{ijkl} f_k (\lambda g_{jl} - \mathrm{Rc}_{jl}) \\
&= -\int_M \nabla_i \mathrm{W}_{ijkl} f_k \mathrm{Rc}_{jl} = -\frac{1}{2} \int_M (\delta \mathrm{W})_{jkl} M_{klj} \\
&= -\int_M \langle \delta \mathrm{W}, M \rangle .
\end{aligned}$$

Next, we compute the second term,

$$\begin{aligned}
\int_M \mathrm{W}_{ijkl} f_k \nabla_i f_{jl} &= -\int_M \mathrm{W}_{ijlk} f_k \nabla_i (g_{jl} - \mathrm{Rc}_{jl}) = \int_M \mathrm{W}_{ijlk} f_k \nabla_i \mathrm{Rc}_{jl} \\
&= \frac{1}{2} \int_M \mathrm{W}_{ijlk} f_k P_{ijl} = -\int_M \left\langle i_{\nabla f} \mathrm{W}, P + \frac{Q}{2(n-1)} \right\rangle \\
&= -\frac{n-2}{n-3} \int_M \left\langle \delta \mathrm{W}, i_{\nabla f} \mathrm{W} \right\rangle \\
&= \frac{n-2}{n-3} \int_M \left\langle \delta \mathrm{W}, -P + \frac{M}{n-2} \right\rangle .
\end{aligned}$$

It is noted that we have used Corollary 7.3.6 repeatedly to manipulate Q and N.

To conclude, we combine equations above,

$$\int_M W_{ijkl} f_{ik} f_{jl} = \int_M \langle \delta W, M \rangle - \frac{n-2}{n-3} \int_M \left\langle \delta W, -P + \frac{M}{n-2} \right\rangle$$
$$= \frac{1}{n-3} \int_M \langle \delta W, (n-2)P + (n-4)M \rangle .$$

If $n = 4$, then

$$\int_M W_{ijkl} f_{ik} f_{jl} = \int_M 2 \langle \delta W, P \rangle = \int_M 2 \left\langle \delta W, P + \frac{Q}{6} \right\rangle$$
$$= \int_M 2 \langle \delta W, 2\delta W \rangle = 4 \int_M |\delta W|^2.$$

□

Remark 7.3.8. *The formula in dimension four is also a consequence of the divergence-free property of the Bach tensor. We omit the details here.*

Moreover, in dimension four, we have similar results for $W^\pm$.

Lemma 7.3.7. *Let* (M^4, g, f, λ) *be a GRS, then at each point, we have*

$$0 = \left\langle Q^\pm, i_{\nabla f} W^\pm \right\rangle = \langle Q^\pm, \delta W^\pm \rangle = \left\langle N^\pm, i_{\nabla f} W^\pm \right\rangle = \langle N^\pm, \delta W^\pm \rangle . \tag{7.36}$$

Proof. It suffices to show the statements is true for the self-dual part.

Let $\{e_i\}_{i=1}^4$ be a normal orthonormal local frame and let $\{\alpha_i\}_{i=1}^4$ be an orthonormal basis for Λ_2^+ . Then

$$\left\langle Q^+, i_{\nabla f} W^+ \right\rangle = \sum_i \sum_j Q(\alpha_i, e_j) W(\nabla f \wedge e_j, \alpha_i)$$
$$= -2 \left\langle \alpha_i(e_j), \mathrm{Rc}(\nabla f) \right\rangle W(\nabla f \wedge e_j, \alpha_i).$$

Furthermore, we can choose a special basis, namely the normal form as in (2.14). Then α_i's diagonalize W^+ with eigenvalues λ_i's. Consequently,

$$W(\nabla f \wedge e_j, \alpha_i) = \lambda_i \alpha_i (\nabla f \wedge e_j) = \lambda_i \left\langle \nabla f, \alpha_i(e_j) \right\rangle .$$

Thus,

$$\left\langle Q^{+}, i_{\nabla f}\mathrm{W}^{+}\right\rangle = -2\lambda_i \left\langle \alpha_i(e_j), \mathrm{Rc}(\nabla f)\right\rangle \left\langle \alpha_i(e_j), \nabla f\right\rangle$$
$$= -2\eta_k \langle e_k, \mathrm{Rc}(\nabla f)\rangle \langle e_k, \nabla f\rangle,$$
$$\text{for } \eta_k = \sum_{i,j:\alpha_i(e_j)=\pm e_k} \lambda_i.$$

Now by (2.2), it is easy to see that each $\eta_k = 0$ because W^{+} is traceless.

Claim: $\langle P^{+}, Q^{+}\rangle = -\frac{1}{4}|\nabla \mathrm{S}|^2$.

To prove this claim, we choose $\{\alpha_i\}$ as in (2.11) and observe that,

$$P(\alpha_1, e_j)Q(\alpha_1, e_j) = \frac{1}{2}P(e_{12}+e_{34}, e_j)Q(e_{12}+e_{34}, e_j)$$
$$= -(P_{12j}+P_{34j})\left\langle (e_{12}+e_{34})e_j, \mathrm{Rc}(\nabla f)\right\rangle$$
$$= -(\nabla_1 \mathrm{Rc}_{2j} - \nabla_2 \mathrm{Rc}_{1j} + \nabla_3 \mathrm{Rc}_{4j} - \nabla_4 \mathrm{Rc}_{3j})\left\langle (e_{12}+e_{34})e_j, \mathrm{Rc}(\nabla f)\right\rangle.$$

Similarly,

$$P(\alpha_2, e_j)Q(\alpha_2, e_j) = -(\nabla_1 \mathrm{Rc}_{3j} - \nabla_3 \mathrm{Rc}_{1j} - \nabla_2 \mathrm{Rc}_{4j} + \nabla_4 \mathrm{Rc}_{2j})\left\langle (e_{13}-e_{24})e_j, \mathrm{Rc}(\nabla f)\right\rangle,$$
$$P(\alpha_3, e_j)Q(\alpha_3, e_j) = -(\nabla_1 \mathrm{Rc}_{4j} - \nabla_4 \mathrm{Rc}_{1j} + \nabla_2 \mathrm{Rc}_{3j} - \nabla_3 \mathrm{Rc}_{2j})\left\langle (e_{14}+e_{23})e_j, \mathrm{Rc}(\nabla f)\right\rangle.$$

Thus,

$$\langle P^{+}, Q^{+}\rangle = \sum_{i,j} P(\alpha_i, e_j)Q(\alpha_i, e_j)$$
$$= -\sum_k \zeta_k \langle e_k, \mathrm{Rc}(\nabla f)\rangle,$$
$$\text{for } \zeta_k = \sum_{i,j:\alpha_i(e_j)=e_k} \sqrt{2}P(\alpha_i, e_j) - \sum_{i,j:\alpha_i(e_j)=-e_k} \sqrt{2}P(\alpha_i, e_j).$$

Using (2.2), we can compute,

$$\begin{aligned}\zeta_1 = &\sqrt{2}\Big(P(\alpha_1, e_2) + P(\alpha_2, e_3) + P(\alpha_3, e_4)\Big)\\ =&\nabla_1 \mathrm{Rc}_{22} - \nabla_2 \mathrm{Rc}_{12} + \nabla_3 \mathrm{Rc}_{42} - \nabla_4 \mathrm{Rc}_{32}\\ &+ \nabla_1 \mathrm{Rc}_{33} - \nabla_3 \mathrm{Rc}_{13} - \nabla_2 \mathrm{Rc}_{43} + \nabla_4 \mathrm{Rc}_{23}\\ &+ \nabla_1 \mathrm{Rc}_{44} - \nabla_4 \mathrm{Rc}_{14} + \nabla_2 \mathrm{Rc}_{34} - \nabla_3 \mathrm{Rc}_{24}\\ =&\nabla_1(\mathrm{S} - \mathrm{Rc}_{11}) - (\frac{1}{2}\nabla_1 \mathrm{S} - \nabla_1 \mathrm{Rc}_{11}) = \frac{1}{2}\nabla_1 \mathrm{S}.\end{aligned}$$

Similarly we have $\zeta_k = \frac{1}{2}\nabla_k \mathrm{S}$. We also have $\mathrm{Rc}(\nabla f) = \frac{1}{2}\nabla \mathrm{S}$. This proves our claim.

In addition, it is easy to see that

$$\langle Q^+, Q^+\rangle = \frac{3}{2}|\nabla \mathrm{S}|^2.$$

Since $\delta \mathrm{W}^+ = \frac{P^+}{2} + \frac{Q^+}{12}$, it follows that

$$\langle Q^+, \delta \mathrm{W}^+\rangle = 0.$$

The statements involved N follow from analogous calculations as

$$N(\alpha_i, e_j) = \left\langle \alpha_i(e_j), \nabla f\right\rangle.$$

□

By manipulation as in the proof of Theorem 7.0.9, using Remark 7.3.4 (replacing Lemmas 7.3.2 and 7.3.3) and Lemma 7.3.7 (replacing Lemma 7.3.6), we immediately obtain the following result.

Corollary 7.3.8. *Let (M, g, f, λ) be a four-dimensional closed GRS. Then we have the following identity:*

$$\int_M \langle \mathrm{W}^+, \mathrm{Rc} \circ \mathrm{Rc}\rangle = 4\int_M |\delta \mathrm{W}^+|^2. \tag{7.37}$$

7.4 Rigidity Results

In this section, we present conditions that imply the rigidity of a GRS using the analysis on the framework discussed in the previous section.

First, Proposition 7.4.10 provides a geometrical way to understand tensor D defined in (7.34). In particular, it says that $D \equiv 0$ is equivalent to a special condition, namely, the normalization of ∇f (if not trivial) is an eigenvector of the Ricci tensor, and all other eigenvectors have the same eigenvalue. Such a structure will imply rigidity as the geometry of the level surface (of f) being well-described.

On the other hand, Theorem 7.0.9 reveals an interesting connection between the Ricci tensor and the Weyl tensor in dimension four. That allows us to obtain rigidity results using only the structure of the Ricci curvature for a GRS.

Theorem 7.4.1. *Let (M^4, g, f, λ) be a closed four-dimensional GRS. Assume that at each point the Ricci curvature has one eigenvalue of multiplicity one and another of multiplicity three, then the GRS is rigid, hence Einstein.*

We also find conditions that imply the vanishing of tensor D.

Theorem 7.4.2. *Let (M^n, g, f, τ), $n > 3$, be a GRS. Assuming one of these conditions holds:*

1. $i_{\nabla f}\mathrm{Rc} \circ g \equiv 0$;
2. $i_{\nabla f}\mathrm{W} \equiv 0$ *and* $\delta\mathrm{W}(.,.,\nabla f) = 0$.

Then at the point $\nabla f \neq 0$, $D = 0$.

Remark 7.4.1. *$D \equiv 0$ can be derived from other conditions such as the vanishing of the Bach tensor (cf. [18, Lemma 4.1]).*

Remark 7.4.2. *For GRS's, condition (2) is a slight improvement of [31], where the author characterizes generalized quasi-Einstein manifolds with $\delta \mathrm{W} = i_{\nabla f}\mathrm{W} = 0$.*

In dimension four, the result can be improved significantly.

Theorem 7.4.3. *Let (M, g, f, λ) be a four-dimensional GRS. At points where $\nabla f \neq 0$, then $\mathrm{W}^+(\nabla f, ., ., .) = 0$ implies $\mathrm{W}^+ = 0$.*

As discussed in the last section, there are some similarities between taking the divergence and interior product $i_{\nabla f}$ of the Weyl tensor, for example, see Corollary 7.3.6. The following theorem is inspired by condition (1) of Theorem 7.4.2.

Theorem 7.4.4. *Let (M^n, g, f, τ), $n > 3$, be a GRS. Then $\delta(\mathrm{Rc} \circ g) \equiv 0$ if and only if the Weyl tensor is harmonic and the scalar curvature is constant.*

An immediate consequence of the results above (plus known classifications discussed in the Introduction) is to obtain rigidity results.

Corollary 7.4.5. *Let (M^n, g, f, λ), $n \geq 4$, be a complete shrinking GRS.*

i. *If $i_{\nabla f}\mathrm{Rc} \circ g \equiv 0$, then (M^n, g, f, λ) is Einstein;*

ii. *If $i_{\nabla f}\mathrm{W} = 0$ and $\delta \mathrm{W}(., ., \nabla f) = 0$, then (M^n, g, f, λ) is rigid of rank $k = 0, 1, n$;*

iii. *If $\delta(\mathrm{Rc} \circ g) = 0$, then (M^n, g, f, λ) is rigid of rank $0 \leq k \leq n$.*

In particular, when the dimension is four, we have the following result.

Corollary 7.4.6. *Let (M, g, f, λ) be a four-dimensional complete GRS. If*

$$\mathrm{W}^{+}(\nabla f, ., ., .) = 0,$$

then the GRS is either Einstein or has $\mathrm{W}^{+} = 0$. *Furthermore, in the second case, it is isometric to a Bryant soliton or Ricci flat manifold if $\lambda = 0$; or is a finite quotient of* $\mathbb{R}^4$, $\mathrm{S}^3 \times \mathbb{R}$, S^4 *or* CP^2 *if $\lambda > 0$.*

The general strategy to prove aforementioned statements is to use the framework to study the structure of the Ricci tensor.

7.4.1 Eigenvectors of the Ricci curvature

Here we study various interconnections between the eigenvectors of the Ricci curvature, the Weyl tensor, and the potential function. First, we observe the following lemma.

Lemma 7.4.7. *Let (M, g) be a Riemannian manifold. Assume that, at each point, the Ricci curvature has one eigenvalue of multiplicity one and another of multiplicity $n-1$. Then we have,*

$$\langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = 0.$$

Proof. Without loss of generality, we can choose a basis $\{e_i\}_{i=1}^{n}$ of T_pM consisting of eigenvectors of Rc, namely $\mathrm{Rc}_{11} = \eta$ and $\mathrm{Rc}_{ii} = \zeta$ for $i = 2, ..., n$. Then,

$$\langle \mathrm{W}, \mathrm{Rc} \circ \mathrm{Rc} \rangle = \sum_{i<j;k<l} \mathrm{W}_{ijkl}\mathrm{Rc}_{ik}\mathrm{Rc}_{jl} \tag{7.38}$$

$$= \sum_{i<j} \mathrm{W}_{ijij}\mathrm{Rc}_{ii}\mathrm{Rc}_{jj} = \eta\zeta \sum_{j} \mathrm{W}_{1j1j} + \zeta^2 \sum_{1<i<j} \mathrm{W}_{ijij}. \tag{7.39}$$

We observe that,

$$\sum_{j>1} \mathrm{W}_{ijij} = -\mathrm{W}_{1i1i}, \tag{7.40}$$

$$2\sum_{1<i<j} \mathrm{W}_{ijij} = \sum_{i>1}\sum_{j>1} \mathrm{W}_{ijij} = -\sum_{i} \mathrm{W}_{1i1i} = 0. \tag{7.41}$$

The result then follows. □

Next, a consequence of our previous framework (on P, Q, M, and N) is the following characterization about the condition $\mathrm{Rc}(\nabla f) = \mu\nabla f$.

Lemma 7.4.8. *Let (M, g, f, λ) be a GRS. Then the followings are equivalent:*

1. $\mathrm{Rc}(\nabla f) = \mu\nabla f$;
2. $Q(.,.,\nabla f) = 0$;
3. $M(.,.,\nabla f) = 0$;
4. $\delta\mathrm{W}(\nabla f,.,.) = 0$;
5. $\delta\mathrm{H}(\nabla f,.,.) = 0$.

Proof. We'll show that (1) ↔ (2), (1) ↔ (3), (2) ↔ (4), and (2) ↔ (5).

For (2) → (1): Let $\alpha \in \Lambda_2$, we have $0 = Q(\alpha, \nabla f) = -2(\alpha(\nabla f), \mathrm{Rc}(\nabla f))$. Since α can be arbitrary, $\alpha(\nabla f)$ can realize any vector in the complement of ∇f in TM. Therefore, $\mathrm{Rc}(\nabla f) = \mu\nabla f$.

For (1) → (2): $Q(\alpha, \nabla f) = -2(\alpha(\nabla f), \mathrm{Rc}(\nabla f)) = -2(\alpha(\nabla f), \mu\nabla f) = 0$ because $\alpha(\nabla f) \perp \nabla f$.

(1) being equivalent to (3) follows from an identical argument as above.

(2) being equivalent to (4) follows from

$$\delta \mathrm{W}(X, Y, Z) = \frac{n-3}{n-2} P(Y, Z, X) + \frac{n-3}{2(n-1)(n-2)} Q(Y, Z, X),$$
$$P(Y, Z, \nabla f) = -\mathrm{R}(Y, Z, \nabla f, \nabla f) = 0.$$

(2) being equivalent to (5) follows from

$$\delta \mathrm{H}(X, Y, Z) = -P(Y, Z, X) + \frac{1}{2} Q(Y, Z, X),$$
$$P(Y, Z, \nabla f) = -\mathrm{R}(Y, Z, \nabla f, \nabla f) = 0.$$

□

Furthermore, the rigidity of these operators Q, M, N is captured by the following result.

Proposition 7.4.9. *Let* (M^n, g, f, τ), $n > 3$, *be a GRS and* $T = aQ + bM + cN$ *for some real numbers a,b,c.*

i. *Assume that* $T \equiv 0$. *If* $a \neq 0$ *then* $\mathrm{Rc}(\nabla f) = \mu \nabla f$; *moreover, if* $\nabla f \neq 0$ *and* $b \neq 0$, *then all other eigenvectors must have the same eigenvalue;*

ii. *In dimension four, if* $T_{|\Lambda_2^+ \otimes TM} \equiv 0$ *then* $T \equiv 0$.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis which consists of eigenvector of Rc with corresponding eigenvalues λ_i. Then we have

$$\begin{aligned} T(\alpha, e_i) &= aQ(\alpha, e_i) + bM(\alpha, e_i) + cN(\alpha, e_i) \\ &= -2a \langle \alpha(e_i), \mathrm{Rc}(\nabla f) \rangle - b \langle \alpha(\nabla f), \mathrm{Rc}(e_i) \rangle + c \langle \alpha(e_i), \nabla f \rangle \\ &= -2a \langle \alpha(e_i), \mathrm{Rc}(\nabla f) \rangle + b \langle \nabla f, \alpha(\lambda_i e_i) \rangle + c \langle \alpha(e_i), \nabla f \rangle \\ &= \langle \alpha(e_i), -2a\mathrm{Rc}(\nabla f) + b\lambda_i \nabla f + c \nabla f \rangle . \end{aligned} \quad (7.42)$$

i. Without loss of generality, we can assume $\nabla f \neq 0$. Since $T(\alpha, e_i) = 0$ for arbitrary α and e_i,

$$T(\alpha, \nabla f) = 0 = \langle \alpha(\nabla f), \mathrm{Rc}(\nabla f) \rangle = Q(\alpha, \nabla f).$$

By Lemma 7.4.8, $e_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector of Rc. Plugging into (7.42) yields,

$$T(\alpha, e_i) = (-2a\lambda_1 + b\lambda_i + c) \langle \alpha(e_i), \nabla f \rangle .$$

Therefore, $-2a\lambda_1 + b\lambda_i + c = 0$. Hence, as $b \neq 0$, all other eigenvectors have the same eigenvalue.

ii. In dimension four, fix a unit vector e_i and note that $T(\alpha, e_i) = 0$ for any $\alpha \in \Lambda_2^+$. By Lemma 2.3.1 and Remark **??**, $T(\beta, e_i) = 0$ for all $\beta \in \Lambda_2^-$. As e_i is arbitrary the result then follows. □

Recall that tensor D is a special linear combination of M, N, Q. Therefore, we obtain the following geometric characterization.

Proposition 7.4.10. *Let (M^n, g), $n > 3$, be a Riemannian manifold and D defined as in (7.34). Then the followings are equivalent:*

1. *$D \equiv 0$;*
2. *The Weyl tensor under the conformal change $\tilde{g} = e^{\frac{-2f}{n-2}} g$ is harmonic;*
3. *Either $\nabla f = 0$ and Cotton tensor $C_{ijk} = 0$, or ∇f is an eigenvector of* Rc *and all other eigenvectors have the same eigenvalue.*

Proof. We shall show (1) ↔ (2), (1) → (3) and (3) → (1).

For (1) ↔ (2) : By equation (7.34) and (7.33), we have

$$D_{ijk} = C_{ijk} + \mathrm{W}_{ijkp}\nabla^p f = \frac{n-2}{n-3}(\delta \mathrm{W})_{kij} - \mathrm{W}(\nabla f, e_k, e_i, e_j).$$

Thus, $D \equiv 0$ is equivalent to

$$\delta \mathrm{W}(X, Y, Z) - \frac{n-3}{n-2} \mathrm{W}(\nabla f, X, Y, Z) = 0.$$

Under the conformal transofrmation $\tilde{g} = u^2 g$ (see the appendix), $\widetilde{\mathrm{W}} = u^2 \mathrm{W}$, and

$$\delta \widetilde{\mathrm{W}}(X, Y, Z) = \delta \mathrm{W}(X, Y, Z) + (n-3) \mathrm{W}(\frac{\nabla u}{u}, X, Y, Z).$$

The result then follows from the last two equation.

The statement (1) → (3) follows from [18, Proposition 3.2 and Lemma 4.2].

For (3) → (1)**:** $\forall a, b, c$, let $T = aQ + bM + cN$. For any $\alpha \in \Lambda_2$ and e_i a unit tangent vector, by (7.42), we have

$$T(\alpha, e_i) = \langle \alpha(e_i), -2a\mathrm{Rc}(\nabla f) + b\lambda_i \nabla f + c\nabla f \rangle .$$

For the tensor D,

$$\begin{aligned} a =& \frac{-1}{2(n-1)(n-2)}, \\ b =& \frac{1}{n-2}, \\ c =& \frac{-\mathrm{S}}{(n-1)(n-2)}. \end{aligned}$$

If $\nabla f = 0$ then $T \equiv 0$, hence $D \equiv 0$. If $\nabla f \neq 0$, then there exist $e_1 = \frac{\nabla f}{|\nabla f|}$ and $\{e_i\}_{i=2}^n$, eigenvectors of Rc, with eigenvalues ζ, η, respectively. Then,

$$T(\alpha, e_i) = \langle \alpha(e_i), (-2a\zeta + b\eta + c)\nabla f \rangle .$$

Since $\zeta + (n-1)\eta = \mathrm{S}$, with given values of a, b, c above, it follows that $-2a\zeta + b\eta + c = 0$. Thus, $D \equiv 0$. □

Remark 7.4.3. *Our formulas are different from [39, 2.19] by a sign convention.*

Remark 7.4.4. *Under that conformal change of the metric, the Ricci tensor is given by*

$$\begin{aligned}\widetilde{\mathrm{Rc}} =& \mathrm{Rc} + Hess f + \frac{1}{n-2} df \otimes df + \frac{1}{n-2}(\Delta f - |\nabla f|^2) g \\ =& \frac{1}{n-2} df \otimes df + \frac{1}{n-2}(\Delta f - |\nabla f|^2 + (n-2)\lambda) g.\end{aligned}$$

Therefore, at each point, $\widetilde{\mathrm{Rc}}$ *has at most two eigenvalues. Furthermore, since* $\tilde{g}$ *has harmonic Weyl tensor, its Schouten tensor*

$$\widetilde{Sc} = \frac{1}{n-2}(\widetilde{\mathrm{Rc}} - \frac{1}{2(n-1)} \tilde{\mathrm{S}} \tilde{g})$$

is a Codazzi tensor with at most two eigenvalues. Using the splitting results for Riemannian manifolds admitting such a tensor gives another proof of results in [18]. This method is inspired by [31].

Now we investigate several conditions which will imply that $\mathrm{Rc}(\nabla f) = \mu \nabla f$.

Proposition 7.4.11. *Let* (M^n, g, f, τ)*,* $n > 3$*, be a GRS. Assuming one of these conditions holds:*

1. $i_{\nabla f} \mathrm{W} \equiv 0$;
2. $\delta \mathrm{W}^+ = 0$ *if* $n = 4$.

Then $\mathrm{Rc}(\nabla f) = \mu \nabla f$.

Proof. The idea is to find a connection of each condition with Lemma 7.4.8.

Assuming (1): We claim that $\delta \mathrm{W}(\nabla f, ., .) = 0$.

Choosing a normal local frame $\{e_i\}_{i=1}^n$, we have:

$$
\begin{aligned}
\delta \mathrm{W}(\nabla f, e_k, e_l) &= \sum_i (\nabla_i \mathrm{W})(e_i, \nabla f, e_k, e_l) \\
&= \sum_i \nabla_i \mathrm{W}(e_i, \nabla f, e_k, e_l) - \sum_i \mathrm{W}(e_i, \nabla_i \nabla f, e_k, e_l) \\
&= 0 - \mathrm{W}(\mathrm{Hess} f, e_k, e_l).
\end{aligned}
$$

Since $\mathrm{Hess} f$ is symmetric and W is anti-symmetric, $\delta \mathrm{W}(\nabla f, ., .) = 0$. The result then follows.

Assuming (2): First recall

$$\delta \mathrm{W}(X, Y, Z) = \frac{1}{2} C(Y, Z, X) = \frac{1}{2} P(Y \wedge Z, X) + \frac{1}{12} Q(Y \wedge Z, X).$$

$\forall \alpha \in \Lambda^2_+$, since

$$\delta \mathrm{W}^-(X, \alpha) = \nabla_i \mathrm{W}^-(e_i \wedge X, \alpha) = 0,$$

we have

$$\delta(\mathrm{W})(X, \alpha) = \delta(\mathrm{W}^+)(X, \alpha) = \frac{1}{2} P(\alpha, X) + \frac{1}{12} Q(\alpha, X).$$

Since $0 = \mathrm{R}(Y, Z, \nabla f, \nabla f) = -P(Y \wedge Z, \nabla f)$ and $\delta \mathrm{W}^+ = 0$, hence $Q(\alpha, \nabla f) = 0$. The desired statement follows from Lemmas 2.3.1 and 7.4.8.

□

7.4.2 Proofs of Rigidity Theorems

Proof. **(Theorem 7.4.1)**

By Lemma 7.4.7, we have

$$\int_M \mathrm{W}(\mathrm{Rc}\circ\mathrm{Rc}) = 0.$$

Theorem 7.0.9, hence, implies that $\delta \mathrm{W} \equiv 0$. Then by the rigidity result for harmonic Weyl tensor discussed in the Introduction, the result follows. □

Proof. **(Theorem 7.4.2)**.

Assuming (1): We observe that

$$\mathrm{Rc}\circ g(X, Y, Z, \nabla f) = \frac{1}{2}Q(X, Y, Z) - M(X, Y, Z).$$

Therefore, the result follows from Lemma 7.4.9 and Proposition 7.4.10.

Assuming (2): By Proposition 7.4.11, $e_1 = \frac{\nabla f}{|\nabla f|}$ is a unit eigenvector. Let $\{e_i\}_{i=1}^n$ be an orthonomal basis of Rc with eigenvalues λ_i. By (7.29) and $\mathrm{W}(\nabla f, ., ., .) = 0$,

$$P = -\frac{Q}{2(n-2)} + \frac{M}{(n-2)} - \frac{\mathrm{S}N}{(n-1)(n-2)}.$$

Thefore,

$$\begin{aligned} P(i, j, k) &= \frac{|\nabla f|}{n-2}\Big[\lambda_1(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1}) - \lambda_k(\delta_{j1}\delta_{ik} - \delta_{i1}\delta_{jk}) - \frac{S}{n-1}(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1})\Big] \\ &= \frac{|\nabla f|}{n-2}(\delta_{jk}\delta_{1i} - \delta_{ik}\delta_{j1})(\lambda_1 + \lambda_k - \frac{S}{n-1}). \end{aligned} \tag{7.43}$$

Using the assumption $\delta \mathrm{W}(., ., \nabla f) = 0$, we obtain that

$$(P + \frac{1}{2(n-1)}Q)(\nabla f, ., .) = 0.$$

Combining with (7.43) yields,

$$P(1, k, k) = -\frac{1}{2(n-1)}Q(1, k, k) = \frac{\lambda_1|\nabla f|}{(n-1)} = \frac{|\nabla f|}{n-2}(\lambda_1 + \lambda_k - \frac{S}{n-1}).$$

Thus $\lambda_2 = \lambda_3 = \lambda_4 = \frac{S-\lambda_1}{n-1}$. Proposition 7.4.10 then concludes the argument.

□

The proof of Theorem 7.4.4 follows from a similar argument.

Proof. **(Theorem 7.4.4)**

By equation (7.32), $\delta(\mathrm{Rc} \circ g) = 0$ implies $P - \frac{Q}{2} = 0$. Thus, by Lemma 7.3.5,

$$2|P|^2 = 2\left\langle P, \frac{Q}{2}\right\rangle = -\frac{|\nabla \mathrm{S}|^2}{2}.$$

Hence $P = 0 = \nabla \mathrm{S}$. It then follows from Corollary 7.3.6 that $\delta \mathrm{W} = \delta \mathrm{S} = 0$. The converse is obvious. □

Proof. (**Theorem 7.4.3)**

Using a normal local frame, we can rewrite the assumption as,

$$\sum_i f_i \mathrm{W}^+_{ijkl} = 0.$$

We pick an arbitrary index a and multiply both sides with W_{ajkl} to arrive at,

$$\sum_i f_i \mathrm{W}^+_{ijkl} \mathrm{W}^+_{ajkl} = 0.$$

Applying identity (2.15) yields,

$$\begin{aligned} 0 &= \sum_{jkl} \sum_i f_i \mathrm{W}^+_{ijkl} \mathrm{W}^+_{ajkl} \\ &= \sum_i f_i \sum_{jkl} \mathrm{W}^+_{ijkl} \mathrm{W}^+_{ajkl} \\ &= \sum_i f_i |\mathrm{W}^+|^2 g_{ia} = f_a |\mathrm{W}^+|^2. \end{aligned}$$

Since index a is arbitrary, we have $\nabla f = 0$ or $|\mathrm{W}^+| = 0$. □

Proof. (**Corollary 7.4.5**)

By Theorem 7.4.2 and Theorem 7.4.4, each condition implies $D \equiv 0$. Then, [18, Lemma 4.2] further implies that $\delta \mathrm{W} = 0$. It follows, from classification results for harmonic Weyl tensor as discussed in the Introduction, that the manifold must be rigid. We now look at each case closely and observe that not all ranks can arise.

i. In this case, Lemma 7.4.9 reveals that $\lambda_0 - \lambda_i = 0$ with $\mathrm{Rc}(\nabla f) = \lambda_0 \nabla_f$, and λ_i is any other eigenvalue of Rc. Therefore, the manifold structure must be Einstein.

ii. In this case, since $D \equiv 0$ implies Rc has at most two eigenvalues with one of multiplicity 1 and another of $n - 1$. So k can only be $0, 1, n$.

iii. In this case, there is no obvious obstruction, so all rank can arise.

□

Proof. (**Corollary 7.4.6**)

The statement follows immediately from Theorem 7.4.3, [34, Theorems 1.1, 1.2], and the analyticity of a GRS with bounded curvature [4]. □

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