

# HEAT KERNEL ESTIMATES ON GLUED SPACES

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# HEAT KERNEL ESTIMATES ON GLUED SPACES

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In this thesis, we prove heat kernel estimates in two main contexts: (1) manifolds with ends with mixed Dirichlet and Neumann boundary condition and (2) infinite (countable) graphs satisfying certain properties, which we call book-like graphs. In both of these settings, we start with “sufficiently nice” pieces (pieces satisfying two-sided Gaussian heat kernel estimates) that are “glued” together in some sufficiently nice way. The results in setting (1) extend previous results of Grigor’yan and Saloff-Coste in the case of manifolds with ends with Neumann (or no) boundary condition. In setting (2), we are in the discrete case, where there is not direct prior work. This thesis extends some of the continuous setting results of Grigor’yan and Saloff-Coste mentioned above to the discrete setting, and the results here are also related to results of Grigor’yan and Ishiwata regarding gluing two copies of  $\mathbb{R}^n$  via a surface of revolution. In both settings, the results of this thesis rely heavily on the  $h$ -transform technique and understanding particular harmonic functions and hitting probabilities. In the setting of (1), we show the existence of a global harmonic function satisfying particular properties. In the setting of (2), we give estimates on certain hitting probabilities that naturally arise from considering subgraphs of larger graphs. All work in this thesis is joint with Laurent Saloff-Coste.

## **BIOGRAPHICAL SKETCH**

Emily Dautenhahn was born in Lake Charles, Louisiana in 1996 and has since traveled northward, finally landing in graduate school at Cornell University after completing undergraduate degrees in history and mathematics at the University of Kentucky. Emily will finally be changing direction this fall by heading south to start a job as Assistant Professor of Mathematics at Murray State University in Kentucky.

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To my parents: I know you don't understand why I'm doing math without any clear immediate applications, but thank you for putting up with me being so far away and all those long drives to pick me up from the airport.

To all of my friends, near and far: I would have never finished this degree without you. You know who you are.

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# CHAPTER 1

## INTRODUCTION

Since its introduction by Joseph Fourier in the 1800s, the heat equation  $\Delta u = \partial_t u$  has been of interest to mathematicians in a variety of fields. It is the quintessential example of a parabolic partial differential equation, and it is also deeply related to probability theory through its connection to Brownian motion and random walks. Part of the broad appeal and importance of the heat equation comes from the fact that it can be studied on many different kinds of spaces, and, moreover, the behavior of the heat kernel is closely related to the geometry of the space and other interesting properties. Examples of spaces where the heat equation can be studied include manifolds [28, 47, 57], graphs [5, 29, 65], Lie groups [19, 43, 56], groups [9, 41, 52], and fractals [2, 4, 44]; the references given here are by no means exhaustive or even representative. The position of this thesis, then, is to add to the already extensive literature surrounding the heat equation and its solutions. The main results proved in this thesis concern heat kernel estimates in spaces that can broadly be construed as a finite number of nice pieces that are glued together in a nice fashion. In particular, we obtain results for a certain class of manifolds and for a certain class of graphs.

Recall that from the perspective of partial differential equations, the heat kernel is the fundamental solution of the heat equation, while, from a probability perspective, the heat kernel is the transition density of Brownian motion. A first natural space to study the heat equation on is  $\mathbb{R}^n$ , where there are many classical results and the heat kernel is given by the explicit formula

$$p(t, x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{d(x, y)^2}{4t}\right), \quad (1.1)$$

where  $d(x, y)$  denotes the Euclidean distance between  $x$  and  $y$ .

However, the heat equation and heat kernel can be studied on any metric measure space nice enough to admit a version of the Laplacian  $\Delta$ . The theory of Dirichlet forms (see [8, 22]) provides a framework to make sense of the previous sentence. While we (mostly) avoid appealing to such abstraction in this thesis, this theory helps to explain why the two main settings considered here, manifolds and graphs, are not as different as they may first appear. That said, we mostly try to write results and proofs in the more concrete language of each setting, and hence the notation and definitions used vary between Chapter 2 and Chapters 3 and 4.

In spaces that are more abstract or less symmetric than  $\mathbb{R}^n$  (or its discrete counterpart  $\mathbb{Z}^d$ ), we can no longer hope to have a precise formula for the heat kernel  $p(t, x, y)$ . As with many problems in analysis and probability, the question then becomes one of obtaining good upper and lower bounds. In this thesis, we are particularly interested in “matching” upper and lower bounds, which means the upper and lower bounds have the same general form and the same kinds of terms. Many existing results of this type are of the form of classical two-sided Gaussian heat kernel estimates, which resemble the formula (1.1), and, in fact, such heat kernel estimates are known to be equivalent to nice functional inequalities. For more details, see Sections 2.3 and 3.1.2 below.

Settings where such nice two-sided *Gaussian* estimates are less likely to hold include situations where moving between different parts of a space is difficult (when so-called “bottleneck effects” are present) and when there is some sort of killing or Dirichlet boundary condition. In this thesis, our primary concern is with “glued” spaces, that is, spaces that can be thought of as being made up of certain nice pieces, which do satisfy two-sided Gaussian heat kernel estimates,

that are glued together somehow. We in general do not expect such glued spaces to have Gaussian estimates. Initial work on this type of gluing program was first begun by Alexander Grigor'yan and Laurent Saloff-Coste around three decades ago. The series of papers [34, 35, 36, 37, 38] builds up the theory of heat kernel estimates on manifolds with ends. In these initial papers, the ends satisfy two-sided Gaussian estimates and are glued via a compact set; no (or Neumann) boundary condition is taken. Of importance is that the overall manifold is transient (has a Green function). Work in the case where the underlying manifold is parabolic continues to this day in papers of Grigor'yan, Ishiwata, and Saloff-Coste [30, 31, 32]. Other recent work includes that of Chen and Lou [10], who do not constrain the gluing operation to stay in the class of smooth manifolds and consequently give results in the Dirichlet space setting. In this thesis, we continue work in this vein of gluing problems by giving heat kernel estimates on manifolds with ends with mixed boundary condition and heat kernel estimates for certain graphs, which we will call book-like graphs.

In Chapter 2 (which contains work from [14]), we consider heat kernel estimates on manifolds with ends with mixed Dirichlet and Neumann boundary condition. This builds off of previous work of Grigor'yan and Saloff-Coste, who considered the case of manifolds with ends with no, or Neumann, boundary condition (see [37] and references therein). Much of this chapter involves describing a setting in which previous results can be applied and combined. In particular, we wish to use the results of Gyrya and Saloff-Coste in [40], which deal with Dirichlet and mixed boundary condition, along with the results of [37] on gluing manifolds. A main tool appearing in this chapter is the use of an appropriate  $h$ -transform, and one of the main new results is Theorem 2.4.1, which constructs a sufficiently nice global harmonic function on the manifold.

The general heat kernel estimates in this setting are given in Theorem 2.5.1.

Chapters 3 (which contains work from [15]) and 4 together ultimately describe heat kernel estimates on what we call book-like graphs (Theorem 4.3.1). In particular, Chapter 4 aims to begin the study of heat kernel estimates on spaces made up of gluing “nice” graphs (graphs satisfying two-sided Gaussian heat kernel estimates) together via a possibly infinite set of vertices. These are the first results in the discrete case, and we obtain the discrete version of (some) results of [37] as a special case (Corollary 4.4.1). Chapter 3 grew out of a desire and need to obtain good estimates for certain hitting probabilities in the context of this kind of gluing problem, but may also be of independent interest.

This shift to the discrete case was inspired by considering the problem of gluing manifolds along non-compact sets, for example, along submanifolds or hypersurfaces. Presently, the only results in this direction can be found in work of Grigor’yan and Ishiwata in [39], which addresses gluing two copies of  $\mathbb{R}^n$  via a paraboloid of revolution. This paper relies heavily upon the symmetry of both  $\mathbb{R}^n$  and of the paraboloid of revolution and is far from the general abstract results found in the compact case. In this sort of problem, certain hitting probability calculations are very important, and in the discrete time and space case, these calculations simplify significantly. Hopefully, Chapter 4 will prove a useful blueprint for future work in the continuous setting. However, the discrete case is also interesting in its own right, and, while the hypotheses given in Chapter 4 allow for a finite number of pieces (“ends”) and do not rely on precise symmetry, these hypotheses are still fairly restrictive. In future work, we hope to be able to treat a somewhat more general case using these same methods.

CHAPTER 2  
HEAT KERNELS ON MANIFOLDS WITH ENDS WITH MIXED  
BOUNDARY CONDITION

## 2.1 Introduction

This chapter deals with the continuous setting, that of manifolds with ends. Here the gluing takes place over a compact set, and we allow for Dirichlet boundary condition or mixed Neumann and Dirichlet boundary condition. This chapter contains joint work with Laurent Saloff-Coste from [14]. The particulars are quite technical, so we begin with motivation and gradually build up the necessary notation.

### 2.1.1 Motivation

In [33], Alexander Grigor'yan and Laurent-Saloff Coste initiated the study of two-sided heat kernel estimates on weighted complete Riemannian manifolds with finitely many nice ends,  $M = M_1 \# \cdots \# M_k$ , where the notation  $\#$  denotes the connected sum operation. The components  $M_i$  of this connected sum are, themselves, assumed to be weighted complete Riemannian manifolds. The main assumption is that, on each  $M_i$ , the heat kernel  $p_{M_i}(t, x, y)$ , is well understood in the sense that it satisfies a classical-looking two-sided Gaussian estimate, uniformly at all times and locations. Equivalently ([23, 54, 55]), the volume functions of these manifolds,  $M_i$ ,  $1 \leq i \leq k$ , are uniformly doubling at all scales and locations **and** their geodesic balls satisfy a Neumann-type Poincaré inequality, uniformly at all all scales and locations. These are very strong hypotheses, and,

in certain cases, additional more technical hypotheses are needed. The results of [33] are sharp two-sided estimates on the heat kernel of  $M$ . The most basic case illustrating these results is when  $M_i = \mathbb{R}^N$  for some  $N$ , and, more generally,  $M_i = \mathbb{R}^{n_i} \times \mathbb{S}^{N-n_i}$ , for some  $N$  and  $n_i$ ,  $1 \leq n_i \leq N$ . These basic cases were new and already plenty challenging at the time [33] was published. They are richer than they appear if one takes into consideration the variation afforded by the weight functions. In addition, the results hold without change when the term “complete Riemannian manifold” is interpreted in the context of manifolds with boundary. Complete, then, means metrically complete, and the heat equations and heat kernels on  $M$  and on the  $M_i$ ,  $1 \leq i \leq k$ , are all taken with Neumann boundary condition.

The aim of the present chapter is to initiate the study of the case when the heat equation on the complete manifold  $M$  (with boundary) is taken with mixed boundary condition: Neumann on some part of the boundary and Dirichlet on the rest of the boundary. (Of course, restrictive assumptions will be made on the nature of the set on which Dirichlet boundary condition holds.) Here, as usual, Neumann boundary condition refers to the requirement that the normal derivative of the solution vanishes at the boundary, whereas Dirichlet boundary condition refers to the vanishing of the solution itself at the boundary. Even the simplest possible instances of this problem, such as the planar domain depicted in Figure 2.1, present interesting challenges.

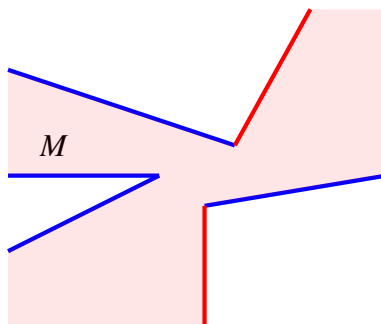


Figure 2.1: Sketch of a planar, unbounded, complete manifold  $M$  (light red) with three conic ends. Dirichlet boundary is depicted in blue, Neumann boundary in red. Corners should be rounded so that  $M$  is a manifold with boundary. However, the presence of a finite number of corners actually does not matter; see Section 2.7.

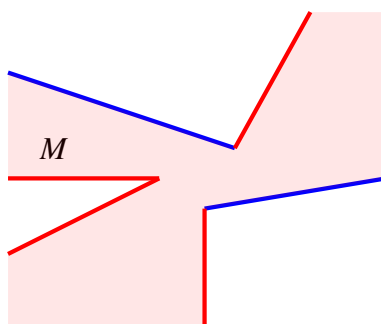


Figure 2.2: Same  $M$  as in Figure 2.1, but with different boundary conditions.

Describing the behavior of the heat kernel  $p(t, x, y)$  in the domain depicted in Figures 2.1 and 2.2 (with the given boundary conditions) will require the introduction of a fair bit of notation. Ultimately, in our main result of Theorem 2.5.1, we give upper and lower bounds, valid for all time  $t > 0$  and pairs  $(x, y) \in M$ , which are essentially “matching bounds” in the sense used widely in the literature on heat kernel bounds. For the purpose of this introduction, we focus on the following particular case: Fix a point  $o$  in  $M$  not on the Dirichlet boundary. What is the behavior of  $p(t, o, o)$  as  $t$  tends to infinity when  $M$  is the domain

depicted in Figure 2.1 with the given boundary conditions?

To answer this question, starting from upper left and continuing counter-clockwise, denote by  $M_1, M_2, M_3$  the three cones whose connected sum is  $M$ . Note that  $M_1$  carries Dirichlet boundary condition on both sides whereas  $M_2$  and  $M_3$  carry Dirichlet boundary condition on one side and Neumann on the other. Let  $\alpha_i$  be the apertures of  $M_i$ ,  $1 \leq i \leq 3$ . We will show that, because each  $\alpha_i$  is positive, there are constants  $0 < c_o \leq C_o < +\infty$  such that, for all  $t > 1$ ,

$$c_o t^{-a} \leq p(t, o, o) \leq C_o t^{-a}$$

with

$$a = 1 + \min \left\{ \frac{\pi}{\alpha_1}, \frac{\pi}{2\alpha_2}, \frac{\pi}{2\alpha_3} \right\}.$$

Now consider the case where there is a cone of positive aperture carrying Neumann boundary condition on both sides and (at least) one other cone carrying Dirichlet boundary condition on at least one side as in Figure 2.2. We will show that this situation leads to the behavior

$$c_o (t \log^2 t)^{-1} \leq p(t, o, o) \leq C_o (t \log^2 t)^{-1}.$$

To give yet another variation, consider the domain depicted in Figure 2.3.

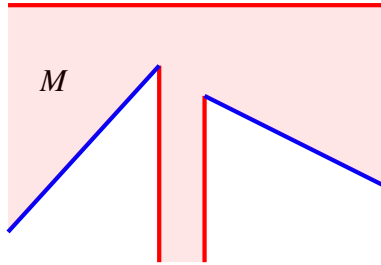


Figure 2.3: An  $M$  with an end that is a cone of aperture zero.

In this case,

$$c_o t^{-3/2} \leq p(t, o, o) \leq C_o t^{-3/2}.$$

## 2.1.2 Description of the method

These results will be obtained via a general method based on a combination of the basic ideas developed in [34, 35, 36, 37, 38] and [40] (the results in [37] make heavy use of those in [34, 35, 36, 38]). Reference [37] provides a very general line of attack to reconstruct heat kernel estimates on a connected sum from heat kernel estimates on and related knowledge of the parts forming that sum. Reference [40] provides the ideas that make the technique of [37] applicable to the case when Dirichlet boundary condition is present. Namely, after an appropriate  $h$ -transform (also known as Doob's transform after Joseph Doob), the Dirichlet condition disappears, and one can apply the technique of [37] straightforwardly even though the set-up is not quite that of [37]. (Section 2.3.4 contains the relevant adaptations of the results from [37].) Further connections with earlier results are described in Section 2.8.

The rest of this chapter proceeds as follows. Section 2.2 introduces the specific objects and setting under consideration. Section 2.3 introduces notation, definitions, and summaries of relevant past work (with some additional connections given in the final section of this chapter, Section 2.8), as well as a framework that is slightly more general. Section 2.4 constructs a harmonic function with special properties to be used as the key function  $h$  in the  $h$ -transform technique. Section 2.5 then implements the  $h$ -transform technique to obtain the desired heat kernel estimates, and Section 2.6 applies the main theorem from Sec-

tion 2.5 to various examples. Section 2.7 describes how to extend the results of this chapter to manifolds with simple corners using the framework from Section 2.3.

## 2.2 Set-up for our problem

Here we describe the notation and set-up most important to us in this chapter; we postpone some more technical definitions until Section 2.3 and refer to them as necessary.

### 2.2.1 The underlying complete manifold $M$

We start with a smooth manifold with boundary,  $(M, \delta M)$ , equipped with a Riemannian structure  $g$  and a positive, smooth weight  $\sigma : M \rightarrow (0, +\infty)$ . It will sometimes be useful to set  $M^\bullet = M \setminus \delta M$ . We let  $d$  be the geodesic distance on  $(M, g)$  and assume that  $(M, d)$  is a complete metric space. We call  $M$  a weighted, complete Riemannian manifold with boundary (by definition, a manifold is connected). Hence  $M$  comes equipped with a number of additional objects we briefly describe.

- The Riemannian measure,  $dx$ , and its weighted version  $\mu(dx) = \sigma(x)dx$ . We view  $(M, d, \mu)$  as our main metric measure space.
- Geodesic balls in  $M$ , which are denoted by  $B_M(x, r)$ ,  $x \in M$ ,  $r > 0$ . The  $\mu$  volume of  $B_M(x, r)$  is  $V(x, r) := \mu(B_M(x, r))$ .

- The gradient  $\nabla f$  defined on smooth functions by

$$df|_x(X) = g_x(\nabla f(x), X)$$

for any tangent vector  $X$  at  $x \in M$ .

- The divergence  $\operatorname{div} X = \operatorname{div}_\mu X$  defined on smooth vector fields by

$$\int_M \operatorname{div}(X)f \, d\mu = - \int_M g(X, \nabla f) \, d\mu$$

for any smooth compactly supported function  $f$  on  $M$ .

- The Laplace operator  $\Delta = \Delta_\mu$  defined on smooth functions on  $M^\bullet = M \setminus \delta M$  by  $\Delta f = \operatorname{div}(\nabla f)$ .

**Definition 2.2.1** (The Sobolev space  $W_0^1(V)$ ). The (local) Sobolev space  $W_{\text{loc}}(M^\bullet)$  is the space of distributions on  $M^\bullet$  which can be represented locally by an  $L^2$  function and whose first partial derivatives in any precompact local chart of  $M^\bullet$  can also be represented by  $L^2$  functions. For any open set  $U^\bullet \subset M^\bullet$ , we may define  $W_{\text{loc}}(U^\bullet)$  in the same way by replacing  $M^\bullet$  with  $U^\bullet$ . For any open subset  $V \subset M$ , the Sobolev space  $W_0(V) = W_0^1(V)$  is the subspace of  $L^2(V) = L^2(V, \mu|_V)$  obtained by closing the space of smooth compactly supported functions on  $V$ ,  $C_c^\infty(V)$ , under the norm  $(\int_V |f|^2 \, d\mu + \int_V |\nabla f|^2 \, d\mu)^{1/2}$ .

**Definition 2.2.2** (Heat equation on  $M$ ). The heat semigroup  $P_t^M$  is the semigroup associated with the Dirichlet form  $(W_0(M), \int_M g(\nabla f, \nabla f) \, d\mu)$ . It is given on  $L^2(M)$  by

$$P_t^M f(x) = \int_M p_M(t, x, y) f(y) \, d\mu(y), \quad t > 0, \quad x \in M,$$

where the heat kernel  $p_M$ , viewed as a function of  $t$  and  $x$ , satisfies the heat equation  $(\partial_t - \Delta)p(t, x, y) = 0$  on  $M \setminus \delta M$  with Neumann boundary condition along the boundary  $\delta M$  and the initial condition  $p_M(0, x, \cdot) = \delta_x(\cdot)$  (when the distribution/smooth function pairing is given by the extension of  $\langle \phi, \psi \rangle = \int \phi \psi \, d\mu$ ).

The infinitesimal generator associated with this Dirichlet form will be referred to as  $\Delta_M$ .

## 2.2.2 Our main objects of study

The complete manifold  $M$  and its heat kernel are not the main objects of interest in the present chapter. Instead, we consider an open subset  $\Omega$  of  $M$  such that the closed set  $M \setminus \Omega$  is a subset of  $\delta M$ . Hence, the topological boundary of  $\Omega$  in  $M$  is  $\partial\Omega = M \setminus \Omega$ .

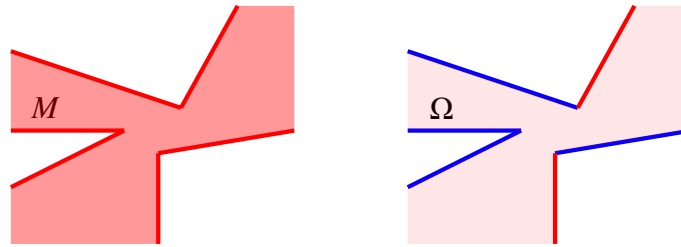


Figure 2.4: Sketch (corners should be rounded) of the complete manifold  $M$  (dark red) and its submanifold  $\Omega$  (light red and red boundary) with “Dirichlet boundary”  $\partial\Omega \subseteq \delta M$ , not part of  $\Omega$ , highlighted in blue.

We can view  $\Omega$  as a manifold with boundary  $\delta\Omega = \delta M \cap \Omega$ , but it is not metrically complete if  $\partial\Omega \neq \emptyset$ . The metric completion of  $\Omega$  is (isometric to)  $M$ . We will use the following notation:

- Geodesic balls in  $\Omega$  are denoted by  $B(x, r) = B_\Omega(x, r)$ . The  $\mu$ -volume of  $B(x, r)$  is  $V(x, r) = \mu(B(x, r)) = \mu(B_M(x, r))$ . By abuse of language and notation, if  $x \in \partial\Omega$ , we write  $B_\Omega(x, r) = B_M(x, r) \cap \Omega$ .
- The heat semigroup  $P_t = P_t^\Omega$  and its kernel  $p(t, x, y) = p_\Omega(t, x, y)$ ,  $(t, x, y) \in$

$(0, +\infty) \times \Omega \times \Omega$ , are given on  $L^2(\Omega)$  by

$$P_t f(x) = \int_{\Omega} p(t, x, y) f(y) d\mu(y), \quad t > 0, x \in M,$$

and are associated with the Dirichlet form  $(W_0(\Omega), \int_{\Omega} g(\nabla f, \nabla f) d\mu)$ , which has infinitesimal generator  $\Delta_{\Omega}$ . By definition, the heat kernel  $p = p_{\Omega}$ , viewed as a function of  $t$  and  $x$ , satisfies the heat equation  $(\partial_t - \Delta)p(t, x, y) = 0$  on  $\Omega \setminus \delta\Omega$  with Neumann boundary condition along the boundary  $\delta\Omega$  and Dirichlet boundary condition (in the weak sense) along  $\partial\Omega$ .

Condition (\*): Unless specified otherwise, we make the simplifying assumptions that the closed set  $\partial\Omega \subseteq \delta M$  has countably many connected components, each of which is a smooth codimension 1 manifold with boundary, and that any point in  $M$  has a neighborhood in  $M$  containing at most finitely many connected components of  $\partial\Omega$ .

### 2.2.3 Finitely many nice ends

In this subsection, we describe the main additional hypotheses we make on the global geometric structure of  $M$  (and hence  $\Omega$ ). Namely, we assume that  $M$  is the connected sum of  $k$  complete Riemannian manifolds with boundary  $(M_1, \delta M_1), \dots, (M_k, \delta M_k)$ ,

$$M = M_1 \# M_2 \# \dots \# M_k.$$

With  $\sqcup$  denoting disjoint union, this means that

$$M = K \sqcup (E_1 \sqcup \dots \sqcup E_k),$$

where  $K$  is a compact subset of  $M$  with the property that  $M \setminus K$  has  $k$  connected components  $E_1, \dots, E_k$  and each  $E_i$  is isometric to a connected subset of

$M_i$  with compact complement  $K_i$  (hence,  $M_i = K_i \sqcup E_i$ ). The explicit decomposition  $M = K \sqcup (E_1 \sqcup \cdots \sqcup E_k)$  is, of course, not unique, and we will assume this decomposition possesses additional nice properties. We assume that the metric closure of each  $E_i$  is, itself, a manifold with boundary. This is a somewhat constraining hypothesis, but it has the advantage of simplifying exposition by restricting our attention to smooth manifolds with boundary. The weight  $\sigma$  on  $M$  is assumed to be compatible with a weight  $\sigma_i$  on each  $M_i$  in the sense that  $\sigma|_{E_i} = \sigma_i$ .

Next, we consider an open set  $\Omega \subset M$  with  $M \setminus \Omega \subseteq \delta M$  and set

$$U_i = \Omega \cap E_i, \quad i = 1, \dots, k.$$

The open sets  $U_i$  are important to us. Each  $U_i$  is a weighted Riemannian manifold with boundary  $\delta U_i = \delta M \cap U_i$  and whose topological boundary in  $M$ , denoted by  $\partial U_i$ , is the union of its “lateral” boundary or “side” boundary  $\partial^{\text{side}} U_i = \overline{E_i} \cap \partial \Omega$  and its “inner” boundary  $\partial^{\text{inner}} U_i = \partial E_i$ . The inner boundary  $\partial^{\text{inner}} U_i = \partial E_i$  is also a subset of  $K$ . It is compact with finitely many connected components, which are co-dimension 1 submanifolds with boundary. The union  $\partial^{\text{side}} U_i \cup \partial^{\text{inner}} U_i$  is not necessarily disjoint, but the intersection  $\partial^{\text{side}} U_i \cap \partial^{\text{inner}} U_i$  is of co-dimension at least 2. We make the following strong hypotheses:

(H1) Each  $(M_i, \sigma_i)$  is a Harnack manifold (Definition 2.3.11). Equivalently, each  $M_i$  is doubling and the Poincaré inequality holds, both uniformly (see Definitions 2.3.9 and 2.3.10).

(H2) Each  $U_i$  is uniform in  $M_i$  (Definition 2.3.13).

We now collect a list of important known consequences of these hypotheses for future reference.

- (C1) The condition that each  $U_i$  is uniform in  $M_i$  implies that  $U_i$  satisfies (RCA) from Definition 2.3.1. This can be seen directly from the definitions.
- (C2) The condition that each  $U_i$  is uniform in the Harnack manifold  $M_i$  implies that the elliptic boundary Harnack inequality holds uniformly in  $U_i$  (Definition 2.3.6). Further details on the elliptic boundary Harnack inequality and situations in which it holds can be found in [6, 40, 50] and the references therein.
- (C3) The condition that each  $U_i$  is uniform in the Harnack manifold  $M_i$  implies that  $U_i$  admits a harmonic profile  $u_i$ , that is, a positive harmonic function vanishing along  $\partial U_i$  (see Definition 2.3.3). This profile is unique up to a positive multiplicative constant. This follows from Theorem 4.1 of [40].
- (C4) The weighted Riemannian manifold  $(U_i, \sigma u_i^2)$  is a Harnack manifold. This is given by Theorem 5.9 of [40].

### 2.3 Generalities and notation

In this section, we elaborate on terms that appeared above as well as detail a slightly more general set-up. Let  $(\Omega, \delta\Omega)$  be a weighted Riemannian manifold with boundary. If the associated metric space  $(\Omega, d)$  is not complete, let  $\widetilde{\Omega}$  be its completion and  $\partial\Omega = \widetilde{\Omega} \setminus \Omega$ . This set-up is more general than that which will be considered in the main part of this chapter, where we require the following condition (\*) to hold:

- (\*)  $\Omega$  is a submanifold of the weighted complete Riemannian manifold with boundary  $(M, \delta M)$  with  $M \setminus \Omega \subseteq \delta M$ , and  $\partial\Omega$  has countably many connected

components such that every point in  $M$  has a neighborhood containing only finitely many of these connected components, and these components are themselves codimension 1 submanifolds (with boundary) of  $M$ .

Indeed, in general,  $\widetilde{\Omega}$  may not be a manifold with boundary. One more subtle difference of importance to us here is that the weight  $\sigma$  on  $\Omega$ , which is smooth and positive in  $\Omega$ , might have a variety of behaviors when one approaches the boundary  $\partial\Omega$ . Under hypothesis (\*), the weight  $\sigma$  is smooth and positive up to the boundary  $\delta M$ .

Recall that  $\Omega^\bullet = \Omega \setminus \delta\Omega$ , that  $W_{\text{loc}}(\Omega^\bullet)$  is the local Sobolev space on  $\Omega^\bullet$ , and that  $W_0^1(\Omega)$  is the closure of  $C_c^\infty(\Omega)$  under the norm  $\left(\int_\Omega |f|^2 d\mu + \int_\Omega |\nabla f|^2 d\mu\right)^{1/2}$  (Definition 2.2.1). For  $U \subset \Omega$ , the space  $W^1(U)$  is the set of functions  $f \in W_{\text{loc}}(U^\bullet)$  such that  $\left(\int_U |f|^2 d\mu + \int_U |\nabla f|^2 d\mu\right)^{1/2} < +\infty$ . Then the space  $W_{\text{loc}}(U)$  is defined as the set of functions where for any open relatively compact  $V \subset U$ , there is a function  $f^V \in W^1(U)$  such that  $f = f^V$  almost everywhere on  $V$ . We also let  $\text{Lip}(V)$  be the space of bounded Lipschitz functions on  $V$ .

Recall the following definition used in [37]:

**Definition 2.3.1** (Relatively connected annuli property). A metric space  $(M, d)$  satisfies the relatively connected annuli property ((RCA), for short) with respect to a point  $o \in M$  if there exists a constant  $C_A$  such that for any  $r \geq C_A^2$  and all  $x, y \in M$  such that  $d(o, x) = d(o, y) = r$ , there exists a continuous path  $\gamma : [0, 1] \rightarrow M$  with  $\gamma(0) = x, \gamma(1) = y$  whose image is contained in  $B(o, C_A r) \setminus B(o, C_A^{-1} r)$ .

### 2.3.1 Local and global harmonic functions

Throughout, we choose to use appropriate weak definitions of solutions of the Laplace or heat equation even though, in the special set-up of interest to us, because of various simplifying hypotheses made in the main parts of the chapter, such solutions are, in fact, classical solutions, including with respect to the boundary conditions (see, for instance, [48]).

**Definition 2.3.2** (Harmonic function in an open set  $U$  of  $\Omega$ ). Let  $U$  be an open subset of  $\Omega$ . A function  $u$  defined in  $U$  is a (local) harmonic function in  $U$  if  $u \in W_{\text{loc}}(U)$  and, for any  $\phi \in C_c^\infty(U)$ ,

$$\int_U g(\nabla u, \nabla \phi) d\mu = 0.$$

In classical terms,  $u \in C_{\text{loc}}^\infty(U)$ ,  $\Delta u = 0$  in  $U \cap \Omega^\bullet = U \cap M^\bullet$ , and  $u$  has vanishing normal derivative along  $U \cap \delta\Omega$  (and no condition along  $\overline{U} \cap \partial\Omega$ ).

**Definition 2.3.3** (Harmonic function in  $U \subset \Omega$  open, vanishing on  $\partial\Omega$ ). Let  $U$  be an open subset of  $\Omega$ . A function  $u$  defined in  $U$  is a (local) harmonic function in  $U$  with Dirichlet boundary condition along  $\partial\Omega$  (i.e., vanishing along  $\partial\Omega$ ) if  $u$  is locally harmonic in  $U$  and, for any  $\psi \in C_c(U^\#) \cap \text{Lip}(U^\#)$ ,  $u\psi \in W_0^1(U)$ . Here  $U^\#$  is the largest open set in  $\widetilde{\Omega}$  such that  $U^\# \cap \Omega = U$ . In classical terms, under condition (\*),  $u \in C_{\text{loc}}^\infty(U)$ ,  $\Delta u = 0$  in  $U \cap \Omega^\bullet = U \cap M^\bullet$ ,  $u$  has vanishing normal derivative along  $U \cap \delta\Omega$ , and  $u$  can be extended continuously by setting  $u(x) = 0$  at any point  $x \in \partial\Omega$  which is at positive distance from  $\Omega \setminus U$ .

**Definition 2.3.4** (Global harmonic function in  $\Omega$ ). A global harmonic function in  $\Omega$  is a function  $u$  in  $\Omega$  which is locally harmonic in  $\Omega$  and vanishes along  $\partial\Omega$ .

**Remark 2.3.1.** This last definition applies to the case  $M = \Omega$ , providing the definition of global harmonic function in  $M$ . In that case, there is no Dirichlet boundary condition as  $\partial M = \emptyset$ .

**Definition 2.3.5** (Elliptic Harnack inequality). We say that:

- The elliptic Harnack inequality holds locally in a subset  $V$  of  $\Omega$  if for any compact set  $K \subset V$  there exist  $H_K$  and  $r_K > 0$  such that, for all  $(x, r) \in K \times (0, r_K)$ , and any positive harmonic function  $u$  in  $B_\Omega(x, 2r)$ ,

$$\sup_{B_\Omega(x,r)} \{u\} \leq H_K \inf_{B_\Omega(x,r)} \{u\}.$$

- The elliptic Harnack inequality holds up to scale  $r_0$  over a subset  $K$  of  $\Omega$  if there is a constant  $H_{K,r_0}$  such that, for all  $(x, r) \in K \times (0, r_0)$  and any positive harmonic function  $u$  in  $B_\Omega(x, 2r)$ ,

$$\sup_{B_\Omega(x,r)} \{u\} \leq H_{K,r_0} \inf_{B_\Omega(x,r)} \{u\}.$$

- The elliptic Harnack inequality holds uniformly in an open subset  $U$  of  $\Omega$  if there is a constant  $H_U$  such that for all  $(x, r) \in U \times (0, +\infty)$  such that  $B_\Omega(x, 2r) \subset U$  and any positive harmonic function  $u$  in  $B_\Omega(x, 2r)$ ,

$$\sup_{B_\Omega(x,r)} \{u\} \leq H_U \inf_{B_\Omega(x,r)} \{u\}.$$

**Remark 2.3.2.** The elliptic Harnack inequality always holds locally on  $\Omega$ . It holds up to scale  $r_0 = d(K, \partial\Omega) > 0$  on any compact subset  $K$  of  $\Omega$ . Under condition (\*), the elliptic Harnack inequality always holds locally on  $\Omega$  and on  $M$ . (They do not mean the same thing.) It also holds up to scale  $r_0$  for any fixed  $r_0$  on any compact subset of  $M$ .

**Definition 2.3.6** (Boundary elliptic Harnack inequality). The following definitions are only useful when  $\partial\Omega \neq \emptyset$ .

- The boundary elliptic Harnack inequality holds locally on a subset  $V$  of  $\partial\Omega$  if for any compact set  $K \subset V$  there exist  $H_K$  and  $r_K > 0$  such that,

for all  $(x, r) \in K \times (0, r_K)$ , and any two positive harmonic functions  $u, v$  in  $B_\Omega(x, 2r) = \Omega \cap B_{\bar{\Omega}}(x, 2r)$  vanishing along  $\partial\Omega$ ,

$$\sup_{B_\Omega(x,r)} \{u/v\} \leq H_K \inf_{B_\Omega(x,r)} \{u/v\}.$$

- The boundary elliptic Harnack inequality holds up to scale  $r_0$  over a subset  $K$  of  $\partial\Omega$  if there is a constant  $H_{K,r_0}$  such that, for all  $(x, r) \in K \times (0, r_0)$  and any two positive harmonic functions  $u, v$  in  $B_\Omega(x, 2r) = \Omega \cap B_{\bar{\Omega}}(x, 2r)$  vanishing along  $\partial\Omega$ ,

$$\sup_{B(x,r)} \{u/v\} \leq H_{K,r_0} \inf_{B(x,r)} \{u/v\}.$$

- The boundary elliptic Harnack inequality holds uniformly in an open subset  $U$  of  $\Omega$  if there is a constant  $H_U$  such that for all  $(x, r) \in \partial\Omega \times (0, \infty)$  such that  $B(x, 2r) \subset U$  and any two positive harmonic functions  $u, v$  in  $B(x, 2r)$  vanishing along  $\partial\Omega$ ,

$$\sup_{B(x,r)} \{u/v\} \leq H_U \inf_{B(x,r)} \{u/v\}.$$

**Remark 2.3.3.** The validity of a boundary elliptic Harnack inequality depends on the nature of the boundary  $\partial\Omega$ . Under assumption (\*), the boundary elliptic Harnack inequality always holds locally on  $\Omega$  and, for each  $r_0 > 0$ , up to scale  $r_0$  on any compact subset  $K$  of  $\partial\Omega$ .

### 2.3.2 Local and global solutions of the heat equation

To save space, we refer the reader to [20, 40, 59, 60, 61] for the definition of local weak solutions of the heat equation in an open cylindrical domain  $(a, b) \times U \subset \mathbb{R} \times \Omega$ , in the context of the strictly local regular Dirichlet space  $(W_0^1(\Omega), \int_\Omega g(\nabla f, \nabla f) d\mu)$ . Because such weak solutions are automatically smooth

in time, one can be a bit cavalier with the details of such definitions. In fact, these local weak solutions are always smooth in  $(a, b) \times U$ , including up to  $\delta\Omega \cap U$  where they satisfy the Neumann boundary condition.

For the definition of weak solutions in an open set  $U$  of  $\Omega$  vanishing along  $\partial\Omega$ , we refer the reader to [40]. Simply put, given that weak solutions are smooth in time and at any point in  $U$ , the condition that the solution vanishes along the relevant part of  $\partial\Omega$  can be captured as in Definition 2.3.3 by requiring that, for any  $t \in (a, b)$  and any  $\psi \in C_c(U^\#) \cap \text{Lip}(U^\#)$ ,  $u\psi \in W_0^1(U)$ . In fact, under condition (\*), such a solution will vanish continuously along the relevant part of  $\partial\Omega$  [48].

**Definition 2.3.7** (Global solution of the heat equation in  $(a, b) \times \Omega$ ). A global solution of the heat equation function in  $(a, b) \times \Omega$  is a function  $u$  in  $(a, b) \times M$  which is smooth in  $(a, b) \times \Omega$ , satisfies  $(\partial_t - \Delta)u = 0$  in  $(a, b) \times M^\bullet$ , has vanishing normal derivative on  $\delta\Omega$  and vanishes along  $\partial\Omega$ .

Given a time-space cylinder  $Q = (s, s + 4r^2) \times B(x, 2r)$ , set  $Q_- = (s + r^2, s + 2r^2) \times B(x, r)$  and  $Q_+ = (s + 3r^2, s + 4r^2) \times B(x, r)$ .

**Definition 2.3.8** (Parabolic Harnack inequality). We say that:

- The parabolic Harnack inequality holds locally in a subset  $V$  of  $\Omega$  if for any compact set  $K \subset U$  there exist  $H_K$  and  $r_K > 0$  such that, for all  $s \in \mathbb{R}$ ,  $(x, r) \in K \times (0, r_K)$ , and any local solution  $u \geq 0$  of the heat equation in  $Q = (s, s + 4r^2) \times B_\Omega(x, 2r)$ ,

$$\sup_{Q_-} \{u\} \leq H_K \inf_{Q_+} \{u\}.$$

- The parabolic Harnack inequality holds up to scale  $r_0$  over a subset  $K$  of  $\Omega$  if there is a constant  $H_{K, r_0}$  such that, for all  $s \in \mathbb{R}$ ,  $(x, r) \in K \times (0, r_0)$  and any

local solution  $u \geq 0$  of the heat equation in  $Q = (s, s + 4r^2) \times B_\Omega(x, 2r)$ ,

$$\sup_{Q_-} \{u\} \leq H_{K,r_0} \inf_{Q_+} \{u\}.$$

- The parabolic Harnack inequality holds uniformly in an open subset  $U$  of  $\Omega$  if there is a constant  $H_U$  such that, for all  $s \in \mathbb{R}$  and  $(x, r) \in U \times (0, +\infty)$  such that  $B(x, 2r) \subset U$  and any local solution  $u \geq 0$  of the heat equation in  $(s, s + 4r^2) \times B_\Omega(x, 2r)$ ,

$$\sup_{Q_-} \{u\} \leq H_U \inf_{Q_+} \{u\}.$$

### 2.3.3 Doubling and Poincaré

**Definition 2.3.9** (Doubling). Very generally, doubling refers to the volume function property that  $V(x, 2r) \leq CV(x, r)$  where  $(x, r)$  belong to some specific subset of  $\Omega \times (0, +\infty)$ .

- A set  $V$  is locally doubling if for any compact set  $K \subset V$  there exists  $r_0(K) > 0$  such that  $K$  is doubling up to scale  $r_0(K)$ .
- An arbitrary set  $K$  is doubling up to scale  $r_0$  if there is a constant  $C_{K,r_0}$  such that for all  $(x, r) \in K \times (0, r_0)$ ,  $V(x, r) \leq C_{K,r_0} V(x, 2r)$ .
- An open subset  $U$  of  $\Omega$  or  $\tilde{\Omega}$  is uniformly doubling (or doubling for short) if there is a constant  $C_U$  such that for all  $(x, r) \in U \times (0, +\infty)$  such that  $B(x, 2r) \subset U$ ,  $V(x, 2r) \leq C_U V(x, r)$ .

**Remark 2.3.4.** A manifold with boundary is always locally doubling. It may or not be doubling up to scale  $r_0$  for some  $r_0 > 0$ . It may or not be uniformly doubling. Euclidean space  $\mathbb{R}^n$  is doubling, as is any complete Riemannian manifold without boundary with non-negative Ricci curvature. Convex domains in

$\mathbb{R}^n$  are doubling. Hyperbolic space is doubling up to any fixed scale  $r_0$ , but it is not doubling. A complete Riemannian manifold without boundary with Ricci curvature bounded below is doubling up to any fixed scale  $r_0$ .

A Poincaré inequality is an inequality of the form

$$\forall f \in C^\infty(B(x, r)), \quad \int_{B(x, r)} |f - f_B|^2 d\mu \leq Pr^2 \int_{B(x, r)} |\nabla f|^2 d\mu,$$

where  $f_B$  is the average value of  $f$  over  $B = B(x, r)$ .

**Definition 2.3.10** (Poincaré inequality). Consider the following three versions:

- The Poincaré inequality holds locally in a subset  $V$  of  $\Omega$  or  $\tilde{\Omega}$  if for any compact set  $K \subset V$  there exists  $r_0(K) > 0$  and a constant  $P_K$  such that, for all  $(x, r) \in K \times (0, r_0(K))$ ,

$$\forall f \in C^\infty(B(x, r)), \quad \int_{B(x, r)} |f - f_B|^2 d\mu \leq P_K r^2 \int_{B(x, r)} |\nabla f|^2 d\mu.$$

- The Poincaré inequality holds up to scale  $r_0$  over a subset  $K$  of  $\Omega$  or  $\tilde{\Omega}$  if there is a constant  $P_{K, r_0}$  such that, for all  $(x, r) \in K \times (0, r_0)$ ,

$$\forall f \in C^\infty(B(x, r)), \quad \int_{B(x, r)} |f - f_B|^2 d\mu \leq P_{K, r_0} r^2 \int_{B(x, r)} |\nabla f|^2 d\mu.$$

- The Poincaré inequality holds uniformly in an open subset  $U$  of  $\Omega$  or  $\tilde{\Omega}$  if there is a constant  $P_U$  such that for all  $(x, r) \in U \times (0, +\infty)$  such that  $B(x, r) \subset U$ ,

$$\forall f \in C^\infty(B(x, r)), \quad \int_{B(x, r)} |f - f_B|^2 d\mu \leq P_U r^2 \int_{B(x, r)} |\nabla f|^2 d\mu.$$

**Remark 2.3.5.** A Poincaré inequality always holds locally on any manifold with boundary. A Poincaré inequality up to scale  $r_0$  for some  $r_0 > 0$  may hold or not on a manifold with boundary. A Poincaré inequality may hold uniformly

or not on a manifold with boundary. A Poincaré inequality holds uniformly on Euclidean space  $\mathbb{R}^n$ , and it also holds uniformly on any complete Riemannian manifold without boundary with non-negative Ricci curvature. A Poincaré inequality up to scale  $r_0$  for any fixed  $r_0 > 0$  holds on hyperbolic space, but it does not hold uniformly at all scales. A Poincaré inequality up to scale  $r_0$  for any fixed  $r_0 > 0$  holds on any complete Riemannian manifold without boundary with bounded Ricci curvature.

### 2.3.4 Harnack weighted manifolds

As above, let  $(\Omega, \delta\Omega)$  be a Riemannian manifold. **We do not assume it is complete.** Let  $\tilde{\Omega}$  be its metric completion and  $\partial\Omega = \tilde{\Omega} \setminus \Omega$ . Let  $\sigma$  be a smooth positive weight on  $\Omega$ . We consider the (local regular) Dirichlet space  $(W_0^1(\Omega), \int_{\Omega} |\nabla f|^2 d\mu)$  and the associated heat equation (see, e.g., [40, 60, 61] for details).

**Definition 2.3.11** (Harnack manifold). We call a weighted Riemannian manifold  $\Omega$  is a Harnack manifold if the parabolic Harnack inequality holds uniformly in  $\Omega$ .

Under relatively mild conditions on  $\Omega$ ,  $\tilde{\Omega}$ , and the weight  $\sigma$ , this condition is known to be equivalent ([23, 40, 54, 61]) to the validity of the volume doubling condition and Poincaré inequality, uniformly in  $\tilde{\Omega}$ . It is also equivalent to the validity of the two-sided (Gaussian) heat kernel estimate

$$\frac{c_1 e^{-c_2 \frac{d^2}{t}}}{V(x, \sqrt{t})} \leq p_{\Omega}(t, x, y) \leq \frac{C_1 e^{-c_2 \frac{d^2}{t}}}{V(x, \sqrt{t})}, \quad d = d(x, y). \quad (2.1)$$

**Remark 2.3.6.** The best known large class of Harnack manifolds is the class of complete Riemannian manifolds with non-negative Ricci curvature (see [55])

and the references therein). In this case the weight is the constant weight 1. Reference [33] discusses how to obtain examples with non-trivial weights. We are interested in the case when  $\widetilde{\Omega}$  is a (smooth) manifold satisfying condition (\*). In this case, assuming that  $\sigma$  has a continuous extension to  $\partial\Omega$ , it is necessary for the weight  $\sigma$  to vanish at the boundary in order for the weighted manifold  $\Omega$  to have a chance to be a Harnack manifold. One of the simplest examples of Harnack manifold of this type is the upper-half Euclidean space  $\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$  equipped with the weight  $\sigma(x) = x_n^2$ . See [40] for many more examples.

We will make use of the following key theorems. See [40] for a discussion of more general versions of these theorems.

**Theorem 2.3.1.** *Let  $(\Omega, \sigma)$  be a weighted Riemannian manifold with boundary. Assume that  $\widetilde{\Omega} = M$  is a manifold with boundary and that  $\partial\Omega = M \setminus \Omega$  satisfies condition (\*). Assume that the weight  $\sigma$  has a continuous extension to  $M$ , vanishing on  $\partial\Omega$  and such that the restriction to  $\Omega$  of any Lipschitz function compactly supported in  $M$  is in  $W_0^1(\Omega)$ . Then the weighted manifold  $(\Omega, \sigma)$  is Harnack if and only if  $(\Omega, \sigma)$  is uniformly doubling and the Poincaré inequality holds uniformly.*

This is a slight extension of the results in [23, 54], which essentially cover the case  $\Omega = M$ . This extension is contained in the more general Dirichlet space version given in [40].

The following important theorem follows from Section 5 of [40].

**Theorem 2.3.2.** *Assume that  $(M, \sigma)$  is a weighted complete Riemannian manifold which is uniformly Harnack. Let  $\Omega$  be an open subset of  $M$  such that  $\partial\Omega = M \setminus \Omega$  is a subset of the boundary  $\delta M$  and satisfies (\*). Assume that  $\Omega$  is a uniform subset of*

$M$  (Definition 2.3.13) and let  $h$  be a positive harmonic function vanishing along  $\partial\Omega$  (a harmonic profile for  $\Omega$ ).

Then there are constants  $0 < c \leq C < +\infty$  such that, for any  $x \in M, r > 0$ , and any  $x_r$  such that  $x_r \in B(x, Ar)$  and  $d(x_r, \partial\Omega) \geq ar$ , we have

$$ch(x_r)^2V(x, r) \leq V_h(x, r) := \int_{B(x,r)} h^2 d\mu \leq Ch(x_r)^2V(x, r)$$

where, as usual,  $V(x, r) = \mu(B(x, r))$ .

Moreover, the Riemannian manifold  $\Omega$  weighted by  $\sigma_h = \sigma h^2$  is a Harnack manifold. In particular, if  $p_{\Omega, h^2}$  indicates the heat kernel for  $(\Omega, \sigma_h)$ , there exist constants  $c_1, c_2, c_3, c_4 > 0$  such that  $\forall t > 0, x, y \in \Omega$

$$\frac{c_1}{h(x_{\sqrt{t}})^2V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)}{c_2t}\right) \leq p_{\Omega, h^2}(t, x, y) \leq \frac{c_3}{h(x_{\sqrt{t}})^2V(x, \sqrt{t})} \exp\left(-\frac{d(x, y)}{c_4t}\right).$$

We will also need an extension of a particular case of the main result of [37] that holds on a certain class of manifolds, some of which may be incomplete.

**Theorem 2.3.3.** *Let  $(\Omega, \sigma)$  be a weighted Riemannian manifold with boundary such that  $\tilde{\Omega} = M$  is a manifold with boundary and  $(\Omega, \sigma)$  satisfies (\*). Assume that the weight  $\sigma$  has a continuous extension to  $M$ , vanishing on  $\partial\Omega$  and such that the restriction to  $\Omega$  of any Lipschitz function with compact support in  $M$  belongs to  $W_0^1(\Omega)$ . If  $\Omega$  has ends  $U_1, \dots, U_k$ , further assume that each  $U_i \cup \partial^{inner} U_i, 1 \leq i \leq k$ , is Harnack in the sense of Theorem 2.3.1 and non-parabolic (see Section 2.3.6). Then for all  $x, y \in \Omega$  and  $t > 1$ ,*

$$p(t, x, y) \approx C \left[ \frac{1}{\sqrt{V_{i_x}(x, \sqrt{t})V_{i_y}(y, \sqrt{t})}} \exp\left(-c \frac{d_\emptyset^2(x, y)}{t}\right) + \left( \frac{H(x, t)H(y, t)}{V_0(\sqrt{t})} + \frac{H(y, t)}{V_{i_x}(\sqrt{t})} + \frac{H(x, t)}{V_{i_y}(\sqrt{t})} \right) \exp\left(-c \frac{d_+^2(x, y)}{t}\right) \right],$$

where the constants  $C, c$  take different values in the upper and lower bounds.

Here

$$i_x = \begin{cases} i, & \text{if } x \in U_i \\ 0, & \text{if } x \in K, \end{cases}$$

and, so that  $|x|$  is bounded below away from zero, we set

$$|x| := \sup_{y \in K} d(x, y), \quad x \in M.$$

Then if  $B_i(x, r)$  denotes a ball in  $U_i$  centered at  $x$  with radius  $r$  and  $o_i$  is a fixed reference point on  $\partial^{\text{inner}} U_i$ ,  $1 \leq i \leq k$ ,

$$V_i(r) := V_i(o_i, r) = \int_{B_i(o_i, r)} \sigma_i(x) dx.$$

We further set  $V_0(r) = \min_{1 \leq i \leq k} V_i(r)$ . The notation  $d_+(x, y)$  refers to the distance between  $x$  and  $y$  when passing through the compact middle  $K$ , whereas  $d_\emptyset(x, y)$  refers to the distance between  $x$  and  $y$  if we avoid  $K$ . Finally, we define

$$H(x, t) = \min \left\{ 1, \frac{|x|^2}{V_{i_x}(|x|)} + \left( \int_{|x|^2}^t \frac{ds}{V_{i_x}(\sqrt{s})} \right)_+ \right\}.$$

**Remark 2.3.7.** If  $V_{i_x}(r)$  satisfies the condition that for some  $c, \varepsilon > 0$ ,

$$\frac{V_{i_x}(R)}{V_{i_x}(r)} \geq c \left( \frac{R}{r} \right)^{2+\varepsilon} \quad \text{for all } R > r \geq 1, \quad (2.2)$$

then, as in Section 4.4 of [37], we have the estimate

$$H(x, t) \approx \frac{|x|^2}{V_{i_x}(|x|)}.$$

*Proof of Theorem 2.3.3:* Recall  $d\mu = \sigma dx$ . Since the restriction to  $\Omega$  of any Lipschitz function with compact support in  $M$  belongs to  $W_0^1(\Omega, \mu)$ , in fact  $W_0^1(\Omega, \mu) = W^1(\Omega, \mu)$ . Hence the Dirichlet forms given by  $(W_0^1(\Omega, \mu), \int_\Omega g(\nabla f, \nabla f) d\mu)$  and  $(W^1(\Omega, \mu), \int_\Omega g(\nabla f, \nabla f) d\mu)$  coincide. Therefore we can think of  $\partial\Omega$  as having no boundary condition, which amounts to considering the heat kernel on the complete manifold (with boundary)  $M = \widetilde{\Omega}$ , which has Harnack, non-parabolic ends. Hence the result follows from repeating the proofs of Theorems 4.9 and 5.10 in [37].  $\square$

### 2.3.5 Uniform domains

There are several definitions of uniform domains which are equivalent under certain circumstances (see [40], [49], and the references therein). In this section we need only assume we have a length metric space  $(M, d)$ , that is, a metric space such that  $d(x, y)$  is equal to the infimum of the lengths of all continuous curves joining  $x$  to  $y$  in  $M$ . We recall a few definitions as in [40].

**Definition 2.3.12** (Length of a Curve). Let  $\gamma : I = [a, b] \mapsto M$  be a continuous curve. Then the length of  $\gamma$  is given by

$$L(\gamma) = \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : n \in \mathbb{N}, a \leq t_0 < \cdots < t_n \leq b \right\}.$$

**Definition 2.3.13** (Uniform domain). Let  $U \subset M$  be open and connected. We say  $U$  is *uniform* in  $M$  if there exist positive, finite constants  $c_u, C_U$  such that for any  $x, y \in U$  there exists a continuous curve  $\gamma_{x,y} : [0, 1] \rightarrow U$  with  $\gamma(0) = x, \gamma(1) = y$  that satisfies

1.  $L(\gamma_{x,y}) \leq C_U d(x, y)$
2. For any  $x \in \gamma_{x,y}([0, 1])$ ,

$$d(x, \partial U) \geq c_u \frac{L(\gamma_{[x,z]})L(\gamma_{[z,y]})}{L(\gamma_{x,y})}, \quad (2.3)$$

where for any  $z = \gamma_{x,y}(s), z' = \gamma_{x,y}(s'), 0 \leq s \leq s' \leq 1, L(\gamma_{[z,z']}) = L(\gamma|_{[s,s']})$ .

**Remark 2.3.8.** A set  $U$  satisfying Definition 2.3.13 is sometimes instead referred to as a *length uniform domain*. In this context, a domain may be called uniform if the length  $L$  of curves is replaced by the distance  $d$  in  $M$  everywhere in (2.3). However, under a relatively mild condition on balls, these notions are equivalent. (See Theorem 2.7 of [51] and Proposition 3.3 of [40], noting that the proof

of Proposition 3.3 contains some errors.) In our case of interest, this condition follows from the doubling assumption.

**Remark 2.3.9.** If we replace both the distance  $d$  in  $M$  in (a) and the lengths of curves  $L$  in (2.3) with  $d_U$ , the distance in  $U$ , we obtain the definition of an *inner uniform* domain. In the situations considered in the main part of this chapter, one can easily check a uniform domain is also inner uniform and all relevant results apply. In fact, in the case where the closure of a set  $U$  is nice,  $U$  being uniform in its closure is equivalent to  $U$  being inner uniform.

### 2.3.6 Green function, parabolic versus non-parabolic

**Definition 2.3.14** (Parabolic/Non-parabolic manifolds). Let  $\Omega$  be a weighted Riemannian manifold with minimal heat kernel  $p(t, x, y)$  associated with the Dirichlet form  $(W_0^1(\Omega), \int_{\Omega} g(\nabla f, \nabla f) d\mu)$ . Consider

$$G(x, y) = \int_0^{\infty} p(t, x, y) dt, \quad x \neq y \in \Omega.$$

If this (extended) function of  $x \neq y$  is identically  $+\infty$ , then we say  $\Omega$  is *parabolic*. If  $G(x, y)$  is finite at some pair  $x \neq y$ , then it is finite for all  $x \neq y$ , and we say that  $\Omega$  is *non-parabolic*. In the second case, we call  $G$  the *Green function* on  $\Omega$ ; it is a global harmonic function on  $\Omega$ .

There are many characterizations of parabolicity. One of them is that the constant function  $\mathbf{1} : \Omega \rightarrow (0, +\infty)$  is the limit of a sequence of smooth functions  $\phi_n$  with compact support for the norm  $(\int_V |f| d\mu + \int_{\Omega} |\nabla f|^2 d\mu)^{1/2}$  where  $V$  is one (any) fixed non-empty relatively compact open set in  $\Omega$ .

Let  $(M, \delta M)$  be a complete weighted Riemannian manifold with boundary and assume  $M$  has a strictly positive weight  $\sigma$ . Then using the above characterization of parabolicity, if  $\Omega$  is a submanifold of  $M$  such that  $M \setminus \Omega$  contains a non-empty hypersurface of codimension 1, then it easily follows that the weighted manifold  $\Omega$  is non-parabolic. See [27] for an extensive discussion and references.

When the weighted Riemannian manifold  $\Omega$  is a Harnack weighted manifold, parabolicity boils down to the volume integral condition

$$\int_1^\infty \frac{ds}{V(x, \sqrt{s})} = +\infty. \quad (2.4)$$

This should be satisfied for one (equivalently, all)  $x \in \Omega$ . Moreover, when  $\Omega$  is a Harnack weighted manifold that is non-parabolic, its Green function  $G$  satisfies

$$c_\Omega \int_{d(x,y)^2}^{+\infty} \frac{ds}{V(x, \sqrt{s})} \leq G(x, y) \leq C_\Omega \int_{d(x,y)^2}^{+\infty} \frac{ds}{V(x, \sqrt{s})}. \quad (2.5)$$

## 2.4 Construction of a profile for $\Omega$

The main result of this section is to construct a sufficiently nice harmonic function on the manifold  $\Omega$ . We assume the set-up of Section 2.2 and all hypotheses given in Section 2.2.3 throughout this section.

### 2.4.1 Harmonic profiles for $\Omega$

In order to use the technique of [37], we need to apply an appropriate  $h$ -transform. The effect of this will be to “hide” the Dirichlet boundary and take us to the setting of a connected sum of Harnack manifolds. The goal of this

section is to construct a positive global harmonic function in  $\Omega$  (Definition 2.3.4) that grows at least as fast in each end as the profile for that end; we may refer to this function as a profile for  $\Omega$ . While the profiles for the ends  $U_i$ ,  $1 \leq i \leq k$ , are unique up to constant multiples (see (C3)), this is in general not the case for  $\Omega$ , even with the additional restriction on the growth of the function. Our main result in this section is that  $\Omega$  always possesses a harmonic function of this type.

**Theorem 2.4.1.** *Assuming  $\partial\Omega \neq \emptyset$ , there exists a positive harmonic function  $h$  on  $\Omega$ , vanishing along  $\partial\Omega$ , such that  $h \geq cu_i$  for some constant  $0 < c < +\infty$ , where  $u_i$  denotes the profile for  $U_i$ ,  $1 \leq i \leq k$ , as in (C3).*

If  $\partial\Omega = \emptyset$ , then  $\Omega = M$  is complete and this case is covered by [37], provided  $\Omega$  is non-parabolic (Definition 2.3.14). The theorem is proved by using the profiles  $u_i$ ,  $1 \leq i \leq k$ , to construct a global harmonic function on  $\Omega$ , which we then show satisfies all of the desired further properties. However, we first gather some additional consequences of our hypotheses.

## 2.4.2 Behavior of Green functions

The proof of the theorem relies heavily on the behavior of the Green function  $G$  of  $\Omega$ , which exists since  $\partial\Omega \neq \emptyset$  (recall Section 2.3.6). The behavior of  $G$  is closely related to the behavior of the Green functions for the ends  $U_i$ ,  $G_{U_i}$ ,  $1 \leq i \leq k$ , which exist since all ends  $U_i$  are non-parabolic as  $\partial^{\text{inner}}U_i \neq \emptyset$ . In turn, what we can say about the behavior of  $G_{U_i}$  relies on the strong hypotheses we require of the ends, as well as whether the underlying manifolds  $M_i$  are parabolic or non-parabolic.

**Definition 2.4.1.** We say that a continuous function  $f$  on  $\Omega$  tends to zero at infinity in an end  $U_i$  if, for all  $\varepsilon > 0$ , there exists a compact set  $K_\varepsilon \subset M$  such that  $|f(x)| \leq \varepsilon$  for all points  $x \in U_i \setminus K_\varepsilon$ .

Similarly, we say  $f$  tends to zero at infinity in  $\Omega$  if, for all  $\varepsilon > 0$ , there exists a compact set  $K_\varepsilon \subset M$  such that  $|f(x)| \leq \varepsilon$  for all points  $x \in \Omega \setminus K_\varepsilon$ .

**Definition 2.4.2.** Fix points  $o_i \in U_i$ ,  $1 \leq i \leq k$ . (Generally, we think of  $o_i$  as being near  $\partial^{\text{inner}} U_i$ ). We say that  $x$  tends to infinity in  $U_i$  if the distance between  $x$  and  $o_i$  (taken in  $U_i$ ) tends to infinity.

**Theorem 2.4.2.** *The following dichotomy takes place regarding the behaviors of each of the Green functions  $G_{U_i}$ ,  $1 \leq i \leq k$  :*

(E1) *If  $M_i$  is non-parabolic, then  $G_{U_i}(x, y) \rightarrow 0$  as  $x \rightarrow \infty$  in  $U_i$ , uniformly for all  $y$  in a fixed compact set.*

(E2) *If  $M_i$  is parabolic, there exists an increasing, unbounded function  $f$  taking the positive reals to the positive reals such that for all  $R > 0$  sufficiently large, there exists a point  $x_R$  satisfying  $R/2 < d(o_i, x_R) < 3R/2$  and  $u_i(x_R) \geq f(R)$ .*

*Moreover, if  $x$  (or, equivalently,  $y$ ) is in a fixed compact set, then  $G_{U_i}(x, y)$  is bounded above uniformly, provided  $d(x, y) \geq \delta > 0$  for some fixed  $\delta > 0$ .*

*Proof.* Fix  $i \in \{1, \dots, k\}$  and a point  $o_i \in U_i$ . Throughout this proof, we will assume that  $d$  refers to the distance in  $U_i$  and  $B(x, r) = B_{U_i}(x, r)$ ,  $V(x, r) = V_{U_i}(x, r)$  refer to balls and their volumes in  $U_i$ .

*Proof of (E1):* Since  $M_i$  is non-parabolic, it possesses a Green function  $G_{M_i}$ , which satisfies the estimate (2.5) and condition (2.4) found in Appendix 2.3.6. As the heat kernel is an increasing function of sets, so that  $p_V(t, x, y) \leq p_U(t, x, y)$

if  $V \subseteq U$  for all  $t > 0$ ,  $x, y \in V$ , the Green function is also an increasing function of its domain. Hence

$$G_{U_i}(x, y) \leq G_{M_i}(x, y) \leq C \int_{d^2(x, y)}^{\infty} \frac{dt}{V_{M_i}(y, \sqrt{t})} < +\infty, \quad \forall x, y \in U_i, x \neq y.$$

Since this last integral converges, it tends to zero as  $d(x, y)$  tends to infinity. As  $M_i$  is doubling,  $V_{M_i}(y, \sqrt{t})$  and  $V_{M_i}(y^*, \sqrt{t})$  are comparable for all  $y, y^*$  in a fixed compact set. It follows that  $G_{U_i}(x, y)$  tends to zero as  $x \rightarrow \infty$  uniformly for all  $y$  in this fixed compact set.

*Proof of (E2), first statement:* The situation is more complicated when  $M_i$  is parabolic, since in this case  $M_i$  possesses no Green function. Consider instead  $E_i$ , which possesses a Green function  $G_{E_i}$  since  $\partial E_i \neq \emptyset$ . Recall  $E_i$  differs from  $M_i$ , a Harnack manifold, by a compact set  $K_i$ , which is the setting of [34].

As  $U_i$  is uniform in  $M_i$  and  $\bar{E}_i = \bar{U}_i$ ,  $E_i$  is itself uniform in  $M_i$  and hence possesses a harmonic profile  $w_i$ . The weighted space  $(U_i, \sigma w_i^2)$  remains uniform and hence has a profile  $v_i$ . Consider the product  $w_i v_i$ . This function must vanish along *all* of  $\partial U_i$  since  $w_i$  is a harmonic function vanishing on  $\partial E_i = \partial^{\text{inner}} U_i$  and  $v_i$  must vanish along  $\partial^{\text{side}} U_i$ . Moreover,  $w_i v_i$  has vanishing normal derivative along  $\delta U_i$  as this is true of both  $w_i$  and  $v_i$ . Since  $v_i = (w_i v_i)/w_i$  is the profile of  $(U_i, \sigma w_i^2)$ , the function  $w_i v_i$  must be locally harmonic in  $(U_i, \sigma)$ ; this can be seen by considering the unitary map  $T : L^2(U_i, \sigma w_i^2) \rightarrow L^2(U_i, \sigma)$  given by  $g \mapsto g w_i$ . Therefore  $w_i v_i$  is a profile of  $(U_i, \sigma)$ . Such profiles are unique up to constant multiples, so we may take  $u_i = w_i v_i$ .

Uniformity of  $U_i$  in  $M_i$  also guarantees, as in [40, Lemma 3.20], that for any  $R > 0$ , there exists a point  $x_R \in U_i$  such that  $d(o_i, x_R)$  and  $d(x_R, \partial U_i)$  are both of scale  $R$ . In particular, we can take  $x_R$  such that  $R/2 < d(o_i, x_R) < 3R/2$  and  $d(x_R, \partial U_i) \geq c_0 R/8$  for some fixed constant  $c_0$ . It follows from the proof of Theo-

rem 4.17, [40], that there exists a constant  $C > 0$  such that

$$v_i(y) \leq C v_i(x_R) \quad \forall r > 0, y \in B(x_R, r).$$

Additionally, from the construction of  $u_i$  in [40], there exists a point  $y^* \in U_i$  such that  $v_i(y^*) = 1$ . Thus  $C^{-1} \leq v_i(x_R)$  for all  $R > 0$ .

Moreover, the proof of Lemma 4.5 in [34] implies that for  $d(o_i, x) = r$

$$w_i(x) \approx \int_{r_0}^r \frac{s ds}{V(o_i, s)},$$

for any sufficiently large  $r_0 > 0$ , where  $f \approx g$  means there exist constants  $0 < c^* \leq C^* < +\infty$  such that  $c^* f \leq g \leq C^* f$ . Since  $M_i$  is Harnack and parabolic, the above integral tends to infinity as  $r$  tends to infinity. Hence  $w_i$  tends to infinity in  $U_i$ .

Therefore  $u_i(x_R) = v_i(x_R)w_i(x_R) \rightarrow \infty$  as  $R \rightarrow \infty$  since this is true of  $w_i$  and  $v_i$  is bounded below at the points  $x_R$ . Hence we can construct a function  $f(R)$  of the type necessary to satisfy (E2).

*Proof of (E2), second statement:* Since  $G_{U_i} \leq G_{E_i}$ , it suffices to prove the statement for  $G_{E_i}$ .

By Theorem 5.13 of [40],

$$G_{E_i}(x, y) \approx h(x)h(y) \int_{d(x,y)^2}^{\infty} \frac{dt}{V_h(x, \sqrt{t})},$$

where  $h$  is the profile for  $E_i$ , and

$$V_h(x, \sqrt{t}) := \int_{B(x, \sqrt{t})} h^2(z) d\mu(z),$$

where  $B(x, \sqrt{t}) = \{y \in U_i : d_{U_i}(x, y) < \sqrt{t}\}$ .

Moreover, by Theorem 2.3.2 (see also [40, Theorem 4.17]),

$$V_h(x, \sqrt{t}) \approx [h(x_{\sqrt{t}})]^2 V(x, \sqrt{t}),$$

where  $V$  is the usual volume in  $U_i$ , and  $x_{\sqrt{t}}$  is any point in  $U_i$  satisfying

$$d(x, x_{\sqrt{t}}) \leq \frac{\sqrt{t}}{4} \quad \text{and} \quad d(x_{\sqrt{t}}, M_i \setminus E_i) \geq c_0 \frac{\sqrt{t}}{8}.$$

Therefore

$$G_{E_i}(x, y) \approx h(x)h(y) \int_{d(x,y)^2}^{\infty} \frac{dt}{[h(x_{\sqrt{t}})]^2 V(x, \sqrt{t})}.$$

On the other hand, we recall that [34] implies

$$h(z) \approx \int_{r_0}^{d(o_i, z)} \frac{t dt}{V(o_i, t)},$$

where  $r_0$  is such that  $B_{M_i}(o_i, r_0)$  contains  $\partial E_i$ .

Therefore, by adding a correction term near zero, we can write

$$h(x_{\sqrt{t}}) \approx \int_0^{d(o_i, x_{\sqrt{t}})} \frac{se^{-1/s}}{V(s)} ds,$$

where  $V(s) := V(o_i, s)$ .

For any  $x \in E_i$ , set  $|x| := d(o_i, x)$ . Hence, for  $x$  fixed,

$$\begin{aligned} G_{E_i}(x, y) &\approx h(x)h(y) \int_{d(x,y)^2}^{\infty} \frac{dt}{\left[ \int_0^{|x_{\sqrt{t}}|} \frac{se^{-1/s}}{V(s)} ds \right]^2 V(x, \sqrt{t})} \\ &\approx h(x) \left( \int_0^{|y|} \frac{se^{-1/s}}{V(s)} ds \right) \left( \int_{d(x,y)^2}^{\infty} \frac{dt}{\left[ \int_0^{|x_{\sqrt{t}}|} \frac{se^{-1/s}}{V(s)} ds \right]^2 V(x, \sqrt{t})} \right). \end{aligned}$$

Let  $R := d(x, y)$ . Since  $x$  is fixed  $|x_{\sqrt{t}}| \approx \sqrt{t}$ , as  $d(x, x_{\sqrt{t}})$  is of order  $\sqrt{t}$ . Additionally,  $|y| \approx d(x, y) \approx R$  for  $R$  sufficiently large. Also,  $V(x, R) \approx V(R)$ .

We define two functions,  $f, g : [0, \infty) \rightarrow \mathbb{R}$ , as follows:

$$F(R) := \int_{R^2}^{\infty} \frac{dt}{\left[ \int_0^{\sqrt{t}} \frac{se^{-1/s}}{V(s)} ds \right]^2 V(\sqrt{t})}; \quad G(R) := \frac{1}{\int_0^R \frac{se^{-1/s}}{V(s)} ds}.$$

Then

$$F'(R) = \frac{-2R}{\left[\int_0^R \frac{se^{-1/s}}{V(s)} ds\right]^2 V(R)}; \quad G'(R) = \frac{-Re^{-1/R}}{\left[\int_0^R \frac{se^{-1/s}}{V(s)} ds\right]^2 V(R)}$$

so that  $F'(R) = 2e^{1/R}G'(R)$ .

Thus, for  $R$  large enough,  $F'(R), G'(R) < 0$ . Hence both  $F$  and  $G$  are decreasing, and  $F$  is decreasing at least as quickly as  $G$ , which implies  $F(R) \leq G(R)$  for large  $R$ . This plus the estimates above yield

$$G_{E_i}(x, y) \approx h(x) \frac{F(R)}{G(R)} \approx \frac{F(R)}{G(R)},$$

since  $x$  is fixed. Therefore  $G_{E_i}$  is largest for small  $R = d(x, y)$  and decreases with  $R$ . Moreover, for  $x$  in some fixed compact set, we can again treat  $h(x)$  as constant, which implies  $G_{E_i}$  is bounded in the desired fashion.  $\square$

**Lemma 2.4.1.** *Under the hypotheses of Theorem 2.4.1, if  $G_{U_i}$  satisfies (E1), then (E1) holds for  $G$  in  $U_i$ , and the same holds for (E2).*

*Proof.* Let  $O_1, O_2$  be two precompact sets with smooth boundary in  $M$  such that  $K \subset O_1 \subset O_2$  and  $\bar{O}_1 \subset O_2$ . Let  $K_i = \bar{O}_i$ ,  $i = 1, 2$ . We will show there exist constants  $0 < c \leq C < +\infty$  such that, for any  $y \in U_i \cap K_1$ ,

$$cG_{U_i}(\cdot, y) \leq G(\cdot, y) \leq CG_{U_i}(\cdot, y)$$

on  $U_i \setminus K_2$ .

This implies  $G_{U_i}, G$  are comparable in the end  $U_i$  sufficiently far away from the middle. Since  $K_2$  is compact, the behavior of  $G$  near the middle is determined by the boundary of  $\Omega$  and its similarity to  $G_{U_i}$  in the ends, as local elliptic and boundary Harnack inequalities hold. (See Section 3.5 for statements of Harnack inequalities.) Thus proving  $G_{U_i} \approx G$  as above suffices to prove the lemma.

Since  $U_i \subset \Omega$ ,  $G_{U_i}(x, y) \leq G(x, y)$  for all  $x, y \in U_i$ , proving the first inequality with  $c = 1$ . The other inequality is more challenging.

Let  $\{\Omega_j\}_{j=1}^\infty$  be an exhaustion of  $\Omega$  by pre-compact open sets and  $G_{\Omega_j}$  be the Green function for  $\Omega_j$ ,  $j = 1, 2, 3, \dots$ . As  $G_{\Omega_j} \nearrow G$ , it suffices to show there exists  $0 < C < +\infty$  such that  $G_{\Omega_j}(\cdot, y) \leq CG_{U_i}(\cdot, y)$  in the desired range, where  $C$  does not depend on  $j$ .

Fix  $\varepsilon > 0$ . Let  $o_i$  be a fixed reference point in  $U_i^\bullet \cap K_1$ . Assume  $o_i \in \Omega_j$  and  $K_2 \subset \Omega_j$  for all  $j$ . As in the previous theorem, let  $d, B$ , and  $V$  refer to distance, balls, and volumes taken in  $U_i$ . Locally in a coordinate chart neighborhood of  $o_i$ , the functions  $G_{\Omega_j}$  behave like the Green function of  $\mathbb{R}^n$ , where  $n$  is the dimension of  $M$ . For some  $r_0 > 0$ , we may take  $W = B(o_i, r_0) \subset K_1$  to be our coordinate chart neighborhood. Then for all  $z$  such that  $d(o_i, z) = r_0/2$  and  $d(z, \partial\Omega_j) \geq \varepsilon$ , there exist constants  $0 < c_0 \leq C_0 < +\infty$  independent of  $j$  such that

$$c_0 \leq G_{\Omega_j}(o_i, z) \leq C_0. \quad (2.6)$$

By construction of  $K_1, K_2$ , there exists  $\delta > 0$  such that if  $x \in U_i \setminus K_2$ ,  $y \in K_1$ , then  $d(x, y) \geq \delta$ . As the elliptic Harnack inequality holds locally in  $\Omega$ , since  $G_{\Omega_j}$  is harmonic,  $K_2$  is compact, and (2.6) holds, there exist constants  $0 < c_1 \leq C_1 < +\infty$  which do not depend on  $j$  such that

$$c_1 \leq G_{\Omega_j}(x, y) \leq C_1$$

for all  $x \in \partial K_2 \cap U_i$ ,  $y \in K_1$ ,  $d(x, \partial\Omega_j) \geq \varepsilon$ ,  $d(y, \partial\Omega_j) \geq \varepsilon$ .

Using the boundary Harnack inequality to compare  $G_{\Omega_j}$  to  $G_{U_i}$  along points of  $\partial K_2 \cap U_i$  at distance less than  $\varepsilon$  from  $\partial\Omega_j$  and to push  $y \in K_1 \cap U_i$  toward  $\partial\Omega$ , and using the elliptic Harnack inequality to gain control of  $G_{U_i}$  away from the

Dirichlet boundary, we see there exists a constant  $0 < C < +\infty$  such that

$$G_{\Omega_j}(\cdot, y) \leq CG_{U_i}(\cdot, y)$$

on  $\partial K_2 \cap U_i$  for any  $y \in K_1 \cap U_i$ .

We now use a comparison principle, since both  $G_{\Omega_j}, G_{U_i}$  are harmonic in  $\Omega_j \cap (U_i \setminus K_2)$ . Along  $\partial K_2 \cap U_i$ , we showed  $G_{\Omega_j}(\cdot, y) \leq CG_{U_i}(\cdot, y)$ . Also,  $G_{\Omega_j}$  vanishes along the inner boundary of  $\Omega_j$  that lies in  $U_i$ , while  $G_{U_i}$  is positive there. Both functions vanish along  $\partial\Omega_j \cap \partial U_i$  and have vanishing normal derivative along  $\delta\Omega_j \cap U_i$ . The Hopf boundary lemma guarantees that minimums of harmonic functions cannot occur solely at points where the normal derivative vanishes, and therefore

$$G_{\Omega_j}(\cdot, y) \leq CG_{U_i}(\cdot, y)$$

on  $\Omega_j \cap (U_i \setminus K_2)$ , where  $y \in K_1 \cap U_i$ . Taking  $j \rightarrow \infty$  finishes the proof.  $\square$

### 2.4.3 Construction of the profile for $\Omega$

We now prove the main theorem in this section. The construction of the profile  $h$  of  $\Omega$  closely follows the method of [62].

*Proof of Theorem 2.4.1.* For clarity, the proof is divided into a series of steps.

*Step 1 (Construct a global harmonic function on  $\Omega$ ):* Fix a point  $o \in K$  and take precompact open sets  $O_1, O_2 \subset M$  with smooth boundary such that  $K \subset O_1 \subset O_2$  and no points in  $\overline{\delta\Omega} \cap \partial\Omega$  belong to the set  $O_2 \setminus O_1$ , which is possible since every point in  $M$  possesses a neighborhood containing only finitely many components of  $\partial\Omega$ .

Construct a smooth function  $\psi$  on  $\Omega$  such that  $\psi \equiv 1$  on  $\Omega \setminus \mathcal{O}_2$ ,  $\psi \equiv 0$  on  $\mathcal{O}_1$ , and  $\psi$  has vanishing normal derivative on  $\delta\Omega$ . Let  $u$  be defined on  $\Omega \setminus K$  by  $u|_{U_i} = u_i, 1 \leq i \leq k$ .

Define

$$h(x) = (u\psi)(x) + \int_{\Omega} G(x, y)\Delta(u\psi)(y) d\mu(y), \quad \forall x \in \Omega.$$

Here  $\Delta(u\psi)$  is a smooth function with compact support on  $M$  by the construction of  $\psi$  and elliptic regularity theory (it extends smoothly to the boundary by construction of  $\mathcal{O}_1, \mathcal{O}_2$ ). The relevant weak definitions regarding harmonic functions may be found in Section 2.3.1; here, for simplicity, we write the proof in terms of the corresponding classical definitions.

For any smooth, compactly supported function  $\alpha$  on  $M$ , set

$$G(\alpha) := \int_{\Omega} G(x, y)\alpha(y) d\mu(y).$$

Then for  $\alpha = \Delta(u\psi)$ ,  $h = u\psi + G(\alpha)$ .

We compute

$$\Delta G(\alpha) = \int_{\Omega} \Delta G(x, y)\alpha(y) d\mu(y) = \int_{\Omega} \Delta_{\Omega} G(x, y)\alpha(y) d\mu(y) = -\alpha(x),$$

as we may replace  $\Delta$  (which applies to smooth functions) by  $\Delta_{\Omega}$  (the infinitesimal generator associated with  $(W_0(\Omega), \int g(\nabla f, \nabla f) d\mu)$ ) since  $G$  is the Green function for  $\Omega$ .

As  $\delta\Omega$  is smooth, a direct calculation shows the normal derivative of  $h$  on  $\delta\Omega$  vanishes, since this is true of all of  $u, \psi$ , and  $G$  by definition. Similarly, as  $u = 0$  on  $\partial\Omega \setminus K$  and  $G = 0$  on all of  $\partial\Omega$ , it follows that  $h = 0$  on  $\partial\Omega$  as well. Thus  $h$  is a global harmonic function on  $\Omega$ .

*Step 2 (On each end  $U_i$ ,  $h$  behaves similarly to  $u_i$ ):* For the remainder of the proof, we need to make use of the two possible cases of the behavior of  $G$  on each  $U_i$ ,  $1 \leq i \leq k$ , given by Theorem 2.4.2 and Lemma 2.4.1. Crucially, these two conditions imply that  $h$  behaves similarly to  $u_i$  in (at least) one of two (non-equivalent) ways.

First suppose  $U_i$  satisfies (E1). Note  $\alpha = \Delta(u\psi)$  is bounded and the integral in  $G(\alpha)$  is only over the compact set  $\overline{O}_2$ . Additionally, (E1) implies  $G$  tends to zero with  $x$ . These three facts imply  $G(\alpha) \rightarrow 0$  at infinity in the end  $U_i$ . Since  $\psi \rightarrow 1$  at infinity,  $h - u_i \rightarrow 0$  at infinity in  $U_i$ .

If  $U_i$  satisfies (E2), then we claim  $h/u_i \rightarrow 1$  at infinity in the end  $U_i$ . To see this, take  $R > 0$  sufficiently large and consider the annulus  $A_R = \{x \in U_i : R/2 < d(o, x) < 3R/2\}$ . The key step in proving this claim is to show that  $u_i$  is actually not too small in the entire annulus  $A_R$  by obtaining a lower bound for  $u_i$  depending on  $G$  and  $R$ .

We will make use of the technique of remote and anchored balls found in [36], to which we refer the reader for more details. In brief, an anchored ball is simply a ball whose center belongs to  $\partial\Omega$ , while a remote ball (in  $\Omega$ ) is one whose double is precompact in  $\Omega$ . The import of this is that elliptic Harnack inequalities hold in remote balls, whereas boundary elliptic Harnack inequalities hold in anchored balls.

By assumption (E2), there exists a point  $x_R \in A_R$  and an increasing, unbounded real function  $f$  such that  $u_i(x_R) \geq f(R)$  for all  $R > 0$  sufficiently large. Since  $U_i$  satisfies (RCA) by (C1), for any  $x^* \in A_R$  and any fixed  $\varepsilon > 0$ , there exists a sequence of at most  $Q_\varepsilon$  balls connecting  $x^*$  and  $x_R$  where each ball is either re-

remote of radius  $\frac{\varepsilon R}{4}$  or anchored to the boundary of radius  $\varepsilon R$ . Label this sequence of balls  $B_0, \dots, B_l$  in such a way that  $x_R \in B_0, x^* \in B_l$  and  $B_j \cap B_{j+1} \neq \emptyset, 0 \leq j \leq l-1$ .

Recall that (C2) indicates an elliptic boundary Harnack inequality holds uniformly in  $U_i$ , while the hypothesis (H1) implies an elliptic Harnack inequality holds uniformly in  $M_i$ . Let  $C_H$  be the uniform elliptic Harnack inequality constant for  $M_i$  and  $C_B$  denote the uniform elliptic boundary Harnack constant for  $U_i$ . We show that  $u_i$  remains relatively large in all of  $B_0$ .

If  $B_0$  is a remote ball and  $y \in O_2$ , applying the elliptic Harnack inequality to the non-negative harmonic functions  $G$  and  $u_i$  in  $B_0$ ,

$$G(x, y) \leq C_H^2 \frac{G(x_R, y)}{u_i(x_R)} u_i(x) \leq \frac{C_H^2 L}{f(R)} u_i(x) \quad \forall x \in B_0,$$

where  $L$  is an upper bound on  $G(x_R, y)$  for large  $R$  given by Theorem 2.4.2.

Similarly, if  $B_0$  is anchored to the boundary, we may compare  $G$  and  $u_i$  using the elliptic boundary Harnack inequality. Although this inequality a priori applies only in the ball of half the radius, by covering  $B_0$  by a finite number of remote or anchored balls and chaining appropriate Harnack inequalities, the boundary Harnack inequality actually holds in all of  $B_0$ , albeit with a potentially different constant, which we continue to call  $C_B$ . Thus

$$\frac{G(x, y)}{u_i(x)} \leq C_B \frac{G(x_R, y)}{u_i(x_R)} \leq \frac{C_B L}{f(R)} \quad \forall x \in B_0.$$

In either case, we obtain a lower bound for  $u_i$ , involving  $G$ , in the entire ball  $B_0$ . We then chain between the balls  $B_0, \dots, B_l$ , obtaining a similar inequality with an additional constant at each stage. Since there are at most  $Q_\varepsilon$  balls, there exists a constant  $0 < C < +\infty$  depending only on  $Q_\varepsilon, C_H, C_B$ , and  $L$  such that

$$G(x^*, y) \leq \frac{C}{f(R)} u_i(x^*) \quad \forall x^* \in A_R.$$

Recall  $h = u\psi + G(\alpha)$ , where  $\alpha$  is bounded. Hence for  $d(o, x) = |x| \approx R$  large enough in  $U_i$ ,

$$u_i(x) - \delta_R u_i(x) \leq h(x) \leq u_i(x) + \delta_R u_i(x)$$

for some  $\delta_R > 0$  that tends to zero as  $R$  tends to infinity. Thus  $h/u_i \rightarrow 1$  as  $x \rightarrow \infty$  in  $U_i$  as claimed.

*Step 3 (h is non-negative):* Let  $\varepsilon > 0$ . Then there exists a compact set  $K_\varepsilon \subset M$  such that on  $\Omega \setminus K_\varepsilon$ ,  $-\varepsilon \leq u_i - \varepsilon \leq h$  on ends  $U_i$  where  $U_i$  satisfies (E1) and  $0 < (1/2)u_i \leq h$  at points in ends  $U_i$  satisfying (E2), and every end falls in (at least one) of these two cases. Recall  $h = 0$  on all of  $\partial\Omega$ , and by the Hopf boundary lemma, a minimum of  $h$  cannot occur only on  $\delta\Omega$ . Hence a minimum principle implies  $-\varepsilon < h$  on all of  $\Omega$ . Since  $\varepsilon$  was arbitrary, we conclude  $h \geq 0$  on  $\Omega$ .

*Step 4 (h ≥ cu<sub>i</sub> on U<sub>i</sub>):* We again employ a minimum principle. For fixed  $i$ , first assume that  $U_i$  satisfies (E1). For every  $\varepsilon > 0$ , since  $h - u_i \rightarrow 0$  at infinity in  $U_i$ , there exists a compact set  $K_\varepsilon$  such that  $u_i - \varepsilon \leq h$  in  $U_i \setminus K_\varepsilon$ . Since  $h$  is non-negative and  $u_i$  vanishes along  $\partial^{\text{inner}}U_i$  by definition,  $u_i \leq h$  there. Both  $u_i$  and  $h$  vanish along  $\partial^{\text{side}}U_i$ , and both have vanishing normal derivative along  $\delta U_i$ . Take a sequence of balls  $B(o, R_l)$  such that  $R_l \rightarrow \infty$  as  $l \rightarrow \infty$  and  $K_{1/l} \setminus \partial\Omega$  is contained in  $B(o, R_l)$  for  $l = 1, 2, 3, \dots$ . Then on  $U_i \cap \partial B(o, R_l)$ ,  $u_i - 1/l \leq h$ . Thus on  $U_i \cap B(o, R_l)$  the weak minimum principle combined with the Hopf boundary lemma gives  $u_i - 1/l \leq h$ . Sending  $l \rightarrow \infty$  yields  $u_i \leq h$  on  $U_i$ .

Assume  $U_i$  satisfies (E2) instead. Then we may choose  $R_0 > 0$  such that  $(1/2)u_i \leq h$  in  $U_i \setminus \overline{B(o, R_0)}$ . Then for  $R$  sufficiently large,  $h = u_i = 0$  on  $\partial^{\text{side}}U_i \cap B(o, R)$  and, along  $\partial^{\text{inner}}U_i$  and  $\partial B(o, R) \cap U_i$ ,  $(1/2)u_i \leq h$ . Therefore the Hopf boundary lemma and a minimum principle give  $(1/2)u_i \leq h$  on  $B(o, R) \cap U_i$ . Consequently  $(1/2)u_i \leq h$  on all of  $U_i$ .

Combining the two cases above,  $(1/2)u_i \leq h$  on  $U_i$ ,  $1 \leq i \leq k$ , so in each end,  $h$  grows at least as fast as the harmonic profile for that end.

*Step 5 ( $h$  is positive on  $\Omega$ ):* As a local elliptic Harnack inequality holds in  $\Omega$ , either  $h \equiv 0$  on  $\Omega$  or  $h > 0$  on  $\Omega$ . Since  $u_i > 0$  in  $U_i$ , the previous step implies  $h \neq 0$ . Thus  $h$  is positive and hence is a profile for  $\Omega$ .  $\square$

#### 2.4.4 Relationship between $h$ and the $u_i$

By virtue of Theorem 2.4.1,  $\Omega$  possesses a profile  $h$  which must grow at least as fast as the profile  $u_i$  in  $U_i$ ,  $1 \leq i \leq k$ . In fact, outside of a compact set in  $U_i$ , the profiles  $h$  and  $u_i$  are comparable. This comparison is crucial for our main result; the existence of a non-negative harmonic function on  $\Omega$  satisfying the appropriate boundary conditions (but no other properties) follows from [49] due to the smoothness of the boundaries  $\partial\Omega, \delta\Omega$ .

**Theorem 2.4.3.** *Assume  $\partial\Omega \neq \emptyset$  and let  $h$  be a harmonic profile for  $\Omega$  as constructed in Theorem 2.4.1. Then there exist constants  $0 < c_i \leq C_i < \infty$  and compact sets  $\widehat{K}_i \subset M$ ,  $1 \leq i \leq k$  such that*

$$c_i u_i \leq h \leq C_i u_i$$

on  $U_i \setminus \widehat{K}_i$ .

*Proof.* For simplicity, we drop the subscript  $i$ . The desired lower bound holds with  $c = 1/2$  on all of  $U$  as in Theorem 2.4.1. Once again the proof depends on the relative behavior of the Green function.

Assume  $U$  satisfies (E1). Let  $\widehat{K}$  be a compact subset of  $M$  such that the inner boundary of  $U$  is contained in the interior of  $\widehat{K}$ . Recall an elliptic Harnack

inequality holds locally on  $\Omega$ , and since  $U$  is uniform in a Harnack manifold, by consequence (C2) an elliptic boundary Harnack inequality holds uniformly in  $U$ . For  $\widehat{U} := U \setminus \widehat{K}$ , consider  $\partial^{\text{inner}}\widehat{U}$ . This is a compact set and hence it can be covered by a finite number of balls that are either far away from  $\partial^{\text{side}}\widehat{U}$  or that are near this boundary. In balls far away from  $\partial^{\text{side}}\widehat{U}$ , the elliptic Harnack inequality implies both  $h$  and  $u$  are relatively constant. As we approach  $\partial^{\text{side}}\widehat{U}$ , since an elliptic boundary Harnack inequality holds uniformly,  $h$  and  $u$  decay at the same rate. Hence there exists  $C \geq 1$  such that  $h \leq Cu$  along  $\partial^{\text{inner}}\widehat{U}$ .

Since  $h - u \rightarrow 0$  at infinity in  $U$ , there exists a sequence of balls  $B(o, R_l)$  such that  $h \leq u + 1/l$  on  $\widehat{U} \setminus B(o, R_l)$ . It follows from a minimum principle and the Hopf boundary lemma that  $h \leq Cu + 1/l$  on  $B(o, R_l) \cap \widehat{U}$ . Sending  $l \rightarrow \infty$  gives  $h \leq Cu$  on  $\widehat{U}$ .

If instead  $U$  satisfies (E2), the desired statement follows immediately from the fact that  $h/u \rightarrow 1$  at infinity in  $U$ . □

## 2.5 Heat kernel estimates

This section explains how the profile  $h$  of  $\Omega$  (from Theorem 2.4.1) can be used to estimate the mixed boundary condition heat kernel on  $\Omega$ . The key techniques used here are those of [33, 37] (dealing with manifolds with ends) and of [40] (dealing with mixed Dirichlet and Neumann boundary conditions). Section 2.8 explains additional connections with the existing literature.

## 2.5.1 The $h$ -transform space

Assume  $\partial\Omega \neq \emptyset$  and let  $h$  be a harmonic profile for  $\Omega$  as constructed in Theorem 2.4.1. Consider the weighted manifold  $(\Omega, h^2\sigma)$ . Notice this change of measure is related to the unitary map  $T : L^2(\Omega, h^2\sigma) \rightarrow L^2(\Omega, \sigma)$  defined by  $T(f) = hf$  for all  $f \in L^2(\Omega, h^2\sigma)$ . The heat kernel  $p_{\Omega, h^2}(t, x, y) = p_h(t, x, y)$  for  $\Omega$  after  $h$ -transform (that is,  $(\Omega, h^2\sigma)$ ) is related to the heat kernel  $p(t, x, y)$  for  $(\Omega, \sigma)$  by the following simple formula [34, 40]:

$$p(t, x, y) = h(x)h(y)p_h(t, x, y).$$

Hence in order to estimate  $p(t, x, y)$ , it suffices to estimate  $p_h(t, x, y)$ . To estimate this quantity, we will use Theorem 2.3.3. The results in this section rely heavily upon Section 2.3.4.

Let  $K^* \subset M$  be compact such that  $K$  is a subset of the (topological) interior of  $K^*$ . Then  $\Omega = K^* \sqcup U_1^* \sqcup \cdots \sqcup U_k^*$ , where  $U_i^*$  and  $U_i$  differ by a compact set for  $1 \leq i \leq k$ .

Consider the manifolds  $(U_i^*, \sigma h^2)$ , where we put Neumann boundary condition on  $\partial^{\text{inner}} U_i^*$  (this amounts to the abuse of notation " $U_i^* = U_i^* \cup \partial^{\text{inner}} U_i^*$ "). We first show these manifolds are in fact Harnack and non-parabolic.

**Proposition 2.5.1.** *The manifolds  $(U_i^*, \sigma h^2)$ ,  $1 \leq i \leq k$ , described in the preceding paragraphs are Harnack.*

*Proof.* For appropriate choice of  $K^*$ , Theorem 2.4.3 guarantees  $cu_i \leq h \leq Cu_i$  on  $U_i^*$ . By consequence (C4) of our hypotheses, the manifold  $(U_i, \sigma u_i^2)$  is a Harnack manifold. We can also choose  $K^*$  such that the boundaries of the manifolds  $U_i^*$ ,  $1 \leq i \leq k$ , satisfy condition (\*). Moreover,  $h$  vanishes on

$\partial U_i^* = \partial\Omega \cap (\partial U_i \setminus K^*)$  by construction. The function  $h$  is harmonic in the interior of  $U_i^*$  but does not have vanishing normal derivative along  $\partial^{\text{inner}} U_i^*$ . However, since such points are part of  $U_i^*$ , they do not affect whether functions in  $\text{Lip}_c(\overline{U}_i^*)$  belong to  $W_0^1(U_i^*, h^2\sigma)$ . Thus it follows from Proposition 5.8 of [40] that  $\text{Lip}_c(\overline{U}_i^*) \subset W_0^1(U_i^*, h^2\sigma)$ . Moreover,  $(U_i^*, \sigma h^2)$  must be uniformly doubling and satisfy the Poincaré inequality uniformly since this is true of  $(U_i, u_i^2\sigma)$  and  $h \approx u_i$  in  $U_i^*$ . Therefore by Theorem 2.3.1,  $(U_i^*, h^2\sigma)$  is a Harnack manifold.  $\square$

**Proposition 2.5.2.** *The manifolds  $(U_i^*, \sigma h^2)$ ,  $1 \leq i \leq k$ , are non-parabolic.*

*Proof.* One of the equivalent definitions of non-parabolicity is that the space  $(U_i^*, \sigma h^2)$  possesses a non-constant, positive superharmonic function [27]. Such a function must satisfy Definition 2.3.2 where we consider  $U_i^*$  as an open subset of  $\overline{U}_i^*$ , but with the equality replaced by a less than or equal to. In other words, we need a smooth function  $u$  such that  $\Delta_{U_i^*, h^2\sigma} u \leq 0$  in the (geometric) interior of  $U_i^*$  and  $u$  has vanishing normal derivative along the boundary points of  $U_i^*$  as a manifold.

Consider the function  $1/h$  on  $(U_i^*, \sigma h^2)$ . By the correspondence between  $L^2(U_i^*, \sigma)$  and  $L^2(U_i^*, \sigma h^2)$ ,  $1/h$  is harmonic in the geometric interior of  $U_i^*$ . It also has vanishing normal derivative on  $\delta U_i^* \cap \delta U_i$ , since this is true of  $h$ . However,  $h$  does not have vanishing normal derivative on the inner boundary of  $U_i^*$ , so  $1/h$  is not itself superharmonic. Nonetheless,  $1/h$  is a local harmonic function in  $U_i^*$  in the desired sense outside of a compact set containing  $\partial^{\text{inner}} U_i^*$ , so  $1/h$  can be extended to a positive superharmonic function in  $U_i^*$ . Since  $h$  behaves like  $u_i$ , if this modified version of  $1/h$  is constant, then  $u_i$  must be relatively constant on  $U_i^*$ . However, if this is the case, then  $\overline{U}_i$  must have been non-parabolic to start with, so  $U_i^*$  must also be non-parabolic.  $\square$

We now come to the main result of this chapter. In Theorem 2.5.1 below, all notation will be as in Theorem 2.3.3 where subscripts  $h$  indicate we have applied the theorem to the manifold  $(\Omega, h^2\sigma)$ . For explicit examples of the estimates in Theorem 2.5.1, see Section 2.6.

Before the main result, we recall some notation (see also Theorem 2.3.3). For  $x \in \Omega$ , the notation  $i_x$  indicates to select the index  $i$  such that  $x$  belongs to the end  $U_i$ . If  $x \in K$ , we set  $i_x = 0$ . Also let  $|x| := \sup_{y \in K} d(x, y)$ . The distance  $d_+$  indicates distance passing through the compact middle  $K$ , whereas  $d_\emptyset$  indicates distance avoiding  $K$ . We will also need the following two functions involving the  $h$ -transform.

**Definition 2.5.1.** If  $B_i(x, r)$  denotes a ball in  $U_i$  centered at  $x$  with radius  $r$  and  $o_i$  is a fixed reference point on  $\partial^{\text{inner}} U_i$ ,  $1 \leq i \leq k$ , then

$$V_{i,h}(r) = V_{i,h}(o_i, r) = \mu_{i,h}(B_i(o_i, r)) := \int_{B_i(o_i, r)} h^2(x) \sigma_i(x) dx \quad (2.7)$$

and  $V_{0,h}(r) = \min_{1 \leq i \leq k} V_{i,h}(r)$ .

Furthermore, we set

$$H_h(x, t) = \min \left\{ 1, \frac{|x|^2}{V_{i_x, h}(|x|)} + \left( \int_{|x|^2}^t \frac{ds}{V_{i_x, h}(\sqrt{s})} \right)_+ \right\}. \quad (2.8)$$

**Theorem 2.5.1.** Let  $(M, \sigma)$  be a weighted complete Riemannian manifold with boundary such that  $M = M_1 \# \cdots \# M_k$ . Let  $\Omega \subset M$  be open such that  $\partial\Omega = M \setminus \Omega \subset \delta M$  satisfies condition (\*) and assume  $\partial\Omega \neq \emptyset$ .

Let  $K$  be a compact set such that  $M = K \sqcup (E_1 \sqcup \cdots \sqcup E_k)$  and  $U_i = \Omega \cap E_i$ ,  $1 \leq i \leq k$ . Assume (H1) and (H2) so that  $(M_i, \sigma_i)$ ,  $1 \leq i \leq k$  are Harnack manifolds and each  $U_i$  is uniform in  $M_i$ . Let  $h$  denote a profile for  $\Omega$  as constructed in Theorem 2.4.1.

Then for any  $t > 1$ ,  $x, y \in \Omega$ , we have the estimate

$$p(t, x, y) \approx Ch(x)h(y) \left[ \frac{1}{\sqrt{V_{i_x, h}(x, \sqrt{t})V_{i_y, h}(y, \sqrt{t})}} \exp\left(-c \frac{d_0^2(x, y)}{t}\right) + \left( \frac{H_h(x, t)H_h(y, t)}{V_{0, h}(\sqrt{t})} + \frac{H_h(y, t)}{V_{i_x, h}(\sqrt{t})} + \frac{H_h(x, t)}{V_{i_y, h}(\sqrt{t})} \right) \exp\left(-c \frac{d_+^2(x, y)}{t}\right) \right],$$

where the constants  $C, c$  are different in the upper and lower bounds.

*Proof.* By Propositions 2.5.1 and 2.5.2, the ends  $U_i^*$  are non-parabolic and Harnack in the sense of Theorem 2.3.1. Moreover, the restriction of any Lipschitz function with compact support in  $M$  will lie in  $W_0^1(\Omega, \sigma h^2)$  by Proposition 5.8 of [40] since  $h$  is a positive harmonic function in  $\Omega$ , vanishing on  $\partial\Omega$ .

Therefore we apply Theorem 2.3.3 to  $(\Omega, \sigma h^2)$ , with ends  $U_i^*$ ,  $1 \leq i \leq k$ , giving us an estimate for  $p_h(t, x, y)$ . The theorem follows once we recall  $p(t, x, y) = h(x)h(y)p_h(t, x, y)$ .  $\square$

**Remark 2.5.1.** We now indicate how to compute some of the quantities in Theorem 2.5.1 in practice. In fact, all such quantities can be computed based solely on information about the ends  $x$  and  $y$  belong to *except* for the quantity  $V_{0, h}(r)$ .

By Theorem 2.4.3, in each end  $U_i$ , the profile  $h$  of  $\Omega$  is comparable to the profile  $u_i$  of that end, away from some compact set  $K^* \supset K$ . Hence we can compute  $h$  using these profiles. As  $h$  is harmonic, inside of  $K^*$  it is roughly constant away from points of  $\partial\Omega$  and vanishes linearly as it approaches such points. Frequently, given an end  $U_i$ , it may be easier to compute the profile of some set  $V_i$  that is close to  $U_i$  in the sense that their difference is a compact subset of  $K^*$ . Using Harnack inequalities and maximum principles as in Section 2.4, we see the profiles of such  $U_i, V_i$  are comparable.

A useful technique for computing quantities  $V_{i,x,h}(x, \sqrt{t})$  in the theorem above is the use of points  $x_{\sqrt{t}}$ , which were encountered in the proof of Theorem 2.4.2. The spirit is the same as that of Theorem 2.3.2, which does not directly apply. For any  $t > 0$  and any point  $x \in U_i$ ,  $1 \leq i \leq k$ , there exists a point  $x_{\sqrt{t}} \in U_i$  such that  $d(x, x_{\sqrt{t}}) \leq \sqrt{t}/4$  and  $d(x_{\sqrt{t}}, M_i \setminus E_i) \geq c_0 \sqrt{t}/8$  for some constant  $c_0 > 0$  [40, Lemma 3.20]. As the  $U_i$ 's are uniform, by Theorem 4.17 of [40] (or Theorem 2.3.2), we have

$$\int_{B_i(x, \sqrt{t})} u_i^2(y) \sigma_i(y) dy \approx u_i(x_{\sqrt{t}})^2 V_i(x, \sqrt{t}) \quad \forall x \in U_i.$$

As  $u_i \approx h$  in  $U_i \setminus K^*$  for  $K^*$  compact, it follows that

$$V_{i,x,h}(x, \sqrt{t}) \approx h(x_{\sqrt{t}})^2 V_i(x, \sqrt{t}) \quad \forall x \in U_i.$$

In the simplest examples, the integral in the definition (2.8) of  $H_h(x, t)$  does not contribute and the computation reduces to

$$H_h(x, t) \approx \min \left\{ 1, \frac{|x|^2}{V_{i,x,h}(|x|)} \right\} \approx \min \left\{ 1, \frac{|x|^2}{h^2(x_{|x|}) V_{i,x}(|x|)} \right\}.$$

If  $V_{i,x,h}$  grows fast enough, then the second term above is always less than 1 and the computation simplifies further. See Remark 2.3.7 and (2.2) for the appropriate condition on the volume.

Obtaining heat kernel estimates for small times  $t \leq 1$  is much simpler and follows from the fact that the parabolic Harnack inequality (Definition 2.3.8) holds for small scales in  $(\Omega, \sigma h^2)$ .

**Theorem 2.5.2.** *Under the hypotheses of Theorem 2.5.1, for any  $0 < t \leq 1$  and  $x, y \in \Omega$ ,*

$$p(t, x, y) \approx h(x)h(y) \frac{1}{V_h(x, \sqrt{t})} \exp \left( -c \frac{d(x, y)^2}{t} \right),$$

where  $V_h$  denotes the volume in  $(M, \sigma h^2)$  and  $d$  denotes distance in  $M$ .

*Proof.* Since  $(\Omega, \sigma h^2)$  is a connected sum of the Harnack manifolds  $(\Omega_i, \sigma h^2)$ , the parabolic Harnack inequality holds up to scale  $r_0$  for any  $r_0 > 0$  as in [37, Lemma 5.9]. Thus for any  $0 < t < 1$ ,  $x, y \in \Omega$ ,

$$p_h(t, x, y) \approx \frac{1}{V_h(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t}\right)$$

and the result follows from the relation between  $p(t, x, y)$  and  $p_h(t, x, y)$ .  $\square$

**Remark 2.5.2.** In fact, the estimate in Theorem 2.5.2 can be replaced by that in Theorem 2.5.1 as is explained in [37].

## 2.6 Examples

**Example 2.6.1.** Suppose  $M$  is a connected sum of three cones in  $\mathbb{R}^2$  with apertures  $\alpha_1, \alpha_2, \alpha_3 \in [0, 2\pi)$  such that  $\alpha_1 + \alpha_2 + \alpha_3 < 2\pi$ . (While we should round the corners to stay in the category of smooth manifolds, this changes nothing significant.) For simplicity, we assume that the vertex of each cone of positive aperture is the origin. We consider  $\Omega \subset M$  that encodes boundary conditions on these cones; for each cone of positive aperture, we assign one of the following three boundary conditions: either both sides of the cone carry Neumann boundary condition, both sides carry Dirichlet boundary condition, or one side carries each boundary condition. A cone of zero aperture is represented by a strip with Neumann condition on both sides. A typical example of this situation is found in Figures 2.5 and 2.6.

The above six pieces of information (the three apertures of the cones and what boundary conditions they carry on their sides) are all that is necessary to determine the behavior of  $p(t, o, o)$  in such domains (where naturally we take  $o$  to be the origin).

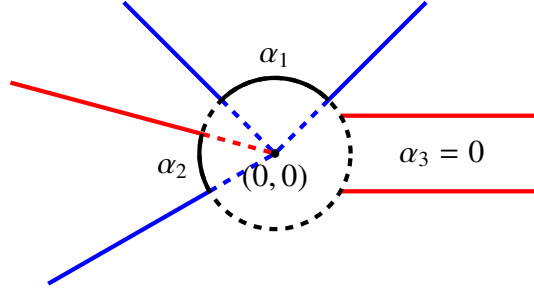


Figure 2.5: An example of a connected sum of three cones whose vertices lie at the origin and which are placed around the unit circle.

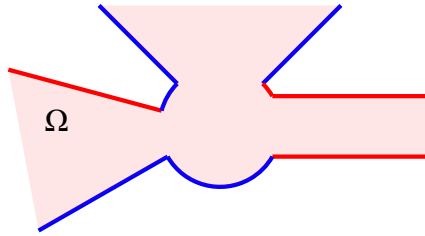


Figure 2.6: The manifold  $\Omega$  associated with Figure 2.5.

We will assume at least one cone has some Dirichlet boundary condition to ensure that  $\Omega$  is non-parabolic (recall Section 2.3.6). If there exists a cone of positive aperture with Neumann condition on both sides, then for  $t > 1$ ,

$$c_o(t \log^2 t)^{-1} \leq p(t, o, o) \leq C_o(t \log^2 t)^{-1}.$$

Let  $U_1, U_2, U_3$  denote the ends of  $\Omega$  with respect to the closure of the unit circle, and let  $V_1, V_2, V_3$  denote the actual cones. Consider the map  $A : \{V_1, V_2, V_3\}$  given by

$$V_i \mapsto \begin{cases} \frac{3}{2}, & \alpha_i = 0 \\ 1 + \frac{\pi}{\alpha_i}, & \alpha_i > 0 \text{ and the cone has Dirichlet boundary condition} \\ 1 + \frac{\pi}{2\alpha_i}, & \alpha_i > 0 \text{ and the cone has both boundary conditions} \end{cases}$$

for  $i = 1, 2, 3$ .

Then for  $t > 1$ ,

$$c_0 t^{-a} \leq p(t, o, o) \leq C_o t^{-a},$$

where

$$a = \min\{A(V_1), A(V_2), A(V_3)\}.$$

This naturally generalizes for any finite number of cones. Furthermore, the requirement  $\sum_{i=1}^3 \alpha_i < 2\pi$  can be removed by considering moving the vertices of the cones farther away from the center of the origin so that each cone takes up less arc length of the unit circle or by noting that we need not require the resulting manifold to be embedded in the plane.

With slightly more information, we can give more precise estimates on  $p(t, x, y)$  for any  $t \geq 1$ ,  $x, y \in \Omega$ . For simplicity of notation, we will assume all cones have positive aperture and Dirichlet boundary condition on both sides. Let  $\phi_i$  denote the angle between the positive  $x$ -axis and the edge of the cone such that when continuing counter-clockwise from this edge, we lie inside of the cone  $V_i$ ,  $1 \leq i \leq 3$ . (The other edge of the cone is at angle  $\phi_i + \alpha_i$  as measured from the positive  $x$ -axis; note  $\phi_i$  maybe be negative.) In polar coordinates, the profile of a cone with aperture  $\alpha$  and edge at angle  $\phi$  as above with Dirichlet boundary condition on both sides is given by  $h_{\text{cone}}(r, \theta) = r^{\pi/\alpha} \sin(\frac{\pi}{\alpha}(\theta - \phi))$ . (In the case of cones with Dirichlet condition on one side and Neumann condition on the other side,  $h_{\text{cone}}(r, \theta) = r^{\pi/2\alpha} \sin(\frac{\pi}{2\alpha}(\theta - \phi))$  if the first edge has Dirichlet condition and a similar formula holds if the first edge instead carries Neumann condition.) The desired estimate depends on whether or not the points  $x, y$  lie in the same end  $U_i$ .

Let  $x = (|x|, \theta_x), y = (|y|, \theta_y)$  denote  $x, y$  written in polar coordinates. Previously,  $|x|$  was defined as  $\sup_{y \in K} d(x, y)$  to be bounded below away from zero. Below, taking  $|x| = d(0, x)$  as is needed for polar coordinates will not be a problem since in all such instances the point  $x$  lies in  $U_i, 1 \leq i \leq 3$ , and hence  $|x| \geq 1$ . Above, we have already seen what occurs if both points lie in the middle (and away from any Dirichlet boundary) by examining  $p(t, o, o)$ . We have the following further cases, where we continue to assume  $t > 1$ :

*Case 1:* Suppose  $x$  and  $y$  are in different ends; without loss of generality assume  $x \in U_1, y \in U_2$ . Then

$$p(t, x, y) \approx \sin\left(\frac{\pi}{\alpha_1}(\theta_x - \phi_1)\right) \sin\left(\frac{\pi}{\alpha_2}(\theta_y - \phi_2)\right) \cdot \left[ \frac{1}{t^{\frac{\pi}{\alpha^*} + 1} |x|^{\frac{\pi}{\alpha_1}} |y|^{\frac{\pi}{\alpha_2}}} + \frac{|x|^{\frac{\pi}{\alpha_1}}}{t^{\frac{\pi}{\alpha_1} + 1} |y|^{\frac{\pi}{\alpha_2}}} + \frac{|y|^{\frac{\pi}{\alpha_2}}}{t^{\frac{\pi}{\alpha_2} + 1} |x|^{\frac{\pi}{\alpha_1}}} \right] \exp\left(-c \frac{d_+^2(x, y)}{t}\right),$$

where  $\alpha^* = \max_{1 \leq i \leq 3} \alpha_i$ . For fixed  $x, y$ , we obtain the same decay rate as above for  $p(t, o, o)$ , and if  $|x| \approx |y| \approx \sqrt{t}$ , then  $p(t, x, y)$  decays like

$$\sin\left(\frac{\pi}{\alpha_1}(\theta_x - \phi_1)\right) \sin\left(\frac{\pi}{\alpha_2}(\theta_y - \phi_2)\right) t^{-\frac{\pi}{2\alpha_1} - \frac{\pi}{2\alpha_2} - 1}.$$

*Case 2:* Suppose  $x, y$  are in the same end,  $U_1$ . Then

$$p(t, x, y) \approx \sin\left(\frac{\pi}{\alpha_1}(\theta_x - \phi_1)\right) \sin\left(\frac{\pi}{\alpha_1}(\theta_y - \phi_1)\right) \cdot \left( \frac{|x|^{\frac{\pi}{\alpha_1}} |y|^{\frac{\pi}{\alpha_1}}}{t h(x/\sqrt{t}) h(y/\sqrt{t})} \exp\left(-c \frac{d_\theta^2(x, y)}{t}\right) + \left[ \frac{1}{t^{\frac{\pi}{\alpha^*} + 1} |x|^{\frac{\pi}{\alpha_1}} |y|^{\frac{\pi}{\alpha_1}}} + \frac{|x|^{\frac{\pi}{\alpha_1}}}{t^{\frac{\pi}{\alpha_1} + 1} |y|^{\frac{\pi}{\alpha_1}}} + \frac{|y|^{\frac{\pi}{\alpha_1}}}{t^{\frac{\pi}{\alpha_1} + 1} |x|^{\frac{\pi}{\alpha_1}}} \right] \exp\left(-c \frac{d_+^2(x, y)}{t}\right) \right).$$

Further, we can compute the quantity  $h(x/\sqrt{t})$  described following the proof of Theorem 2.5.1. For any  $x \in U_1$ ,

$$h(x/\sqrt{t}) \approx \begin{cases} t^{\frac{\pi}{\alpha_1}}, & \text{if } 1 \leq |x|^2 \leq t \\ |x|^{\frac{2\pi}{\alpha_1}} \sin^2\left(\frac{\pi}{\alpha_1}(\theta_x + \frac{\sqrt{t}}{|x|} - \phi_1)\right), & \text{if } 1 \leq t \leq |x|^2. \end{cases}$$

*Case 3:* Suppose one point lies in the middle; assume this point is  $o$ , the origin. The other point  $x$  lies in some end, say  $U_1$ . Then

$$p(t, o, x) \approx \sin\left(\frac{\pi}{\alpha_1}(\theta_x - \phi_1)\right) \left[ \frac{1}{t^{\frac{\pi}{\alpha_1} + 1}} + \frac{|x|^{\frac{\pi}{\alpha_1}}}{|x|^{\frac{\pi}{\alpha_1} + 1}} \right] \exp\left(-c \frac{|x|^2}{t}\right).$$

**Example 2.6.2.** The previous example of unions of cones can also be considered in dimensions other than two. In general, a cone is a subset of  $\mathbb{R}^n$  of the form  $U = \mathbb{R}_+ \times \Sigma$ , where  $\Sigma$  is a subset of  $\mathbb{S}^{n-1}$ , the  $(n-1)$ -dimensional unit sphere. If  $\Sigma$  has smooth boundary, then  $U$  is uniform in  $\mathbb{R}^n$ .

The profile for such a cone  $U$  with Dirichlet boundary condition everywhere (see [3, 40]) is given by

$$h_U(x) = |x|^\alpha \phi\left(\frac{x}{|x|}\right),$$

where  $\lambda$  is the first Dirichlet eigenvalue of the spherical Laplacian,  $\phi$  is its corresponding eigenfunction, and

$$\alpha = \frac{\sqrt{(n-2)^2 + 4\lambda} - (n-2)}{2}. \quad (2.9)$$

If we take a union of such cones with Dirichlet boundary condition everywhere, smoothing corners as necessary, then Theorem 2.5.1 applies. Everything is as in the previous two-dimensional example, except we may now be unable to compute  $\alpha$  and  $\phi$ .

In particular, consider a union of  $k$  such cones, all with Dirichlet boundary condition. Define a map  $A$  on the ends  $U_1, \dots, U_k$  corresponding to the cones by  $A(U_i) = n/2 + \alpha_i$ , where  $\alpha_i$  is given by (2.9) and indicates the power of  $|x|$  appearing in  $h_{U_i}$ . Then, as above, for all  $t > 1$ ,

$$c_0 t^{-a} \leq p(t, o, o) \leq C_0 t^{-a},$$

where  $a = \min\{A(U_1), \dots, A(U_k)\}$ . Here  $o$  is a fixed point in  $M$  and the constants  $c_0, C_0$  depends on  $o$ . Theorem 2.5.1 gives a two-sided estimate over all  $t > 1$  and  $x, y \in \Omega$  but it is more complicated to write down explicitly.

We can also consider the case  $n \geq 3$  where at least one of the cones, say  $U_1$ , carries Neumann boundary condition instead of Dirichlet boundary condition. Then  $h_{U_1} \approx 1$ , and, for any fixed  $o$ , there are constant  $c_1, C_1$  such that, for all  $t > 1$ , we have

$$c_1 t^{-n/2} \leq p(t, o, o) \leq C_1 t^{-n/2}.$$

Note this is the same behavior as for  $p_{\mathbb{R}^n}(t, o, o)$ .

**Example 2.6.3.** Consider the three dimensional body consisting of a solid disk to which we have attached a solid cone on top and a solid cylinder on bottom, as shown in Figure 2 of [37]. With Neumann condition everywhere, this figure was considered in Example 6.15 of [37]. If  $p_N(t, x, y)$  denotes the heat kernel for this figure with Neumann boundary (here,  $N$  in  $p_N$  stands for Neumann) everywhere and  $o$  is a fixed point, then, for  $t > 1$ ,

$$c_0(t \log^2 t)^{-1} \leq p_N(t, o, o) \leq C_0(t \log^2 t)^{-1}.$$

Here we consider this example with some Dirichlet boundary condition. The most natural place to add Dirichlet boundary condition is on the three dimensional cone. The cone with Dirichlet boundary everywhere has a profile with growth of power  $\alpha > 0$  by the previous example, so that the volume of the cone weighted by its profile is approximately  $r^{2\alpha+3}$ . Thus volume in the cone grows faster than  $r^2 \log^2 r$ , which describes how volume grows after  $h$ -transform in the infinite solid disk. Hence  $p_D(t, o, o)$  has the same long-term decay in time as  $p_N(t, o, o)$ .

In fact, the previous paragraph still holds true when we impose *any* Dirichlet boundary condition on the cone in such a way that condition (\*) holds, as the following lemma demonstrates that profiles cannot decrease volume in some sense.

**Lemma 2.6.1.** *Let  $(U, \sigma)$  be an unbounded weighted Riemannian manifold that is uniform in its closure  $\bar{U}$ , which is a Harnack manifold. Let  $u$  denote the profile for  $U$ . Then there exists  $C > 0$  such that*

$$\int_{B_U(x,r)} u^2(y)\sigma(y)dy =: V_u(x,r) \geq CV(x,r)$$

for all  $x \in \bar{U}$  and all  $r > 0$  sufficiently large (where  $r$  may depend on  $x$ ).

*Proof.* It is not possible that  $u(x) \rightarrow 0$  as  $x \rightarrow \infty$  since if this were the case, the maximum principle combined with the Hopf boundary lemma imply  $u \equiv 0$ . Therefore there exists a sequence of points  $\{z_j\}_{j=1}^\infty$  in  $U$  and a number  $\varepsilon > 0$  such that for any fixed point  $o \in U$ ,  $d(z_j, o) \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $u(z_j) \geq \varepsilon$ .

Then by Theorem 2.3.2 there exist constants  $0 < c_0 < +\infty$  and  $0 < c_1 \leq C_1 < +\infty$  such that

$$c_1 u(x_r)^2 V(x,r) \leq V_u(x,r) \leq C_1 u(x_r)^2 V(x,r)$$

for all  $x \in \bar{U}$ ,  $r > 0$ , and  $x_r$  such that  $d(x, x_r) \geq r/4$  and  $d(x_r, \partial U) \geq c_0 r/8$ . Moreover, by the proof of Theorem 4.7 in [40], there exists a constant  $0 < C_2 < +\infty$  such that

$$u(y) \leq C_2 u(x_r) \quad \forall x \in \bar{U}, y \in B(x,r), r > 0.$$

Given  $x \in \bar{U}$ , let  $r > 0$  be sufficiently large so that  $z_j \in B(x,r)$  for some  $j = 1, 2, \dots$ . Then

$$V_u(x,r) \geq \frac{c_1}{C_2} u(z_j)^2 V(x,r) \geq \frac{c_1 \varepsilon^2}{C_2} V(x,r)$$

as claimed. □

**Example 2.6.4.** Consider two copies of the exterior of a parabola in  $\mathbb{R}^2$ . Put Dirichlet condition along each parabola and glue the two copies via a collar, as in Figure 2.7. If  $K$  indicates the compact collar, then this manifold  $\Omega$  has two ends, both of which are the exterior of a parabola in  $\mathbb{R}^2$ , minus a disk.

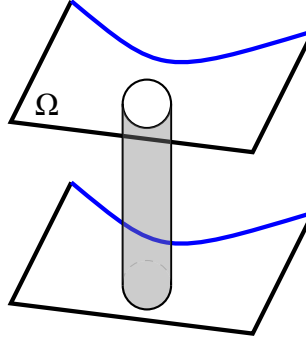


Figure 2.7: A connected sum of the exterior of two parabolas in  $\mathbb{R}^2$ .

As  $\mathbb{R}^2$  minus a parabola is the complement of a convex set, it is uniform in its closure [40, Proposition 6.16], and removing a disk of fixed radius will not change this property. Further, with Neumann condition along both parabola and disk, this manifold is Harnack [40, Theorem 3.10]. Thus hypotheses (H1) and (H2) are satisfied.

The global profile  $h$  for  $\Omega$  will behave like the profile for  $\mathbb{R}^2$  minus a parabola and a disk in each end. Denote the profile for the exterior of a parabola by  $h_{EP}$ . Consider the exterior of the parabola weighted by  $h_{EP}^2$ . Then this space is transient and satisfies the parabolic Harnack inequality, so removing the disk, a compact set, has little effect [34, 40]. What is important to us here is that  $\hat{h}$ , the profile for  $\mathbb{R}^2$  minus a parabola and a disk, weighted by  $h_{EP}^2$ , is essentially constant away from the disk. As the profile for the ends we are interested in is

the product of  $h_{EP}$  and  $\hat{h}$ , it behaves like  $h_{EP}$  when away from the disk. Thus the global profile  $h$  for  $\Omega$ , which appears in Theorem 2.5.1, also behaves like  $h_{EP}$ .

The profile for the exterior of the parabola  $x_2 = x_1^2$  in  $\mathbb{R}^2$ , that is, the space  $EP = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < x_1^2\}$ , is given by

$$h_{EP}(x) = \sqrt{2\left(\sqrt{x_1^2 + (1/4 - x_2)^2} + 1/4 - x_2\right)} - 1. \quad (2.10)$$

The profile for the exterior of any parabola can be found by making an appropriate change of variables in this formula.

Using (2.10) to compute quantities appearing in Theorem 2.5.1, for any fixed point  $o$ , there exist constants  $0 < c \leq C < +\infty$  such that for all  $t > 1$ ,

$$ct^{-3/2} \leq p(t, o, o) \leq Ct^{-3/2}.$$

Now fix  $0 < r_1 \leq r_2$ . Assume that  $x, y$  lie in different copies of the exterior of the parabola and are both at distance approximately  $\sqrt{t}$  from the collar, that is,  $r_1 \sqrt{t} \leq |x|, |y| \leq r_2 \sqrt{t}$ . Since  $V_{i,h}(r)$  satisfies (2.2),

$$H_h(x, t) \approx \frac{|x|^2}{V_{i,h}(|x|)} \approx \frac{t}{V_{i,h}(\sqrt{t})} \approx t^{-1/2}.$$

Likewise,  $H_h(y, t) \approx t^{-1/2}$ . Thus there exists constants  $0 < c_1 \leq C_1 < +\infty$  such that, for  $t$  sufficiently large and all such  $x, y$ ,

$$c_1 h(x)h(y)t^{-2} \leq p(t, x, y) \leq C_1 h(x)h(y)t^{-2}.$$

Depending on the location of  $x, y$  relative to the parabola,  $h(x), h(y)$  can range from zero to behaving like  $t^{1/4}$ . See Figure 2.8. Note if  $h(x) \approx h(y) \approx t^{1/4}$ , then  $p(t, x, y) \approx t^{-3/2}$ .

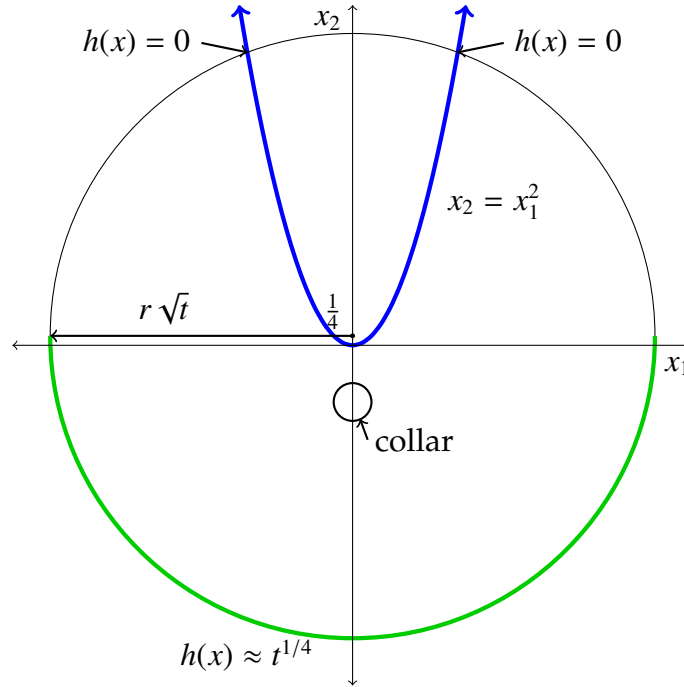


Figure 2.8: If  $|x| \approx \sqrt{t}$ , then for  $t$  sufficiently large,  $x$  is also approximately at distance  $\sqrt{t}$  from the focus of the parabola; denote this distance by  $r\sqrt{t}$  so that  $x$  lies on the circle depicted above. In the bottom half of this circle, colored green,  $h(x) \approx t^{1/4}$  for large  $t$ . As  $x$  travels along the circle toward the parabola,  $h(x)$  decreases to zero.

**Example 2.6.5.** Consider Example 2.6.4, except remove a parabola with Dirichlet condition from only *one* plane. Then the profile  $h(x)$  for this manifold behaves like  $h_{EP}(x)$ , which is given by (2.10), in the end with the parabola removed, and, in the plane without the parabola removed,  $h(x)$  behaves like  $\log(|x|)$ , as this is the harmonic profile for the plane minus a disk. Thus, for  $o$  fixed, the presence of the plane without the parabola removed results in heat kernel decay of the form

$$c(t \log^2 t)^{-1} \leq p(t, o, o) \leq C(t \log^2 t)^{-1} \quad \text{for all } t > 1.$$

Again, fix  $0 < r_1 \leq r_2$  and take  $x$  in the plane minus the parabola and  $y$  in

the plane, both so that their distance to the collar lies between  $r_1 \sqrt{t}$  and  $r_2 \sqrt{t}$ . We still have  $H(x, t) \approx t^{-1/2}$  as in the previous example, but  $V_{i,y,h}$  does not satisfy (2.2) for  $y$  in the plane without the parabola removed. Working directly with the definition of  $H_h(y, t)$  (see formula (2.8)), we find  $H_h(y, t) \approx (\log^2 t)^{-1}$  for all such  $y$ . Therefore, for  $t > 1$ , we obtain the estimate

$$c_1 h(x) (t^{3/2} \log t)^{-1} \leq p(t, x, y) \leq C_1 h(x) (t^{3/2} \log t)^{-1}.$$

Again, the behavior of  $h(x)$  depends on where  $x$  is relative to the parabola as in Figure 2.8. If  $h(x) \approx t^{1/4}$ , then  $p(t, x, y) \approx (t^{5/4} \log t)^{-1}$ .

**Example 2.6.6.** Now consider an analog of Example 2.6.4 or 2.6.5, but in higher dimensions. For instance, remove a paraboloid of revolution from a copy of  $\mathbb{R}^3$  and impose Dirichlet boundary condition on the resulting boundary. Take two copies of this space and glue them via a collar. Then, in theory, we may apply the technique of the previous two examples to estimate the heat kernel of this space. However, estimates for the profile of  $\mathbb{R}^3$  minus a paraboloid are not known. Thus, in practice, we cannot compute explicit decay rates of this heat kernel.

## 2.7 Allowing for corners

We chose to write our main results in the category of Riemannian manifolds with boundary, but there are no serious difficulties other than notational and expository to apply the same method under various levels of generalization. Because allowing some corners is very natural in the context of connected sums, we feel compelled to describe briefly a restrictive but simple set of hypotheses

that can replace the basic assumption that all our manifolds are smooth manifolds with boundary whose metric closures are also smooth manifolds with boundary satisfying condition (\*).

Let us start with  $M^\bullet$ , a smooth Riemannian  $n$ -manifold without boundary and its metric closure  $M$ . Let  $\Omega$  be an open subset of  $M$  with  $M$ -topological boundary  $\partial\Omega$  contained in  $M \setminus M^\bullet$ . In our results up to this point, we were assuming that  $M$  was a smooth manifold with boundary, and that  $\Omega$  was a manifold with boundary satisfying the extra condition (\*).

Let consider instead the assumption that, for any point  $x$  of  $M \setminus M^\bullet$ , there is a neighborhood  $N_x$  of  $x$  in  $M$ , a Lipschitz map  $\Phi_x : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$  defining  $R_x = \{(x_1, \dots, x_n) : x_n \geq \Phi_x(x_1, \dots, x_{n-1})\}$  and a one-to-one Lipschitz map  $\phi_x : N_x \rightarrow R_x$  which is bi-Lipschitz on its image  $V_x$ . The Lipschitz constants associated to  $\Phi_x$  and  $\phi_x$ ,  $\phi_x^{-1}$  may depend on  $x$  but are locally bounded on  $M \setminus M^\bullet$ . The (minimal) heat kernel on the (weighted) smooth manifold  $(M^\bullet, \mu)$  is well defined as usual. The (“Neumann type”) heat kernel on  $(M, \mu)$  is also easily defined, being associated with the regular strictly local Dirichlet form  $\int_M |\nabla f|^2 d\mu$  with domain the set of all functions  $f$  in  $W_{\text{loc}}(M^\bullet)$  such that  $\int_M (|f|^2 + |\nabla f|^2) d\mu < +\infty$ . In this Dirichlet space whose underlying space is  $M$ , solutions of the heat equation satisfy the local parabolic Harnack inequality. Any open set  $\Omega$  with  $\partial\Omega \subseteq M \setminus M^\bullet$  is locally inner-uniform in  $M$  (see [49, Section 3.2] for details on local inner-uniformity). By [49, 50], it follows that harmonic functions in  $\Omega$  which vanish on  $\partial\Omega$  satisfy the local version of the boundary elliptic Harnack inequality. Together, these facts allow for the generalization of the results of this chapter in this context. The key difference lies in the way in which positive harmonic functions vanish at the boundary. On smooth manifolds with boundary, positive harmonic func-

tions vanishing at the boundary vanish linearly. In the more general context described above, the best one can say is already contained in the boundary Harnack inequality, and vanishing of the type  $d(x, \partial\Omega)^{\eta_{x_0}}$  when  $x$  tends to  $x_0 \in \partial\Omega$ , with  $\eta_{x_0} \in [0, 1]$ , is typical. Without entering into all the details necessary to make the above line of reasoning precise, it can easily be implemented to cover the very basic examples with corners shown in Figures 2.1–2.3 of the introduction.

## 2.8 Connection with earlier results

To help the reader understand the techniques and estimates discussed above and relate them to the existing literature, we illustrate how they include some known results. Even though our focus is on manifolds with finitely many nice ends, this section discusses the simpler case when there is only one end.

Consider a complete Harnack Riemannian manifold  $M$  (e.g., a complete manifold with non-negative Ricci curvature) and a domain  $\Omega = M \setminus K$ . When  $K$  is a bounded  $C^{1,1}$ -domain and Dirichlet condition is assumed on the boundary of  $\Omega$  (assume for simplicity that  $\Omega$  is a domain, hence connected), [67] gives global two-sided heat kernel estimates for  $p_\Omega(t, x, y)$  at all times and locations. In the case where  $M$  is non-parabolic, the estimates of [67, Theorem 1.1(a)] compare  $p_\Omega(t, x, y)$  (at all times  $t$  and locations  $x, y$ ) to expressions of the form

$$C \left( \frac{d(x, K)}{\sqrt{t} \wedge 1} \wedge 1 \right) \left( \frac{d(y, K)}{\sqrt{t} \wedge 1} \wedge 1 \right) \frac{\exp\left(-c \frac{d(x, y)^2}{t}\right)}{V(x, \sqrt{t})}.$$

The key ingredients in [67] are (a) near boundary estimates based on [21] and the  $C^{1,1}$  nature of the boundary (see also [66]) and (b) global estimates away from

the boundary from [34] treating the case when  $d(x, K)$  and  $d(y, K)$  are greater than 1.

The validity of such two sided global heat kernel estimates are extended in several different directions in [40, Theorem 5.11]. Theorem 5.11 of [40] applies to a domain  $\Omega = M \setminus K$  in a complete Harnack manifold whenever  $\Omega$  is uniform (in fact, inner-uniform suffices—see, e.g., [40, 49]). In such cases (and without the hypothesis of non-parabolicity), there exists a positive harmonic function  $h$  on  $\Omega$ , vanishing appropriately when reaching  $K$ , such that the heat kernel  $p_\Omega(t, x, y)$  compares (at all times and locations  $x, y$ ) to expressions of the form

$$C \frac{h(x)h(y)}{V_h(x, \sqrt{t})} \exp\left(-c \frac{d(x, y)^2}{t}\right).$$

Here  $V_h(x, r) = \int_{B(x, r)} h^2(z) dz \approx h(x_r)^2 V(x, r)$  where  $x_r$  is a point at distance at most  $Ar$  from  $x$  and at least  $ar$  from  $K$  for some appropriate fixed constants  $a, A$  (see Theorem 2.3.2). When  $\Omega = M \setminus K$  is connected and  $K$  is a bounded  $C^{1,1}$ -domain,  $\Omega$  is automatically uniform and, by classical theory,  $h$  vanishes linearly near the boundary of  $\Omega$ . If, in addition,  $M$  is non-parabolic then  $h(x) \approx d(x, K) \wedge 1$  and, by simple computation, one recovers the estimates of [67]. This yields a different proof of the results in [67], independent from the earlier results in [21, 34].

In addition, [40, Theorem 5.11] allows for  $K$  to be unbounded and non-smooth as long as the key hypothesis that  $\Omega \setminus K$  is uniform (in fact, inner-uniform) remains. In fact, because it is stated in the setting of Dirichlet spaces, [40, Theorem 5.11] allows for the treatment of mixed boundary condition. For instance, as in Example 2.6.6, one can take  $M = \mathbb{R}^d$ ,  $K = \{x = (x_1, \dots, x_d) : x_1^2 + \dots + x_{d-1}^2 \leq x_d\}$  (a paraboloid of revolution), and  $\Omega = M \setminus K$ . Moreover, one can impose mixed boundary condition along  $\partial K = \partial\Omega$ . The technique of [40] is to obtain estimates on  $p_\Omega(t, x, y)$  via intermediate heat kernel estimates on a

related heat kernel, the heat kernel  $p_{\Omega, h^2}(t, x, y)$  of the weighted manifold  $(\Omega, h^2)$  where  $h$  is the harmonic profile of the domain  $\Omega$ . The key point is that when  $\Omega$  is uniform, one can prove that the profile  $h$  has good properties that imply  $(\Omega, h^2)$  is a Harnack manifold. This implies that classical two-sided heat kernel bounds hold for  $p_{\Omega, h^2}(t, x, y)$  (see Theorem 2.3.2 above). The estimates for  $p_{\Omega}(t, x, y)$  then follow simply because  $p_{\Omega}(t, x, y) = h(x)h(y)p_{\Omega, h^2}(t, x, y)$ . One important aspect of this approach is that it resolves all at once the problems related to the global geometric structure of the manifold  $M$  and domain  $\Omega$ , and the local problems related to the presence of boundary conditions.

The strategy from [40] just explained is implemented in the above chapter to prove the main result, Theorem 2.5.1, using the function  $h$  constructed in Theorem 2.4.1, and Theorem 2.3.3 applied to the weighted manifold  $(M, \sigma h^2)$ .

CHAPTER 3  
HITTING PROBABILITIES AND UNIFORMLY  $S$ -TRANSIENT  
SUBGRAPHS

In this chapter and the subsequent one, we now consider the discrete time and space setting. This chapter develops some background and estimates that will be used to address gluing graphs in Chapter 4; on its own, this chapter is about subgraphs of graphs. This chapter contains work appearing in [15], which is joint with Laurent Saloff-Coste.

### 3.1 Introduction

In the study of Markov chains, questions about hitting times (or exit times) of certain subsets are natural. In this chapter, we are interested in discrete time random walks on countable graphs such as the square grid  $\mathbb{Z}^d$ . Namely, we are motivated by the problem of studying random walks on graphs that are obtained by gluing simpler graphs along particular subsets of vertices (as an example, think of  $\mathbb{Z}^4$  and  $\mathbb{Z}^5$  glued along their respective first coordinate axes). With this in mind, we investigate hitting times, hitting probabilities, and a related notion of transience for subgraphs of a larger graph (think  $\mathbb{Z}^4 \setminus Z$  where  $Z = \mathbb{Z}$  is embedded nicely into  $\mathbb{Z}^4$ ) when the random walk on the underlying larger graph is assumed to have an iterated transition kernel satisfying (discrete) two-sided Gaussian estimates. We will call a graph satisfying such two-sided Gaussian estimates a Harnack graph (see Theorem 3.1.1). There is much literature discussing two-sided Gaussian estimates on graphs and equivalent properties. See e.g., [5, 11, 12, 16] and the references therein.

The examples we consider here stem from our goal to apply these results to the problem of gluing graphs along infinite subsets. In such settings, we are interested in whether it is certain the random walk will hit a subset  $K$  or not. We consider this as a sort of recurrence/transience question, although it is important to be careful with what is meant by these definitions. Here we define a notion of “ $S$ -transience” based on the probability  $\psi_K(x)$  that a random walk started at vertex  $x$  hits  $K$  before time infinity. This probability is of course  $1 - \text{Esc}_K(x)$ , where  $\text{Esc}_K(x)$  is the probability  $K$  is never hit. Considering the quantity  $\text{Esc}_K(x)$  is related to the harmonic measure from infinity,  $H_K(x)$ , particularly in the case where  $K$  is finite. Such questions are addressed for  $\mathbb{Z}^d$  in Chapter 2 of [46] and Section 25 of [58]. Work of Boivin and Rau [7] considers the harmonic measure from infinity on weighted graphs; see also the references therein. Moreover, questions of recurrence/transience are related to Wiener’s test. However, none of these related results cover the precise situation of interest to us.

One of the main theorems, Theorem 3.2.1, gives an upper bound on the hitting probability of a subset of a Harnack graph in terms of a ratio of volumes. Although this bound is not always optimal, it makes no assumptions about the geometry of the set we want to hit. This bound can be computed using only volume functions (which in practice are often easier to compute than other quantities).

The other main theorem, Theorem 3.3.4, gives two-sided bounds on the hitting probability in terms of volumes and the harmonic profile  $h$  (a special harmonic function). This theorem requires an additional significant geometric assumption (inner uniformity). We then obtain a partial analog to a well-known theorem that states that a Harnack space is transient (in the classical sense) if

and only if

$$\int^{\infty} [V(x, \sqrt{r})]^{-1} < +\infty$$

for some/all points  $x$ . Further, we can apply the same ideas as in the proof of Theorem 3.3.4 to get two-sided bounds on related quantities, such as the harmonic measure.

This chapter proceeds as follows. The rest of this section describes the setting of interest and introduces notation. Section 3.2 carefully defines what we mean by transience and gives an upper bound for the hitting probability of a set (Theorem 3.2.1). It concludes with several examples of applying the theorem to demonstrate its practicality. Section 3.3 introduces the well-known notion of  $h$ -transform which is used to give an upper and lower bound on the hitting probability in Theorem 3.3.4. One can compare the discrete  $h$ -transform here with the continuous version encountered in Chapter 2. We also state several related corollaries and apply the theorem and corollaries to examples. Section 3.4 gives remarks on the relation between our results and Wiener's test.

### 3.1.1 General graph notation and random walks

Let  $\Gamma = (V, E)$  be a connected graph, where  $E$  is a subset of the pairs of elements in  $V$ . In other words,  $\Gamma$  is a simple graph that does not contain loops or multiple edges. Any graphs appearing will be assumed to be simple and connected unless stated otherwise.

On  $\Gamma$  we take the usual graph distance  $d$  based on the shortest path of edges between vertices, and we consider closed balls with respect to this distance:

$$B(x, r) = \{y \in V : d(x, y) \leq r\} \quad \forall x \in V, r > 0.$$

We also assume  $\Gamma$  has a random walk structure given by edge weights (conductances)  $\mu_{xy}$  and vertex weights (measure)  $\pi(x)$  with the following properties:

- $\mu_{xy} \neq 0 \iff \{x, y\} \in E$  and  $\mu_{xy} = \mu_{yx}$  (the edge weights are *adapted to the edges* and *symmetric*)
- $\sum_{y \sim x} \mu_{xy} \leq \pi(x) \quad \forall x \in V$  (the edge weights are *subordinate* to the measure/vertex weights).

The notation  $y \sim x$  means that the unordered pair  $\{x, y\}$  belongs to the edge set  $E$ . The notation  $y \simeq x$  means either  $y \sim x$  or  $y = x$ . We will use  $V(x, r)$  to denote the volume (with respect to  $\pi$ ) of  $B(x, r)$ .

Given a graph, we can always impose such a random walk structure on it; for example, we can take  $\mu_{xy} \equiv 1 \quad \forall \{x, y\} \in E$  and  $\pi(x) = \deg(x) \quad \forall x \in V$ . We will refer to this particular structure as *simple weights*.

Under the above assumptions, we define a Markov kernel  $\mathcal{K}$  on  $\Gamma$  via:

$$\mathcal{K}(x, y) = \begin{cases} \frac{\mu_{xy}}{\pi(x)}, & x \neq y \\ 1 - \sum_{z \sim x} \frac{\mu_{xz}}{\pi(x)}, & x = y. \end{cases} \quad (3.1)$$

Hence loops are not allowed in  $\Gamma$ , but the random walk is allowed to stay in place. Note that the Markov kernel  $\mathcal{K}$  is *reversible* with respect to the measure  $\pi$ , that is,

$$\mathcal{K}(x, y)\pi(x) = \mathcal{K}(y, x)\pi(y) \quad \forall x, y \in \Gamma.$$

The random walk structure on  $\Gamma$  may be equivalently defined by either a given  $\mu, \pi$ , in which case  $\mathcal{K}$  is as in (3.1), or by a given Markov kernel  $\mathcal{K}$  with reversible measure  $\pi$ .

Let  $\mathcal{K}^n(x, y)$  denote the  $n$ -th convolution power of  $\mathcal{K}(x, y)$ . Then if  $(X_n)_{n \geq 0}$  denotes the random walk on  $\Gamma$  driven by  $\mathcal{K}$ , we have  $\mathcal{K}^n(x, y) = \mathbb{P}^x(X_n = y)$ . The quantity  $\mathcal{K}^n(x, y)$  is not symmetric (in particular,  $\mathcal{K}$  itself need not be symmetric), so we will often be interested in studying instead its *transition density*, the *heat kernel* of the random walk, given by

$$p(n, x, y) = p_n(x, y) = \frac{\mathcal{K}^n(x, y)}{\pi(y)}.$$

There are various hypotheses one may make about the weights that have nice consequences. Here we will make the hypothesis of controlled weights.

**Definition 3.1.1.** We say a graph  $\Gamma$  has *controlled weights* if there exists a constant  $C_c > 1$  such that

$$\frac{\mu_{xy}}{\pi(x)} \geq \frac{1}{C_c} \quad \forall x \in \Gamma, y \sim x. \quad (3.2)$$

This assumption implies that  $\Gamma$  is locally uniformly finite (that is, there is a uniform bound on the degree of any vertex) and that for  $x \sim y$ , we have  $\mu_{xy} \approx \pi(x) \approx \pi(y)$ ; we prove these straightforward consequences in the two lemmas below.

**Lemma 3.1.1.** *Let  $(\Gamma, \pi, \mu)$  be a graph with controlled weights. For any  $x \in \Gamma$  and  $y \sim x$  we have  $\mu_{xy} \approx \pi(x) \approx \pi(y)$ , where  $\approx$  means we have upper and lower bounds involving constants from the hypotheses above.*

*As a consequence, if  $d(x, x') \leq A$ , then  $\pi(x) \approx \pi(x')$  and  $\mu_{xy} \approx \mu_{x'y'}$  for all  $y \sim x, y' \sim x'$  (where the constants in the inequalities depend on  $A$ ).*

*Proof.* For  $y \sim x$ ,  $\mu_{xy} \leq \pi(x)$  follows from the assumption that weights are subordinate to the measure and having controlled weights implies  $\pi(x) \leq C_c \mu_{xy}$ .

Hence  $\pi(x) \approx \mu_{xy} = \mu_{yx} \approx \pi(y)$ . Iterating this argument at most  $A$  times gives the desired result.  $\square$

**Lemma 3.1.2.** *Let  $(\Gamma, \pi, \mu)$  be a graph with controlled weights. Then  $\Gamma$  there is a uniform bound on the number of neighbors of any vertex.*

*Proof.* Indeed, for any fixed  $x \in \Gamma$ ,

$$\#\{y \in \Gamma : y \sim x\} = \sum_{y \sim x} 1 = \sum_{y \sim x} \frac{\mu_{xy}}{\mu_{xy}} \leq \frac{C_c}{\pi(x)} \sum_{y \sim x} \mu_{xy} \leq C_c \quad \forall x \in V.$$

$\square$

**Unless stated otherwise, we will assume all graphs appearing have controlled weights.**

### 3.1.2 Harnack graphs

In this section we describe several further properties graphs  $(\Gamma, \pi, \mu)$  may possess and some of the consequences of these properties. The reader may wish to compare and contrast these properties to those described in Section 2.3.

**Definition 3.1.2.** A graph is said to be *doubling* if there exists a constant  $D$  such that for all  $r > 0$ ,  $x \in \Gamma$ ,

$$V(x, 2r) \leq DV(x, r). \quad (3.3)$$

**Definition 3.1.3.** We say that  $\Gamma$  satisfies the *Poincaré inequality* if there exists constants  $C_p > 0$ ,  $\kappa \geq 1$  such that for all  $r > 0$ ,  $x \in \Gamma$ , and functions  $f$  supported in  $B(x, \kappa r)$ ,

$$\sum_{y \in B(x, r)} |f(y) - f_B|^2 \pi(y) \leq C_p r^2 \sum_{y, z \in B(x, \kappa r)} |f(y) - f(z)|^2 \mu_{yz}, \quad (3.4)$$

where  $f_B$  is the (weighted) average of  $f$  over the ball  $B = B(x, r)$ , that is,

$$f_B = \frac{1}{V(x, r)} \sum_{y \in B(x, r)} f(y) \pi(y).$$

Under doubling, the Poincaré inequality with constant  $\kappa \geq 1$  (which appears in  $B(x, \kappa r)$  on the right hand side) is equivalent to the Poincaré inequality with  $\kappa = 1$ .

**Definition 3.1.4.** We say a pair  $(\mu, \pi)$  is *uniformly lazy* if there exists  $C_e \in (0, 1)$  such that

$$\sum_{y \sim x} \mu_{xy} \leq (1 - C_e) \pi(x) \quad \forall x \in V, y \sim x.$$

We say a Markov kernel  $\mathcal{K}$  is *uniformly lazy* if there exists  $C_e \in (0, 1)$  such that

$$\mathcal{K}(x, x) = 1 - \sum_{z \sim x} \frac{\mu_{xz}}{\pi(x)} \geq C_e \quad \forall x \in \Gamma.$$

These two conditions are equivalent. In this case, the Markov chain is aperiodic. **Unless stated otherwise, we will consider all random walk structures appearing to be uniformly lazy.**

**Definition 3.1.5.** A function  $h : \Gamma \rightarrow \mathbb{R}$  is *harmonic* (with respect to  $\mathcal{K}$ ) if

$$h(x) = \sum_{y \in \Gamma} \mathcal{K}(x, y) h(y) \quad \forall x \in \Gamma. \quad (3.5)$$

Given a subset  $\Omega$  of  $\Gamma$  (usually a ball),  $h$  is harmonic on that set if (3.5) holds for all points in  $\Omega$ ; this requires that  $h$  be defined on  $\{v \in \Gamma : \exists \omega \in \Omega, v \simeq \omega\}$ .

As  $\mathcal{K}(x, y) = 0$  unless  $y \simeq x$ , the sum over  $y \in \Gamma$  in (3.5) can be replaced by a sum over  $y \simeq x$ .

**Definition 3.1.6.** A graph  $(\Gamma, \pi, \mu)$  satisfies the *elliptic Harnack inequality* if there exist  $\eta \in (0, 1)$ ,  $C_H > 0$  such that for all  $r > 0$ ,  $x_0 \in \Gamma$ , and all non-negative

harmonic functions  $h$  on  $B(x_0, r)$ , we have

$$\sup_{B(x_0, \eta r)} h \leq C_H \inf_{B(x_0, \eta r)} h.$$

**Definition 3.1.7.** A function  $u : \mathbb{Z}_+ \times \Gamma \rightarrow \mathbb{R}$  solves the discrete heat equation if

$$u(n+1, x) - u(n, x) = \sum_{y \in \Gamma} \mathcal{K}(x, y)[u(n, y) - u(n, x)] \quad \forall n \geq 1, x \in \Gamma. \quad (3.6)$$

Given a discrete space-time cylinder  $Q = I \times B$ ,  $u$  solves the heat equation on  $Q$  if (3.6) holds there (this requires that for each  $n \in I$ ,  $u(n, \cdot)$  is defined on  $\{z \in \Gamma : \exists x \in B, z \simeq x\}$ ).

**Definition 3.1.8** (Parabolic Harnack inequality). A graph  $(\Gamma, \pi, \mu)$  satisfies the (discrete time and space) *parabolic Harnack inequality* if: there exist  $\eta \in (0, 1)$ ,  $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$  and  $C_P > 0$  such that for all  $s, r > 0$ ,  $x_0 \in \Gamma$ , and every non-negative solution  $u$  of the heat equation in the cylinder  $Q = (\mathbb{Z}_+ \cap [s, s + \theta_4 r^2]) \times B(x_0, r)$ , we have where  $Q_- = (\mathbb{Z}_+ \cap [s + \theta_1 r^2, s + \theta_2 r^2]) \times B(x_0, \eta r)$  and  $Q_+ = (\mathbb{Z}_+ \cap [s + \theta_3 r^2, s + \theta_4 r^2]) \times B(x_0, \eta r)$ .

The parabolic Harnack inequality obviously implies the elliptic version. The following theorem relates several of the above concepts.

**Theorem 3.1.1** (Theorem 1.7 in [16]). *Given  $(\Gamma, \pi, \mu)$  (or  $(\Gamma, \mathcal{K}, \pi)$ ) where  $\Gamma$  has controlled weights and  $\mathcal{K}$  is uniformly lazy, the following are equivalent:*

- (a)  $\Gamma$  is doubling and satisfies the Poincaré inequality
- (b)  $\Gamma$  satisfies the parabolic Harnack inequality
- (c)  $\Gamma$  satisfies two-sided Gaussian heat kernel estimates, that is there exists constants  $c_1, c_2, c_3, c_4 > 0$  such that for all  $x, y \in V$ ,  $n \geq d(x, y)$ ,

$$\frac{c_1}{V(x, \sqrt{n})} \exp\left(-\frac{d^2(x, y)}{c_3 n}\right) \leq p(n, x, y) \leq \frac{c_3}{V(x, \sqrt{n})} \exp\left(-\frac{d^2(x, y)}{c_4 n}\right). \quad (3.7)$$

**Definition 3.1.9.** If  $(\Gamma, \pi, \mu)$  satisfies any of the conditions in Theorem 3.1.1, we call  $\Gamma$  a *Harnack graph*.

**Remark 3.1.1.** The uniformly lazy assumption avoids problems related to parity (such as those that appear in bipartite graphs). Without this assumption, it may be that (a) holds but  $p(n, x, y) = 0$  for some  $n \geq d(x, y)$ , and then (b) and the lower bound in (c) do not hold. Here we avoid such difficulties by assuming the graph is uniformly lazy; another solution to this problem is to state (b) and (c) for the sum over two consecutive discrete times  $n, n + 1$ , e.g., for (c),  $p(n, x, y) + p(n + 1, x, y)$ .

**Definition 3.1.10** (The notation  $\approx$ ). For two functions of a variable  $x$ , the notation  $f \approx g$  means there exist constants  $c_1, c_2$  (independent of  $x$ ) such that

$$c_1 f(x) \leq g(x) \leq c_2 g(x).$$

**Definition 3.1.11** (Abuse of  $\approx$ ). We will often abuse the notation  $\approx$  in the case of heat kernel and hitting probability estimates to write formulas more compactly. For instance, we will write (3.7) as

$$p(n, x, y) \approx \frac{1}{V(x, \sqrt{n})} \exp\left(-\frac{d^2(x, y)}{n}\right).$$

Note this use of  $\approx$  means there are different constants in the upper and lower bounds both inside *and* outside the exponential. In the event that we chain such notations together, the all constants may change from line to line.

**Definition 3.1.12** ((Lazy) Simple Random Walk). Given any graph  $\Gamma$  with a uniform bound on the degree of any vertex, we can define the *simple random walk* on  $\Gamma$  (“take *simple weights* on  $\Gamma$ ”) by setting  $\pi(x) = \deg(x)$  for all  $x \in \Gamma$  and  $\mu_{xy} = 1$  for all  $x, y \in E$ . In other words, at each time step, the random walk moves to a neighbor of the current vertex with equal probability.

The *lazy* simple random walk stays in place with probability  $1/2$  and otherwise moves to a neighbor of the current vertex with equal probability. This corresponds to taking weights  $\pi(x) = 2\deg(x)$  for all  $x \in \Gamma$  and  $\mu_{xy} = 1$  for all  $x, y \in E$ . The lazy simple random walk satisfies our hypotheses of having controlled and uniformly lazy weights.

### 3.1.3 Subgraphs of a larger graph

Sometimes we think of  $\Gamma$  as a subgraph of a larger graph  $\widehat{\Gamma} = (\widehat{V}, \widehat{E})$ . If given  $\widehat{\Gamma}$ , then for any subset of  $V$  of  $\widehat{V}$ , we can construct a graph  $\Gamma$  with vertex set  $V$  and edge set  $E$  where  $\{x, y\} \in E$  if and only if  $x, y \in V$  and  $\{x, y\} \in \widehat{E}$ . On occasion, we will abuse notation and use the same symbol to denote both a subset of  $\widehat{V}$  and its associated subgraph.

Further, a subgraph  $\Gamma$  inherits a random walk structure from  $\widehat{\Gamma}$ . We set  $\pi_{\Gamma}(x) = \pi_{\widehat{\Gamma}}(x)$  and  $\mu_{xy}^{\Gamma} = \mu_{xy}^{\widehat{\Gamma}}$  for all  $x, y \in V$ ,  $\{x, y\} \in E$ . (Hence we may simply use  $\pi, \mu$  without indicating the whole graph versus the subgraph, provided that  $x, y \in \Gamma$ .)

Then we may define a Markov kernel on  $\Gamma$  as in (3.1); we call this Markov kernel the Neumann Markov kernel of  $\Gamma$  (with respect to  $\widehat{\Gamma}$ ) and denote it by  $\mathcal{K}_{\Gamma, N}$ . To be precise,

$$\mathcal{K}_{\Gamma, N}(x, y) = \begin{cases} \frac{\mu_{xy}^{\Gamma}}{\pi(x)} = \frac{\mu_{xy}^{\widehat{\Gamma}}}{\pi(x)}, & x \neq y, x, y \in V \\ 1 - \sum_{z \sim x} \frac{\mu_{xz}^{\Gamma}}{\pi(x)} = 1 - \sum_{z \sim x, z \in V} \frac{\mu_{xz}^{\widehat{\Gamma}}}{\pi(x)}, & x = y \in V. \end{cases} \quad (3.8)$$

We can also define the Dirichlet Markov kernel of  $\Gamma$  (with respect to  $\widehat{\Gamma}$ ) by

$$\mathcal{K}_{\Gamma,D}(x,y) = \mathcal{K}_{\widehat{\Gamma}}(x,y)\mathbb{1}_V(x)\mathbb{1}_V(y) = \begin{cases} \frac{\mu_{xy}^\Gamma}{\pi(x)}, & x \neq y, x, y \in V \\ 1 - \sum_{z \sim x, z \in \widehat{V}} \frac{\mu_{xz}^{\widehat{\Gamma}}}{\pi(x)}, & x = y \in V, \end{cases} \quad (3.9)$$

where  $\mathbb{1}_V(x) = 1$  if  $x \in V$  and zero otherwise. When  $V \neq \widehat{V}$ ,  $\mathcal{K}_{\Gamma,D}$  is only a sub-Markovian kernel.

A subgraph  $\Gamma$  comes with its own notion of distance  $d_\Gamma$ , where  $d_\Gamma(x,y)$  is the length of the shortest path between  $x$  and  $y$  of edges contained in  $\Gamma$ . It is always true that  $d_{\widehat{\Gamma}}(x,y) \leq d_\Gamma(x,y)$ .

There are two natural notions for the boundary of  $\Gamma$ , both of which are useful to us here.

**Definition 3.1.13.** The (*exterior*) *boundary* of  $\Gamma$  is

$$\partial\Gamma = \{y \in \widehat{\Gamma} \setminus \Gamma : \exists x \in \Gamma \text{ s.t. } d_{\widehat{\Gamma}}(x,y) = 1\},$$

in other words, the set of points that do not belong to  $\Gamma$  with neighbors in  $\Gamma$ .

The *inner boundary* of  $\Gamma$  is the set of points inside  $\Gamma$  with neighbors outside of  $\Gamma$ ,

$$\partial_I\Gamma = \{x \in \Gamma : \exists y \notin \Gamma \text{ s.t. } d_{\widehat{\Gamma}}(x,y) = 1\}.$$

For  $x \in \Gamma$  and  $y \in \partial\Gamma$ , we extend  $d_\Gamma(x, \cdot)$  by setting

$$d_\Gamma(x,y) = 1 + \min_{z \in \Gamma: z \sim y} d_\Gamma(z,x).$$

However, this extension is *not* a distance function as it need not satisfy the triangle inequality. The correct way to think of adding points in  $\partial\Gamma$  to  $\Gamma$  may be to think of them as multiple points. If the boundary points are duplicated appropriately, this extension can be made into a distance function.

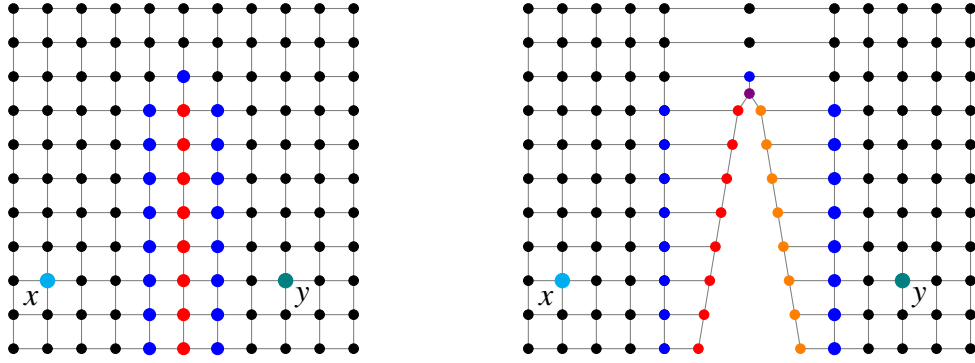


Figure 3.1: Example of interior and exterior boundary points and duplicating the boundary points.

For instance, consider Figure 3.1. Let  $\widehat{\Gamma}$  be the full ten edges by ten edges square as on the left. Take  $\Gamma$  to be  $\widehat{\Gamma}$  minus the red points. The red points are  $\partial\Gamma$ , and the blue points are  $\partial_I\Gamma$ . Then  $d(x, \partial\Gamma) = 4$  and  $d(y, \partial\Gamma) = 3$ , and both of these distances are achieved by the same point in  $\partial\Gamma$ , call it  $z$ . Note  $d_\Gamma(x, y) = 19 > d_\Gamma(x, z) + d_\Gamma(y, z) = 7$ . The correct way to think of this is duplicating the red points of  $\partial\Gamma$  as shown in the right figure.

## 3.2 Hitting probabilities and $S$ -transient graphs

### 3.2.1 Hitting probability upper bound

For this section, consider a graph  $\widehat{\Gamma} = (\widehat{V}, \widehat{E})$  with controlled and uniformly lazy weights  $(\mu, \pi)$ . Let  $K$  be a subset of  $\widehat{V}$ , where we abuse notation to let  $K$  indicate both this set of vertices and the subgraph of  $\widehat{\Gamma}$  induced by these vertices. Set  $\Gamma := \widehat{\Gamma} \setminus K$ , that is, we think of  $\Gamma$  as the subgraph of  $\widehat{\Gamma}$  with vertex set  $\widehat{V} \setminus K$ . We are interested in transience properties of  $\Gamma$  and the hitting probability of  $K$ . We will assume  $\widehat{\Gamma}, \Gamma$  to be infinite and connected;  $K$  may be either finite or infinite

and connected or disconnected.

We are used to thinking of Markov (or subMarkovian) kernels as recurrent if random walks return to their starting points infinitely often and transient if they do not. However, in the present setting of a subgraph which inherits a random walk structure from a larger graph, there are several natural ways to think of transience/recurrence.

**Definition 3.2.1** (*N-transience*). We call a subgraph  $\Gamma$  of  $(\widehat{\Gamma}, \mathcal{K}, \pi)$  *N-transient* (“Neumann”-transient) if  $(\Gamma, \mathcal{K}_{\Gamma, N}, \pi)$  is transient, that is, with probability one, a random walk on  $\Gamma$  only returns to its starting point finitely often.

Being *N-transient* is an intrinsic property of the subgraph  $\Gamma$ , which is in some sense independent of  $\widehat{\Gamma}$ . A similar definition could be given using  $(\Gamma, \mathcal{K}_{\Gamma, D}, \pi)$  instead to define *D-transience*. The killing present in the subMarkovian kernel  $\mathcal{K}_{\Gamma, D}$  makes *D-transience* more likely.

However, in this thesis, the main definition of transience we will be concerned with is *S-transience*, or “survival”-transience, defined below. The explanation for the name is that a subgraph of a larger graph is survival-transient if there is positive probability that a random walk started inside the subgraph never sees vertices that do not belong to the subgraph.

**Definition 3.2.2** (*Hitting time, hitting probability*). Consider a graph  $(\widehat{\Gamma}, \mathcal{K}, \pi)$  with random walk denoted by  $(X_n)_{n \geq 0}$ . Then we denote the first hitting time of a subset of vertices  $K$  by  $\tau_K := \min\{n \geq 0 : X_n \in K\}$  and the first return time to  $K$  by  $\tau_K^+ := \min\{n \geq 1 : X_n \in K\}$ . If  $X_0 \notin K$ , then  $\tau_K$  and  $\tau_K^+$  are the same. Further, denote the hitting probability of  $K$  by  $\psi_K(x) = \mathbb{P}^x(\tau_K < +\infty)$ .

**Definition 3.2.3** (*S-transience*). Let  $(\widehat{\Gamma}, \mathcal{K}, \pi)$  be a connected graph with con-

trolled weights and  $K$  be a subset of  $\widehat{\Gamma}$  such that  $\Gamma := \widehat{\Gamma} \setminus K$  is connected. We say the subgraph  $\Gamma$  is *S-transient* (“survival”-transient) or that the graph  $\widehat{\Gamma}$  is *S-transient with respect to the set  $K$*  if there exists  $x \in \widehat{\Gamma}$  such that  $\psi_K(x) < 1$ .

If this is not the case, then we say  $\Gamma$  is *S-recurrent* (or that  $\widehat{\Gamma}$  is *S-recurrent with respect to  $K$* ).

**Lemma 3.2.1.** *Let  $(\widehat{\Gamma}, \mathcal{K}, \pi)$  be a connected graph with controlled weights and  $K$  be a subset of  $\widehat{\Gamma}$  such that  $\Gamma := \widehat{\Gamma} \setminus K$  is connected. Then the following are equivalent*

1. *There exists  $x \in \widehat{\Gamma}$  such that  $\psi_K(x) < 1$ .*
2. *For all  $x \in \Gamma$ ,  $\psi_K(x) < 1$ .*
3. *For all  $y \in \partial\Gamma$ ,  $\mathbb{P}^y(\tau_K^+ < +\infty) < 1$ .*

*Proof.* Clearly (b) implies (a). That (a) implies (b) follows from the maximum principle: By the definition of a hitting probability,  $\psi_K$  is a harmonic function on  $\Gamma := \widehat{\Gamma} \setminus K$ ; thus so is  $1 - \psi_K$ , which is non-negative. By the maximum principle, if there exists some  $x \in \Gamma$  such that  $1 - \psi_K(x) = 0$ , then  $1 - \psi_K \equiv 0$  on  $\Gamma$ . Hence if  $\psi_K(x) < 1$  for a single  $x \in \Gamma$ , this must be true of all  $x \in \Gamma$ .

We now show the equivalence of (a)-(b) and (c). If  $y \in \partial\Gamma$ , then using the Markov property,

$$\mathbb{P}^y(\tau_K^+ < +\infty) = \sum_{x \approx y} \mathbb{P}^y(\tau_K^+ < +\infty, X_1 = x) = \sum_{x \approx y} \mathbb{E}^y(\mathbf{1}_{\{X_1=x\}} \mathbb{E}^{X_1}(\mathbf{1}_{\{\tau_K^+ < +\infty\}})) \quad (3.10)$$

$$= \sum_{x \approx y} \mathcal{K}(y, x) \mathbb{P}^x(\tau_K^+ < +\infty) = \sum_{x \approx y} \psi_K(x) \mathcal{K}(y, x). \quad (3.11)$$

Since  $y \in \partial\Gamma$ , there exists  $z \in \Gamma$  such that  $z \sim y$ . If (b) holds, then  $\psi_K(z) < 1$  so

$$\mathbb{P}^y(\tau_K^+ < +\infty) < \sum_{x \approx y} \mathcal{K}(y, x) = 1$$

and (c) holds. Conversely, if (c) holds, then

$$\sum_{x \approx y} \psi_K(x) \mathcal{K}(x, y) < 1 = \sum_{x \approx y} \mathcal{K}(x, y).$$

Thus there exists some  $z \sim y$  such that  $\psi_K(z) < 1$ , so (a) holds.  $\square$

Note the lemma does not contain some of the other usual equivalent definitions of transience as allowing for  $K$  to be infinite can cause difficulties. For example, whether  $K$  is hit infinitely often or not can depend upon the precise choice of  $K$  as well as upon if we start inside or outside of  $K$ . A graph is transient in the classical sense if and only if it is  $S$ -transient with respect to any finite set.

**Example 3.2.1** (Lattices  $\mathbb{Z}^m$ ). The lattice  $\mathbb{Z}^m$  with simple weights is (classically) transient/ $S$ -transient with respect to any finite set if and only if  $m \geq 3$ .

**Example 3.2.2** (Lattices in lattices,  $\mathbb{Z}^m \setminus \mathbb{Z}^k$ ). Consider the copy of a  $k$ -dimensional lattices  $\mathbb{Z}^k$  inside of  $\mathbb{Z}^m$ , where we assume  $k \leq m$  and  $m \geq 3$ .

If  $k \leq m - 3$ , then  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is connected,  $N$ -transient, and  $S$ -transient.

If  $k = m - 2$ , then  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is connected and  $N$ -transient, but it is not  $S$ -transient, since the set  $\mathbb{Z}^k$  will be visited infinitely often with probability one (and hence certainly  $\psi_{\mathbb{Z}^k} \equiv 1$ ).

If  $k = m - 1$ , then  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is disconnects into two half-spaces (see Example 3.2.3 below).

**Example 3.2.3** (Half-space  $\mathbb{Z}_+^m$ ). Let  $\Gamma = \mathbb{Z}_+^m = \{(x_1, \dots, x_{m-1}, x_m) \in \mathbb{Z}^m : x_m > 0\}$  denote the upper half-space inside of  $\mathbb{Z}^m$  with simple weights. Then  $\mathbb{Z}_+^m$  is  $N$ -transient if and only if  $m \geq 3$ , and it is always transient if we kill the walk along the set  $\{x_m = 0\}$ . However,  $\mathbb{Z}_+^m$  is *never*  $S$ -transient, since in any dimension the walk on  $\mathbb{Z}_+^m$  hits the set  $\{x_m = 0\}$  with probability one.

**Definition 3.2.4** (Uniform  $S$ -transience). Let  $(\widehat{\Gamma}, \pi, \mu)$  be a graph and  $\Gamma := \widehat{\Gamma} \setminus K$  a subgraph. We say  $\Gamma$  is *uniformly  $S$ -transient*, or that  $\widehat{\Gamma}$  is *uniformly  $S$ -transient with respect to  $K$* , if there exist  $L, \varepsilon > 0$  such that  $d(x, K) \geq L$  implies that  $\psi_K(x) \leq 1 - \varepsilon$ .

The following theorem gives an upper bound on the hitting probability of  $K$ . This bound can be useful for showing  $S$ -transience.

**Theorem 3.2.1.** *Let  $(\widehat{\Gamma}, \mu, \pi)$  have controlled weights that are uniformly lazy. Let  $K$  be a subset/subgraph of  $\widehat{\Gamma}$ . Set  $\Gamma := \widehat{\Gamma} \setminus K$  and note  $\partial\Gamma \subseteq K$ . Assume that  $\widehat{\Gamma}$  is Harnack. Define*

$$B_{\partial\Gamma}(x, r) = B_{\widehat{\Gamma}}(x, r) \cap \partial\Gamma \quad \text{and} \quad V_{\partial\Gamma}(x, r) = \pi(B_{\partial\Gamma}(x, r)) \quad \forall x \in \widehat{\Gamma},$$

that is,  $V_{\partial\Gamma}$  is the volume of traces of  $\widehat{\Gamma}$ -balls in  $\partial\Gamma$ .

For any  $x \in \Gamma = \widehat{\Gamma} \setminus K$ , set

$$W(x, r) := \frac{V_{\widehat{\Gamma}}(x, r)}{V_{\partial\Gamma}(x, r)}.$$

Then, if  $d_x := d(x, K)$ , there exists a constant  $C$  (depending on the constants appearing in the controlled weights, uniformly lazy, and Harnack assumptions) such that

$$\psi_K(x) \leq \sum_{n \geq d_x^2} \frac{C}{W(x, \sqrt{n})} \quad \forall x \in \Gamma \setminus \partial\Gamma. \quad (3.12)$$

The theorem does not discuss  $x \in \partial\Gamma$ , since in this case  $\psi_K(x) \approx 1$ , independently of  $x$ , due to the controlled weights hypothesis.

Using the theorem, it is easy to verify that  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is uniformly  $S$ -transient when  $k \leq m - 3$  (see Example 3.2.4 below).

*Proof.* For any  $x \in \Gamma \setminus \partial_I \Gamma$ ,  $d(x, K) \geq 2$ , and we have

$$\begin{aligned} \psi_K(x) &:= \mathbb{P}^x(\tau_K < +\infty) = \sum_{n=1}^{\infty} \mathbb{P}^x(\tau_K = n) = \sum_{n \geq 1} \sum_{v \in K} \mathbb{P}^x(\tau_K = n, X_{\tau_K} = v) \\ &= \sum_{n \geq 2} \sum_{v \in \partial \Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \mathcal{K}_{\Gamma, D}^{n-1}(x, y) \mathcal{K}_{\widehat{\Gamma}}(y, v). \end{aligned} \tag{3.13}$$

Controlled weights mean that  $\mathcal{K}_{\widehat{\Gamma}}(y, v)$  is roughly constant. The Dirichlet Markov kernel on  $\Gamma$  is less than the Markov kernel on all of  $\widehat{\Gamma}$ , which is Harnack. Hence

$$\begin{aligned} \psi_K(x) &= \sum_{n \geq 2} \sum_{v \in \partial \Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \mathcal{K}_{\Gamma, D}^{n-1}(x, y) \mathcal{K}_{\widehat{\Gamma}}(y, v) \leq \sum_{n \geq 2} \sum_{v \in \partial \Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \mathcal{K}_{\widehat{\Gamma}}^{n-1}(x, y) \\ &\leq C \sum_{n \geq 2} \sum_{v \in \partial \Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \frac{\pi(y)}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d_{\widehat{\Gamma}}^2(x, y)}{cn}\right). \end{aligned}$$

Now  $\pi(y) \approx \pi(v)$  and the number of  $y \sim v$  is uniformly bounded above. Moreover, as any of the above  $y$ 's belong to  $\partial_I \Gamma$  while  $x$  does not, it is impossible that  $x = y$ . Hence  $d_{\widehat{\Gamma}}(x, y) \geq 1$ . Thus we can replace  $d_{\widehat{\Gamma}}(x, y)$  via  $d_{\widehat{\Gamma}}(x, v)$  since

$$d_{\widehat{\Gamma}}(x, v) \leq d_{\widehat{\Gamma}}(x, y) + 1 \leq 2d_{\widehat{\Gamma}}(x, y).$$

We have

$$\psi_K(x) \leq C \sum_{n \geq 2} \sum_{v \in \partial \Gamma} \frac{\pi(v)}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d_{\widehat{\Gamma}}^2(x, v)}{cn}\right)$$

We sum first in time. First we split the sum into two, noting that the exponential is large for large  $n$ ; recall the notation  $\approx$  is as in Definition 3.1.11 and means we have matching upper/lower bounds with different constants inside

and outside the exponential:

$$\begin{aligned} \sum_{n \geq 2} \frac{1}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d_{\widehat{\Gamma}}^2(x, v)}{n}\right) &\approx \sum_{n \leq d_{\widehat{\Gamma}}^2(x, v)} \frac{1}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d_{\widehat{\Gamma}}^2(x, v)}{n}\right) \\ &+ \sum_{n \geq d_{\widehat{\Gamma}}^2(x, v)} \frac{1}{V_{\widehat{\Gamma}}(x, \sqrt{n})}. \end{aligned}$$

We compute the first piece of the sum by arranging a dyadic decomposition with  $d/\sqrt{n} \asymp 2^l$ , where  $d := d_{\widehat{\Gamma}}(x, v)$ . Here the quantity  $d/\sqrt{n}$  ranges from 1 to  $d$ .

Let  $l_{x,v}$  be the integer such that  $d(x, v) \asymp 2^{l_{x,v}}$ . Then

$$\begin{aligned} \sum_{n \leq d_{\widehat{\Gamma}}^2(x, v)} \frac{C}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d_{\widehat{\Gamma}}^2(x, v)}{cn}\right) &\leq \sum_{l=0}^{l_{x,v}} \sum_{\sqrt{n} \geq d2^{-l}} \frac{C}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \exp\left(-\frac{d^2}{cn}\right) \\ &\leq \sum_{l=0}^{l_{x,v}} \frac{C}{V_{\widehat{\Gamma}}(x, d/2^{l+1})} \frac{d^2}{4^l} \exp\left(-\frac{4^l}{c}\right) \\ &\leq C \frac{d^2}{V_{\widehat{\Gamma}}(x, d)} \sum_{l=0}^{l_{x,v}} \exp\left(-\frac{4^l}{c}\right) \leq C \frac{d^2}{V_{\widehat{\Gamma}}(x, d)}, \end{aligned}$$

where in the last line we used the doubling of  $\widehat{\Gamma}$  (a consequence of the assumption that  $\widehat{\Gamma}$  is Harnack).

It is easy to see the sum we just computed is dominated by the tail sum ( $n \geq d^2(x, v)$ ) as

$$\sum_{n=d^2}^{4d^2} \frac{1}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \approx \frac{d^2}{V_{\widehat{\Gamma}}(x, d)}$$

due to the doubling of  $\widehat{\Gamma}$ . Recall here the exponential is large.

Let  $d_x := d(x, K)$  and note  $d(x, K) \leq d(x, v)$  for all  $v \in \partial\Gamma$ . Switching the order of summation in the upper bound above, we find

$$\begin{aligned} \psi_K(x) &\leq C \sum_{v \in \partial\Gamma} \sum_{n \geq d^2(x, v)} \frac{\pi(v)}{V_{\widehat{\Gamma}}(x, \sqrt{n})} = C \sum_{n \geq d_x^2} \sum_{\substack{v \in \partial\Gamma: \\ d(x, v)^2 \leq n}} \frac{\pi(v)}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \\ &= C \sum_{n \geq d_x^2} \frac{V_{\partial\Gamma}(x, \sqrt{n})}{V_{\widehat{\Gamma}}(x, \sqrt{n})} = \sum_{n \geq d_x^2} \frac{C}{W(x, \sqrt{n})}. \end{aligned}$$

□

In the case the volume of traces of  $\widehat{\Gamma}$ -balls in  $\partial\Gamma$  are doubling, the theorem simplifies. It is important that in the corollary we only consider traces whose centers belong to  $\partial\Gamma$ .

**Definition 3.2.5** (Doubling Boundary). Consider a graph  $(\widehat{\Gamma}, \mu, \pi)$  and subgraph  $\Gamma$  of  $\widehat{\Gamma}$  with exterior boundary  $\partial\Gamma$ . We say  $V_{\partial\Gamma}$  is doubling if there exists a constant  $D > 0$  such that for all  $z \in \partial\Gamma$ ,  $r > 0$ ,  $V_{\partial\Gamma}(z, 2r) \leq DV_{\partial\Gamma}(z, r)$ .

**Corollary 3.2.1.** *Under the assumptions of Theorem 3.2.1 and the additional assumption that  $V_{\partial\Gamma}$  is doubling (as in Definition 3.2.5), then the upper bound of Theorem 3.2.1 has the following simplified form. Let  $v_x \in \partial\Gamma$  achieve  $d(x, K)$  and set  $\widetilde{W}(x, r) := V_{\widehat{\Gamma}}(x, r)/V_{\partial\Gamma}(v_x, r)$ .*

*Then there exists a constant  $C$  (depending on  $D$  from Definition 3.2.5 and the constants from the assumptions as in Theorem 3.2.1) such that*

$$\psi_K(x) \leq \sum_{n \geq d_x^2} \frac{C}{\widetilde{W}(x, \sqrt{n})}.$$

*Proof.* Return to the point in the proof of Theorem 3.2.1 where

$$\psi_K(x) \leq C \sum_{v \in \partial\Gamma} \sum_{n \geq d_{\widehat{\Gamma}}^2(x, v)} \frac{\pi(v)}{V_{\widehat{\Gamma}}(x, \sqrt{n})}.$$

Then  $d_{\widehat{\Gamma}}(x, v_x) \leq d_{\widehat{\Gamma}}(x, v)$  by definition of  $v_x$  and  $d_{\widehat{\Gamma}}(v_x, v) \leq d_{\widehat{\Gamma}}(v_x, x) + d_{\widehat{\Gamma}}(x, v) \leq 2d_{\widehat{\Gamma}}(x, v)$ . Therefore  $d_{\widehat{\Gamma}}(v, x) \geq \frac{1}{3}(d_{\widehat{\Gamma}}(x, v_x) + d_{\widehat{\Gamma}}(v_x, v))$ , and we can replace  $d_{\widehat{\Gamma}}(v, x)$  with this sum, that is,

$$\psi_K(x) \leq C \sum_{v \in \partial\Gamma} \sum_{n \geq \frac{1}{9}(d_{\widehat{\Gamma}}^2(x, v_x) + d_{\widehat{\Gamma}}^2(v_x, v))} \frac{\pi(v)}{V_{\widehat{\Gamma}}(x, \sqrt{n})}.$$

Again interchanging the order of summation, noting that the time sum for a particular  $v$  requires  $n \geq cd_{\widehat{\Gamma}}^2(v_x, v)$ , we have

$$\begin{aligned} \psi_K(x) &\leq C \sum_{n \geq \frac{1}{9}d_{\widehat{\Gamma}}^2(x, v_x)} \frac{1}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \sum_{\substack{v \in \partial\Gamma: \\ \frac{1}{9}d_{\widehat{\Gamma}}^2(v_x, v) \leq n}} \pi(v) \\ &\leq C \sum_{n \geq d_{\widehat{\Gamma}}^2(x, v_x)} \frac{V_{\partial\Gamma}(v_x, c'\sqrt{n})}{V_{\widehat{\Gamma}}(x, \sqrt{n})} \leq C \sum_{n \geq d_{\widehat{\Gamma}}^2(x, v_x)} \frac{V_{\partial\Gamma}(v_x, \sqrt{n})}{V_{\widehat{\Gamma}}(x, \sqrt{n})}, \end{aligned}$$

where we used the doubling of both  $V_{\partial\Gamma}$  and  $V_{\widehat{\Gamma}}$  in the last line.  $\square$

**Remark 3.2.1.** If  $\{\sum_{n \geq 1} (W(x, \sqrt{n}))^{-1}\}_{x \in \Gamma}$  (or the sum with  $\widetilde{W}$ ) converges uniformly, that is, for all  $\varepsilon > 0$ , there exists  $N$  (independent of  $x$ ) such that

$$\sum_{n \geq N} \frac{1}{W(x, \sqrt{n})} < \varepsilon, \quad (3.14)$$

then it follows from Theorem 3.2.1 that  $\widehat{\Gamma}$  is uniformly  $S$ -transient with respect to  $K$ . In fact, in this case,  $\psi_K(x) \rightarrow 0$  uniformly as  $d_{\widehat{\Gamma}}(x, K) \rightarrow \infty$ .

In the  $S$ -recurrent case,  $\sum_{n \geq 1} (W(x, \sqrt{n}))^{-1}$  need not converge. We will also see examples where this sum converges, but not uniformly in  $x$ . In certain regimes of the latter type of example, we may be able to use (3.12) to see that  $\psi_K < 1$  for some  $x$  (and hence show transience). In other regimes and in the recurrent case, this bound is not useful. (Recall  $\psi(x) \leq 1 \forall x$  since  $\psi$  is a probability.) Observe that the above theorem is not strong enough to conclude that if  $\sum_{n \geq 1} (W(x, \sqrt{n}))^{-1}$  converges for some (any)  $x \in \widehat{\Gamma} \setminus K$  then  $\widehat{\Gamma}$  is  $S$ -transient with respect to  $K$ . This is because in the bound given by the theorem, the start of tail of the sum depends on the point  $x$ , so although  $\sum_{n \geq 1} (W(x, \sqrt{n}))^{-1}$  may converge, that does not guarantee that  $\sum_{n \geq d_x^2} (W(x, \sqrt{n}))^{-1}$  is sufficiently small to make  $\psi_K(x) < 1$ .

### 3.2.2 Examples

In this section we give examples of applying Theorem 3.2.1 or Corollary 3.2.1 to demonstrate  $S$ -transience or uniform  $S$ -transience. Below we frequently use  $\approx$  from Definition 3.1.10.

**Example 3.2.4** (Lattices in lattices). We verify that  $\mathbb{Z}^m$  is uniformly  $S$ -transient with respect to  $\mathbb{Z}^k$  whenever  $m - k \geq 3$ .

Consider  $\widehat{\Gamma} = \mathbb{Z}^m$  with  $K = \partial\Gamma = \mathbb{Z}^k = \{(x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{Z}^m : x_1, \dots, x_k \in \mathbb{Z}\}$  the  $k$ -dimensional sublattice made up of the first  $k$ -coordinates. Assume  $m - k \geq 3$ . Suppose  $\mathbb{Z}^m$  has simple weights or a variation thereof (such as taking bounded weights or taking the lazy simple random walk on  $\mathbb{Z}^m$ ). With these weights,  $\mathbb{Z}^m$  is Harnack, and  $B_{\partial\Gamma}(z, r) = B_{\mathbb{Z}^m}(z, r) \cap \mathbb{Z}^k = B_{\mathbb{Z}^k}(z, r)$  for all  $z \in \mathbb{Z}^k$ , so it is clear  $V_{\mathbb{Z}^k}$  is doubling. Thus all the hypotheses of Theorem 3.2.1 and Corollary 3.2.1 are satisfied. We compute

$$\widetilde{W}(x, r) := \frac{V_{\mathbb{Z}^m}(x, r)}{V_{\mathbb{Z}^k}(v_x, r)} \approx \frac{r^m}{r^k} = r^{m-k}.$$

Hence,

$$\psi_K(x) \leq \sum_{n \geq d_x^2} \frac{1}{\widetilde{W}(x, \sqrt{n})} \approx \sum_{n \geq d^2} \frac{1}{n^{(m-k)/2}} \approx \frac{1}{d^{m-k-2}} \rightarrow 0 \quad \text{as } d \rightarrow \infty$$

since  $(m - k)/2 > 1$  as  $m - k \geq 3$ . Here there is only dependence on  $d$ , the distance to  $K = \mathbb{Z}^k$ , and not on  $x$  itself. Hence  $\mathbb{Z}^m$  is uniformly  $S$ -transient with respect to  $\mathbb{Z}^k$ . In fact, in this case  $\{\sum_{n \geq 1} (W(x, \sqrt{n}))^{-1}\}_{x \in \widehat{\Gamma} \setminus K}$  converges uniformly.

Note that if  $m - k < 3$ , then we get a series that fails to converge, and Theorem 3.2.1 gives the pointless bound of  $\psi \leq \infty$ .

**Remark 3.2.2.** In our examples, it is common that  $K = \partial\Gamma$ . Theorem 3.2.1 is useful for showing that  $\widehat{\Gamma}$  is transient with respect to a subset  $K$  of its vertices, and

we have the idea that transient sets tend to be “thin” or of smaller dimension, as we saw above, so the set  $K$  doesn’t have a much of an “interior” in the larger graph.

However, for future applications involving gluing graphs, thinking of the set  $K$  as having some “thickness” may be useful. Consider  $\widehat{\Gamma} = \mathbb{Z}^m$  and take  $K$  to be a cylinder, say  $K = \{\vec{x} \in \mathbb{Z}^m : |x_m| \leq r\}$ , so that  $K$  is the set of all points within distance  $r$  from the  $x_m$ -axis. Then if  $r \geq 1$  and  $\Gamma = \mathbb{Z}^m \setminus K$ ,  $\partial\Gamma \neq K$ . However, the chance we hit  $K$  is essentially the same as the chance we hit a single line in  $\mathbb{Z}^m$ , so  $\Gamma$  is  $S$ -transient if and only if  $m \geq 4$ .

**Example 3.2.5** (A sparse line in  $\mathbb{Z}^3$ ). In Example 3.2.4, we could not use Theorem 3.2.1 to decide the  $S$ -transience/ $S$ -recurrence of a copy of  $\mathbb{Z}$  in  $\mathbb{Z}^3$ . The same can be said for a half-line in  $\mathbb{Z}^3$ . Now consider the sparse half-line in  $\mathbb{Z}^3$  given by dyadic points:  $K = \partial\Gamma = \{(2^k, 0, 0) \in \mathbb{Z}^3 : k \in \mathbb{Z}_{\geq 0}\}$ . Take the lazy SRW on  $\mathbb{Z}^3$ .

The hypotheses of Theorem 3.2.1/Corollary 3.2.1 are satisfied; to verify doubling of  $\partial\Gamma$ , note that  $V_{\partial\Gamma}(z, r) \approx \log_2(r)$  for  $z \in \partial\Gamma$ . Further note for any  $v_x \in \partial\Gamma$ , we have  $V_{\partial\Gamma}(v_x, r) \leq V_{\partial\Gamma}(0, r)$ . We now compute, for any  $x \in \mathbb{Z}^3$ ,

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{\widetilde{W}(x, \sqrt{n})} &= \sum_{n \geq 1} \frac{V_{\partial\Gamma}(v_x, \sqrt{n})}{V_{\mathbb{Z}^3}(x, \sqrt{n})} \leq \sum_{n \geq 1} \frac{V_{\partial\Gamma}(0, \sqrt{n})}{V_{\mathbb{Z}^3}(x, \sqrt{n})} \approx \sum_{l \geq 0} \sum_{\sqrt{n} \approx 2^l} \frac{V_{\partial\Gamma}(0, \sqrt{n})}{n^{3/2}} \\ &\approx \sum_{l \geq 0} \sum_{\sqrt{n} \approx 2^l} \frac{l+1}{2^{3l}} \leq \sum_{l \geq 0} \frac{l+1}{2^{3l}} 2^{2l} = \sum_{l \geq 0} \frac{l+1}{2^l}. \end{aligned}$$

This is a convergent sum that is independent of  $x$ . Therefore  $\mathbb{Z}^3$  is uniformly  $S$ -transient with respect to the sparse line  $K$ .

**Example 3.2.6** (Weighted half-spaces). Consider  $\widehat{\Gamma} = \mathbb{Z}_+^m = \{(x_1, \dots, x_m) \in \mathbb{Z}^m : x_m \geq 0\}$  where  $\pi(x_1, \dots, x_m) = (1 + x_m)^\alpha$  and  $\alpha > 1$ . Let  $K = \partial\Gamma = \{(x_1, \dots, x_m) \in \mathbb{Z}_+^m : x_m = 0\}$ . Let  $\mathcal{K}_{\mathbb{Z}_+^m}$  denote the Markov kernel for the lazy SRW on  $\mathbb{Z}_+^m$ , that is at each

step we stay in place with probability  $1/2$  or move to a neighbor uniformly at random (at points on the edge, we have a higher probability of staying in place).

Define a new Markov kernel on  $\mathbb{Z}_+^m$  by

$$\mathcal{M}_{\mathbb{Z}_+^m}(x, y) = \begin{cases} \mathcal{K}_{\mathbb{Z}_+^m}(x, y) \min\{1, \frac{\pi(y)}{\pi(x)}\}, & x \neq y \\ 1 - \sum_{z \sim x} \mathcal{M}_{\mathbb{Z}_+^m}(x, z), & x = y. \end{cases}$$

Then this is a Markov kernel, and we consider the graph  $(\mathbb{Z}_+^m, \pi, \mathcal{M}_{\mathbb{Z}_+^m})$ . Since  $\mathcal{K}_{\mathbb{Z}_+^m}$  is symmetric with respect to the vertex measure that is identically 1, it is easy to verify  $\mathcal{M}_{\mathbb{Z}_+^m}$  is symmetric with respect to  $\pi$ .

The appropriate edge weights that give the same Markov kernel are

$$\mu_{xy} = \pi(x) \mathcal{M}_{\mathbb{Z}_+^m}(x, y).$$

As  $\mathcal{K}_{\mathbb{Z}_+^m}$  was uniformly lazy and had controlled weights,  $\mathcal{M}_{\mathbb{Z}_+^m}$  inherits these properties.

That  $\mathbb{Z}_+^m$  is Harnack with respect to this random walk structure can be verified by directly showing it is doubling and satisfies the Poincaré inequality or by using arguments similar to those in Section 4.3 of [36]. On  $K$ , we have  $\pi \equiv 1$  and  $V_{\partial\Gamma}((x_1, \dots, x_{m-1}, 0), r) \approx r^{m-1}$  is doubling. Further,

$$V_{\Gamma}((x_1, \dots, x_m), r) \approx \begin{cases} |x_m|^\alpha r^m, & r \leq |x_m| \\ r^{m+\alpha}, & r \geq |x_m|. \end{cases}$$

The above computation can be seen as follows: If  $r \leq |x_m|$ , there are approximately  $r^m$  points in  $B((x_1, \dots, x_m), r)$ , each of which has weight approximately  $|x_m|^\alpha$ . It is clear we can get such an upper bound; for the lower bound, note the ball of radius  $r$  contains the ball of radius  $r/2$  which again has approximately  $r^m$  points and since  $r \leq |x_m|$ , all such points have weight approximately  $|x_m|^\alpha$ . On the

other hand, if  $r \geq |x_m|$ , we can consider a ball of radius  $r$  with center on  $\partial\Gamma$ . As  $\pi$  is constant except in the  $x_m$  direction, the volume of such a ball is approximately  $r^{m-1}(1^\alpha + \dots + r^\alpha)$ . As  $(\frac{r}{2})^\alpha \frac{r}{2} \leq 1^\alpha + \dots + r^\alpha \leq r^\alpha \cdot r$ , the desired volume follows.

Therefore

$$\widetilde{W}((x_1, \dots, x_m), r) \approx \begin{cases} |x_m|^\alpha r, & r \leq |x_m| \\ r^{1+\alpha}, & r \geq |x_m|. \end{cases}$$

Note that in this example, the family of sums  $\{\sum_{n \geq 1} (\widetilde{W}(x, \sqrt{n}))^{-1}\}_{x \in \mathbb{Z}_+^d \setminus K}$  does not converge uniformly as

$$\begin{aligned} \sum_{n \geq 1} \frac{1}{\widetilde{W}(x, \sqrt{n})} &= \sum_{n=1}^{|x_m|^2} \frac{1}{|x_m|^\alpha n^{1/2}} + \sum_{n > |x_m|^2} \frac{1}{n^{(1+\alpha)/2}} \\ &\leq \frac{1}{|x_m|^\alpha} [2|x_m| - 2] + \left(\frac{2}{\alpha - 1}\right) \frac{1}{|x_m|^{\alpha-1}} \quad (\text{since } \alpha > 1), \end{aligned}$$

which depends on  $x$ .

However, this example is still uniformly  $S$ -transient due to the relationship between  $d_x = d(x, K) = |x_m|$  and the condition which determines the form of  $\widetilde{W}$ .

We see

$$\sum_{n \geq d_x^2} \frac{1}{\widetilde{W}(x, \sqrt{n})} = \sum_{n \geq |x_m|^2} \frac{1}{n^{(1+\alpha)/2}} = \frac{1}{d_x^{\alpha-1}} \rightarrow 0 \quad \text{as } d_x \rightarrow \infty.$$

Therefore for any  $\varepsilon > 0$ , we can choose  $L > 0$  such that whenever  $d(x, K) \geq L$ , we have

$$\psi_K(x) \leq \sum_{n > d_x^2} \frac{1}{\widetilde{W}(x, \sqrt{n})} = \frac{1}{d_x^{\alpha-1}} \leq \frac{1}{L^{\alpha-1}} < \varepsilon,$$

that is, the weighted half-space  $\mathbb{Z}_+^m$  (with  $\alpha > 1$ ) is uniformly  $S$ -transient with respect to  $K$  for all  $m$ .

### 3.3 Harmonic profiles and hitting probability estimates

The previous section obtained an upper bound for the hitting probability  $\psi_K$ . In this section, we obtain two-sided bounds on  $\psi_K$ . Getting a lower bound requires a better estimate on  $\mathcal{K}_{\Gamma,D}$ , which we will give in terms of a nice harmonic function (a harmonic profile) on  $\Gamma$ . To guarantee such harmonic functions exist, we will make geometric assumptions about  $\Gamma$ .

Recall all graphs  $(\Gamma, \pi, \mu)$  are assumed to be uniformly lazy and have controlled weights.

**Definition 3.3.1.** A subgraph  $\Gamma$  of a graph  $\widehat{\Gamma}$  is *uniform* in  $\widehat{\Gamma}$  if there exist constants  $0 < c_u, C_U < +\infty$  such that for any  $x, y \in \Gamma$  there is a path  $\gamma_{xy} = (x_0 = x, x_1, \dots, x_k = y)$  between  $x$  and  $y$  in  $\Gamma$  such that

1.  $k \leq C_U d_{\widehat{\Gamma}}(x, y)$
2. For any  $j \in \{0, \dots, k\}$ ,

$$d_{\widehat{\Gamma}}(x_j, \partial\Gamma) = d_{\widehat{\Gamma}}(x_j, \widehat{\Gamma} \setminus \Gamma) \geq c_u(1 + \min\{j, k - j\}).$$

**Definition 3.3.2.** A subgraph  $\Gamma$  of  $\widehat{\Gamma}$  is *inner uniform* in  $\widehat{\Gamma}$  if there exist constants  $0 < c_u, C_U < +\infty$  such that for any  $x, y \in \Gamma$  there is a path  $\gamma_{xy} = (x_0 = x, x_1, \dots, x_k = y)$  between  $x$  and  $y$  in  $\Gamma$  such that

1.  $k \leq C_U d_{\widehat{\Gamma}}(x, y)$
2. For any  $j \in \{0, \dots, k\}$ ,

$$d_{\widehat{\Gamma}}(x_j, \partial\Gamma) = d_{\widehat{\Gamma}}(x_j, \widehat{\Gamma} \setminus \Gamma) \geq c_u(1 + \min\{j, k - j\}).$$

The only difference between a uniform domain and an inner uniform domain is that uniform domains require the length of the path in  $\Gamma$  to be comparable to the distance between  $x$  and  $y$  in the larger graph  $\widehat{\Gamma}$ , while an inner uniform domain requires the length of the path to be comparable to distance in  $\Gamma$ . This somewhat subtle difference is key. Recall  $d_\Gamma(x_j, \partial\Gamma) = d_{\widehat{\Gamma}}(x_j, \partial\Gamma)$  if we extend  $d_\Gamma$  to  $\partial\Gamma$ . We refer the reader to [18] and the references therein for more details on such geometric assumptions on domains in the discrete space setting; in particular, Section 8.1 gives many examples of finite (inner) uniform domains. Condition (b) in these definitions can be thought of as a “banana” or “cigar” condition and says that it must be possible to fit a linearly growing “banana” (with respect to the distance to the boundary) around all paths from  $x$  to  $y$ . This “banana” must stay inside the domain.

All uniform domains are inner uniform. Domains above Lipschitz functions in  $\mathbb{Z}^d$  are uniform. A slit two-dimensional lattice is the typical example of a domain that is inner uniform but not uniform. Similarly, the complement of a discrete parabola in  $\mathbb{Z}^2$  is inner uniform but not uniform. In general, slits and “bottlenecks” are obstacles to uniformity. An example of a domain that is neither inner uniform nor uniform would be  $\{(x, y) \in \mathbb{Z}^2 : x \leq -2\} \cup \{(x, y) \in \mathbb{Z}^2 : x \geq 2\} \cup \{(-1, 0), (0, 0), (1, 0)\}$ , considered as a subgraph of  $\mathbb{Z}^2$ . There is no criterion to determine (inner) uniformity, and proving whether a given set is (inner) uniform is difficult.

Inner uniform domains are useful because they allow us to transfer the Harnack inequality from a larger graph to a subgraph.

**Theorem 3.3.1** (Theorem 1.10 of [42]). *Let  $(\widehat{\Gamma}, \mathcal{K}, \pi)$  be a Harnack graph and  $\Gamma$  be an inner uniform subgraph of  $\widehat{\Gamma}$ . Then  $(\Gamma, \mathcal{K}_{\Gamma, N}, \pi)$  is also a Harnack graph.*

**Remark 3.3.1.** The converse of Theorem 3.3.1 is not true. For instance, consider the traces of two parabolas in  $\mathbb{Z}^2$  (with the lazy simple random walk) connected by a finite number of edges. One such example is  $\Gamma = \{(x, y) \in \mathbb{Z}^2 : y \geq x^2 + 1\} \cup \{(x, y) \in \mathbb{Z}^2 : y \leq -x^2 - 1\} \cup \{(0, 0)\}$ , where this denotes the vertex set of a subgraph of  $\mathbb{Z}^2$ . The continuous version of this example is Harnack by Theorem 7.1 of [36]. Therefore, the discrete version is also Harnack by results of [13]. This is an example of a subgraph of a Harnack graph where the subgraph is neither uniform nor inner uniform but  $(\Gamma, \mathcal{K}_{\Gamma, N}, \pi)$  is nonetheless Harnack.

**Definition 3.3.3.** A function  $h$  is an *harmonic profile* for an infinite graph  $\Gamma$  that is a subgraph of  $\widehat{\Gamma}$  if it satisfies the following properties:

1.  $h > 0$  in  $\Gamma$
2.  $h = 0$  on the exterior boundary of  $\Gamma$
3.  $h$  is harmonic in  $\Gamma$ , that is,

$$h(x) = \sum_{y \in \Gamma} \mathcal{K}_{\widehat{\Gamma}}(x, y)h(y) = \sum_{y \in \Gamma} \mathcal{K}_{\Gamma, D}(x, y)h(y) \quad \forall x \in \Gamma. \quad (3.15)$$

(Note  $\mathcal{K}_{\Gamma, D}(x, y) = 0$  unless  $y \simeq x$  and  $h(y) = 0$  if  $y \notin \Gamma$ .)

On finite graphs, there is no such function satisfying properties 1., 2., and 3. above since any harmonic function that is zero on the exterior boundary of  $\Gamma$  (which we assume to be non-empty) is zero everywhere.

We would like to appeal to a variety of pre-existing results about the existence of harmonic profiles and their properties in inner uniform domains. In the continuous space setting, the desired results are found in [40]. These results were transferred to the graph setting in the case of infinite graphs in [42, Chapter 5]; see also [18, Chapter 8]. In general, the technique of [42] is to associate

with any given graph its cable process. The cable process takes place in a continuous space with a nice Dirichlet form, so the results of [40] apply to it, and there is a one-to-one correspondence between a profile of the cable process and a profile of the graph.

**Proposition 3.3.1** (Prop. 5.1, Corollary 5.3 of [42]). *Suppose  $\Gamma$  is a proper infinite subgraph of  $(\widehat{\Gamma}, \mu, \pi)$ . Then there exists a harmonic profile  $h$  for  $\Gamma$ . Moreover, if  $\Gamma$  is inner uniform in  $\widehat{\Gamma}$  and  $\widehat{\Gamma}$  is Harnack, then the profile  $h$  of  $\Gamma$  is unique up to multiplication by a constant.*

The existence of  $h$  is straightforward. The uniqueness of  $h$  is more subtle and can be obtained from [40, Theorem 4.1] via the cable process.

### 3.3.1 $h$ -transform on graphs

The existence of the profile  $h$  for a graph  $\Gamma$  (considered as a subgraph of a graph  $\widehat{\Gamma}$ ) allows us to consider a reweighted version of  $\Gamma$ , which we will refer to as the  $h$ -transform space. Recall a graph and the random walk structure on it may be given by triples of the form  $(\Gamma, \pi, \mu)$  or  $(\Gamma, \mathcal{K}, \pi)$ .

Reweight the measure  $\pi$  on  $\Gamma$  by  $h^2$  to obtain the measure  $\pi_h(x) = h^2(x)\pi(x)$ . As  $h > 0$  on  $\Gamma$ ,  $\pi_h$  remains a positive function on the vertices of  $\Gamma$ . Define a Markov kernel by

$$\mathcal{K}_h(x, y) = \frac{1}{h(x)} \mathcal{K}_{\Gamma, D}(x, y) h(y). \quad (3.16)$$

That this is a Markov kernel follows since  $h$  is harmonic as, for all  $x \in \Gamma$ ,

$$\sum_{y \in \Gamma} \mathcal{K}_h(x, y) = \sum_{y \in \Gamma} \frac{1}{h(x)} \mathcal{K}_{\Gamma, D}(x, y) h(y) = \frac{h(x)}{h(x)} = 1.$$

Notice  $\mathcal{K}_h$  is a Markov kernel, despite that not being the case for  $\mathcal{K}_{\Gamma,D}$ . Thus one effect of the  $h$ -transform is to return us to the setting of Markov kernels as opposed to subMarkovian ones.

Moreover,  $\mathcal{K}_h$  is reversible with respect to  $\pi_h$  since  $\mathcal{K}_{\Gamma,D}$  is reversible with respect to  $\pi$ :

$$\begin{aligned}\mathcal{K}_h(x, y)\pi_h(x) &= \mathcal{K}_h(x, y)h(x)^2\pi(x) = h(x)h(y)\mathcal{K}_{\Gamma,D}(x, y)\pi(x) \\ &= h(x)h(y)\mathcal{K}_{\Gamma,D}(y, x)\pi(y) = \mathcal{K}_h(y, x)h^2(y)\pi(y) \\ &= \mathcal{K}_h(y, x)\pi_h(y).\end{aligned}$$

Directly defining  $\mathcal{K}_h$  as in (3.16) is equivalent to taking reweighted conductances  $\mu_{xy}^h = h(x)h(y)\mu_{xy}$  on  $\Gamma$  and then defining the Markov kernel as in Section 3.1.3. Note that  $\mu_{xy}^h = 0$  if at least one of  $x, y \notin \Gamma$ , so it does not matter whether we think of the Neumann or Dirichlet kernel. Considering the graph this way, the weights  $\mu_{xy}^h$  are subordinate to the measure  $\pi_h$  due to the harmonicity of  $h$ . Further, if  $\widehat{\Gamma}$  has controlled weights, the same holds for the  $h$ -transform space since  $h(y)/h(x)$  is bounded below for  $x \sim y$  ( $x, y \in \Gamma$ ).

The heat kernel  $p_h(n, x, y)$  on the  $h$ -transform of  $\Gamma$  is the transition density of  $\mathcal{K}_h$  and is given by  $\mathcal{K}_h^n(x, y)/\pi_h(y)$ . The  $h$ -transform heat kernel on  $\Gamma$  and the Dirichlet heat kernel on  $\Gamma$  have the following relationship:

$$p_h(n, x, y) = \frac{\mathcal{K}_h^n(x, y)}{\pi_h(y)} = \frac{\mathcal{K}_{\Gamma,D}^n(x, y)}{h(x)h(y)\pi(y)} = \frac{1}{h(x)h(y)}p_{\Gamma,D}(n, x, y).$$

Under certain conditions, we have good two-sided estimates for the heat kernel of the  $h$ -transform of  $\Gamma$ , which is the content of the next theorem.

**Theorem 3.3.2** (Theorem 1.11 and Corollary 5.8 of [42]). *Suppose that  $(\widehat{\Gamma}, \mu, \pi)$  is a Harnack graph and  $\Gamma$  is an inner uniform subgraph of  $\widehat{\Gamma}$ . Then  $(\Gamma, \mathcal{K}_h, \pi_h)$  is also a*

*Harnack graph. Consequently, there exist constants  $c_1, c_2, c_3, c_4 > 0$  such that, for all  $x, y \in \Gamma$  and  $n \geq d_\Gamma(x, y)$ ,*

$$\frac{c_1}{V_h(x, \sqrt{n})} \exp\left(-\frac{c_2 d_\Gamma^2(x, y)}{n}\right) \leq p_h(n, x, y) \leq \frac{c_3}{V_h(x, \sqrt{n})} \exp\left(-\frac{c_4 d_\Gamma^2(x, y)}{n}\right)$$

*or, equivalently,*

$$\frac{c_1 h(x)h(y)}{V_h(x, \sqrt{n})} \exp\left(-\frac{c_2 d_\Gamma^2(x, y)}{n}\right) \leq p_{\Gamma, D}(n, x, y) \leq \frac{c_3 h(x)h(y)}{V_h(x, \sqrt{n})} \exp\left(-\frac{c_4 d_\Gamma^2(x, y)}{n}\right)$$

*Here  $V_h$  denotes the volume in  $\Gamma$  with respect to the measure  $\pi_h$ .*

The following lemma is useful for computing  $V_h$ .

**Lemma 3.3.1** ([42, Proposition 5.5]). *Let  $\Gamma$  be inner uniform in a Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . For any  $x \in \Gamma$ ,  $r > 0$ , let  $x_r \in \Gamma$  be a point such that  $d_\Gamma(x, x_r) \leq r/4$  and  $d(x_r, \partial\Gamma) \geq c_u r/8$  (recall  $c_u$  is one of the inner uniformity constants). Then there exist constants  $c, C$  (independent of  $x, r$ ) such that*

$$ch(x_r)^2 V_\Gamma(x, r) \leq V_h(x, r) \leq Ch(x_r)^2 V_\Gamma(x, r).$$

**Remark 3.3.2.** The existence of points  $x_r$  as in the lemma above is a relatively straightforward consequence of the inner uniform assumption (see [40, Lemma 3.20], [42, Lemma 4.7]).

**Remark 3.3.3.** The definition of such points  $x_r$  is motivated by the following key property of  $h$ : There exists a constant  $A$  such that

$$h(y) \leq Ah(x_r) \quad \forall r > 0, y \in B_\Gamma(x, r).$$

This property is called a Carleson estimate, and it follows from arguments given in Section 4.3.3 (in particular (4.28)) of [40], Chapter 8 of [18], and Theorem 2 of [1]. This property is crucial to Lemma 3.3.1.

Moreover, due to the harmonicity of  $h$  and the inner uniform property,  $h(x_{2r}) \approx h(x_r)$ , and  $V_\Gamma$  is doubling.

**Remark 3.3.4.** In the situation where we can compute  $h$ , the above abstract examples become concrete. For example, if  $\widehat{\Gamma} = \mathbb{Z}^m$  and  $\Gamma = \{(x_1, \dots, x_m) \in \mathbb{Z}^m : x_m > 0\}$  is the upper half-space, then  $h(x_1, \dots, x_m) = x_m$ . It is easy to verify the above claims about  $h$  for this example. However, there are only a few situations where exact formulas for  $h$  are known, and, in general, estimating  $h$  is a hard problem.

The following theorem holds for continuous spaces and is discussed in Chapter 4 of [40]. Once again, the theorem can be transferred to the discrete setting using the cable process (see [42]).

**Theorem 3.3.3** (Boundary Harnack Principle [40, Theorem 4.18]). *Let  $\Gamma$  be an inner uniform subgraph of the Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . Then there exist constants  $A_0, A_1 \in (1, \infty)$  such that for any  $\xi \in \partial_I \Gamma$  and any two positive harmonic functions  $f, g$  on  $B_\Gamma(\xi, A_0 r)$  that are zero along  $\partial \Gamma \cap B_\Gamma(\xi, A_0 r)$ , we have*

$$\frac{f(x)}{f(x')} \leq A_1 \frac{g(x)}{g(x')} \quad \forall x, x' \in B_\Gamma(\xi, r).$$

### 3.3.2 Hitting probabilities and Dirichlet kernels in the inner uniform case

Theorem 3.3.2 gave two-sided estimates of  $p_{\Gamma, D}$  in terms of  $h$ ; whenever we can estimate  $h$  on part (or all) of  $\Gamma$ , the abstract estimate of  $p_{\Gamma, D}$  becomes more concrete.

**Lemma 3.3.2.** *Let  $\Gamma := \widehat{\Gamma} \setminus K$  be inner uniform in the Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . If  $\widehat{\Gamma}$  is  $S$ -transient with respect to  $K$ , then the profile  $h$  of  $\Gamma$  is given by  $1 - \psi_K$ . If  $\widehat{\Gamma}$  is uniformly  $S$ -transient with respect to  $K$ , then  $h \approx 1$ .*

*Proof.* Since  $\psi_K$  is the hitting probability of  $K$ , and the exterior boundary of  $\Gamma$  is contained in  $K$ ,  $\psi_K$  is harmonic in  $\Gamma$ . Further,  $0 \leq \psi_K \leq 1$  on  $\Gamma$  and  $\psi_K \equiv 1$  on  $K$ . Hence  $h = 1 - \psi_K$  is a harmonic function inside of  $\Gamma$  that is zero on the exterior boundary of  $\Gamma$ . Since  $\widehat{\Gamma}$  is  $S$ -transient with respect to  $K$ , there exists some  $y \in \Gamma$  such that  $\psi_K(y) < 1$ . Thus  $h(y) > 0$ , and, by the maximum principle,  $h(x) > 0$  for all  $x \in \Gamma$ . Therefore  $h$  is the profile of  $\Gamma$ .

Now suppose  $\widehat{\Gamma}$  is uniformly  $S$ -transient with respect to  $K$ . Then there exist  $L, \varepsilon > 0$  such that  $\psi_K(x) \leq 1 - \varepsilon$  whenever  $d(x, K) \geq L$ . (Note the distance from  $x$  to  $K$  is the same whether considered in all of  $\widehat{\Gamma}$  or only in  $\Gamma$ .) Hence for  $d(x, K) \geq L$ , we have  $\varepsilon \leq 1 - \psi_K(x) = h(x)$ . From the definition of a harmonic function,  $h(x) \geq (1/C_c)h(y)$  for  $x \sim y$ ,  $x, y \in \Gamma$ , where  $C_c$  is the constant for controlled weights. Applying this inequality a finite number of times (since  $d(x, K) \geq 1 \forall x \in \Gamma$ ), there exists  $\varepsilon_* > 0$  such that  $\varepsilon_* \leq h(x) \leq 1$  for all  $x \in \Gamma$ .  $\square$

**Corollary 3.3.1.** *Assume that  $\widehat{\Gamma}$  is a Harnack graph that is uniformly  $S$ -transient with respect to  $K$  and that  $\Gamma := \widehat{\Gamma} \setminus K$  is inner uniform. Then there exist constants  $0 < c, C < +\infty$  such that*

$$c p_{\Gamma, N}(Cn, x, y) \leq p_{\Gamma, D}(n, x, y) \leq p_{\Gamma, N}(n, x, y).$$

*If  $\widehat{\Gamma}$  is  $S$ -transient with respect to  $K$ , then the Neumann and Dirichlet heat kernels are comparable in the region where  $h \approx 1$ .*

In other words, adding killing along  $\Gamma$  does not significantly alter the behavior of the heat kernel in this setting.

*Proof.* The upper bound is immediate. Since we are in the setting where  $\Gamma$  is an inner uniform subgraph of a Harnack  $\widehat{\Gamma}$ , by Theorem 3.3.1,  $(\Gamma, \mathcal{K}_{\Gamma, N}, \pi)$  is a

Harnack graph. Thus there exist constants  $c_1, c_2, c_3, c_4 > 0$  such that for all  $x, y \in \Gamma$  and all  $n \geq d(x, y)$ ,

$$\frac{c_1}{V(x, \sqrt{n})} \exp\left(-\frac{c_2 d_\Gamma^2(x, y)}{n}\right) \leq p_{\Gamma, N}(n, x, y) \leq \frac{c_3}{V(x, \sqrt{n})} \exp\left(-\frac{c_4 d_\Gamma^2(x, y)}{n}\right).$$

From Theorem 3.3.2, we also know that the  $h$ -transform of  $\Gamma$  is Harnack. In the uniformly  $S$ -transient setting,  $h \approx 1$  by Lemma 3.3.2. Therefore  $h(x) \approx h(y) \approx 1$  and  $V_h \approx V$ . Therefore there exist constants  $b_1, b_2, b_3, b_4 > 0$  such that

$$\frac{b_1}{V(x, \sqrt{n})} \exp\left(-\frac{b_2 d_\Gamma^2(x, y)}{n}\right) \leq p_{\Gamma, D}(n, x, y) \leq \frac{b_3}{V(x, \sqrt{n})} \exp\left(-\frac{b_4 d_\Gamma^2(x, y)}{n}\right).$$

Hence  $p_{\Gamma, N}, p_{\Gamma, D}$  satisfy two-sided Gaussian estimates and we obtain the desired lower bound. This argument holds whenever  $h(x), h(y) \approx 1$ , so the statement about the transient case follows.  $\square$

**Theorem 3.3.4.** *Suppose that  $\Gamma := \widehat{\Gamma} \setminus K$  is inner uniform in the Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ .*

*Then, where the constants for  $\approx$  depend on the constants appearing in the inner uniform, Harnack, controlled weights, and uniformly lazy assumptions,*

$$\psi_K(x) \approx \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \sum_{\substack{y \in \partial_I \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{h(x)}{h(x, \sqrt{n})} \frac{h(y)}{h(y, \sqrt{n})} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})} \quad \forall x \in \Gamma \setminus \partial_I \Gamma. \quad (3.17)$$

*If, in addition,  $\Gamma$  is uniformly  $S$ -transient, then the two-sided bound*

$$\psi_K(x) \approx \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \sum_{\substack{y \in \partial_I \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})} \quad (3.18)$$

*holds, where the constants in  $\approx$  are as above and also depend upon  $L, \varepsilon$  from the uniformly  $S$ -transient assumption.*

**Remark 3.3.5.** In Theorem 3.2.1, the main step of the proof that resulted in an upper bound (without a matching lower bound) came from using the inequality

$\mathcal{K}_{\Gamma,D}^{n-1} \leq \mathcal{K}_{\Gamma,N}^{n-1}$ . In Theorem 3.2.1, no assumptions about the geometry of  $\Gamma$  (or  $K$ ) were made, and the proof uses  $d_{\widehat{\Gamma}}$ . Theorem 3.3.4 instead uses the distance  $d_{\Gamma}$ , so while these theorems are similar, the main objects differ. If  $\Gamma$  is *uniform* in  $\widehat{\Gamma}$ , then  $d_{\Gamma} \approx d_{\widehat{\Gamma}}$ . However, even under uniformity, the upper bounds of Theorem 3.2.1 and Theorem 3.3.4 only clearly agree up to a constant if there is also some sort of doubling of the set  $\partial\Gamma$  as in Corollary 3.2.1.

*Proof.* For  $x \in \Gamma \setminus \partial\Gamma$ ,  $d_{\Gamma}(x, K) \geq 2$  and

$$\begin{aligned} \psi_K(x) &= \mathbb{P}^x(\tau_K < +\infty) = \sum_{n \geq 2} \sum_{v \in \partial\Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \mathcal{K}_{\Gamma,D}^{n-1}(x, y) \mathcal{K}_{\widehat{\Gamma}}(y, v) \\ &= \sum_{n \geq 2} \sum_{v \in \partial\Gamma} \sum_{\substack{y \sim v \\ y \in \Gamma}} \frac{h(x)}{h(y)} \mathcal{K}_h^{n-1}(x, y) \mathcal{K}_{\widehat{\Gamma}}(y, v) \\ &\approx \sum_{n \geq 2} \sum_{v \in \partial\Gamma} \sum_{y \sim v} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{n})} \exp\left(\frac{-cd_{\Gamma}^2(x, y)}{n}\right) \\ &\approx \sum_{n \geq 2} \sum_{y \in \partial\Gamma} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{n})} \exp\left(\frac{-cd_{\Gamma}^2(x, y)}{n}\right), \end{aligned}$$

where we have used that  $\mathcal{K}_{\widehat{\Gamma}}(y, v)$  is roughly constant (by the controlled weight hypothesis), the result of Theorem 3.3.2 for  $\mathcal{K}_h^{n-1}$ , and that each  $y \in \partial\Gamma$  is adjacent to at least one, but at most finitely many,  $v \in \partial\Gamma$  (uniformly over  $y$ ).

Since  $(\Gamma, \mathcal{K}_h, \pi_h)$  is a Harnack graph, it must be doubling and satisfy the Poincaré inequality. Taking the sum in time  $n$  and using Lemma 3.3.1 to estimate  $V_h$ ,

$$\sum_{n \geq 2} \frac{1}{V_h(x, \sqrt{n})} \exp\left(-\frac{d_{\Gamma}^2(x, y)}{n}\right) \approx \sum_{n \geq d_{\Gamma}^2(x, y)} \frac{1}{V_h(x, \sqrt{n})} \approx \sum_{n \geq d_{\Gamma}^2(x, y)} \frac{1}{h(x_{\sqrt{n}})^2 V_{\Gamma}(x, \sqrt{n})},$$

where the upper bound follows from the same argument as in Theorem 3.2.1 and the lower bound comes from forgetting the earlier terms of the sum.

If  $d_\Gamma(x, y) \leq \sqrt{n}$ , then  $h(x_{\sqrt{n}}) \approx h(y_{\sqrt{n}})$ . This follows from the inequality

$$\begin{aligned} h(y_{\sqrt{n}})^2 V_\Gamma(y, \sqrt{n}) &\leq C V_h(y, \sqrt{n}) \leq C V_h(x, 2\sqrt{n}) \leq C h(x_{\sqrt{n}})^2 V_\Gamma(x, 2\sqrt{n}) \\ &\leq C h(x_{\sqrt{n}})^2 V_\Gamma(y, \sqrt{n}), \end{aligned}$$

where we have used the relationship between  $V_h$  and  $V_\Gamma$  and that both of these are doubling.

Hence

$$\psi_K(x) \approx \sum_{y \in \partial_I \Gamma} \sum_{n \geq d_\Gamma^2(x, y)} \frac{h(x)}{h(x_{\sqrt{n}})} \frac{h(y)}{h(y_{\sqrt{n}})} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})}.$$

Now interchange the order of summation. Noting the set  $\{y \in \partial_I \Gamma : d_\Gamma^2(x, y) \leq n\}$  is nonempty if and only if  $n \geq d_\Gamma^2(x, y_x)$ ,

$$\sum_{y \in \partial_I \Gamma} \sum_{n \geq d_\Gamma^2(x, y)} \longleftrightarrow \sum_{n \geq d_\Gamma^2(x, y_x)} \sum_{\substack{y \in \partial_I \Gamma: \\ d_\Gamma^2(x, y) \leq n}}.$$

Thus

$$\psi_K(x) \approx \sum_{n \geq d_\Gamma^2(x, y_x)} \sum_{\substack{y \in \partial_I \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{h(x)}{h(x_{\sqrt{n}})} \frac{h(y)}{h(y_{\sqrt{n}})} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})}$$

The result for the uniformly  $S$ -transient case follows from the above and Lemma 3.3.2.  $\square$

**Remark 3.3.6.** Recall from Remark 3.3.3 that the Carleson estimate  $h(z) \leq Ah(x_r)$  holds for all  $r > 0$ ,  $z \in B_\Gamma(x, r)$ . Therefore the terms  $h(x)/h(x_{\sqrt{n}})$  and  $h(y)/h(y_{\sqrt{n}})$  are bounded and, essentially, add additional decay to the sum. Thus Theorem 3.3.4 has additional decay that is not present in Theorem 3.2.1. However, if  $h \approx 1$ ,  $d_\Gamma \approx d_\Gamma^-$ , and the boundary is doubling, then these bounds are the same.

In the  $S$ -recurrent case,  $\psi_K \equiv 1$ , so the two-sided bound above should yield constants. Example 3.3.4 demonstrates this.

**Theorem 3.3.5.** *Suppose that  $\Gamma := \widehat{\Gamma} \setminus K$  is an inner uniform subgraph of the Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ .*

Let  $B_{\partial_I}(x, r) := B_\Gamma(x, r) \cap \partial_I \Gamma$  denote the trace of  $\Gamma$ -balls in  $\partial_I \Gamma$  and  $V_{\partial_I}(x, r) = \pi(B_{\partial_I}(x, r))$  for any  $x \in \Gamma$ . Define

$$W_{\partial_I}(x, r) := \frac{V_\Gamma(x, r)}{V_{\partial_I}(x, r)} \quad \forall x \in \Gamma.$$

Then:

1. *If  $\Gamma$  is uniformly  $S$ -transient, there exists some  $L', \varepsilon' > 0$  such that*

$$d(x, K) \geq L' \implies \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \frac{1}{W_{\partial_I}(x, \sqrt{n})} \leq \varepsilon'.$$

2. *If for any  $\varepsilon > 0$ , there exists  $L_\varepsilon > 0$  such that*

$$d(x, K) \geq L_\varepsilon \implies \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \frac{1}{W_{\partial_I}(x, \sqrt{n})} \leq \varepsilon,$$

*then  $\Gamma$  is uniformly  $S$ -transient.*

*Proof.* (1): Suppose that  $\Gamma$  is uniformly  $S$ -transient, so there exist  $\varepsilon, L > 0$  such that  $\psi_K(x) \leq 1 - \varepsilon$  whenever  $d(x, K) \geq L$ . By Lemma 3.3.2, we have  $h \approx 1$ . Using the result of Theorem 3.3.4, we have

$$\sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \frac{1}{W_{\partial_I}(x, \sqrt{n})} = \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \sum_{\substack{y \in \partial_I \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})} \leq C\psi_K(x) \leq C(1 - \varepsilon)$$

whenever  $d(x, K) \geq L$ . Setting  $\varepsilon' = C(1 - \varepsilon)$  gives the result.

(2): Now suppose that for any  $\varepsilon > 0$ , there exists  $L_\varepsilon > 0$  such that  $\sum_{n \geq d_\Gamma^2(x, \partial \Gamma)} (W_{\partial \Gamma}(x, \sqrt{n}))^{-1} < \varepsilon$  whenever  $d(x, K) \geq L_\varepsilon$ . Using Theorem 3.3.4 and the fact that  $h(x)/h(x_{\sqrt{n}}) \leq 1$  for all  $x \in \Gamma$ ,  $\sqrt{n} \geq 1$ ,

$$\begin{aligned} \psi_K(x) &\leq C \sum_{n \geq d_\Gamma^2(x, \partial \Gamma)} \sum_{\substack{y \in \partial \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{h(x)}{h(x_{\sqrt{n}})} \frac{h(y)}{h(y_{\sqrt{n}})} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})} \leq C \sum_{n \geq d_\Gamma^2(x, \partial \Gamma)} \sum_{\substack{y \in \partial \Gamma: \\ d_\Gamma^2(x, y) \leq n}} \frac{\pi(y)}{V_\Gamma(x, \sqrt{n})} \\ &= \sum_{n \geq d_\Gamma^2(x, \partial \Gamma)} \frac{C}{W_{\partial \Gamma}(x, \sqrt{n})} \leq C\varepsilon. \end{aligned}$$

Taking  $\varepsilon$  sufficiently small, there exists  $\varepsilon', L' > 0$  such that  $\psi_K(x) \leq 1 - \varepsilon'$  whenever  $d(x, K) \geq L'$ , which is precisely the definition of  $\Gamma$  being uniformly  $S$ -transient.  $\square$

**Remark 3.3.7.** Theorem 3.3.5 relies upon the lower bound of Theorem 3.3.4 in (1) and the upper bound in (2). An analogous statement of (2) could be obtained in the setting of Theorem 3.2.1 using the function  $W$  as opposed to the function  $W_{\partial \Gamma}$ .

### 3.3.3 Two-sided bounds on hitting probabilities accounting for time or vertex hit

When  $\Gamma$  is an inner uniform subgraph of a Harnack graph  $\widehat{\Gamma}$ , Theorem 3.3.4 gives matching upper and lower bounds on the probability of leaving  $\Gamma$  (i.e. the probability of hitting  $\Gamma^c$ .) Other questions of natural interest include the likelihood of exiting  $\Gamma$  at a particular point  $v \in \partial \Gamma$ , or the chance of exiting  $\Gamma$  (in general, or at a particular point), at or before time  $n$ . While in the recurrent case  $\psi_K(x) \equiv 1$ , this is not the case for the probabilities in the previous sentence, and these questions remain interesting. Bounds on these probabilities can be given

using the same ideas and reasoning we have already seen. We collect these results below as corollaries. In particular, Corollaries 3.3.3 and 3.3.4 can be seen as discrete versions of the results of [35].

**Definition 3.3.4** (Various Hitting Probabilities). Recall that given  $(\widehat{\Gamma}, \mu, \pi)$  with subgraph  $\Gamma = \widehat{\Gamma} \setminus K$ ,  $\tau_K$  denotes the first hitting time of  $K$ /first exist time of  $\Gamma$ . Define the following hitting probabilities, where  $x \in \Gamma$ ,  $v \in \partial\Gamma$ , and  $n \geq d_\Gamma(x, v)$  :

- $\psi_K(x) = \mathbb{P}^x(\tau_K < +\infty)$ , the chance of hitting  $K$ , given the walk starts at  $x$
- $\psi_K(x, v) = \mathbb{P}^x(X_{\tau_K} = v, \tau_K < +\infty)$ , the chance of hitting  $K$  for the first time at the point  $v$ , given the walk starts at  $x$
- $\psi_K(n, x, v) = \mathbb{P}^x(X_{\tau_K} = v, \tau_K \leq n)$ , the chance of hitting  $K$  for the first time at the point  $v \in \partial\Gamma$ , and doing so in time less than or equal to  $n$
- $\psi'_K(m, x, v) = \psi_K(m, x, v) - \psi_K(m-1, x, v) = \mathbb{P}^x(X_{\tau_K} = v, \tau_K = m)$ , the chance of hitting  $K$  for the first time at  $v$  at the time  $m$
- $\psi_K(n, x) = \mathbb{P}^x(\tau_K \leq n)$  the chance of hitting  $K$  at time less than or equal to  $n$
- $\psi'_K(m, x) = \psi_K(m, x) - \psi_K(m-1, x) = \mathbb{P}^x(\tau_K = m)$ , the chance of hitting  $K$  for the first time at time  $m$ .

There are various relationships between these quantities, for example

$$\begin{aligned}\psi_K(n, x, v) &= \sum_{m=0}^n \psi'_K(m, x, v) \\ \psi_K(n, x) &= \sum_{m=0}^n \psi'_K(m, x) = \sum_{v \in \partial\Gamma} \sum_{m=d(x,v)}^n \psi'_K(m, x, v) = \sum_{v \in \partial\Gamma} \psi_K(n, x, v).\end{aligned}$$

Our theorems above dealt with  $\psi_K(x)$ ; the corollaries below provide estimates for some of these other quantities. These corollaries use the symbol  $\approx$  from Definitions 3.1.10 and 3.1.11 where constants are allowed both inside and

outside exponentials. These constants depend on the constants appearing in the definitions of controlled weights, uniformly lazy, inner uniform, and Harnack graph, and, in the assumption of uniform  $S$ -transience, on  $L, \varepsilon$ .

**Corollary 3.3.2.** *Assume  $\Gamma := \widehat{\Gamma} \setminus K$  is an inner uniform subgraph of a Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . Then*

$$\psi_K(x, v) \approx \sum_{\substack{y \in \Gamma: \\ y \sim v}} h(x)h(y)\pi(y) \sum_{n \geq d_\Gamma^2(x, y)} \frac{1}{V_h(x, \sqrt{n})} \quad \forall x \in \Gamma \setminus \partial_I \Gamma, v \in \partial \Gamma. \quad (3.19)$$

*In the event that  $\Gamma$  is uniformly  $S$ -transient, then*

$$\psi_K(x, v) \approx \sum_{n \geq d_\Gamma^2(x, v)} \frac{\pi(v)}{V_\Gamma(x, \sqrt{n})}. \quad (3.20)$$

*Proof.* Reasoning as in Theorem 3.3.4, but without summing over *all* points of the boundary of  $\Gamma$  yields

$$\psi_K(x, v) \approx \sum_{n \geq 2} \sum_{\substack{y \in \Gamma: \\ y \sim v}} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{n})} \exp\left(-\frac{d_\Gamma^2(x, y)}{n}\right) \approx \sum_{\substack{y \in \Gamma: \\ y \sim v}} h(x)h(y)\pi(y) \sum_{n \geq d_\Gamma^2(x, y)} \frac{1}{V_h(x, \sqrt{n})}.$$

If  $\Gamma$  is uniformly transient, the result follows as  $h \approx 1$  by Lemma 3.3.2.  $\square$

**Remark 3.3.8.** In (3.20), a sum over the neighbors of  $v$  that belong to  $\partial_I \Gamma$  appears. For any  $x, v$  there is always a point  $y_{x,v} \in \partial_I \Gamma$  such that  $d_\Gamma(x, y_{x,v}) + 1 = d_\Gamma(x, v)$ , but there may be multiple points that achieve this or other neighbors of  $v$  that are further away from  $x$  in  $\Gamma$ . In the lower bound, we may keep only the point  $y_{x,v}$ , but, in the upper bound, we do not know a relationship that would allow us to replace a generic  $h(y)$  by  $h(y_{x,v})$ . If  $h \approx 1$ , or if we know all  $y \sim v$  are close in  $\Gamma$  (not just in  $\widehat{\Gamma}$ ), this is not a problem and only one  $y_{x,v}$  counts. However, if  $v$  can be approached from multiple “sides,” this is not the case, and in fact  $h$  may be very different on the different sides. (Consider a slit domain or two sides of a boundary with a “corner.”)

We are, however, always free to replace  $\pi(y)$  by  $\pi(v)$  due to the assumption of controlled weights.

**Corollary 3.3.3.** *Assume  $\Gamma := \widehat{\Gamma} \setminus K$  is an inner uniform subgraph of a Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . Then*

$$\psi'_K(m, x, v) \approx \sum_{y \in \Gamma: y \sim v} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right) \quad \forall x \in \Gamma \setminus \partial_I \Gamma, v \in \partial \Gamma, m \geq d_\Gamma(x, v) \quad (3.21)$$

and for  $n \geq d_\Gamma^2(x, v)$ ,

$$\psi_K(x, v) - \psi_K(n, x, v) \approx \sum_{y \in \Gamma: y \sim v} h(x)h(y)\pi(y) \sum_{m=n}^{\infty} \frac{1}{V_h(x, \sqrt{m})} \quad (3.22)$$

If  $\Gamma$  is uniformly  $S$ -transient, then

$$\psi'_K(m, x, v) \approx \frac{\pi(v)}{V_\Gamma(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, v)}{m}\right), \quad (3.23)$$

and when  $n \geq d_\Gamma^2(x, v)$ ,

$$\psi_K(x, v) - \psi_K(n, x, v) \approx \pi(v) \sum_{m=n}^{\infty} \frac{1}{V_\Gamma(x, \sqrt{m})}. \quad (3.24)$$

*Proof.* Proceeding as in the proof of Theorem 3.3.4, but summing in neither time nor space, for  $x \in \Gamma \setminus \partial_I \Gamma, v \in \partial \Gamma$ ,

$$\begin{aligned} \psi'_K(m, x, v) &= \mathbb{P}^x(X_{\tau_K} = v, \tau_K = m) = \sum_{y \in \Gamma: y \sim v} \mathcal{K}_{\Gamma, D}^{m-1}(x, y) \mathcal{K}_\Gamma(y, v) \\ &\approx \sum_{y \in \Gamma: y \sim v} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right). \end{aligned}$$

Note points  $y$  in the sum above will only appear if  $m \geq d_\Gamma(x, y)$  (there is always at least one such  $y$  since  $m \geq d_\Gamma(x, v)$  by assumption).

To obtain (3.22), use (3.21) to find

$$\begin{aligned}\psi_K(x, v) - \psi_K(n, x, v) &= \sum_{m=n}^{\infty} \psi'_K(m, x, v) \\ &\approx \sum_{y \in \Gamma: y \sim v} h(x)h(y)\pi(y) \sum_{m=n}^{\infty} \frac{1}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right).\end{aligned}$$

When  $n \geq d_\Gamma^2(x, y)$ , the exponential does not count.

In the uniformly  $S$ -transient case, we know  $h \approx 1$ . For the lower bounds, discard any inconvenient terms; for the upper bounds, recall the number of neighbors  $y$  of  $v$  is bounded above and all such neighbors satisfy  $d_\Gamma(x, y) + 1 \geq d_\Gamma(x, v) \geq 2$ .  $\square$

**Corollary 3.3.4.** *Assume  $\Gamma := \widehat{\Gamma} \setminus K$  is an inner uniform subgraph of a Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . For all  $x \in \Gamma \setminus \partial_I \Gamma$ ,  $v \in \partial \Gamma$ ,  $n \geq d_\Gamma(x, v)$ ,*

$$\begin{aligned}\psi_K(n, x, v) &\approx \sum_{y \in \Gamma: y \sim v} \left[ \frac{h(x)h(y)\pi(y)d_\Gamma^2(x, y)}{V_h(x, d_\Gamma(x, y))} \exp\left(-\frac{d_\Gamma^2(x, y)}{n}\right) \right. \\ &\quad \left. + \sum_{m=d_\Gamma(x, y)^2}^n \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \right].\end{aligned}\tag{3.25}$$

*In the uniformly  $S$ -transient case,*

$$\psi_K(n, x, v) \approx \frac{\pi(v)d_\Gamma^2(x, v)}{V_\Gamma(x, d_\Gamma(x, v))} \exp\left(-\frac{d_\Gamma^2(x, v)}{n}\right) + \sum_{m=d_\Gamma(x, v)^2}^n \frac{\pi(v)}{V_\Gamma(x, \sqrt{m})}.\tag{3.26}$$

*Proof.* This quantity is like that of Theorem 3.3.4, except that the sum in time stops at a value  $n$  instead of continuing to infinity. We are forced to consider several cases about the relationship between the size of  $n$  and  $d_\Gamma(x, v)$ . As before, the uniformly  $S$ -transient case will follow by recalling  $h \approx 1$  and that only one  $y \sim v$  counts.

In all cases, using Corollary 3.3.3,

$$\psi_K(n, x, v) = \sum_{m=0}^n \psi'_K(m, x, v) \approx \sum_{\substack{y \in \Gamma: \\ y \sim v}} \sum_{m=d_\Gamma(x,y)}^n \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right).$$

We now compute the inner sum in time  $m$  above. For simplicity, we will often abbreviate  $d_\Gamma(x, y)$  by  $d$  in the rest of the proof.

Case 1: Total time  $n$  is small compared to distance, that is  $d_\Gamma(x, y) \approx n$ ; say  $d_\Gamma(x, y) \leq n \leq 2d_\Gamma(x, y)$ .

Then the inner sum is roughly

$$h(x)h(y)\pi(y) \sum_{m=d}^n \frac{1}{V_h(x, \sqrt{d})} \exp\left(-\frac{d^2}{d}\right) \approx \frac{h(x)h(y)\pi(y)}{V_h(x, d)} \exp(-d).$$

In this situation the exponential is very small, so any powers of  $d$  that appear by taking the sum or adjusting the radius of  $V_h$  can be fed to the exponential by changing the constant. Recall  $V_h$  is doubling (see Theorem 3.3.2).

Case 2: The intermediate case,  $2d_\Gamma(x, y) \leq n < d_\Gamma^2(x, y)$ .

We use a dyadic decomposition and cut the sum into pieces where  $d2^{-l-1} \leq \sqrt{m} \leq d2^{-l}$ . Let  $a$  denote the integer such that  $\sqrt{n} \approx d2^{-a}$ , or  $a \approx \log_2(d/\sqrt{n})$  and  $b$  be the integer such that  $\sqrt{d} \approx d2^{-b}$  or  $b \approx \log_2(\sqrt{d})$ . Since  $d/\sqrt{n} \leq \sqrt{d}$  in this case, we have  $a \leq b$ . Hence using the same tools to compute the sum as above, where

the constants  $C, c$  can change from line to line,

$$\begin{aligned}
\sum_{m=d_{\Gamma}(x,y)}^n \frac{Ch(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_{\Gamma}^2(x,y)}{cm}\right) \\
\leq Ch(x)h(y)\pi(y) \sum_{l=a}^b \sum_{\sqrt{m} \approx d2^{-l}} \frac{1}{V_h(x, d2^{-l})} \exp\left(-\frac{4^l}{c}\right) \\
\leq Ch(x)h(y)\pi(y) \sum_{l=a}^b \frac{d^2}{4^l} \frac{1}{V_h(x, d2^{-l})} \exp\left(-\frac{4^l}{c}\right) \\
\leq \frac{Ch(x)h(y)\pi(y)d^2}{V_h(x, d)} \sum_{l=a}^b \exp\left(-\frac{4^l}{c}\right) \\
\leq \frac{Ch(x)h(y)\pi(y)d^2}{V_h(x, d)} \exp\left(-\frac{4^a}{c}\right).
\end{aligned}$$

The last step follows from bounding the sum from above by  $\sum_{l \geq a}$ , and recalling  $4^a \approx d^2/n$  gives the desired upper bound. For the lower bound, repeat the same series of steps, except in the last step keep only the first term  $l = a$ .

We have found

$$\sum_{m=d_{\Gamma}(x,y)}^n \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_{\Gamma}^2(x,y)}{m}\right) \approx \frac{h(x)h(y)\pi(y)d^2}{V_h(x, d)} \exp\left(-\frac{d^2}{n}\right).$$

Case 3: The case where time is large compared to the distance squared,  $n \geq d_{\Gamma}^2(x, y)$ .

Cut the sum into two pieces: where  $m < d^2$  and where  $m \geq d^2$ . For the first piece, apply the previous case. Here the exponential is large, so we may always ignore it. We find

$$\sum_{m=d_{\Gamma}(x,y)}^n \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_{\Gamma}^2(x,y)}{m}\right) \approx \frac{h(x)h(y)\pi(y)d^2}{V_h(x, d)} + \sum_{m=d^2}^n \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})}.$$

To finish estimating  $\psi_{\kappa}(n, x, \nu)$ , take the sum in points  $y$ . Different points  $y \sim \nu$  may fall into different cases above, but in all cases the expression found matches that of (3.25).  $\square$

**Corollary 3.3.5.** *Assume  $\Gamma := \widehat{\Gamma} \setminus K$  is an inner uniform subgraph of a Harnack graph  $(\widehat{\Gamma}, \mu, \pi)$ . For all  $x \in \Gamma \setminus \partial_I \Gamma$  and all  $m \geq d_\Gamma(x, K)$ ,*

$$\psi'_K(m, x) \approx \sum_{\substack{y \in \partial_I \Gamma \\ d_\Gamma(x, y) \leq m}} \frac{h(x)h(y)\pi(y)}{V_h(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right). \quad (3.27)$$

*If  $\Gamma$  is uniformly  $S$ -transient, then*

$$\psi'_K(m, x) \approx \sum_{\substack{y \in \partial_I \Gamma \\ d_\Gamma(x, y) \leq m}} \frac{\pi(y)}{V_\Gamma(x, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(x, y)}{m}\right). \quad (3.28)$$

The corollary follows from similar arguments as above. There are no particularly nice simplifications for any of these expressions since the sums in space rely on  $h, \pi$ , and  $d_\Gamma(x, y)$ .

Note either of the previous two corollaries could be used to get an estimate on  $\psi_K(n, x)$ .

### 3.3.4 Examples

In this section we apply the results of previous sections to various examples.

Recall we have already seen that  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is uniformly  $S$ -transient when  $k \leq m - 3$ . Further, it is not too difficult to verify that  $\mathbb{Z}^m \setminus \mathbb{Z}^k$  is uniform if and only if  $k \leq m - 2$ , so that the results of the previous section also apply. This example generalizes as follows.

**Example 3.3.1** (Examples with regular volume growth). Let  $\Gamma := \widehat{\Gamma} \setminus K$  be inner uniform inside the Harnack graph  $(\widehat{\Gamma}, \pi, \mu)$ . Assume there exists  $\alpha > 0$  such that  $V_\Gamma(x, r) \approx r^\alpha$  for all  $x \in \Gamma, r > 0$ . Further assume that  $V_{\partial_I \Gamma}$  is doubling in the sense

of Definition 3.2.5 and that  $\partial_I \Gamma$  is regular in the sense that there exists  $\beta > 0$  such that  $V_{\partial_I \Gamma}(y, r) \approx r^\beta$  for all  $y \in \partial_I \Gamma, r > 0$ . Assume  $\alpha - \beta > 2$ .

Then we may use Corollary 3.2.1 to justify that  $\Gamma$  is uniformly  $S$ -transient, in which case  $h \approx 1$  and (3.18) gives us a two-sided bound on  $\psi_K$  as a function of  $x$ :

$$\psi_K(x) \approx \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \frac{V_{\partial_I \Gamma}(y_x, \sqrt{n})}{V_\Gamma(x, \sqrt{n})} \approx \sum_{n \geq d_\Gamma^2(x, \partial_I \Gamma)} \frac{1}{n^{(\alpha-\beta)/2}} \approx \frac{1}{d_\Gamma(x, \partial_I \Gamma)^{\alpha-\beta-2}}.$$

**Example 3.3.2** (Half-space,  $\mathbb{Z}^m \setminus \mathbb{Z}^{m-1}$ ). Consider upper half-space  $\Gamma = \mathbb{Z}_+^m = \{(x_1, \dots, x_m) \in \mathbb{Z}^d : x_m > 0\}$  inside of  $\mathbb{Z}^m$  with the lazy simple random walk (Definition 3.1.12). Let  $\vec{x} = (x_1, \dots, x_m) \in \Gamma$ . We consider the chance we hit  $\vec{v} = (v_1, \dots, v_{m-1}, 0)$  from  $\vec{x}$ . Clearly  $\Gamma$  is inner uniform in  $\mathbb{Z}^m$ , which is Harnack. In this case,  $h(\vec{x}) = x_m$ . Let  $\vec{y}_v := (v_1, \dots, v_{m-1}, 1)$ .

Let  $x = (x_1, \dots, x_{m-1}), v = (v_1, \dots, v_{m-1})$  and  $d(x, v) = |x_1 - v_1| + \dots + |x_{m-1} - v_{m-1}|$ . Applying various corollaries from the previous section (and assuming  $n \geq d_\Gamma^2(\vec{x}, \vec{v})$  where sensible),

$$\begin{aligned} \psi_K(\vec{x}, \vec{v}) &\approx \frac{x_m}{[d(\vec{x}, \vec{y}_v)]^m} = \frac{x_m}{(d(x, v) + |x_m - 1|)^m} \\ \psi'_K(n, \vec{x}, \vec{v}) &\approx \frac{x_m}{(x_m + \sqrt{n})^2 n^{m/2}} \exp\left(-\frac{([d(x, v)]^2 + |x_m|^2)}{n}\right) \\ \psi_K(\vec{x}, \vec{v}) - \psi_K(n, \vec{x}, \vec{v}) &\approx \frac{x_m}{n^{m/2}} \\ \psi_K(n, \vec{x}, \vec{v}) &\approx \frac{x_m}{[d_\Gamma(\vec{x}, \vec{v})]^m} \exp\left(-\frac{d_\Gamma^2(\vec{x}, \vec{v})}{n}\right) + x_m \left[\frac{1}{[d_\Gamma(\vec{x}, \vec{v})]^m} - \frac{1}{n^{m/2}}\right] \\ \psi'_K(n, \vec{x}) &\approx \frac{x_m}{n^{3/2}} \\ \psi_K(\vec{x}) - \psi_K(n, \vec{x}) &\approx \frac{x_m}{n^{1/2}} \\ \psi_K(n, \vec{x}) &\approx x_m \left[\frac{1}{x_m^{1/2}} - \frac{1}{n^{1/2}}\right]. \end{aligned}$$

The above estimate for  $\psi_K(\vec{x}, \vec{v})$  is essentially a (multivariate) Cauchy distribution as expected. This is clearer to see if we take  $m = 2, \vec{x} = (0, 2)$ , and  $\vec{v} = (v, 0)$

so that  $\psi_K(\vec{x}, \vec{v}) \approx \frac{1}{(1+|\vec{v}|)^2} \approx \frac{1}{1+v^2}$ . Further, notice that the rate of convergence of  $\psi_K(\vec{x}, \vec{v})$  in time is dependent on the dimension, but convergence to  $\psi_K(\vec{x}) \equiv 1$  has the same rate in all dimensions.

**Example 3.3.3** (Cones in  $\mathbb{Z}^2$ ). Let  $\widehat{\Gamma} = \mathbb{Z}^2$  and  $\Gamma$  be the lattice points lying inside of a cone of aperture  $\alpha \in (0, 2\pi)$  with vertex at  $(0, 0)$  and one side of the cone lying along the  $x$ -axis. Note this is a case where  $K \neq \partial\Gamma$ . As the cone is inner uniform, the results of the previous section apply.

In the continuous case, it is known that the profile of such a cone is  $h(r, \theta) = r^{\pi/\alpha} \sin(\frac{\pi}{\alpha}(\theta))$ , where  $(r, \theta) \in \mathbb{R}^2$  are polar coordinates. Since a cone can be thought of as the graph above a Lipschitz domain, a result of Varopoulos [64] says harmonic functions in the discrete (lattice) and continuous versions should be similar away from the boundary; for further discussion of harmonic functions in cones see [17] and references therein.

Therefore, assuming  $\vec{x} \in \mathbb{Z}^2$  is away from the boundary of our discrete cone and  $\vec{v} \in \mathbb{Z}^2$  lies along the boundary of the cone, by Corollary 3.3.2,

$$\psi_K(\vec{x}, \vec{v}) \approx \frac{|\vec{x}|^{\pi/\alpha} \sin(\frac{\pi}{\alpha}(\theta_{\vec{x}})) |\vec{y}_{\vec{v}}|^{\pi/\alpha} \sin(\frac{\pi}{\alpha}(\theta_{\vec{y}_{\vec{v}}}))}{[d_{\Gamma}(\vec{x}, \vec{y}_{\vec{v}})]^{2\pi/\alpha}},$$

where  $\vec{y}_{\vec{v}} \sim \vec{v}$  and belongs to  $\Gamma$ . One can verify this result matches that of the half-plane in the previous example ( $\alpha = \pi, m = 2$ ).

We can also express  $\psi_K(\vec{x}, \vec{v})$  in terms of distances to the edges of the cone. Let the edge of the cone that lies along the  $x$ -axis be  $L_0$  and the other edge be  $L_1$ . Then  $|\vec{x}| \approx d(\vec{x}, L_0) + d(\vec{x}, L_1)$  and, for  $\alpha$  fixed,  $\sin(\frac{\pi}{\alpha}(\theta_{\vec{x}})) \approx \frac{d(\vec{x}, L_0)}{|\vec{x}|} \frac{d(\vec{x}, L_1)}{|\vec{x}|}$ . (Note one of these factors is always roughly constant.) Thus

$$\psi_K(\vec{x}, \vec{v}) \approx \frac{[d(\vec{x}, L_0) + d(\vec{x}, L_1)]^{\frac{\pi}{\alpha}-2} [d(\vec{v}, L_0) + d(\vec{v}, L_1)]^{\frac{\pi}{\alpha}-1} d(x, L_0), d(x, L_1)}{[d_{\Gamma}(\vec{x}, \vec{y}_{\vec{v}})]^{2\pi/\alpha}}.$$

**Example 3.3.4** (A line in  $\mathbb{Z}^3$ , e.g.  $\mathbb{Z}^m \setminus \mathbb{Z}^{m-2}$ ). Consider  $\widehat{\Gamma} = \mathbb{Z}^3$  and  $K = \{(0, 0, x_3) : x_3 \in \mathbb{Z}\}$ , the  $x_3$ -axis. The arguments below apply more generally to  $\mathbb{Z}^m \setminus \mathbb{Z}^{m-2}$ . The harmonic profile is the same as the harmonic profile of a single point in  $\mathbb{Z}^2$ , and consequently  $h(x_1, x_2, x_3) \approx \log(|x_1| + |x_2| + 1)$  (see e.g. [58], Section 11). Given  $\vec{x} = (x_1, x_2, x_3) \in \Gamma := \mathbb{Z}^3 \setminus K$ , then  $d_{\vec{x}} := d(\vec{x}, \partial_I \Gamma) = |x_1| + |x_2| - 1$ . We can use Theorem 3.3.4 to check that  $\psi_K(\vec{x}) \approx 1$ :

$$\begin{aligned} \psi_K(\vec{x}) &\approx \sum_{n \geq d_{\vec{x}}^2} \sum_{\substack{\vec{y} \in \partial_I \Gamma, \\ d_{\Gamma}^2(\vec{x}, \vec{y}) \leq n}} \frac{h(\vec{x})h(\vec{y})}{h(\vec{x}_{\sqrt{n}})h(\vec{y}_{\sqrt{n}})} \frac{\pi(\vec{y})}{V_{\Gamma}(\vec{x}, \sqrt{n})} \\ &\approx \sum_{n \geq d_{\vec{x}}^2} \sum_{\vec{y}: d_{\Gamma}^2(\vec{x}, \vec{y}) \leq n} \frac{\log(d_{\vec{x}})}{\log(d_{\vec{x}} + \sqrt{n}) \log(\sqrt{n}) n^{3/2}} \\ &\approx \sum_{n \geq d_{\vec{x}}^2} \frac{\log(d_{\vec{x}})}{(\log(n))^2 n} \approx \frac{\log(d_{\vec{x}})}{\log(d_{\vec{x}})} = 1. \end{aligned}$$

The above calculation used that the number of  $\vec{y}$ 's in the  $x_3$ -axis at distance less than  $\sqrt{n}$  from  $\vec{x}$  is about  $\sqrt{n} - d_{\vec{x}} \approx \sqrt{n}$ . This is sensible if we replace the exterior sum  $n \geq d_{\vec{x}}^2$  by  $n \geq cd_{\vec{x}}^2$ ; for the lower bound, we can throw this away, and, in the upper bound, the sum over  $d_{\vec{x}}^2 \leq n \leq cd_{\vec{x}}^2$  is can be controlled by later pieces of the sum. This might seem simple, but the fact that we can make manipulations like this in our calculations relies on the fact that in this case our boundary is doubling. (See Remark 3.3.9 below.)

It is more interesting to compute  $\psi_K(\vec{x}, \vec{v})$  where  $\vec{v} = (0, 0, v) \in K$ . Then via Corollary 3.3.2:

$$\begin{aligned} \psi_K(\vec{x}, \vec{v}) &\approx \sum_{\vec{y} \in \Gamma: \vec{y} \sim \vec{v}} h(\vec{x})h(\vec{y})\pi(w) \sum_{n \geq d_{\Gamma}^2(\vec{x}, \vec{y})} \frac{1}{V_h(\vec{x}, \sqrt{n})} \approx \log(d_{\vec{x}}) \sum_{n \geq d_{\Gamma}^2(\vec{x}, \vec{y})} \frac{1}{(\log(n))^2 n^{3/2}} \\ &\approx \frac{\log(d(\vec{x}, \vec{v}_{\vec{x}}))}{(d_{\Gamma}(\vec{x}, \vec{v}_{\vec{x}}) + d_{\Gamma}(\vec{v}_{\vec{x}}, \vec{v}))(\log(d_{\Gamma}(\vec{x}, \vec{v}_{\vec{x}}) + d_{\Gamma}(\vec{v}_{\vec{x}}, \vec{v}))^2)}. \end{aligned}$$

**Remark 3.3.9.** Given  $x \in \Gamma$ ,  $y \in \partial_I \Gamma$ , it is always true that  $d_{\Gamma}(x, y) \approx d_{\Gamma}(x, y_x) + d_{\Gamma}(y_x, y)$ , where  $y_x \in \partial_I \Gamma$  achieves  $d_{\Gamma}(x, \partial_I \Gamma)$  (and that  $y_x \sim v_x \in \partial \Gamma$  that achieves

$d(x, \partial\Gamma)$ ). Provided changing the “radius” by a constant does not really change how many points  $y \in \partial_I\Gamma$  are at a particular distance from  $y_x \in \partial_I\Gamma$ , then when  $n$  is sufficiently large, the inner sums in our theorems/corollaries can be taken over  $y \in \partial_I\Gamma : d^2(y, y_x) \leq n$ . This remark is similar in spirit to Corollary 3.2.1; Example 3.3.6 below gives an example where such assumptions do not hold.

**Example 3.3.5** (Weighted half-spaces). This example is a continuation of Example 3.2.6. Once again we consider  $\Gamma = \{\vec{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m : x_m > 0\}$  inside  $\mathbb{Z}_{\geq 0}^m$  with weight  $(1 + x_m)^\alpha$ . Provided  $\alpha > -m$ , then  $\mathbb{Z}_{\geq 0}^m$  with this weight is Harnack, which can be shown using similar arguments to those given in Section 4.3 of [36]. The profile for such a space clearly only depends on the  $x_m$  coordinate and reduces to computing the profile on the weighted half-line. Using the definition of harmonic and choosing the scaling by setting  $h(x_1, \dots, x_{m-1}, 1) = 1$ , we can compute

$$h(x_1, \dots, x_m) = \begin{cases} \sum_{l=1}^{x_m} \frac{1}{l^\alpha}, & \alpha \geq 0 \\ \sum_{n=2}^{x_m+1} \frac{2^\alpha}{n^\alpha}, & \alpha \in (-N, 0) \end{cases} \approx x_m^{1-\alpha}.$$

If  $\alpha > 1$ , then  $1-\alpha < 0$  and it is clear  $h$  is uniformly bounded above and below. In Example 3.2.6, we already saw that  $\mathbb{Z}_+^m$  was uniformly  $S$ -transient with such weights. Using Theorem 3.3.4 gives us a lower bound that matches the upper bound found in Example 3.2.6, and we can also find  $\psi_K(\vec{x}, \vec{v})$ :

$$\psi_K(\vec{x}) \approx \frac{1}{x_m^{\alpha-1}}$$

$$\psi_K(\vec{x}, \vec{v}) \approx \frac{1}{[d(\vec{x}, \vec{v})]^{m+\alpha-2}} \approx \frac{1}{[|x_1 - v_1| + \dots + |x_{m-1} - v_{m-1}| + |x_m|]^{m+\alpha-2}}.$$

Now consider  $\alpha \in (-N, 1]$ . Using Theorem 3.3.4, we find that  $\psi_K(\vec{x})$  is roughly

constant, and we can compute  $\psi_K(\vec{x}, \vec{v})$  :

$$\begin{aligned}\psi_K(\vec{x}) &\approx \sum_{n \geq d^2(\vec{x}, \partial\Gamma)} \frac{x_m^{1-\alpha} n^{(m-1)/2}}{n^{1-\alpha} n^{(m+\alpha)/2}} \approx \sum_{n \geq x_m^2} \frac{x_m^{1-\alpha}}{n^{(3-\alpha)/2}} \approx \frac{x_m^{1-\alpha}}{x_m^{1-\alpha}} \approx 1 \\ \psi_K(\vec{x}, \vec{v}) &\approx \frac{x_m^{1-\alpha}}{[d(\vec{x}, \vec{v})]^{m-\alpha}} = \frac{x_m^{1-\alpha}}{[|x_1 - v_1| + \cdots + |x_{m-1} - v_{m-1}| + |x_m|]^{m-\alpha}}.\end{aligned}$$

Substituting  $\alpha = 0$  into the expression for  $\psi_K(\vec{x}, \vec{v})$  above, we recover the formula from Example 3.3.2.

In general knowing  $\psi_K(\vec{x}) \approx 1$  is not sufficient to conclude a subgraph is  $S$ -recurrent, as this does not necessarily imply  $\psi_K(\vec{x}) = 1$ . However, in this specific case, we can use symmetry to argue that the half-space cannot be  $S$ -recurrent and have  $\psi_K(\vec{x})$  uniformly bounded below away from zero. First note that clearly  $\psi_K(\vec{x})$  only depends on  $x_m = d(\vec{x}, \partial\Gamma)$ . Also, by using repeated applications of the Markov property, if  $x_m = d$ , then in order for the random walk to hit the set  $\{x_m = 0\}$ , it must first hit the set  $\{x_m = d - 1\}$ , then the set  $\{x_m = d - 2\}$ , and so on, so the probability of hitting  $\{x_m = 0\}$  decomposes into a product of probabilities of hitting a set that is distance 1 away from the starting point. Although the weights are different if we consider hitting  $\{x_m = 0\}$  from a point where  $x_m = 1$  in the usual half-space versus hitting  $\{x_m = k\}$  from a point where  $x_m = k + 1$  in the half-space  $\{x_m \geq k\}$ , the weights will be uniformly comparable. Since  $\psi_K$  is the chance of hitting  $K$  before time  $\infty$ , a bounded change of weights will not change it. Hence if  $\psi_K(\vec{x}) < 1$  everywhere, there must be points where  $\psi_K$  is arbitrarily close to zero. Hence knowing  $\psi_K(\vec{x}) \approx 1$  shows that these weighted half-spaces are in fact  $S$ -recurrent.

**Example 3.3.6** (“Flyswatter”). In  $\mathbb{Z}^4$ , consider  $K$  to be a two-dimensional infinite “flyswatter” as in the Figure 3.2 below. A key point is that the flyswatter has long “handles” and “mesh parts” at every scale; this causes  $K = \partial\Gamma$  to fail to

be doubling in  $\mathbb{Z}^4$ . However,  $\Gamma = \mathbb{Z}^4 \setminus K$  is uniform as one can always use the extra two dimensions to move away from the flyswatter, and  $d_\Gamma \approx d_{\mathbb{Z}^4}$  since the flyswatter is either thin or has frequent holes. While Theorem 3.3.4 and associated corollaries apply to this example, we do not know how to compute  $h$ . This situation is typical.

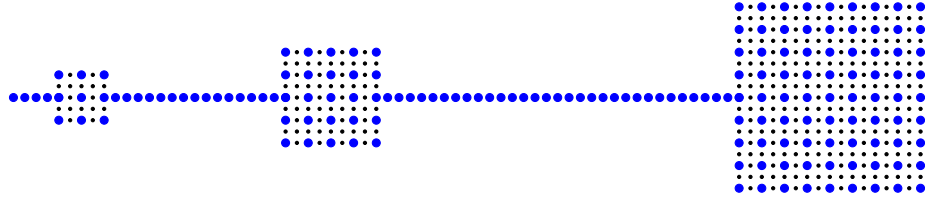


Figure 3.2: The blue “flyswatter,” which we imagine continues infinitely in both directions in a similar manner. Although this picture is in two dimensions, we think of this in a higher dimensional space. Note how there are black points in-between the blue points, and it is easy to see distance in  $\mathbb{Z}^d$  would not be changed significantly by avoiding the blue points when  $d \geq 4$ .

### 3.3.5 Example: A set that is $S$ -transient but not uniformly so

In this section, we discuss an example that turns out to be  $S$ -transient, but not uniformly so, illustrating the distinction between these notions. We apply both Theorems 3.2.1 and 3.3.4 and discuss what we can say about its harmonic profile  $h$ .

Let  $\widehat{\Gamma} = \mathbb{Z}^4$ . Think of  $\mathbf{x} \in \mathbb{Z}^4$  as  $\mathbf{x} = (x_1, x_2, x_3, x_4)$ . In the  $x_1x_2$ -plane, let  $K = \partial\Gamma$  be the set of lattice points that lie inside the graph of  $x_2 = \pm x_1^\alpha$  for  $\alpha \in (0, 1)$ ,  $x_1 \in \mathbb{Z}_{\geq 0}$ . In the case  $\alpha = 1/2$ , we have a parabola whose axis of symmetry is the  $x_1$ -axis; we may often refer to the points of  $K$  as a “parabola” regardless of the value of  $\alpha$  (or the fact that we are only considering a discrete analog of a parabola). Note

that  $K$  is a two-dimensional object in four-dimensional space, so  $\Gamma := \mathbb{Z}^4 \setminus K$  is inner uniform.

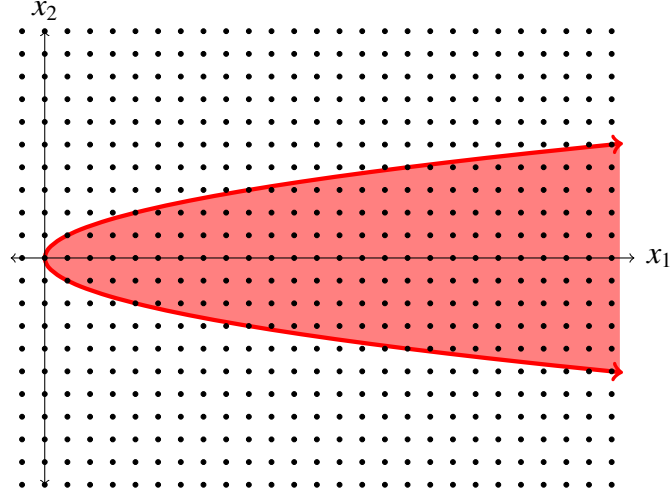


Figure 3.3: For  $\alpha = 1/2$ , we take the lattice points inside of the parabola  $x_1 = x_2^2$  as our set  $K$ . This figure is the  $x_1x_2$ -plane that lives inside  $\mathbb{Z}^4$ .

On  $\mathbb{Z}^4$  consider the lazy simple random walk, which stays in place at each step of the Markov kernel with probability  $1/2$  and otherwise moves to any of the eight neighboring vertices with equal probability. Then  $\mathbb{Z}^4$  has controlled weights and is Harnack. Hence we can apply any of our results to this example.

We first use Theorem 3.2.1 to show that  $\mathbb{Z}^4$  is  $S$ -transient with respect to  $K$ . Doubling of traces of balls in  $\partial\Gamma$  can be seen by the following formula for  $V_{\partial\Gamma}$  :

$$V_{\partial\Gamma}(\mathbf{x}, r) \approx \begin{cases} r^2, & r \leq |x_1|^\alpha \\ |x_1|^\alpha r, & |x_1|^\alpha < r < |x_1| \\ r^{\alpha+1}, & r \geq |x_1| \end{cases} \quad \text{for } \mathbf{x} = (x_1, x_2, 0, 0) \in K.$$

For any point  $\mathbf{x} \in \widehat{\Gamma}$ , we have  $V_{\widehat{\Gamma}}(\mathbf{x}, r) \approx r^4$ . For any  $\mathbf{x} \in \Gamma := \mathbb{Z}^4 \setminus K$ , let  $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$  denote the unique point in  $K$  that achieves  $d(\mathbf{x}, K)$ . Thus for

any  $r > 0$  and any  $\mathbf{x} \in \Gamma$ ,

$$\widetilde{W}(\mathbf{x}, r) \approx \begin{cases} r^2, & r \leq |x_1^*|^\alpha \\ \frac{r^3}{|x_1^*|^\alpha}, & |x_1^*|^\alpha < r < |x_1^*| \\ r^{3-\alpha}, & r \geq |x_1^*|. \end{cases}$$

**Lemma 3.3.3.** *The graph  $\mathbb{Z}^4$  is  $S$ -transient with respect to the parabola  $K$ . Moreover, for a given  $\varepsilon > 0$ , we can pick  $L_\varepsilon = L$  sufficiently large so that in the regime where  $d_{\mathbf{x}} := d_{\mathbb{Z}^4}(\mathbf{x}, K) \geq |x_1^*| \geq L$ , we have  $\psi_K(\mathbf{x}) \leq 1 - \varepsilon$ .*

*Proof.* Recall  $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$  is the point that achieves  $d_{\mathbb{Z}^4}(\mathbf{x}, K)$ . When  $d_{\mathbf{x}} \geq |x_1^*|$ ,

$$\psi_K(\mathbf{x}) \leq \sum_{n \geq d_{\mathbf{x}}^2} \frac{1}{\widetilde{W}(\mathbf{x}, \sqrt{n})} \approx \sum_{n \geq d_{\mathbf{x}}^2} \frac{1}{n^{(3-\alpha)/2}} \approx -\frac{1}{t^{(1-\alpha)/2}} \Big|_{t=d_{\mathbf{x}}^2}^{\infty} \approx \frac{1}{d_{\mathbf{x}}^{1-\alpha}} \leq \frac{1}{L^{1-\alpha}}.$$

Thus  $\psi_K(\mathbf{x}) < 1 - \varepsilon$  for  $L_\varepsilon$  sufficiently large.  $\square$

A key component of the proof of the above lemma was the assumption that  $d_{\mathbf{x}} \geq |x_1^*|$ . If instead  $|x_1^*|^\alpha \leq d_{\mathbf{x}} \leq |x_1^*|$ , then we have the bound

$$\psi_K(\mathbf{x}) \leq \sum_{n \geq d_{\mathbf{x}}^2} \frac{1}{\widetilde{W}(\mathbf{x}, \sqrt{n})} = \sum_{n=d_{\mathbf{x}}^2}^{|x_1^*|^2} \frac{|x_1^*|^\alpha}{n^{3/2}} + \sum_{n \geq |x_1^*|^2} \frac{1}{n^{(3-\alpha)/2}} = \frac{c_a |x_1^*|^\alpha}{d_{\mathbf{x}}} - \frac{c_a}{|x_1^*|^{1-\alpha}} + \frac{c_b}{|x_1^*|^{1-\alpha}}, \quad (3.29)$$

where the constants  $c_a, c_b$  depend on the approximation of  $\widetilde{W}$  and on the estimation of the sums above. We only write these constants to emphasize that the  $|x_1^*|^{\alpha-1}$  terms do not cancel. If instead  $d_{\mathbf{x}} < |x_1^*|^\alpha$ , then there is a third term appearing in the estimate for  $\psi_K$  given by Theorem 3.2.1/Corollary 3.2.1.

Lemma 3.3.3 does not show that  $\mathbb{Z}^4$  is uniformly  $S$ -transient with respect to the parabola  $K$  since  $d_{\mathbf{x}}$  and  $|x_1^*|$  are related. Indeed, it is possible to pick a sequence of points  $\{\mathbf{x}^m\}_{m \geq 0}$  such that  $d_{\mathbf{x}^m} \rightarrow \infty$ , but the bound in Theorem 3.2.1 does

not give useful information. To that end, consider points  $\mathbf{x}^m = (x_1^m, x_2^m, x_3^m, x_4^m)$  that lie directly above the parabola so that  $d_{\mathbf{x}^m} \approx x_3^m + x_4^m$ , which is independent of  $x_1^m = (x_1^m)^*$ . Further, take  $d_{\mathbf{x}^m} = |(x_1^m)^*|^\alpha$  for all  $m$ . Then we are in the situation of (3.29) so that

$$\psi_K(\mathbf{x}^m) \leq \frac{c_a |(x_1^m)^*|^\alpha}{d_{\mathbf{x}^m}} - \frac{c_a}{|(x_1^m)^*|^{1-\alpha}} + \frac{c_b}{|(x_1^m)^*|^{1-\alpha}} = c_a - \frac{c_a}{|(x_1^m)^*|^{1-\alpha}} + \frac{c_b}{|(x_1^m)^*|^{1-\alpha}}.$$

Thus  $\psi_K(\mathbf{x}^m) \rightarrow c_a$  as  $d_{\mathbf{x}^m} = |(x_1^m)^*|^\alpha \rightarrow \infty$ . From this, we cannot conclude that  $\psi_K(\mathbf{x}^m)$  tends to zero as  $d_{\mathbf{x}^m} \rightarrow \infty$ , and, if  $c_a \geq 1$ , this tells us no information on  $\psi_K$  at all. Indeed, the appearance of the constant  $c_a$  (essentially “1”) in the computation of the above sum indicates that Theorem 3.2.1 will not give a useful bound in this regime.

From Lemma 3.3.3, we know that  $\Gamma$  is  $S$ -transient and that  $h \approx 1$  in the region where  $d(\mathbf{x}, K) \gg |x_1^*|$ , since  $\psi_K(\mathbf{x}) \leq 1 - \varepsilon$  in this region. The two lemmas below capture how the results of Section 3.3 can improve our knowledge of  $h$  as we approach the parabola in certain ways.

**Lemma 3.3.4.** *For any  $\mathbf{x} \in \Gamma$  satisfying  $d_{\mathbf{x}} \gg |x_1^*|^\alpha \geq L$ , we have  $h(\mathbf{x}) \approx 1$ .*

**Lemma 3.3.5.** *Let  $\mathbf{u}^* = (u_1, 0, 0, 0) \in K$  and  $B = B_\Gamma(\mathbf{u}^*, \frac{1}{2}|u_1|^\alpha)$ . Then there exists a constant  $0 < a < 1$  such that*

$$h(\mathbf{x}) \approx c \frac{\log(d_{\mathbf{x}})}{\log(\hat{c}|u_1|^\alpha)} \quad \forall \mathbf{x} \in B_\Gamma(\mathbf{u}^*, a|u_1|^\alpha).$$

*Proof of Lemma 3.3.4.* We already know this result for  $\mathbf{x} \in \Gamma$  satisfying  $d_{\mathbf{x}} \gg |x_1^*|$  due to Lemmas 3.3.3 and 3.3.2. Therefore it suffices to consider  $\mathbf{x} \in \Gamma$  such that  $d_{\mathbf{x}} \approx \hat{c}|x_1^*|^\alpha$  for some constant  $\hat{c}$ . In this region, by (3.29) and Lemma 3.3.2,

$$h(\mathbf{x}) = 1 - \psi_K(u) \geq 1 + \frac{c_a - c_b}{|x_1^*|^{1-\alpha}} - c_a \frac{|x_1^*|^\alpha}{d_{\mathbf{x}}}.$$

If  $c_a > c_b$  we can ignore the middle term; otherwise, assume  $|x_1^*|^\alpha \geq L$  where  $L$  is large enough to ensure  $(c_a - c_b)/|x_1^*|^{1-\alpha} \geq -1/4$ . Also choose  $\hat{c}$  so that  $c_a/\hat{c} < 1/4$ . With these choices, for  $\mathbf{x}$  satisfying  $d_{\mathbf{x}} \approx |x_1^*|^\alpha \geq L$ ,

$$h(\mathbf{x}) \geq \frac{3}{4} - \frac{c_a|x_1^*|^\alpha}{\hat{c}|x_1^*|^\alpha} \geq \frac{1}{2}.$$

Thus  $h \approx 1$  whenever  $d_{\mathbf{x}} \gg |x_1^*|^\alpha \geq L$  as desired.  $\square$

*Proof of Lemma 3.3.5.* As this result is about points near the parabola  $K$ , we use the boundary Harnack inequality. Given  $\mathbf{u}^* = (u_1, 0, 0, 0) \in K$ , take  $\mathbf{u} = (u_1, 0, u_3, u_4)$  such that  $d_{\mathbf{u}} \approx |u_1|^\alpha$  and  $h(\mathbf{u}) \approx 1$  as in Lemma 3.3.4. Note  $\mathbf{u}^*$  is the projection of  $\mathbf{u}$  onto  $K$ .

As  $\widehat{\Gamma}$  is Harnack and  $h$  is harmonic inside  $\Gamma$ , by applying the elliptic Harnack inequality a finite number of times, we find a point (which we continue to call  $\mathbf{u}$ ) such that  $h(\mathbf{u}) \approx 1$  and  $\mathbf{u}$  lies in  $B = B_{\widehat{\Gamma}}(\mathbf{u}^*, \frac{1}{2}|u_1|^\alpha)$ .

From the perspective of  $B$ , we cannot tell that  $K$  is not the entire  $x_1x_2$ -plane. As in  $B$  we are looking at a two-dimensional ball inside of four-dimensional space, we know there is a positive harmonic function  $f$  in  $B$  that is zero on the intersection of  $B$  with  $K$  such that  $f(\mathbf{x}) \approx \log(|x_3|^2 + |x_4|^2) \approx \log(d(\mathbf{x}, K)^2)$ .

Therefore, by the boundary Harnack inequality (Theorem 3.3.3),

$$\frac{f(\mathbf{x})}{f(\mathbf{u})} \leq A_1 \frac{h(\mathbf{u})}{h(\mathbf{x})} \implies c \frac{\log(d_{\mathbf{x}}^2)}{\log(d_{\mathbf{u}}^2)} = c \frac{\log(d_{\mathbf{x}})}{\log(\hat{c}|u_1|^\alpha)} \leq h(\mathbf{x}) \quad \forall \mathbf{x} \in B(\mathbf{u}^*, \frac{1}{2A_0}|u_1|^\alpha).$$

As we may also apply boundary Harnack in the other direction, we conclude

$$h(\mathbf{x}) \approx c \frac{\log(d_{\mathbf{x}})}{\log(\hat{c}|u_1|^\alpha)}$$

on a ball of radius strictly smaller than that of  $B$  (but comparable to  $|u_1|^\alpha$ ).  $\square$

The two lemmas above give a wide region where we understand  $h$ . However, there is still a “bad region” of points where the behavior of  $h$  remains unknown. For any  $\mathbf{x} \in \Gamma$ , recall  $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*)$  is its projection on to  $K$ . The behavior of  $h$  is not known for points  $\mathbf{x} \in \Gamma$  where  $d_{\mathbf{x}} \ll |x_1^*|^\alpha$  and  $|x_2^*| \gg |x_1^*|^\alpha$ , that is for points that neither Lemma 3.3.4 nor Lemma 3.3.5 apply to. (Lemma 3.3.5 can be applied to points near the parabola but that do not get too close to its “edge.” Repeated applications of boundary Harnack could get similar results to hold in balls with centers of the form  $\mathbf{u}^* = (u_1, u_2, 0, 0)$  as long as  $u_2$  is sufficiently small compared to  $u_1$ .) These bad points lie in a tube around the parabola of radius comparable to  $|x_1^*|^\alpha$ .

Lemma 3.3.5 shows that  $h \neq 1$  for points close to the middle of the parabola, so along with Lemma 3.3.2, this shows  $\mathbb{Z}^4$  is not uniformly  $S$ -transient with respect to the parabola.

### 3.4 Connections with Wiener’s test

In many situations, Wiener’s test gives an optimal way for determining classical transience/recurrence of a set  $S \subset \Gamma$ , where transience is taken to mean  $\mathbb{P}^x(X_n \in S \text{ i.o.}) = 0$  and recurrence means  $\mathbb{P}^x(X_n \in S \text{ i.o.}) > 0$ . In many cases (such as for the simple random walk on  $\mathbb{Z}^d$ ), a 0 – 1 law holds for these probabilities, but such a 0 – 1 law does not hold in the general setting considered in this thesis.

Below we give the version of Wiener’s test in the case of interest to us. See, for example, [5, 45, 53, 63] for statements of Wiener’s test in various settings.

**Theorem 3.4.1** (Wiener’s test for Harnack Graphs). *Assume  $(\Gamma, \mathcal{K}, \pi)$  is a Harnack graph with controlled weights. Let  $(X_n)_{n \geq 0}$  denote the process on the graph.*

Assume that  $\Gamma$  is transient in the sense that  $\mathbb{P}^x(X_n = x \text{ i.o.}) = 0$  for some/all  $x \in \Gamma$ .

Fix  $o \in \Gamma$  and let  $A_k := B_\Gamma(o, a^{k+1}) \setminus B_\Gamma(o, a^k)$  for some constant  $a$ .

Then there exists  $a > 1$  such that for any set  $S \subset \Gamma$ ,

$$\mathbb{P}^o(X_n \in S \text{ i.o.}) = 0 \iff \sum_{k=1}^{\infty} \frac{\text{Cap}(S \cap A_k)}{\text{Cap}(A_k)} < +\infty. \quad (3.30)$$

Here  $\text{Cap}$  denotes the capacity, defined as

$$\text{Cap}(S) = \sum_{y \in S} e_S(y),$$

where

$$e_S(y) = \begin{cases} \mathbb{P}^y(\forall n \geq 1, X_n \notin S), & y \in S \\ 0, & y \notin S \end{cases}$$

is the equilibrium potential of  $S$ .

Further, if  $V$  denotes the volume function on  $\Gamma$ , and  $y \in A_k$  such that  $d(y, \partial A_k) \approx a^k$ ,

then

$$\text{Cap}(A_k) \approx \left[ \sum_{n=a^{2k}}^{\infty} \frac{1}{V(y, \sqrt{n})} \right]^{-1}. \quad (3.31)$$

This theorem follows by repeating the proof of Theorem 7.23 in [5] with a few modifications to account for the different form of assumed heat kernel bounds on  $\Gamma$  here.

There are key differences between Wiener's test and the questions we addressed in the main part of this chapter. First, the definitions of transience used do not align. In this chapter, we defined transience as  $\psi_K(x) < 1$  for some/all  $x \in \Gamma := \widehat{\Gamma} \setminus K$ . Wiener's test takes transience to be  $\mathbb{P}^x(X_n \in K \text{ i.o.}) = 0$  for all  $x \in \widehat{\Gamma}$ . These may not be the same, and Wiener's test does not account for uniform  $S$ -transience (see Example 3.4.1 below), which is of much interest to us.

Further, Wiener's test does not care about where the walk is started. However, we are only interested in starting the walk outside of the set  $K$ . There may be cases where the random walk started well inside of  $K$  is unlikely to ever leave  $K$ , but a random walk started outside of  $K$  may have a positive chance to never visit  $K$ .

**Example 3.4.1** (Applying Wiener's test to the parabola example). Let  $K$  be the "parabola" inside a lattice  $\mathbb{Z}^4$  as in Section 3.3.5.

In the case of the lattice  $\mathbb{Z}^d$ , we can take  $a = 2$ . We do this here to emphasize Theorem 3.4.1 is a generalization of the classical formulation of Wiener's test for  $\mathbb{Z}^d$ .

First, by (3.31), we have

$$\text{Cap}(A_k) \approx \left[ \sum_{n=2^{2k}}^{\infty} \frac{1}{V(y, \sqrt{n})} \right]^{-1} \approx \left[ \sum_{n=2^{2k}}^{\infty} \frac{1}{n^2} \right]^{-1} \approx 2^{2k}.$$

The intersection of the parabola and the (4-dimensional) annulus,  $K \cap A_k$ , is contained inside a two-dimensional rectangle  $R_k$  of length approximately  $2^k$  and width approximately  $2^{\alpha k}$ , where  $\alpha$  determines the shape of the parabola, i.e.  $\alpha = 1/2$  for an actual parabola. Since in  $\mathbb{Z}^4$  the capacity of a point is a positive constant, if  $|R_k|$  denotes the number of points in  $R_k$ , then

$$\text{Cap}(S \cap A_k) \leq \text{Cap}(R_k) \leq c|R_k| \leq c2^{k+k\alpha}.$$

Therefore

$$\sum_{k=0}^{\infty} \frac{\text{Cap}(S \cap Q_k)}{2^{2k}} \leq c \sum_{k=0}^{\infty} \frac{2^{k+k\alpha}}{2^{2k}} = c \sum_{k=0}^{\infty} \frac{1}{2^{(1-\alpha)k}} < \infty \quad \text{since } \alpha \in (0, 1).$$

Therefore  $\mathbb{Z}^4 \setminus K$  is transient in the sense of Wiener's test and is  $S$ -transient,

but it is not uniformly  $S$ -transient. This shows that Wiener's test is not sufficient for our purposes.

## CHAPTER 4

### HEAT KERNEL ESTIMATES ON BOOK-LIKE GRAPHS

In this chapter, we return to the setting of a gluing problem. In particular, the goal of this chapter is to obtain heat kernel estimates for a certain class of graphs which can be thought of as nice pieces (“pages”) glued together in a sufficiently nice way via a gluing “spine”. The results we obtain can handle the case of gluing pieces satisfying the parabolic Harnack inequality via a finite set of vertices (the discrete version of some of the results of Grigor’yan and Saloff-Coste in [37]), as well as gluing such graphs via an infinite set of vertices—under a specific set of hypotheses. The only existing work in a similar vein to this latter situation is work of Grigor’yan and Ishiwata [39], which considers gluing copies of  $\mathbb{R}^n$  via a paraboloid of revolution. While the hypotheses we make about the gluing set of vertices in this chapter are fairly restrictive, they are different in flavor from those of [39], and our work does have the advantage of not needing precise symmetry. For example, our results which apply to gluing lattices  $\mathbb{Z}^d$  equally apply to gluing graphs that are lattice-like (say quasi-isometric to lattices).

The main motivation for considering the discrete case at the moment, as opposed to its continuous analog, is the results of Chapter 3. A key idea from that chapter was that in order to hit a set at time  $n$ , the random walk must be at a neighbor of that set at time  $n - 1$ , which leads to a nice decomposition of the hitting probability. While these results should have their appropriate analogs in the continuous setting, the proof will necessarily require a somewhat different strategy. It should also be emphasized that, in general, having heat kernel estimates in the discrete setting does not directly imply the same result in the

corresponding continuous setting, and vice versa. Such results are true in certain cases, for instance, in the case of spaces satisfying the parabolic Harnack inequality (see e.g. [13]), because in this case heat kernel estimates are related to functional inequalities that are stable under perturbations. For the sort of estimates we obtain here, that is not the case. Nonetheless, we expect the results to be the same and hope this chapter can provide a guide for how to treat the continuous case.

The rest of this chapter proceeds as follows. Section 4.1 describes the particular construction of cutting/gluing graphs we consider here, as well as the key hypotheses we will assume. Section 4.2 describes the general framework for estimating the heat kernel on the kind of graph we consider using gluing formulas from Appendix A. Terms appearing in these gluing formulas can be estimated by Chapter 3 and by results from Appendix B. Section 4.3 then applies this method to obtain general heat kernel estimates for the kind of book-like graph we consider here, with the main result being Theorem 4.3.1. The last three sections of this chapter apply Theorem 4.3.1 to several examples where we can get more concrete estimates. Section 4.4 addresses the case of a finite gluing set, and Corollary 4.4.1 is the discrete version of some continuous setting results of [37]. Section 4.5 addresses the main example that motivated this chapter: gluing lattices of varying dimensions along a smaller dimensional lattice. The heat kernel estimate in this case is given in Corollary 4.5.1; this discrete setting estimate is related to some continuous setting estimates found in [37, 39]. Our results also hold in the case where the graphs in question are not precisely lattices, a case which the corresponding continuous results do not handle. Section 4.6 gives additional examples of gluing lattices along a half-space or along a cone.

## 4.1 Set-up: Cutting and gluing graphs, book-like graphs

In this section, we describe the class of graphs we consider in the rest of this chapter, both in terms of their construction and various hypotheses we wish to assume. We explain the construction from the perspective of both cutting and gluing. In general, we assume the notation and general set-up of Chapter 3, that is, we consider connected graphs with random walk structure given by  $(\Gamma, \pi, \mu)$  or  $(\Gamma, \mathcal{K}, \pi)$  that have controlled weights and are uniformly lazy. All definitions of terms such as Harnack graphs, (inner) uniform set, harmonic functions, etc., remain as given in Chapter 3.

### 4.1.1 Gluing graphs together

Start with graphs  $(\Gamma_i, \pi_i, \mu^i)$  for  $i = 1, \dots, l$ . These graphs will be referred to as “pages” (occasionally “pieces”) and play the role of “ends” in the manifold with ends setting.

Further assume we have a graph  $(\Gamma_0, \pi_0, \mu^0)$ . We will refer to  $\Gamma_0$  as the “spine” or “gluing set”. In the language of the manifolds with ends setting, the spine plays the role of the central compact set. We will allow for  $\Gamma_0$  to be disconnected. We will require that weights on  $\Gamma_0$  be adapted, subordinate, controlled, and uniformly lazy (see Sections 3.1.1 and 3.1.2).

We glue together the pages  $\Gamma_i$ ,  $1 \leq i \leq l$  along the spine  $\Gamma_0$  as follows. Assume each  $\Gamma_i$  comes with a marked set of vertices (a “margin”); label this set of vertices  $G_i$ . Further, assume that for each  $i \in \{1, \dots, l\}$  the spine  $\Gamma_0$  contains a set of vertices  $G'_i$  that is a copy of  $G_i$ . (Note the  $G'_i$  need not be disjoint.) For

each  $i$ , we assume there is an identification map (bijection) between  $G_i$  and  $G'_i$ . Moreover, this identification map should satisfy the condition that there exist constants  $c_I, C_I$  such that if  $x \in G_i, x' \in G'_i$  are identified, then

$$c_I \pi_0(x') \leq \pi_i(x) \leq C_I \pi_0(x'). \quad (4.1)$$

In other words, the weights at vertices that we glue together should be comparable, with uniform constants that apply to the whole graph. Due to the controlled weights condition, this also means the weights on any edges we glue together are comparable. We glue the pages to the spine according to these identification maps and call the resulting graph  $\Gamma$ .

**Remark 4.1.1.**

- (1) The identification maps above are between vertices. We could think of these identification maps as literally “identifying” vertices as being the same, or we could think of them as telling us to put an edge between the vertices that are identified. Either way should make no real difference. However, for the sake of being consistent, here we will think of the identification as literally identifying the vertices, which ensures the following makes sense.
- (2) **In the instance that a vertex  $x \in \Gamma_0$  has no neighbors (in  $\Gamma_0$ ), it may be convenient on occasion to allow for  $\pi_0(x) = 0$ .** In terms of the Markov kernel, nothing changes; if  $\pi_0(x) = 0$ , then  $\mathcal{K}(x, y) = 0$  for all  $y \neq x$  and  $\mathcal{K}(x, x) = 1$ . In this setting, obviously (4.1) cannot hold, and the appropriate condition should be instead that there exist (uniform) constants  $c_I, C_I$  such that for all  $x_i \in \Gamma_i$  and  $x_j \in \Gamma_j$  where there exists  $x' \in \Gamma_0$  such that both  $x_i, x_j$  are identified with  $x'$ , we have

$$c_I \pi_j(x_j) \leq \pi_i(x_i) \leq C_I \pi_j(x_j). \quad (4.2)$$

The random walk structure on  $\Gamma = (V, E)$  is given by a pair of weights  $(\pi, \mu)$  defined from the weights on the pages and spine via

$$\pi(x) = \sum_{i=0}^l \pi_i(x) \quad \text{and} \quad \mu_{xy} = \sum_{i=0}^l \mu_{xy}^i \quad \forall x, y \in V,$$

where we take the convention that  $\pi_i(x) = 0$  if  $x \notin V_i$  (the vertex set of  $\Gamma_i$ ) and  $\mu_{xy}^i = 0$  if  $\{x, y\} \notin E_i$  (the edge set of  $\Gamma_i$ ).

We ask that the following be true of  $\Gamma$ :

1.  $\Gamma$  is connected.
2. There exists a number  $\alpha > 0$  such that  $\Gamma_0$ , seen as a subgraph of  $\Gamma$ , is  $\alpha$ -connected, that is,  $[\Gamma_0]_\alpha$ , the  $\alpha$ -neighborhood of  $\Gamma_0$  in  $\Gamma$ , is connected.
3. When seen as a subgraph of  $\Gamma$ , for all  $i = 1, \dots, l$ , we have  $\partial_I \Gamma_i = G_i$ .
4. The identification between  $G_i, G'_i$  satisfies the description given above, i.e. is a bijection between vertices with compatible weights.

**Lemma 4.1.1.** *The graph  $(\Gamma, \pi, \mu)$  obtained by the gluing procedure above is of the type we consider. That is,  $\Gamma = (V, E)$  is a simple connected graph with edge weights  $\mu_{xy}$  that are both symmetric and adapted to the edges, and which are subordinate to the vertex weight  $\pi$ . Further, the weights on  $\Gamma$  are both controlled and uniformly lazy.*

*Proof.* That  $\Gamma$  is connected is requirement 1. above, and if any edges are “doubled” from gluing, we will still think of them as one edge, so the graph remains simple.

That  $\mu_{xy}$  is symmetric follows from the symmetry of the  $\mu_{xy}^i$ . If  $\mu_{xy} \neq 0$ , then there exists some  $i \in \{0, \dots, l\}$  such that  $\mu_{xy}^i \neq 0$ , which implies  $\{x, y\} \in E_i$ , and hence that  $\{x, y\} \in E$ . Moreover, if  $\{x, y\} \in E$ , then it must be that  $\{x, y\} \in E_i$  for

some  $i \in \{0, \dots, l\}$ , and hence  $\mu_{xy} \geq \mu_{xy}^i > 0$ . Therefore  $\mu$  is adapted to the edges of  $\Gamma$ .

Further, the edge weights are subordinate to the vertex weights since that is true of the spine and pages:

$$\sum_{y \sim x} \mu_{xy} = \sum_{y \sim x} \sum_{i=0}^l \mu_{xy}^i \leq \sum_{i=0}^l \pi_i(x) = \pi(x).$$

That  $\Gamma$  has controlled weights follows from the assumed compatibility condition (4.1). Consider any edge  $\{x, y\} \in \Gamma$ . Then there must exist at least one  $i \in \{0, \dots, l\}$  such that  $\{x, y\} \in E_i$ , since  $\Gamma$  is connected and the gluing process does not add new edges (except those that occur from identifying vertices). Then, if  $C_c^i$  denotes the constant for controlled weights of  $\Gamma_i$ , we have:

$$\frac{\mu_{xy}}{\pi(x)} = \frac{\sum_{j=0}^l \mu_{xy}^j}{\sum_{j=0}^l \pi_j(x)} \geq \frac{\mu_{xy}^i}{C_l \pi_i(x)} \geq \frac{1}{C_c^i C_l} \geq \frac{1}{C_l l \max_i C_c^i}.$$

That  $\Gamma$  is uniformly lazy again follows from that assumption on the pages and the spine:

$$\begin{aligned} \sum_{y \sim x} \mu_{xy} &= \sum_{i=0}^l \sum_{y \sim x} \mu_{xy}^i = \sum_{i=0}^l (1 - C_e^i) \pi_i(x) = \sum_{i=0}^l \pi_i(x) - \sum_{i=0}^l C_e^i \pi_i(x) \\ &\leq \pi(x) - \sum_{i=0}^l (\min_i C_e^i) \pi_i(x) = (1 - (\min_i C_e^i)) \pi(x). \end{aligned}$$

□

## 4.1.2 Cutting graphs into pieces

The gluing operation described in the previous subsection is relatively natural for graphs. It is also natural to instead approach the question from the point of view of cutting a larger graph  $(\Gamma, \pi, \mu)$  apart into pages and a spine.

More precisely, assume we have  $(\Gamma, \pi, \mu)$ . Identify a set of vertices (and their induced subgraph) as  $\Gamma_0$ . Remove the **vertices** in  $\Gamma_0$  from  $\Gamma$ . This splits the graph into connected components with trailing edges left from removing the vertices in  $\Gamma_0$ .

Assume  $\Gamma_0$  is such that doing the above procedure produces a finite number of connected components,  $\Gamma_1, \dots, \Gamma_l$ , all of which are infinite graphs. We “cap” the trailing edges in a given  $\Gamma_i$  with vertices and call these cap vertices the set  $G_i$ . Then  $G_i$  has a natural identification with a subset of vertices  $G'_i$  of  $\Gamma_0$ .

On  $\Gamma_0, \Gamma_1, \dots, \Gamma_l$  recall the random walk structure inherited from  $\Gamma$  with Neumann Markov kernel is obtained by taking the vertex and edge weights from  $\Gamma$  and then staying in place with the appropriate probability to make this a Markov kernel. When cutting, we add some additional flexibility to this notion by saying that weights on the subgraphs should be comparable to weights on  $\Gamma$ , that is, there exist constants  $C_B, c_B$  such that for all  $x \in \Gamma_i$ ,  $0 \leq i \leq l$ ,

$$c_B \pi(x) \leq \pi_i(x) \leq C_B \pi(x),$$

and a similar inequality holds for edge weights.

All nice properties of the random walk structure on  $\Gamma$  are inherited by these subgraphs.

**Remark 4.1.2.**

- (1) With the notion of adding “cap” vertices, the question of do we add edges between two such vertices arises. If we set  $\mu_{xy}^i = \mu_{xy}$  if  $x, y \in \Gamma_i$  and zero else, then this adds back in edges we may have removed.
- (2) While it might seem most natural to simply take the random walk structure inherited from  $\Gamma$  on the spine and pages, allowing some flexibility

in changing the weights enables us to ensure that it is possible to cut a graph apart in such a way that it can be glued back together into *exactly* the graph we began with. Under the set of hypotheses we will assume (see Section 4.1.3 below), everything about the pages  $\Gamma_1, \dots, \Gamma_l$  is stable under such perturbations of weights.

We require the following of the above cutting construction:

1. As already mentioned, removing  $\Gamma_0$  should create a finite number of connected components, all of which are infinite.
2. In  $\Gamma$ , the graph  $\Gamma_0$  is  $\alpha$ -connected.
3. The set  $\Gamma_0$  should have the property that any vertex  $x \in \Gamma_0$  either has (1) all neighbors also in  $\Gamma_0$  or (2) has neighbors in  $\Gamma_i$  and  $\Gamma_j$  with  $i \neq j$  and  $i, j \in \{0, 1, \dots, l\}$ .

Property 3. above ensures that  $\partial_I \Gamma_i = G_i$  and prevents selecting vertices for  $\Gamma_0$  that are surrounded by other vertices from only one page and is the analog of property 3. for the gluing operation.

We illustrate the above construction with a simple example and postpone further examples until later, after we have described additional hypotheses.

**Example 4.1.1** (Gluing half-planes). Consider  $(\Gamma_i, \pi_i, \mu_{xy}^i)$  for  $i = 1, 2$ , where  $\Gamma_1 = \{(x, y) \in \mathbb{Z}^2 : y \geq 0\}$  is the discrete upper half-plane and  $\Gamma_2 = \{(x, y) \in \mathbb{Z}^2 : y \leq 0\}$  is the discrete lower half-plane. Assume both graphs have the lazy simple random walk (Definition 3.1.12). Take  $\Gamma_0$  to be a totally disconnected copy of  $\mathbb{Z}$  (vertices are  $\mathbb{Z}$ , no edges) with  $\pi_0(x) \equiv 0$ .

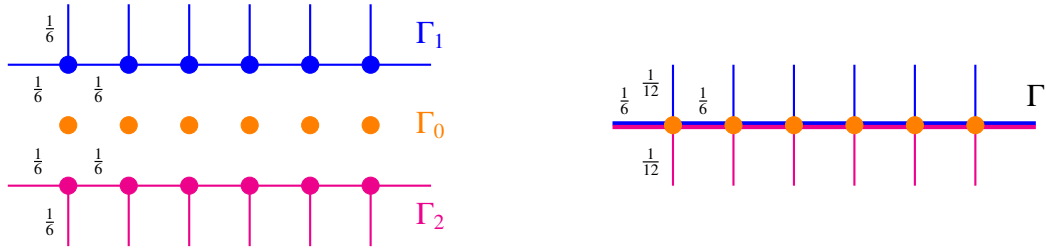


Figure 4.1: On the left, we have  $\Gamma_1, \Gamma_2$  with the lazy simple random walk. Vertices in  $G_i$ ,  $i = 1, 2$  have three neighbors, so the walk moves to any neighbor with probability  $1/6$ . When gluing over the spine  $\Gamma_0$ , we obtain the graph on the right. Using the construction above, vertical edges leading away from  $\Gamma_0$  now have probability  $1/12$ , but horizontal edges are “doubled” from the gluing and still have probability  $1/6$ , since the orange vertices now have weight 2. At all other vertices in the graph, we will move to any of the four neighbors with probability  $1/8$  and stay in place with probability  $1/2$ .

Let  $G_i$  be the set  $\{y = 0\}$  and  $G'_i = \Gamma_0$  for both  $i = 1, 2$ . Then all  $G_i, G'_i$  are copies of  $\mathbb{Z}$  and  $G_i, G'_i$  have a natural identification for  $i = 1, 2$ . Figure 4.1 illustrates this scenario and how the random walk changes along the gluing spine  $\Gamma_0$ ; away from the gluing spine, the random walk structure is still that of the lazy simple random walk (lazy SRW).

We could also think of taking  $\Gamma = \mathbb{Z}^2$  with lazy SRW and cutting it apart along the set  $\{y = 0\}$ ; we re-derive the above picture if we “cap” the vertices and add back in edges between these capped vertices. However, the probabilities in this case are slightly different: if we take the Neumann random walk on the pages, the probability of all edges is  $1/8$ . In this setting, gluing the pages back together does not give us  $\mathbb{Z}^2$  with the lazy SRW. On the other hand, instead cut  $\mathbb{Z}^2$  apart with the rule that we give each vertex in  $G_i$  or  $\Gamma_0$  weight  $1/3$  and that we give edges along the  $y = 0$  axis weight  $1/24$  in  $\Gamma_i$ ,  $i = 0, 1, 2$ . In this case  $\Gamma_0$  is a connected copy of  $\mathbb{Z}$ . We leave all other edges with weight  $1/8$  and all other

vertices with weight 1. This cutting/gluing operation satisfies the condition that the weights be comparable, and when gluing this graph back together, we do in fact obtain  $\mathbb{Z}^2$  with the lazy SRW.

### 4.1.3 Book-like graphs and further assumptions

Above, we described a construction for cutting and gluing graphs. In this section, we give additional hypotheses we want to impose upon such graphs in order to obtain heat kernel estimates.

**Definition 4.1.1** (Book-like graph). Assume we have a construction  $(\Gamma, \pi, \mu)$  as in the previous subsections with pages  $\Gamma_1, \dots, \Gamma_l$  and gluing spine  $\Gamma_0$ . If there exists a number  $\delta > 0$  such that for all  $x \in \Gamma_0$  and for all  $1 \leq i \leq l$ , we have  $d(x, \Gamma_i) \leq \delta$ , we call  $\Gamma$  a  $\delta$ -book-like graph (or simply a book-like graph).

In other words,  $\Gamma$  is book-like if each vertex in the spine  $\Gamma_0$  is at distance at most  $\delta$  from *all* of the pages.

**Definition 4.1.2** (Augmented pages). Let  $(\Gamma, \pi, \mu)$  be a  $\delta$ -book-like graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$ . For each  $i \in \{1, \dots, l\}$ , the *augmented page* associated with  $\Gamma_i$  will be denoted  $\widehat{\Gamma}_i$  and is defined as

$$\widehat{\Gamma}_i := [\Gamma_i]_\delta \cap (\Gamma_i \cup \Gamma_0),$$

where  $[\Gamma_i]_\delta := \{y \in \Gamma : d(y, \Gamma_i) \leq \delta\}$  is the  $\delta$ -neighborhood of  $\Gamma_i$ .

Note that  $\Gamma_0 \subseteq \widehat{\Gamma}_i$  for all  $i \in \{1, \dots, l\}$ . The  $\delta$ -book-like hypothesis ensures the augmented page  $\widehat{\Gamma}_i$  consists of “page  $i$ ” and the entire spine  $\Gamma_0$ .

We can define  $\widehat{\pi}_i, \widehat{\mu}^i$  as just taking the values of  $\pi, \mu$ ; in the event that the weights on  $\Gamma_i, \Gamma_0$  are not precisely the same as that on  $\Gamma$  (due to the compatible weights condition), we could instead apply the weights from  $\Gamma_i$  and  $\Gamma_0$  (seen as graphs separate from  $\Gamma$ ). Any such variation will always produce comparable weights and will not affect any results (except up to constants).

The following notion is often useful for thinking about “perturbations” of graphs or comparing two similar graphs.

**Definition 4.1.3** (Quasi-isometry). Consider two graphs as metric measure spaces,  $(\Gamma_1, d_1, \pi_1)$  and  $(\Gamma_2, d_2, \pi_2)$ , where  $d_i$  denotes the graph distance and  $\pi_i$  denotes a measure on vertices. (For this definition, we do not need a stochastic process/random walk.) We say  $\Gamma_1$  and  $\Gamma_2$  are *quasi-isometric* if there exists a function  $\Phi : \Gamma_1 \rightarrow \Gamma_2$  such that

1. There exists  $\varepsilon > 0$  such that the  $\varepsilon$ -neighborhood of the image of  $\Phi$  is equal to  $\Gamma_2$ .
2. There exist constants  $a, b$  such that

$$a^{-1}d_1(x, y) - b \leq d_2(\Phi(x), \Phi(y)) \leq ad_1(x, y) \quad \forall x, y \in \Gamma_1.$$

3. There exists a constant  $C_q > 0$  such that

$$\frac{1}{C_q}\pi_1(x) \leq \pi_2(\Phi(x)) \leq C_q\pi_1(x).$$

Further remarks on quasi-isometry can be found in Appendix B.1.1.

The following theorem indicates that Harnack graphs are stable under quasi-isometry and will be useful to us due to the ambiguity present in our cutting/gluing construction.

**Theorem 4.1.1** (see [13]). *Assume  $(\Gamma_1, \mathcal{K}_1, \pi_1)$  is a Harnack graph and is quasi-isometric to  $(\Gamma_2, \mathcal{K}_2, \pi_2)$ . Then  $\Gamma_2$  is also Harnack.*

Below we give the key hypotheses that will be assume throughout the rest of the chapter.

Key Hypotheses:

- (B1)  $(\Gamma, \pi, \mu)$  is a  $\delta$ -book-like graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$ .
- (B2) Assume that each  $\Gamma_i$ ,  $1 \leq i \leq l$  is a Harnack graph (Definition 3.1.9). In other words, assume the heat kernel associated with  $\mathcal{K}_{\Gamma_i, N}$  is Harnack for all  $1 \leq i \leq l$ .
- (B3) Assume that each  $\Gamma_i$  is inner uniform (Definition 3.3.2) and is uniformly  $S$ -transient when considered as a subgraph of  $\widehat{\Gamma}_i$  in the sense of Definition 3.2.4.
- (B4) Assume that each  $\Gamma_i$ ,  $1 \leq i \leq l$  is uniform (Definition 3.3.1) in  $\Gamma$ .

**Lemma 4.1.2.** *Under hypotheses (B1)-(B4) above, each page  $\Gamma_i$  is quasi-isometric to both its augmented version  $\widehat{\Gamma}_i$  and to its 1-neighborhood  $[\Gamma_i]_1 = \{x \in \Gamma : d(x, \Gamma_i) \leq 1\}$ .*

*Proof.* We aim to show there exists a quasi-isometry  $\Phi : \Gamma_i \rightarrow \widehat{\Gamma}_i$  (or to  $[\Gamma_i]_1$ ). The proof is the same in both instances. Let  $\Phi$  be the inclusion map. It is obvious that by taking  $\varepsilon = \delta$  (or  $\varepsilon = 1$ ) that an  $\varepsilon$ -neighborhood of the image of  $\Phi$  contains the whole set.

Given any two vertices  $x, y \in \Gamma_i$ , we know that there exists a constant  $c_U$  such that  $c_U d_i(x, y) \leq d_\Gamma(x, y)$  since  $\Gamma_i$  is uniform in  $\Gamma$  (hypothesis (B4)). Consequently,

since the graph distance can only get smaller in a larger graph,

$$c_U d_i(x, y) \leq d_\Gamma(x, y) \leq d_{\widehat{\Gamma}_i}(x, y) \leq d_i(x, y),$$

which satisfies hypothesis 2. of being a quasi-isometry.

Hypothesis 3. of the definition of a quasi-isometry follows from the assumption that vertex weights are comparable when we do a gluing and the fact that we know vertex weights in any ball of a fixed radius are comparable.  $\square$

An important consequence of hypothesis (B2), Lemma 4.1.2, and Theorem 4.1.1 is that  $\widehat{\Gamma}_i$  and  $[\Gamma_i]_1$  are also Harnack graphs.

#### 4.1.4 Examples

We describe several examples and how they fit in with the hypotheses from the previous section.

**Example 4.1.2** (Gluing lattices along a shared lattice). The motivating or prototypical example considered in this chapter is that of gluing lattices along a lower dimensional lattice. For  $i = 1, \dots, l$ , let  $\Gamma_i = \mathbb{Z}^{D_i}$  with the lazy simple random walk, so that the pages of the graph are lattices of varying dimensions. Let the spine  $\Gamma_0$  be a disconnected (no edges) copy of  $\mathbb{Z}^k$ , where we require  $\min_{1 \leq i \leq l} D_i - k \geq 3$ . Then we may set  $G_i$  to be equal to a copy of  $\mathbb{Z}^k$  in  $\mathbb{Z}^{D_i}$ , for example,  $G_i = \{(x_1, \dots, x_k, 0, \dots, 0) \in \mathbb{Z}^{D_i}\}$  and  $G'_i = \Gamma_0$  for all  $i = 1, \dots, l$ . This is a  $\delta$ -book-like graph for any  $\delta > 0$  since the spine is part of all of the pages.

Moreover,  $\mathbb{Z}^{D_i}$  is always a Harnack graph, and the condition  $D_i - k \geq 3$  guarantees each  $\Gamma_i$  is uniform in its augmented page and uniformly  $S$ -transient, as

seen in Example 3.2.4. It is also clear that distances in  $\Gamma_i$  versus in  $\Gamma$  remain unchanged, so each  $\Gamma_i$  is uniform in  $\Gamma$  for  $i = 1, \dots, l$ . Thus such graphs satisfy all of the key hypotheses (B1)-(B4).

We obtain heat kernel estimates for this example in Section 4.5.

**Remark 4.1.3.** We could also consider  $\Gamma_0$  to be made up of several copies of  $\mathbb{Z}^k$  so that there are disjoint copies of  $\mathbb{Z}^k$  to identify with each page, i.e. so the  $G'_i$  are disjoint (and, in fact, it is often useful to do so—see Remark 4.2.2). These disjoint  $G'_i = \mathbb{Z}^k$ 's should then all link to a central copy of  $\mathbb{Z}^k$  that belongs to the spine  $\Gamma_0$  but not to any page. In this situation, we may take the lazy simple random walk on  $\Gamma_0$ . Such a graph is clearly still book-like and satisfies hypotheses (B1)-(B4).

**Example 4.1.3** (Cutting/gluing over a finite set). Start with a connected graph  $(\Gamma, \mu, \pi)$  with controlled and uniformly lazy weights. Pick a finite set of vertices  $\Gamma_0$  in  $\Gamma$ . We require the three properties of a cutting construction above; since  $\Gamma_0$  is finite, it must be  $\alpha$ -connected in  $\Gamma$ , but the other two cutting hypotheses are not immediately satisfied. In terms of the additional hypotheses (B1)-(B4), that this construction is book-like, i.e. satisfies hypothesis (B1), is automatic since  $\Gamma_0$  is finite. Whether (B2)-(B4) hold depends upon the specifics of this construction.

On the other hand, we could instead begin with Harnack graphs  $(\Gamma_i, \mu^i, \pi_i)$  for  $1 \leq i \leq l$ . Let  $\Gamma_0$  be a finite set of size  $K$ . In each page  $\Gamma_i$ , let  $G_i$  be a finite set of vertices of size  $K$ . Then we may set  $G'_i = \Gamma_0$  for all  $1 \leq i \leq l$ , and assume we have a set bijection between each  $G_i$  and  $\Gamma_0$ .

We may assume  $\Gamma_0$  is connected and comes with the simple lazy random walk. Then certainly  $\Gamma$  is connected, and  $\Gamma_0$  remains connected in  $\Gamma$ . Since  $\Gamma_0$  is finite, the vertex weights are automatically comparable, and, moreover, it is impossible to make a graph that is not book-like via this construction.

Whether the key hypotheses (B3), (B4) are satisfied depends upon the nature of the gluing and graphs in question. In this case, the questions of  $S$ -transience becomes equivalent to the question of transience in the usual sense. Whether or not the pages are uniform depends on how the gluing is done; if it is done in a reasonable sense where  $G_i$  is made up of nearby vertices, then property (B4) holds.

See Section 4.4 below for a return to this example and some additional discussion of these hypotheses.

**Example 4.1.4** (A graph that is not book-like). Suppose  $l = 3$  and  $\Gamma_i = \mathbb{Z}^4$  for all  $1 \leq i \leq 3$  with lazy simple random walk. Identify the  $x_1$ -axes of  $\Gamma_1$  and  $\Gamma_2$ , and identify the  $x_2$ -axes of  $\Gamma_2$  and  $\Gamma_3$ . Then this example fits the given gluing construction, where  $\Gamma_0$  consists of two axes that share a single vertex (we can think of  $\Gamma_0$  as connected or not). However, this example is not a book-like graph since points far along the shared  $x_1$ -axis of  $\Gamma_1, \Gamma_2$  are far from  $\Gamma_3$ .

This is the simplest sort of example that is not book-like but that does satisfy our other key hypotheses, since the pages are Harnack, lines are  $S$ -transient in 4-dimensional space, and the described gluing does not change distances in individual pages. Moreover, this example does have a “fixed width” spine as in Definition B.2.1 in Appendix B. While we would ultimately like to obtain heat kernel estimates for such graphs, that is beyond the scope of this chapter.

## 4.2 Gluing heat kernels

In this section, we first discuss some abstract estimates for gluing heat kernels in a discrete setting that hold with minimal hypotheses. After that, we consider

specifically the setting of the construction of the previous section with graphs satisfying hypotheses (B1)-(B4) and what we can say about certain hitting probabilities. This section builds up the main tools and ingredients that will be used to actually obtain heat kernel estimates in Section 4.3.

### 4.2.1 Abstract gluing estimates

We start with an abstract theorem about gluing heat kernels. This theorem is very general and does not require any of the strong hypotheses of the graph, that is, we do not need (B1)-(B4). It is a discrete version of Theorem 3.5 of [34].

**Theorem 4.2.1** ([34, Theorem 3.5]). *Let  $U_1, U_2$  be two subgraphs of  $(\Gamma, \pi, \mu)$  satisfying one of the following two conditions:*

1.  $U_1 \cap U_2 = \emptyset$  in such a way that  $\partial U_1 \cap U_2$  and  $\partial U_2 \cap U_1$  are also empty
2.  $U_2 \subset U_1$ .

Then for all  $x \in U_1, y \in U_2$ , and  $n \geq d(x, y)$ ,

$$p(n, x, y) \leq p_{U_1, D}(n, x, y) + 2 \underbrace{\sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} p(m, v, w) \psi_{\partial U_1}(n, x, v) \psi_{\partial U_2}(n, y, w)}_{\text{TERM A}} \quad (4.3)$$

$$+ \underbrace{\sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_2}(m, y, w) \psi_{\partial U_1}(n, x, v) \sum_{l=d(v, w)}^n p(l, v, w)}_{\text{TERM B}_x} \quad (4.4)$$

$$+ \underbrace{\sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_1}(m, x, v) \psi_{\partial U_2}(n, y, w) \sum_{l=d(v, w)}^n p(l, v, w)}_{\text{TERM B}_y} \quad (4.5)$$

and

$$2p(n, x, y) \geq p_{U_1, D}(n, x, y) + \underbrace{2 \sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} p(m, v, w) \psi_{\partial U_1}(\lfloor \frac{n}{4} \rfloor, x, v) \psi_{\partial U_2}(\lfloor \frac{n}{4} \rfloor, y, w)}_{\text{TERM A}} \quad (4.6)$$

$$+ \underbrace{\sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_2}(m, y, w) \psi_{\partial U_1}(\lfloor \frac{n}{4} \rfloor, x, v) \sum_{l=d(v, w)}^{\lfloor \frac{n}{4} \rfloor - 1} p(l, v, w)}_{\text{TERM B}_x} \quad (4.7)$$

$$+ \underbrace{\sum_{v \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_1}(m, x, v) \psi_{\partial U_2}(\lfloor \frac{n}{4} \rfloor, y, w) \sum_{l=d(v, w)}^{\lfloor \frac{n}{4} \rfloor - 1} p(l, v, w)}_{\text{TERM B}_y}. \quad (4.8)$$

Here the notation of hitting probabilities is as in Definition 3.3.4, that is  $\psi_{\partial U_1}(n, x, v)$  is the chance of hitting  $\partial U_1$  for the first time at  $v$  in time less than or equal to  $n$ , while  $\psi'_{\partial U_1}(m, x, v)$  is the chance of hitting  $\partial U_1$  for the first time at vertex  $v$  in time exactly  $m$ . The proof of Theorem 4.2.1 is essentially the same as in the continuous and compact case of [37] and is based on a series of gluing lemmas; for completeness, the details of this proof are given in Appendix A.

**Remark 4.2.1.** Although the sums appearing in Theorem 4.2.1 look as if they are over the full (possibly infinite) sets  $\partial U_1$ ,  $\partial U_2$ , in reality each of the three “colorful” terms is only non-zero when all of the following are true:  $d_\Gamma(x, v) \leq n$ ,  $d_\Gamma(y, w) \leq n$ , and  $d_\Gamma(v, w) \leq n$ . (In the lower bound, replace  $n$  with  $\lfloor \frac{n}{4} \rfloor$ .) Therefore the sums will always be finite, and the distance between any points appearing is at most of order  $n$ .

The additional hypotheses (B1)-(B4) and gluing structure we have described will be useful for evaluating the objects appearing in the estimate above. We will in general apply the above theorem with either  $U_1, U_2$  being distinct pages

or with  $U_1$  being a neighborhood of the page and  $U_2$  being the page itself. We would like to have  $\partial U_1, \partial U_2 \subset \Gamma_0 \setminus (\Gamma_1 \cup \dots \cup \Gamma_l)$ ; this is possible with an appropriate construction (see Remark 4.2.2 below). If this holds, then there are two kinds of objects above: (1) estimates involving the heat kernel between two points in the gluing spine, e.g.  $p(m, v, w)$ , and (2) certain hitting probabilities of the gluing set/exit probabilities of a page. It is worth noting that in [39], Grigor'yan and Ishiwata use a continuous version of Lemma A.1.2, as opposed to the more precise theorem above, which suffices as they only ever consider two ends (pages).

To address (1), we will use a Faber-Krahn estimate. This is where the hypothesis that the graph is book-like, or that the gluing spine always sees all ends, comes into play. To address (2), we use results from Chapter 3.

**Remark 4.2.2.** While it is natural to wish to use Theorem 4.2.1 with say  $U_1 = \Gamma_1, U_2 = \Gamma_2$ , this is only possible if  $\partial\Gamma_1 \cap \Gamma_2 = \emptyset$  (and vice versa). That is, we need the boundary of each page to avoid touching the boundary of another page. While this hypothesis is not necessarily satisfied by some of our descriptions (e.g. “identify” the shared axes), it is always possible to make this hypothesis hold by “fattening” the spine with additional vertices that belong only to the spine.

For instance, suppose  $z \in \partial\Gamma_i \cap \Gamma_j$  with  $i, j \in \{1, \dots, l\}, i \neq j$ . By construction, it must be that  $z \notin \Gamma_i$  and  $d(z, \Gamma_0) \leq 1$ . We may replace  $z$  by two (or more) vertices  $z_1, z_2$  such that  $z_1 \in \partial\Gamma_i, z_1 \notin \Gamma_j, z_2 \in \Gamma_j$ , and  $z_1 \sim z_2$ . Essentially, we turn  $z$  into two vertices (that are neighbors) in such a way that separates  $\partial\Gamma_i$  and  $\Gamma_j$  but preserves the overall geometric structure. Since we have a finite number of ends, by repeating this process we eventually arrive at a description of  $\Gamma_0$  so that taking any two pages will satisfy hypothesis (a) of Theorem 4.2.1. Using

this same idea, it is also possible to make say  $[\Gamma_i]_2$ , the 2-neighborhood of  $\Gamma_i$  such that  $[\Gamma_i]_2 \setminus \Gamma_i \subseteq \Gamma_0$  for all  $1 \leq i \leq j$ .

If we are given a graph  $\Gamma$ , it is obvious that we can choose  $\Gamma_0$  in such a way as to make the spine satisfy the properties above (say by taking a 1-neighborhood of a given  $\Gamma_0$  that is not “fat”). This may alter the pages, but under our hypotheses (B1)-(B4), this only changes things up to a quasi-isometry and everything of interest is stable.

## 4.2.2 The $\Gamma$ heat kernel from $\Gamma_0$ to $\Gamma_0$

In order to apply Theorem 4.2.1, we need good estimates on  $p(m, z, w)$  where  $z, w \in \Gamma_0$ . We also need to understand finite sums of these quantities in the time variable. We treat the upper bound using Faber-Krahn functions and the lower bound using a local parabolic Harnack inequality. In general, obtaining good  $\Gamma_0$  to  $\Gamma_0$  heat kernel estimates, particularly in the upper bound, is a major obstacle to proving more general results about graphs that are not book-like.

First we consider the upper bound.

Set  $B = B(z, r)$  and define

$$V_{\min}(z, r) := \min_{1 \leq i \leq l} \min_{y \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} V_i(y, r). \quad (4.9)$$

Then the results of Appendix B, in particular Lemma B.2.1 and Theorem B.3.1 give the following theorem:

**Theorem 4.2.2.** *Let  $(\Gamma, \mu, \pi)$  be a book-like graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$*

satisfying hypotheses (B1)-(B4). Then for any  $v, w \in \Gamma_0$ ,

$$p(m, v, w) \leq \frac{c_1}{\sqrt{V_{\min}(v, \sqrt{m})V_{\min}(w, \sqrt{m})}} \exp\left(-\frac{d_{\Gamma}^2(v, w)}{c_2 m}\right). \quad (4.10)$$

There are several simplifications in Theorem 4.2.2 as opposed to the results of Appendix B because here we assume  $\Gamma$  is  $\delta$ -book-like (instead of merely having a fixed width spine) and we are only interested in the estimate for points along the spine. The main simplification is that  $\Gamma_0 \subseteq \widehat{\Gamma}_i$  for all  $1 \leq i \leq l$ , which is a direct consequence of hypothesis (B1).

We now turn to the lower bound. Since  $(\Gamma, \mu, \pi)$  is a connected graph with controlled and uniformly lazy weights,  $\Gamma$  satisfies a parabolic Harnack inequality at scale 1 (in fact, at any finite scale). Let  $m \in \mathbb{Z}_+$  and  $v, w \in \Gamma_0$ . Let  $v_i, w_i$  denote the closest points to  $v$  and  $w$ , respectively, in  $\Gamma_i$  for  $i = 1, \dots, l$ .

**Theorem 4.2.3.** *Let  $(\Gamma, \mu, \pi)$  be a book-like graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$  satisfying hypotheses (B1)-(B4). Then there exist constants  $c_1, c_2$  such that for all  $v, w \in \Gamma_0$  and all  $m \gg d_{\Gamma}(v, w) + \delta$ ,*

$$p(m, v, w) \geq \min_{1 \leq i \leq l} \frac{c_1}{V_i(v_i, \sqrt{m})} \exp\left(-\frac{d_{\Gamma}^2(v, w)}{c_2 m}\right). \quad (4.11)$$

The condition  $m \gg d_{\Gamma}(v, w) + \delta$  just means  $m \geq C(d_{\Gamma}(v, w) + \delta)$  for some fixed constant  $C$ ; we add the  $\delta$  to handle the event that  $v = w$ . We need this condition to ensure the random walk has enough time to see all pages; for small values of  $m$  bounds on  $p(m, v, w)$  are not so interesting.

*Proof.* By hypothesis (B1),  $\Gamma$  is  $\delta$ -book-like, so  $d(v_i, v), d(w_i, w) \leq \delta$  for all  $i$ . By the small scale parabolic Harnack inequality,

$$p(m, v, w) \approx p(m', v_i, w) \approx p(m'', v_i, w_i),$$

where  $m \approx m' \approx m''$ .

By hypotheses (B2) and (B3) we have that each page  $\Gamma_i$  is uniformly  $S$ -transient and inner uniform in  $\widehat{\Gamma}_i$ , a Harnack graph (since  $\widehat{\Gamma}_i$  is quasi-isometric to  $\Gamma_i$ , which is Harnack). Therefore, by Corollary 3.3.1,  $p_{\Gamma_i, D} \approx p_{\Gamma_i, N}$  for all  $1 \leq i \leq l$ . As by definition the Dirichlet heat kernel of any subgraph is less than the heat kernel on the entire graph, we have

$$p_{\Gamma}(m'', v_i, w_i) \geq p_{\Gamma_i, D}(m'', v_i, w_i) \geq c p_{\Gamma_i, N}(Cm'', v_i, w_i) \quad \forall 1 \leq i \leq l.$$

Moreover, by assumption (B2), each page  $\Gamma_i$  is in fact Harnack, so we have a Gaussian lower-bound on  $p_{\Gamma_i, N}$ . Consequently

$$p_{\Gamma}(m, v, w) \geq \frac{c_1}{V_i(v_i, \sqrt{m})} \exp\left(-\frac{d_{\Gamma_i}^2(v_i, w_i)}{c_2 m}\right) \quad \forall 1 \leq i \leq l. \quad (4.12)$$

This is almost exactly what we wanted to prove, except we want to replace  $d_{\Gamma_i}(v_i, w_i)$  by  $d_{\Gamma}(v, w)$ . First, by hypothesis (B4), each page  $\Gamma_i$  is uniform in  $\Gamma$ , so  $d_{\Gamma_i}(v_i, w_i) \approx d_{\Gamma}(v_i, w_i)$ . Then, via the triangle inequality and  $\Gamma$  being  $\delta$ -book-like,

$$d_{\Gamma}(v_i, w_i) \leq d_{\Gamma}(v_i, v) + d_{\Gamma}(v, w) + d_{\Gamma}(w_i, w) \leq 2\delta + d_{\Gamma}(v, w).$$

As  $(a + b)^2 \approx a^2 + b^2$ , we can square both sides, recall  $\delta$  is constant, and note that this inequality goes precisely the way we want to replace  $d_{\Gamma_i}^2(v_i, w_i)$  by  $d_{\Gamma}^2(v, w)$  in the exponential above and get a lower bound (at the price of changing the values of  $c_1, c_2$ ).

Since (4.12) holds for all  $i \in \{1, \dots, l\}$  (with  $d_{\Gamma}(v, w)$  in the exponential) it holds for the minimum over all such  $i$ , which is precisely what we wanted to prove.

□

We now want to know that the upper and lower bounds we found for  $p(m, v, w)$  in (4.10) and (4.11) are in fact matching, which is the content of the next lemma.

**Lemma 4.2.1.** *Let  $(\Gamma, \mu, \pi)$  be a book-like graph satisfying (B1)-(B4). With  $V_{\min}$  given by (4.9), then for all  $v, w \in \Gamma_0$  and  $m \gg d_\Gamma(v, w) + \delta$ ,*

$$p(m, v, w) \approx \frac{c_1}{V_{\min}(v, \sqrt{m})} \exp\left(-\frac{d_\Gamma^2(v, w)}{c_2 m}\right). \quad (4.13)$$

*Proof.* First we show that we can replace the  $\min_{y \in [B(v, r)]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta}$  in  $V_{\min}$  by simply the point  $v_i$  defined above (i.e.  $v_i \in \Gamma_i$  is a closed point in  $\Gamma_i$  to  $v$ ). Note  $v_i$  belongs to the set  $[B(v, r)]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta$  for all  $r \geq 0$ , so obviously  $\min_{y \in [B(v, r)]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} V_i(y, r) \leq V_i(v_i, r)$ .

To get the other inequality, suppose  $y \in [B(v, r)]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta$ . Then if  $r \geq 1$ ,  $v_i \in B(y, C\delta r)$  for some fixed constant  $C$ . Since  $V_i$  is doubling, we know  $V_i(v_i, r) \leq V_i(y, C\delta r) \leq \widehat{C}V_i(y, r)$  for some constant  $\widehat{C}$  independent of  $y, r$ . Hence  $V_i(v_i, r) \leq \widehat{C} \min_{y \in [B(v, r)]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} V_i(y, r)$ .

We have shown that

$$V_{\min}(v, r) \approx \min_{1 \leq i \leq l} V_i(v_i, r). \quad (4.14)$$

From this it is clear that  $V_{\min}$  is a doubling function since each  $V_i$ , so standard arguments show that the quantity  $\sqrt{V_{\min}(v, \sqrt{m})V_{\min}(w, \sqrt{m})}$  appearing in the upper bound (4.10) can be replaced by either  $V_{\min}(v, \sqrt{m})$  or  $V_{\min}(w, \sqrt{m})$  at the price of changing the constants slightly. The lower bound in (4.13) follows immediately from (4.11) given (4.14).  $\square$

**Remark 4.2.3.** The motivation for the assumption (B1), that the graph under consideration be book-like, is exactly so that Lemma 4.2.1 holds. In general, we

would prefer to replace being book-like with  $\Gamma_0$  having the fixed width property encountered in Appendix B. However, in that case, the set  $J_B$  is not longer all possible indices and it is not longer obvious that  $V_{\min}(z, r)$  is increasing in  $r$  in some reasonable sense. It is also no longer clear that the upper bound given by the Faber-Krahn function argument and the lower bound obtained using the local parabolic Harnack inequality are the same except in some special cases (such as when all pages are lattices of the same dimension).

Finally, we want to consider sums of  $p(m, v, w)$  in time, where we still have  $v, w \in \Gamma_0$ . Since  $V_{\min}$  is doubling, we can compute this sort of sum using exactly the same kind of arguments as given in Chapter 3, in particular, see the proof of Corollary 3.3.4. Using the notation  $\approx$  as in Definition 3.1.11, we find

$$\begin{aligned} \sum_{l=d_{\Gamma}(v,w)}^n p(l, v, w) &\approx \sum_{l=d_{\Gamma}(v,w)}^n \frac{1}{V_{\min}(v, \sqrt{l})} \exp\left(-\frac{d_{\Gamma}^2(v, w)}{l}\right) \\ &\approx \frac{d_{\Gamma}^2(v, w)}{V_{\min}(v, d_{\Gamma}(v, w))} \exp\left(-\frac{d_{\Gamma}^2(v, w)}{n}\right) + \sum_{l=d_{\Gamma}^2(v,w)}^n \frac{1}{V_{\min}(v, \sqrt{l})}. \end{aligned} \quad (4.15)$$

We might worry that the above bounds only hold for  $l$  sufficiently large. However, in the lower bound we can always throw away terms where  $l$  is too small or simply estimate the heat kernel by  $\exp(-d(v, w))$ , and the upper bound does not actually have the same restriction in  $l$ .

As seen in the proof of Lemma 4.2.1,  $V_{\min}$  is a doubling function. In the event that  $V_{\min}$  satisfies the condition that there exists  $\varepsilon > 0$  such that

$$\frac{V_{\min}(v, R)}{V_{\min}(v, r)} \geq c\left(\frac{R}{r}\right)^{2+\varepsilon} \quad \forall v \in \Gamma_0, R \geq r, \quad (4.16)$$

then the bound in (4.15) simplifies to

$$\sum_{l=d_{\Gamma}(v,w)}^n p(l, v, w) \approx \frac{d_{\Gamma}^2(v, w)}{V_{\min}(v, d_{\Gamma}(v, w))} \exp\left(-\frac{d_{\Gamma}^2(v, w)}{n}\right). \quad (4.17)$$

This is obvious in the lower bound and in the case that  $n \leq d_\Gamma^2(v, w)$ . In the remaining case, one uses the doubling property of  $V_{\min}$  to compute the “tail” sum in (4.15) using arguments very similar to those seen multiple times in Chapter 3. The condition (4.16) guarantees that we get a finite sum.

### 4.2.3 Hitting probabilities in Harnack graphs

Let us recall some results of Chapter 3, applied to the present setting.

**Lemma 4.2.2** (Corollaries 3.3.3 and 3.3.4). *Let  $(\widehat{\Gamma}, \pi, \mu)$  be a connected graph that is uniformly lazy with controlled weights. Assume that  $(\widehat{\Gamma}, \pi, \mu)$  is Harnack and  $\Gamma$  is an inner uniform subgraph of  $\widehat{\Gamma}$  that is uniformly  $S$ -transient. Then,  $\forall x \in \Gamma \setminus \partial_\Gamma \Gamma$ ,  $v \in \partial\Gamma$ ,  $n \geq d_\Gamma(x, v)$ , with  $\approx$  as in Definition 3.1.11,*

$$\psi'_{\partial\Gamma}(n, x, v) \approx \frac{\pi(v)}{V_\Gamma(x, \sqrt{n})} \exp\left(-\frac{d_\Gamma^2(x, v)}{n}\right) \quad (4.18)$$

$$\psi_{\partial\Gamma}(n, x, v) \approx \frac{\pi(v)d_\Gamma^2(x, v)}{V_\Gamma(x, d_\Gamma(x, v))} \exp\left(-\frac{d_\Gamma^2(x, v)}{n}\right) + \sum_{m=d_\Gamma^2(x, v)}^n \frac{\pi(v)}{V_\Gamma(x, \sqrt{m})}. \quad (4.19)$$

If, in addition,  $V_\Gamma$  satisfies condition (4.16), then in fact

$$\psi_{\partial\Gamma}(n, x, v) \approx \frac{\pi(v)d_\Gamma^2(x, v)}{V_\Gamma(x, d_\Gamma(x, v))} \exp\left(-\frac{d_\Gamma^2(x, v)}{n}\right). \quad (4.20)$$

We will in general apply Lemma 4.2.2 in the case where  $\Gamma$  is a page and  $\widehat{\Gamma}$  is the augmented page or a neighborhood of the page; the precise details will depend on various cases as we will see below in Section 4.3.

## 4.2.4 Summary of Estimates

Theorem 4.2.1 provides a way to obtain heat kernel estimates with upper and lower bounds that are matching (modulo some supremums/infimums). Above, we got estimates for all of the abstract terms appearing in Theorem 4.2.1 in terms of quantities such as distances and volumes. In particular, (4.13), (4.15), (4.18), and (4.19) give matching upper and lower bounds for all terms in Theorem 4.2.1. Some care must be taken as to what the definitions of the sets  $U_1, U_2$  are, but in general, the idea is clear. In the following section we explain the details under the additional assumption (4.16), that is, that for each  $1 \leq i \leq l$  there exists  $c_i, \varepsilon_i$  such that

$$\frac{V_i(x, R)}{V_i(x, r)} \geq c_i \left(\frac{R}{r}\right)^{2+\varepsilon_i} \quad \forall x \in \Gamma_i, R \geq r.$$

There is no real need to make this additional assumption, other than the fact the formulas are already complicated and long enough in the simpler case.

## 4.3 General heat kernel estimates

The objective of this section is to carry out the strategy mentioned in the previous section to obtain somewhat more concrete heat kernel estimates that still broadly apply. As mentioned above, we assume all volumes grow fast enough to simplify formulas. Due to the assumption that each page is uniformly  $S$ -transient, most easy to think of examples satisfy this volume growth condition.

**Theorem 4.3.1.** *Assume  $(\Gamma, \mu, \pi)$  is a book-like graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$  satisfying hypotheses (B1)-(B4). Further assume the construction of  $\Gamma$  is such that the spine  $\Gamma_0$  is thick; see Remark 4.2.2. Finally, assume that  $V_{\min}, V_i$  satisfy the condition (4.16).*

Given  $v, w, x, y$ , set

$$d_3^2 := d_\Gamma^2(x, v) + d_\Gamma^2(v, w) + d_\Gamma^2(w, y).$$

We have the following heat kernel estimates for  $x, y \in \Gamma$  and  $n \gg d_\Gamma(x, y) + d(x, \Gamma_0) + d(y, \Gamma_0) + \delta$ :

1. If  $x \in \Gamma_i, y \in \Gamma_j$  where  $i \neq j$ , and neither point is near  $\Gamma_0$  (within distance say  $4\delta$ ) then we have upper and lower bounds of  $p(n, x, y)$  of the form

$$\begin{aligned} & \sum_{\substack{v \in \partial\Gamma_i: \\ d(x, v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y, w) \leq n, \\ d(v, w) \leq n}} \frac{1}{V_{\min}(v, \sqrt{n})} \frac{\pi(v)d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{\pi(w)d_j^2(y, w)}{V_j(y, d_j(y, w))} \exp\left(-\frac{d_3^2}{n}\right) \\ & + \sum_{\substack{v \in \partial\Gamma_i: \\ d(x, v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y, w) \leq n, \\ d(v, w) \leq n}} \frac{\pi(w)}{V_j(y, \sqrt{n})} \frac{\pi(v)d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \exp\left(-\frac{d_3^2}{n}\right) \\ & + \sum_{\substack{v \in \partial\Gamma_i: \\ d(x, v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y, w) \leq n, \\ d(v, w) \leq n}} \frac{\pi(v)}{V_i(x, \sqrt{n})} \frac{\pi(w)d_j^2(y, w)}{V_j(y, d_j(y, w))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \exp\left(-\frac{d_3^2}{n}\right). \end{aligned} \quad (4.21)$$

2. If both  $x, y$  are near the spine  $\Gamma_0$ , then

$$p(n, x, y) \approx \frac{1}{V_{\min}(x, \sqrt{n})} \exp\left(-\frac{d_\Gamma^2(x, y)}{n}\right). \quad (4.22)$$

3. If  $x, y$  belong to the same page  $\Gamma_i$  and are away from  $\Gamma_0$ , then we have upper and lower bounds for  $p(n, x, y)$  of the form

$$\begin{aligned} & \frac{1}{V_i(x, \sqrt{n})} \exp\left(-\frac{d_i^2(x, y)}{n}\right) + \sum_{v, w \in \partial\Gamma_i} \left[ \frac{1}{V_{\min}(v, \sqrt{n})} \frac{\pi(v)d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{\pi(w)d_i^2(y, w)}{V_i(y, d_i(y, w))} \right. \\ & + \sum_{v, w \in \partial\Gamma_i} \frac{\pi(w)}{V_i(y, \sqrt{n})} \frac{\pi(v)d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \\ & \left. + \sum_{v, w \in \Gamma_i} \frac{\pi(v)}{V_i(x, \sqrt{n})} \frac{\pi(w)d_i^2(y, w)}{V_i(y, d_i(y, w))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \right] \exp\left(-\frac{d_3^2}{n}\right), \end{aligned} \quad (4.23)$$

where the sums are only over  $v, w \in \partial\Gamma_i$  such that  $d(x, v), d(y, w), d(v, w) \leq n$ .

4. If one of  $x, y$  is near the spine (say  $x \in \Gamma_i$  away from  $\Gamma_0$  and  $y \in [\Gamma_0]_4$ ), for  $1 \leq k \leq l$ , let  $y_k$  denote a point in  $\Gamma_k$  as close as possible to  $y$  and that satisfies  $d(y_i, \Gamma_0) > 4$ . Then (4.23) holds with  $y = y_i$  and (4.21) holds with  $y = y_j$  for all  $j \neq i$ . Consequently, these estimates must necessarily be the same in this case.

As we will see in concrete examples later, the various cases in the above theorem can typically be captured by a single formula/expression. However, in practice, the way to compute estimates is to consider these various cases, which is part of why we list them above.

*Proof.* We treat each of the given cases.

Case 1:  $x, y$  are in distinct pages away from the gluing set

Assume  $x \in \Gamma_i, y \in \Gamma_j$  where  $i \neq j$  and  $d(x, \Gamma_0), d(y, \Gamma_0) \geq 4$ . We apply Theorem 4.2.1 with  $U_1 = \Gamma_i, U_2 = \Gamma_j$ ; due to our assumption on the construction of  $\Gamma$ , hypothesis (a) of the theorem is satisfied. As  $x$  and  $y$  are in different pages,  $p_{\Gamma_i, D}(n, x, y) = 0$  for all  $n$ . We are left with the three “colorful” terms to estimate.

By (4.13) and the fact that  $V_{\min}$  is doubling, taking supremums or infimums of the estimate of  $p(m, v, w)$  with  $\lfloor \frac{n}{4} \rfloor \leq m \leq n$  does not change the estimate. Moreover, we can estimate  $\psi_{\partial\Gamma_i}(n, x, v)$  using Lemma 4.2.2 since  $\Gamma_i$  is inner uniform and uniformly  $S$ -transient inside of the Harnack graph  $\widehat{\Gamma}_i$  by hypotheses (B2), (B3). Further, any distances that should be taken in  $\Gamma_i, \Gamma_j$  can be replaced by distances in the full graph  $\Gamma$  due to hypothesis (B4). The notation  $V_i, d_i$  stands for  $V_{\Gamma_i}, d_{\Gamma_i}$ , and recall  $d_3^2 = d_{\Gamma}^2(x, v) + d_{\Gamma}^2(v, w) + d_{\Gamma}^2(w, y)$ . Thus regardless of the term

we get matching upper/lower bounds by simply using these earlier estimates.

$$\begin{aligned}
\text{TERM A} &\approx 2 \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} p(m, v, w) \psi_{\partial\Gamma_i}(n, x, v) \psi_{\partial\Gamma_j}(n, y, w) \\
&\approx \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \frac{1}{V_{\min}(v, \sqrt{n})} \frac{\pi(v) d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{\pi(w) d_j^2(y, w)}{V_j(y, d_j(y, w))} \exp\left(-\frac{d_3^2}{n}\right)
\end{aligned}$$

$$\begin{aligned}
\text{TERM } B_x &\approx \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_2}(m, y, w) \psi_{\partial U_1}(n, x, v) \sum_{l=d(v,w)}^n p(l, v, w) \\
&\approx \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \frac{\pi(w)}{V_j(y, \sqrt{n})} \frac{\pi(v) d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \exp\left(-\frac{d_3^2}{n}\right)
\end{aligned}$$

$$\begin{aligned}
\text{TERM } B_y &\approx \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_1}(m, x, v) \psi_{\partial U_2}(n, y, w) \sum_{l=d(v,w)}^n p(l, v, w) \\
&\approx \sum_{\substack{v \in \partial\Gamma_i: \\ d(x,v) \leq n}} \sum_{\substack{w \in \partial\Gamma_j: \\ d(y,w) \leq n, \\ d(v,w) \leq n}} \frac{\pi(v)}{V_i(x, \sqrt{n})} \frac{\pi(w) d_j^2(y, w)}{V_j(y, d_j(y, w))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \exp\left(-\frac{d_3^2}{n}\right)
\end{aligned}$$

These are exactly the three terms we wanted.

### Case 2: $x, y$ are both near the spine

If  $x, y$  are in  $\Gamma_0$ , then (4.22) is exactly the same as (4.13), since we already estimated the  $\Gamma_0$  to  $\Gamma_0$  heat kernel in the previous section. If  $d(x, \Gamma_0) \leq 4$ , then via the small scale parabolic Harnack inequality,  $p(m, x, y) \approx p(m', v_x, y)$  where  $v_x \in \Gamma_0$  achieves  $d(x, \Gamma_0)$  and  $m' \approx m$ . We again recover (4.22).

### Case 3: $x, y$ are away from the same spine and in the same end

Assume  $x, y \in \Gamma_i$  and  $d(x, \Gamma_0), d(y, \Gamma_0) \geq 4$ . We apply Theorem 4.2.1 with  $U_2 = \Gamma_i$  and  $U_1 = [\Gamma_i]_1$ , sets which satisfy condition (b). Since  $\Gamma_i$  is inner uniform and uniformly  $S$ -transient inside a Harnack graph  $\widehat{\Gamma}_i \supseteq [\Gamma_i]_1$ , it is also true that  $\Gamma_i$  is inner uniform and uniformly  $S$ -transient inside the Harnack graph  $[\Gamma_i]_1$ . Therefore, by Corollary 3.3.1,

$$p_{\Gamma_i, D}(n, x, y) \approx p_{\Gamma_i, N}(n, x, y) \approx \frac{1}{V_i(x, \sqrt{n})} \exp\left(-\frac{d_i^2(x, y)}{n}\right),$$

since  $\Gamma_i$  is also itself Harnack with Neumann condition.

Estimating the “colorful” terms is very similar to as in Case 1. Since  $\Gamma_i, [\Gamma_i]_1$  are basically the same and the boundary points present are still in the spine, nothing particularly changes about the estimates from the previous section. Again, recall hypothesis (B4) enables us to turn distances in a page into distance in the whole graph  $\Gamma$ . Therefore:

$$\begin{aligned} \text{TERM A} &\approx 2 \sum_{v \in \partial[\Gamma_i]_1} \sum_{w \in \partial\Gamma_i} p(n, v, w) \psi_{\partial[\Gamma_i]_1}(n, x, v) \psi_{\partial\Gamma_i}(n, y, w) \\ &\approx \sum_{v, w \in \partial\Gamma_i} \frac{1}{V_{\min}(v, \sqrt{n})} \frac{\pi(v) d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{\pi(w) d_i^2(y, w)}{V_i(y, d_i(y, w))} \exp\left(-\frac{d_3^2}{n}\right). \end{aligned}$$

Note quantities like  $d_{\Gamma_i}(x, v)$  make sense when interpreted as extending  $d_i$  to  $[\Gamma_i]_2$ ; again, due to uniformity, we can also replace all such quantities with  $d_{\Gamma}(x, v)$ . While technically the double sum is over  $v \in \partial[\Gamma_i]_1, w \in \partial\Gamma_i$ , since  $[\Gamma_i]_1$  is quasi-isometric to  $\Gamma_i$  and none of the estimates change significantly for terms at distance 1 from each other, there is no trouble replacing this with the double sum over  $v, w \in \Gamma_i$ . The other terms are similar:

$$\begin{aligned} \text{TERM } B_x &\approx \sum_{v \in \partial[\Gamma_i]_1} \sum_{w \in \partial\Gamma_i} \psi'_{\partial\Gamma_i}(m, y, w) \psi_{\partial[\Gamma_i]_1}(n, x, v) \sum_{l=d(v, w)}^n p(l, v, w) \\ &\approx \sum_{v, w \in \partial\Gamma_i} \frac{\pi(w)}{V_i(y, \sqrt{n})} \frac{\pi(v) d_i^2(x, v)}{V_i(x, d_i(x, v))} \frac{d_{\Gamma}^2(v, w)}{V_{\min}(v, d_{\Gamma}(v, w))} \exp\left(-\frac{d_3^2}{n}\right) \end{aligned}$$

$$\begin{aligned}
\text{TERM } B_y &\approx \sum_{v \in \partial[\Gamma_i]_1} \sum_{w \in \partial\Gamma_i} \psi'_{\partial[\Gamma_i]_1}(n, x, v) \psi_{\partial\Gamma_i}(n, y, w) \sum_{l=d(v,w)}^n p(l, v, w) \\
&\approx \sum_{v, w \in \Gamma_i} \frac{\pi(v)}{V_i(x, \sqrt{n})} \frac{\pi(w) d_i^2(y, w)}{V_i(y, d_i(y, w))} \frac{d_\Gamma^2(v, w)}{V_{\min}(v, d_\Gamma(v, w))} \exp\left(-\frac{d_3^2}{n}\right).
\end{aligned}$$

Again, these are exactly the terms from 4.23.

#### Case 4: Only one of $x, y$ is near the gluing set

Using the local parabolic Harnack inequality, we have  $p(n, x, y) \approx p(n, x, y_k)$  for all  $1 \leq k \leq l$  where  $d(y_k, \Gamma_0) \geq 4$ . Depending on whether  $k = i$  or not gives the various possible cases. It is sensible that the formulas in these two different cases should match up in this case as since  $y$  is near the gluing set, so hitting  $y$  is like hitting the gluing set, which is close to all of the ends.  $\square$

Another approach to Case 4 would be to use Lemma A.1.2 instead of Theorem 4.2.1, which would give yet another estimate that should be the same as the ones above.

## 4.4 Example: $\Gamma_0$ is finite

**Corollary 4.4.1.** *Assume the hypotheses of Theorem 4.3.1. In addition, assume the gluing spine  $\Gamma_0$  is finite. Fix a point  $o \in \Gamma_0$  (all points in  $\Gamma_0$  are essentially identical). Let  $|x| := \max\{1, d(x, \Gamma_0)\}$  and  $i_x$  denote the index  $i$  of the page  $x$  belongs to (if  $x$  is in the spine  $\Gamma_0$ , set  $i_x = 0$  and  $V_{i_x} = V_{\min}$ ). Then for all  $x, y \in \Gamma$  and sufficiently large  $n$ ,*

$$p(n, x, y) \approx \frac{\pi(y)}{V_{i_x}(x, \sqrt{n})} \exp\left(-\frac{d_{i_x}^2(x, y)}{n}\right) \quad (4.24)$$

$$+ \left[ \frac{|x|^2|y|^2}{V_{\min}(o, \sqrt{n})V_{i_x}(x, |x|)V_{j_y}(y, |y|)} + \frac{|x|^2}{V_{j_y}(y, \sqrt{n})V_{i_x}(x, |x|)} \right] \quad (4.25)$$

$$+ \frac{|y|^2}{V_{i_x}(x, \sqrt{n})V_{j_y}(y, |y|)} \exp\left(-\frac{(|x|^2 + |y|^2)}{n}\right). \quad (4.26)$$

In the first term, we take the convention that  $d_{i_x}(x, y) = +\infty$  if  $x, y$  do not belong to the same page, in which case this term disappears. In this case, we also have  $|x| + |y| \approx d_\Gamma(x, y)$ . In the case  $x, y$  are in the same end, then  $d_{i_x}(x, y) = d_\Gamma(x, y)$  and  $|x| + |y|$  is essentially the distance between  $x$  and  $y$  requiring a visit to  $\Gamma_0$ . The corollary is an immediate consequence of Theorem 4.3.1, with significant simplifications coming from the fact that  $\Gamma_0$  is finite, so all points of  $\Gamma_0$  can be treated as identical and all sums can be taken over all points of  $\Gamma_0$ . The (unwritten) constants will contain dependences on  $\Gamma_0$ , such as its diameter and volume. Moreover, it is easy to see all four cases in Theorem 4.3.1 reduce to the terms above and vice versa.

For instance, if  $x \in \Gamma_0$ , then  $|x| \approx 1$  and  $V_{i_x}(x, |x|) \approx V_{\min}(o, 1)$ , a constant. In the situation both  $x, y \in \Gamma_0$ , the theorem reduces to

$$p(n, x, y) \approx \frac{1}{V_{\min}(o, \sqrt{n})} \exp\left(-\frac{\text{diam}(\Gamma_0)^2}{n}\right),$$

as expected from Section 4.2.2.

**Remark 4.4.1.** The estimates in the corollary are exactly the discrete analog of heat kernel estimates in the case of gluing transient manifolds over compact sets found in the work of Grigor'yan and Saloff-Coste [37]. However, while the estimates are the same, the hypotheses are phrased differently. Namely, in [37]

the assumption is that each end is Harnack and transient (in the classical sense). Note the lack of uniformity hypotheses. However, in [34] Grigor'yan and Saloff-Coste also show that a transient Harnack manifold can only have one end. The proof should work equally well in the discrete case, and this result should imply some sort of uniformity. We do not aim to resolve this question fully here, but merely to point out that one should be careful that these results may not be exactly the same. In addition, the results of [37] cover the case where some ends can be recurrent, provided at least end one is transient. We do not discuss the case of only one page being transient here.

#### 4.5 Example: Gluing pages $\mathbb{Z}^{n_i}$ along a spine $\mathbb{Z}^m$

The prototypical example of a “book-like” graph is that of gluing pages which are lattices over a spine of a lower-dimensional lattice as described in Example 4.1.2. Instead of precisely gluing lattices over a single shared copy of  $\mathbb{Z}^k$ , as in Remark 4.2.2 we “fatten” the spine by thinking of it as  $\mathbb{Z}^k$  with some thickness so that the boundaries do not overlap. To summarize,  $(\Gamma, \mu, \pi)$  is a book-like graph made up of pages  $\Gamma_1 = \mathbb{Z}^{D_1}, \dots, \Gamma_l = \mathbb{Z}^{D_l}$  with a spine  $\Gamma_0 = \mathbb{Z}^k \times \{1, \dots, A\}$  for some fixed constant  $A$ . We may take the lazy simple random walk on all such quantities and we identify distinct “slices”  $\mathbb{Z}^k$  in  $\Gamma_0$  with the first  $k$ -coordinates of  $\Gamma_i$  in such a way that these slices don't overlap. We have the following concrete heat kernel estimates for this example.

**Corollary 4.5.1.** *Let  $(\Gamma, \mu, \pi)$  be the book-like graph with pages  $\mathbb{Z}^{D_1}, \dots, \mathbb{Z}^{D_l}$  and spine  $\Gamma_0$  a thick version of  $\mathbb{Z}^k$  as described above. In particular, we assume  $D_{\min} := \min_{1 \leq i \leq l} D_i \geq k + 3$ . For any  $x \in \Gamma$ , let  $D_x$  denote the dimension of the page  $x$  belongs to, with  $D_x := D_{\min}$  in the case that  $d(x, \Gamma_0) \leq 4$ , and let  $i_x$  denote the index of the page*

$x$  belongs to, again with  $i_x = 0$  if  $x$  is near  $\Gamma_0$ . Also let  $|x| := \max\{1, d(x, \Gamma_0)\}$  and  $d_+(x, y)$  denote the minimum distance between  $x$  and  $y$  where the path must pass through the gluing spine  $\Gamma_0$ .

Then for all  $x, y \in \Gamma$  and  $n$  sufficiently large,

$$p(n, x, y) \approx \frac{1}{n^{\frac{D_x}{2}}} \exp\left(-\frac{d_{i_x}^2(x, y)}{n}\right) + \left[ \frac{1}{n^{\frac{D_{\min}}{2}} |x|^{D_x-k-2} |y|^{D_y-k-2}} + \frac{1}{n^{\frac{D_y}{2}} |x|^{D_x-k-2}} + \frac{1}{n^{\frac{D_x}{2}} |y|^{D_y-k-2}} \right] \exp\left(-\frac{d_+^2(x, y)}{n}\right). \quad (4.27)$$

Corollary 4.5.1 follows from applying Theorem 4.3.1 to this example. The rest of this section provides the details of doing so, which are non-trivial.

*Proof.* First note many quantities appearing in Theorem 4.3.1 can be easily computed. We have  $V_i(x, r) \approx r^{D_i}$  for all  $x \in \Gamma_i$ ,  $1 \leq i \leq l$ , and  $V_{\min}(v, r) \approx r^{D_{\min}}$  for all  $v \in \Gamma_0$ . Further, since the global weight function  $\pi$  is uniformly bounded, we may treat all appearances of it as a constant. Also, recall that the uniformity hypothesis (B4) means we can always replace the distance in any end with the global distance in  $\Gamma$ . We go through the cases in Theorem 4.3.1, and, as we will see, all such cases are captured by estimate (4.27).

Case 1:  $x, y$  are in distinct pages and away from the gluing spine  $\Gamma_0$

This is the main case and is the one that will take us the most time to prove.

By estimate (4.21) from Theorem 4.3.1,

$$\begin{aligned}
p(n, x, y) \approx & \sum_{\substack{v \in \mathbb{Z}^k: \\ d(x, v) \leq n}} \sum_{\substack{w \in \mathbb{Z}^k: \\ d(y, w) \leq n, \\ d(v, w) \leq n}} \left[ \frac{1}{n^{\frac{D_{\min}}{2}} d(x, v)^{D_x-2} d(y, w)^{D_y-2}} + \frac{1}{n^{\frac{D_y}{2}} d(x, v)^{D_x-2} d(v, w)^{D_{\min}-2}} \right. \\
& \left. + \frac{1}{n^{\frac{D_x}{2}} d(y, w)^{D_y-2} d(v, w)^{D_{\min}-2}} \right] \exp\left(-\frac{[d^2(x, v) + d^2(v, w) + d^2(w, y)]}{n}\right).
\end{aligned} \tag{4.28}$$

Assume  $x \in \Gamma_i \setminus [\Gamma_0]_4$  and  $v \in \partial\Gamma_i$ . We claim that  $d(x, v) \approx |x| + d(v_x, v)$ , where we recall  $|x| := \max\{1, d(x, \Gamma_0)\}$  and where  $v_x$  is a vertex in  $\partial\Gamma_i$  achieving  $\min_{v \in \partial\Gamma_i} d(x, v)$  (if there are multiple such vertices, pick one). The upper bound easily follows from the triangle inequality, as  $d(x, v) \leq d(x, v_x) + d(v_x, v) = |x| + d(v_x, v)$ . The lower bound follows from the fact that  $|x| = d(x, v_x) \leq d(x, v)$  by definition of  $v_x$ , and also  $d(v_x, v) \leq d(v_x, x) + d(x, v) \leq 2d(x, v)$ , using the triangle inequality and the definition of  $v_x$  again. Hence (4.28) becomes

$$\begin{aligned}
p(n, x, y) \approx & \sum_{\substack{v \in \mathbb{Z}^k: \\ d(x, v) \leq n}} \sum_{\substack{w \in \mathbb{Z}^k: \\ d(y, w) \leq n, \\ d(v, w) \leq n}} \left[ \frac{1}{n^{\frac{D_{\min}}{2}} [|x| + d(v_x, v)]^{D_x-2} [|y| + d(w_y, w)]^{D_y-2}} \right. \\
& \left. + \frac{1}{n^{\frac{D_y}{2}} [|x| + d(v_x, v)]^{D_x-2} d(v, w)^{D_{\min}-2}} + \frac{1}{n^{\frac{D_x}{2}} [|y| + d(w_y, w)]^{D_y-2} d(v, w)^{D_{\min}-2}} \right] \\
& \cdot \exp\left(-\frac{[d^2(x, v) + d^2(v, w) + d^2(w, y)]}{n}\right).
\end{aligned} \tag{4.29}$$

We would like to actually compute the double sums above. At this point we need to use different arguments for the upper and lower bounds.

Upper bound: Here the main idea is we can ignore the restrictions on the sum and simply take them over entire copies of  $\mathbb{Z}^k$ . Further, the distances in the exponential are clearly controlled by  $d(x, y)$  due to the triangle inequality. We require a some facts from calculus which we collect in the following lemma.

**Lemma 4.5.1.** *Assume  $a \geq 1$  and  $A, D \geq k + 3$ . Then if  $d(\cdot, \cdot)$  denotes the distance between two points in the lattice  $\mathbb{Z}^k$  (or in some fattened version of the lattice as we have seen above),*

$$\sum_{v' \in \mathbb{Z}^k} \frac{1}{[a + d(v^*, v')]^{D-2}} \leq \frac{1}{a^{D-k-2}}, \quad \forall v^* \in \mathbb{Z}^k \quad (4.30)$$

$$\sum_{w \in \mathbb{Z}^k} \frac{1}{[1 + d(v, w)]^{D-2}} \leq 1, \quad \forall v \in \mathbb{Z}^k \quad (4.31)$$

$$\sum_{v \in \mathbb{Z}^k} \frac{1}{(a + d(v, v))^{D-2}} \sum_{w \in \mathbb{Z}^k} \frac{1}{[1 + d(v, w)]^{A-2}} \leq \frac{1}{a^{D-k-2}}. \quad (4.32)$$

Here the value “1” in the upper bounds represents a constant (independent of  $a$ ).

*Proof of Lemma 4.5.1.* We begin by proving (4.30). Arrange  $\mathbb{Z}^k$  so that  $v^*$  is the origin and  $d(v^*, v') = |v'_1| + \dots + |v'_k|$ , where  $v' = (v'_1, \dots, v'_k)$ . Then

$$\sum_{v' \in \mathbb{Z}^k} \frac{1}{[a + d(v^*, v')]^{D-2}} = \sum_{v'_1 \in \mathbb{Z}} \dots \sum_{v'_k \in \mathbb{Z}} \frac{1}{[a + |v'_1| + \dots + |v'_k|]^{D-2}}. \quad (4.33)$$

As we sum in each coordinate, the other coordinates are fixed, so it suffices to consider a one-dimensional sum. In that case,

$$\begin{aligned} \sum_{v' \in \mathbb{Z}} \frac{1}{[a + |v'|]^{D-2}} &= 2 \sum_{v'=0}^{\infty} \frac{1}{[a + v']^{D-2}} \leq \frac{1}{a^{D-2}} + \int_0^{\infty} \frac{dx}{(a+x)^{D-2}} \\ &\leq \frac{1}{a^{D-2}} + \int_a^{\infty} \frac{du}{u^{D-2}} = \frac{1}{a^{D-2}} + \left[ -\frac{1}{u^{D-3}} \right]_a^{\infty} \\ &\leq \frac{1}{a^{D-2}} + \frac{1}{a^{D-3}} \leq \frac{1}{a^{D-3}}, \end{aligned}$$

where we have used  $a \geq 1$ .

Therefore, continuing equation (4.33) yields (4.30):

$$\sum_{v' \in \mathbb{Z}^k} \frac{1}{[a + d(v^*, v')]^{D-2}} \leq \sum_{v'_1 \in \mathbb{Z}} \dots \sum_{v'_{k-1} \in \mathbb{Z}} \frac{1}{[a + |v'_1| + \dots + |v'_{k-1}|]^{D-3}} \leq \frac{1}{a^{D-k-2}}.$$

We now prove (4.31). The copies of  $\mathbb{Z}^k$  that  $v, w$  live in need not be exactly the same, but for any  $v \in \mathbb{Z}^k$ , there exists  $w_v$  in the the “ $w$ ” copy of  $\mathbb{Z}^k$  that achieves the minimum distance between  $v$  and that copy. We have  $d(v, w) \approx d(v, w_v) + d(w_v, w)$  and we know  $0 \leq d(v, w_v) \leq \delta$  by the book-like graph assumption (B1). The “ $1 + d(v, w)$ ” in (4.31) accounts for the situation that  $v = w_v$ . Then the desired result follows by making the denominator smaller and using (4.30):

$$\sum_{w \in \mathbb{Z}^k} \frac{1}{[1 + d(v, w)]^{D-2}} \leq \sum_{w \in \mathbb{Z}^k} \frac{1}{[1 + d(w_v, w)]^{D-2}} \leq 1.$$

The inequality (4.32) follows by applying (4.31) to the innermost sum, followed by applying (4.30) to the outer sum.  $\square$

We now wish to apply Lemma 4.5.1 to (4.29). The only thing to worry about is whether it is sensible to consider  $d(v, w) \approx 1 + d(v, w)$ . However, this must clearly be the case, as we have assumed the spine  $\Gamma_0$  is “fat” so it is impossible  $v = w$ , and, even if we had not done so, then we should have been more careful with our earlier estimates as if  $v = w$ , writing  $d(v, w)$  in the denominator is nonsensical. (In this setting, we can think of the ball of radius zero as still containing the point  $w$ , and in this case the “volume” of that ball is approximately 1.) Consequently, we find

$$p(n, x, y) \leq \left[ \frac{1}{n^{\frac{D_{\min}}{2}} |x|^{D_x-k-2} |y|^{D_y-k-2}} + \frac{1}{n^{\frac{D_y}{2}} |x|^{D_x-k-2}} + \frac{1}{n^{\frac{D_x}{2}} |y|^{D_y-k-2}} \right] \exp\left(-\frac{d^2(x, y)}{n}\right). \quad (4.34)$$

In this case,  $d_{i_x}(x, y) = +\infty$ , so the first term of (4.27) vanishes and  $d(x, y) = d_+(x, y)$ , so this is precisely the desired estimate.

Lower bound: Here the idea is that we can throw away terms appearing in the sums that are not useful to us, as opposed to adding more terms. In the end,

we will see that only the terms in a certain “window” matter, as we will find a lower bound that matches the upper bound we already have. The key idea is we will only look at  $v, w$  in the gluing spine at distance on the scale of  $d(x, y)$  away from  $x, y$ , respectively (or closer).

The full sum appearing is over  $d(x, v), d(y, w), d(v, w) \leq \lfloor \frac{n}{4} \rfloor$ . For the lower bound, instead of looking at the full sum, we look at a particular part of this sum; we will call the part where we look a “window.” Given  $x, y$  in distinct pages, let  $v_x, w_y$  denote a choice of closest points in  $\Gamma_0$  to  $x, y$ , respectively. Also select a geodesic path between  $x$  and  $y$ . Since  $x, y$  are in distinct pages, such a geodesic path must cross both  $\partial\Gamma_{i_x}$  and  $\partial\Gamma_{i_y}$ , so we may select vertices  $v_g \in \partial\Gamma_{i_x}, w_g \in \partial\Gamma_{i_y}$  such that  $d(x, y) = d(x, v_g) + d(v_g, w_g) + d(w_g, y)$ . (We use the subscript  $g$  to denote “geodesic”.)

Define  $W_{x,y} := \{v \in \Gamma_{i_x} : d(v, v_g) \leq 4d(x, y)\}$ ; we could analogously define  $W_{y,x}$ . The constant 4 here is not particularly important.

The set  $W_{x,y}$  has the property that it contains  $v_x$  since  $d(v_x, v_g) \leq d(v_g, x) + d(x, v_x) \leq 2d(x, y)$ . Moreover, if  $v \in W_{x,y}$  then  $d(x, v) \leq d(x, v_x) + d(v_x, v) \leq d(x, y) + 4d(x, y) \leq 5d(x, y)$ . By assumption,  $n \gg d(x, y)$  which means  $d(x, v) \leq \lfloor \frac{n}{4} \rfloor$  for all  $v \in W_{x,y}$ ; that is, all  $v \in W_{x,y}$  appear in the sum present in the lower bound.

If  $v \in W_{x,y}$  and  $w \in W_{y,x}$ , then  $d(v, w) \leq d(v, v_g) + d(v_g, w_g) + d(w_g, w) \leq 9d(x, y)$ , so for  $n$  sufficiently large the double sums appearing in Theorem 4.3.1 contain  $\sum_{v \in W_{x,y}, w \in W_{y,x}}$ . As this is the lower bound, we can throw away all other terms.

Above, we justified that  $d(x, v), d(v, w), d(y, w)$  are all controlled by  $d(x, y)$  (times some fixed constant), so we may again replace the distances in the exponential in (4.29) with  $d(x, y)$ .

We now need an analog of Lemma 4.5.1.

**Lemma 4.5.2.** *Let  $\hat{C}, C$  be fixed constants. Assume  $0 \leq a \leq \hat{C}d(x, y)$  and  $D \geq 3 + k$ .*

*Then:*

$$\sum_{v \in \mathbb{Z}^k: d(v, v') \leq Cd(x, y)} \frac{1}{[a + d(v', v)]^{D-2}} \geq \frac{1}{a^{D-k-2}}, \quad \forall v' \in \mathbb{Z}^k. \quad (4.35)$$

*Proof.* We can always think of  $v'$  as being the origin. Then if  $d(v, v') = |v_1| + \dots + |v_k| \leq Cd(x, y)$ , we have  $|v_1|, \dots, |v_k| \leq Cd(x, y)$ . Hence it suffices to prove the following lower bound on a one-dimensional sum:

$$\sum_{v=0}^{Cd(x, y)} \frac{1}{[a + v]^{D-2}} \geq \frac{1}{a^{D-3}}. \quad (4.36)$$

We may need to adjust  $\hat{C}, C$  at each step to ensure we always have terms that show up in the full sum, but this poses no problem. There are two cases to consider in order to prove (4.36).

Case 1:  $a \leq Cd(x, y)$  : In this case,

$$\begin{aligned} \sum_{v=0}^{Cd(x, y)} \frac{1}{(a + |v|)^{D-2}} &= \sum_{v=0}^a \frac{1}{(a + |v|)^{D-2}} + \sum_{v=a+1}^{Cd(x, y)} \frac{1}{(a + |v|)^{D-2}} \geq \sum_{v=0}^a \frac{1}{(a + |v|)^{D-2}} \\ &\geq \sum_{v=0}^a \frac{1}{a^{D-2}} = \frac{a}{a^{D-2}} = \frac{1}{a^{D-3}}. \end{aligned}$$

Case 2:  $Cd(x, y) \leq a \leq \hat{C}d(x, y)$  : In this case  $a \approx d(x, y)$  and  $\tilde{c}a = Cd(x, y)$  for some value of  $\tilde{c}$  (which is bounded above/below). Thus, since  $a \leq a + |v| \leq (C + 1)a$ ,

$$\sum_{v=0}^{Cd(x, y)} \frac{1}{(a + |v|)^{D-2}} \geq \sum_{v=0}^{\tilde{c}a} \frac{1}{a^{D-2}} = \frac{1}{a^{D-3}}.$$

□

The above lemma is sufficient to deal with the first term appearing in 4.29. However, we also are interested in sums of the form

$$\sum_{\substack{v, w \in \mathbb{Z}^k: \\ d(x, v), d(y, w), d(v, w) \leq n}} \frac{1}{[|x| + d(v_x, v)]^{D_x - 2} d(v, w)^{D_{\min} - 2}}. \quad (4.37)$$

To deal with such terms, instead of reducing to taking the sum over the windows  $W_{x,y}, W_{y,x}$ , we take the sum over a slightly different set of windows  $W_{x,y}, \tilde{W}_{y,x}$ . The only difference between  $\tilde{W}_{y,x}$  and  $W_{y,x}$  is that we change the constant “4” to a constant sufficiently large so that for any  $v \in W_{x,y}$ , the closest point in  $\partial\Gamma_{i_y}$  to  $v$ , denoted by  $w_v$ , belongs to  $\tilde{W}_{y,x}$ . This change is possible since

$$d(w_v, w_g) \leq d(w_v, v) + d(v, v_g) + d(v_g, w_g) \leq \delta + 4d(x, y) + d(x, y) \leq (5 + \delta)d(x, y).$$

In other words, to define  $\tilde{W}$ , replace the constant 4 by the constant  $5 + \delta$ . Again, since  $n \gg d(x, y)$ , we can take the double sum in (4.29) to be over  $W_{x,y}, \tilde{W}_{y,x}$ . Then

$$\begin{aligned} \sum_{v \in W_{x,y}} \frac{1}{|x| + d(v_x, v)^{D_x - 2}} \sum_{w \in \tilde{W}_{y,x}} \frac{1}{d(v, w)^{D_{\min} - 2}} &\geq \sum_{v \in W_{x,y}} \frac{1}{|x| + d(v_x, v)^{D_x - 2}} \frac{1}{d(v, w_v)^{D_{\min} - 2}} \\ &\geq \frac{1}{\delta^{D_{\min} - 2}} \frac{1}{|x|^{D_x - k - 2}} = \frac{1}{|x|^{D_x - k - 2}}, \end{aligned}$$

where in the last line we used Lemma 4.5.2 and recalled  $\delta$  is a fixed constant.

Applying the same line of reasoning as above to all of the terms in (4.29) gives a lower bound that is of exactly the same form as the upper bound (4.34), which matches the estimates in the corollary.

Case 2:  $x, y$  are both near the gluing spine  $\Gamma_0$

By (4.22), we have

$$p(n, x, y) \approx \frac{1}{n^{D_{\min}/2}} \exp\left(-\frac{d_{\Gamma}^2(x, y)}{n}\right),$$

which is exactly the same as (4.27) since the first term is zero,  $d_\Gamma(x, y) \approx |x| + |y|$ ,  $|x|, |y| \approx 1$ , and  $d_\Gamma(x, y) + |x| + |y| \approx d_\Gamma(x, y)$ .

Case 3:  $x, y$  are in the same end away from the gluing spine  $\Gamma_0$

The arguments are very similar to the distinct ends case. Applying estimate (4.23) from Theorem 4.3.1 and recalling  $D_x = D_y$  gives

$$p(n, x, y) \approx \frac{1}{n^{\frac{D_x}{2}}} \exp\left(-\frac{d_{i_x}^2(x, y)}{n}\right) + \sum_{\substack{v, w \in \partial\Gamma_i: \\ d(x, v), d(y, w), d(v, w) \leq n}} \left[ \frac{1}{n^{\frac{D_{\min}}{2}} d(x, v)^{i_x-2} d(y, w)^{i_x-2}} \right. \\ \left. + \frac{1}{n^{\frac{D_x}{2}} d(x, v)^{i_x-2} d(v, w)^{D_{\min}-2}} + \frac{1}{n^{\frac{D_x}{2}} d(y, w)^{i_x-2} d(v, w)^{D_{\min}-2}} \right] \\ \cdot \exp\left(-\frac{[d^2(x, v) + d^2(v, w) + d^2(y, w)]}{n}\right).$$

The first term is precisely the first term of (4.27). For the upper bound, it is obvious that the exponential is controlled by  $d_+(x, y)$ , since this quantity can be thought of as minimizing over all paths between  $x$  and  $y$  that hit a  $v$  and a  $w$  (it could be that  $v = w$ ). Arguing exactly as in the distinct ends case and taking the sum over all of  $\mathbb{Z}^k$  gives the desired upper bound.

For the lower bound, again as in the distinct ends case we use the idea of only taking the sum over a certain “window.” Take a path between  $x$  and  $y$  that hits  $\Gamma_0$  and achieves  $d_+(x, y)$ . Then  $d_+(x, y) = d(x, v_g) + d(v_g, y)$  for some  $v_g \in \Gamma_0$ . We take windows of scale  $d_+(x, y)$  around  $v_g$ . Then we can repeat the arguments given in the distinct ends case (note  $|x|, |y|$  are controlled by  $d_+$ ).

Case 4: One of  $x, y$  is near the gluing spine  $\Gamma_0$

In this case, we should be able to use either the distinct ends case or the same ends case to get the estimate (4.27). Assuming  $y$  is near  $\Gamma_0$ , then  $|y| \approx 1$ . Using

(4.34), we get

$$p(n, x, y) \approx \left[ \frac{1}{n^{\frac{D_{\min}}{2}} |x|^{D_x - k - 2}} + \frac{1}{n^{\frac{D_x}{2}}} \right] \exp\left(-\frac{d^2(x, y)}{n}\right).$$

If we used the estimate where both points are in the same end instead, we get the same terms as above; the extra term appearing is the same as one of the ones we already have. The estimate is above is also what (4.27) reduces to in this situation.  $\square$

**Remark 4.5.1.** The results of Corollary 4.5.1 also extend to the case where the pages and spine are only quasi-isometric to lattices. We already allowed for the spine  $\Gamma_0$  to be a “fat” lattice, but it also need not have such strict symmetry. This extension is important because it demonstrates that our arguments are sufficiently robust as to allow for certain perturbations. It is easier to think about this about cutting apart a graph  $\Gamma$  into pages and a spine all of which are quasi-isometric to the appropriate lattices; below we describe a gluing that would result in such a graph.

As above, let  $k$  be the dimension of the gluing spine and  $D_1, \dots, D_l$  be positive integers greater than or equal to  $k + 3$  be the dimensions of our  $l$  pages. Let  $M = \max\{D_1, \dots, D_l\}$ . Consider pages  $\Gamma_1, \dots, \Gamma_l$  such that  $\Gamma_i$  is quasi-isometric to  $\mathbb{Z}^{D_i}$  for all  $1 \leq i \leq l$  in the sense of Definition 4.1.3.

Defining the gluing spine in this situation is more subtle. In  $\mathbb{R}^M$ , take a (possibly affine) subspace of dimension  $k$ ; call it  $S$ . Fix  $\varepsilon_s, \varepsilon_r > 0$ . To construct  $\Gamma_0$ , proceed as follows. First take the  $\varepsilon_s$  neighborhood of  $S$ , which we denote  $N(S, \varepsilon_s)$ . Then take a discrete subset  $\Gamma_0$  of points in  $N(S, \varepsilon_s)$ . We turn  $\Gamma_0$  into a graph (also denoted  $\Gamma_0$ ) by connecting two vertices (elements) in  $\Gamma_0$  via an edge if and only if they are at distance less than  $\varepsilon_r$  from each other in  $\mathbb{R}^M$ . We require  $\Gamma_0$  to be such

that, as a graph, it is quasi-isometric to  $N(S, \varepsilon_s)$ . (While Definition 4.1.3 referred only to quasi-isometries between graphs, it has a natural extension to general metric measure spaces, though we now need a “+b” on the right hand side of hypothesis 2.)

As usual we need identification maps (bijections) between subsets of  $\Gamma_0$  and  $\Gamma_i$ . Let us further require that these subsets (like  $\Gamma_0$  itself) also be quasi-isometric to  $N(S, \varepsilon_s)$ .

While this description is somewhat complicated, essentially nothing should change about the proof of Corollary 4.5.1. Even though the pages are only quasi-isometric to a lattice, the volumes are still the same, up to a constant, and distances still resemble those of a lattice. The most worrying part of the above proofs may be the computation of sums over  $\mathbb{Z}^k$ , but again, there is sufficient flexibility in the quasi-isometry and our computation of the sums to result in no change except in some constants.

This remark is particularly important since the results of Corollary 4.5.1 in the case of exactly gluing lattices along a lattice match those of [37, 39] by reducing the dimension. Again, there is the caveat that these continuous setting results do not imply the discrete setting results. However, the technique of reducing the dimension fails if we are not gluing over *exactly* lattices, whereas Corollary 4.5.1 is stable under quasi-isometry as discussed above.

## 4.6 Further examples

In this section we briefly describe a few more examples to which we could apply Theorem 4.3.1 and obtain more concrete results.

**Example 4.6.1** (Gluing two  $\mathbb{Z}^4$ 's via a half-line). Consider two copies of the lattice  $\mathbb{Z}^4$ . Identify the non-negative part of the  $x_n$ -axis in each copy (fattening as necessary). In this case, we have two pages, each a copy of  $\mathbb{Z}^4$ , and  $\Gamma_0$  is a (thick) half-line. This clearly satisfies hypotheses (B1)-(B4) and the additional volume growth condition (4.16) so that Theorem 4.3.1 applies. In fact, the computations of a more concrete estimate in this setting are essentially the same as those in Corollary 4.5.1, except that the quantities  $|x|, |y|$  appearing in (4.27) now represent the distance of  $x, y$  to the  $x_4$ -half axis (as opposed to just the  $x_4$ -axis). This difference also comes into play in computing  $d_+$ . So, while the estimate has the same *form* as (4.27), the terms appearing are not quite the same. In particular, if  $x = (x_1, x_2, x_3, x_4)$  where  $x_4$  is large and negative, then  $|x|$  is much larger when gluing over the half-line than when gluing over the line.

**Remark 4.6.1.** There is nothing special here about the dimension 4 (or even that the dimensions be the same). The same estimates hold for any number of copies of  $\mathbb{Z}^d$  of varying dimensions where in each one we select half of a coordinate axis and identify all of these axes together. One could also consider half-lattices of other dimensions, provided the pages have high enough dimension that hypothesis (B3) holds.

**Example 4.6.2** (Gluing lattices via a two-dimensional cone). Consider taking a set of points in a copy of  $\mathbb{Z}^2$  that corresponds to a cone. For instance, take a cone of aperture  $\alpha$  in  $\mathbb{R}^2$ , and then take the set of lattice points lying inside of that cone. Consider  $l$  lattices  $\mathbb{Z}^{D_i}$  with  $D_i \geq 5$  for all  $1 \leq i \leq l$  and identify the chosen

cone points across the last two coordinate axes. This gives a book-like graph with  $\Gamma_i = \mathbb{Z}^{D_i}$  and  $\Gamma_0$  the chosen cone. Once again we can apply both Theorem 4.3.1 and the reasoning of Corollary 4.5.1 to get heat kernel estimates of the form (4.27), with distances interpreted appropriately.

## APPENDIX A

### GLUING ESTIMATE IN ABSTRACT TERM

This appendix proves Theorem 4.2.1, which is based on a series of lemmas. We first remind ourselves of some notation. Let  $(\Gamma, \mathcal{K}, \pi)$  be an infinite connected graph with controlled weights and heat kernel  $p_\Gamma(n, x, y)$ . **In this appendix, we need not assume the weights on  $\Gamma$  are uniformly lazy.** This section is general and does not in general require hypotheses about the geometry of the graph. Let notation of hitting times and probabilities be as in Chapter 3 (see Definition 3.3.4) so that if  $U$  is a subset/subgraph of  $\Gamma$  and  $x \in U$ ,

$$\tau_{\partial U} = \tau_{U^c} = \inf\{n \geq 0 : X_n \in U^c\} = \inf\{n \geq 0 : X_n \in \partial U\}$$

$$\psi_{\partial U}(n, x) = \mathbb{P}^x(\tau_{\partial U} \leq n)$$

$$\psi_{\partial U}(n, x, z) = \mathbb{P}^x(\tau_{\partial U} \leq n, X_{\tau_{\partial U}} = z)$$

$$\psi'_{\partial U}(n, x, z) = \psi_{\partial U}(n, x, z) - \psi_{\partial U}(n-1, x, z) = \mathbb{P}^x(\tau_{\partial U} = n, X_{\tau_{\partial U}} = z)$$

$$\mathcal{K}_{U,D}^n(x, y) := \mathbb{P}_{U,D}^x(X_n = y) := \mathbb{P}^x(X_n = y, \tau_{\partial U} > n)$$

$$p_{U,D}(n, x, y) := \frac{\mathcal{K}_{U,D}^n(x, y)}{\pi_y}.$$

While in Chapter 3 we tended to use notation along the lines of  $\tau_{U^c}$ ,  $\psi_{U^c}$  as opposed to using  $\tau_{\partial U}$ ,  $\psi_{\partial U}$ , there is no difference between these notions since a random walk started in  $U$  must necessarily first hit  $U^c$  along  $\partial U$ .

### A.1 Gluing lemmas

This section adapts gluing lemmas from Section 3 of [37] to the discrete case, as well as allowing for an infinite gluing set. These lemmas and the ideas contained in them will be combined in the next section to prove Theorem 4.2.1.

**Lemma A.1.1** ([37, Lemma 3.1]). *Let  $(\Gamma, \pi, \mu)$  be an infinite connected graph with controlled weights and  $U \subset \Gamma$  denote both a subset of vertices and their associated subgraph. Then for all  $x \in U$ ,  $y \in \Gamma$ , and  $n > 0$ , we have:*

$$p(n, x, y) \leq p_{U,D}(n, x, y) + \sum_{z \in \partial U} \sup_{0 \leq m \leq n} p(m, z, y) \psi_{\partial U}(n, x, z). \quad (\text{A.1})$$

We may also refine the estimate to

$$\begin{aligned} p(n, x, y) &\leq p_{U,D}(n, x, y) + \sum_{z \in \partial U} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U}(\lfloor \frac{n}{2} \rfloor, x, z) \\ &\quad + \sum_{z \in \partial U} \sup_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \psi'_{\partial U}(m, x, z) \sum_{l=0}^{n - \lfloor \frac{n}{2} \rfloor - 1} p(l, z, y) \end{aligned} \quad (\text{A.2})$$

with corresponding lower bound

$$\begin{aligned} p(n, x, y) &\geq p_{U,D}(n, x, y) + \sum_{z \in \partial U} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U}(\lfloor \frac{n}{2} \rfloor, x, z) \\ &\quad + \sum_{z \in \partial U} \inf_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \psi'_{\partial U}(m, x, z) \sum_{l=0}^{n - \lfloor \frac{n}{2} \rfloor - 1} p(l, z, y). \end{aligned} \quad (\text{A.3})$$

Note if  $n < d(x, y)$ , then all quantities appearing above are zero and there is nothing interesting to say. Also, the lower bound reduces to the obviously true inequality  $p(n, x, y) \geq p_{U,D}(n, x, y)$  if  $n - \lfloor \frac{n}{2} \rfloor < d(x, y)$ .

*Proof.* We begin by noting

$$\begin{aligned} p(n, x, y) &= \frac{\mathcal{K}^n(x, y)}{\pi_y} = \frac{1}{\pi_y} \mathbb{P}^x(X_n = y) \\ &= \frac{1}{\pi_y} (\mathbb{P}^x(X_n = y, \tau_{\partial U} > n) + \mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n)) \\ &= \frac{1}{\pi_y} \mathcal{K}_{U,D}^n(x, y) + \frac{1}{\pi_y} \mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n) \\ &= p_{U,D}(n, x, y) + \frac{1}{\pi_y} \mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n). \end{aligned}$$

In other words, any path traveling from  $x$  to  $y$  in time  $n$  either stays within  $U$  or leaves it.

Let  $\mathbb{1}_A$  denote the characteristic function of a set  $A$  and  $(\mathcal{F}_n)_{n \geq 0}$  denote the filtration on the underlying probability space so that  $\mathcal{F}_{\tau_{\partial U}}$  is the  $\sigma$ -algebra associated to  $(X_n)_{n \geq 0}$  with respect to the stopping time given by  $\tau_{\partial U}$ . By the strong Markov property,

$$\begin{aligned}
\mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n) &= \mathbb{E}^x(\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathbb{1}_{\{y\}}(X_n)) = \mathbb{E}^x\left[\mathbb{E}^x(\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathbb{1}_{\{y\}}(X_n) | \mathcal{F}_{\tau_{\partial U}})\right] \\
&= \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathbb{E}^x(\mathbb{1}_{\{y\}}(X_n) | \mathcal{F}_{\tau_{\partial U}})\right] = \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathbb{E}^{X_{\tau_{\partial U}}}(\mathbb{1}_{\{y\}}(X_{n-\tau_{\partial U}}))\right] \\
&= \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathcal{K}^{n-\tau_{\partial U}}(X_{\tau_{\partial U}}, y)\right] \\
&= \sum_{z \in \partial U} \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\partial U} \leq n\}} \mathbb{1}_{\{X_{\tau_{\partial U}} = z\}} \mathcal{K}^{n-\tau_{\partial U}}(X_{\tau_{\partial U}}, y)\right] \\
&\leq \sum_{z \in \partial U} \sup_{0 \leq m \leq n} \mathcal{K}^m(z, y) \mathbb{P}^x(\tau_{\partial U} \leq n, X_{\tau_{\partial U}} = z) \\
&= \sum_{z \in \partial U} \sup_{0 \leq m \leq n} \mathcal{K}^m(z, y) \psi_{\partial U}(n, x, z).
\end{aligned}$$

Dividing both sides of the above by  $\pi_y$ ,

$$\frac{1}{\pi_y} \mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n) \leq \sum_{z \in \partial U} \sup_{0 \leq m \leq n} p(m, z, y) \psi_{\partial U}(n, x, z)$$

and hence

$$p(n, x, y) \leq p_{U,D}(n, x, y) + \sum_{z \in \partial U} \sup_{0 \leq m \leq n} p(m, z, y) \psi_{\partial U}(n, x, z),$$

which is precisely inequality (A.1).

To obtain inequality (A.2), we estimate the second term from above more carefully by cutting it into two pieces:

$$\mathbb{P}^x(X_n = y, \tau_{\partial U} \leq n) = \mathbb{E}^x(\mathbb{1}_{\{0 \leq \tau_{\partial U} \leq \lfloor \frac{n}{2} \rfloor\}} \mathbb{1}_{\{y\}}(X_n)) + \mathbb{E}^x(\mathbb{1}_{\{\lfloor \frac{n}{2} \rfloor < \tau_{\partial U} \leq n\}} \mathbb{1}_{\{y\}}(X_n)). \quad (\text{A.4})$$

For the first term on the right hand side, we use the strong Markov property as above and divide by  $\pi_y$  to obtain

$$\frac{1}{\pi_y} \mathbb{E}^x(\mathbb{1}_{\{0 \leq \tau_{\partial U} \leq \lfloor \frac{n}{2} \rfloor\}} \mathbb{1}_{\{y\}}(X_n)) \leq \sum_{z \in \partial U} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U}\left(\left\lfloor \frac{n}{2} \right\rfloor, x, z\right).$$

For the second term on the right hand side of (A.4), we can make use of  $\tau_{\partial U}$  being bounded below away from zero. Again using the strong Markov property and dividing by  $\pi_y$ ,

$$\begin{aligned}
\frac{1}{\pi_y} \mathbb{E}^x(\mathbb{1}_{\{\lfloor \frac{n}{2} \rfloor < \tau_{\partial U} \leq n\}} \mathbb{1}_{\{y\}}(X_n)) &= \sum_{z \in \partial U} \mathbb{E}^x(\mathbb{1}_{\{\lfloor \frac{n}{2} \rfloor < \tau_{\partial U} \leq n\}} \mathbb{1}_{\{X_{\tau_{\partial U}} = z\}} p(n - \tau_{\partial U}, X_{\tau_{\partial U}}, y)) \\
&= \sum_{z \in \partial U} \sum_{l = \lfloor \frac{n}{2} \rfloor + 1}^n \mathbb{E}^x(\mathbb{1}_{\{\tau_{\partial U} = l\}} \mathbb{1}_{\{X_{\tau_{\partial U}} = z\}} p(n - l, z, y)) \\
&\leq \sum_{z \in \partial U} \sup_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \mathbb{P}^x(\tau_{\partial U} = m, X_{\tau_{\partial U}} = z) \sum_{l = \lfloor \frac{n}{2} \rfloor + 1}^n p(n - l, z, y) \\
&= \sum_{z \in \partial U} \sup_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \psi'_{\partial U}(m, x, z) \sum_{l = \lfloor \frac{n}{2} \rfloor + 1}^n p(n - l, z, y).
\end{aligned}$$

Combining the above estimates gives

$$\begin{aligned}
p(n, x, y) &\leq p_{U,D}(n, x, y) + \sum_{z \in \partial U} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U}\left(\left\lfloor \frac{n}{2} \right\rfloor, x, z\right) \\
&\quad + \sum_{z \in \partial U} \sup_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \psi'_{\partial U}(m, x, z) \sum_{l=0}^{n - \lfloor \frac{n}{2} \rfloor - 1} p(l, z, y)
\end{aligned}$$

as claimed in (A.2).

To treat the lower bound, we return to (A.4). Using similar arguments as for the upper bound above,

$$\frac{1}{\pi_y} \mathbb{E}^x(\mathbb{1}_{\{0 \leq \tau_{\partial U} \leq \lfloor \frac{n}{2} \rfloor\}} \mathbb{1}_{\{y\}}(X_n)) \geq \sum_{z \in \partial U} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U}\left(\left\lfloor \frac{n}{2} \right\rfloor, x, z\right)$$

and

$$\frac{1}{\pi_y} \mathbb{E}^x(\mathbb{1}_{\{\lfloor \frac{n}{2} \rfloor < \tau_{\partial U} \leq n\}} \mathbb{1}_{\{y\}}(X_n)) \geq \sum_{z \in \partial U} \inf_{\lfloor \frac{n}{2} \rfloor + 1 \leq m \leq n} \psi'_{\partial U}(m, x, z) \sum_{l=0}^{n - \lfloor \frac{n}{2} \rfloor - 1} p(l, z, y),$$

which proves (A.3).

□

**Lemma A.1.2** ([34, Lemma 3.3]). *Let  $U_1, U_2$  be two subgraphs of  $(\Gamma, \pi, \mu)$  that satisfy one of the following conditions:*

1.  $U_1 \cap U_2 = \emptyset$  in such a way that  $\partial U_1 \cap U_2$  and  $\partial U_2 \cap U_1$  are also empty.
2.  $U_2 \subset U_1$ .

*Then for all  $x \in U_1$ ,  $y \in U_2$ , and  $n > 0$ , we have the lower bound*

$$\begin{aligned}
2p(n, x, y) &\geq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1} \left( \lfloor \frac{n}{2} \rfloor, x, z \right) \\
&\quad + \sum_{w \in \partial U_2} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, w, x) \psi_{\partial U_2} \left( \lfloor \frac{n}{2} \rfloor, y, w \right)
\end{aligned} \tag{A.5}$$

*and matching upper bound*

$$\begin{aligned}
p(n, x, y) &\leq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1} \left( \lfloor \frac{n}{2} \rfloor, x, z \right) \\
&\quad + \sum_{w \in \partial U_2} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, w, x) \psi_{\partial U_2} \left( \lfloor \frac{n}{2} \rfloor, y, w \right).
\end{aligned} \tag{A.6}$$

*Moreover, we can refine the upper bound (A.6) to*

$$\begin{aligned}
p(n, x, y) &\leq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1} \left( \lfloor \frac{n}{2} \rfloor, x, z \right) \\
&\quad + \sum_{w \in \partial U_2} \sup_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} \widehat{p}_{U_1}(m, w, x) \psi_{\partial U_2} \left( \lfloor \frac{n}{2} \rfloor, y, w \right),
\end{aligned} \tag{A.7}$$

*where*

$$\widehat{p}_{U_1}(m, w, x) := p(m, w, x) - p_{U_1, D}(m, w, x). \tag{A.8}$$

*The only difference in the refinement is we have replaced  $p$  with  $\widehat{p}_{U_1}$  in the last term.*

*Proof.* To prove (A.5), we use the lower bound (A.3) twice: once for  $U_1$ , where we neglect the term with the derivative of the hitting probability, and then once

for  $U_2$ , taking advantage of the fact that  $p(n, x, y) = p(n, y, x)$  and neglecting both the term with the derivative of the hitting probability and the  $p_{U_2, D}$  term. This yields precisely the bound we wanted:

$$2p(n, x, y) \geq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1}(\lfloor \frac{n}{2} \rfloor, x, z) \\ + \sum_{w \in \partial U_2} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, w, x) \psi_{\partial U_2}(\lfloor \frac{n}{2} \rfloor, y, w).$$

The upper bound is more challenging because it requires a time reversal argument. Recall  $\Gamma = (V_\Gamma, E_\Gamma)$  as a graph. Let  $\Omega_N$  denote the space of all paths on  $\Gamma$  of length  $N + 1$  (for  $N > 0$  fixed), that is  $\Omega_N = \{\omega : \{0, \dots, N\} \rightarrow V_\Gamma \text{ with } \omega(i) \sim \omega(i + 1) \forall 0 \leq i \leq N - 1\}$ . We can think of  $\mathbb{P}^x$  as a measure on  $\Omega_N$ , where  $\mathbb{P}^x$  sits on the subset  $\Omega_{N, x} := \{\omega \in \Omega_N : \omega(0) = x\}$  of  $\Omega_N$ .

Then for any  $A \subset V_\Gamma$  with  $\pi(A) = \sum_{x \in A} \pi_x < \infty$ , we define a measure  $\mathbb{P}_A$  on  $\Omega_N$  via

$$\mathbb{P}_A(\mathcal{A}) = \sum_{v \in A} \mathbb{P}^v(\mathcal{A}) \pi_v$$

for any event  $\mathcal{A}$  in  $\Omega_N$ . Note this is **not** a probability measure, since if  $\mathcal{A} = \Omega_N$ , then  $\mathbb{P}_A(\Omega_N) = \pi(A)$ .

It will also sometimes be useful for us to consider the measure

$$\tilde{\mathbb{P}}_v(\mathcal{A}) = \mathbb{P}_v(\mathcal{A}) \pi_v = \mathbb{P}^v(\mathcal{A}) \pi_v$$

for any event  $\mathcal{A}$  as above. As shown above, for a vertex  $v \in V_\Gamma$ , we have  $\mathbb{P}_v(\mathcal{A}) = \mathbb{P}^v(\mathcal{A})$ . The above definition lets us write

$$\mathbb{P}_A(\mathcal{A}) = \sum_{v \in A} \tilde{\mathbb{P}}_v(\mathcal{A}).$$

Let us define a **probability** measure on  $\Omega_N$  via

$$\mathbb{P}_{N, A, B}(\mathcal{A}) := \frac{\mathbb{P}_A(\mathcal{A} \cap (X_N \in B))}{\mathbb{P}_A(X_N \in B)} \tag{A.9}$$

for any  $A, B \subset V_\Gamma$  such that  $\mathbb{P}_A(X_N \in B) \neq 0$  and any event  $\mathcal{A}$  in  $\Omega_N$ .

If  $\omega$  is a path in  $\Omega_N$ , we denote its time reversal by  $\omega^*$ , that is  $\omega^*(n) = \omega(N - n)$ .

Similarly, if  $\mathcal{A}$  is an event in  $\Omega_N$ , let  $\mathcal{A}^* := \{\omega^* : \omega \in \mathcal{A}\}$ .

Claim:  $\mathbb{P}_{N,A,B}(\mathcal{A}) = \mathbb{P}_{N,B,A}(\mathcal{A}^*)$  for all  $N, A, B$  such that this quantity makes sense.

First, notice

$$\begin{aligned} \mathbb{P}_A(X_N \in B) &= \sum_{v \in A} \mathbb{P}^v(X_N \in B) \pi_v = \sum_{v \in A} \sum_{w \in B} \mathcal{K}^N(v, w) \pi_v \\ &= \sum_{v \in A} \sum_{w \in B} \mathcal{K}^N(w, v) \pi_w = \sum_{w \in B} \mathbb{P}^w(X_N \in A) \pi_w \\ &= \mathbb{P}_B(X_N \in A). \end{aligned}$$

Thus to prove the claim, it remains to show

$$\mathbb{P}_A(\mathcal{A} \cap (X_N \in B)) = \mathbb{P}_B(\mathcal{A}^* \cap (X_N \in A)). \quad (\text{A.10})$$

Assume  $\mathcal{A} = (X_{n_1} \in C_1, \dots, X_{n_l} \in C_l)$ , for  $0 < n_1 < \dots < n_l < N$  and  $C_k \subset V_\Gamma$  for  $1 \leq k \leq l$ . It suffices to prove (A.10) for such events.

For such an event  $\mathcal{A}$ , the Markov property gives

$$\begin{aligned} \mathbb{P}_A(\mathcal{A} \cap (X_N \in B)) &= \sum_{v \in A} \mathbb{P}_v(X_{n_1} \in C_1, \dots, X_{n_l} \in C_l, X_N \in B) \pi_v \\ &= \sum_{v \in A} \sum_{w \in B} \sum_{z_l \in C_l} \dots \sum_{z_1 \in C_1} \mathcal{K}^{n_1}(v, z_1) \mathcal{K}^{n_2 - n_1}(z_1, z_2) \dots \mathcal{K}^{N - n_l}(z_l, w) \pi_v \end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}_B(\mathcal{A}^* \cap (X_N \in A)) &= \sum_{w \in B} \mathbb{P}_w(X_{N-n_1} \in C_1, \dots, X_{N-n_l} \in C_l, X_N \in A) \pi_w \\
&= \sum_{w \in B} \sum_{v \in A} \sum_{z_1 \in C_1} \cdots \sum_{z_l \in C_l} \mathcal{K}^{N-n_l}(w, z_l) \mathcal{K}^{n_l-n_{l-1}}(z_l, z_{l-1}) \cdots \mathcal{K}^{n_1}(z_1, v) \pi_w \\
&= \sum_{w \in B} \sum_{v \in A} \sum_{z_1 \in C_1} \cdots \sum_{z_l \in C_l} \mathcal{K}^{N-n_l}(z_l, w) \frac{\pi_{z_l}}{\pi_w} \cdots \mathcal{K}^{n_1}(v, z_1) \frac{\pi_v}{\pi_{z_1}} \pi_w \\
&= \sum_{w \in B} \sum_{v \in A} \sum_{z_1 \in C_1} \cdots \sum_{z_l \in C_l} \mathcal{K}^{N-n_l}(z_l, w) \cdots \mathcal{K}^{n_1}(v, z_1) \pi_v.
\end{aligned}$$

Rearranging the sums above yields the equality (A.10), finishing the proof of the claim.

Now consider a path  $\omega \in \Omega_N$ . Let  $\tau_1 = \tau_{\partial U_1}$  and  $\tau_2 = \tau_{\partial U_2}$  denote the first hitting times of  $\partial U_1, \partial U_2$ . We will consider  $\mathbb{P}_{N,A,B}$  as above with  $A = \{x\}, B = \{y\}$ . Recall  $x \in U_1, y \in U_2$ . (In the continuous setting of Lemma 3.3 of [37], this part of the proof requires a limiting argument, which is not necessary here in the discrete setting.)

Notice  $\mathbb{P}_{N,x,y}$  sits on the set of paths  $\omega \in \Omega_N$  satisfying  $\omega(0) = x$  and  $\omega(N) = y$ . Call this set of paths  $\Omega_{N,x,y}$ . If  $\omega \in \Omega_{N,x,y}$ , then by the hypotheses on  $U_1, U_2$ , either  $\omega$  always stays in  $U_1$  (this is only possible if condition 2. holds) or  $\omega$  hits both  $\partial U_1$  and  $\partial U_2$  before time  $N$  (this is possible in either case). In the latter case, then  $\tau_{\partial U_1}(\omega), \tau_{\partial U_2}(\omega^*)$  are finite and  $\tau_{\partial U_1}(\omega) + \tau_{\partial U_2}(\omega^*) \leq N$  so that at least one of  $\tau_{\partial U_1}(\omega), \tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor$ .

Therefore

$$\begin{aligned}
1 = \mathbb{P}_{N,x,y}(\Omega_{N,x,y}) &\leq \mathbb{P}_{N,x,y}(\omega \text{ stays in } U_1) + \mathbb{P}_{N,x,y}(\tau_{\partial U_1}(\omega) \leq \lfloor N/2 \rfloor) \\
&\quad + \mathbb{P}_{N,x,y}(\tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor).
\end{aligned} \tag{A.11}$$

If  $\mathcal{A} = \{\omega : \tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor\}$ , then

$$\mathcal{A}^* = \{\omega^* : \omega \in \mathcal{A}\} = \{\omega^* : \tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor\} = \{\omega : \tau_{\partial U_2}(\omega) \leq \lfloor N/2 \rfloor\}. \quad (\text{A.12})$$

Multiplying both sides of the inequality in (A.11) by  $\widetilde{\mathbb{P}}_x(X_N = y) = \mathcal{K}^N(x, y)\pi_x$  and using a time reversal argument along with (A.12) on the last term yields

$$\begin{aligned} \mathcal{K}^N(x, y)\pi_x &\leq \widetilde{\mathbb{P}}_x(\tau_{\partial U_1} > N, X_N = y) + \widetilde{\mathbb{P}}_x(\tau_{\partial U_1}(\omega) \leq \lfloor N/2 \rfloor, X_N = y) \\ &\quad + \widetilde{\mathbb{P}}_y(\tau_{\partial U_2}(\omega) \leq \lfloor N/2 \rfloor, X_N = x). \end{aligned} \quad (\text{A.13})$$

Note

$$\widetilde{\mathbb{P}}_x(\tau_{\partial U_1} > N, X_N = y) = \mathcal{K}_{U_1, D}^N(x, y)\pi_x.$$

and dividing by both by  $\pi_x\pi_y$  above gives  $p_{U_1, D}(N, x, y)$  on the left. As

$$\widetilde{\mathbb{P}}_x(\tau_1(\omega) \leq \lfloor N/2 \rfloor, X_N = y) = \mathbb{P}_x(\tau_1 \leq \lfloor N/2 \rfloor, X_N = y)\pi_x,$$

dividing by  $\pi_x\pi_y$  and using estimates as in the proof of Lemma A.1.1 gives

$$\frac{1}{\pi_x\pi_y}\widetilde{\mathbb{P}}_x(\tau_1(\omega) \leq \lfloor N/2 \rfloor, X_N = y) \leq \sum_{z \in \partial U_1} \sup_{N - \lfloor \frac{N}{2} \rfloor \leq m \leq N} p(m, z, y) \psi_{\partial U_1}(\lfloor \frac{N}{2} \rfloor, x, z)$$

and

$$\frac{1}{\pi_x\pi_y}\widetilde{\mathbb{P}}_y(\tau_2(\omega) \leq \lfloor N/2 \rfloor, X_N = x) \leq \sum_{w \in \partial U_2} \sup_{N - \lfloor \frac{N}{2} \rfloor \leq m \leq N} p(m, w, x) \psi_{\partial U_2}(\lfloor \frac{N}{2} \rfloor, y, w).$$

Hence dividing both sides of (A.13) by  $\pi_x\pi_y$  and using the above estimates, we find

$$p(N, x, y) \leq p_{U_1, D}(N, x, y) + \sum_{z \in \partial U_1} \sup_{N - \lfloor \frac{N}{2} \rfloor \leq m \leq N} p(m, z, y) \psi_{\partial U_1}(\lfloor \frac{N}{2} \rfloor, x, z) \quad (\text{A.14})$$

$$+ \sum_{w \in \partial U_2} \sup_{N - \lfloor \frac{N}{2} \rfloor \leq m \leq N} p(m, w, x) \psi_{\partial U_2}(\lfloor \frac{N}{2} \rfloor, y, w), \quad (\text{A.15})$$

which is exactly the upper bound (A.6).

In order to obtain the desired refinement of the upper bound, we need to show the expression  $p(m, w, x)$  in (A.15) above can be replaced by  $\widehat{p}_{U_1}(m, w, x) = p(m, w, x) - p_{U_1, D}(m, w, x)$ .

If we assume that  $U_1, U_2$  satisfy condition 1., then since  $x \in U_1, w \in \partial U_2$ , and  $\partial U_2 \cap U_1 = \emptyset$ , it follows that  $p_{U_1, D}(m, w, x) = 0$  and there is nothing to show.

Now assume  $U_1, U_2$  satisfy condition 2. instead. In this case, any paths that do not stay in  $U_1$  must cross  $\partial U_1$ , so (A.11) becomes

$$\begin{aligned} 1 &= \mathbb{P}_{N,x,y}(\Omega_{N,x,y}) \leq \mathbb{P}_{N,x,y}(\omega \text{ stays in } U_1) + \mathbb{P}_{N,x,y}(\tau_{\partial U_1}(\omega) \leq \lfloor N/2 \rfloor) \\ &\quad + \mathbb{P}_{N,x,y}(\tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor, \omega^* \text{ crosses } \partial U_1). \end{aligned}$$

Only the last term is different from before, as it now takes into account our assumptions on  $U_1, U_2$ . Again using time reversal,

$$\begin{aligned} &\mathbb{P}_{N,x,y}(\tau_{\partial U_2}(\omega^*) \leq \lfloor N/2 \rfloor, \omega^* \text{ crosses } \partial U_1) \\ &= \mathbb{P}_{N,y,x}(\tau_{\partial U_2}(\omega) \leq \lfloor N/2 \rfloor, \omega \text{ crosses } \partial U_1) \\ &= \mathbb{P}_{N,y,x}(\tau_{\partial U_2}(\omega) \leq \lfloor N/2 \rfloor) - \mathbb{P}_{N,y,x}(\tau_{\partial U_2}(\omega) \leq \lfloor N/2 \rfloor, \omega \text{ does not cross } \partial U_1). \end{aligned}$$

Multiplying the last expression above by  $\widetilde{\mathbb{P}}_y(X_N = x)$  gives

$$\mathbb{P}_y(\tau_{\partial U_2} \leq \lfloor N/2 \rfloor, X_N = x) \pi_y - \mathbb{P}_y(\tau_{\partial U_2} \leq \lfloor N/2 \rfloor, \omega \text{ does not hit } \partial U_1, X_N = x) \pi_y. \tag{A.16}$$

Then using the strong Markov property as in Lemma A.1.1 (conditioning with

respect to  $\mathcal{F}_{\tau_{\partial U_2}}$ , we find

$$\begin{aligned}
(A.16) &= \sum_{w \in \partial U_2} \mathbb{E}^y \left[ \mathbb{1}_{\{\tau_{\partial U_2} \leq \lfloor N/2 \rfloor\}} \mathbb{1}_{\{X_{\tau_{\partial U_2}} = w\}} \mathcal{K}^{N-\tau_{\partial U_2}}(X_{\tau_{\partial U_2}}, x) \right] \pi_y \\
&\quad - \sum_{w \in \partial U_2} \mathbb{E}^y \left[ \mathbb{1}_{\{\tau_{\partial U_2} \leq \lfloor N/2 \rfloor\}} \mathbb{1}_{\{X_{\tau_{\partial U_2}} = w\}} \mathcal{K}_{U_1, D}^{N-\tau_{\partial U_2}}(X_{\tau_{\partial U_2}}, x) \right] \pi_y \\
&= \sum_{w \in \partial U_2} \mathbb{E}^y \left[ \mathbb{1}_{\{\tau_{\partial U_2} \leq \lfloor N/2 \rfloor\}} \mathbb{1}_{\{X_{\tau_{\partial U_2}} = w\}} \left( \mathcal{K}^{N-\tau_{\partial U_2}}(X_{\tau_{\partial U_2}}, x) - \mathcal{K}_{U_1, D}^{N-\tau_{\partial U_2}}(X_{\tau_{\partial U_2}}, x) \right) \right] \pi_y.
\end{aligned} \tag{A.17}$$

Again dividing by  $\pi_x \pi_y$ ,

$$(A.17) \leq \sum_{w \in \partial U_2} \sup_{N-\lfloor N/2 \rfloor \leq m \leq N} \widehat{p}_{U_1}(m, w, x) \psi_{\partial U_2} \left( \lfloor \frac{N}{2} \rfloor, y, w \right).$$

This is exactly the estimate needed to prove the refinement (A.7).  $\square$

## A.2 Proof of Theorem 4.2.1

*Proof of Theorem 4.2.1* We begin with the upper bound. Applying (A.7) and using that hitting probabilities  $\psi$  are increasing in  $n$  gives

$$\begin{aligned}
p(n, x, y) &\leq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \sup_{n-\lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1}(n, x, z) \\
&\quad + \sum_{w \in \partial U_2} \sup_{n-\lfloor \frac{n}{2} \rfloor \leq m \leq n} \widehat{p}_{U_1}(m, w, x) \psi_{\partial U_2}(n, y, w).
\end{aligned} \tag{A.18}$$

Now applying (A.2) from Lemma 1 (using  $U = U_1$ ) to  $p(m, w, x)$  and monotonicity of the hitting probability (again) yields

$$\begin{aligned}
\widehat{p}_{U_1}(m, w, x) &= p(m, x, w) - p_{U_1, D}(m, x, w) \\
&\leq \sum_{z \in \partial U_1} \sup_{m-\lfloor \frac{m}{2} \rfloor \leq j \leq m} p(j, z, w) \psi_{\partial U_1}(m, x, z) \\
&\quad + \sum_{z \in \partial U_1} \sup_{\lfloor \frac{m}{2} \rfloor + 1 \leq j \leq m} \psi'_{\partial U_1}(j, x, z) \sum_{l=0}^{m-\lfloor \frac{m}{2} \rfloor - 1} p(l, z, w).
\end{aligned}$$

For  $z \in \partial U_1$ ,  $y \in U_2$ , the assumptions on  $U_1, U_2$  guarantee that  $p_{U_2, D}(m, z, y) = 0$ . Therefore now applying (A.2) from Lemma 1 (using  $U = U_2$ ) to  $p(m, z, y)$  gives

$$p(m, z, y) = p(m, y, z) \leq \sum_{w \in \partial U_2} \sup_{m - \lfloor \frac{m}{2} \rfloor \leq j \leq m} p(j, w, z) \psi_{\partial U_2}(m, y, w) \\ + \sum_{w \in \partial U_2} \sup_{\lfloor \frac{m}{2} \rfloor + 1 \leq j \leq m} \psi'_{\partial U_2}(j, y, w) \sum_{l=0}^{m - \lfloor \frac{m}{2} \rfloor - 1} p(l, w, z).$$

Plugging these estimates into (A.18), using the symmetry and positivity of the heat kernel, the monotonicity of hitting probabilities, and collecting like terms, we obtain

$$p(n, x, y) \leq p_{U_1, D}(n, x, y) + 2 \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} p(m, z, w) \psi_{\partial U_1}(n, x, z) \psi_{\partial U_2}(n, y, w) \\ + \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_2}(m, y, w) \psi_{\partial U_1}(n, x, z) \sum_{l=0}^n p(l, z, w) \\ + \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \sup_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_1}(m, x, z) \psi_{\partial U_2}(n, y, w) \sum_{l=0}^n p(l, z, w).$$

The lower bound is proved similarly. First, we apply the bound (A.5) from Lemma A.1.2:

$$2p(n, x, y) \geq p_{U_1, D}(n, x, y) + \sum_{z \in \partial U_1} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, z, y) \psi_{\partial U_1}(\lfloor \frac{n}{2} \rfloor, x, z) \\ + \sum_{w \in \partial U_2} \inf_{n - \lfloor \frac{n}{2} \rfloor \leq m \leq n} p(m, w, x) \psi_{\partial U_2}(\lfloor \frac{n}{2} \rfloor, y, w). \quad (\text{A.19})$$

Then we apply (A.3) from Lemma A.1.1 to both  $p(m, z, y)$  (using  $U = U_2$ ) and  $p(m, w, x)$  (using  $U = U_1$ ), and, in both cases, forget about the  $p_{U, D}$  term. This yields

$$p(m, z, y) = p(m, y, z) \geq \sum_{w \in \partial U_2} \inf_{m - \lfloor \frac{m}{2} \rfloor \leq j \leq m} p(j, w, z) \psi_{\partial U_2}(\lfloor \frac{m}{2} \rfloor, y, w) \\ + \sum_{w \in \partial U_2} \inf_{\lfloor \frac{m}{2} \rfloor + 1 \leq j \leq m} \psi'_{\partial U_2}(j, y, w) \sum_{l=0}^{m - \lfloor \frac{m}{2} \rfloor - 1} p(l, w, z)$$

and

$$\begin{aligned}
p(m, w, x) = p(m, x, w) &\geq \sum_{z \in \partial U_1} \inf_{m - \lfloor \frac{m}{2} \rfloor \leq j \leq m} p(j, z, w) \psi_{\partial U_1}(\lfloor \frac{m}{2} \rfloor, x, z) \\
&+ \sum_{z \in \partial U_1} \inf_{\lfloor \frac{m}{2} \rfloor + 1 \leq j \leq n} \psi'_{\partial U_1}(j, x, z) \sum_{l=0}^{m - \lfloor \frac{m}{2} \rfloor - 1} p(l, z, w).
\end{aligned}$$

Substituting these estimates into (A.19) and using monotonicity gives the desired

$$\begin{aligned}
2p(n, x, y) &\geq p_{U_1, D}(n, x, y) + 2 \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} p(m, z, w) \psi_{\partial U_1}(\lfloor \frac{n}{4} \rfloor, x, z) \psi_{\partial U_2}(\lfloor \frac{n}{4} \rfloor, y, w) \\
&+ \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_2}(m, y, w) \psi_{\partial U_1}(\lfloor \frac{n}{4} \rfloor, x, z) \sum_{l=0}^{\lfloor \frac{n}{4} \rfloor - 1} p(l, z, w) \\
&+ \sum_{z \in \partial U_1} \sum_{w \in \partial U_2} \inf_{\lfloor \frac{n}{4} \rfloor \leq m \leq n} \psi'_{\partial U_1}(m, x, z) \psi_{\partial U_2}(\lfloor \frac{n}{4} \rfloor, y, w) \sum_{l=0}^{\lfloor \frac{n}{4} \rfloor - 1} p(l, z, w).
\end{aligned}$$

Note that quantities like  $\lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{2} \rfloor$  and  $n - \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n - \lfloor \frac{n}{2} \rfloor}{2} \rfloor$  are equal to either  $\lfloor \frac{n}{4} \rfloor$  or  $\lfloor \frac{n}{4} \rfloor + 1$  depending on the value of  $n \bmod 4$ . Taking supremums over a larger set always results in a larger value (and taking infimums over a larger set results in a smaller value), so using these and various monotonicity properties gives the precise indices here.  $\square$

## APPENDIX B

### FABER-KRAHN FUNCTIONS AND HEAT KERNEL ESTIMATES

Here we discuss gluing Faber-Krahn functions on glued spaces with the aim to generalize the results of [38].

#### B.1 Faber-Krahn functions and quasi-isometry

In this section, we provide clarifying remarks and comments about the concept of quasi-isometry (Definition 4.1.3), define (relative) Faber-Krahn functions, and explain the relationship between Faber-Krahn functions of two graphs that are quasi-isometric in Lemma B.1.1.

##### B.1.1 Further comments on quasi-isometry

Recall that quasi-isometry is a property of metric measure spaces. In the present setting, this means considering graphs with their graph distance and a vertex weight.

**Remark B.1.1.**

- Suppose  $(\Gamma_1, \pi_1, \mu^1)$  and  $(\Gamma_2, \pi_2, \mu^2)$  are connected graphs with controlled and uniformly lazy weights, as has been assumed throughout this thesis. Suppose  $\Gamma_1, \Gamma_2$  are quasi-isometric as in Definition 4.1.3 with a quasi-isometry  $\Phi : \Gamma_1 \rightarrow \Gamma_2$ . Hypothesis 3., which says that the quasi-isometry takes vertices to vertices with comparable weights, can also be thought of as a condition on the volume of small balls. In particular, hypothesis 3.,

along with Lemmas 3.1.1 and 3.1.2, imply

$$V_1(x, 1) \approx V_2(\Phi(x), 1) \text{ and } \mu_{xy}^1 \approx \mu_{\Phi(x)z}^2 \quad \forall y \sim x \in \Gamma_1, z \sim \Phi(x) \in \Gamma_2.$$

- Suppose  $\Gamma_1, \Gamma_2$  are quasi-isometric and there is a quasi-isometry  $\Phi : \Gamma_1 \rightarrow \Gamma_2$ . Then we may always define a quasi-isometry  $\Phi^{-1} : \Gamma_2 \rightarrow \Gamma_1$  as follows: for every  $z \in \Gamma_2$ , set  $\Phi^{-1}(z)$  to be an element  $\beta \in \Gamma_1$  such that  $d_2(\Phi(\beta), z) \leq \varepsilon$ . (Such a  $\beta$  always exists as  $\Phi$  is a quasi-isometry.) While  $\Phi^{-1}$  is not literally the inverse of  $\Phi$ , for such a choice of  $\Phi^{-1}$  we always have

$$d_2(z, \Phi(\Phi^{-1}(z))) = d_2(z, \Phi(\beta)) \leq \varepsilon.$$

Note the quasi-isometry constants of  $\Phi^{-1}$  need not be the same as those for  $\Phi$  (though they are related).

## B.1.2 Faber-Krahn functions

Essentially, a Faber-Krahn function is a function that provides a lower bound on the first Dirichlet eigenvalue of the Laplacian; this can also be considered as a kind of isoperimetric inequality. There are several common definitions or “kinds” of Faber-Krahn functions, where the differences in definition are related to precisely which Laplacian we are considering and whether we want the bound to be local or global (in some appropriate sense). Faber-Krahn inequalities are very closely related to heat kernel upper bounds. The theory of using Faber-Krahn estimates to obtain heat kernel estimates or vice versa is well-developed in the work of Grigor’yan (see e.g. [24, 25, 26]). In this section, we define the kind of relative Faber-Krahn function useful to us in this thesis and show a relationship between relative Faber-Krahn functions of quasi-isometric graphs.

**Definition B.1.1** (Relative Faber-Krahn function). Let  $(\Gamma, \pi, \mu)$  be an infinite connected graph with controlled and uniformly lazy weights. Let  $B = B(z, r)$ ,  $\nu \in (0, +\infty)$ . We say that a function  $\Lambda$  where  $(B, \nu) \mapsto \Lambda(B, \nu)$  is a *relative Faber-Krahn function* of  $\Gamma$  if the following two properties hold:

1.  $\nu \mapsto \Lambda(B, \nu)$  is non-increasing in  $\nu$
2. For all balls  $B \subset \Gamma$ , and any  $\Omega \subset B$ ,

$$\lambda_1(\Omega) \geq \Lambda(B, \pi(\Omega)),$$

where  $\lambda_1(\Omega)$  denotes the first Dirichlet eigenvalue of the Laplacian in  $\Omega$ .

**Remark B.1.2.** We can write  $\lambda_1(\Omega)$  using the variational definition

$$\lambda_1(\Omega) = \inf_{\text{supp } f \subseteq \Omega} \frac{\|\nabla f\|_2^2}{\|f\|_2^2}$$

where

$$\begin{aligned} |\nabla f|(x) &= \left( \frac{1}{2} \sum_{y \in \Gamma} |f(x) - f(y)|^2 \mathcal{K}(x, y) \right)^{1/2} \\ \|\nabla f\|_2^2 &= \sum_{x \in \Gamma} [|\nabla f|(x)]^2 \pi(x) = \frac{1}{2} \sum_{x, y \in \Gamma} |f(x) - f(y)|^2 \mu_{xy} \\ \|f\|_2^2 &= \sum_{x \in \Gamma} |f(x)|^2 \pi(x). \end{aligned}$$

**Lemma B.1.1** (Relative Faber-Krahn Functions of quasi-isometric Graphs). *Assume that  $\Gamma = (\Gamma, d, \pi, \mu)$ ,  $\widehat{\Gamma} = (\widehat{\Gamma}, \widehat{d}, \widehat{\pi}, \widehat{\mu})$  are connected, countably infinite graphs with controlled and uniformly lazy weights. Further, assume  $\Gamma, \widehat{\Gamma}$  are quasi-isometric and fix a quasi-isometry  $\Phi : \Gamma \rightarrow \widehat{\Gamma}$  and a choice of its inverse  $\Phi^{-1}$ .*

*Then there exist constants  $c_1, c_2, c_3$  (depending on choice of quasi-isometry and the constants controlling the weights) such that if  $\Lambda(B, \nu)$  is a relative Faber-Krahn function for  $\Gamma$ , then a relative Faber-Krahn functions for  $\widehat{\Gamma}$  is given by*

$$\widehat{\Lambda}(\widehat{B}(z, r), \nu) = c_1 \Lambda(B(\Phi^{-1}(z), c_2 r), c_3 \nu).$$

*Proof.* Let the quasi-isometry constants of  $\Phi$  be  $a, b, \varepsilon, C_q, C_w$  as in Definition 4.1.3. In this proof, we use the Latin alphabet to denote elements of  $\widehat{\Gamma}$  and the Greek alphabet to denote elements of  $\Gamma$ .

Let  $r > 0$ ,  $z \in \widehat{\Gamma}$ . Consider  $\widehat{\Omega} \subset \widehat{B}(z, r)$  and fix a function  $f$  supported on  $\widehat{\Omega}$ . Since we know the eigenfunction of  $\lambda_1(\Omega)$  has a sign, we may assume  $f \geq 0$ .

We will also assume  $r \geq 1$  : If  $r < 1$  then  $\widehat{B}(z, r) = \{z\}$  and the only choice is  $\Omega = B$  so  $f(z) = C$  and  $f(x) = 0$  for  $x \in \widehat{\Gamma}$  where  $x \neq z$ . Then, if  $\widehat{C}_c$  is the constant for controlled weights in  $\widehat{\Gamma}$ ,

$$\|\nabla f\|_2^2 = C^2 \sum_{y \sim z} \widehat{\mu}_{zy} \geq C^2 \frac{\widehat{\pi}(z)}{\widehat{C}_c} = \frac{1}{\widehat{C}_c} \|f\|^2.$$

Thus  $\lambda_1(\widehat{\Omega}) \approx 1$ . This estimate is independent of  $z$ , and similarly,  $\lambda_1(\{\alpha\}) \approx 1$  for any  $\alpha \in \Gamma$ . Thus these Faber-Krahn functions evaluated at such balls are always similar and we can focus on the more interesting case where  $r \geq 1$ .

Let  $f_\varepsilon$  denote the weighted average of  $f$  on balls of radius  $\varepsilon$ , that is,

$$f_\varepsilon(x) = \frac{1}{\widehat{V}(x, \varepsilon)} \sum_{y \in \widehat{B}(x, \varepsilon)} f(y) \widehat{\pi}(y).$$

Support of  $f_\varepsilon \circ \Phi$ :

Consider  $f_\varepsilon \circ \Phi$ , which is a function on  $\Gamma$ .

By definition,  $\text{supp } f \subseteq \widehat{\Omega} \subseteq \widehat{B}(z, r)$ . Then  $\text{supp } f_\varepsilon \subseteq [\widehat{\Omega}]_\varepsilon \subseteq \widehat{B}(z, r + \varepsilon) \subseteq \widehat{B}(z, (1 + \varepsilon)r)$ , where  $[\widehat{\Omega}]_\varepsilon$  denotes the  $\varepsilon$ -neighborhood of  $\widehat{\Omega}$ .

We claim that if  $\alpha \in \text{supp}(f_\varepsilon \circ \Phi)$ , then  $\alpha \in B(\Phi^{-1}(z), c_2 r)$ , where  $c_2 := a(1 + 2\varepsilon + b)$ .

Suppose  $\alpha \in \text{supp } f_\varepsilon \circ \Phi$ . Then  $\Phi(\alpha) \in \widehat{B}(z, (1 + \varepsilon)r)$  and hence:

$$\begin{aligned} d(\alpha, \Phi^{-1}(z)) &\leq a\widehat{d}(\Phi \circ \Phi^{-1}(z), \Phi(\alpha)) + ab \leq a[\widehat{d}(\Phi \circ \Phi^{-1}(z), z) + \widehat{d}(z, \Phi(\alpha))] + ab \\ &\leq a[\varepsilon + (1 + \varepsilon)r] + ab \leq a(1 + 2\varepsilon + b)r. \end{aligned}$$

Moreover, let  $\Omega := \text{PreIm}_\Phi([\widehat{\Omega}]_\varepsilon) \subset B(\Phi^{-1}(z), c_2r)$ . Note  $\alpha \in \Omega \iff \Phi(\alpha) \in [\widehat{\Omega}]_\varepsilon$ . Hence  $\text{supp}(f_\varepsilon \circ \Phi) \subset \Omega \subset B(\Phi^{-1}(z), c_2r)$ .

Using a Faber-Krahn Function on  $\Gamma$ :

Since  $\Lambda$  is a relative Faber-Krahn function on  $\Gamma$ , we have

$$\|\nabla(f_\varepsilon \circ \Phi)\|_2^2 \geq \Lambda(B(\Phi^{-1}(z), c_2r), \pi(\Omega)) \|f_\varepsilon \circ \Phi\|_2^2. \quad (\text{B.1})$$

The result will follow if we can prove the following three claims:

1.  $\pi(\Omega) \leq c_3 \widehat{\pi}(\widehat{\Omega})$  for some constant  $c_3$  (since  $\Lambda$  is decreasing in  $\nu$  by definition)
2.  $\|\nabla(f_\varepsilon \circ \Phi)\|_2^2 \leq C_A \|\nabla f\|_2^2$  for some constant  $C_A$
3.  $\|f_\varepsilon \circ \Phi\|_2^2 \geq C_B \|f\|_2^2$  for some constant  $C_B$ .

(Note above we must be careful to choose the appropriate  $L^2$  norms with respect to either  $\Gamma$  or  $\widehat{\Gamma}$ .)

Claim 1:  $\pi(\Omega) \leq c_3 \widehat{\pi}(\widehat{\Omega})$  for some constant  $c_3$ .

As  $\widehat{\Gamma}$  is locally uniformly finite, each point in  $\widehat{\Gamma}$  has at most  $\widehat{N}$  neighbors. Further, suppose  $w \in [\widehat{\Omega}]_\varepsilon \setminus \widehat{\Omega}$ . Then there exists some point  $x \in \widehat{\Omega}$  such that  $\widehat{d}(x, w) < \varepsilon$  and hence  $\widehat{\pi}(x) \approx \widehat{\pi}(w)$  (with constants independent of  $x, w$ ). The contribution of  $w$  to  $\widehat{\pi}([\widehat{\Omega}]_\varepsilon)$  is  $\widehat{\pi}(w)$ . For each point  $x \in \widehat{\Omega}$ , there are at most finitely many  $w$ 's in

$[\widehat{\Omega}]_\varepsilon \setminus \widehat{\Omega}$  satisfying  $\widehat{d}(x, w) < \varepsilon$ . Hence there is some constant  $C$  (depending on  $\varepsilon$  and the constants of the hypotheses on the graphs) such that

$$\widehat{\pi}(\widehat{\Omega}) \leq \widehat{\pi}([\widehat{\Omega}]_\varepsilon) \leq C \widehat{\pi}(\widehat{\Omega}).$$

By definition,  $\Omega = \text{PreIm}_\Phi([\widehat{\Omega}]_\varepsilon)$ . Each point in  $[\widehat{\Omega}]_\varepsilon$  can be mapped to by a finite number of points in  $\Omega$  since  $\Gamma_1$  is locally uniformly finite and

$$ad(\alpha, \beta) - b \leq \widehat{d}(\Phi(\alpha), \Phi(\beta)) = 0 \implies d(\alpha, \beta) \leq \frac{b}{a}.$$

Then, where  $c$  is a constant (depending on the graph structure and quasi-isometry) whose value changes in each step, the above and earlier remarks imply

$$\pi(\Omega) = \sum_{\alpha \in \Omega} \pi(\alpha) \leq c \sum_{\alpha \in \Omega} \widehat{\pi}(\Phi(\alpha)) \leq c \sum_{v \in [\widehat{\Omega}]_\varepsilon} \widehat{\pi}(v) \leq c \widehat{\pi}(\widehat{\Omega}).$$

We may take  $c_3$  to be any value of  $c$  that achieves this upper bound.

Claim 2:  $\|\nabla(f_\varepsilon \circ \Phi)\|_2^2 \leq C_A \|\nabla f\|_2^2$  for some constant  $C_A$

Let  $\alpha, \beta \in \Gamma$  satisfy  $\alpha \sim \beta$ . Then  $\widehat{d}(\Phi(\alpha), \Phi(\beta)) \leq ad(\alpha, \beta) = a$ . Let  $x = \Phi(\alpha)$ ,  $y = \Phi(\beta)$ ; by the previous sentences,  $y \in \widehat{B}(x, a)$ . Consequently, for  $w \in \widehat{B}(y, \varepsilon)$ , we have  $w \in \widehat{B}(x, \varepsilon + a)$ . Hence

$$\begin{aligned} |f_\varepsilon(x) - f_\varepsilon(y)| &= |f_\varepsilon(x) - f(x) + f(x) - f_\varepsilon(y)| \\ &\leq \frac{1}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon)} |f(v) - f(x)| \widehat{\pi}(v) + \frac{1}{|\widehat{V}(y, \varepsilon)|} \sum_{w \in \widehat{B}(y, \varepsilon)} |f(x) - f(w)| \widehat{\pi}(w) \\ &\leq \frac{C}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon + a)} |f(v) - f(x)| \widehat{\pi}(v). \end{aligned}$$

Using Jensen's inequality,

$$\begin{aligned} |f_\varepsilon(x) - f_\varepsilon(y)|^2 &\leq \left| \frac{C}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon+a)} |f(v) - f(x)| \widehat{\pi}(v) \right|^2 \\ &\leq \frac{C}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon+a)} |f(v) - f(x)|^2 \widehat{\pi}(v) \end{aligned}$$

Now, for each  $v \in \widehat{B}(x, \varepsilon + a)$ , take a path of vertices  $x_0 = x, x_1, \dots, x_{\widehat{d}(x,v)} = v$  between  $x$  and  $v$  such that  $\widehat{d}(x_i, x_{i+1}) = 1$  for all  $i$ . Then since such a path has a bounded length,

$$|f(v) - f(x)|^2 \leq \left| \sum_{i=0}^{\widehat{d}(x,v)-1} |f(x_i) - f(x_{i+1})| \right|^2 \leq C \sum_{i=0}^{\widehat{d}(x,v)-1} |f(x_i) - f(x_{i+1})|^2.$$

Moreover  $\widehat{\pi}(v) \approx \widehat{\pi}(x_i)$  for any  $i = 0, \dots, \widehat{d}(x, v) - 1$  since  $\widehat{d}(x_i, v) \leq \varepsilon + a$ , a fixed value. Putting the above estimates together,

$$|f_\varepsilon(x) - f_\varepsilon(y)|^2 \leq \frac{C}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon+a)} \sum_{i=0}^{\widehat{d}(x,v)-1} |f(x_i) - f(x_{i+1})|^2 \widehat{\pi}(x_i).$$

Recall  $x = \Phi(\alpha)$  (and  $y = \Phi(\beta)$ ). Using Remark B.1.1 and the controlled weights of the graphs,

$$\mu_{\alpha\beta} \approx \widehat{\mu}_{\Phi(\alpha)x_1} \approx \widehat{\pi}(\Phi(\alpha)) \approx \widehat{\pi}(x_i) \approx \widehat{\mu}_{x_i x_{i+1}}.$$

Now, as at most finitely points of  $\Gamma$  can map to the same point in  $\widehat{\Gamma}$ ,

$$\begin{aligned} \|\nabla(f_\varepsilon \circ \Phi)\|_2^2 &= \frac{1}{2} \sum_{\alpha, \beta \in \Gamma} |f_\varepsilon \circ \Phi(\alpha) - f_\varepsilon \circ \Phi(\beta)|^2 \mu_{\alpha\beta} = \frac{1}{2} \sum_{\alpha, \beta \in \Gamma} |f_\varepsilon(x) - f_\varepsilon(y)|^2 \mu_{\alpha\beta} \\ &\leq C \sum_{\alpha \in \Gamma} \frac{1}{|\widehat{V}(x, \varepsilon)|} \sum_{v \in \widehat{B}(x, \varepsilon+a)} \sum_{i=0}^{\widehat{d}(x,v)-1} |f(x_i) - f(x_{i+1})|^2 \widehat{\pi}(x_i) \widehat{\mu}(x_i, x_{i+1}) \\ &\leq C \sum_{w \in \widehat{\Gamma}} \frac{1}{|\widehat{V}(w, \varepsilon)|} \sum_{v \in \widehat{B}(w, \varepsilon+a)} \sum_{i=0}^{\widehat{d}(w,v)-1} |f(x_i) - f(x_{i+1})|^2 \widehat{\pi}(x_i) \widehat{\mu}(x_i, x_{i+1}) \\ &\leq C \sum_{w \in \widehat{\Gamma}} \sum_{v \in \widehat{B}(w, \varepsilon+a)} \sum_{i=0}^{\widehat{d}(w,v)-1} |f(x_i) - f(x_{i+1})|^2 \widehat{\mu}(x_i, x_{i+1}), \end{aligned}$$

where in the last line we used that  $\widehat{\pi}(x_i) \approx \widehat{\pi}(w) \approx \widehat{V}(w, \varepsilon)$  for all  $x_i \in \widehat{B}(w, \varepsilon + a)$ .

Further, suppose  $u_1, u_2 \in \widehat{\Gamma}$  and  $u_1 \sim u_2$ . Then there are a finite number of elements  $w \in \widehat{\Gamma}$  such that  $u_1, u_2 \in \widehat{B}(w, \varepsilon + a)$ . Further, in each such ball, there are also a finite number of elements  $v$  where  $u_1, u_2$  could appear on a shortest path between  $w$  and  $v$ . Hence in the above sum, each pair  $u_1, u_2$  appears at most finitely many times. Thus

$$\|\nabla(f_\varepsilon \circ \Phi)\|_2^2 \leq C \sum_{u_1 \in \widehat{\Gamma}} \sum_{u_2 \sim u_1} |f(u_1) - f(u_2)|^2 \widehat{\mu}_{u_1 u_2} = C_A \|\nabla f\|_2^2.$$

Claim 3:  $\|f_\varepsilon \circ \Phi\|_2^2 \geq C_B \|f\|_2^2$  for some constant  $C_B$ .

By our earlier arguments about the support of  $f_\varepsilon \circ \Phi$ , we have

$$\|f_\varepsilon \circ \Phi\|_2^2 = \sum_{\alpha \in \Gamma} |f_\varepsilon \circ \Phi(\alpha)|^2 \pi(\alpha) = \sum_{\alpha \in B(\Phi^{-1}(z), c_2 r)} |f_\varepsilon \circ \Phi(\alpha)|^2 \pi(\alpha).$$

Take a set of points  $\{w_i\}_{i=1}^{i_\varepsilon} \subset \widehat{\Gamma}$  such that

- $w_i = \Phi(\alpha_i)$  for some  $\alpha_i \in \Gamma$  (as the  $w_i$  are distinct, so are the  $\alpha_i$ )
- $\widehat{B}(z, r) \cap \widehat{B}(w_i, \varepsilon) \neq \emptyset$
- $\widehat{B}(z, r) \subseteq \bigcup_{i=1}^{i_\varepsilon} \widehat{B}(w_i, \varepsilon)$ .

Such a set of points exists since every point in  $x \in \widehat{\Gamma}$  must be at distance at most  $\varepsilon$  from a point  $w = \Phi(\alpha)$ .

This choice forces each  $\alpha_i \in B(\Phi^{-1}(z), c_2 r)$ , so that all of the  $w_i$ 's appear in the  $L^2$  norm of  $f_\varepsilon \circ \Phi$ . Hence, recalling  $\pi(\alpha) \approx \widehat{\pi}(w_i)$ ,

$$\|f_\varepsilon \circ \Phi\|_2^2 \geq c \sum_{i=1}^{i_\varepsilon} |f_\varepsilon(w_i)|^2 \widehat{\pi}(w_i).$$

For each  $x \in \widehat{B}(z, r)$ , there exists at least one  $j$  such that  $x \in \widehat{B}(w_j, \varepsilon)$ . Thus

$$\frac{f(x)\widehat{\pi}(x)}{|\widehat{V}(w_j, \varepsilon)|} \leq \frac{1}{|\widehat{V}(w_j, \varepsilon)|} \sum_{y \in \widehat{B}(w_j, \varepsilon)} f(y)\widehat{\pi}(y) = f_\varepsilon(w_j).$$

Up to a constant, the left hand side above is just  $f(x)$ , since  $\widehat{\pi}(x) \approx \widehat{\pi}(w_j) \approx \widehat{V}(w_j, \varepsilon)$ .

As at most finitely many elements  $x \in \widehat{B}(z, r)$  are in the same  $\widehat{B}(w_j, \varepsilon)$  (and, in fact, we have a uniform bound on how many such elements there are), and these small balls cover the entire larger ball, we find

$$\|f_\varepsilon \circ \Phi\|_2^2 \geq c \sum_{i=1}^{i_\varepsilon} |f_\varepsilon(w_i)|^2 \widehat{\pi}(w_i) \geq c \sum_{x \in \Omega} |f(x)|^2 \widehat{\pi}(x) = C_B \|f\|_2^2.$$

□

## B.2 Gluing relative Faber-Krahn functions

Recall all graphs  $(\Gamma, \mu, \pi)$  are assumed to be connected with controlled and uniformly lazy weights. In this section, we will assume  $\Gamma$  is a graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$  as in Chapter 4; however, here we do not assume all of the key hypotheses and instead assume (B4) and a weaker version of (B1).

**Definition B.2.1** (Fixed width spine). As in Chapter 4, let  $(\Gamma, \mu, \pi)$  be a graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$ . We say the spine  $\Gamma_0$  has a *fixed width* of  $\delta > 0$  if for all  $v \in \Gamma_0$ , there exists  $i \in \{1, \dots, l\}$  such that  $d(v, \Gamma_i) \leq \delta$ .

In particular, assuming  $\Gamma_0$  has fixed width  $\delta$  means there exists a page  $\Gamma_i$  and  $x \in \Gamma_i$  such that  $d(v, x) \leq \delta$ . Note the difference between a fixed width spine, where each point of the spine  $\Gamma_0$  is at distance at most  $\delta$  from *some* page, as

opposed to a  $\delta$ -book-like graph, where each point of the spine  $\Gamma_0$  is at distance at most  $\delta$  from *all* pages.

As in Definition 4.1.2, we may still define augmented pages via

$$\widehat{\Gamma}_i := [\Gamma_i]_\delta \cap (\Gamma_i \cup \Gamma_0).$$

Note the augmented pages now need not contain the *entire* spine. It is still the case that the inclusion map  $i : \Gamma_i \rightarrow \widehat{\Gamma}_i$  is a quasi-isometry due to the uniformity hypothesis and controlled weights.

The following theorem gives a relative Faber-Krahn function for such graphs  $\Gamma$  in terms of the relative Faber-Krahn functions on the pages.

**Lemma B.2.1** (Gluing Faber-Krahn functions). *Let  $(\Gamma, \mu, \pi)$  be a graph with pages  $\Gamma_1, \dots, \Gamma_l$  and spine  $\Gamma_0$ . Assume the spine  $\Gamma_0$  has fixed width (Definition B.2.1) and hypothesis (B4) from Section 4.1.3 (that each page  $\Gamma_i$  is uniform in  $\Gamma$ ). Moreover, assume each page  $\Gamma_i$ ,  $1 \leq i \leq l$  has a relative Faber-Krahn function  $\Lambda_i$ .*

*Let  $B = B(z, r) \subset \Gamma$ . Then  $\Gamma$  has a relative Faber-Krahn function given by*

$$\Lambda(B, \nu) := \begin{cases} a_1 \Lambda_i(B, a_2 \nu) & \text{if } B \subset \Gamma_i \text{ for some } i = 1, \dots, l \\ \overline{\Lambda}(B, \nu) & \text{else,} \end{cases}$$

where

$$\overline{\Lambda}(B, \nu) := c_1 \min_{i \in J_B} \min_{\alpha \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} \Lambda_i(B_i(\alpha, c_2 r), c_3 \nu).$$

with  $J_B := \{1 \leq i \leq l : B \cap \widehat{\Gamma}_i \neq \emptyset\}$ .

*Proof.* Let  $B = B(z, r) \subset \Gamma$  and let  $\Omega \subset B$ . Take a function  $f$  supported in  $\Omega$ . Notice for a fixed  $B$ , the definition of  $\Lambda$  in the lemma is decreasing in  $\nu$  since each  $\Lambda_i$  is.

If  $B \subset \Gamma_i$  for some  $1 \leq i \leq l$ , then  $B = B_i$ . Recalling the cutting/gluing construction that makes  $\Gamma$  ensures compatible weights, we compute:

$$\begin{aligned} \|\nabla f\|_2^2 &= \frac{1}{2} \sum_{x,y \in \Gamma} |f(x) - f(y)|^2 \mu_{xy}^\Gamma \geq \frac{1}{2} \sum_{x,y \in \Gamma_i} |f(x) - f(y)|^2 \mu_{xy}^i \\ &\geq \Lambda_i(B_i, \pi_i(\Omega)) \sum_{x \in \Gamma_i} |f(x)|^2 \pi_i(x) \\ &\geq a_1 \Lambda_i(B_i, \pi_i(\Omega)) \sum_{x \in \Gamma} |f(x)|^2 \pi_\Gamma(x). \end{aligned}$$

Above  $a_1$  is some constant depending on the compatible weights. Additionally, due to the compatible weight condition and the fact that  $\Omega \subset \Gamma_i$ , we have  $\pi_i(\Omega) \approx \pi_\Gamma(\Omega)$ , proving the first part of the lemma for some appropriate constant  $a_2$ , which again depends on the compatible weights hypothesis.

Now suppose that  $B \not\subset \Gamma_i$  for some  $1 \leq i \leq l$ . Consider the augmented pages  $\widehat{\Gamma}_i$ . Set  $\widehat{\Omega}_i := \widehat{\Gamma}_i \cap \Omega$  and define

$$\widehat{I}_i := \sum_{x \in \widehat{\Omega}_i} |f(x)|^2 \widehat{\pi}_i(x) \left( = \sum_{x \in \widehat{\Gamma}_i} |f(x)|^2 \widehat{\pi}_i(x) \right).$$

(Recall there is some ambiguity in how  $\widehat{\pi}_i$  is defined, but that all such definitions are comparable to each other and to  $\pi_\Gamma$  up to constants; see Definition 4.1.2.)

Since the spine  $\Gamma_0$  has fixed width,  $\Gamma \subseteq \bigcup_{i=1}^l \widehat{\Gamma}_i$ . Then, where  $c_B, C_B$  are the constants controlling compatible weights in the cutting/gluing,

$$l \sum_{x \in \Gamma} |f(x)|^2 \pi_\Gamma(x) \geq \frac{1}{C_B} \sum_{i=1}^l \widehat{I}_i \geq \frac{c_B}{C_B} \sum_{x \in \Gamma} |f(x)|^2 \pi_\Gamma(x),$$

since each point in  $\Gamma$  appears in at most all  $l$  of the sets  $\widehat{\Gamma}_i$  and as the weights  $\widehat{\pi}_i$  are all comparable to  $\pi = \pi_\Gamma$ .

Fix  $\varepsilon < (c_B/l)$ . Then there exists at least on  $j \in \{1, \dots, l\}$  such that  $\widehat{I}_j \geq \varepsilon \|f\|_2^2$ . In particular, for such  $j$ , this means  $f$  is not identically zero on  $\widehat{\Gamma}_j$ . Hence  $B$  must

intersect  $\widehat{\Gamma}_j$  and  $j \in J_B$ . Further,  $B \cap (\widehat{\Gamma}_j \setminus \Gamma_j) \neq \emptyset$ , since  $B$  is connected and is not contained only in  $\Gamma_j$ . Take  $y \in B \cap (\widehat{\Gamma}_j \setminus \Gamma_j)$ .

We claim there exists a constant  $C^*$  such that  $\widehat{\Omega}_j \subseteq \widehat{B}_j(y, C^*, r)$ .

First note that  $\widehat{\Omega}_j \subseteq \Omega \subseteq B_\Gamma(z, r) \subset B_\Gamma(y, 4r)$  since  $y \in B_\Gamma(z, r)$ . Let  $w \in \widehat{\Omega}_j$ . Recall  $\Gamma_j$  and  $\widehat{\Gamma}_j$  are quasi-isometric (with constant  $a$  relating their distances, constant  $\varepsilon = \delta$  relating neighborhoods) and  $\Gamma_j$  is uniform in  $\Gamma$  (with a constant  $C_U$ ). Then, since  $r \geq 1$ ,

$$\begin{aligned} \widehat{d}_j(w, y) &\leq ad_j(\Phi^{-1}(w), \Phi^{-1}(y)) \leq ac_U \widehat{d}_i(\Phi^{-1}(w), \Phi^{-1}(y)) \\ &\leq ac_U (\widehat{d}_i(\Phi^{-1}(w), w) + \widehat{d}_i(w, y) + \widehat{d}_i(y, \Phi^{-1}(y))) \\ &\leq ac_U(2\delta + 4r) \leq ac_U(2\delta + 4)r. \end{aligned}$$

Thus  $w \in \widehat{B}_j(y, C^*r)$  for  $C^* = ac_U(2\delta + 4)$ .

Let  $\widehat{\Lambda}_j$  denote a Faber-Krahn function for  $\widehat{\Gamma}_j$ . Assume the comparable condition of the edge weights is given by  $c_m \mu_{xy}^\Gamma \leq \mu_{xy}^i \leq C_M \mu_{xy}^\Gamma$ . Then, by the above, our choice of the index  $j$ , and Lemma B.1.1,

$$\begin{aligned} \|\nabla f\|_2^2 &= \frac{1}{2} \sum_{x, y \in \Gamma} |f(x) - f(y)|^2 \mu_{xy}^\Gamma \geq \frac{1}{C_M} \sum_{x, y \in \widehat{\Gamma}_j} |f(x) - f(y)|^2 \widehat{\mu}_{xy}^j \\ &\geq \frac{1}{C_M} \widehat{\Lambda}_j(\widehat{B}_j(y, C^*r), \widehat{\pi}_j(\widehat{\Omega}_j)) \sum_{x \in \widehat{\Gamma}_j} |f(x)|^2 \widehat{\pi}_j(x) \\ &\geq \frac{\varepsilon}{C_M} \widehat{\Lambda}_j(\widehat{B}_j(y, C^*r), \widehat{\pi}_j(\widehat{\Omega}_j)) \sum_{x \in \Gamma} |f(x)|^2 \pi_\Gamma(x) \\ &\geq c_1 \Lambda_j(B_j(\Phi^{-1}(y), c_2r), c_3 \widehat{\pi}_j(\widehat{\Omega}_j)), \|f\|_2^2. \end{aligned}$$

In the last line, we combined some constants. Now,  $\widehat{\pi}_j(\widehat{\Omega}_j) \leq C_B \pi(\Omega)$ , and Faber-Krahn functions are non-increasing in  $v$ , so we may replace  $c_3 \widehat{\pi}_j(\widehat{\Omega}_j)$  by  $c_3 C_B \pi(\Omega)$  in  $\Lambda_j$  above, we then call  $c_3 C_B$  as  $c_3$  again.

It remains to show that we can control the expression involving  $\Lambda_j$  above by  $\bar{\Lambda}$ . The vertex  $\Phi^{-1}(y)$  belongs to  $\Gamma_j$  and lies within the  $\delta$ -neighborhoods of both  $\Gamma_0$  and  $B$  since  $y \in \Gamma_0 \cap B$ . Therefore it is clear

$$c_1 \Lambda_j(B_j(\Phi^{-1}(y), c_2 r), c_3 \pi(\Omega)) \geq \bar{\Lambda}(B, \nu).$$

□

### B.3 Relation between Faber-Krahn functions and heat kernel estimates

Let  $\Gamma$  be a graph satisfying all the hypotheses of the previous section and Lemma B.2.1. Assume in addition that hypothesis (B2) holds, that is, that each page  $\Gamma_i$  is Harnack. A consequence of this assumption is that the relative Faber-Krahn function of each  $\Gamma_i$  has a nice form. In particular, there exist constants  $a_i, \alpha_i > 0$  such that for all balls  $B_i(z, r)$  in  $\Gamma_i$  and all  $\nu > 0$ ,

$$\Lambda_i(B_i(z, r), \nu) = \frac{a_i}{r^2} \left( \frac{V_i(z, r)}{\nu} \right)^{\alpha_i}. \quad (\text{B.2})$$

Assuming that  $\Gamma$  satisfies all hypotheses of Lemma B.2.1 *and* hypothesis (B2), then we can use the above to write out  $\bar{\Lambda}$  in terms of volume functions of the pages:

$$\begin{aligned} \bar{\Lambda}(B, \nu) &= c_1 \min_{i \in J_B} \min_{\alpha \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} \Lambda_i(B_i(\alpha, c_2 r), c_3 \nu) \\ &= c_1 \min_{i \in J_B} \min_{\alpha \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} \frac{a_i}{c_2^2 r^2} \left( \frac{V_i(y, c_2 r)}{c_2 \nu} \right)^{\alpha_i} \\ &\approx \min_{i \in J_B} \min_{\alpha \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} \frac{C}{r^2} \left( \frac{V_i(y, r)}{\nu} \right)^{\alpha_i}. \end{aligned}$$

In the last line, we have used the fact that each  $V_i$  is doubling (and we have a finite number of pages), so  $V_i(y, c_2 r) \approx V_i(y, r)$  since  $c_2$  is a fixed constant.

Define

$$V_{\min}(z, r) := \min_{i \in J_B} \min_{y \in [B]_\delta \cap \Gamma_i \cap [\Gamma_0]_\delta} V_i(y, r) \quad (\text{B.3})$$

and set

$$F(z, r) := \begin{cases} V_i(z, r) = V_\Gamma(z, r), & \text{if } B \subset \Gamma_i \text{ for some } 1 \leq i \leq l \\ V_{\min}(z, r), & \text{else.} \end{cases} \quad (\text{B.4})$$

**Corollary B.3.1.** *Let  $(\Gamma, \mu, \pi)$  be a graph with pages  $\Gamma_1, \dots, \Gamma_l$  and a fixed width spine  $\Gamma_0$  satisfying hypotheses (B2), and (B4). Set  $\alpha = \min_{1 \leq i \leq l} \alpha_i$ , where  $\alpha_i$  is the exponent appearing in the relative Faber-Krahn function of  $\Gamma_i$  as in (B.2). Let  $V_{\min}$ ,  $F$  be defined as in (B.3), (B.4), respectively. Then a relative Faber-Krahn function for  $\Gamma$  is given by*

$$\Lambda(B(z, r), \nu) = \frac{C}{r^2} \left( \frac{F(z, r)}{\nu} \right)^\alpha. \quad (\text{B.5})$$

The corollary follows directly from Lemma B.2.1 and the above.

**Theorem B.3.1.** *Assume  $\Gamma$  is a graph with a relative Faber-Krahn function given by (B.5) as in Corollary B.3.1 above. Then*

$$p(n, x, y) \leq \frac{c_1}{\sqrt{F(x, \sqrt{n})F(y, \sqrt{n})}} \exp\left(-\frac{d^2(x, y)}{c_2 n}\right). \quad (\text{B.6})$$

There is a significant amount of literature on the relationship between Faber-Krahn functions and heat kernel upper bounds. However, the literature does not quite cover the case given here. As such, we require several lemmas and small modifications of existing work, which we give in the next section for completeness.

### B.3.1 Proof of Theorem B.3.1

There are some very general results about heat kernel upper bounds and Faber-Krahn functions in the continuous case in the work of Grigor'yan; see for example Theorem 5.2 of [25]. However, there are not quite such general results in the discrete case. In [11], Coulhon and Grigor'yan prove the equivalence of  $\Gamma$  being Harnack and having relative Faber-Krahn function of form (B.2). In particular, this implies that if  $\Gamma$  has a relative Faber-Krahn function of form (B.2), then  $p(n, x, y)$  satisfies the upper bound

$$p(n, x, y) \leq \frac{C}{\sqrt{V(x, \sqrt{n})V(y, \sqrt{n})}} \exp\left(-\frac{d_\Gamma^2(x, y)}{cn}\right).$$

We would instead like to prove that if  $\Gamma$  satisfies (B.5) that  $p(n, x, y)$  has the upper bound given by (B.6); this amounts to replacing  $V_\Gamma(z, r)$  by the function  $F(z, r)$ . It is not possible for us to simply replace  $V$  with  $F$  in their entire argument, in part because having a relative Faber-Krahn inequality of form (B.2) would imply  $\Gamma$  is doubling, which we need not be true in our case. Nonetheless, we can still follow most of the same general approach of both [25, 11]. We begin by collecting together some lemmas.

We start with an  $L^2$  mean value inequality as in Section 4 of [11].

**Lemma B.3.1.** *Let  $(\Gamma, \mu, \pi)$  be a graph with a relative Faber-Krahn function of the form (B.5) and let  $u(T, z)$  be a non-negative sub-solution of the discrete heat equation. Then*

$$u^2(T, z) \leq \frac{C}{F(z, r) \min\{T^{1/\alpha+1}R^{-2/\alpha}, R^2\}} \sum_{k=0}^{2T} \sum_{x \in B(z, R)} u^2(k, x) \pi(x). \quad (\text{B.7})$$

*Proof.* We can essentially copy the proof given in [11], Sections 4.4 and 4.5. Mostly, they treat

$$\beta = \frac{a}{R^2} V(z, R)^\alpha$$

as a constant. All of the difference in the situation here enters into the quantity  $\beta$ , as we have instead

$$\tilde{\beta} = \frac{a}{R^2} F(z, R)^\alpha.$$

The only time the volume is explicitly used is to prove that

$$\beta \leq c\pi(z)^\alpha \implies \beta^{\alpha^{-1}\gamma^{-N}} \leq c\pi(z)^{\gamma^{-N}}, \quad (\text{B.8})$$

an equality which seems to “magically” make certain factors in the somewhat involved iteration proof cancel. That the inequality in (B.8) holds for  $\tilde{\beta}$  is true for essentially the same reasons. The desired inequality will follow if we can show

$$\frac{F(z, R)}{\pi(z)} \leq cR^{2/\alpha}.$$

Consider  $B = B(z, R)$  and let  $\Omega = \{z\}$ , which is a subset of  $B(z, R)$ . Let  $f$  be the function on  $\Gamma$  that is 1 at  $z$  and zero everywhere else. Therefore, by the form (B.5) of the Faber-Krahn function on  $\Gamma$ ,

$$\begin{aligned} \frac{1}{2} \sum_{x, y \in \Gamma} |f(y) - f(x)|^2 \mu_{xy} &\geq \Lambda(B, \pi_z) \sum_{x \in B(z, R)} |f(x)|^2 \pi_x \\ \implies \sum_{y \sim z} \mu_{yz} &\geq \frac{a}{R^2} \left( \frac{F(z, r)}{\pi_z} \right)^\alpha \pi_z. \end{aligned}$$

Since the weights are subordinated to the measure, it follows that  $\sum_{y \sim z} \mu_{yz} \leq \pi_z$ .

Therefore we get precisely the desired inequality

$$\left( \frac{F(z, r)}{\pi_z} \right)^\alpha \leq CR^2.$$

Otherwise than this modification, the proof goes through exactly as in [11].

□

We can think of Lemma B.3.1 as an  $L^2 \rightarrow L^\infty$  type estimate. We would like to obtain an  $L^1$  version of this result. There are several ways we can proceed.

The approach we take here is to try to directly repeat the arguments given in the continuous setting in [25]. One of the main obstacles present in the discrete work [11] is that at the time of that paper's publication, there was not yet a discrete version of the integrated maximum principle, which is a key component of the arguments of [25]. Hence the methods in [11] necessarily avoid appealing to this principle. However, Coulhon, Grigor'yan, and Zucca give a discrete version of the integrated maximum principle in [12]. Hence we may make use of several of their results to follow the method of [25].

**Lemma B.3.2** (Discrete integrated maximum principle, Theorem 2.2 [12]). *Let  $(\Gamma, \mu, \pi)$  be connected graph with controlled and uniformly lazy weights, where  $\alpha$  is a constant such that  $\mathcal{K}(x, x) \geq \alpha$  for all  $x \in \Gamma$ . Let  $f$  be a strictly positive function on  $[0, n] \times \Gamma$  such that for all  $x \in \Gamma$ ,  $k \in [0, n)$ ,*

$$\partial_k f(x) + \frac{|\nabla f_{k+1}|^2}{4\alpha f_{k+1}}(x) \leq 0. \quad (\text{B.9})$$

*Then, for any solution  $u$  of the heat equation in  $[0, n] \times \Gamma$ , the quantity*

$$J_k = J_k(u) := \sum_{x \in \Gamma} u_k^2(x) f_k(x) \pi(x) \quad (\text{B.10})$$

*is non-increasing in  $k$ , that is,  $J_{k+1} \leq J_k$  for all  $k \in [0, n)$ .*

**Lemma B.3.3** (Proposition 2.5 of [12]). *Let  $(\Gamma, \mu, \pi)$  be a connected graph with controlled and uniformly lazy weights. Let  $\rho$  be a 1-Lipschitz function on  $\Gamma$  such that  $\inf \rho \geq 1$ . Then there exists a positive number  $D_\alpha$  (depending only on  $\alpha$  from the uniformly lazy condition) such that for all  $D \geq D_\alpha$ , the weight function*

$$f_k(x) = f_k^D(x) := \exp\left(-\frac{\rho^2(x)}{D(n+1-k)}\right) \quad (\text{B.11})$$

*satisfies (B.9) for all  $x \in \Gamma$ ,  $k \in [0, n)$ .*

**Lemma B.3.4** (Proposition 5.3 of [12]). *Again let  $(\Gamma, \mu, \pi)$  be a connected graph with controlled and uniformly lazy weights. Define the quantity*

$$E_D(k, x) := \sum_{z \in \Gamma} p^2(k, x, z) \exp\left(\frac{d_1^2(x, z)}{Dk}\right) \pi(z), \quad (\text{B.12})$$

where  $d_1(x, z) := \max\{d(x, z), 1\}$ .

Then for all  $x, y \in \Gamma$ ,  $k \in \mathbb{N}$ , and all  $D > 0$ ,

$$p(2k, x, y) \leq \sqrt{E_D(k, x)E_D(k, y)} \exp\left(-\frac{d^2(x, y)}{4Dk}\right). \quad (\text{B.13})$$

We now use the above lemmas to prove a discrete version of Theorem 4.1 of [25]. The proof is essentially the same and simply requires appealing to the above discrete setting lemmas (instead of their continuous analogs).

**Lemma B.3.5.** *As above, let  $(\Gamma, \mu, \pi)$  be a connected graph with controlled and uniformly lazy weights with relative Faber-Krahn function of form (B.5). Let  $B = B(z, R)$ ,  $\tilde{d}(x, y) = \max\{1, d(x, y)\}$ , and  $\tilde{d}(x, B) = \max\{1, d(x, B(z, R))\} = \max\{1, (d(x, z) - R)_+\}$ . Then*

$$\begin{aligned} \sum_{y \in \Gamma} p(k, y, z)^2 \exp\left(\frac{\tilde{d}(y, B)^2}{\tilde{c}(T+1)}\right) \pi(y) &\leq \frac{\tilde{c}T}{F(z, R) \min\{R^2, T^{1+1/\alpha}R^{-2/\alpha}\}} \\ &= \frac{c}{F(z, R) \min\{(R^2/T), (T/R^2)^{1/\alpha}\}}. \end{aligned}$$

*Proof.* Fix  $B = B(z, R)$ . Let  $\varphi$  be a function on  $\Gamma$  with finite support, and take  $\Omega \subset \Gamma$  finite and containing both  $\text{supp } \varphi$  and  $B(z, R)$ .

Set

$$u_\Omega(k, x) := \sum_{y \in \Omega} p_{\Omega, D}(k, x, y) \varphi(y) \pi(y). \quad (\text{B.14})$$

By properties of the heat kernel  $p$ ,  $u_\Omega$  is a solution of the heat equation on  $\mathbb{N} \times B(z, R)$ . Without loss of generality, we may assume that  $\varphi \geq 0$ , and, consequently,

the same is true of  $u_\Omega$ . Applying Lemma B.3.1 to  $u_\Omega$  yields

$$u_\Omega^2(T, z) \leq \frac{C}{\underbrace{F(z, R) \min\{T^{1/\alpha+1}R^{-2/\alpha}, R^2\}}_A} \sum_{k=0}^{2T} \sum_{x \in B(z, R)} u_\Omega^2(k, x) \pi(x). \quad (\text{B.15})$$

Now consider the function

$$f(s, x) := \exp\left(-\frac{\tilde{d}(x, B)^2}{\hat{c}(T+1-s)}\right).$$

For  $\hat{c}$  sufficiently large,  $f(s, x)$  is the sort of function considered Lemma B.3.3.

When  $x \in B$ ,  $\tilde{d}(x, B) = 1$  and  $f(x, s) = \exp(-1/(\hat{c}(T+1-s)))$ , which is largest when  $s$  is smallest. In particular, when  $x \in B$ ,  $\exp(-1/\hat{c}) \leq f(s, x) \leq \exp(-1/(\hat{c}(T+1)))$ .

Therefore

$$\begin{aligned} u_\Omega^2(T, z) &\leq A \sum_{k=0}^{2T} \sum_{x \in B(z, R)} u_\Omega^2(k, x) \pi(x) \exp\left(-\frac{1}{\hat{c}}\right) \exp\left(\frac{1}{\hat{c}}\right) \\ &\leq A \exp\left(\frac{1}{\hat{c}}\right) \sum_{k=0}^{2T} \sum_{x \in B(z, R)} u_\Omega^2(k, x) f(k, x) \pi(x) \\ &\leq AC(\hat{c}) \underbrace{\sum_{k=0}^{2T} \sum_{x \in \Omega} u_\Omega^2(k, x) f(k, x) \pi(x)}_{B_k} \end{aligned}$$

We now want to apply the discrete integrated maximum principle. As mentioned above,  $u_\Omega$  solves the heat equation in  $\Omega$ , and, by choice of  $\hat{c}$ , we also have that  $f$  satisfies (B.9). Therefore,  $B_k$  is decreasing in  $k$ , which means it is largest when  $k = 0$ . Consequently

$$\begin{aligned} u_\Omega^2(T, z) &\leq 2AC(\hat{c})T \sum_{x \in \Omega} u_\Omega^2(0, x) f(0, x) \pi(x) \\ &= 2AC(\hat{c})T \sum_{x \in \Omega} \left[ \sum_{y \in \Omega} p_\Omega(0, x, y) \varphi(y) \pi(y) \right]^2 \exp\left(-\frac{\tilde{d}(x, B)^2}{\hat{c}(T+1)}\right) \pi(x) \\ &= 2AC(\hat{c})T \sum_{x \in \Omega} \varphi^2(x) \exp\left(-\frac{\tilde{d}(x, B)^2}{\hat{c}(T+1)}\right) \pi(x). \end{aligned}$$

All sums above were finite and well-defined since  $\Omega$  is itself finite. As  $\Omega \rightarrow \Gamma$ , then  $u_\Omega \rightarrow u_\Gamma$  and

$$u_\Gamma^2(T, z) = \left[ \sum_{y \in \Gamma} p(T, z, y) \varphi(y) \pi(y) \right]^2 \leq 2AC(\hat{c})T \sum_{x \in \Gamma} \varphi(x)^2 \exp\left(-\frac{\tilde{d}(x, B)^2}{\hat{c}(T+1)}\right) \pi(x).$$

Now think of  $T, z$  as fixed and consider a map  $\Phi : L^2(\Gamma) \rightarrow \mathbb{R}$  given by

$$\Phi(\eta) := \sum_{y \in \Gamma} p(T, y, z) \exp\left(\frac{\tilde{d}(y, B)^2}{2\hat{c}(T+1)}\right) \eta(y) \pi(y).$$

If  $\eta(y)$  has the form  $\exp\left(-\frac{\tilde{d}(y, B)^2}{2\hat{c}(T+1)}\right) \varphi(y)$ , then  $\Phi(\eta) = u(T, z)$ . In this case,

$$\Phi(\eta)^2 \leq AC(\hat{c})T \|\eta\|_{L^2(\Gamma)}^2.$$

Since functions  $\eta$  of the form  $\exp\left(-\frac{\tilde{d}(y, B)^2}{2\hat{c}(T+1)}\right) \varphi(y)$  (where  $\varphi$  has compact support) are dense in  $L^2(\Gamma)$ ,

$$\|\Phi(\eta)\|_{\mathbb{R}}^2 = |\Phi(\eta)|^2 \leq AC(\hat{c})T \|\eta\|_{L^2(\Gamma)}^2, \quad (\text{B.16})$$

which implies  $\|\Phi\|^2 \leq AC(\hat{c})T$  and that  $\Phi$  is well-defined.

On the other hand,  $\|\Phi\| = \sup\{\|\Phi(\eta)\| : \|\eta\|_{L^2} = 1\}$ , so by Cauchy-Schwartz,

$$\begin{aligned} \|\Phi(\eta)\| &\leq \left[ \sum_{y \in \Gamma} p^2(T, y, z) \exp\left(\frac{\tilde{d}(y, B)^2}{\hat{c}(T+1)}\right) \pi(y) \right]^{1/2} \left[ \sum_{y \in \Gamma} \eta^2(y) \pi(y) \right]^{1/2} \\ &= \left[ \sum_{y \in \Gamma} p^2(T, y, z) \exp\left(\frac{\tilde{d}(y, B)^2}{\hat{c}(T+1)}\right) \pi(y) \right]^{1/2}. \end{aligned}$$

The supremum occurs when the above inequality is an equality, and hence

$$\|\Phi\|^2 = \sum_{y \in \Gamma} p^2(T, y, z) \exp\left(\frac{\tilde{d}(y, B)^2}{\hat{c}(T+1)}\right) \pi(y) \leq AC(\hat{c})T.$$

Recalling the definition of  $A$  and simplifying, we get the desired inequality

$$\sum_{y \in \Gamma} p^2(T, y, z) \exp\left(\frac{\tilde{d}(y, B)^2}{\hat{c}(T+1)}\right) \pi(y) \leq \frac{Ce^{-1/\hat{c}}}{F(z, R) \min\{(T/R^2)^{1/\alpha}, R^2/T\}}.$$

□

We are at last ready to prove the theorem.

*Proof of Theorem B.3.1.* We begin with the result of Lemma B.3.4:

$$p(2k, x, y) \leq \sqrt{E_D(k, x)E_D(k, y)} \exp\left(\frac{-d^2(x, y)}{4Dk}\right).$$

Recall

$$E_D(k, x) = \sum_{w \in \Gamma} p^2(k, x, w) \exp\left(\frac{\tilde{d}(x, w)^2}{Dk}\right) \pi(w),$$

where  $\tilde{d}(x, w) := \max\{d(x, w), 1\}$ . The only difference between  $E_D$  and the left hand side of Lemma B.3.5 is in the exponential term. In order to apply Lemma B.3.5 to  $E_D$ , we need to justify that we can replace  $\exp\left(\frac{\tilde{d}(x, w)^2}{Dk}\right)$  with  $\exp\left(\frac{\tilde{d}(w, B)^2}{\hat{c}(k+1)}\right)$ , where we take  $B = B(x, R)$ .

This sort of estimate is only of interest to us if  $k \geq 1$ , so  $k+1 \approx k$ . As in [25] (see the proof of Corollary 4.1), a quadratic inequality holds. In particular, provided that  $D > 2\hat{c}$ ,

$$\frac{2\hat{c}}{D} \tilde{d}(x, y)^2 - \frac{2\hat{c}}{D - 2\hat{c}} R^2 \leq \tilde{d}(w, B)^2. \quad (\text{B.17})$$

Therefore

$$\exp\left(\frac{\tilde{d}(x, w)^2}{Dk}\right) \leq \exp\left(\frac{R^2}{(D - 2\hat{c})k}\right) \exp\left(\frac{\tilde{d}(w, B)^2}{2\hat{c}k}\right) \leq \exp\left(\frac{R^2}{(D - 2\hat{c})k}\right) \exp\left(\frac{\tilde{d}(w, B)^2}{\hat{c}(k+1)}\right).$$

Hence, applying Lemma B.3.5,

$$\begin{aligned} E_D(k, x) &\leq \sum_{w \in \Gamma} p^2(k, x, w) \exp\left(\frac{R^2}{(D - 2\hat{c})k}\right) \exp\left(\frac{\tilde{d}(w, B)^2}{\hat{c}(k+1)}\right) \pi(w) \\ &\leq \exp\left(\frac{R^2}{(D - 2\hat{c})k}\right) \frac{C(\hat{c})}{F(x, R) \min\{(R^2/k), (k/R^2)^{1/\alpha}\}}. \end{aligned}$$

Now choose  $R = \sqrt{2k}$ ; then  $R^2/k$  is the constant 2 and

$$E_D(k, x) \leq \frac{C(\hat{c}, D)}{F(x, \sqrt{2k})}.$$

Chaining inequalities above together gives

$$p(2k, x, y) \leq \frac{C}{\sqrt{F(x, \sqrt{2k})F(y, \sqrt{2k})}} \exp\left(-\frac{d^2(x, y)}{\tilde{c}(2k)}\right), \quad (\text{B.18})$$

where  $C, \tilde{c}$  depend on  $\hat{c}$ . Tracing back through the lemmas,  $\hat{c}$  only depends upon the value  $\alpha$  from the uniformly lazy condition (i.e.  $\mathcal{K}(x, x) \geq \alpha \forall x \in \Gamma$ , and not on  $x, y$ , or  $k$ ). Inequality (B.18) gives the desired result for even times; to get the result for odd times, note one more time step doesn't change much due to the controlled and uniformly lazy weights. There is also no real need in the proof above to take  $k$  an integer, though some modifications should be made with floor/ceiling functions where appropriate.  $\square$

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