CHAPTER II.

POSITION, VELOCITY, AND ACCELERATION.

11. **Velocity.**—While Kinematics in its general sense comprises all kinds of problems dealing with pure motion, the number of such problems falling within the province of the Kinematics of Machines is somewhat limited. We shall consider in this chapter some elementary notions concerning velocity which are applicable to the purposes of the Kinematics of Machines. Methods of studying the position and motion of a point or rigid body from a geometrical point of view have already been indicated; it now remains to investigate not only the amount by which such position is changed during motion, but the rate of such change of position. Going a step farther still, it may be asked, does such velocity increase, diminish, or change in any way as time goes on, and if so, at what rate?

The rate of change of position of a point or body is called its *velocity.* A body, as we have seen, may change its position by a motion of translation, or by one of rotation. Hence we distinguish between linear and angular velocity. The former is measured by the space passed over in unit of time, and is usually expressed in feet per second, although other units, such as miles per hour or knots, are adopted in special cases. The latter is measured by the angle described in unit of time, the natural unit being therefore one radian per second. Engineers, however, commonly measure angular velocity in revolutions per minute. Either kind of velocity may be uniform or variable.

It is important to note that the term velocity involves the ideas of both speed, direction, and sense. In other words, a velocity is a vector quantity, and, like other vector quantities, may be represented by a straight line of definite length, this length being proportional to the speed, or magnitude of the velocity, measured in feet per second, radians per second, or whatever units are to be employed.

In the case of linear velocity the direction of the vector or straight line representing the velocity on the diagram is taken to represent the *direction* of the motion. Thus, for example, we might draw upon a map a line running east and west, and 2 inches in length, and take this line as representing a linear velocity of 2 miles per hour, or 2 feet per second, either from east to west, or from west to east. The *sense* of the motion may be either from east to west, or from west to east. In order to indicate the sense, we place upon the line a small arrow-head so as to show the point towards which the body is moving (see Fig. 12).

In the case of angular velocity the direction of the vector on the diagram would be taken to represent the direction in space of the axis about which the spin or rotation is taking place, and a line similar to that mentioned above would mean a spin of two radians per second, or two revolutions per minute, according to the scale, about an axis lying east and west. This rotation may be either right-handed or left-handed, and it is therefore customary to indicate the sense by placing the arrow-head in such a fashion that the spin will appear to be right-handed, or clockwise, when looking along the axis and following the arrow-head.

It is plain that in this manner a velocity, whether linear or angular, may be completely represented by a vector, having magnitude, direction, and sense.

12. Uniform Velocity.—A body having uniform velocity (whether angular or linear) performs equal changes of position in equal times. If the body has a uniform linear velocity v, it describes a distance vt in time t, where t is any number of units of time. Calling s the space described, we have therefore $s = vt$. Similarly, if the uniform velocity is angular and is denoted by ω, any line on the body in a plane perpendicular to the axis of rotation describes ω radians in each second and therefore ωt radians in t seconds. Hence, calling θ the angle described in t seconds, we have

$$\theta = \omega t.$$

If a point, at distance r from the centre about which it moves in a circular path, has a linear velocity v, its angular velocity is measured by the angle subtended at the centre by the path it describes in one second. Hence

$$\omega = \frac{v}{r} \quad \text{or} \quad v = \omega r.$$

13. Variable Velocity. — In general a moving body varies its speed as well as its direction of motion. It is easy by observing the time taken to travel over a known distance, for example in a train, to calculate the *average speed* of the train during the interval considered. This does not tell us, however, the *actual speed of the train at any instant* during the interval of time, which may be quite different from the average speed.

The velocity at any instant, or instantaneous velocity, is measured by the space (or angle, as the case may be) which would have been described in a unit of time if the motion had continued uniformly, during that interval, at the same rate as at the instant considered. The word instant is here used to mean an indefinitely small interval of time.

We are not able to measure the distance (or angle) described during an indefinitely small interval of time, and therefore have to obtain the value of the instantaneous velocity of a body in another manner.

This will be best understood by a numerical example. Suppose that a man in a street-car at 12 o'clock finds that in 10 seconds the car traverses a distance of 200 feet. This gives 20 feet per second as the average speed during the 10 seconds after 12 o'clock. Suppose that other observations taken during the first $1\frac{1}{5}$, 2, and 4 seconds showed that during these times the car travelled 30, 48, and 100 feet, corresponding to average speeds of 25, 24, and 21.75 feet per second. It is evident that the speed must really have been continually diminishing, and that the shorter the time during which the observation was made, the more nearly do we

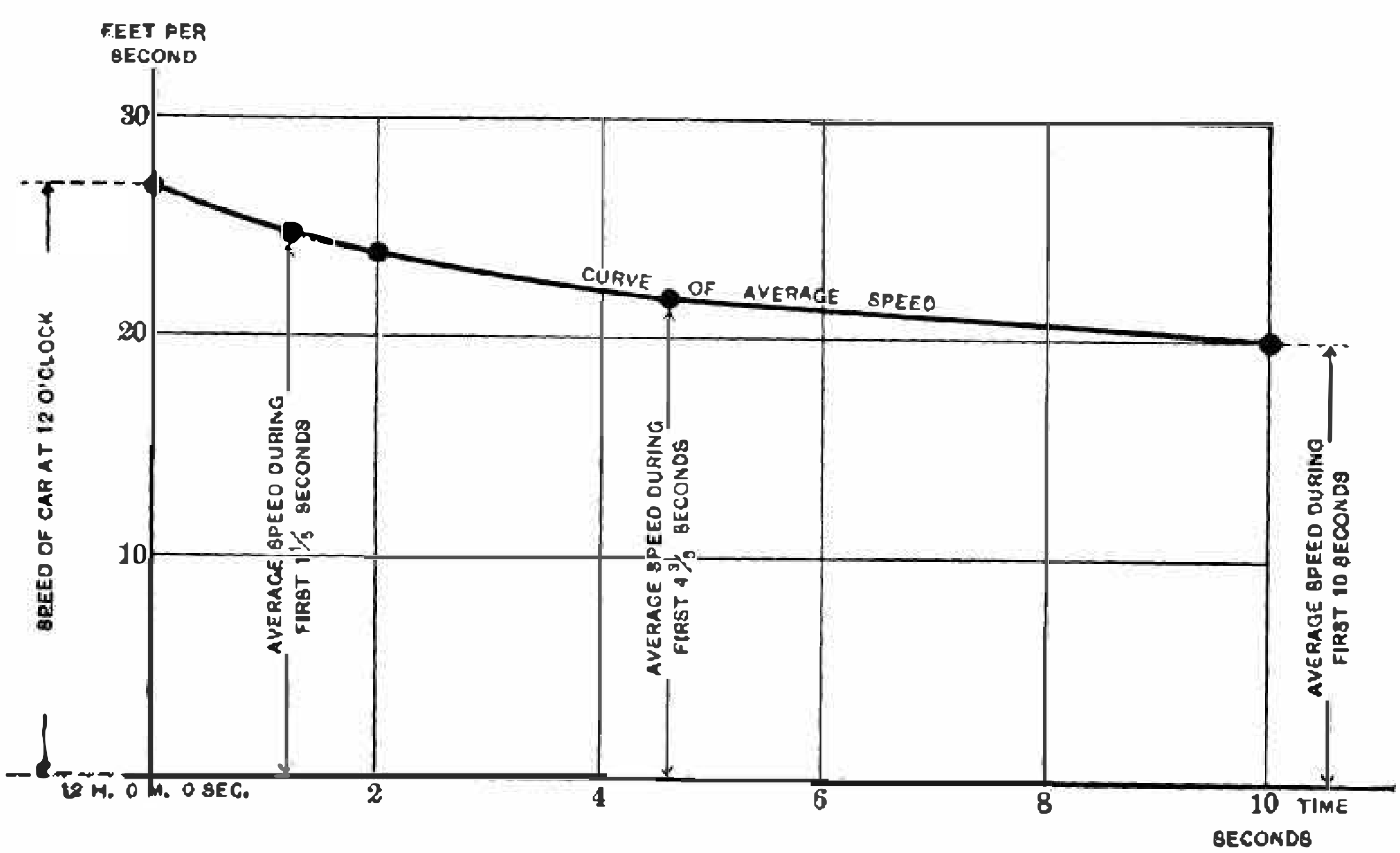


FIG. 10.

obtain the speed at which the car must have been travelling at 12 o'clock. To arrive at this more exactly, since we cannot measure the distance passed over in an infinitely small interval of time, we plot a curve from our observations, as in Fig. 10, and see that the speed at 12 o'clock must have been 27 feet per second. In mathematical language, if Δs be the distance traversed in a small interval of time Δt, the average velocity during that small interval is $\frac{\Delta s}{\Delta t}$, while the velocity at the instant beginning the interval is measured by diminishing Δt indefinitely, and finding the limiting value of $\frac{\Delta s}{\Delta t}$, or, in the language of the calculus $\frac{ds}{dt}$. Thus

$$v = \frac{ds}{dt}.$$

The same reasoning, of course, applies in the case of angular velocity, where we should write

$$\omega = \frac{d\theta}{dt}.$$

Compare these with the corresponding expressions in the case of uniform velocity.

14. **Uniform Acceleration.**—A body moving with uniform acceleration changes its velocity by equal amounts in equal times. Thus suppose that in time t the velocity changes from v_1 to v_2; we have, if a is the acceleration,

$$a = \frac{v_2 - v_1}{t}. \qquad (1)$$

Again, the average velocity during the time t is $\frac{v_2 + v_1}{2}$, the arithmetical mean between the initial and final velocities; hence if s be the space described,

$$s = \frac{v_2 + v_1}{2} t. \qquad (2)$$

From these two expressions we find

$$s = \frac{v_2^2 - v_1^2}{2a}. \qquad (3)$$

But $v_2 = v_1 + at$. Substituting in (3), we get

$$s = v_1 t + \frac{a}{2} t^2. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

In the case of angular velocity precisely similar relations hold, so that, calling α the uniform angular acceleration, ω the angular velocity at the beginning of the time t, and θ the angle described, we have, instead of (4),

$$\theta = \omega_1 t + \frac{\alpha}{2} t^2. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4a)$$

To express the velocity in terms of distance (or angle) and initial velocity we shall have instead of (3)

$$\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta, \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3a)$$

while the expression connecting velocity, acceleration, and time is

$$\omega_2 = \omega_1 + \alpha t. \qquad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1a)$$

As an example of the use of these expressions, suppose a wheel is revolving thirty times per second and comes to rest in 12 seconds. How many revolutions will it make in coming to rest if uniformly retarded?

We have $\omega_2 = \omega_1 + \alpha t$; hence

$$12\alpha + 30 \times 2\pi = 0,$$

and $\alpha = -\dfrac{60\pi}{12} = -15.71$ radians per second per second.

Again, $\omega_2{}^2 = \omega_1{}^2 + 2\alpha\theta$; hence

$$(60\pi)^2 - 10\pi\theta = 0,$$

and $\theta = \dfrac{(60\pi)^2}{10\pi} = 360\pi$ radians.

Hence the wheel comes to rest in 180 revolutions.

Again, a train starting from rest has a uniform acceleration of half a mile per hour per second. How far will it have travelled before attaining a speed of 30 miles per hour, and in what time will this occur?

In the equation (4) above we have $s = v_1 t + \frac{1}{2}at^2$.

Here $v_1 = 0$, t evidently will be 60 seconds, and $a = \frac{2640}{3600} =$ 0.733 feet per second per second. Thus

$$s = \frac{0.733 \times 3600}{2} = 1218 \text{ feet.}$$

It should be noted that it is as incorrect to speak of an acceleration of so many *feet per second* as it would be to say that a body has a velocity of so many *feet*, without mentioning the unit of time.

15. Acceleration in General.—The determination of velocity and acceleration in the case of non-uniform or non-

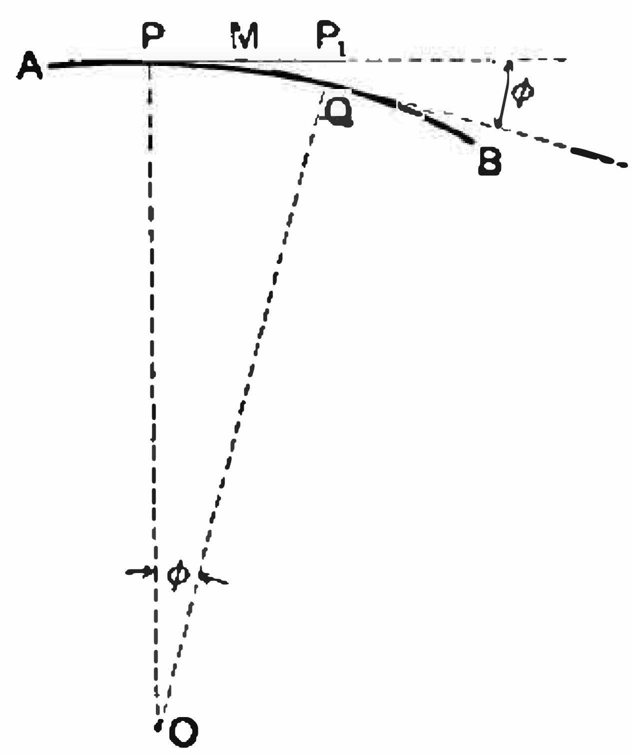


FIG. 11.

uniformly accelerated motion will be discussed later. Acceleration is defined generally as rate of change of velocity with regard to time, and so far we have used the term as meaning change in the *magnitude* of the velocity, whether linear or angular. Strictly speaking, however, a change in the *direction* of motion in linear velocity, or in the position of the axis of rotation in angular velocity, is also an acceleration. In fact, a point travelling in a circular path around a fixed point has an acceleration impressed upon it, although its angular velocity may be uniform, and such acceleration is called *radial*, for reasons which will presently be seen.

In Fig. 11 let AB represent a portion of the curved path alng which a point is travelling with a linear velocity v, whose direction is continually changing. Let ρ be the radius of curvature OP of a very small portion PQ, and O the centre of curvature, φ being the very small angle between the tangents at P and Q, an angle so small that the arc PQ is not sensibly different from its chord.

Consider the acceleration in a direction parallel to PO. The time taken for the particle to travel from P to Q will be $\frac{PQ}{v}$. But during this time the distance traversed under acceleration a is P_1Q parallel to PO. Hence (if a is constant)

$$P_1Q = \tfrac{1}{2}a\left(\frac{PQ}{v}\right)^2 \text{ and } a = v^2\frac{2P_1Q}{(PQ)^2}.$$

It is known that for very small angles the numerical value of the sine of an angle is sensibly the same as the angle itself (of course expressed in circular measure). Also in the figure if we make φ small enough, $MQ = \frac{1}{2}PQ$, the error in this statement diminishing as φ diminishes. Hence *for an indefinitely small value of* φ we may say that

$$\frac{2P_1Q}{PQ^2} = \frac{P_1Q}{PQ.MQ} = \phi \times \frac{1}{PQ} = \frac{PQ}{OP} \times \frac{1}{PQ} = \frac{1}{\rho} \text{ and hence } a = \frac{v^2}{\rho}.$$

The statement is exactly correct, and a is the radial acceleration at P, because we have taken φ as being the angle described during an indefinitely small interval of time.

The earth's equatorial radius is 4000 miles, and the earth makes one rotation on its axis in about 86,200 seconds. What is the radial acceleration of a particle on the earth's surface at the equator?

Linear velocity of a point at equator

$$=\frac{2\pi\times 4000\times 5280}{86200}\text{ feet per second.}$$

$$\text{Thus }\frac{v^2}{\rho}=\frac{(2\pi)^2\times 4000\times 5280}{86200^2}$$

$$=0.112\text{ feet per second per second.}$$

It is often necessary to find the acceleration of a body along its virtual radius; this is of course determined in exactly the same way as the radial acceleration with regard to a permanent centre.

16. Composition of Velocities and Accelerations.—It has been already pointed out that velocities, whether linear or angular, can be represented by straight lines of definite length, sense, and direction, and are in fact *vector quantities*, as distinguished from *scalar quantities*, such as mass, energy, and so on which have simply numerical values. Accelerations are also vector quantities.

The *resolved part* of a vector in any new direction is found by projecting its original length on the new direction.

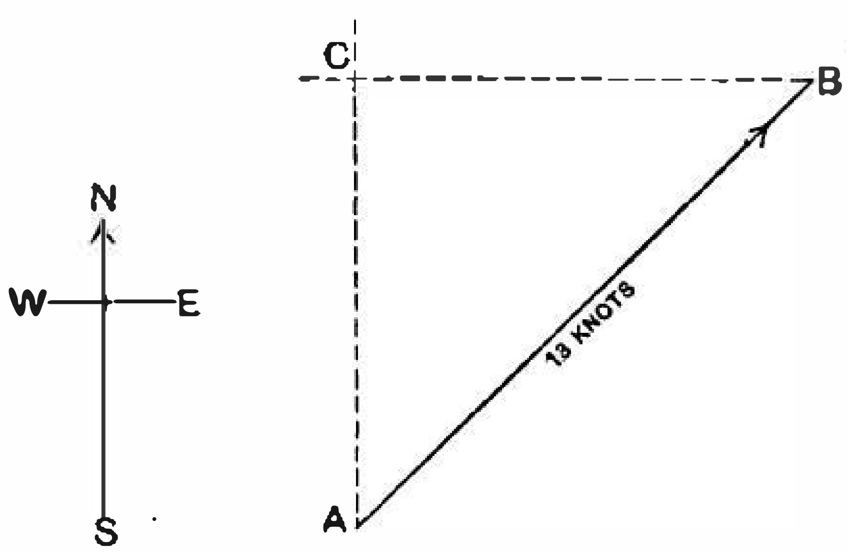


FIG. 12.

If, for example, a ship is proceeding northeast at a speed of 13 knots, represented by the vector AB (1 knot being a speed of 6080 feet, or 1 nautical mile, per hour), its resolved velocity in a northerly direction is represented by

$$AC = AB \cos 45^\circ = 13 \times 0.707 = 9.19 \text{ knots.}$$

This shows that each hour the position of the ship is 9.19 nautical miles farther to the northward.

Again, suppose that the ship, still steering N.E. at the same speed, runs into a current whose speed is 4 knots due east, what will be the real velocity of the ship relatively to the earth? *Relatively to the water* its speed is still 13 knots

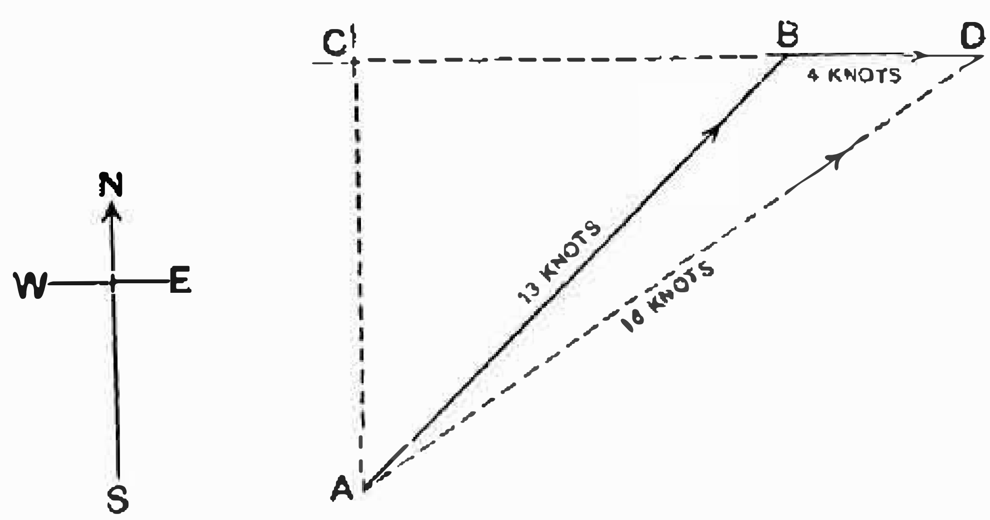


FIG. 13.

in a N.E. direction, but the water is itself moving, and at the end of the hour the ship will evidently be at D, a position obtained by measuring 4 nautical miles east from B. On calculation it will be found that at any time during the hour the ship has been moving relatively to the earth along the line AD, and its real speed over the ground (about 16 knots) will be measured by the length of AD, the third side of a triangle, whose other two sides represent respectively the velocity of the ship relatively to the water, and the velocity of the water relatively to the earth. We say, then, that the vector AD represents the *resultant* of the two vectors AB and BD, obtained by the process of vector addition.

The above example deals with plane motion in a straight line only. But if we are treating of the motion in space of a body having six degrees of freedom, its motion may be considered as made up of three motions of simple translation and three motions of rotation, which, when compounded according to the method just explained, constitute the actual motion of the body.

It must not be forgotten that the resultant of two or more angular velocities can be found in exactly the same way as for linear velocities. As already explained, it is customary to indicate an angular velocity by a vector (as in Fig. 14), representing the numerical value of the velocity by the length AB, the direction of the axis by the direction of AB, and the sense of rotation by drawing AB in such a manner that the rotation is clockwise, or right-handed, when looking from A to B. It is often necessary to compound or to resolve spins or angular velocities according to the method of vector addition, which will be already familiar to most readers under the name of the triangle of velocities, or the parallelogram law for the composition of vectors.

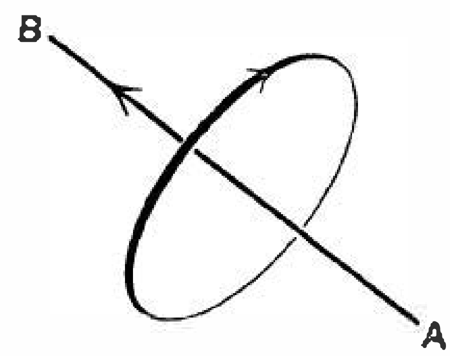


FIG. 14.

17. Resultant Acceleration.—In Fig. 15 let AB represent the original velocity of a particle, and suppose that accelerations represented by BC, BD are impressed upon the particle. Then BC and BD may be taken to represent the velocities generated in one second, corresponding respectively to the two accelerations.

If now the acceleration BC had alone acted on the point, its velocity at the end of one second would have been AC. Again, if AC had been the original velocity and an acceleration BD had been impressed, the final velocity at the end of one second would have been AD', where CD' is equal and parallel to BD. The two accelerations, therefore, have changed the original velocity from AB to AD'. But this effect would have been produced by compounding with AB *for one second* a velocity BD', and we may therefore look on BD' as representing the change of velocity in one second, due to the action of the accelerations BC and BD. In

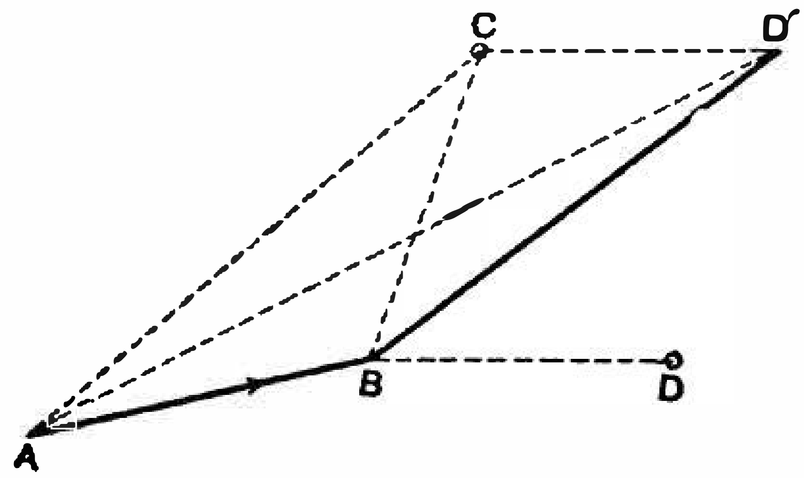


FIG. 15.

other words, BD' is the resultant of the accelerations BC and BD.

The general rule for the composition or addition of vectors, then, is that the resultant of two vectors is the diagonal of a parallelogram (or the third side of a triangle) of which the two components form the two adjacent sides. In this way we can find the resultant of any number of velocities, or of accelerations, either linear or angular. The same rules apply to the composition of any other vector quantities.

18. Diagrams of Displacement and Velocity.—In studying the motion of a body, whether linear or angular, it is necessary to know the position of the body at every instant during the motion, if we desire full information as to its velocity and acceleration. We have seen that if we only know the position of the body at certain times we can obtain the value of the average velocity between those times, but cannot tell exactly how the real velocity has changed.

It would of course be very cumbersome to have to state in words or figures a sufficient number of particulars to give us a practically complete knowledge of the position, velocity, and acceleration of a body, and therefore in such cases graphic methods of representation and calculation are generally adopted.

For example, in order to determine the velocity of a body whose changes of position are known, such a diagram as Fig. 16 is constructed. Two axes, OA, OB, are drawn at right angles, and distances measured parallel to OA according to any convenient scale are considered to represent time,

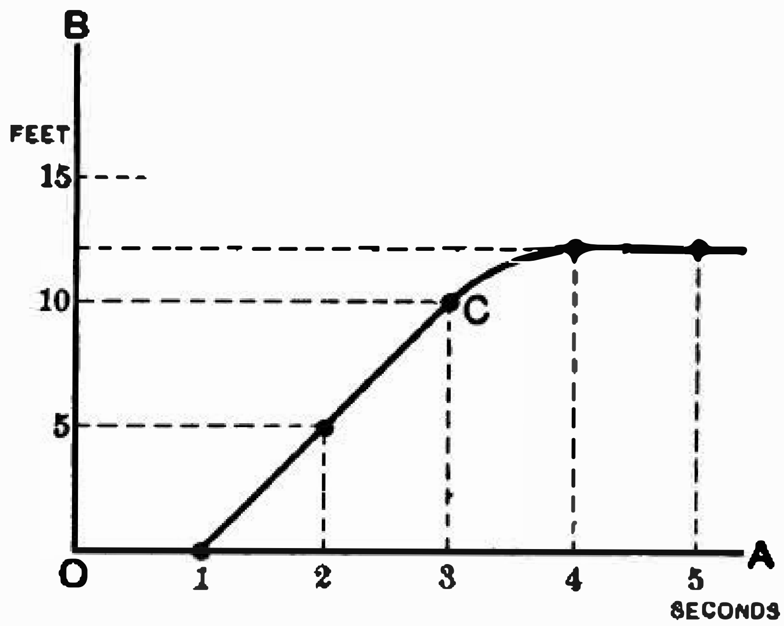


FIG. 16.

while lengths measured parallel to OB represent either the distance that has been traversed by the body, reckoning from some known position, or the angle turned through by the body at any given instant. This quantity we may call the displacement of the body, and it may be either linear or angular.

For instance, from the figure we see that after the lapse of 3 seconds the body in question has moved 10 feet from its original position, and we might give the information contained in the diagram in a less complete form in the shape of a table, thuse

Time........	0	1	2	3	4	5 seconds.
Displacement.	0	0	5	10	12	12 feet.

From the diagram, however, we are enabled to gather further particulars, for it is plain that the curve of displacement during the second and third seconds is straight, i.e., distance is increasing proportionally to time, or the velocity is uniform and the speed 5 feet per second. During the fourth second the distance increases more and more slowly, and then remains constant, hence the velocity diminishes and finally ceases. Thus plainly, in order to find out the velocity at any instant from such a diagram, we have only to determine the rate at which distance (or angle) is increasing or diminishing at that instant. We shall see that this information can be obtained from the form of the curve.

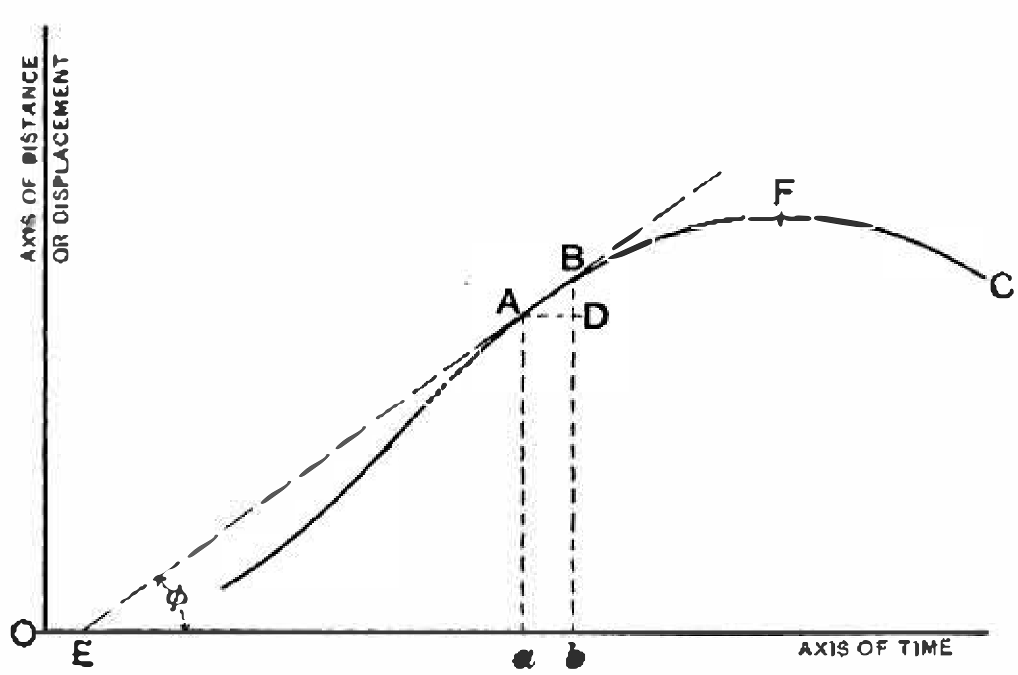


FIG. 17.

In Fig. 17 let ABC be any curve of displacement on a time base, and let Aa, Bb represent the distances (or angles) corresponding to the times Oa, Ob. We wish to determine the velocity at the point A, i.e., after the lapse of the time Oa.

It has already been pointed out that to find the instantaneous value of a velocity, or the velocity at any instant, we must take what is really the average velocity during an infinitely small time, or, if Δs be a small change in position and Δt the corresponding interval of time, the instantaneous velocity is the limiting value of $\frac{\Delta s}{\Delta t}$. In the figure let A and B be very close together; evidently $BD = Bb - Aa = \Delta s$, the change in distance during the time Δt. The small interval of time Δt is represented by the length $ab = AD$. Hence the ratio

$$\frac{BD}{AD} = \frac{\Delta s}{\Delta t}.$$

Draw the straight line AB and produce it to cut the axis of time in E, making an angle $AEa = \varphi$. Then $\frac{\Delta s}{\Delta t} = \frac{BD}{AD} = \frac{Aa}{Ea}$ $= \tan\varphi$. Now suppose we diminish Δt, making B approach A more and more nearly, as is shown on a larger scale in

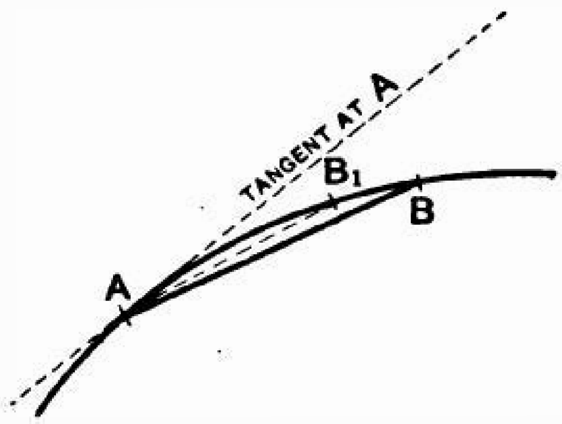


FIG. 18.

Fig. 18. The chord AB becomes AB_1 and, as Δt diminishes, approaches more and more closely to the tangent to the curve at A, and, if Δt is made infinitely small, AB will coincide with that tangent. Still, however, the value of $\frac{\Delta s}{\Delta t}$ is shown by the numerical value of $\tan\varphi$, and in the limit

$$\text{velocity} = \tan\varphi = \frac{ds}{dt}.*$$

Accordingly we may say that to find the velocity of a body at any instant from its diagram of displacement *drawn on a time base*, we have only to draw at the point corresponding to that instant a tangent to the curve. The slope of that tangent, as measured by the tangent of the angle it makes with the axis of time, is proportional to the velocity, and indeed represents the velocity numerically if the distances are measured to their proper scales. For example, at C in Fig. 16, to the scales marked, $\tan\varphi$ has the value $\frac{10 \text{ feet}}{2 \text{ sec.}}$ and shows, therefore, that at 3 seconds the body in

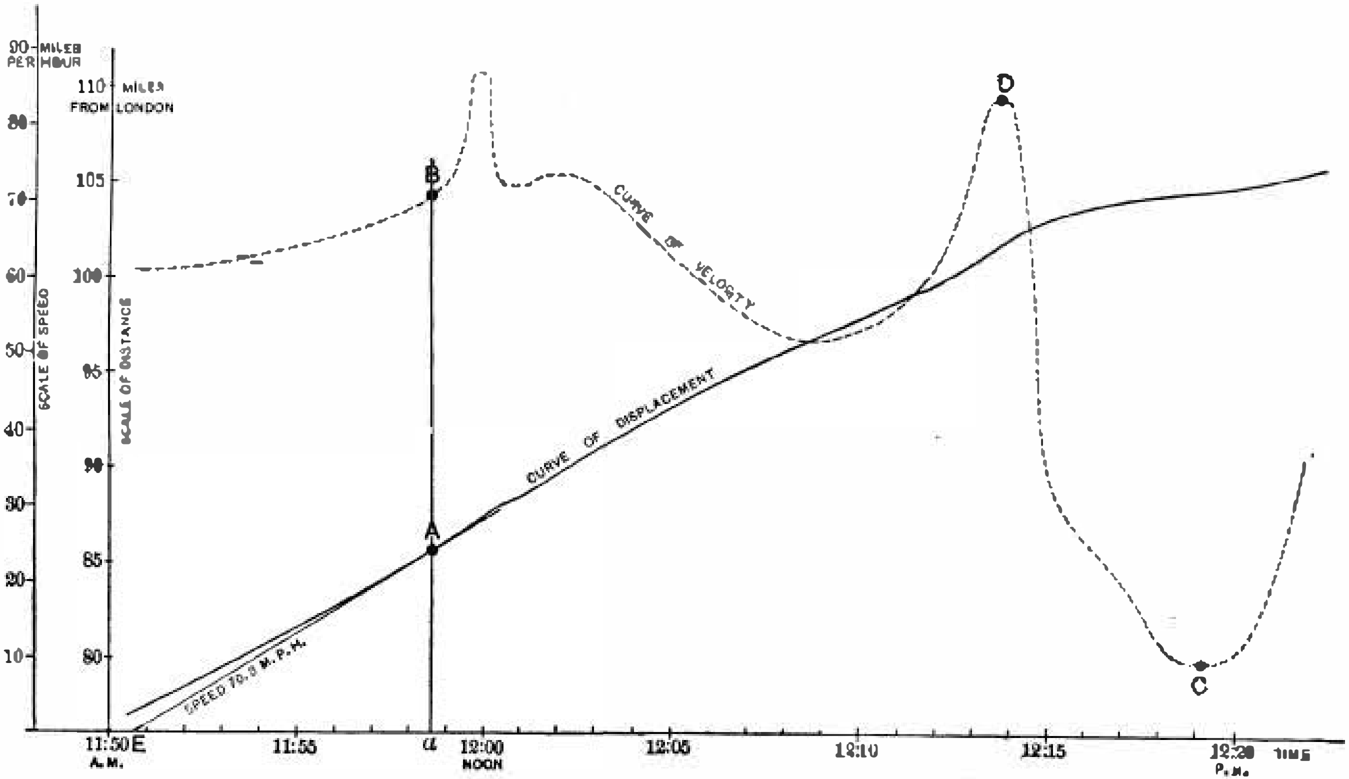
* See § 13.

question had a velocity of 5 feet per second. If a scale of miles had been marked along the axis of distance, while hours had been measured along the axis of time, our velocity would have been read in miles per hour.

The reader should notice that at the point F in Fig. 17 the displacement has ceased to increase with increase of time, and is about to decrease; the body has in fact reached its maximum distance from its starting-point. At this point the body must of course cease to be moving for an instant, which is shown by the fact that the tangent to the curve at F is horizontal, corresponding to zero velocity. After F the velocity will of course have to be reckoned negative, since distance is now diminishing as time goes on.

If the velocity at every instant could be measured in this way and a new curve drawn on a time base, having ordinates at each instant proportional in length to the velocity at that instant, we should obtain a curve or *diagram of velocity*. Actually we obtain only a sufficient number of values to give us a series of points on the curve, through which the curve can be drawn. This has been done as an example in Fig. 19. The full curve is a diagram showing the distance from London at times between 11.50 A.M. and 12.20 P.M., on a certain date, of the London and Exeter express on the Great Western Railway, the times of passing various stations and mile-posts having been carefully noted. From this full curve has been drawn a dotted one, the height of which at any point is proportional to the speed of the train at that time; this dotted line in fact is a curve of velocity.

It will be seen that the curve of displacement slopes continually upward, showing that the train did not stop during the time considered. Its speed, however, was very variable. The train performed the whole journey from London to Exeter, 194 miles, in 3 hours 38 minutes without a stop; thus the average speed was 53.4 miles per hour. At about 12.14, however, the speed for a very short time is seen to have exceeded 80 miles per hour, as shown at D, while



90 MILES PER HOUR
110 MILES FROM LONDON
SCALE OF SPEED
SCALE OF DISTANCE
CURVE OF VELOCITY
CURVE OF DISPLACEMENT
B
A
D
C
E
a
11:50 A.M.
11:55
12:00 NOON
12:05
12:10
12:15
12:20 P.M.
TIME

FIG. 19.

shortly afterwards, at about 12.19, a signal-check caused the speed to be reduced to about 10 miles per hour, as shown at C. The method of finding points on the curve of velocity is shown at B. The ordinate at B is proportional to the ratio $Aa : Ea$, where AE is a tangent to the displacement curve at A.

By the construction of such curves we can trace out the whole history of the motion, and if a sufficient number of points have been taken, and our drawing has been accurate, the results will be trustworthy for all practical purposes.

It is often difficult to draw the tangents to the curve correctly enough, especially if the slope of the curve is small, and it is usual to adopt another construction, to be explained later, which avoids the necessity of drawing the tangents by guesswork.

It is to be noted that diagrams of displacement may quite well be drawn, in which ordinates represent angles instead of distances, and from such diagrams angular velocities can be obtained exactly as described above.

19. Diagrams of Acceleration.—If a diagram of velocity on a time base be drawn, the curve of acceleration can be obtained from it by an application of the same method adopted for getting the velocity curve from that of displacement. In Fig. 20 let OAB be a curve of velocity. At any point A the rate of change of velocity is the limiting value of the ratio of the small change Δv in velocity to the small interval of time Δt in which such small change occurs, and by similar reasoning to that in § 18 it will be seen that this ratio is numerically equal to the tangent of the angle of slope of the curve at the point considered. In the figure, then, the acceleration at A is represented by tan α. Plainly at such a point as C, where the velocity has a maximum value (having ceased to increase and being about to decrease), there will be no acceleration, and the acceleration curve will pass through the time axis, as at c. Further, it will be noticed that where the velocity is increasing uniformly (as between D and E), and the curve of velocity is therefore straight, the acceleration curve becomes a horizontal straight line, as at *de.*

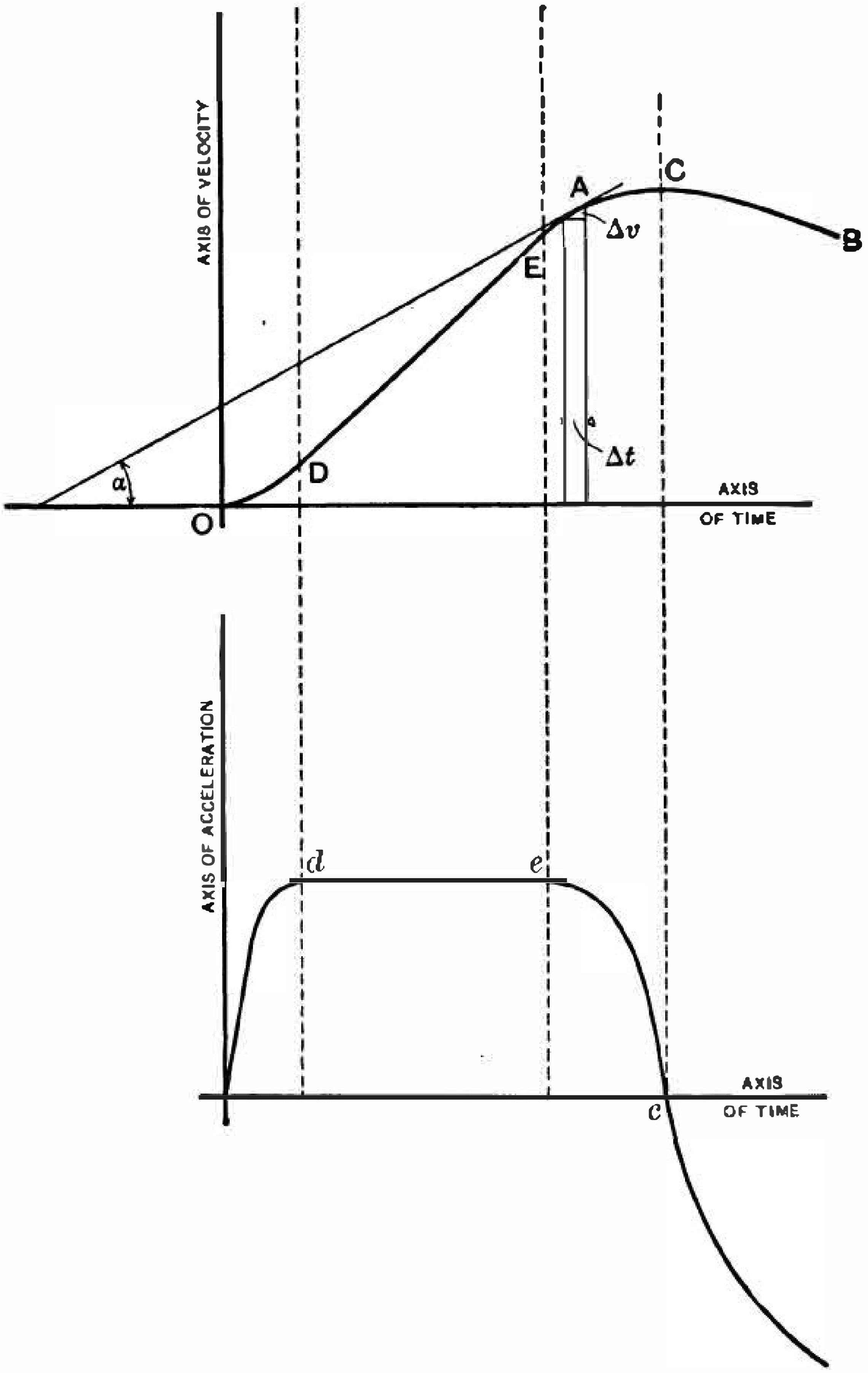
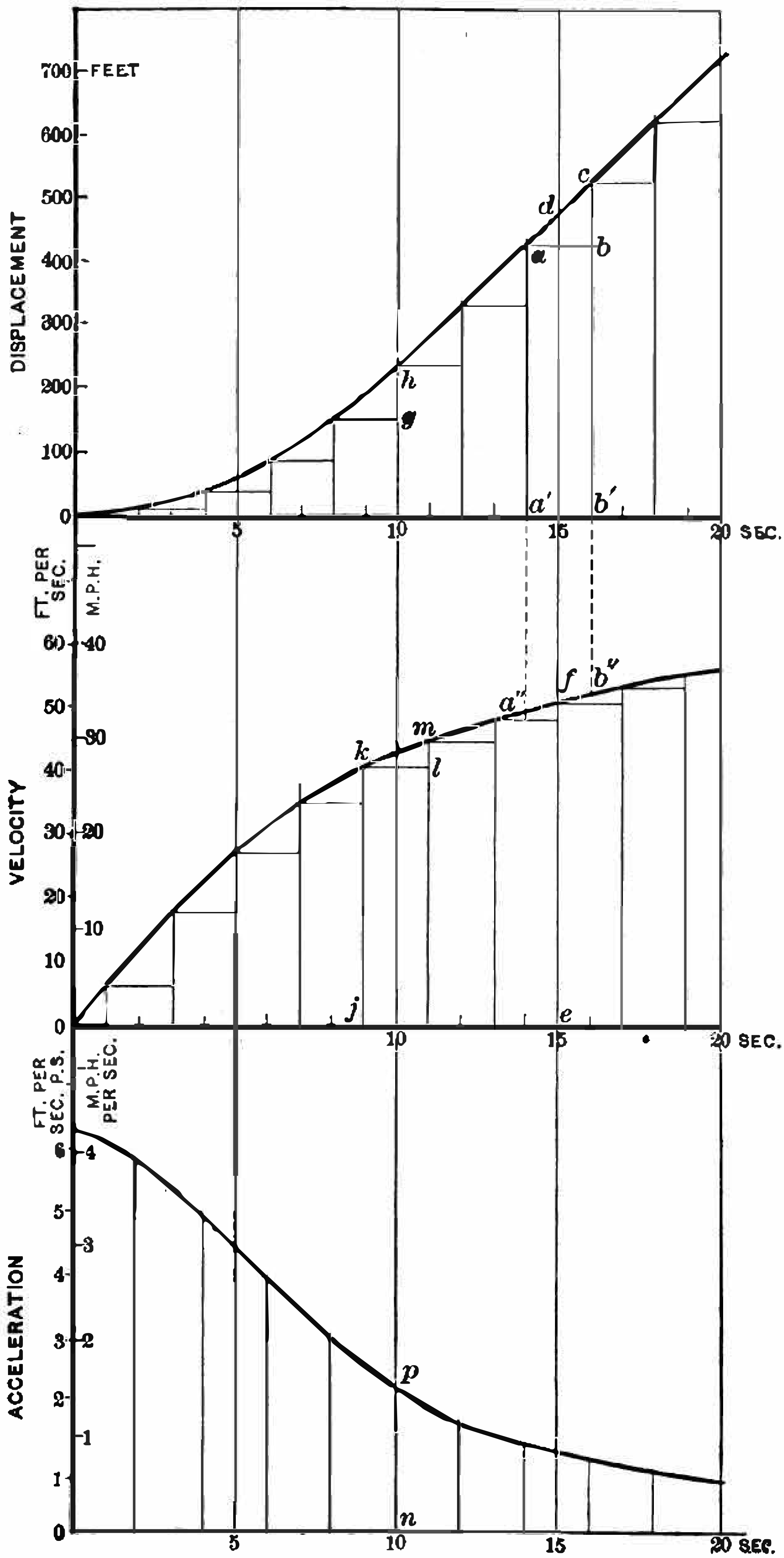


FIG. 20.

In practice, when such diagrams have to be drawn either for curves of velocity or of acceleration, a somewhat different method is adopted (as shown in Fig. 21), based on



Distances from Rest.

5 sec.	66 ft.
10	245
15	490
20	750

Velocity.

5 sec.	19 m. p.h.
10	30
15	36
20	39

Acceleration.

5 sec.	3 m. p. h. per [sec.
10	1.4
15	.9
20	.6

FIG. 21.

the same principle, but avoiding the necessity of drawing a number of tangents to the curve, many of which can only be determined approximately.

The diagram shows a curve of displacement* for an electric street-car starting from rest. The data were obtained from tests of a special car designed for an initial acceleration of 3 miles per hour per second.

To construct from such a curve the corresponding curve of velocity, the time shown by the diagram is divided into a number of small intervals, in this case of 2 seconds each, as at ab. On measurement the length bc is found to represent 100 feet and is the difference between aa', or bb', and cb'. Now aa' is the distance moved by the car during the first 14 seconds. Thus cb represents the distance traversed during the fifteenth and sixteenth seconds, and accordingly the *average velocity* for those two seconds was $\frac{100}{2}$, or 50 feet per second.

In the figure an ordinate ef of length five times bc has been marked off at the point corresponding to 15 seconds from the start, and its extremity gives one point on our curve of velocity. In the same way jk has been made $5 \times gh$, and so on.

A curve drawn through the points thus found shows approximately what was the velocity at any time after the start. We say approximately, because the *actual velocity* at the middle of a 2-second interval would only be equal to the *average velocity* during such an interval if the straight line joining ac (for instance) had been parallel to the tangent to the displacement curve at d; that is, if the points a'', f, b'' had been in a straight line. We know that actually this may or may not be the case, but by taking sufficiently small intervals of time we can reduce the error from this cause to any desired extent, until in fact it becomes negligible.

In order to obtain from the velocity curve that of accel-

* Taken from *Engineering News*, Oct. 14, 1897.

eration, an exactly similar procedure is employed. The length lm, for example, has been set up and exaggerated fivefold at np, and is proportional to the change of velocity during the tenth and eleventh seconds.

Having drawn our curves, it becomes necessary to determine their *scales*, that of the original diagram of displacement being known.

The displacement diagram was drawn originally* on such a scale that bc (representing 100 feet) was actually half an inch; the scale of distance was then 200 feet to the inch. The scale of time was $1'' = 5$ seconds, or 1 second $= \frac{1}{5}$ of an inch. A length bc transferred to the velocity diagram then represented (if $ab = 2$ seconds) a velocity of 50 feet per second, giving a scale of $\frac{1}{2}'' = 50$ feet per second, or 100 feet per second to the inch. For clearness this was exaggerated in the figure, so that 20 feet per second = 1 inch. In the same way the scale of the acceleration curve was made such that five inches = 10 feet per second per second. Scales of miles per hour and miles per hour per second have also been marked for comparison.

In general, then, the scales of such diagrams may be determined as follows:

Let the displacement diagram be drawn to a distance scale of 1 inch $= l$ feet, and suppose the short intervals of time during which the average velocities are estimated are each n seconds.

An ordinate of 1 inch in length on the displacement diagram when transferred to the velocity diagram then represents a velocity of l feet in n seconds, i.e., l/n feet per second. The scale of the velocity diagram is then n/l inches $= 1$ foot per second

Thus in the figure above we should have, if the ordinates were not exaggerated, a velocity scale of $\frac{1}{100}$ inch = 1 foot per second, for $l = 200$ and $n = 2$. It was drawn actually to a scale of $\frac{1}{20}$ inch = 1 foot per second.

* It is of course reproduced here to a smaller scale.

Considering next the scale of the acceleration diagram, suppose that on the velocity diagram the velocity scale is 1 inch $= m$ feet per second, while the small intervals of time are as before n seconds each; then an ordinate of 1 inch on the acceleration diagram will represent an acceleration of m/n feet per second per second.

In the figure $m = 20$, $n = 2$, so that the acceleration scale would naturally have been $\frac{20}{2} = 10$ feet per second per second to the inch, had it not been exaggerated for clearness to 2 feet per second per second to the inch.

20. Diagrams on a Displacement Base.— Diagrams of velocity and acceleration may also be drawn *on a displacement base*, in which case lengths measured horizontally are proportional to distance traversed or angle described; such a construction is frequently very useful.

We have seen that on a time base the velocity curve for a body having uniform acceleration will be a straight line, passing through the origin of the two axes if the body has no velocity when time is reckoned zero. Velocity is then proportional to time. Suppose, however, that we consider the way in which velocity varies with regard to displacement in such a case.

Let a be the constant acceleration, v the velocity attained from rest after moving for a time t; then by our definition $a = \frac{v}{t}$. Now the distance moved by the body will be numerically equal to the average velocity $\left(\frac{v}{2}\right)$ multiplied by the time: thus $s = \frac{vt}{2}$ or $v = \frac{2s}{t}$.

But $v = at$; hence

$$v^2 = \frac{2ast}{t} = 2as.$$

Therefore, the acceleration being constant, the displacement varies as the square of the velocity. In the figure the acceleration is 1.5 feet per second per second, and the distance traversed in the first two seconds (during which a

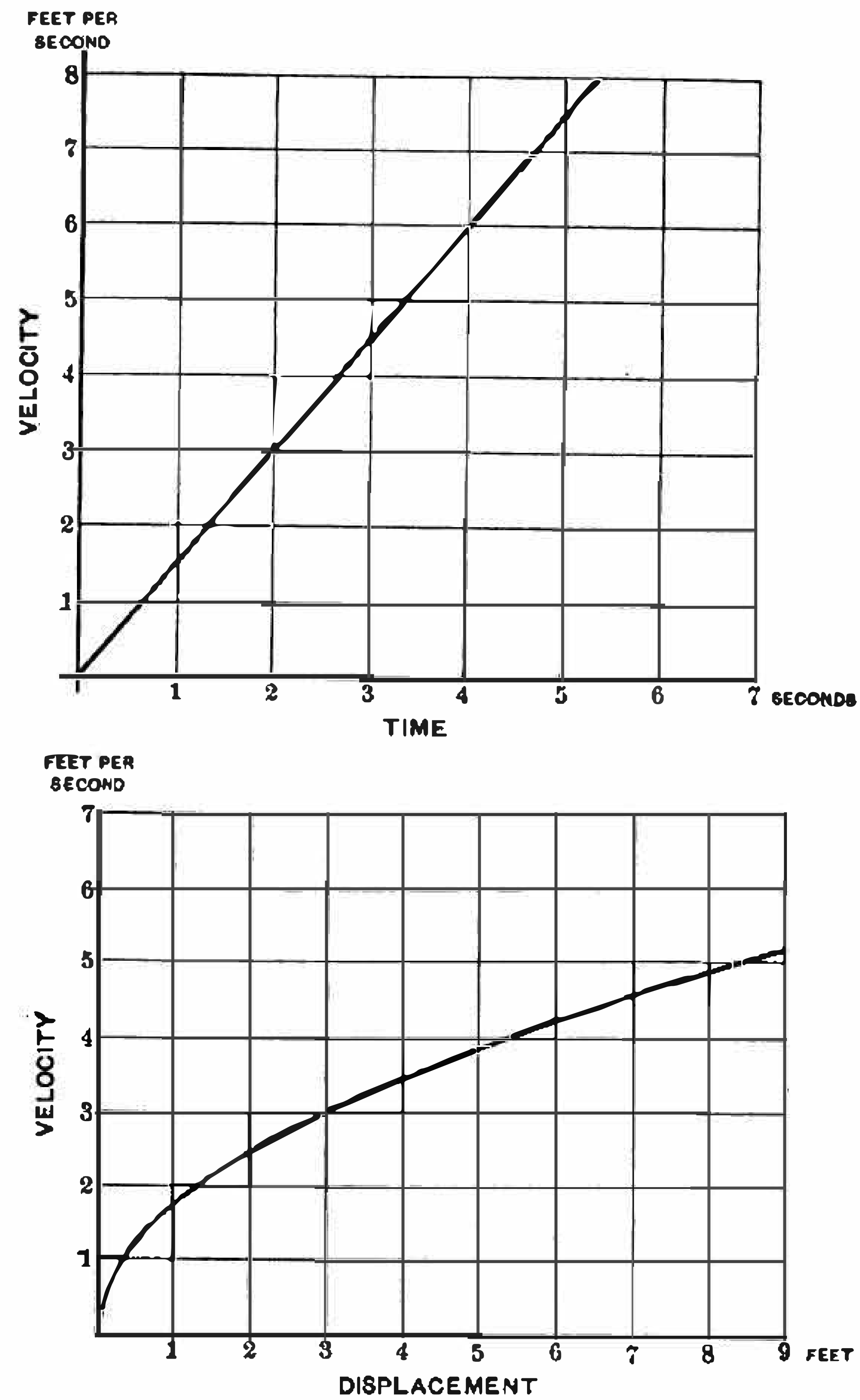


FIG. 22.

velocity of 3 feet per second has been attained) is $\frac{3}{2} \times 2 =$ 3 feet. In 3 seconds, the velocity being 4.5 feet per second, a distance of $\frac{4.5}{2} \times 3 = 6.75$ feet will have been described, and in 1 second, the velocity being 1.5 feet per second, a distance of $\frac{1.5}{2} \times 1 = 0.75$ feet will have been covered. The diagram of velocity and displacement shown in the lower part of Fig. 22 will be found to express these relations, and the velocity curve on a displacement base is not a straight line, but a curve whose ordinates are proportional to the square roots of the abscissæ.

21. Acceleration Curves on a Displacement Base.—Let a, c, (Fig. 23), be two points on a velocity curve drawn on a displacement base. The difference of the ordinates bc represents the change of velocity. Draw the straight line ac, bisect

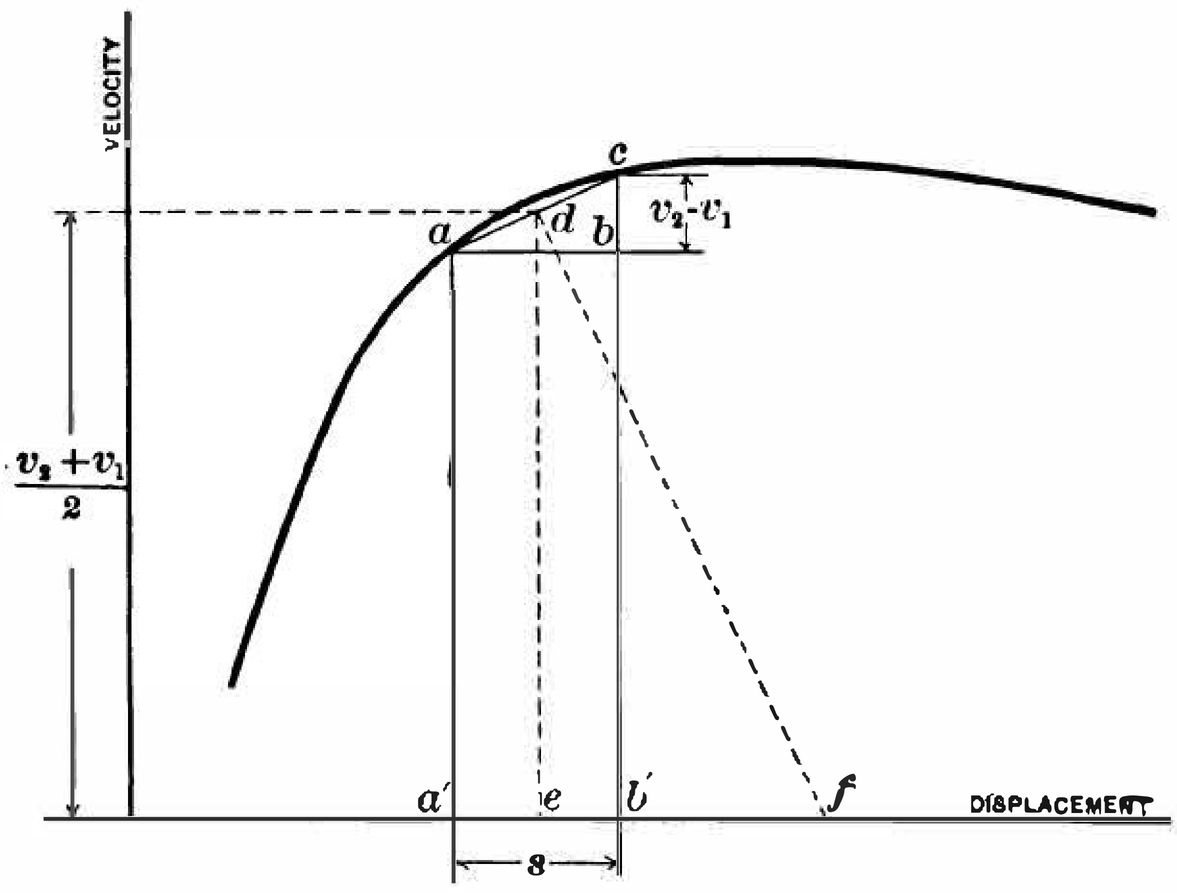


FIG. 23.

it by the line df at right angles, and draw de perpendicular to the axis of displacement. Then $\frac{ef}{ed} = \frac{bc}{ab}$, since the triangles abc, def are similar.

Assuming that the velocity changes uniformly, while the moving body describes the distance represented by $a'b'$, let s be the space described in time t, while the velocity changes from v_1 to v_2. Then we know that

$$s = \frac{v_2 + v_1}{2} t \quad \text{and} \quad t = \frac{2s}{v_2 + v_1}.$$

Again, the acceleration

$$a=\frac{v_2-v_1}{t};$$

hence

$$a=\frac{(v_2-v_1\ (v_2+v_1)}{2s}.$$

But in the figure $aa'=v_1$, $cb'=v_2$, so that $v_2-v_1=bc$, $\frac{v_2+v_1}{2}=de$, and $s=ab$; thus

$$\text{acceleration}=\frac{bc\times de}{ab}=ef.$$

Actually, of course, the velocity in general does not change uniformly; we may, however, take a indefinitely close to c, so that the straight line ac becomes a tangent to the velocity curve, and df becomes the normal to that curve, while ef is the subnormal at d and represents the acceleration.

We find, then, that in the case of a velocity curve drawn on a displacement base the subnormal at any point represents the acceleration. It only remains to determine the scale on which, for example, ef represents the acceleration at c.

Let the scale for velocity be 1 inch $=m$ feet per second, while the distance scale is one inch $=n$ feet. Let ab, bc, de, and ef be measured in inches. Then, *numerically*, $v_2-v_1=bc\times m$ feet per second, $\frac{v_2+v_1}{2}=de\times m$ feet per second, and $s=ab\times n$ feet. But we have just seen that

$$\begin{aligned}\text{acceleration}&=\frac{(v_2-v_1)(v_1+v_2)}{s\times 2}\\&=\frac{bc\times m\times de\times m}{ab\times n}\\&=ef\times\frac{m^2}{n},\end{aligned}$$

ef being also measured in inches.

In Fig. 24 is given the curve of velocity for a cable-car from starting to stopping on a run of 200 feet, and from it the curve of acceleration has been drawn, the construction for one ordinate being shown. Note the high positive acceleration at the start, indicating a considerable jerk, followed (after the first 35 feet) by a small and variable value of the acceleration, sometimes positive, sometimes negative. The stoppage of the car is accomplished in the last 35 feet, and is shown by the negative acceleration or retardation during that portion of its travel.

As originally drawn, the velocity scale was 1 inch$\models$5 miles per hour $=7.33$ feet per second, while the distance scale was one inch$\models$50 feet. Accordingly the acceleration scale was 1 inch$\models\frac{7.33\times 7.33}{50}=1.075$ feet per second per second, or 0.732 mile per hour per second. The greatest acceleration is then about 3.3 miles per hour per second, and the greatest retardation about 2.2 miles per hour per second.

It should be noted that the *curve of velocity on a displacement base, when acceleration is constant, is a parabola,* this being the curve one of whose characteristic properties is the constancy of the subnormal.*

22. Polar Diagrams of Displacement, Velocity, and Acceleration.—Besides employing the methods just given for drawing curves of displacement, etc., it is often useful, especially in considering periodic motion (in which the same circumstances or conditions as to velocity or displacement repeat themselves at regular intervals of time), to draw diagrams in which a radius vector represents displacement, velocity, or acceleration, while the angle turned through by such a radius is proportional to time.

In Fig. 25 suppose that a line ON turns about the point O. starting from the initial position OM; the angle θ it has described, when it has reached any position such as OA, being proportional to an interval of time t_1, during which a body

* See Fig. 22.

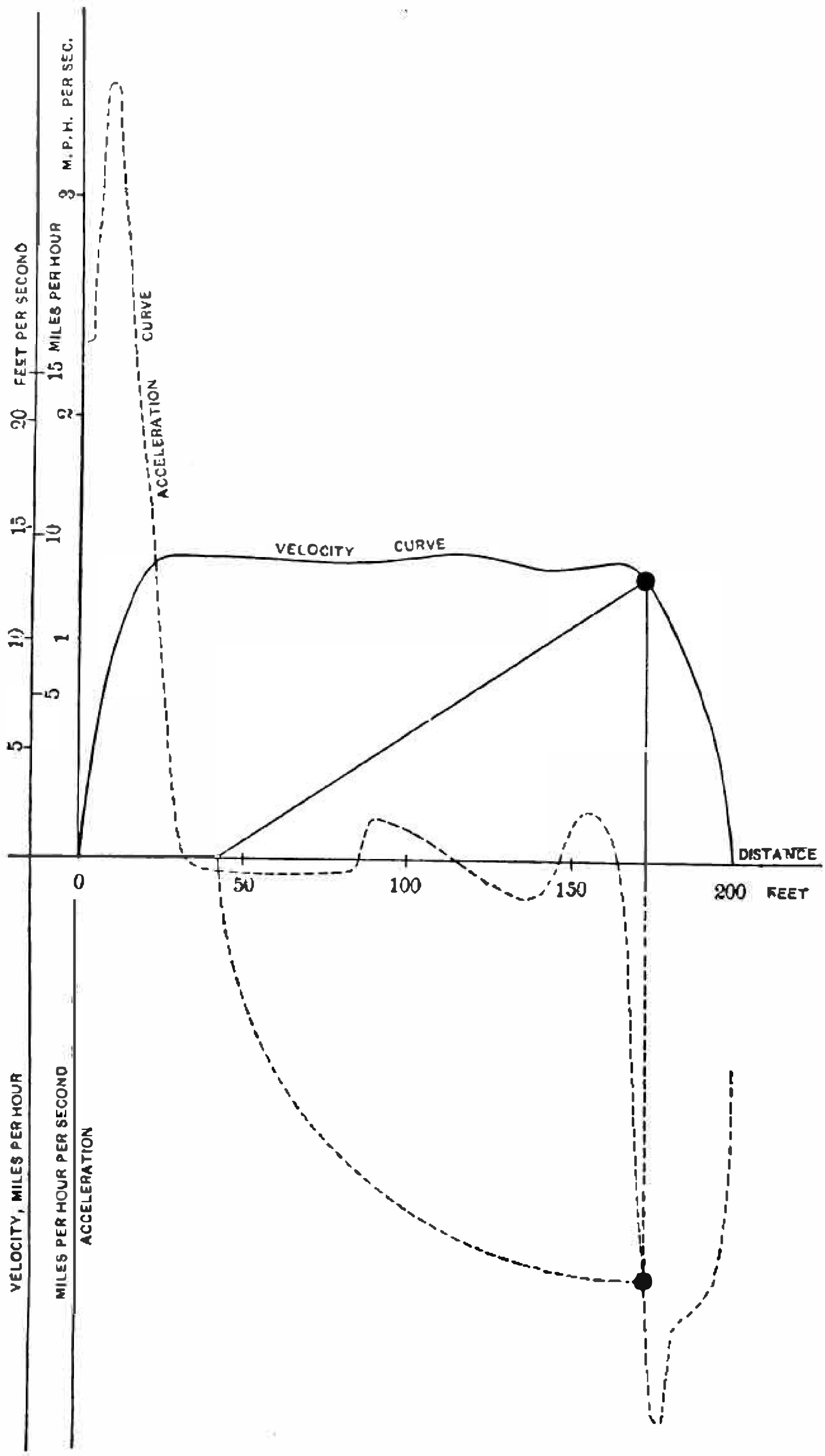

FEET PER SECOND
MILES PER HOUR
M.P.H. PER SEC.
ACCELERATION CURVE
VELOCITY CURVE
DISTANCE
FEET
0
50
100
150
200
VELOCITY, MILES PER HOUR
MILES PER HOUR PER SECOND
ACCELERATION

FIG. 24.

whose motion we are considering has moved to a distance s_1 from its starting-point. Mark off a distance $OA = s_1$. Then a curve, such as NAB, drawn through successive positions

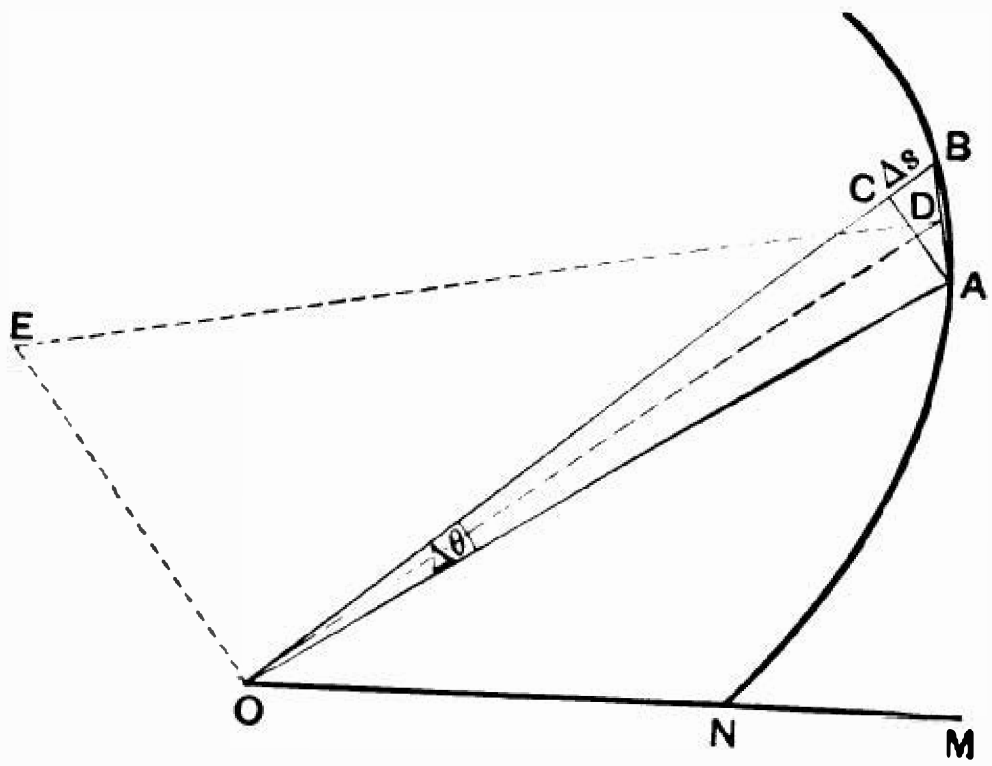


FIG. 25.

of A, is a *polar diagram of displacement* for the body considered. Let $OB = s_2$, while the angle $BOM = t_2$.

Then the average velocity between the times t_1 and t_2 will be represented by $\dfrac{BO - AO}{\text{angle } BOA} = \dfrac{s_2 - s_1}{t_2 - t_1}$. Making $OC = OA$, we have, therefore,

$$\text{average velocity} = \frac{BC}{\text{angle } BOA} = \frac{\Delta s}{\Delta\theta}.$$

Join AC, and draw the straight line BA. Now if $\Delta\theta$ is a *very small angle*, the length of the straight line AC does not differ sensibly from that of the arc of a circle of radius OA, so that we may sav that if $\Delta\theta$ is small,

$$\Delta\theta = \frac{AC}{AO};$$

$$\frac{\Delta s}{\Delta\theta} = \frac{BC.AO}{AC}.$$

Bisect AB in D, and draw OD, DE, OE, respectively perpendicular to AC, AB, BC, so that the triangles BCA, EOD are similar; then $\frac{BC}{AC}=\frac{EO}{DO}$ and

$$\frac{\Delta s}{\Delta\theta e}=\frac{OE}{DO}\times AO.$$

Next suppose that the interval of time represented by $\Delta\theta$ becomes indefinitely small, so that the points B and A coincide, and AB becomes a tangent to the curve. Δs and $\Delta\theta$ will both become infinitesimal in magnitude, but their ratio (which now represents the velocity the body has when a time represented by θ has elapsed) is finite, and its value is measured by the limiting value of $\frac{OE.AO}{DO}$ when $AO=DO$.

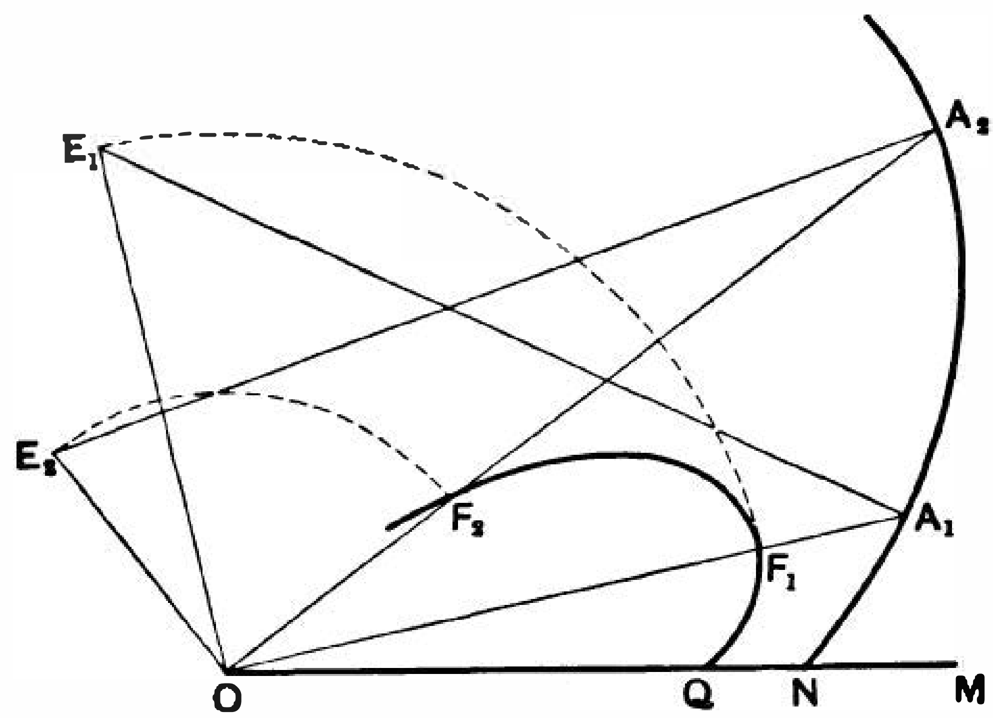


FIG. 26.

This is of course OE. Hence the velocity is measured by the length of OE when taken to the proper scale and drawn perpendicular to OA, and to find the velocity corresponding to any point A on a polar displacement diagram, we draw a normal to the curve, and find its intercept OE on a line drawn perpendicular to the radius vector OA. By carrying out this construction for a number of points and marking off values of OE along OA (as at OF, Fig. 26) in each case, we find points on a new curve, QF_1F_2, which is in fact a polar velocity-time diagram.

The reader will now be in a position to see that by repeating the same construction with the new velocity curve a further diagram is obtained, that of acceleration; since acceleration has exactly the same relation to velocity that velocity has to displacement. In this case, of course, the quantity represented by a line drawn in the same way as OE above is the rate of change of *velocity* with regard to time.

We may next study some examples of such diagrams.

Fig. 27 represents a polar diagram or curve of displacement, in which distance increases uniformly with time, as shown by the fact that $QO - PO = PO - NO$, if the angle QOP = angle PON, and so on. The curve $ONPQ$ is of course an Archimedean spiral. It may be proved that the length of the intercept OE by the normal DE on a line OE through O perpendicular to the radius vector is constant whatever the position of D * on the curve. This length OE has been shown to represent the velocity, and accordingly in this case the velocity diagram will be a circle with radius OE, and the acceleration will be zero, since the velocity is uniform. In the drawing, if the displacement scale is as shown, and the time scale is $90° = 10$ seconds, the velocity of the body is 1 foot per second. To determine the velocity scale of such a diagram we need only reflect that

---

* The polar equation to the spiral of Archimedes is $r = a\theta$;

$$\text{hence } \theta = \frac{r}{a} \text{ and } \frac{d\theta}{dr} = \frac{1}{a}.$$

If ϕ be the angle which the tangent makes with the radius vector,

$$\tan \phi = r\frac{d\theta}{dr}.$$

Thus $\tan \phi = \dfrac{r}{a}$.

Now the angle between the normal DE and a line OE perpendicular to the radius vector is ϕ. Hence $OE = \dfrac{r}{\tan \phi} = r \cdot \dfrac{a}{r} = a =$ constant.

the velocity is numerically $\frac{ds}{d\theta}$. If the displacement scale were 1 inch = 3 feet, a velocity of 1 foot per second would be represented by a length of one third of an inch divided by the angle representing 1 second. In this case

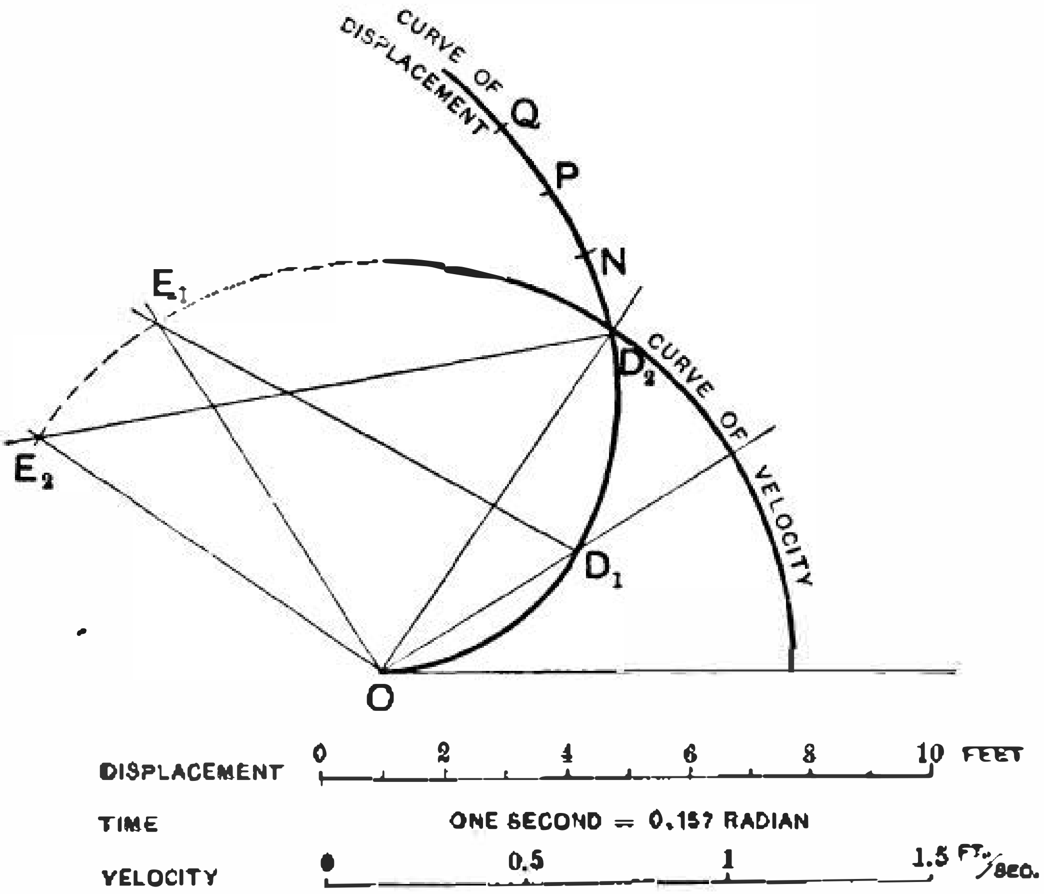


FIG. 27.

90° represents 10 seconds of time, so that 1 second of time would be denoted by $\frac{\pi}{20}$ or 0.157 radian. Hence unit velocity would be represented by $\frac{1}{3 \times 0.157} = 2.12$ inches, and the velocity scale would be 1 inch = 0.471 foot per second.

23. Diagrams for Simple Harmonic Motion. — In the following chapter we shall frequently meet with cases in which bodies have periodic motion in a straight line, either exactly or approximately harmonic in character. We define *simple harmonic motion* as the motion of a point which is the orthogonal projection on a straight line of another point moving uniformly in a circle, termed the *auxiliary circle*. The radius of the auxiliary circle is called the *amplitude* of the motion. Such motion can be conveniently studied by means of polar diagrams; the engineer, for example, often employs such diagrams to elucidate the action of the slide-valve of a steam-engine, for the valve has very nearly such a motion as has been defined above. In Fig. 28 let AOB represent the path of a point having

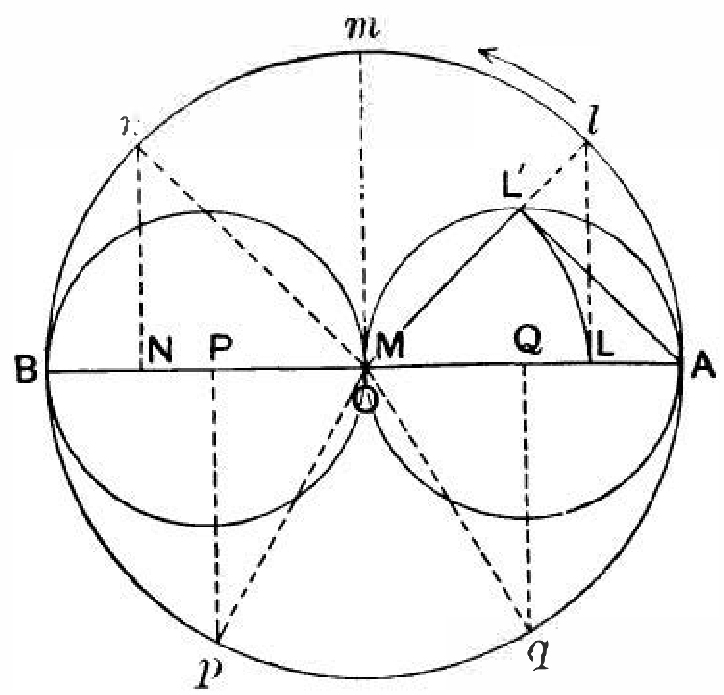


FIG. 28.

simple harmonic motion; we wish to draw diagrams of displacement, velocity, and acceleration for this point. AO is the amplitude of the motion, $AmBq$ is the auxiliary circle, and, according to our definition, $LMNP$, etc., will be positions of the point after the lapse of times proportional to the angles AOl, AOm, AOn, AOp, and so on. It is plain, then, that a polar time-displacement diagram can be readily drawn by marking off along each radius a distance equal to the corresponding displacement of the vibrating point from its mid-position. Thus OL' has been made equal to OL. It should be noticed that during each revolution of the rotating point round the auxiliary circle, its projection travels from A to B and back again, so that twice during each period the displacement of the vibrating point will be zero, while at the same time the velocity will have its greatest value.

The *period* is of course the time of a complete revolution of the point round the auxiliary circle. During one half of this time the vibrating point has a positive velocity, i.e., is moving in one sense, say from right to left, while during the remainder of the period the velocity will have a negative value.

The displacement diagram is drawn by obtaining a series of such points as L', where $OL' = OL$; the locus of such points is easily shown to be a pair of circles touching the auxiliary circle and each other. Taking the point L', for example, join $L'A$. Then in the triangles OlL, OAL' we have $OL' = OL$ and $Ol = OA$, while the angle lOA is common to both. Hence the triangles are equal in all respects, and the angle $OL'A$ is a right angle. Therefore the point L' lies on a semicircle drawn on OA as a diameter, and the complete locus of L' is a pair of circles, as shown. The radius vector OL' represents the displacement of the vibrating point at a time represented by the angle between the radius vector and the initial line OA.

Having given such a diagram of displacement (Fig. 29), let us apply to it our construction for determining velocity. At any point D the line DE is drawn normal to the curve of displacement, and is cut by OE where the angle EOD is a

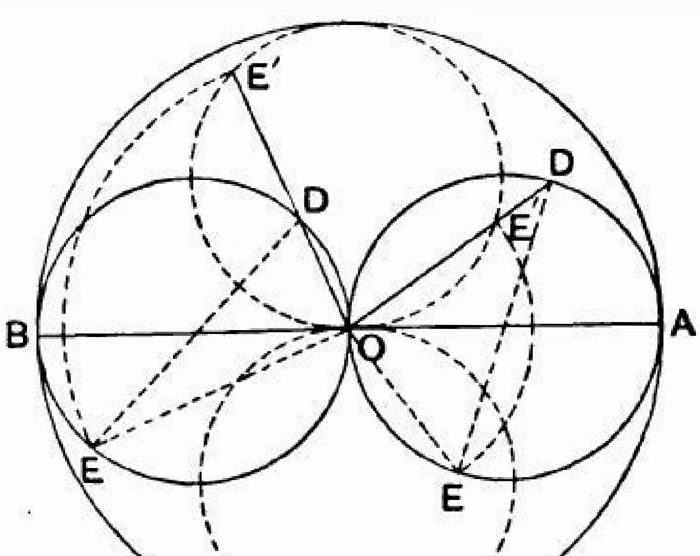


FIG. 29.

right angle. Accordingly, when the length OE is marked off at E' along OD we get one point on the velocity diagram. Plainly, in this particular case, any radius vector of the velocity diagram, such as OE', is equal in length to the radius vector of the displacement diagram which makes an angle of 90° with it, and it follows that the velocity diagram will also be a pair of circles, as shown in the figure. Their axis is, however, at right angles to the axis of the displacement diagram, the maximum velocity being reached when the body is at the middle point of its path.

On constructing the diagram of acceleration, we find that it also takes the form of a pair of circles and coincides with that of displacement. The scales of the two diagrams are of course not the same, but it follows that *in simple harmonic motion the acceleration at any instant is proportional to the distance of the vibrating point from its mid-position*, a fact which can also be readily proved analytically.*

Fig. 30 gives the linear and polar diagrams of displacement, velocity, and acceleration for a simple harmonic motion of which the amplitude is 0.75 foot, while the period is $\frac{1}{5}$ second. This corresponds approximately, but by no means exactly, to the motion of the piston of a steam-engine, 18 inches stroke, and making 300 revolutions per minute.

The linear diagrams have been plotted on a straight line base from the radial diagrams; they might have been drawn by the methods of §§ 18 and 19, in which case their scales would have been different from those shown. In general, polar diagrams are easier to draw than linear diagrams, and are more useful for cases of periodic motion such as this; the linear diagrams are added for the sake of comparison.

With regard to the scales of these figures, the scale of time as originally drawn was $180° = 0.1$ second (i.e., 1 second $= \frac{2\pi}{0.2} = 31.416$ radians), while that of distance was 1 foot $=$ 1 inch. Accordingly unit velocity was represented by a length of $\frac{1}{31.416}$ inch. The velocity scale was thus 1 inch $= 31.416$ feet per second. Unit acceleration was represented by

* See Perry, Applied Mechanics, p. 549.

$\dfrac{1 \text{ foot per second}}{1 \text{ second}}$, or $\dfrac{.0318}{31.416} = .001011$ inch, hence the acceleration scale was 1 inch = 988 feet per second per second.

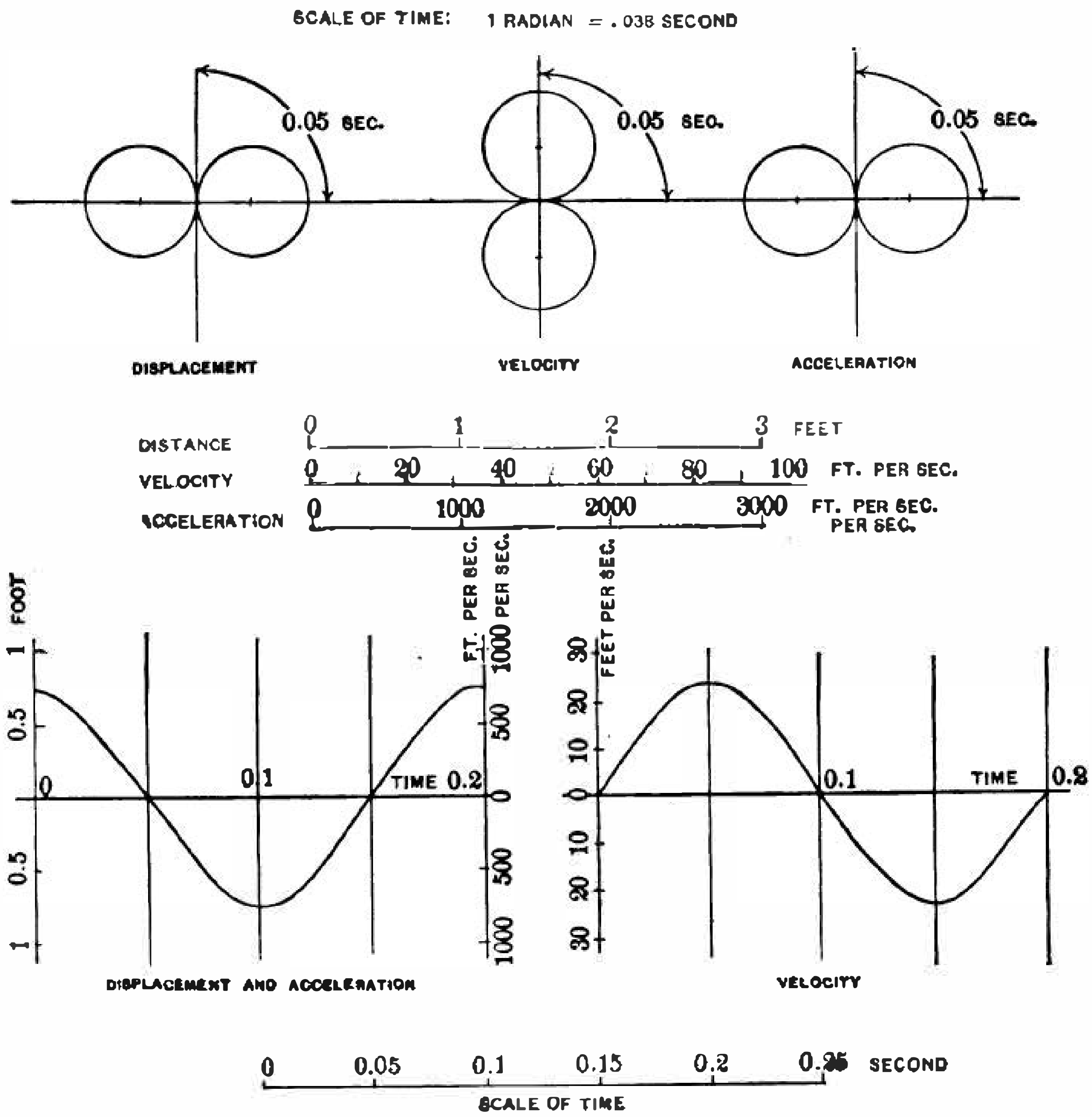


FIG. 30.

Note that the velocity of the rotating point round the auxiliary circle is $\dfrac{1.5\pi}{0.2} = 23.6$ feet per second. The maximum velocity of the vibrating point as measured from the diagram is $0.75 \times 31.416 = 23.6$ feet per second, thus agreeing with our definition of simple harmonic motion. The maximum acceleration of the vibrating point will be found to be the same as the radial acceleration of the point travelling round the auxiliary circle, namely $\frac{v^2}{r}$, or $\frac{23.6 \times 23.6}{0.75} = 741$ feet per second per second.

If an acceleration diagram on a displacement base is required, this can easily be drawn from the acceleration-time and displacement-time diagrams, and it will be found to take the form of a straight line, since, as has previously been remarked, acceleration in simple harmonic motion is proportional to displacement.

24. Relative Motion of Two Bodies each having S. H. M.— Cases arise in which it is necessary to find the relative

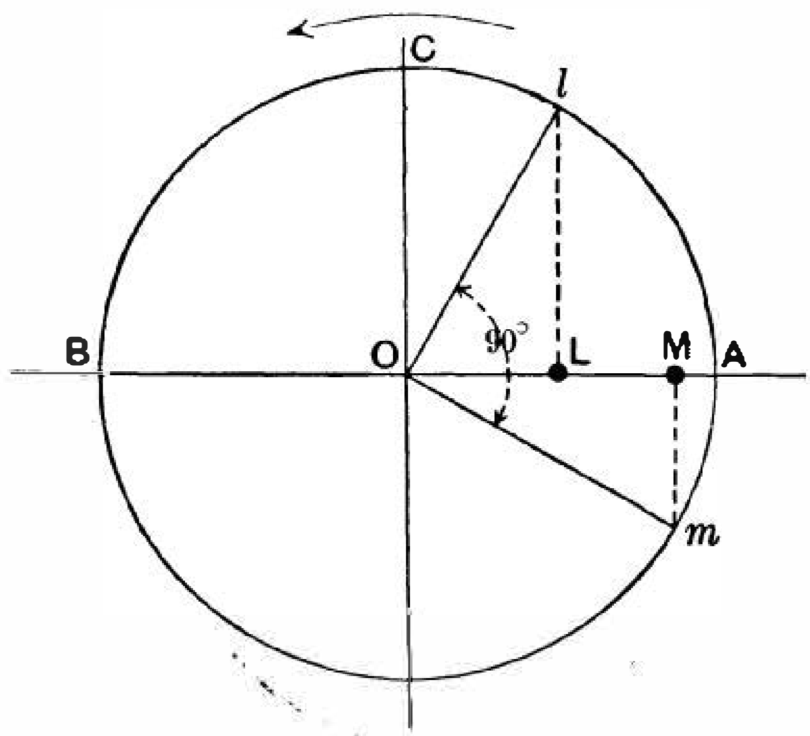


Fig. 31.

motion of two or moree bodies having simple harmonic motion, or we may wish to determine the motion resulting from the combination of two or more simple harmonic motions. We proceed to show how this is done.

It has been seen that every simple harmonic motion is capable of being referred to a corresponding uniform motion round a circle called the auxiliary circle. At any instant, for example, the simple harmonic motion of the point L (Fig. 31) corresponds to the uniform motion round the circle of the point l.

Let there be two points, L and M, having simple harmonic motion (of the same or different amplitude), and let l and m be their reference points. If AOB is the line on which the motion of l and m is projected, we define the constant difference between the angles COl and COm as the *difference of phase* in the two simple harmonic motions. In Fig. 31 the point M has a motion differing in phase by $-90°$ from that of the point L; in other words, Om continually lags $90°$ behind Ol, and of course M lags behind L to a corresponding extent.

In Fig. 32 let there be two points, L and M, having simple harmonic motions along AOB, whose amplitudes are

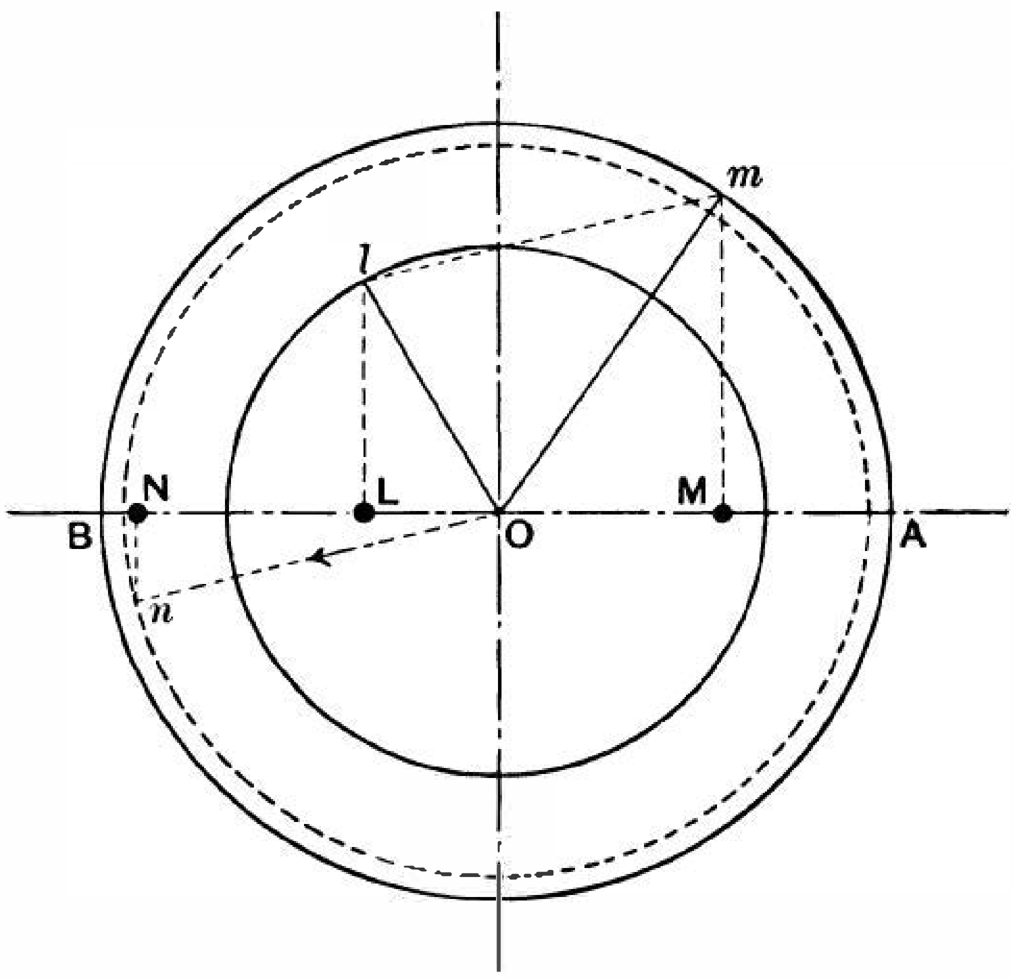


FIG. 32.

Ol and Om, and whose difference of phase is the angle lOm, the period being the same for both motions. We wish to find the relative motion of L and M, compounded of their two simple harmonic motions.

Join lm and draw On equal and parallel to ml. Draw nN perpendicular to AB.

Then the relative displacement of L and M is $OM - OL$, where OM or OL is reckoned negative when measured to the left of O. Now $OM - OL = ML$ is the projection on AB of the line ml. Also, ON is the projection on AB of the line On, equal and parallel to ml. Thus $ON = ML$, since each is equal to $On \cos NOn$.

Hence in any position the relative displacement of M and L is equal to the distance of the point N from the centre O. But N has a simple harmonic motion of the same period as that of M and L, of amplitude $On = ml$, and differing in phase from that of L by the angle lOn. Thus if two points have simple harmonic motions of the same periods and along the same straight line, their relative motion is also a simple harmonic motion, in general differing in amplitude from either of the components, and also differing in phase.

In Fig. 33 diagrams of displacement are drawn for two simple harmonic motions of 2 seconds period, the amplitudes being $\frac{3}{4}$ inch and 1 inch, and the phase difference 60°.

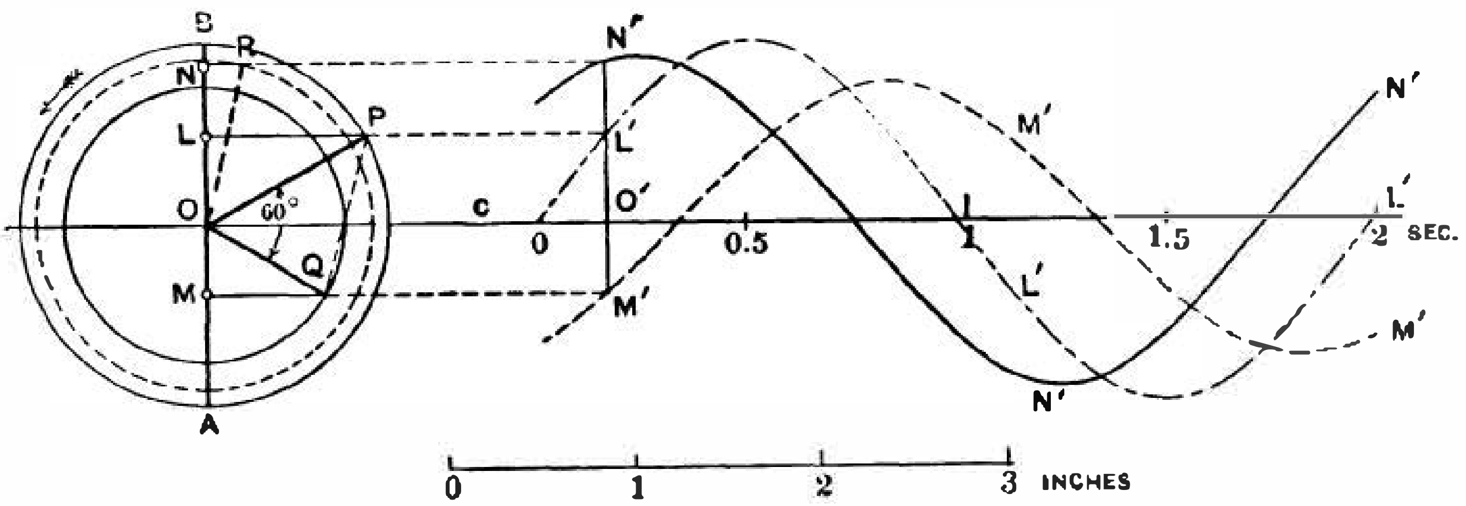


FIG. 33.

The curves L' and M' have ordinates proportional to the displacements of the corresponding points along AB, as represented by the motion of the points PQ round their respective auxiliary circles. The resultant displacement of L relatively to M is shown by the distance $M'L' = O'N'$, N' being above or below the line CD (along which time is measured) according as L is above or below M along the line AB.

Thus the curve N' is drawn through points whose heights above CD are equal to the heights of L above M at different times. It will be found that these displacements are the same as those of a point N moving along AB with a simple harmonic motion of period 2 seconds, amplitude 0.9 inch, whose phase is about 45° behind that of L and 105° behind that of M.

The curves drawn in Fig. 33 are in fact sine curves, since their ordinates are proportional to the sines of angles which are proportional to the abscissæ. In general, on compounding simple harmonic motions of different periods we do not obtain a simple harmonic motion as a result, but a more complex movement which is still, however, *periodic*. A case of this is shown in Fig. 34, where two simple har-

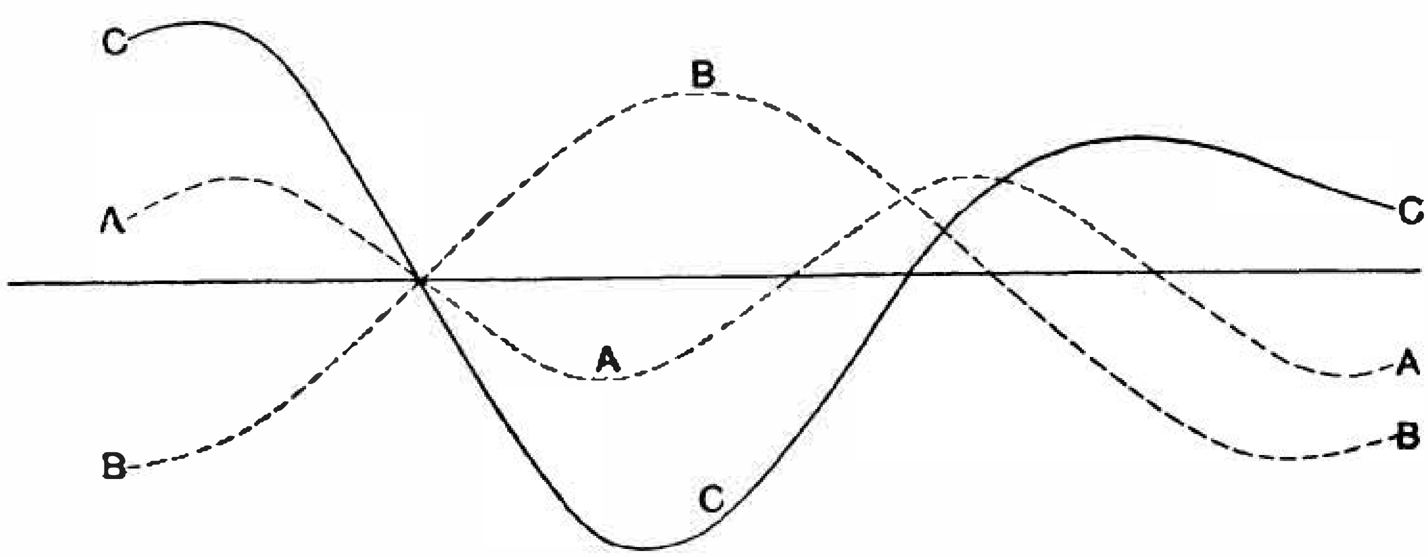


FIG. 34.

monic curves, A and B (shown by dotted lines), differing in phase, amplitude, and period, are compounded, the resultant curve C being shown by a full line. It is possible to resolve the curve representing any periodic function into a number of component sine curves of different phase, amplitude, and period. In this way, for example, the complex curve drawn by a tide-gauge is analyzed, and the periods, amplitudes, and phases of its component tides are determined.

In the figure, for instance, the curve CC might represent the actual rise and fall of the water-level at a certain place, due to the simultaneous effects of two tides, which, acting alone, would respectively produce the fluctuations shown by *AA* and *BB*. Distances measured horizontally represent time as before.

Such diagrams as those given above enable us to study any periodic function, and they find many important applications in scientific work.

25. Composition of S. H. M. not along Same Line.—Imagine that a point, having simple harmonic motion in the direction of a given line *AB*, has impressed upon it another simple harmonic motion in the direction of a line *CD* at right angles to *AB*. The motion of the bob of a simple pendulum is very approximately simple harmonic. An arrangement like that shown in Fig. 35 may be devised, which will give to a pencil, *P*, a motion compounded of those of two pendulums, *Q* and *R*, swinging in planes at right angles to one another.

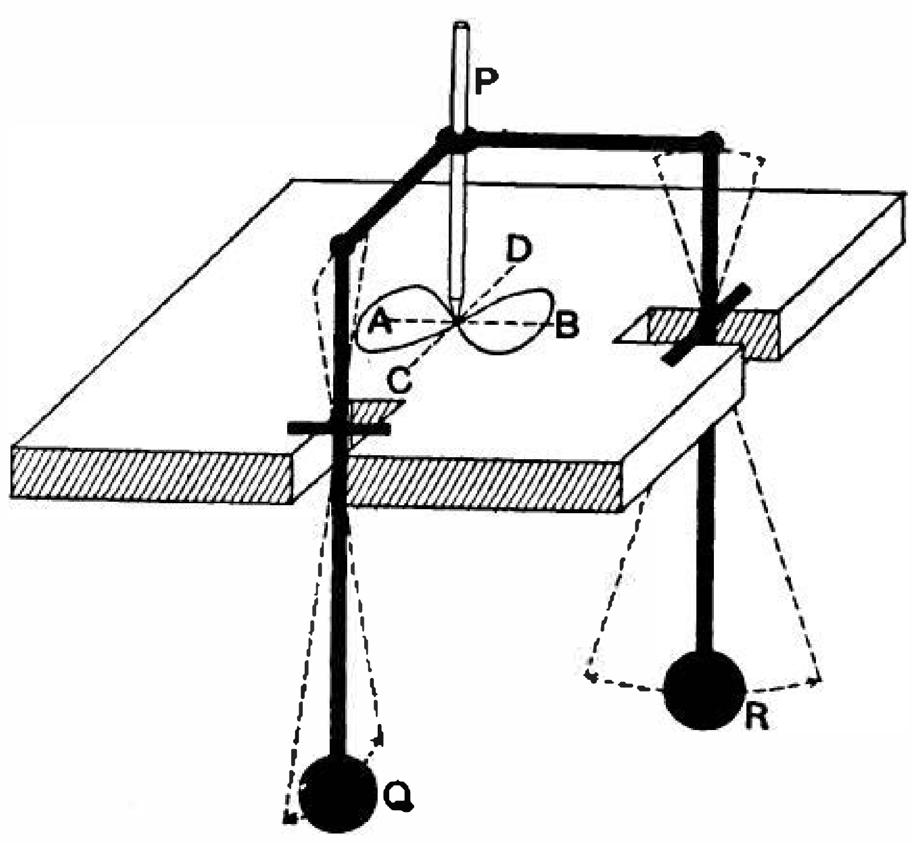
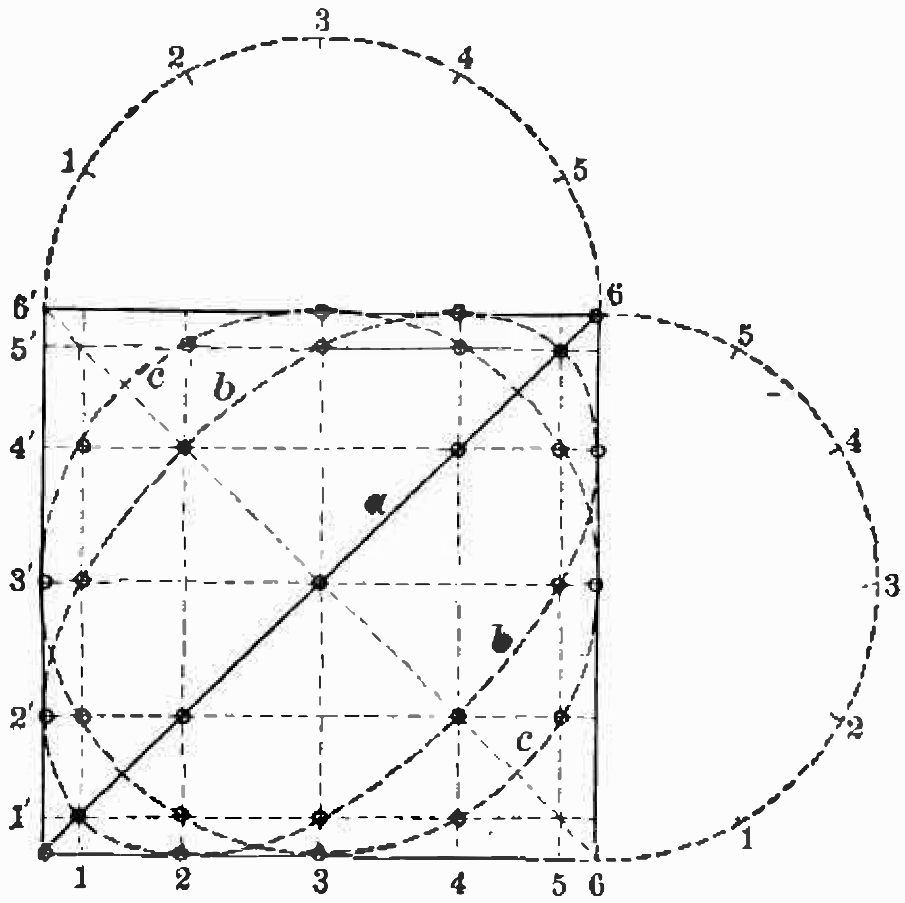


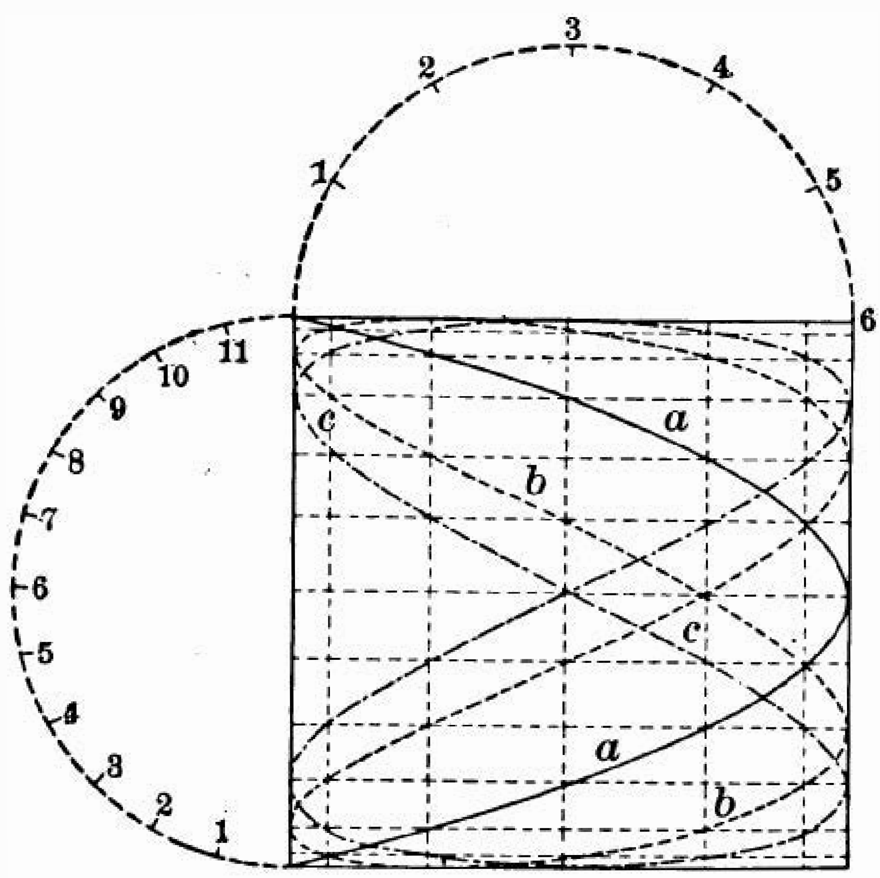
Fig. 35.

It is easy to see what the path of such a pencil will be. In Fig. 36 let 0 1 2 3 4 5 6 represent successive positions of the



PERIODS EQUAL, AMPLITUDES EQUAL
a, SAME PHASE
b, PHASE DIFFERENCE 60°
c, " " 90°

FIG. 36.

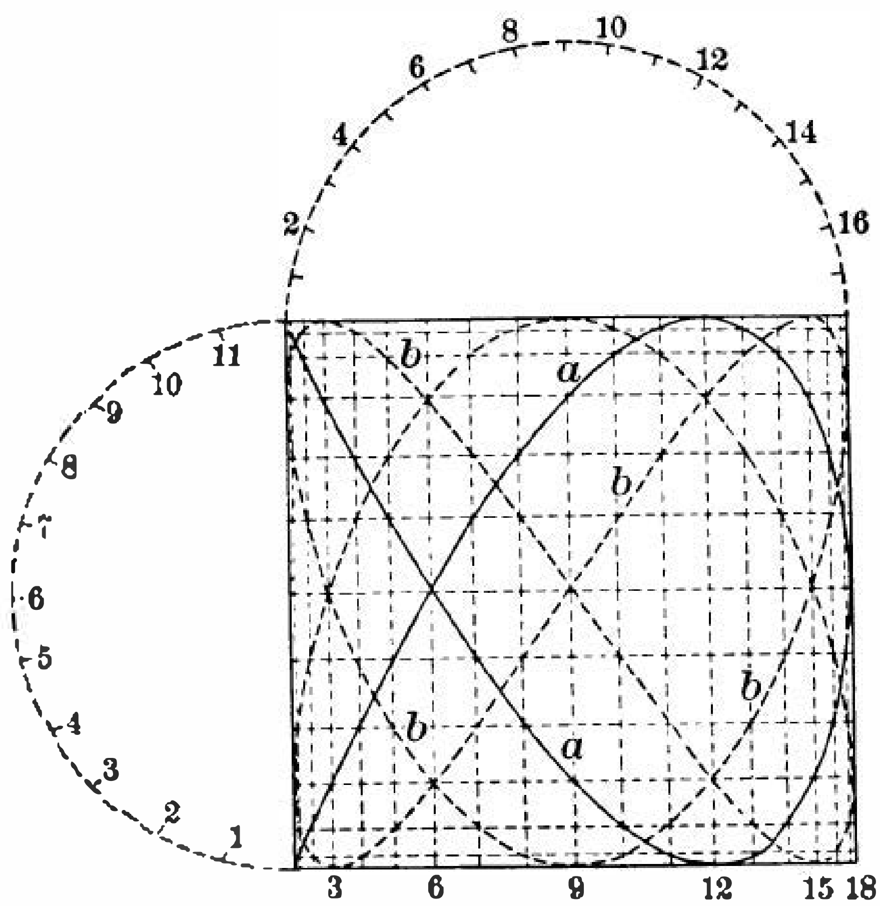


PERIODS 2:1, AMPLITUDES EQUAL
a, INITIAL PHASE DIFFERENCE 0°
b, " " " 60°
c, " " " 90°

FIG. 37.

vibrating point measured along one line, and $0\,1'2'3'4'5'6'$ those measured along the other, equal intervals of time being taken, and the periods of the two motions being equal.

In the figure the movement from 0 to 1 or 1 to 2 is executed in $\frac{1}{12}$ of a complete period. Now if both harmonic motions have the same phase, points of intersection of the lines drawn through 1 and 1′, 2 and 2′, 3 and 3′, etc., parallel to the axes *OA*, *OC*, will give successive positions of the tracing-point. If there is a phase difference of 60°, the tracing-point will have moved as far as 2′ along one line, while it



PERIODS 2:3, AMPLITUDES EQUAL
a, INITIAL PHASE DIFFERENCE 0°
b, " " " 30°

FIG. 38.

is still at 0 on the other; hence the lines drawn through 1 and 3′, 2 and 4′, and so on, now give the path, which is seen to be an ellipse. Again, with a phase difference of 90° the path becomes circulare Figs. 37 and 38 show the curves resulting from the compounding of two simple harmonic motions of equal amplitudes, having periods in the proportion of 2 : 1 and 2 : 3 respectively, and various initial phase differences The combination of motions which have periods in any other ratio can readily be illustrated by the same method, and the reader will find it instructive to plot for himself some of the resulting curves.