HAMILTONIAN TORUS ACTIONS IN EQUIVARIANT COHOMOLOGY AND SYMPLECTIC TOPOLOGY

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The central theme of this work are Hamiltonian torus actions on symplectic manifolds. We investigate the invariants of the action, and use the action to answer questions about the invariants of the manifold itself.

In the first chapter we concentrate on equivariant cohomology ring, a topological invariant for a manifold equipped with a group action. We consider a Hamiltonian action of n-dimensional torus, T^n, on a compact symplectic manifold (M, ω) with d isolated fixed points. There exists a basis $\{a_p\}$ for $H_T^*(M;\mathbb{Q})$ as an $H^*(BT;\mathbb{Q})$ module indexed by the fixed points $p \in M^T$. The classes a_p are not uniquely determined. The map induced by inclusion, $\iota^* : H_T^*(M;\mathbb{Q}) \to H_T^*(M^T;\mathbb{Q}) = \oplus_{j=1}^d \mathbb{Q}[x_1, \ldots, x_n]$ is injective. We will use the basis $\{a_p\}$ to give necessary and sufficient conditions for $f = (f_1, \ldots, f_d)$ in $\oplus_{j=1}^d \mathbb{Q}[x_1, \ldots, x_n]$ to be in the image of ι^*, i.e. to represent an equiviariant cohomology class on M. When the one skeleton is 2-dimensional, we recover the GKM Theorem. Moreover, our techniques give combinatorial description of $H_K^*(M;\mathbb{Q})$, for a subgroup $K \hookrightarrow T$, even though we are then no longer in GKM case.

The second part of the thesis is devoted to a symplectic invariant called the Gromov width. Let G be a compact connected Lie group and T its maximal torus. The Thi orbit $\mathcal{O}_\lambda$ through $\lambda \in \mathfrak{t}^*$ is canonically a symplectic manifold. Therefore a natural question is to determine its Gromov width. In many cases the width is known to be exactly the minimum over the set $\{\langle \alpha_j^\vee, \lambda \rangle; \alpha_j^\vee \text{ a coroot}, \langle \alpha_j^\vee, \lambda \rangle > 0\}$.

We show that the lower bound for Gromov width of regular coadjoint orbits of the unitary group and of the special orthogonal group is given by the above minimum. To prove this result we will equip the (open dense subset of the) orbit with a Hamiltonian torus action, and use the action to construct explicit embeddings of symplectic balls. The proof uses the torus action coming from the Gelfand-Tsetlin system.

BIOGRAPHICAL SKETCH

Milena Dorota Pabiniak was born in Łódź, Poland on March 5th 1982. She studied at the Department of Mathematics of the University of Łódź from 2000 to 2005 and graduated with degree “Magister”. During the years 2005-2008 she was a student at the Department of Mathematics of the George Washington University in Washington DC. There she obtained MA in mathematics. She continued her studies at PhD Program at Cornell University.

To my Friends,

and

to my niece Zosia,

who, as this thesis, starts her life in the summer 2012.

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Several people contributed greatly to my understanding and enjoyment of mathematics.

I would like to thank my advisor Tara Holm for her help and encouragement. I've learnt from her not only a lot of interesting mathematics, but also a lot about a life of a mathematician - how to remain sane in the world of academia, and even how to enjoy it! She is a very friendly person, with a great enthusiasm for mathematics. Thanks to her I had a chance to meet many of my mathematical friends.

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Last, but far from least, I owe my warmest gratitude to my Family and Friends. They are great! To them I owe most of what I have achieved.

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CHAPTER 1

INTRODUCTION

This work consists of two independent results: one about an equivariant cohomology ring, and the second about Gromov width. We study both in the context of Hamiltonian torus actions, but in two very different ways. In this introduction we only give basic definitions and postpone the proper explanation of the problems to the introductions of the main chapters.

Let (M, ω) be a connected symplectic manifold. The action of a (compact) torus $T \cong (S^1)^k$ is called **Hamiltonian** if there exists a T-invariant map $\Phi \colon M \to \mathfrak{t}^*$, called the **momentum map**, such that

$$\iota(\xi_M)\omega = -d\,\langle \Phi, \xi \rangle \quad \forall\, \xi \in \mathfrak{t},$$

where ξ_M is the vector field on M generated by $\xi \in \mathfrak{t}$. Two different sign conventions are commonly used by symplectic geometers. The above one is widely used while working with symplectic toric manifolds. In this convention, the isotropy weights of the induced T-action on the tangent space at a fixed point can be identified with the generators of edges of the polytope $\Phi(M)$ corresponding to the given symplectic toric manifold. It will be convenient to use this convention in Chapter 2 where we talk about the equivariant cohomology of symplectic toric manifolds and more general GKM spaces. The second convention defines the momentum map as a function satisfying

$$\iota(\xi_M)\omega = d\,\langle \Phi, \xi \rangle\,.$$

This choice of sign means that the isotropy weights of the action are pointing outside of the momentum map image. However we will use this convention in Chapters 3 to 5. The reason is that we want to relate the momentum map image of the Gelfand-Tsetlin action with the Gelfand-Tsetlin polytope, which has already appeared in numerous mathematical works.

The notion of a Hamiltonian torus action comes from physics. Every symmetry of a physical system X has a corresponding conserved quantity, such as angular momentum. This conserved quantity is a real-valued function H on the phase space T^*X called the Hamiltonian. We can use the (non-degenerate) symplectic form to turn the differential of H into a vector field, and this provides a flow on the manifold. When this flow is periodic, with same period, it gives rise to a (Hamiltonian) circle action on the symplectic manifold T^*X.

The first example of a Hamiltonian action is a circle acting on a 2-sphere by rotation about the z-axis, presented on Figure 1.1. The north and south poles are fixed, and the momentum map is simply given by the z coordinate (the height function).

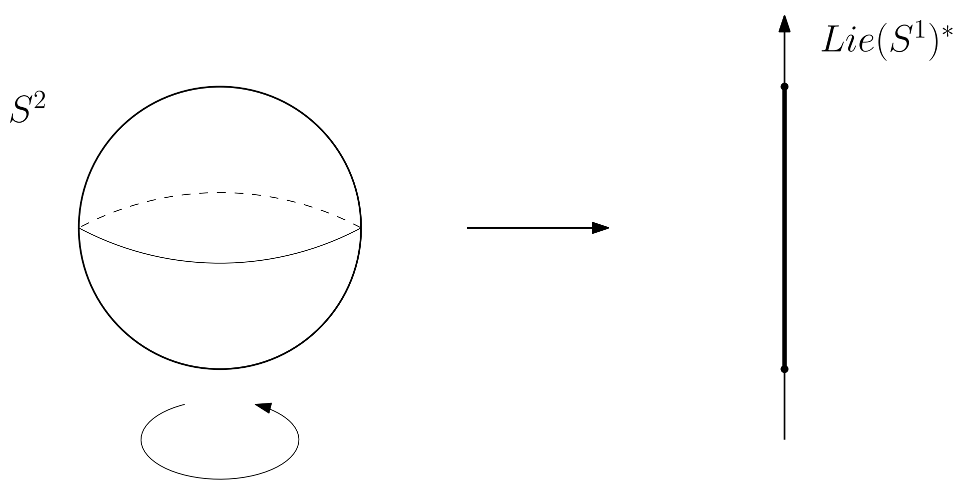


Figure 1.1: Hamiltonian S^1 action on S^2.

An important class of examples of symplectic manifolds is given by coadjoint orbits of Lie groups. A Lie group G acts on $\mathfrak{g}^*$, the dual of its Lie algebra, through the coadjoint action. Each orbit $\mathcal{O}$ of the coadjoint action is naturally equipped with the Kostant-Kirillov symplectic form:

$$\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle, \quad \xi \in \mathcal{O}_\lambda \subset \mathfrak{g}^*,\ X,Y \in \mathfrak{g} \cong T_\xi \mathcal{O}_\lambda.$$

The action of G on an orbit $\mathcal{O}$ is Hamiltonian, and the momentum map is just inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

When $G = U(n)$ the group of (complex) unitary matrices, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. If all eigenvalues are distinct, the orbit is a manifold of full flags in $\mathbb{C}^n$. For example, $U(2)$ orbit through $\mathrm{diag}(\lambda_1 > \lambda_2)$ is a full flag in $\mathbb{C}^2$, i.e. $\mathbb{C}P^1 \cong S^2$, and consists of matrices

$$\left\{ A = \begin{bmatrix} a & c+id \\ c-id & b \end{bmatrix};\ a,b,c,d \in \mathbb{R},\ \chi_A(t) = (t-\lambda_1)(t-\lambda_2) \right\}$$
$$= \left\{ A;\ \ a+b = \lambda_1 + \lambda_2,\ \ (a-b)^2 + 4c^2 + 4d^2 = (\lambda_1 - \lambda_2)^2 \right\}.$$

In Chapter 2 we consider symplectic manifolds with Hamiltonian torus actions. We use information coming from momentum map to find a convenient presentation of their equivariant cohomology ring. In the three subsequent Chapters we concentrate on the symplectic invariant called the Gromov width. We equip a manifold with a Hamiltonian torus action and use it to find lower bounds for its Gromov width.

CHAPTER 2

EQUIVARIANT COHOMOLOGY

2.1 Introduction

Suppose that a compact Lie group G acts on a compact, closed, connected and oriented manifold M. Unless otherwise stated, all the manifolds considered here are assumed to be compact, closed and connected. Let $EG \to BG$ denote the classifying bundle for G. The equivariant cohomology ring $H_G^*(M;R) := H^*(M \times_G EG; R)$, with coefficients in a ring R, encodes topological information about the manifold and the action. In the case of a Hamiltonian action on a symplectic manifold, a variety of techniques has made computing $H_G^*(M;R)$ tractable. The work of Goresky-Kottwitz-MacPherson [11] describes this ring combinatorially when G is a torus, R a field, and the action has very specific form. We give a more general description that has a similar flavor. A theorem of Kirwan [22] states that the inclusion of the fixed points induces an injective map in equivariant cohomology. We quote this result below, following Tolman and Weitsman [36].

Theorem 2.1.1 (Kirwan, [22]). *Let a torus T act on a symplectic compact connected manifold (M, ω) in a Hamiltonian fashion and let $\iota : M^T \to M$ denote the natural inclusion of fixed points into manifold. Then the induced map $\iota^* : H_T^*(M;\mathbb{Q}) \to H_T^*(M^T;\mathbb{Q})$ is injective. If M^T consists of isolated points then also $\iota^* : H_T^*(M;\mathbb{Z}) \to H_T^*(M^T;\mathbb{Z})$ is injective.*

If there are d fixed points then $H_T^*(M^T;\mathbb{Q}) = \oplus_{j=1}^d \mathbb{Q}[x_1, \ldots, x_n]$, where n is the dimension of the torus. Therefore we can think about an equivariant cohomology class in $H_T^*(M^T;\mathbb{Q})$ as a d-tuple of polynomials $f = (f_1, \ldots, f_d)$, with each f_j in

$\mathbb{Q}[x_1, \dots, x_n]$. The goal of this paper is to give necessary and sufficient conditions for a d-tuple of polynomials to be in the image of ι^*, that is to represent an equiviariant cohomology class on M. By abuse of language we will say that a d-tuple of polynomials $f = (f_1, \dots, f_d)$ 'is' an equivariant cohomology class if it is the image under ι^* of an honest (unique) equivariant cohomology class on M. The following result of Chang and Skjelbred [3] guarantees that we only need to consider the case of an S^1 action.

Theorem 2.1.2 (Chang, Skjelbred, [3]). *The image of* $\iota^* : H_T^*(M;\mathbb{Q}) \to H_T^*(M^T;\mathbb{Q})$ *is the set*

$$\bigcap_H \iota_{M^H}^*(H_T^*(M^H;\mathbb{Q})),$$

where the intersection in $H_T^*(M^T;\mathbb{Q})$ *is taken over all codimension-one subtori* H *of* T*, and* ι_{M^H} *is the inclusion of* M^T *into* M^H*.*

In fact the only nontrivial contributions to this intersection are those codimension 1 subtori H which appear as isotropy groups of some elements of M (that is $M^H \neq M^T$). Therefore we will consider a circle acting on a compact, connected and closed symplectic manifold (M, ω) in a Hamiltonian fashion with isolated fixed points and momentum map $\mu : M \to Lie(S^1)^*$. In this Chapter we use the convention where $\iota(\xi_M)\omega = -d\langle \mu, \xi \rangle$ for all $\xi \in Lie(S^1)$.

Recall the Atiyah-Bott, Berline-Vergne (ABBV) localization theorem. For a fixed point p let $e(p)$ be the equivariant Euler class of tangent bundle T_pM, which in this case is equal to the product of weights of the torus action (see for example Lemma 2.2 in [37]).

Theorem 2.1.3 (ABBV Localization, [1][2]). *Let* M *be a compact oriented manifold equipped with an* S^1 *action with isolated fixed points, and let* $\alpha \in H_{S^1}^*(M;\mathbb{Q})$*.*

Then as elements of $H^*(BS^1;\mathbb{Q}) = \mathbb{Q}[x]$,

$$\int_M \alpha = \sum_p \frac{\alpha|_p}{e(p)},$$

where the sum is taken over all fixed points.

Let $Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q})$ denote the $\mathbb{Q}(x)$-vector space of fractions of $H^*_{S^1}(M^{S^1};\mathbb{Q})$. We extend the notion of integration to $Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q})$. Define a $\mathbb{Q}(x)$-linear functional

$$\int : Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q}) \to \mathbb{Q}(x)$$

by

$$\int \alpha = \sum_p \frac{\alpha|_p}{e(p)}.$$

The functional $\int$ agrees with $\int_M$ on $H^*_{S^1}(M;\mathbb{Q})$. Consider the $\mathbb{Q}(x)$-bilinear pairing

$$\langle\, ,\rangle\colon Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q}) \times Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q}) \to \mathbb{Q}(x)$$

given by

$$\langle \alpha, \beta \rangle = \int \alpha \cdot \beta.$$

When restricted to $H^*_{S^1}(M;\mathbb{Q}) \times H^*_{S^1}(M;\mathbb{Q})$, this pairing is the equivariant Poincaré pairing. The pairing induces the map

$$\Phi : H^*_{S^1}(M;\mathbb{Q}) \to\ Hom_{\mathbb{Q}[x]}(H^*_{S^1}(M;\mathbb{Q});\mathbb{Q}[x]),$$

defined by $\Phi(\alpha)(\beta) := \langle \alpha, \beta \rangle$. The Main Theorem is:

Theorem 2.1.4. *Let a circle act on a closed compact connected symplectic manifold* M *in a Hamiltonian fashion, with isolated fixed points. The equivariant Poincaré pairing*

$$\langle\, ,\rangle\colon H^*_{S^1}(M;\mathbb{Q}) \times H^*_{S^1}(M;\mathbb{Q}) \to \mathbb{Q}[x]$$

is a perfect pairing, that is, the map $\Phi : H^*_{S^1}(M;\mathbb{Q}) \to\ Hom_{\mathbb{Q}[x]}(H^*_{S^1}(M;\mathbb{Q});\mathbb{Q}[x])$ *defined by* $\Phi(\alpha)(\beta) := \langle \alpha, \beta \rangle$ *is an isomorphism.*

Knutson in [23, Section 1.3] proved that the equivariant Poincaré pairing is non-degenerate, therefore the map Φ is injective. We recall the proof in Section 2.2. Every fixed point $p \in M^{S^1}$ defines an equivariant cohomology class $[p] \in H^*_{S^1}(M;\mathbb{Q})$. Therefore every $\mathbb{Q}[x]$-homomorphism in $Hom_{\mathbb{Q}[x]}(H^*_{S^1}(M;\mathbb{Q});\mathbb{Q}[x])$ extends uniquely to an $\mathbb{Q}(x)$-linear map from $\mathbb{Q}(x)$-vector space $Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q})$ to $\mathbb{Q}(x)$. All such maps are given by $\beta \to \langle \alpha, \beta \rangle$, for some $\alpha \in Fr\, H^*_{S^1}(M^{S^1};\mathbb{Q})$. To prove surjectivity of Φ we need to show that $\alpha \in H^*_{S^1}(M;\mathbb{Q})$. The fact that $\beta \to \langle \alpha, \beta \rangle$ maps $H^*_{S^1}(M;\mathbb{Q})$ to $\mathbb{Q}[x]$ implies that $\alpha \in H^*_{S^1}(M^{S^1};\mathbb{Q})$, as for any fixed point p, $\langle \alpha, e(p) \rangle = \alpha_{|p}$ must be in $\mathbb{Q}[x]$. Therefore, to prove the surjectivity part of the theorem we only need to show that if an element $\alpha \in H^*_{S^1}(M^{S^1};\mathbb{Q})$ satisfies

$$\forall_{\beta \in H^*_{S^1}(M;\mathbb{Q})} \quad \langle \alpha, \beta \rangle \in \mathbb{Q}[x], \tag{2.1}$$

then $\alpha \in H^*_{S^1}(M;\mathbb{Q})$. We now review some background and reformulate the theorem in a form which is more useful for applications.

Let a circle act on a closed compact connected symplectic manifold M in a Hamiltonian fashion, with isolated fixed points. It turns out that with these assumptions we are in the Morse Theory setting.

Theorem 2.1.5 (Frankel [6], Kirwan [22]). *In the above setting, the momentum map μ is a perfect Morse function on M (for both ordinary and equivariant cohomology). The critical points of μ are the fixed points of M, and the index of a critical point p is precisely twice the number of negative weights of the circle action on T_pM.*

The Morse function is called **perfect** if the number of critical points of index k is equal to the dimension of k-th cohomology group. The action of a torus of higher dimension also carries a Morse function. For $\xi \in \mathfrak{t}$ we define $\Phi^\xi : M \to \mathbb{R}$, the component of momentum map along ξ, by $\Phi^\xi(p) = \langle \Phi(p), \xi \rangle$. We call $\xi \in \mathfrak{t}$ **generic** if $\langle \eta, \xi \rangle \neq 0$ for each weight $\eta \in \mathfrak{t}^*$ of T action on T_pM, for every p in the fixed set M^T. For a generic, rational ξ, Φ^ξ is a Morse function with critical set M^T. This map is a momentum map for the action of a subcircle $S \hookrightarrow T$ generated by $\xi \in \mathfrak{t}$. Using Morse Theory, Kirwan constructed equivariant cohomology classes that form a basis for integral equivariant cohomology ring of M. Then the existence of a basis for rational equivariant cohomology ring of M follows. We quote this theorem with the integral coeficients, and action of T, although here we work mostly with rational coefficients and circle actions.

Theorem 2.1.6 (Kirwan, [22])**.** *Let a torus T act on a symplectic compact manifold M with isolated fixed points, and let $\mu = \Phi^\xi : M \to \mathbb{R}$ be a component of momentum map Φ along generic $\xi \in \mathfrak{t}$. Let p be any fixed point of index $2k$ and let $w_1, \ldots, w_k$ be the negative weights of the T action on T_pM. Then there exists a class $a_p \in H_T^{2k}(M; \mathbb{Z})$ such that*

- $a_p|_p = \Pi_{i=1}^k w_i$;
- $a_p|_{p'} = 0$ *for all fixed points $p' \in M^T \setminus \{p\}$ such that $\mu(p') \leq \mu(p)$.*

Moreover, taken together over all fixed points, these classes are a basis for the cohomology $H_T^*(M; \mathbb{Z})$ as an $H^*(BT; \mathbb{Z})$ module.

In the above theorem we use the convention that empty product is equal to 1. We will call the above classes **Kirwan classes**. These classes may be not unique. Goldin and Tolman give a different basis for the cohomology ring $H_T^*(M; \mathbb{Z})$ in [10]. They require $a_p|_{p'} = 0$ for all fixed points $p' \neq p$ of index less then or equal $2k$ (where $2k$ is index of p). Goldin and Tolman's classes, if they exist, are unique. Therefore they are called **canonical classes**. For our purposes, it is enough to have some basis for the rational equivariant cohomology ring with respect to circle action, and with the following property

($\star$) elements of the basis are in such a bijection with the fixed points that a class corresponding to a fixed point of index $2k$ evaluated at any fixed point is 0 or a homogeneous polynomial of degree k.

We will call elements of a basis satisfying condition ($\star$) **generating classes**. Kirwan classes and Goldin-Tolman canonical classes satisfy the above condition.

The hypothesis of Theorem 2.1.4 is that a circle acts on a closed compact connected symplectic manifold M in a Hamiltonian fashion, with isolated fixed points. Denote the fixed points by $p_1, \ldots, p_d$. Let $\{a_p\}$ be the basis of $H_T^*(M;\mathbb{Q})$, satisfying condition ($\star$). Its existence is guaranteed by Theorem 2.1.6. A choice of basis allows us to restate the surjectivity part of Theorem 2.1.4 (condition 2.1) in more applicable form.

Theorem 2.1.7. (Surjectivity of Φ from Theorem 2.1.4.)
Let $f = (f_1, \ldots, f_d) \in \oplus_{j=1}^d \mathbb{Q}[x] = H_{S^1}^*(M^{S^1};\mathbb{Q})$. Then f is an equivariant cohomology class on M if and only if for every fixed point p of index $2k$, $0 \leq k < n$ we have

$$\sum_{j=1}^{d} \frac{f_j\, a_p(p_j)}{e(p_j)} \in \mathbb{Q}[x], \tag{2.2}$$

where $a_p(p_j)$ denotes $\iota_{p_j}^*(a_p)$, with $\iota_{p_j} : p_j \hookrightarrow M$ the inclusion of the fixed point p_j into M.

Note that if p is a fixed point of index $2n$, this condition is automatically satisfied. This is because a_p is nonzero only at p, and there its value is the Euler class $e(p)$. Therefore it is sufficient to check the above condition only for points of index strongly less then $2n = \dim M$.

Remark 2.1.8. *If f is a cohomology class, then so is $f \cdot a_p$. Applying the Localization Theorem to the class $f \cdot a_p$ we see that conditions (2.2) must be satisfied. The interesting part of the theorem is that they are sufficient to describe $H_T^*(M)$ as a subring of $H_T^*(M^T)$.*

Example 2.1.9. ***Recovering the GKM Theorem.*** *Consider the standard Hamiltonian S^1 action on S^2 by rotation with weight ax. The isolated fixed points are south and north poles which we will denote by p_1 and p_2 respectively. The Goldin-Tolman class associated to p_1 is 1. It exists due to Theorem 1.6 in [10] as the momentum map is index-increasing. Theorem 2.1.7 then says that $f = (f_1, f_2)$ represents equivariant cohomology class if and only if*

$$\frac{f_1\, a_1(p_1)}{e(p_1)} + \frac{f_2\, a_1(p_2)}{e(p_2)} = \frac{f_1}{ax} + \frac{f_2}{-ax} = \frac{f_1 - f_2}{ax} \in \mathbb{Q}[x].$$

The above condition is exactly the same as the condition (1) in [9]. Using the solution for this special case, together with the Chang-Skjelbred Lemma, Goldin and Holm recover the GKM Theorem in Section 1 and 2 of [9].

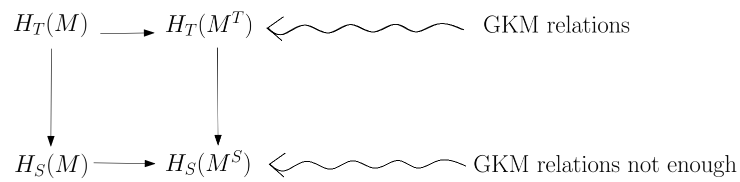
Let M be a compact, connected, symplectic manifold with a Hamiltonian, effective action of a torus T and with finitely many fixed points. Let $N \subset M$ be the set of points whose orbits under the G action are 1-dimensional. The ***one-skeleton*** *of M is the closure $\overline{N}$. The manifold M is called a* ***GKM manifold*** *if N has finitely many connected components N_α.*

Theorem 2.1.10 ([11] and [9],[36])**.** *Let M be a GKM manifold with a Hamiltonian torus action by G. Let M^G be the fixed point set, and $\overline{N}$ be the one-skeleton. Let $r : M^G \hookrightarrow M$ be the inclusion of the fixed point set to M and $j : M^G \hookrightarrow \overline{N}$ be the inclusion to $\overline{N}$. The induced maps $r^* : H_G^*(M) \to H_G^*(M^G)$ and $j^* : H_G^*(\overline{N}) \to H_G^*(M^G)$ on equivariant cohomology have the same image.*

Theorem 2.1.7 is useful only if we know the restrictions to the fixed points of a set of generating classes (whose existence is guaranteed by Theorem 2.1.6). It is not surprising that there is a translation from the values of generating classes at fixed points to relations defining $H_T^*(M) \subset H_T^*(M^T)$. Our translation provides a particularly combinatorial description that is easy to apply in examples. Although we cannot compute these classes in general, there are algorithms that work for a wide class of spaces, for example GKM spaces, including symplectic toric manifolds and flag manifolds (see [38]). For the sake of completeness we will describe an algorithm for obtaining Kirwan classes for symplectic toric manifolds in Section 2.3. The choice of a_p assigned to fixed point p may be not unique, even for symplectic toric manifolds. In the case when momentum map is so called "index increasing" and the manifold is a GKM manifold, uniqueness was proved by Goldin and Tolman in [10].

A particularly interesting application of our theorem is when we want to restrict the action of T to an action of a subtorus $S \hookrightarrow T$ such that $M^S = M^T$, and compute $\iota^*(H_S^*(M)) \subseteq H_S^*(M^S) = H_S^*(M^T)$. We call this process **specialization** of the T action to the action of subtorus S. Having generating classes for T action we can easily compute generating classes for S action using the projection $\mathfrak{t}^* \to \mathfrak{s}^*$. Theorem 2.1.7 gives relations that cut out $\iota^*(H_S^*(M)) \subseteq H_S^*(M^T)$. In particular we can use this method to restrict the torus action on a symplectic toric manifold to a generic circle, i.e. such a circle S for which $M^S = M^T$ (see Example 2.4.2). A priori we only require that M^S is finite, as we still want to describe $H_S^*(M)$ by analyzing the relations on polynomials defining the image $\iota^*(H_S^*(M)) \subseteq H_S^*(M^S) = \oplus \mathbb{Q}[x_1, \dots x_k]$. However it turns out that this requirement implies $M^T = M^S$. We can explain this fact using Morse theory. If $\Phi : M \to \mathfrak{t}^*$ is a momentum map for T action and $\xi \in \mathfrak{t}$ is generic, then Φ^ξ, a component of Φ along ξ, is a perfect Morse function with critical set M^T. Therefore $\sum \dim H^i(M) = |M^T|$. Similarly, taking $\mu = pr_{\mathfrak{s}^*} \circ \Phi$ for the momentum map for S action, and any generic $\eta \in \mathfrak{s}$, we obtain μ^η which is also a perfect Morse function for M. Thus $|M^S| = \sum \dim H^i(M) = |M^T|$. As obviously $M^T \subset M^S$, the sets must actually be equal.

Consider restriction of the GKM action of T to a generic subcircle S:



GKM relations are sufficient to describe the image of $H_T^*(M)$ in $H_T^*(M^T)$, but their "projections" are not sufficient to describe the image of $H_S^*(M)$ in $H_S^*(M^T)$. However projecting generating classes and using Theorem 2.1.7 to construct relations from such a basis will give all the relations we need.

The GKM Theorem is a very powerful tool that allows us to compute the image under ι^* of $H_T^*(M) \hookrightarrow H_T^*(M^T)$. However this theorem cannot be applied if for some codimension 1 subtorus $H \hookrightarrow T$ we have dim $M^H > 2$. Goldin and Holm in [9] provide a generalization of this result to the case where dim $M^H \leq 4$ for all codimension 1 subtori $H \hookrightarrow T$. An important corollary is that, in the case of Hamiltonian circle actions, with isolated fixed points, on manifolds of dimension 2 or 4, the rational equivariant cohomology ring can be computed solely from the isotropy weights of the circle action at the fixed points. In dimension 2 this is given for example by the GKM Theorem. In dimension 4 one can apply the algorithm presented by Goldin and Holm in [9] or use the fact that any such S^1 action is actually a specialization of a toric T^2 action (see [19]). If one wishes to compute the integral equivariant cohomology ring, one will need an additional piece of information, so called "isotropy skeleton" ([8]). Godinho in [8] presents such an algorithm. Information encoded in the isotropy skeleton is essential. There cannot exist an algorithm computing the integral equivariant cohomology only from the fixed points data. Karshon in [18](Example 1), constructs two 4-dimensional S^1 spaces with the same weights at the fixed points but different integral equivariant cohomology rings. This suggests that we should not hope for an algorithm computing the rational equivariant cohomology ring from the isotropy weights at the fixed points for manifolds of dimension greater then 4. More information is needed. Tolman and Weitsman used generating classes to compute the equivariant cohomology ring in case of semifree action in [37]. Their work gave us the idea for constructing necessary relations described in the present paper using information from generating classes. Our proof was also motivated by the work of Goldin and Holm [9] where the Localization Theorem and dimensional reasoning were used.

2.2 Proof of Theorem 2.1.4

Let a circle act on a manifold M in a Hamiltonian fashion with isolated fixed points which we denote $p_1, \ldots, p_d$.

Proof. **Injectivity of Φ.** We show that the map

$$\Phi : H^*_{S^1}(M;\mathbb{Q}) \to \; Hom_{\mathbb{Q}[x]}(H^*_{S^1}(M;\mathbb{Q});\mathbb{Q}[x])$$

is injective, following [23, Section 1.3]. Take an element $\alpha \in H^*_{S^1}(M;\mathbb{Q})$ such that $\Phi(\alpha) = 0$, that is, for any $\beta \in H^*_{S^1}(M;\mathbb{Q})$ one has $\langle \alpha, \beta \rangle = 0$. In particular, for any fixed point p_j we have $0 = \langle \alpha, [p_j] \rangle = \alpha_{|p_j}$. Injectivity of the map $H^*_{S^1}(M;\mathbb{Q}) \to H^*_{S^1}(M^{S^1};\mathbb{Q})$ (see Theorem 2.1.1) implies that $\alpha = 0$.

Surjectivity of Φ (proof of Theorem 2.1.7). As explained in the Introduction, surjectivity of Φ is equivalent to Theorem 2.1.7. Let $\{a_p\}$ be a basis of $H^*_{S^1}(M;\mathbb{Q})$, satisfying condition $(\star)$. We want to show that if $f=(f_1,\ldots,f_d)\in \oplus_{j=1}^d \mathbb{Q}[x] = H^*_{S^1}(M^{S^1})$ satisfies relations (2.2):

$$\sum_{j=1}^{d} \frac{f_j\, a_p(p_j)}{e_{p_j}} \in \mathbb{Q}[x],$$

for every fixed point p, then f is in the image, $\iota^*(H^*_{S^1}(M;\mathbb{Q}))$, of injective, degree preserving map ι^*. By abuse of notation we say such f is an equivariant cohomology class of M. Recall that $\mathbb{Q}[x]$ is a PID. Let R be a submodule of $\oplus_{j=1}^d \mathbb{Q}[x]$ consisting of all d-tuples $f=(f_1,\ldots,f_d)$ satisfying all of the above relations. As a submodule of a free module over PID, R itself is free. Hamiltonian S^1-spaces are equivariantly formal, that is $H^*_{S^1}(M;\mathbb{Q}) \cong H^*(M;\mathbb{Q})\otimes H^*(BS^1;\mathbb{Q})$ as modules. We already noticed that all the above relations are necessary. Therefore $\iota^*(H^*_{S^1}(M;\mathbb{Q}))$ is a free $\mathbb{Q}[x]$ submodule of $R\subset \oplus_{j=1}^d \mathbb{Q}[x]$ We show that for any k the number of generators of degree k part of $\iota^*(H^*_{S^1}(M;\mathbb{Q}))$ is equal to the number of generators of the degree k part of R. It then follows that $\iota^*(H^*_{S^1}(M;\mathbb{Q}))=R$ as needed.

We first analyze $\iota^*(H^*_{S^1}(M;\mathbb{Q}))$. The momentum map is a Morse function. Therefore the idex of a fixed point is well defined. Let b_k be the number of fixed points of index $2k$. Then $d=\sum_{k=0}^n b_k$ is the total number of fixed points. By Theorem 1.3 of Frankel and Kirwan, we know that b_k is also the $2k$-th Betti number of M. The fact $H^*_{S^1}(M;\mathbb{Q}) \cong H^*(M;\mathbb{Q})\otimes H^*(BS^1;\mathbb{Q})$ implies that the equivariant Poincaré polynomial for M is

$$P_M^{S^1}(t) = P_M(t)P_{pt}^{S^1}(t) = (b_0+b_1t^2+\ldots+b_nt^{2n})(1+t^2+t^4+\ldots) =$$

$$= b_0+(b_0+b_1)t^2+\ldots+(b_0+b_1+\ldots+b_k)t^{2k}+\ldots+dt^{2n}+dt^{2(n+1)}+\ldots.$$

Therefore $\iota^*(H^*_{S^1}(M;\mathbb{Q}))$ is a free $\mathbb{Q}[x]$ submodule of R, where degree k piece is a vector space over $\mathbb{Q}$ of dimension $(b_0+b_1+\ldots+b_k)$.

We now analyze and count the relations defining R. For any $f = (f_1, \dots, f_d) \in \oplus_{j=1}^d \mathbb{Q}[x] = H^*_{S^1}(M^{S^1})$ we introduce the notation

$$f_j(x) = \sum_{k=0}^{K_j} r_{jk} x^k,$$

with $r_{jk} \in \mathbb{Q}$. Then r_{jk} are independent variables. Relations of type

$$\sum_{j=0}^{d} s_j r_{jk} = 0$$

for some constants s_j's are called relations of degree k, as they involve the coefficients of x^k. Notice that if $f \in (\oplus_{j=1}^d \mathbb{Q}[x])_k$ is a homogeneous element of degree k then it automatically satisfies all relations of degrees different then k. For any fixed point p of index $2(k-1)$, a generating class a_p associated with it assigns to each fixed point p_j either 0 or a homogeneous polynomial of degree $(k-1)$. Denote by c_j^p the rational number satisfying

$$\frac{a_p(p_j)}{e(p_j)} = c_j^p \, x^{k-1-n}.$$

If f is an equivariant cohomology class of M then $f \cdot a_p$ is also. The Localization Theorem gives the relation

$$\int_M a_p f = \sum_{j=1}^{d} \frac{f_j \, a_p(p_j)}{e_{p_j}} \in \mathbb{Q}[x].$$

We may rewrite this in the following form:

$$\begin{aligned}
\int_M a_p f &= \sum_{j=1}^{d} \frac{f_j \, a_p(p_j)}{e_(p_j)} \\
&= \sum_{j=1}^{d} f_j c_j^p \, x^{k-1-n} \\
&= \sum_{j=1}^{d} c_j^p \left(\sum_{l=0}^{K_j} r_{jl} x^l \right) x^{k-1-n} \\
&= \sum_{j=1}^{d} c_j^p \left(\sum_{l=0}^{K_j} r_{jl} x^{k-1-n+l} \right) \in \mathbb{Q}[x].
\end{aligned}$$

Using the convention $r_{jl} = 0$ for $l > K_j$, we can write

$$\int_M a_p f = \sum_{j=1}^{d} c_j^p \left(\sum_{l=0}^{n-k} r_{jl} x^{k-1-n+l} \right) + \sum_{j=1}^{d} c_j^p \left(\sum_{l=n-k+1}^{K_j} r_{jl} x^{k-1-n+l} \right)$$
$$= \sum_{l=0}^{n-k} \left(\sum_{j=1}^{d} c_j^p r_{jl} \right) x^{k-1-n+l} + \sum_{j=1}^{d} c_j^p \left(\sum_{l=n-k+1}^{K_j} r_{jl} x^{k-1-n+l} \right).$$

The second component is an element of $\mathbb{Q}[x]$ as all the exponents of x are non-negative. Thus $\int_M a_p f$ is in $\mathbb{Q}[x]$ if and only if all the coefficients of x in the first component (that is coefficients of negative powers of x) are 0. Therefore for any fixed point p and any $l = 0, \ldots, n-k$, where $2(k-1)$ is the index of p, we get the following linear relation of degree l:

$$\sum_{j=1}^{d} c_j^p r_{jl} = 0.$$

Note that these relations are independent. We will show this by explicit computation. It is enough to show that for any l all the relations of degree l are independent, as relations of different degree involve different subset of variables $\{r_{jk}\}$. Suppose that in some degree l these relations in r_{jl}'s are not independent. That is, there are rational numbers s_p, not all zero, such that

$$\forall_{r_{jl}} \qquad 0 = \sum_p s_p \left(\sum_{j=1}^{d} c_j^p r_{jl} \right) = \sum_{j=1}^{d} \left(\sum_p s_p c_j^p \right) r_{jl}$$

As r_{jl} are independent variables, we have $\sum_p s_p c_j^p = 0$, for all $j = 1, \ldots, d$. Multiplying both sides by $e(p_j) x^{k-1-n}$ we obtain

$$\sum_p s_p e(p_j) c_j^p x^{k-1-n} = 0.$$

Recall the definition of c_j^p to notice that the above equation is equivalent to

$$\sum_p s_p a_p(p_j) = 0.$$

That means $\sum_p s_p a_p$ vanishes on every fixed point and therefore is the 0 class, although it is a nontrivial combination of classes a_p. This contradicts the independence of the generating classes a_p's.

Now we count the relations just constructed. As noted above, a fixed point of index $2(k-1)$ gives relations of degrees $0, \ldots, n-k$. Therefore a relation of degree $n-k$ is obtained from each fixed point of index $2(k-1)$ or less. That means we get a relation of degree k for each fixed point of index $2(n-k-1)$ or less, in total

$$(b_0 + b_1 + \ldots + b_{n-k-1})$$

relations of degree k. The subspace of $(\oplus^d \mathbb{Q}[x])_k \cong \mathbb{Q}^d$ of elements satisfying all relations of degree k is of dimension $d-(b_0+b_1+\ldots+b_{n-k-1})$. Every homogeneous element $f \in (\oplus^d \mathbb{Q}[x])_k$ satisfying all degree k relations also satisfies all relations of other degrees (as coefficients of x^l are 0 for $l \neq k$). Moreover, the form of conditions (2.2) implies that for any $g \in \mathbb{Q}[x]$, gf also satisfies all the relations (2.2). Therefore degree k part of R is the subspace of $(\oplus^d \mathbb{Q}[x])_k$ of elements satisfying all relations of degree k, and its dimension is $d - (b_0 + b_1 + \ldots + b_{n-k-1})$. By the definition of d and Poincaré duality,

$$d - (b_0 + b_1 + \ldots + b_{n-k-1}) = b_{n-k} + \ldots + b_n = b_0 + b_1 + \ldots + b_k.$$

This means that the degree k part of R, R_k, is a vector space over $\mathbb{Q}$ of dimension $(b_0 + b_1 + \ldots + b_k)$ containing a vector subspace $\iota^*(H^*_{S^1}(M; \mathbb{Q}))_k$, degree k part of $\iota^*(H^*_{S^1}(M; \mathbb{Q}))$, of the same dimension. Therefore they must be equal. The two graded submodules: $\iota^*(H^*_{S^1}(M; \mathbb{Q}))$ and R, are equal in each degree. This implies

$$\iota^*(H^*_{S^1}(M; \mathbb{Q})) = R.$$

□

2.3 Generating classes for Symplectic Toric Manifolds

A **symplectic toric manifold** is a connected symplectic manifold (M, ω) equipped with an effective Hamiltonian action of a torus T of dimension $\dim T = \frac{1}{2}\dim M$. Let M^{2n} be a compact symplectic toric manifold with momentum map image a Delzant polytope $\Phi(M) = P \subset \mathfrak{t}^*$. In particular P is simple, rational and smooth. The Lie algebra dual, $\mathfrak{t}^*$, is isomorphic to $\mathbb{R}^n$, though not canonically. One of the conventions is to identify S^1 with $\mathbb{R}/\mathbb{Z}$. Then the exponential map $Lie(S^1) \cong \mathbb{R} \to S^1$ is of the form $t \to e^{2\pi i t}$. With this identification, the function

$$\mathbb{C} \ni z \to -\pi\, k\, |z|^2 \in \mathbb{R} \cong Lie(S^1)$$

is a momentum map for the S^1 action on $(\mathbb{C}, \omega_{standard})$ by rotation with weight k. Therefore we can think of P as a Delzant polytope in $\mathbb{R}^n$. Denote by M_1 the union of all T-orbits of dimension 1. Closures of connected components of M_1 are spheres, called the isotropy spheres. Denote by V the vertices of P, and by E the 1-dimensional faces of P, also called edges. Vertices correspond to the fixed points of the torus action, while edges correspond to the isotropy spheres. Fix a generic $\xi \in \mathbb{R}^n$, so that for any $p, q \in V$ we have $\langle p, \xi \rangle \neq \langle q, \xi \rangle$. Orient the edges so that $\langle i(e), \xi \rangle < \langle t(e), \xi \rangle$ for any edge e, where $i(e)$, $t(e)$ are initial and terminal points of e. Let $w_{i(e)}(e) = -w_{t(e)}(e)$ denote the isotropy weights of T action on tangent spaces to isotropy sphere $\Phi^{-1}(e)$, $T_{\Phi^{-1}(i(e))}\Phi^{-1}(e)$ and $T_{\Phi^{-1}(t(e))}(\Phi^{-1}(e))$ respectively. Note that $w_{i(e)}(e)$ is the primitive integral vector in direction of $\vec{e}$. We denote it by $prim(\vec{e})$. For any $p \in V$ let G_p denote the smallest face containing p and all points $q \in V$ with $\langle p, \xi \rangle < \langle q, \xi \rangle$ which are connected with p by an edge. We will call G_p the **flow up face** for p. We define the class $a_p \in H^*_{S^1}(M^{S^1})$ by

$$a_p(q) = \begin{cases} 0 & for\ q \in V \setminus G_p \\ \prod_r prim(r - q) & for\ q \in G_p \end{cases}$$

where the product is taken over all $r \in V \setminus G_p$ such that r and q are connected by an edge of P. We use convention that empty product is 1. If k edges terminate at p then the $n-k$ edges starting from p belong to the face G_p (as polytope is simple, exactly n edges meet at each vertex). The smoothness of P implies that these $n-k$ edges span an $(n-k)$ affine hyperplane H_p of $\mathbb{R}^n$ and the face G_p is the intersection $G_p = P \cap H_p$. Moreover, it also implies that for any $q \in G_p$ there are $n-k$ edges meeting q that are contained in the face G_p and k edges connecting q to vertices outside the face G_p. Therefore the class a_p assigns to each fixed point 0 or a homogeneous polynomial of degree k. Such classes satisfy the GKM conditions and thus are in the image of the equivariant cohomology of M. The class a_p constructed this way is the canonical equivariant extension (see [26], Corollary 3.5) of the cohomology class Poincaré dual to the submanifold of M mapping to the face G_p. These two facts can be proved using the notion of the axial function introduced in [16]. The classes we have just defined are also linearly independent, which follows easily from the fact that a_p can be nonzero only at vertices q greater or equal to p in the partial order given by the orientation of edges. Our first example is a set of generating classes for $\mathbb{C}P^2$ presented in Figure 2.1

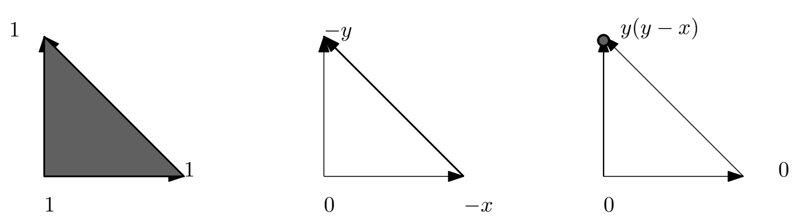


Figure 2.1: Generating classes for $\mathbb{C}P^2$.

Next we give an example where the generating classes are not unique. The above algorithm gives the basis presented on Figure 2.2. However classes in Figure 2.3 also form a basis.

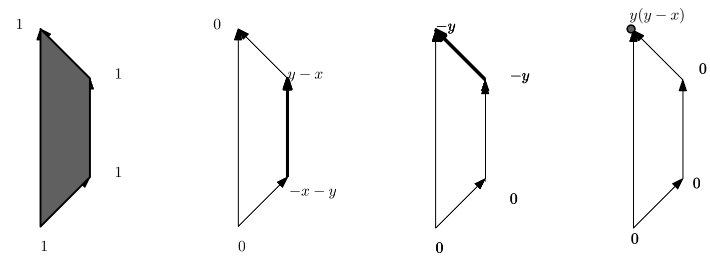


Figure 2.2: The basis of the equivariant cohomology ring given by the above algorithm.

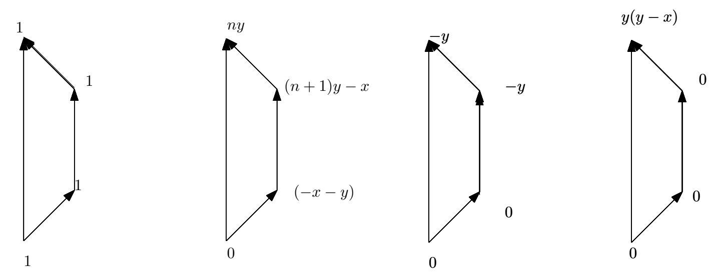


Figure 2.3: Different basis of the equivariant cohomology ring.

This algorithm is also very useful while dealing with specialization, that is while restricting toric T action on M to an action of some subtorus $S \hookrightarrow T$. As explained in the introduction, if S is generic then $M^T = M^S$. Using above algorithm we can find a basis of $H^*_S(M)$ even if we do not have the isotropy weights for the full T^n action. It is enough to know the isotropy weights of S action, the fact that this action is a specialization of some toric action and positions of isotropy spheres for that toric action. These weights are just projections of T weights under $pr : \mathfrak{t}^* \to \mathfrak{s}^*$. That is the S weight on edge e is $pr\,(prim(t(e)-i(e))\,)$. The positions of isotropy spheres for the toric action allow us to find the flow up face G_p for any fixed point p. The above algorithm gives that

$$a_p(q) = \begin{cases} 0 & for\ q \in V \setminus G_p \\ \prod_r pr\,(prim(r-q)\,) & for\ q \in G_p \end{cases}$$

where the product is taken over all $r \in V \setminus G_p$ such that r and q are connected by an isotropy sphere. Having generating classes for S action, we may apply Theorem 2.1.7 to obtain all relations needed to describe $\iota^*(H^*_{S^1}(M))$. This gives us a method for computing equivariant cohomology for a circle action that happens to be part of a toric action.

2.4 Examples

Example 2.4.1. *Consider the product of $\mathbb{C}P^2$ blown up at a point and $\mathbb{C}P^1$*

$$\widetilde{\mathbb{C}P^2} \times \mathbb{C}P^1 = \{([x_1 : x_2][y_0 : y_1 : y_2][z_0 : z_1]) |\, x_1\, y_2 - x_2\, y_1 = 0\},$$

and the following T^3 action on this space:

$$(e^{iu}, e^{iv}, e^{iw}) \cdot ([x_1 : x_2][y_0 : y_1 : y_2][z_0 : z_1]) = ([e^{iu}x_1 : x_2][e^{iv}y_0 : e^{iu}y_1 : y_2][e^{iw}z_0 : z_1]).$$

This is a symplectic toric manifold and its momentum map image is the polytope is shown in Figure 2.4.1. Using the algorithm from Section 2.3 we can compute

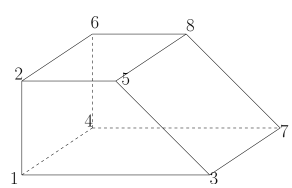
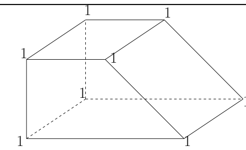
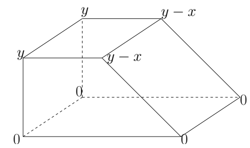


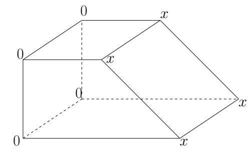
Figure 2.4: Moment polytope for $\widetilde{\mathbb{C}P^2} \times \mathbb{C}P^1$.

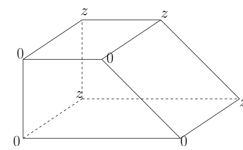
generating classes for the equivariant cohomology with respect to T action. They are presented in the table below.

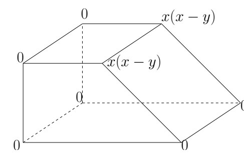
class	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
A_1	1	1	1	1	1	1	1	1
A_2	0	y	0	0	$y-x$	y	0	$y-x$
A_3	0	0	x	0	x	0	x	x
A_4	0	0	0	z	0	z	z	z
A_5	0	0	0	0	$x(x-y)$	0	0	$x(x-y)$
A_6	0	0	0	0	0	yz	0	$(y-x)z$
A_7	0	0	0	0	0	0	xz	xz
A_8	0	0	0	0	0	0	0	$xz(y-x)$

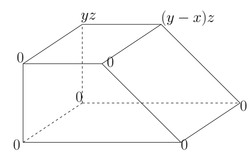


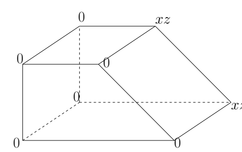


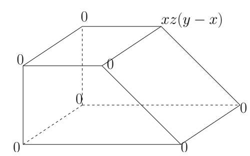












We want to compute equivariant cohomology with respect to the action of $S^1 \hookrightarrow T^3$ *given by* $u \to (u, 2u, u)$*. More precisely, our action is:*

$$e^{iu} \cdot ([x_1 : x_2], [y_0 : y_1 : y_2], [z_0 : z_1]) = ([e^{iu}x_1 : x_2], [e^{i2u}y_0 : e^{iu}y_1 : y_2], [e^{iu}z_0 : z_1]).$$

Note that we still have the same eight fixed points, namely:

$v_1 = ([0:1],[0:0:1],[0:1]),$

$v_2 = ([0:1],[1:0:0],[0:1]),$

$v_3 = ([1:0],[0:1:0],[0:1]),$

$v_4 = ([0:1],[0:0:1],[1:0]),$

$v_5 = ([1:0],[1:0:0],[0:1]),$

$v_6 = ([0:1],[1:0:0],[1:0]),$

$v_7 = ([1:0],[0:1:0],[1:0]),$ *and*

$v_8 = ([1:0],[1:0:0],[1:0]).$

The isotropy weights of this circle actions are:

fixed point	*weights*	*index*
v_1	$u, 2u, u$	*0*
v_2	$u, -2u, u$	*2*
v_3	$-u, u, u$	*2*
v_4	$u, 2u, -u$	*2*
v_5	$-u, -u, u$	*4*
v_6	$u, -2u, -u$	*4*
v_7	$-u, u, -u$	*4*
v_8	$-u, -u, -u$	*6*

We compute generating classes for the S^1 action from the classes for the T action using the projection map $x \mapsto u$, $y \mapsto 2u$, $z \mapsto u$. They are presented in the table below, together with a row with $\frac{2u^3}{e(v_i)}$ that is useful for further computations.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
$\frac{2u^3}{e(v_i)}$	*1*	*-1*	*-2*	*-1*	*2*	*1*	*2*	*-2*
A_1	1	1	1	1	1	1	1	1
A_2	0	$2u$	0	0	u	$2u$	0	u
A_3	0	0	u	0	u	0	u	u
A_4	0	0	0	u	0	u	u	u
A_5	0	0	0	0	u^2	0	0	u^2
A_6	0	0	0	0	0	$2u^2$	0	u^2
A_7	0	0	0	0	0	0	u^2	u^2
A_8	0	0	0	0	0	0	0	u^3

We keep denoting by f_j the restriction of f to a fixed point v_j. The condition that

$$\sum_{j=1}^{8} \frac{f_j\, A_1}{e(V_j)} = \int_M f\, A_1 \in \mathbb{Q}[u],$$

implies that:

$$\frac{f_1}{u^3} + \frac{-f_2}{u^3} + \frac{-2f_3}{u^3} + \frac{-f_4}{u^3} + \frac{2f_5}{u^3} + \frac{f_6}{u^3} + \frac{2f_7}{u^3} + \frac{-2f_8}{u^3} \quad \in \quad \mathbb{Q}[u].$$

Thus

$$f_1 - f_2 - 2f_3 - f_4 + 2f_5 + f_6 + 2f_7 - 2f_8 \quad \in \quad (u^3) \quad \mathbb{Q}[u],$$

Similarly, using class the A_2 we get

$$-f_2 + f_5 + f_6 - f_8 \quad \in \quad (u^2) \quad \mathbb{Q}[u],$$

Other classes give:

$$\begin{aligned} -f_3 + f_5 + f_7 - f_8 \quad &\in \quad (u^2) \quad \mathbb{Q}[u], \\ -f_4 + f_6 + 2f_7 - 2f_8 \quad &\in \quad (u^2) \quad \mathbb{Q}[u], \\ f_5 - f_8 \quad &\in \quad (u) \quad \mathbb{Q}[u], \\ 2f_6 - 2f_8 \quad &\in \quad (u) \quad \mathbb{Q}[u], \textit{ and} \\ f_7 - f_8 \quad &\in \quad (u) \quad \mathbb{Q}[u]. \end{aligned}$$

Therefore $f = (f_1, \dots, f_d)$ *represents equivariant cohomology class if and only if it satisfies:*

- *the degree* 0 *relations:*

  $(f_i - f_j) \in (u)\mathbb{Q}[u]$, *for every* i *and* j,

- *the degree* 1 *relations:*

$$\begin{aligned} -f_3 + f_5 + f_7 - f_8 &\in (u^2) \quad \mathbb{Q}[u] \\ -f_2 + f_5 + f_6 - f_8 &\in (u^2) \quad \mathbb{Q}[u] \\ -f_4 + f_6 + 2f_7 - 2f_8 &\in (u^2) \quad \mathbb{Q}[u] \\ f_1 - f_2 - 2f_3 + 2f_5 &\in (u^2) \quad \mathbb{Q}[u] \end{aligned}$$

- *the degree* 2 *relation:*

  $f_1 - f_2 - 2f_3 - f_4 + 2f_5 + f_6 + 2f_7 - 2f_8 \in (u^3)\mathbb{Q}[u]$.

Example 2.4.2. *In the case of the specialization for a* T^n *action on* M^{2n} *(i.e. a symplectic toric manifold) to the action of some generic* S^1 *(i.e. with* $M^{S^1} = M^T$*), we can proceed using this simple algorithm.*

The isotropy weights of T^n *action are easy to read from moment polytope - they are just primitive integer vectors in the directions of the edges. To get the isotropy weights for our chosen* S^1*-action, we just need to use the appropriate projection* $\pi : \mathfrak{t}^* \to (\mathfrak{s}^1)^*$. *To compute the basis of generating classes we use the method from Section 3 with* ξ *a generator of our* S^1 *to get a* T*-basis, and then we project with* π. *If the fixed points are* $p_1, \dots, p_d$*, we denote by* $a_1, \dots, a_d$ *the generating classes assigned to them and by* $G_1, \dots, G_d$ *the faces of moment polytope that are the flow up faces of the corresponding fixed point. Recall that for any* $v \in (\mathbb{Q}^n)^* \subset (\mathbb{R}^n)^*$ *we denote by* $prim(v) \in (\mathbb{Z}^n)^*$ *the primitive integral vector in direction of* v. *Using this notation, and the construction from Section 2.3, Theorem 2.1.7 states that* $f = (f_1, \dots, f_d) \in \oplus_{j=1}^d \mathbb{Q}[x]$ *is an equivariant cohomology class of* M *if and only* *if for any fixed point p_l we have*

$$\sum_{j=1}^{d} \frac{f_j\, a_l(p_j)}{e(p_j)} = \sum_{\{j\,|\,p_j \in G_l\}} \frac{f_j\, \prod_r \pi(prim(r-p_j))}{e(p_j)} \in \mathbb{Q}[x],$$

where the product is taken over all vertices r not in G_l such that r and p_j are connected by an edge. The equivariant Euler class $e(p_j)$ is a product of all weights at p_j because the representation of S^1 on $T_{p_j}M$ splits as a direct sum of 1-dimensional representations (see for example Lemma 2.2 in [37]). Therefore, up to a multiplication by a rational constant, $e(p_j)$ is equal to

$$\prod_r \pi(prim(r-p_j)),$$

where the product is taken over all vertices r connected to p_j. Thus the above condition is equivalent to

$$\sum_{\{j\,|\,p_j \in G_l\}} \frac{f_j}{\prod_r \pi(prim(r-p_j))} \in \mathbb{Q}[x],$$

where product is taken over all fixed points $r \in G_l$ that are connected with p_j by an edge in G_l.

Consider, for example, vertex v_3 in the Example 2.4.1 above. The face G_3 is the face spanned by v_3, v_5, v_7, v_8. The isotropy weights at v_3 corresponding to edges that are in G_3 are u, u, for v_5: $u, -u$, for v_7: $-u, u$ and for v_8: $-u, -u$. Therefore relation we get is:

$$\frac{f_3}{u^2} + \frac{f_5}{-u^2} + \frac{f_7}{-u^2} + \frac{f_8}{u^2} \in \mathbb{Q}[u].$$

After clearing denominators, we obtain relation $f_3 - f_5 - f_7 + f_8 \in (u^2)\mathbb{Q}[u]$.

CHAPTER 3

LOWER BOUNDS FOR GROMOV WIDTH OF COADJOINT ORBITS

3.1 Introduction

In 1985 Mikhail Gromov proved the nonsqueezing theorem which is one of the foundational results in the modern theory of symplectic invariants. The theorem says that a ball $B^{2N}(r)$ of radius r, in a symplectic vector space $\mathbb{R}^{2N}$ with the usual symplectic structure, cannot be symplectically embedded into $B^2(R)\times\mathbb{R}^{2N-2}$ unless $r \leq R$.

This motivated the definition of the invariant called the Gromov width. Consider the ball of capacity a

$$B^{2N}_a = \Big\{ z \in \mathbb{C}^N \;\Big|\; \pi \sum_{i=1}^{N} |z_i|^2 < a \Big\},$$

with the standard symplectic form $\omega_{std} = \sum dx_j \wedge dy_j$. The **Gromov width** of a $2N$-dimensional symplectic manifold (M, ω) is the supremum of the set of a's such that B^{2N}_a can be symplectically embedded in (M, ω).

In the rest of the thesis we focus on the Gromov width of coadjoint orbits of Lie groups. A Lie group G acts on itself by conjugation

$$G \ni g : G \to G, \quad g(h) = ghg^{-1}.$$

Derivative at the identity element gives the action of G on its Lie algebra $\mathfrak{g}$, called adjoint action. This induces the action of G on $\mathfrak{g}^*$, the dual of its Lie algebra, called the coadjoint action. Each orbit $\mathcal{O}$ of the coadjoint action is naturally equipped

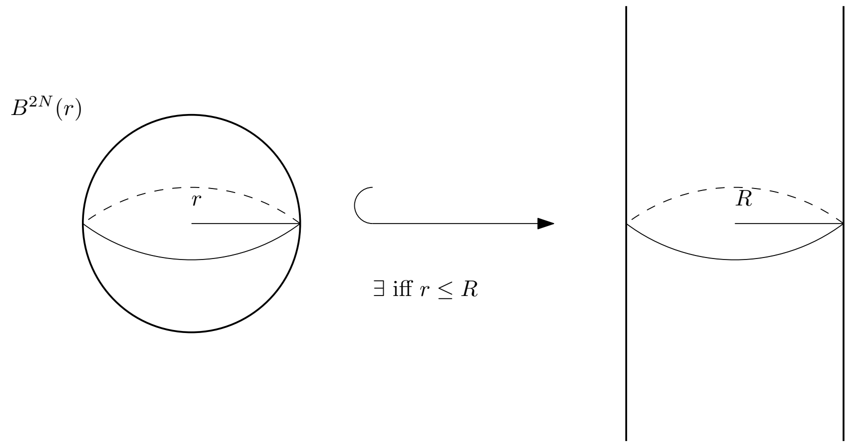


Figure 3.1: Gromov's non-squeezing theorem.

with the Kostant-Kirillov symplectic form:

$$\omega_\xi(X,Y) = \langle \xi, [X,Y] \rangle, \quad \xi \in \mathfrak{g}^*,\ X, Y \in \mathfrak{g}.$$

For example, when $G = U(n)$ the group of (complex) unitary matrices, a coadjoint orbit can be identified with the set of Hermitian matrices with a fixed set of eigenvalues. With this identification, the coadjoint action of G on an orbit $\mathcal{O}$ is simply action by conjugation. It is Hamiltonian, and the momentum map is just inclusion $\mathcal{O} \hookrightarrow \mathfrak{g}^*$.

Choose a maximal torus $T \subset G$ and a positive Weyl chamber $\mathfrak{t}^*_+$. Every coadjoint orbit intersects the positive Weyl chamber in a single point. Therefore there is a bijection between the coadjoint orbits and points in the positive Weyl chamber. Points in the interior of the positive Weyl chamber are called **regular** points.

We prove the following theorem.

Theorem 3.1.1. *Let $M := \mathcal{O}_\lambda$ be the coadjoint orbit of G, $G = U(n)$ or $SO(n)$, through a regular point $\lambda \in \mathfrak{t}^*_+$. The Gromov width of M is at least the minimum*

$$\min\{\, |\langle \alpha^\vee, \lambda \rangle| \ ; \alpha^\vee \ \textit{a coroot}\}.$$

In the case of $U(n)$ and $SO(2n+1)$, this theorem can be strengthened to cover a class of orbits that are not regular (then one needs to take the minimum only over the positive numbers in the above set; see Theorem 4.0.2 and [33, Theorem 7.1]).

This particular lower bound is important because in many known cases it describes the Gromov width, not only its lower bound. Karshon and Tolman in [20] showed that the Gromov width of complex Grassmannians is given by the above formula. Zoghi in [40] analyzed orbits satisfying some additional integrality conditions. He called an orbit $\mathcal{O}_\lambda$ **indecomposable** if there exists a simple root α such that for each root α' there exists a positive integer k (depending on α') such that

$$k\,\langle \alpha^\vee, \lambda\rangle = \langle (\alpha')^\vee, \lambda\rangle.$$

In particular spherically monotone regular orbits are indecomposable. A symplectic manifold (M, ω) is called spherically monotone if there exists $k > 0$ such that for any class X in the image of Hurewicz homomorphism $\pi_2(M) \to H_2(M)$ have that the first Chern class $c_1(TM)[X] = k\,\omega(X)$. For example, the coadjoint orbit of $U(n)$ through $\mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{t}^*_{U(n)}$ is spherically monotone if $\lambda_1 - \lambda_2 = \ldots = \lambda_{n-1} - \lambda_n$. It is indecomposable if there is k such that for any i, j there is an integer m_{ij} such that $\lambda_i - \lambda_j = m_{ij}(\lambda_k - \lambda_{k+1})$. Zoghi proved that for compact connected simple Lie group G the formula $\min\{\, |\langle \alpha^\vee, \lambda\rangle|\,; \alpha^\vee \text{ a coroot}\}$ gives an upper bound for Gromov width of regular indecomposable G-coadjoint orbit through λ ([40, Proposition 3.16]). Combining this result with the results about the lower bound for the $U(n)$ case which he proved in [40] (I just reproved his result), and about lower bounds for the $SO(n)$ case proved here in Chapter 5, we obtain the formula for Gromov width of regular, indecomposable coadjoint orbits of $U(n)$ and $SO(n)$. Table 3.1 summarizes the results about the Gromov width of coadjoint orbits known at the moment.

Table 3.1: Results about the Gromov width of coadjoint orbits.

	lower bound	upper bound	Gromov width
$U(n)$, regular	$\surd$ ([32],[40])		
$U(n)$, regular, indecomposable	$\surd$	$\surd$	$\surd$ ([40])
Grassmannians ($U(n)$, non-regular)	$\surd$	$\surd$	$\surd$ ([20], [27])
a class of non-regular, $U(n)$	$\surd$ ([32])		
any cpt, ctd G, regular indecomposable		$\surd$ ([40])	
$SO(n)$, regular	$\surd$ ([33])		
$SO(n)$, regular indecomposable	$\surd$	$\surd$	$\surd$ ([33]+[40])

To prove Theorem 3.1.1 we recall an action of the Gelfand-Tsetlin torus on an open dense subset of the coadjoint orbit. We then use the theorem of Karshon and Tolman [20] (Proposition 3.2.6) to obtain symplectic embeddings of balls. Coadjoint orbits come equipped with the Hamiltonian action of the maximal torus of the group. One can apply the Karshon and Tolman's result to the region centered with respect to this standard action and obtain a lower bound for Gromov width of the orbit. This is how Zoghi proved in [40] the lower bounds of Gromov width of regular $U(n)$ coadjoint orbits. If the root system is non-simply laced, the lower bound obtained this way is weaker (i.e. lower) then the lower bound we prove here. This phenomenon is explained in the Appendix A. In other words, the lower bounds for $SO(2n+1)$ we prove here could not be obtained using the standard action of maximal torus.

3.2 Centered actions

Centered actions were introduced in [21]. For completeness and to set notation we include the details here following [20]. Let (M, ω) be a connected symplectic manifold, equipped with a symplectic action of a torus $T \cong (S^1)^{\dim T}$. The action of T is called **Hamiltonian** if there exists a T-invariant map $\Phi \colon M \to \mathfrak{t}^*$, called the **momentum map**, such that

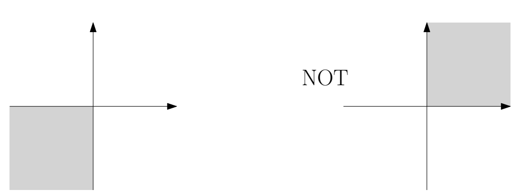
$$\iota(\xi_M)\omega = d\langle \Phi, \xi \rangle \quad \forall\, \xi \in \mathfrak{t}, \tag{3.1}$$

where ξ_M is the vector field on M generated by $\xi \in \mathfrak{t}$. We will identify Lie(S^1) with $\mathbb{R}$ using the convention that the exponential map $exp : \mathbb{R} \cong \text{Lie}(S^1) \to S^1$ is given by $t \to e^{2\pi i t}$, that is $S^1 \cong \mathbb{R}/\mathbb{Z}$.

At a fixed point $p \in M^T$, we may consider the induced action of T on the tangent space T_pM. There exist $\eta_j \in \mathfrak{t}^*$, called the **isotropy weights** at p, such that this action is isomorphic to the action on $(\mathbb{C}^n, \omega_{std})$ generated by the momentum map

$$\Phi_{\mathbb{C}^n}(z) = \Phi(p) + \pi \sum |z_j|^2(-\eta_j).$$

The isotropy weights are uniquely determined up to permutation. Note that with our sign convention in equation 3.1 the isotropy weights are pointing out of the momentum map image. For example, standard S^1 action on $\mathbb{C}^2$ by rotation with speed one gives the following momentum map image:



By the equivariant Darboux theorem, a neighborhood of p in M is equivariantly symplectomorphic to a neighborhood of 0 in $\mathbb{C}^n$. However, this theorem does not tell us how large we may take this neighborhood to be. Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex set which contains $\Phi(M)$. The quadruple $(M, \omega, \Phi, \mathcal{T})$ is a **proper Hamiltonian T-manifold** if the action is effective and Φ is proper as a map to $\mathcal{T}$, that is, the preimage of every compact subset of $\mathcal{T}$ is compact.

For any subgroup K of T, let $M^K = \{m \in M \mid a \cdot m = m \ \forall a \in K\}$ denote its fixed point set.

Definition 3.2.1. *A proper Hamiltonian T-manifold $(M, \omega, \Phi, \mathcal{T})$ is **centered** about a point $\alpha \in \mathcal{T}$ if α is contained in the momentum map image of every component of M^K, for every subgroup $K \subseteq T$.*

We now quote several examples and non-examples, following [20].

Example 3.2.2. *A compact symplectic manifold with a non-trivial T-action is never centered, because it has fixed points with different momentum map images.*

Example 3.2.3. *Let a torus T act linearly on $\mathbb{C}^n$ with a proper momentum map $\Phi_{\mathbb{C}^n}$ such that $\Phi_{\mathbb{C}^n}(0) = 0$. Let $\mathcal{T} \subset \mathfrak{t}^*$ be an open convex subset containing the origin. Then $\Phi_{\mathbb{C}^n}^{-1}(\mathcal{T})$ is centered about the origin.*

A Hamiltonian T action on M is called **toric** if $\dim T = \frac{1}{2} \dim M$.

Example 3.2.4. *Let M be a compact symplectic toric manifold with momentum map $\Phi\colon M \to \mathfrak{t}^*$. Then $\Delta := Im\ \Phi$ is a convex polytope. The orbit type strata in M are the momentum map pre-images of the relative interiors of the faces of Δ. Hence, for any $\alpha \in \Delta$,*

$$\bigcup_{\substack{F \text{ face of } \Delta \\ \alpha \in F}} \Phi^{-1}(\textit{rel-int } F)$$

is the largest subset of M that is centered about α.

When the dimension of the torus acting on a compact symplectic manifold is less then half of the dimension of the manifold, one can easily find a centered region from an x-ray of the Hamiltonian T-space M. The **x-ray** of (M, ω, ϕ) is the collection of convex polytopes $\phi(X)$ over all connected components X of M^K for some subtorus K of T (for more details see [35]). For the toric symplectic manifold, an x-ray is exactly the collection of faces of convex polytope that is the image of momentum map. Figure 3.2 presents some examples of centered regions, that we can see directly from the x-rays of M.

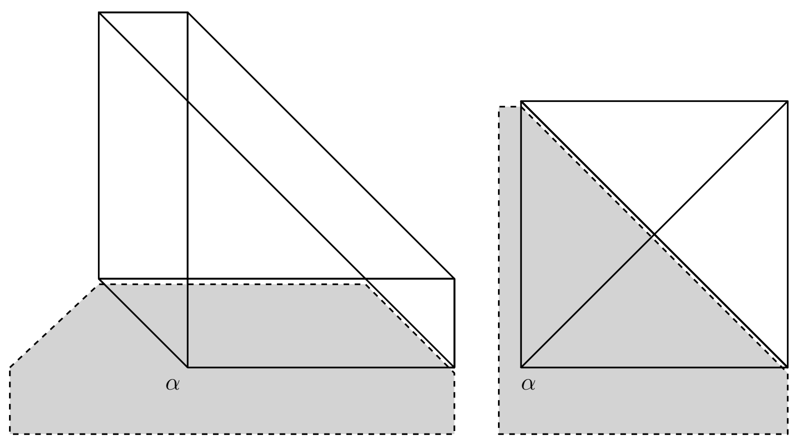


Figure 3.2: The regions centered around α.

Example 3.2.5. *Let $(M, \omega, \Phi, \mathcal{T})$ be a proper Hamiltonian T-manifold. Then every point in $\mathfrak{t}^*$ has a neighborhood whose preimage is centered. This is a consequence of the local normal form theorem and the properness of the momentum map.*

Proposition 3.2.6. *(Karshon, Tolman, [20]) Let $(M, \omega, \Phi, \mathcal{T})$ be a proper Hamiltonian T-manifold. Assume that M is centered about $\alpha \in \mathcal{T}$ and that $\Phi^{-1}(\{\alpha\})$*

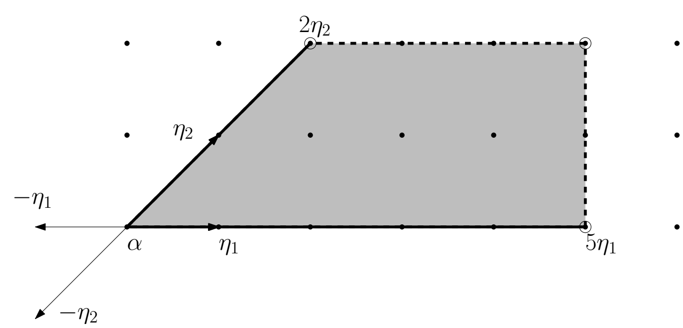
consists of a single fixed point p. Let $-\eta_1, \ldots, -\eta_n$ be the isotropy weights of T action on T_pM. Then M is equivariantly symplectomorphic to

$$\left\{ z \in \mathbb{C}^n \mid \alpha + \pi \sum |z_j|^2 \eta_j \in \mathcal{T} \right\},$$

where T acts on $\mathbb{C}^n$ with weights $-\eta_1, \ldots, -\eta_n$.

Note that the above formulation differs from the one in [20] by a minus sign. This is due to the fact that our definition of momentum map 3.1 also differs by a minus sign from the definition used in [20]. Recall that the definition of a proper Hamiltonian T-manifold includes the assumption that the action is effective.

Example 3.2.7. *Consider a compact symplectic toric manifold M whose momentum map image is the closure of the following region.*



The isotropy weights of the torus action are $(-\eta_1)$ and $(-\eta_2)$, and the lattice lengths of edges starting from α are 5 and 2 (with respect to lattice of isotropy weights). The largest subset of M that is centered about α, as described in Example 3.2.4, maps under the momentum map to the shaded region. The above Proposition tells us that it is equivariantly symplectomorphic to

$$\{z \in \mathbb{C}^2 | \alpha + \pi(|z_1|^2\eta_1 + |z_2|^2\eta_2) \in \text{ shaded region } \}.$$

If $z \in B^4_2 = \{z \in \mathbb{C}^2 \left| \pi(|z_1|^2 + |z_2|^2) < 2\right\}$ then $\alpha + \pi(|z_1|^2\eta_1 + |z_2|^2\eta_2)$ is in the shaded region. Therefore the 4-dimensional ball B^2_2 of capacity 2 embeds into M

and the Gromov width of M is at least the minimum of lattice lengths of edges of the moment polytope, starting at α. Note also that the momentum map image of the embedded ball B_2^2 is the triangle with vertices α, $\alpha + 2\eta_1$ and $\alpha + 2\eta_2$.

3.3 Gelfand-Tsetlin system of action coordinates

In this Subsection we describe the Gelfand-Tsetlin (sometimes spelled Gelfand-Cetlin, or Gelfand-Zetlin) system of action coordinates, which originally appeared in [13]. It is related to the classical Gelfand-Tsetlin polytope introduced in [7]. Let G be a compact, connected Lie group and $\mathcal{O}_\lambda$ its coadjoint orbit. Consider a sequence of subgroups $G = G_k \supset G_{k-1} \supset \ldots \supset G_1$. Inclusion of G_j into G gives an action of G_j on $\mathcal{O}_\lambda$. This action is Hamiltonian with momentum map Φ^j, where Φ^j is the composition of the G-momentum map Φ and a projection $p_j : \mathfrak{g}^* \to \mathfrak{g}_j^*$. Choose maximal tori, T_{G_j}, and positive Weyl chambers for each group G_j in the sequence. Every G_j orbit intersects the positive Weyl chamber $(\mathfrak{t}_{G_j})^*_+$ exactly once. This defines a continuous (but not everywhere smooth) map $s_j : \mathfrak{g}_j^* \to (\mathfrak{t}_{G_j})^*_+$. Let $\Lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_{rk\,G_j}^{(j)})$ denote the composition $s_j \circ \Phi^j$:

$$\begin{array}{ccc} \mathcal{O}_\lambda & \xrightarrow{\Phi^j} & \mathfrak{g}_j^* \\ & \searrow^{\Lambda^{(j)}} & \downarrow{\scriptstyle s_j} \\ & & (\mathfrak{t}_{G_j})^*_+ \end{array}$$

The functions $\{\Lambda^{(j)}\}$, $j = 1, \ldots, k-1$, form the **Gelfand-Tsetlin system** which we denote by $\Lambda : \mathcal{O}_\lambda \to \mathbb{R}^N$.

3.4 Smoothness of the Gelfand-Tsetlin functions

The function $\lambda_k^{(j)}$ need not be smooth on the whole orbit $\mathcal{O}_\lambda$. To identify this subset we will need the following result proved in [4]. This theorem is also true for orbifolds: see [25, Theorem 3.1].

Theorem 3.4.1. *Let G be a compact connected Lie group with a maximal torus T. Suppose G acts on a compact connected symplectic manifold M in a Hamiltonian way, with moment map $\Phi : M \to \mathfrak{g}^*$. Then there exists a unique open wall σ^o of the Weyl chamber $\mathfrak{t}_+^*$ with the properties that $\Phi(M) \cap \mathfrak{t}_+^* \subset \overline{\sigma^o}$ and $\Phi(M) \cap \mathfrak{t}_+^* \cap \sigma^o \neq \emptyset$.*

Let $\sigma^o = \sigma_j^o$ be the unique open wall from the above theorem applied to the $G_j \subset G$ action on $M = \mathcal{O}_\lambda$. We call $\sigma = \overline{\sigma^o}$ the **principal face.** Any wall of positive Weyl chamber $(\mathfrak{t}^j)_+^*$ that contains σ is called a **special wall**, while all the others walls are called **regular walls**. Thus σ is the intersection of all special walls, and $\sigma^o = \sigma \setminus (\cup \text{ regular walls})$. Intersection of $\Lambda^{(j)}(\mathcal{O}_\lambda)$ with a wall of $(\mathfrak{t}^j)_+^*$ is defined by a collection of equations of the form $\lambda_l^{(j)} = \lambda_{l+1}^{(j)}$. If a wall τ is special, i.e. $\sigma \subset \tau$, then its defining equations hold on the whole $\Lambda(\mathcal{O}_\lambda)$. For any regular wall τ, there is at least one of its defining equations, and some $A \in \mathcal{O}_\lambda$ such that $\Lambda(A)$ does not satisfy this equation.

Proposition 3.4.2. *The function $\Lambda^{(j)}$ is smooth on the set $U^{(j)} = (\Lambda^{(j)})^{-1}(\sigma^o)$.*

Proof. To simplify the notation, in this proof we write G for G_j and T for its maximal torus, and π for s_j. Recall that the function $\Lambda^{(j)}$ is a composition of a smooth function Φ^j and a map $\pi = s_j : \mathfrak{g}^* \to \mathfrak{t}_+^*$. Therefore we only need to prove smoothness of the map π on $\Phi^j(U^{(j)}) = \pi^{-1}(\sigma^o)$. Note that all points in σ^o have the same G-stabilizer (under the coadjoint action of G). Denote it by H.

Let S be the subset of $\mathfrak{g}^*$ equal to $\pi^{-1}(\sigma^o)$. This means that $S = (\mathfrak{g}^*)_{(H)}$ is an orbit-type stratum and therefore it is a submanifold of $\mathfrak{g}^*$. Consider the smooth, G-equivariant, surjective map:

$$\begin{aligned} G \times \sigma^o &\rightarrow S \\ (g, x) &\rightarrow g \cdot x \end{aligned}$$

This map induces G-equivariant bijective map

$$\begin{aligned} \Theta : G/H \times \sigma^o &\rightarrow S, \\ ([g], x) &\rightarrow g \cdot x \end{aligned}$$

which is also a diffeomorphism. Notice that the composition, $\pi \circ \Theta$

$$\begin{aligned} G/H \times \sigma^o &\rightarrow \mathfrak{t}^*_+ \\ ([g], x) &\rightarrow x \end{aligned}$$

is just the projection onto second factor, therefore it is smooth. This means that on S, π is smooth, as a composition of Θ^{-1} and a smooth projection. It follows that the function $\Lambda^{(j)}$ is smooth on the set $(\Phi^j)^{-1}(S) = (\Lambda^{(j)})^{-1}(\sigma^o) = U^{(j)}$. $\square$

3.5 The torus action induced by the Gelfand-Tsetlin system

At the points where $\Lambda^{(j)}$ is smooth, it induces a smooth action of $T'_{G_j} \hookrightarrow T_{G_j}$, a subtorus of T_{G_j}. The process of obtaining this new action, which we denote by $*$, is often referred to as the **Thimm trick**. If λ is regular then $T'_{G_j} = T_{G_j}$. An element $t \in T_{G_j}$ acts on a point $A \in \mathcal{O}_\lambda$ by the standard, coadjoint G_j action of $B^{-1}\, t\, B$, where $B \in G_j$ is such that $Ad^*(B)\, \Phi^j(A) \in (\mathfrak{t}_{G_j})^*_+$ is the unique point of intersection of $(\mathfrak{t}_{G_j})^*_+$ and the G_j-coadjoint orbit through $\Phi^j(A)$. That is

$$t * A = Ad^* \left(\left[\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right] \right) (A).$$

In this thesis we consider only matrix groups, and for them the coadjoint action is the action by conjugation. Therefore we will simplify the notation and write conjugation in place of the coadjoint action:

$$t * A = \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right) A \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I \end{array} \right)^{-1}. \tag{3.2}$$

Recall that for regular λ, a matrix $A \in U^{(j)}$ if $B\Phi^j(A)\, B^{-1} \in \mathit{int}\, (\mathfrak{t}_{G_j})^*_+$, so the stabilizer of $B\Phi^j(A)\, B^{-1}$ in G_j is precisely $T_{G_j} = T'_{G_j}$. The fact that T'_{G_j} commutes with the stabilizer of $B\Phi^j(A)\, B^{-1}$ implies that the action is well defined, as explained below.

If λ is not regular then some of the functions $\lambda^{(j)}_*$ may be constant on the whole orbit. Let $T'_{G_j} \hookrightarrow T_{G_j}$ be the subtorus defined by

$$\{(t_1, \ldots, t_{rank\, G_j}) \in T_{G_j};\ t_i = 1 \text{ if } \lambda^{(j)}_i \text{ constant on the whole orbit } \}.$$

(This definition gives $T'_{G_j} = T_{G_j}$ if none of the functions $\lambda^{(j)}_*$ is constant on the whole orbit). Let σ_j be the unique wall of the positive Weyl chamber $(\mathfrak{t}_{G_j})^*_+$ from Theorem 3.4.1. All points in σ^o_j have the same stabilizer. Note that the torus T'_{G_j} commutes with the stabilizer in G_j of points in σ^o_j. Here we analyze only $U(n)$ and $SO(n)$. In the unitary case, the stabilizer in $U(j)$ of points in σ^o_j is a product of circles and of groups $U(m)$ (various $m \le j$ whose sum is at most j), one for each longest sequence $\lambda^{(j)}_i = \lambda^{(j)}_{i+1} = \ldots = \lambda^{(j)}_{i+m-1} \equiv \lambda_i$ of the functions $\lambda^{(j)}_*$ that are constant on the whole orbit. Elements of the torus T'_{G_j} are diagonal matrices with diagonal entries equal to 1 in blocks corresponding to the $U(m)$ factors of the stabilizer. Similarly for the $SO(n)$ case. For example, if $\lambda_1^{(j)} = \lambda_2^{(j)} = \ldots = \lambda_m^{(j)} \equiv \lambda_1$, then the stabilizer in $U(j)$ of points in σ_j^o is $U(m) \times S^1 \times \ldots \times S^1$, while elements of T'_{G_j} are of the form $(1, \ldots, 1, t_{m+1}, \ldots, t_n)$ and thus commute with the stabilizer. The action of $t \in T'_{G_j}$ on $A \in \mathcal{O}_\lambda$ is given by equation (3.2), where $B \in G_j$ is such that $B\Phi^j(A)\, B^{-1} \in \sigma_j \subset (\mathfrak{t}_{G_j})^*_+$. If C is another element of G_j such that $C\,\Phi^j(A)C^{-1} \in (\mathfrak{t}_{G_j})^*_+$, then

$$B\,\Phi^j(A)B^{-1} = C\,\Phi^j(A)C^{-1} = CB^{-1}\,B\,\Phi^j(A)B^{-1}BC^{-1},$$

so $CB^{-1} \in Stab_{G_j}(B\Phi^j(A)B^{-1})$. Therefore for $t \in T'_{G_j}$ have

$$C^{-1}tC = C^{-1}tC\,B^{-1}\,t^{-1}\,t\,B = C^{-1}\,t\,t^{-1}\,C\,B^{-1}\,t\,B = B^{-1}\,t\,B,$$

what implies that the action is well defined.

Proposition 3.5.1. *The new T'_{G_j} action defined above is Hamiltonian on the subset $U^{(j)} = (\Lambda^{(j)})^{-1}(\sigma_j^o)$, with momentum map $\Lambda^{(j)}$. (For non-regular orbits the momentum map consists only of non-constant coordinates of $\Lambda^{(j)}$).*

Proof. To simplify the notation, we will denote $U^{(j)}$ simply by U, T'_{G_j} by T^j, and let $\mathfrak{t}^j$ be the Lie algebra of T^j. Take any $X \in \mathfrak{t}^j$ and denote by X_{new} the vector field on U generated by X with $*$ action, and by X_{std} the vector field on U generated by X using the standard action by conjugation. As usual, for any function $\varphi : \mathcal{O}_\lambda \to \mathfrak{g}_j^*$, and any $X \in \mathfrak{g}_j$, we denote by φ^X a function from $\mathcal{O}_\lambda$ to $\mathbb{R}$ defined by $\varphi^X(p) = \langle \varphi(p), X \rangle$, where $\langle , \rangle$ is the standard G_j invariant pairing between $\mathfrak{g}_j^*$ and $\mathfrak{g}_j$. Take any $A \in U$. We want to prove that for any vector $Y \in T_A\mathcal{O}_\lambda = T_AU$

$$\omega(X_{new}, Y)|_A = d\,(\Lambda^{(j)})^X\,(Y)|_A. \tag{3.3}$$

Denote by N the connected symplectic submanifold $N := (\Phi^j)^{-1}(\sigma^o) \subset \mathcal{O}_\lambda$, where σ is the principal face. We refer to N as the **principal cross-section**. Note that

$U = (\Lambda^{(j)})^{-1}(\sigma^o) = G_j \cdot N$, and so every $A \in U$ can be G_j conjugated to an element of N. We first prove equation (3.3) for $A \in N$.

The proof of theorem 3.8 in [25] implies that

$$T_A \mathcal{O}_\lambda = T_A N \, + \, T_A(G_j \cdot A).$$

This is not a direct sum. Thus to prove the equation (3.3) for $A \in N$, it is enough to consider two cases: when vector Y is tangent to the principal cross-section, and when it is tangent to G_j orbit (for the standard action).

Before we start considering the cases, we fix some notation. For any vector field V on $\mathcal{O}_\lambda$, denote by Ψ^V its flow. Recall that $\Psi^V_{-t} = (\Psi^V_t)^{-1}$. Therefore, for example $\Psi^{X_{std}}_t(Q) = X_t Q X_t^{-1}$ and $\Psi^{X_{std}}_{-t}(Q) = X_t^{-1} Q X_t$.

Case 1: Take $Y \in T_A N \subset T_A \mathcal{O}_\lambda$. We want to compute $\omega(X_{new}, Y)|_A = \langle A, [X_{new}, Y] \rangle$. Notice that on the principal cross section functions Φ^j and $\Lambda^{(j)}$ are equal, and the standard and the new actions of T^j coincide. Therefore the vector fields X_{std} and X_{new} have equal values and flows on N. Using the formula

$$[X_{new}, Y] = \lim_{t \to 0} \frac{(\Psi^{X_{new}}_{-t})_*(Y) - Y}{t} = [X_{std}, Y].$$

we have that, if $Y \in T_A N$, then $\langle A, [X_{new}, Y] \rangle = \langle A, [X_{std}, Y] \rangle$. The fact that functions Φ^j and $\Lambda^{(j)}$ agree on all of the N, means also that for $Y \in T_A N$ we have

$$d(\Phi^j)^X(Y) = d(\Lambda^{(j)})^X(Y).$$

Therefore

$$\begin{aligned}
\omega(X_{new}, Y)|_A &= \langle A, [X_{new}, Y] \rangle = \langle A, [X_{std}, Y] \rangle \\
&= \omega(X_{std}, Y)|_A = d(\Phi^j)^X(Y)|_A \\
&= d(\Lambda^{(j)})^X(Y)|_A.
\end{aligned}$$

Case 2: Take $Y \in T_A(G_j \cdot A)$. That is $Y = Y_{std}$ for some $Y = \frac{d}{dt} Y_t|_{t=0} \in \mathfrak{g}_j$ and the integral curve of Y through A is $\Psi_t^Y(A) = Y_t\, A\, Y_t^{-1}$. As before, we start by analyzing $[X_{new}, Y]$ at A. We have:

$$[X_{new}, Y]|_A = \lim_{t \to 0} \frac{(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} - Y|_A}{t}.$$

The point A is in N, so $\Psi_t^{X_{new}}(A) = X_t \cdot A = X_t\, A\, X_t^{-1}$. Now we need to understand the expression:

$$(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} = \frac{d}{dv}\, \Psi_{-t}^{X_{new}}(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1})\, |_{v=0}.$$

To compute the value of $\Psi_{-t}^{X_{new}}$ on $Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}$, we need to find an element C of G_j that would conjugate $\Phi^j(\Psi_{-t}^{X_{new}})$ to some element in $(\mathfrak{t}^j)^*_+$. We have

$$\begin{aligned} \Phi^j(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}) \quad &= \Phi^j(Y_v\, X_t A\, X_t^{-1}\, Y_v^{-1}) \\ &= Y_v\, X_t\, \Phi^j(A) X_t^{-1} Y_v^{-1}. \end{aligned}$$

Therefore, for

$$C = X_t^{-1}\, Y_v^{-1}$$

we have that

$$C \Phi^j(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1})\, C^{-1} = \; \Phi^j(A) \; \in (\mathfrak{t}^j)^*_+.$$

This means that the new action of X_t at a point $Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}$ is the same as standard action of

$$C^{-1}\, X_t\, C = Y_v X_t\, X_t\, X_t^{-1}\, Y_v^{-1} = Y_v X_t\, Y_v^{-1},$$

so

$$\begin{aligned} &\Psi_{-t}^{X_{new}}(Y_v\, \Psi_t^{X_{new}}(A)\, Y_v^{-1}) \\ &= (Y_v X_t^{-1}\, Y_v^{-1})(\, Y_v\, X_t\, A\, X_t^{-1}\, Y_v^{-1})(Y_v X_t Y_v^{-1}) \\ &= Y_v\, A\, Y_v^{-1}. \end{aligned}$$

Therefore

$$[X_{new}, Y]|_A = \lim_{t \to 0} \frac{(\Psi_{-t}^{X_{new}})_*(Y)|_{\Psi_t^{X_{new}}(A)} - Y|_A}{t} = \lim_{t \to 0} \frac{Y|_A - Y|_A}{t} = 0,$$

and

$$\omega(X_{new}, Y)|_A = \langle A, [X_{new}, Y] \rangle = 0.$$

Notice that the function $\Lambda^{(j)}$ is constant on G_j orbits, because Φ^j is G_j-equivariant and the whole G_j orbit intersects $(\mathfrak{t}^j)^*_+$ in a unique point. Thus, for $Y \in T_A(G_j \cdot A)$,

$$d\,(\Lambda^{(j)})^X\,(Y) = 0.$$

and equation (3.3) for A in N follows.

Now we want to prove equation (3.3) for all $C \in U$. Let B be an element of G_j such that $BCB^{-1} = A \in \mathfrak{t}^*_+$. Take any $X \in \mathfrak{t}$ and $Y \in T_C U$. Using the G_j invariance of ω and of $\Lambda^{(j)}$, and equation (3.3) at the principal cross section, we have

$$\begin{aligned}
\omega(X_{new}, Y)_{|B^{-1}AB} &= \omega(\ \frac{d}{dt}(B^{-1}X_t B \cdot C)|_{t=0},\ \frac{d}{dt}(\Psi^Y_t(C))|_{t=0}\ ) \\
&= \omega(\ \frac{d}{dt}B(B^{-1}X_t B \cdot C)B^{-1}|_{t=0},\ \frac{d}{dt}B(\Psi^Y_t(C))B^{-1}|_{t=0}) \\
&= \omega(\ \frac{d}{dt}(X_t BB^{-1}ABB^{-1}X_t^{-1})|_{t=0},\ \frac{d}{dt}(\Psi^{BYB^{-1}}_t(A))|_{t=0}) \\
&= \omega(X_{new}, BYB^{-1})|_A = d\,(\Lambda^{(j)})^X(BYB^{-1})|_A \\
&= \frac{d}{dt}\,[\,(\Lambda^{(j)})^X(B\Psi^Y_t(C)B^{-1}))\,]|_{t=0} = \frac{d}{dt}\,[\,(\Lambda^{(j)})^X(\Psi^Y_t(C))\,]|_{t=0} \\
&= d\,(\Lambda^{(j)})^X(Y)|_C,
\end{aligned}$$

which is exactly what we needed to show. □

Putting these actions together we obtain a Hamiltonian action of the **Gelfand-Tsetlin torus**

$$T_{GT} := T'_{G_k} \oplus \ldots \oplus T'_{G_1}$$

on the open dense subset,

$$U := \bigcap_j U^{(j)}.$$

We call a wall of $(\mathfrak{t}^N)^*_+$ **special** if there is a j such that the image of this wall under projection $(\mathfrak{t}^N)^* \to (\mathfrak{t}^j)^*$ is a special wall as defined in the Section 3.4. Other walls of $(\mathfrak{t}^N)^*_+$ will be called **regular**.

CHAPTER 4

COADJOINT ORBITS OF THE UNITARY GROUP

In this section we consider coadjoint orbits of $U(n)$. Multiplying by a factor of i, we can identify the Lie algebra $\mathfrak{u}(n)$ with the space of Hermitian matrices. The pairing in $\mathfrak{u}(n)$

$$(A, B) = \mathrm{trace}(AB)$$

gives us the identification of $\mathfrak{u}^*(n)$ with $\mathfrak{u}(n)$. From now on, we will identify $\mathfrak{u}^*(n)$ with the space of Hermitian matrices.

Let $T = T^n$ be the standard maximal torus in $U(n)$ (given by diagonal matrices). We identify its Lie algebra dual, $\mathfrak{t}^*$ with diagonal Hermitian matrices and choose the positive Weyl chamber, $(\mathfrak{t}^*)_+$, to be

$$(\mathfrak{t}^*)_+ := \{\mathrm{diag}(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn});\ \lambda_{11} \geq \lambda_{22} \geq \ldots \geq \lambda_{nn}\}.$$

The coadjoint orbits in $\mathfrak{u}(n)^*$ are in one-to-one correspondence with the points of $(\mathfrak{t}^*)_+$. Precisely, for any $(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn}) \in (\mathfrak{t}^*)_+$ the corresponding coadjoint orbit is the set of all Hermitian matrices with eigenvalues $(\lambda_{11}, \lambda_{22}, \ldots, \lambda_{nn})$. We use coordinates $\{e_{ij}\}$, with e_{ij} corresponding to (i,j)-th entry of a matrix. Then $\Delta = \{e_{ii} - e_{jj} \,|\, i \neq j\}$ is a root system and $\Sigma = \{e_{ii} - e_{i+1,i+1} \,|\, i = 1, 2, \ldots, n-1\}$ is the set of positive roots. The pairing of $\lambda \in \mathfrak{t}^*$ with a coroot $(e_{ii} - e_{jj})^\vee$ gives

$$\langle (e_{ii} - e_{jj})^\vee, \lambda \rangle \ = 2\, \frac{\langle e_{ii} - e_{jj}, \lambda \rangle}{\langle e_{ii} - e_{jj}, e_{ii} - e_{jj} \rangle} = (\lambda_i - \lambda_j).$$

Therefore for λ in our chosen positive Weyl chamber

$$\min\{|\langle \alpha^\vee, \lambda \rangle|\ ; \alpha^\vee \text{ a coroot}\} = \min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n\}.$$

The above observation helps us to translate the condition from Theorem 3.1.1 into explicit condition on eigenvalues λ_j. This Section is devoted to proving Theorem

3.1.1 for $G = U(n)$. In fact we prove even stronger result, covering also some non-regular orbits:

Theorem 4.0.2. *Consider the $U(n)$ coadjoint orbit $M := \mathcal{O}_\lambda$ in $\mathfrak{u}(n)^*$ through a point diag $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ where*

$$\lambda_1 > \lambda_2 > \ldots > \lambda_l = \lambda_{l+1} = \ldots = \lambda_{l+s} > \lambda_{l+s+1} > \ldots > \lambda_n, \ s \geq 0.$$

The Gromov width of M is at least the minimum $\min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$.

Remark 4.0.3. *In fact the hypothesis can be weakened. The only necessary condition is that the Gelfand-Tsetlin polytope associated to $\mathcal{O}_\lambda$ contains at least one good vertex. These notions will be explained in Section 4.3.*

4.1 The standard action of maximal torus

Under our identifications, the coadjoint action of $U(n)$ on $\mathfrak{u}(n)^*$ is by conjugation: $A \cdot \xi = A\xi A^{-1}$. Restricted to an orbit $\mathcal{O}_\lambda$, this action is Hamiltonian with momentum map the inclusion $\mathcal{O}_\lambda \hookrightarrow \mathfrak{u}(n)^*$. The **standard T^n action** on $\mathcal{O}_\lambda$ is the action of the maximal torus $T^n \subset U(n)$. The fixed points of this action are the diagonal matrices. In particular, λ is a fixed point and the isotropy weights of T^n action on $T_\lambda \mathcal{O}_\lambda$ are given by the positive roots Σ. The T^n action is Hamiltonian with momentum map $\mu : \mathcal{O}_\lambda \to (\mathfrak{t}^n)^* \cong \mathbb{R}^n$ that maps a matrix $A = (a_{ij})$ to the diagonal $n \times n$ matrix diag $(a_{11}, \ldots, a_{nn})$. However the dimension of torus acting effectively is less then half of the dimension of the coadjoint orbit, so this action is not toric. If $\mathcal{O}_\lambda$ is regular then this action is effective but $\dim T^n = n$ while $\dim \mathcal{O}_\lambda = \frac{1}{2}n(n-1)$. Let $\mathcal{Q} = \mu(\mathcal{O}_\lambda) \subset (\mathfrak{t}^n)^*$ denote the momentum map image for the standard T^n action. The vertices of $\mathcal{Q}$ correspond to the T^n-fixed points, that is, the diagonal matrices in $\mathcal{O}_\lambda$. If λ is generic, then the vertices correspond exactly to permutations on n elements. Thus there are exactly $n!$ of them. If λ is non-generic, say

$$\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{n-l_s+1} = \ldots = \lambda_n,$$

then the vertices correspond to cosets $S_n/(S_{l_1} \times \ldots \times S_{l_s})$, and there are exactly $\frac{n!}{l_1!\ldots l_s!}$ of them.

Recall that **GKM manifold** is a manifold M equipped with a faithful action of a torus K of dimension $l > 1$ such that the set of zero dimensional orbits in the orbit space M/K is zero dimensional and the set of one dimensional orbits in M/K is one dimensional (see Example 2.1.9 or [11], [12], [36]). The coadjoint orbit $\mathcal{O}_\lambda$ with the standard T^n action is an example of GKM manifold. In particular this means that the closure of every connected component of the set $\{x \in \mathcal{O}_\lambda; \dim(T^n \cdot x) = 1\}$ is a sphere. The closure of $\{x \in \mathcal{O}_\lambda; \dim(T^n \cdot x) = 1\}$ is called **1-skeleton** of $\mathcal{O}_\lambda$. Denote by $\mathcal{Q}_1$ the image of 1-skeleton under the momentum map. The GKM assumption forces $\mathcal{Q}_1$ to be a $(\frac{1}{2}\dim \mathcal{O}_\lambda)$-valent graph with vertices $Vert(\mathcal{Q}_1) = Vert(\mathcal{Q})$ corresponding to T^n-fixed points and edges corresponding to closures of connected components of the 1-skeleton. Note that not all edges in $\mathcal{Q}_1$ are edges of the polytope $\mathcal{Q}$. Images of two fixed points, F and F', are connected by an edge in $\mathcal{Q}_1$ if and only if they differ by one transposition of two different diagonal entries. Therefore there are exactly

$$D := [\, l_1(l_2 + \ldots l_s) + l_2(l_3 + \ldots + l_s) + \ldots + l_{s-1}l_s \,] = \sum_{i<j} l_i l_j$$

edges leaving any vertex of $\mathcal{Q}_1$ and thus dim $\mathcal{O}_\lambda = D$ $\dim(S^2) = 2D$. In the case of generic λ, the moment polytope of $\mathcal{O}_\lambda$ is called a permutahedron.

Denote the diagonal entries of F by $F_{11}, \ldots, F_{nn}$. Let $p < q$ be indices from $\{1, \ldots n\}$ such that $F_{pp} \neq F_{qq}$ and F' is the matrix obtained from F by switching

p-th and q-th entry. The edge joining $\mu(F)$ and $\mu(F')$ is an μ-image of a sphere in $\mathcal{O}_\lambda$. This sphere is the orbit of $SU(2)$ action on F and is obtained in the following way. Denote $F_{pp} = v_i,\ F_{qq} = v_k$. For any $z \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ let I_z be the matrix obtained from the identity matrix by changing four entries (j,k) with $j,k \in \{p,q\}$ in the way presented below and let $F_z = I_z F I_z^{-1}$ be the matrix obtained from F by conjugation with I_z. This means that F_z differs from F only at four entries (j,k) with $j,k \in \{p,q\}$. The matrices have the following shapes

$$I_z = \begin{bmatrix} I & \vdots & & \vdots & \\ \cdots & \frac{1}{Z} & \cdots & \frac{-\bar{z}}{Z} & \cdots \\ & \vdots & I & \vdots & \\ \cdots & \frac{z}{Z} & \cdots & \frac{1}{Z} & \cdots \\ & \vdots & & \vdots & I \end{bmatrix}, \quad F_z = \begin{bmatrix} \ddots & \vdots & 0 & \vdots & 0 \\ \cdots & \frac{(v_i+|z|^2 v_k)}{Z} & \cdots & \frac{\bar{z}(v_i - v_k)}{Z} & \cdots \\ 0 & \vdots & \ddots & \vdots & 0 \\ \cdots & \frac{z(v_i - v_k)}{Z} & \cdots & \frac{(v_k + |z|^2 v_i)}{Z} & \cdots \\ 0 & \vdots & 0 & \vdots & \ddots \end{bmatrix}$$

where $Z = \sqrt{1+|z|^2}$. Then

$$\mu(\{F_z;\ z \in \mathbb{CP}^1\}) = \overline{\mu(F)\,\mu(F')}.$$

Moment image of the standard torus action is also explained in [38],[28].

There are also other natural actions on $\mathcal{O}_\lambda$. For any $j = 1, \dots, n$, we have a natural embedding $\iota_j : U(j) \to U(n)$

$$\iota_j(B) = \left(\begin{array}{c|c} B & 0 \\ \hline 0 & Id \end{array} \right),$$

where $B \in U(j)$. Using this embedding we obtain a $U(j)$ action (and also an action of maximal torus T^j) on $\mathcal{O}_\lambda$: for $B \in U(j)$ and $\xi \in \mathcal{O}_\lambda$, we define

$$B \cdot \xi = \iota_j(B)\, \xi\, (\iota_j(B))^{-1}.$$

To simplify the notation, we will often write B instead of $\iota_j(B)$. Both of these actions are also Hamiltonian. The momentum map for the $U(j)$ action is the projection

$$\Phi^j : \mathcal{O}_\lambda \to \mathfrak{u}(j)^*$$

sending every matrix to its $j \times j$ top left minor.

4.2 Gelfand-Tsetlin system for the unitary group

In this subsection we apply the general construction of the Gelfand Tsetlin system to the case $G = U(n)$. The main reference for this part is the work of Mikhail Kogan [24] (see also [13], [30], [17]). Consider the sequence of subgroups

$$U(n) \supset U(n-1) \supset \ldots \supset U(2) \supset U(1).$$

For each $U(j)$ in the sequence choose the maximal torus T_j to be the set of diagonal matrices in $U(j)$ and the positive Weyl chamber, $(\mathfrak{t}^j)^*_+$, to consist of diagonal Hermitian $j \times j$ matrices with non-increasing diagonal entries. Recall that the momentum map for the $U(j)$ action on $\mathcal{O}_\lambda$ is denoted by Φ^j and maps $A \in \mathcal{O}_\lambda$ to $j \times j$ top left submatrix of A. Denote the eigenvalues of $\Phi^j(A)$, ordered in a non-increasing way, by

$$\lambda_1^{(j)}(A) \geq \lambda_2^{(j)}(A) \geq \ldots \geq \lambda_j^{(j)}(A).$$

We will use the notation $\Lambda^{(j)} = (\lambda_1^{(j)}, \ldots, \lambda_j^{(j)}) : \mathcal{O}_\lambda \to \mathbb{R}^j$, for a function sending A to $(\lambda_1^{(j)}(A), \ldots, \lambda_j^{(j)}(A)) \in \mathbb{R}^j$. For $j = n$, we just get $\Phi^n(A) = A$ and $\lambda_k^{(n)}(A) = \lambda_k$. The **Gelfand -Tsetlin system of action coordinates** is the collection of the functions $\lambda_k^{(j)}$ for $j = 1, \ldots, n-1$ and $k = 1, \ldots, j$. We will denote them by

$$\Lambda : \mathcal{O}_\lambda \to \mathbb{R}^N,$$

where

$$N := (n-1) + (n-2) + \ldots + 1 = \frac{n(n-1)}{2}.$$

Notice that $\Lambda^{(j)}$ is the composition of Φ^j and a map $s_j : \mathfrak{u}(j)^* \to (\mathfrak{t}^j)^*_+ \subset \mathbb{R}^j$ sending a point in $\mathfrak{u}(j)^*$ to the unique point of intersection of its $U(j)$ orbit with the positive Weyl chamber.

$$\begin{array}{ccc} \mathcal{O}_\lambda & \xrightarrow{\Phi^j} & \mathfrak{u}(j)^* \\ & {\scriptstyle\Lambda^{(j)}}\searrow & \downarrow{\scriptstyle s_j} \\ & & (\mathfrak{t}(j))^*_+ \end{array}$$

Here we identify $(\mathfrak{t}^j)^*$ with $\mathbb{R}^j$ by $\mathrm{diag}(a_1, \ldots, a_j) \to (a_1, \ldots, a_j)$.

The components of s_j are $U(j)$ invariant, so they Poisson commute. After precomposing them with Φ^j, we get a family of Poisson commuting functions on $\mathcal{O}_\lambda$ (see Proposition 3.2 in [13]). These are exactly $\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)}$. For $l < j$ denote by $\kappa_{lj} : \mathfrak{u}(j)^* \to \mathfrak{u}(l)^*$ the transpose of the map $\mathfrak{u}(l) \to \mathfrak{u}(j)$ induced by the inclusion. The functions

$$\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)}, \lambda_1^{(l)} \circ \kappa_{lj}, \lambda_2^{(l)} \circ \kappa_{lj}, \ldots, \lambda_l^{(l)} \circ \kappa_{lj}$$

Poisson commute on $\mathfrak{u}(l)^*$ by Proposition 3.2 in [13] and the fact that first j of them are $U(j)$ invariant. Therefore the Gelfand-Tsetlin functions Poisson commute on $\mathcal{O}_\lambda$.

These functions are smooth at points where the eigenvalues do not coincide "unnecessarily", meaning they coincide but not on the whole orbit $\mathcal{O}_\lambda$. If λ is not a regular point, then some of the Gelfand-Tsetlin functions may be forced to coincide by the min-max inequalities. Precisely, the unique open wall from Theorem 3.4.1 applied to the $U(j)$ action is

$$\sigma_j^o = \{x = (x_1, \ldots, x_j) \in \mathfrak{t}^*_{U(j)} |\, x_i \neq x_{i+1} \text{ unless } \lambda_i = \ldots = \lambda_{i+n-j+1}\}.$$

Following notation from Section 3.5, let $T'_{U(j)} \hookrightarrow T_{U(j)}$ be the subtorus consisting of elements $(t_1, \ldots, t_n)$ with $t_i = 1$ if $\lambda_i^{(j)}$ is constant on the orbit. Note that for λ regular σ_j^o is simply the interior of the positive Weyl chamber $\mathrm{int}\mathfrak{t}_+^*$, and $T'_{U(j)} = T_{U(j)}$. Propositions 3.4.2 and 3.5.1 applied to this case give

Proposition 4.2.1. *The function $\Lambda^{(j)}$ is smooth at the preimage $(\Lambda^{(j)})^{-1}(\sigma_j^o) = U^{(j)}$. Moreover, the $*$ action of the torus $T'_{U(j)} \hookrightarrow T_{U(j)}$ on $(\Lambda^{(j)})^{-1}(\sigma_j^o)$ is Hamiltonian and $\Lambda^{(j)}$ is a momentum map.*

Putting the actions together we obtain the Hamiltonian action of the **Gelfand-Tsetlin torus** in $U(n)$ case, $T = T_{GT} = T'_{U(n-1)} \oplus \ldots \oplus T'_{U(1)} \cong (S^1)^D$, $D = \sum_{i=1}^{n-1} \dim(T'_{U(i)})$, on the dense open subset

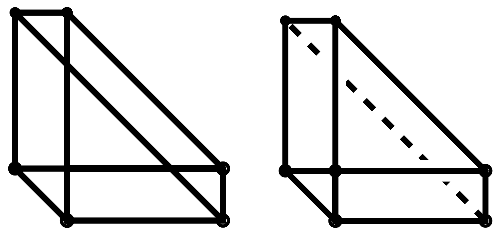
$$U := \cap_j U^{(j)}$$

of the coadjoint orbit $\mathcal{O}_\lambda$ where all functions $\Lambda^{(j)}$ are smooth. This action is called the **Gelfand-Tsetlin action** and its momentum map is Λ. If the orbit is regular then $D = N = \frac{1}{2}n(n-1)$.

Notice that the standard action of T^n, described in the Section 4.1, is a part of the T^N action on U. One can easily compute the T^n-momentum map μ, which mapps a matrix to its diagonal entries, from Λ. Of course $\lambda_1^{(1)}(A) = a_{11}$. Using the fact that the trace of $\Phi^2(A)$ is $a_{11} + a_{22} = \lambda_1^{(2)}(A) + \lambda_2^{(2)}(A)$ we compute the value a_{22}. Continuing this process we obtain all the diagonal entries of A, that is we obtain $\mu(A)$. This defines the projection $pr : (\mathfrak{t}^N)^* \to (\mathfrak{t}^n)^*$, which on the image of Λ is given by the following formula

$$pr(\{\lambda_l^{(j)}\}) = \Big(\lambda_1^{(1)}, (\lambda_1^{(2)} + \lambda_2^{(2)} - \lambda_1^{(1)}), \ldots, \sum_i \lambda_i^{(n-1)} - \sum_i \lambda_i^{(n-2)}, \sum_i \lambda_i^{(n)} - \sum_i \lambda_i^{(n-1)}\Big).$$

This means $\mu = pr \circ \Lambda$. Under this projection, the Gelfand-Tsetlin polytope $\mathcal{P}$, described below, maps to the momentum map image, $\mathcal{Q}$, of the standard maximal torus action. Here is an example for a regular $SU(3)$ coadjoint orbit, $\mathcal{O}_\lambda$.



$\mathcal{Q} = \mu(\mathcal{O}_\lambda) \subset \mathbb{R}^2 \quad \mathcal{P} = \Lambda(\mathcal{O}_\lambda) \subset \mathbb{R}^3$

Figure 4.1: The momentum map images for the standard and Gelfand-Tsetlin actions on a regular $SU(3)$ coadjoint orbit.

Proposition 4.2.2. *The Gelfand-Tsetlin action on a* $U(n)$*-coadjoint orbit* $\mathcal{O}_\lambda$ *is effective for all* λ.

Proof. Suppose that

$$R = (R_{n-1}, \ldots, R_1) \in T'_{U(n-1)} \oplus \ldots \oplus T'_{U(1)} = T_{GT},$$

$R_j = diag(r_{j,1}, \ldots, r_{j,j}, 1, \ldots, 1)$, is a global stabilizer. Let

$$\widetilde{R} := \left(\begin{array}{c|c} R_{n-1} & \\ \hline & 1 \end{array}\right) \cdots \left(\begin{array}{c|c} R_1 & \\ \hline & I_{n-1} \end{array}\right) = \begin{pmatrix} \prod_j r_{1,j} & & & & \\ & \prod_j r_{2,j} & & & \\ & & \ddots & & \\ & & & r_{n-1,n-1} & \\ & & & & 1 \end{pmatrix}.$$

Note that for any $k = 1, \ldots, n-1$

$$(\Phi^{n-1})^{-1}(\sigma^o_{n-1}) \subset (\Phi^k)^{-1}(\sigma^o_k).$$

Therefore for any $A = \left(\begin{array}{c|c} \Phi^{n-1}(A) & X \\ \hline X^* & c \end{array}\right) \in (\Phi^{n-1})^{-1}(\sigma^o_{n-1})$ have

$$R * A = \widetilde{R}\, A\, \widetilde{R}^{-1} = \widetilde{R} \left(\begin{array}{c|c} \Phi^{n-1}(A) & X \\ \hline X^* & c \end{array}\right) \widetilde{R}^{-1} = \left(\begin{array}{c|c} \Phi^{n-1}(A) & \widetilde{R}X \\ \hline X^*\widetilde{R}^{-1} & c \end{array}\right).$$

Let λ be of the form

$$(\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{l_1+\ldots+l_{s-1}+1} = \ldots = \lambda_{l_1+\ldots+l_s}).$$

Denote $\lambda_{l_1+\ldots+l_j}$ by w_j. In this notation

$$\lambda = \begin{pmatrix} w_1 \, I_{l_1} & & \\ & \ddots & \\ & & w_s \, I_{l_s} \end{pmatrix}.$$

Fix any $j = 1, \ldots, s$ and take $\varepsilon > 0$ such that $w_j - \varepsilon > w_{j+1}$. For any

$$X = (0, \ldots, 0, x_{l_1+\ldots+l_{j-1}+1}, \ldots, x_{l_1+\ldots+l_j}, 0, \ldots, 0)^T$$

such that $|x_{l_1+\ldots+l_{j-1}+1}|^2 + \ldots + |x_{l_1+\ldots+l_j}|^2 = l_j(\, w_j^2 - (w_j - \varepsilon)^2 \,)$, the matrix

$$\left(\begin{array}{ccccc|c} w_1 \, I_{l_1} & & & & & \\ & \ddots & & & & \\ & & (w_j - \varepsilon) \, I_{l_j} & & & X \\ & & & \ddots & & \\ & & & & w_s \, I_{l_s} & \\ \hline & & & X^* & & \sum \lambda_i - l_j \varepsilon \end{array}\right)$$

is in $(\Phi^{n-1})^{-1}(\sigma^o_{n-1})$ (see Lemma B.0.1). Therefore R stabilizes this matrix if and only if

$$\widetilde{R} X = X.$$

As R is a global stabilizer, considering similar matrices for other j we see that $\widetilde{R} = I$. In particular this means that in R_{n-1} the coordinate $r_{n-1,n-1}$ must be equal to 1.

Now consider matrices of the form

$$\left(\begin{array}{ccccc|c|c} w_1 I_{l_1} & & & & & & \\ & \ddots & & & & & \\ & & (w_j-\varepsilon) I_{l_j} & & & X & 0 \\ & & & \ddots & & & \\ & & & & w_s I_{l_s} & & \\ \hline & & X^* & & & c_1 & 0 \\ \hline & & 0 & & & 0 & c_2 \end{array}\right).$$

Torus $T'_{U(n-1)}$ acts trivially on such matrices. Therefore $R = (R_{n-1}, \ldots, R_1)$ acts in the same way as $(I, R_{n-2}, \ldots, R_1)$. Using similar argument as above we show that

$$\left(\begin{array}{c|c} R_{n-2} & \\ \hline & I_2 \end{array}\right) \cdots \left(\begin{array}{c|c} R_1 & \\ \hline & I_{n-1} \end{array}\right) = I_n.$$

In particular in R_{n-2} the coordinate $r_{n-2,n-2}$ must be equal to 1. Together with the condition $\widetilde{R} = I_n$ this means that $r_{n-1,n-2} = 1$. Repeating these steps consecutively one shows that $R_i = I$ for all i. Therefore $R = I \in T_{GT}$ is the unique global stabilizer and the action is effective. □

4.3 The Gelfand-Tsetlin polytope for the unitary group

In this subsection we analyze the image $\Lambda(\mathcal{O}_\lambda)$ in $\mathbb{R}^N$, where $N := n(n-1)/2$. The classical mini max principle (see for example Chapter I.4 in [5]) implies that

$$\lambda_j^{(l+1)}(A) \geq \lambda_j^{(l)}(A) \geq \lambda_{j+1}^{(l+1)}(A).$$

We use the following notation for these inequalities:

$$\begin{aligned} A_{l,j}: \quad & \lambda_j^{(l+1)}(A) \geq \lambda_j^{(l)}(A), \\ B_{l,j}: \quad & \lambda_j^{(l)}(A) \geq \lambda_{j+1}^{(l+1)}(A). \end{aligned} \tag{4.1}$$

The inequalities (4.1) cut out a polytope in $\mathbb{R}^N$, which we denoted by $\mathcal{P}$, and $\Lambda(\mathcal{O}_\lambda)$ is contained in this polytope.

Proposition 4.3.1. *The image $\Lambda(\mathcal{O}_\lambda)$ is exactly $\mathcal{P}$.*

Proof. The Proposition follows from successive applications of the following lemma (Lemma 3.5 in [30], see also [14]), as explained below.

Lemma 4.3.2. *For any real numbers $a_1 \geq b_1 \geq a_2 \geq \ldots \geq a_k \geq b_k \geq a_{k+1}$ there exist $x_1, \ldots, x_k$ in $\mathbb{C}$ and x_{k+1} in $\mathbb{R}$ such that the Hermitian matrix*

$$A := \begin{pmatrix} b_1 & & 0 & \bar{x}_1 \\ & \ddots & & \vdots \\ 0 & & b_k & \bar{x}_k \\ x_1 & \ldots & x_k & x_{k+1} \end{pmatrix},$$

has eigenvalues $a_1, \ldots, a_{k+1}$.

Now let $c_1, \ldots, c_{k-1}$ be numbers such that $b_1 \geq c_1 \geq b_2 \ldots \geq b_{k-1} \geq c_{k-1} \geq b_k$. Applying Lemma 4.3.2 again, we get that there exist $y_1, \ldots, y_{k-1}$ in $\mathbb{C}$ and y_k in $\mathbb{R}$ such that the Hermitian matrix

$$B := \begin{pmatrix} c_1 & & 0 & \bar{y}_1 \\ & \ddots & & \vdots \\ 0 & & c_{k-1} & \bar{y}_{k-1} \\ y_1 & \ldots & y_{k-1} & y_k \end{pmatrix},$$

has eigenvalues $b_1, \ldots, b_k$. Therefore there is an invertible matrix $C \in U(k)$ such that $CBC^{-1} = \text{diag}(b_1, \ldots, b_k)$. Denote by X the column vector $(x_1, \ldots, x_k)^T$. Notice that

$$\left(\begin{array}{c|c} C & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \ \ldots \ 0 & 1 \end{array}\right) \left(\begin{array}{c|c} B & C^{-1}\overline{X} \\ \hline X^T C & x_{k+1} \end{array}\right) \left(\begin{array}{c|c} C^{-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \ \ldots \ 0 & 1 \end{array}\right) = \left(\begin{array}{c|c} CBC^{-1} & C\,C^{-1}\overline{X} \\ \hline X^T C\,C^{-1} & x_{k+1} \end{array}\right) = A$$

Therefore the Hermitian matrix

$$\left(\begin{array}{c|c} B & C^{-1}\overline{X} \\ \hline X^T C & x_{k+1} \end{array}\right)$$

has desired values of the Gelfand-Tsetlin functions $\lambda_*^{(k+1)}, \lambda_*^{(k)}, \lambda_*^{(k-1)}$. Continuing this process, we construct a matrix A in $\mathcal{O}_\lambda$ such that $\Lambda(A) = L$, for any chosen point L in the polytope $\mathcal{P}$. □

The polytope $\mathcal{P} \subset \mathbb{R}^N$ is called the **Gelfand-Tsetlin polytope**. We think of $\mathbb{R}^N$ as having coordinates $\{x_k^{(j)}\}$, indexed by pairs (j,k), for $j = 1, \ldots, n-1$, and $k = 1, \ldots, j$, so that $x_k^{(j)}$-th coordinate of $\Lambda(A)$ is $\lambda_k^{(j)}(A)$.

Lemma 4.3.3. *Let $\Lambda(A)$, $A \in \mathcal{O}_\lambda$, be a point in the polytope $\mathcal{P}$, with coordinates $\{\lambda_k^{(j)}(A)\}$. Suppose that for any (j,k), $j = 1, \ldots, n-1$, $k = 1, \ldots, j$, we have that*

$$\lambda_k^{(j)}(A) = \lambda_k^{(j+1)}(A) \text{ or } \lambda_k^{(j)}(A) = \lambda_{k+1}^{(j+1)}(A).$$

Then $\Lambda(A)$ is a vertex of the polytope $\mathcal{P}$.

Proof. For any pair (j,k) pick one equality, $A_{j,k}$ or $B_{j,k}$, that is satisfied by $\Lambda(A)$ (if both are satisfied pick either one of them). Arrange these inequalities to be of the form:

$$(\text{linear combination of variables } x_k^{(j)}) \leq \text{real constant}.$$

Sum all of these N inequalities together, forming the inequality

$$CX \leq Z,$$

where $X = (x_1^{(n-1)}, \dots, x_1^{(1)}) \in \mathbb{R}^N$ is the variable, and $Z, C \in \mathbb{R}^N$ are constants. Every $X \in \mathcal{P}$ has to satisfy $CX \leq Z$, as this is just a sum of N of the $2N$ inequalities defining $\mathcal{P}$. Therefore $\mathcal{P} \cap \{X; CX = Z\}$ is a face of $\mathcal{P}$, (see Definition 2.1 in [39]). Note that $X \in \mathcal{P}$ satisfies $CX = Z$ if and only if all of the N inequalities defining $\mathcal{P}$ we have summed, are equalities for X. This determines the values of all $x_k^{(j)}$ in terms of $\lambda_1, \dots, \lambda_n$. Therefore

$$\mathcal{P} \cap \{X; CX = Z\} = \{\Lambda(A)\}$$

is a 0-dimensional face, in other words a vertex of $\mathcal{P}$. □

To emphasize the main idea of this proof, we give the following example.

Example 4.3.4. *Let $n = 3$, $\lambda = (5, 5, 4)$ and $\Lambda(A) = (\lambda_1^{(2)}(A), \lambda_2^{(2)}(A), \lambda_1^{(1)}(A)) = (5, 4, 5)$. We need to choose inequalities $A_{j,k}, B_{j,k}$, one for each pair (j, k), that are equalities for $\Lambda(A)$. For $\lambda_1^{(2)}(A)$ we have a choice as both of them are equations. Say we pick $B_{2,1}$, $B_{2,2}$ and $A_{1,1}$. The set of rearranged inequalities is*

$$\begin{aligned} -x_1^{(2)} &\leq -\lambda_2 = -5 \\ -x_2^{(2)} &\leq -\lambda_3 = -4 \\ x_1^{(1)} - x_1^{(2)} &\leq 0 \end{aligned}$$

Summing these inequalities together we obtain

$$-2x_1^{(2)} - x_2^{(2)} + x_1^{(1)} \leq -9.$$

This inequality is satisfied on all $\mathcal{P}$. An element $X \in \mathcal{P}$ satisfies $-2x_1^{(2)} - x_2^{(2)} + x_1^{(1)} = -9$ if and only if

$$\begin{aligned} -x_1^{(2)} &= -5 \\ -x_2^{(2)} &= -4 \\ x_1^{(1)} &= x_1^{(2)}. \end{aligned}$$

Thus, we see that $(5,4,5)$ *is the unique solution to these inequalities in* $\mathcal{P}$.

Lemma 4.3.5. *The map* Λ *sends every* T^n*-fixed point to a vertex of* $\mathcal{P}$.

Proof. For a diagonal matrix $F = \mathrm{diag}(F_{1,1}, \ldots, F_{n,n})$, the set of eigenvalues of $F_{j+1} := \Phi^{j+1}(F)$ is obtained from the set of eigenvalues of $F_j := \Phi^j(F)$ by adding $F_{j+1,j+1}$. Let s be such that

$$\lambda_s^{(j)}(F) \geq F_{j+1,j+1} > \lambda_{s+1}^{(j)}(F).$$

Then

$$\begin{aligned} \forall_{l \leq s} \quad \lambda_l^{(j)}(F) &= \lambda_l^{(j+1)}(F) \\ \forall_{l > s} \quad \lambda_l^{(j)}(F) &= \lambda_{l+1}^{(j+1)}(F). \end{aligned}$$

Therefore $\Lambda(F)$ is a vertex of $\mathcal{P}$, by Lemma 4.3.3. □

Lemma 4.3.6. *Let* $\Lambda(A)$*, for* $A \in \mathcal{O}_\lambda$*, be a point in the polytope* $\mathcal{P}$*, with coordinates* $\{\lambda_k^{(j)}(A)\}$*. Suppose that there exists exactly one pair of indices* (j_0, k_0) *such that both inequalities* A_{j_0,k_0} *and* B_{j_0,k_0} *at the point* A *are strict. That is, for all* $(j,k) \neq (j_0,k_0)$*,* $j = 1, \ldots, n-1$*,* $k = 1, \ldots, j$*, we have one of the equalities*

$$\lambda_k^{(j)}(A) = \lambda_k^{(j+1)}(A) \text{ or } \lambda_k^{(j)}(A) = \lambda_{k+1}^{(j+1)}(A).$$

Then $\Lambda(A)$ *is contained in the interior of an edge of* $\mathcal{P}$.

Proof. Proceed similarly as in the proof of Lemma 4.3.3. For any $(j,k) \neq (j_0.k_0)$ choose one of the inequalities $A_{j,k}, B_{j,k}$ that is equality for $\Lambda(A)$. Arrange these inequalities to be of the form:

$$(\text{linear combination of variables } x_k^{(j)}) \leq \text{real constant}.$$

Sum all of these $N-1$ inequalities together forming the inequality

$$CX \leq Z.$$

As before, this gives an inequality valid for $\mathcal{P}$, and $\mathcal{P} \cap \{X; CX = Z\}$ is a face of $\mathcal{P}$. The equation $CX = Z$ determines the values of all $x_k^{(j)}$, with $(j,k) \neq (j_0, k_0)$, in terms of $\lambda_1, \ldots, \lambda_n$ and $x_{k_0}^{(j_0)}$. These uniquely determined values are $x_k^{(j)} = \lambda_k^{(j)}(A)$. For any assignment of the value for $x_{k_0}^{(j_0)}$, the equation $CX = Z$ will still hold. In order to have $X \in \mathcal{P}$ we need to pick the value for $x_{k_0}^{(j_0)}$ in the open interval $(x_{k_0+1}^{(j_0+1)}, x_{k_0}^{(j_0+1)}) = (\lambda_{k_0+1}^{(j_0+1)}(A), \lambda_{k_0}^{(j_0+1)}(A))$. Note that $\lambda_{k_0}^{(j_0+1)}(A) \neq \lambda_{k_0+1}^{(j_0+1)}(A)$ because if they were equal, then they would also be equal to $\lambda_{k_0}^{(j_0)}(A)$ what contradicts our assumptions. Thus we really are choosing the value for $x_{k_0}^{(j_0)}$ from the open, non-degenerate interval $(\lambda_{k_0+1}^{(j_0+1)}(A), \lambda_{k_0}^{(j_0+1)}(A))$. Therefore

$$\mathcal{P} \cap \{X; CX = Z\} \cong (\lambda_{k_0+1}^{(j_0+1)}(A), \lambda_{k_0}^{(j_0+1)}(A))$$

is a 1-dimensional face of $\mathcal{P}$. □

Proposition 4.3.7. *For any λ, the dimension of the polytope $\mathcal{P}$ is half of the dimension of $\mathcal{O}_\lambda$ and $\mathcal{P} \subset (\mathfrak{t}_{GT})^* \subset (\mathfrak{t}^N)^* \cong \mathbb{R}^N$.*

Proof. Fix $\lambda \in (\mathfrak{t}^n)^*_+$, not necessarily generic. Let $l_1, \ldots, l_s$ be the integers such that $l_1 + \ldots + l_s = n$ and

$$\lambda_1 = \ldots = \lambda_{l_1} > \lambda_{l_1+1} = \ldots = \lambda_{l_1+l_2} > \ldots > \lambda_{n-l_s+1} = \ldots = \lambda_n.$$

Consider the coadjoint orbit $M := \mathcal{O}_\lambda$ in $U(n)$. The dimension of $\mathcal{O}_\lambda$ was already computed in Section 4.1 and is equal to

$$2D := 2\,[\,l_1(l_2 + \ldots l_s) + l_2(l_3 + \ldots + l_s) + \ldots + l_{s-1}l_s\,] = 2\sum_{i<j} l_i l_j.$$

If some $l_j > 1$, then the $(l_j - 1)$ functions $\lambda^{(1)}_{l_1+\ldots+l_{j-1}+1} = \ldots = \lambda^{(1)}_{l_1+\ldots+l_j-1}$ have to be equal to $\lambda_{l_1+\ldots+l_{j-1}+1}$ due to inequalities (4.1). Lemma 4.3.2 implies that the image $\Lambda^{(1)}(\mathcal{O}_\lambda)$ in $(\mathfrak{t}^{n-1})^* \cong \mathbb{R}^{n-1}$ has dimension equal to the number of non-constant functions from $\lambda^{(1)}_*$ that is

$$n - 1 - \sum_{j=1}^{s}(l_j - 1).$$

Inequalities (4.1) force also $(l_j - 2)$ of functions $\lambda^{(2)}_*$ to be equal to $\lambda_{l_1+\ldots+l_{j-1}+1}$, as well as $l_j - 3$ of functions $\lambda^{(3)}_*$, etc. The number of our functions λ^*_* that are constant is

$$\frac{l_1(l_1-1)}{2} + \ldots + \frac{l_s(l_s-1)}{2}.$$

The remaining functions form the system of action coordinates, consisting of

$$\frac{n(n-1)}{2} - \left(\frac{l_1(l_1-1)}{2} + \ldots + \frac{l_s(l_s-1)}{2}\right) = \sum_{i<j} l_i l_j = D$$

independent functions (see Proposition 4.3.1 and its proof). Therefore the dimension of the image $\Lambda(\mathcal{O}_\lambda)$ is D. Recall from Section 3.5 that the Gelfand-Tsetlin torus $T_{GT} \cong (S^1)^D$ is a subtorus of $T_{U(n-1)} \oplus \ldots \oplus T_{U(1)} \cong (S^1)^N$ corresponding to D non-constant functions $\lambda^{(*)}_*$. Therefore $\mathcal{P} \subset (\mathfrak{t}_{GT})^* \subset \mathbb{R}^N$. □

If $\mathcal{F}$ is a face of $\mathcal{P}$ containing some $x \in \Lambda(U)$, then, by the definition of U, x is not on any regular wall. Therefore any point of the interior $\mathcal{F}$ also cannot be on any regular wall, so it is in U.

Lemma 4.3.8. *If λ is generic, then the images of fixed points of the standard T^n action are in U. If λ is non generic but there is only one eigenvalue that is repeated - then there is a T^n-fixed point that is in U.*

Proof. If λ is generic, then for any T^n-fixed point F and any k, the matrix $\Phi^j(F)$ is a diagonal matrix with all diagonal entries distinct. Therefore $\Lambda(F)$ is not on any regular wall, so it is in U.

Now assume that λ is of the form

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{l_1} = \lambda_{l_1+1} = \ldots = \lambda_{l_1+s} > \lambda_{l_1+s+1} > \ldots > \lambda_n.$$

Let $\{v_1 > v_2 > \ldots > v_{n-s}\} = \{\lambda_1 > \lambda_2 > \ldots > \lambda_l > \lambda_{l_1+s+1} > \ldots > \lambda_n\}$ be the set of distinct eigenvalues. Consider the T^n-fixed point

$$F = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \lambda_{l_1} \mathrm{Id}_s \end{array} \right)$$

where A is any diagonal $(n-s) \times (n-s)$ matrix with spectrum $\{v_1, v_2, \ldots, v_{n-s}\}$. The figure below presents the values of Gelfand-Tsetlin functions $\lambda_k^{(j)}$ at F, for $j \geq n-s$ For $j \leq n-s$ the values $\lambda_1^{(j)}(F), \ldots, \lambda_j^{(j)}(F)$ are all distinct.

$$\begin{array}{cccccccccccccccc}
v_1 & & \ldots & & v_{l_1-1} & & v_l & & \ldots & & v_{l_1} & & v_{l_1+1} & & \ldots & & v_{n-s} \\
& v_1 & & \ldots & & v_{l_1-1} & & v_l & \ldots & v_{l_1} & & v_{l_1+1} & & \ldots & & v_{n-s} & \\
& & \ddots & & \ddots & & & & \vdots & & & & \cdot^{\cdot^{\cdot}} & & & & \\
& & & v_{l_1} & & \ldots & & v_{l_1-1} & v_{l_1} & v_{l_1+1} & & \ldots & & v_{n-s} & & &
\end{array}$$

Therefore $\lambda_j^{(k)} = \lambda_{j+1}^{(k)}$ at F if and only if this equation is valid for the whole orbit. This shows that the fixed point F of the form described above is in the set U. $\square$

We call Λ images of such T^n-fixed points, $\mathcal{O}_\lambda^{T^n} \cap U$, **good vertices** of $\mathcal{P}$. For example, in the case of regular $SU(3)$ orbit the Gelfand-Tsetlin polytope (see Figure 4.1) has 6 good vertices. The unique vertex with 4 adjacent edges is not a good vertex. In fact, preimage of this vertex is $\mathcal{O}_\lambda \setminus U$.

Now consider a non-regular example: $\lambda = (5, 4, 4, 4, 3, 1)$. Here is the T^n-fixed point that maps to a good vertex, and its Gelfand-Tsetlin functions (the bold ones are constant on the whole orbit)

$$F = \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 5 & & & & \\ & & 3 & & & \\ \hline & & & 4 & & \\ & & & & 4 & \\ & & & & & 4 \end{array}\right), \qquad \begin{array}{ccccccccc} 5 & & \mathbf{4} & & \mathbf{4} & & 3 & & 1 \\ & 5 & & \mathbf{4} & & 3 & & 1 & \\ & & 5 & & 3 & & 1 & & \\ & & & 5 & & 1 & & & \\ & & & & 1 & & & & \end{array}$$

Note that the vertex $\Lambda(\mathrm{diag}(1, 4, 4, 4, 3, 5))$ is not a good vertex.

Proposition 4.3.9. *For any good vertex $V_F = \Lambda(F)$ there are exactly D edges in $\mathcal{P}$ emanating from $\Lambda(F)$.*

Proof. All the Λ preimages of interiors of faces containing $\Lambda(F)$, are also in U. Thus around F we have a smooth, Hamiltonian action of T^D on U. The local normal form theorem, (see for example [21]), gives that, in a suitably chosen basis, the image of momentum map is a D dimensional orthant. In particular this proves that there are exactly D edges starting from this point. □

Note that there may be more than D edges starting from vertices of $\mathcal{P}$ that are not good vertices.

4.4 Proof of the lower bounds for Gromov width of $U(n)$ coadjoint orbits

Let $\mathcal{O}_\lambda$ be a coadjoint orbit such that the Gelfand-Tsetlin polytope $\mathcal{P}$ contains at least one good vertex. In particular λ can be of the form

$$\lambda_1 > \lambda_2 > \ldots > \lambda_{l_1} = \lambda_{l_1+1} = \ldots = \lambda_{l_1+s} > \lambda_{l_1+s+1} > \ldots > \lambda_n, \;\; s \geq 0.$$

Recall that D denotes half of the dimension of $\mathcal{O}_\lambda$, which is equal to the dimension of the Gelfand-Tsetlin torus T_{GT}, and that $\mathcal{P} \subset (\mathfrak{t}_{GT})^* \subset (\mathfrak{t}^N)^* \cong \mathbb{R}^N$. We are to show that the Gromov width of $\mathcal{O}_\lambda$ is at least $\min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$. Let $\Lambda(F) = V_F$ be a good vertex of $\mathcal{P}$ and $\mathcal{T}$ be an open subset of $\mathfrak{t}^*$ such that

$$\Lambda(\mathcal{O}_\lambda) \cap \mathcal{T} = \bigcup_{\substack{\mathcal{F} \text{ face of } \mathcal{P} \\ V_F \in \mathcal{F}}} (\text{rel-int } \mathcal{F})$$

and let $\mathcal{W} = \Phi^{-1}(\mathcal{T})$. Then $\mathcal{W}$ is the largest subset of M centered around V_F (compare with Example 3.2.4). According to the Proposition 3.2.6 there is an equivariant symplectomorphism

$$\Psi : \left\{ z \in \mathbb{C}^D \mid V_F + \pi \sum |z_j|^2 \eta_j \in \mathcal{T} \right\} \xrightarrow{\cong} \mathcal{W},$$

where $-\eta_1, \ldots, -\eta_D$ are the isotropy weights of T^D action on $T_F \mathcal{O}_\lambda$. The vectors $\eta_1, \ldots, \eta_D$ span D edges of $\mathcal{P}$ starting from V_F. We call them the **edge generators**. For the edge in the direction of η_l, there is a number $c_l \in \mathbb{R}$ such that the edge is precisely $c_l\, \eta_l$. This is equivalent to saying that the edge is of lattice length c_l with respect to the weight lattice, because for the coadjoint $U(n)$ action all isotropy weights are primitive with respect to the lattice they span. Let

$$\{v_1 > v_2 > \ldots > v_{n-s}\}$$

be the set of distinct eigenvalues. We prove the main theorem by showing that for any edge, c_l is at least the minimum $\min\{v_i - v_{i+1}\} = \min\{\lambda_i - \lambda_j \,|\, \lambda_i > \lambda_j\}$. Moreover, we will show that for any good vertex there is an edge leaving from this vertex, with the length equal to the minimum of $v_i - v_{i+1}$ times the length of the edge generator. This means that the lower bound we prove is the best possible we can get from this particular action. Let us emphasize that there might exist symplectic embeddings of bigger balls, however this method fails to find them.

Proposition 4.4.1. *The length of any edge in $\mathcal{P}$ starting from a good vertex V_F is at least* $\min\{v_i - v_{i+1}\}$ *times the length of the edge generator. Moreover, there is an edge with length exactly the* $\min\{v_i - v_{i+1}\}$ *times the length of its generator.*

Proof. Recall from Section 3.5 that the momentum maps for the standard and the Gelfand-Tsetlin torus actions are related through projection pr, $\mu = pr \circ \Lambda$. We continue to denote the polytope $\mu(\mathcal{O}_\lambda)$ by $\mathcal{Q}$ and its one-skeleton (image of points whose orbits have dimension at most 1) by $\mathcal{Q}_1$ We will show that for any edge $e \in \mathcal{P}$ starting from V_F there is an edge e' in $\mathcal{Q}_1$ (possibly not and edge but just a line segment in $\mathcal{Q}$) such that $pr(e) \subset e'$. This will help us to analyze edges of $\mathcal{P}$.

Denote the diagonal entries of F by $F_{11}, \ldots, F_{nn}$. Let $p < q$ be indices from $\{1, \ldots, n\}$ such that $v_i = F_{pp} \neq F_{qq} = v_k$ and F' is the matrix obtained from F by switching p-th and q-th entry. There is an edge in $\mathcal{Q}_1$ joining $\mu(F)$ and $\mu(F')$, and it is an μ-image of a sphere $S := \{F_z;\ z \in \mathbb{C} \cup \{\infty\}\}$ in $\mathcal{O}_\lambda$ defined in the Section 4. We will analyze $\Lambda(S)$.

Assume that $v_k < v_i$. The other case is proved in a similar way. First observe that for $j < p$ the matrices $(F_z)_j := \Phi^j(F_z)$ and $(F)_j := \Phi^j(F)$ are both equal to $\operatorname{diag}(F_{1,1}, \ldots, F_{j,j})$. Also for $j \geq q$ the matrices $(F_z)_j$ and F_j have the same eigenvalues. This is because the eigenvalues of this 2×2 matrix

$$\begin{bmatrix} \frac{(v_i+|z|^2 v_k)}{Z} & \frac{\bar{z}(v_i - v_k)}{Z} \\ \frac{z(v_i - v_k)}{Z} & \frac{(v_k + |z|^2 v_i)}{Z} \end{bmatrix},$$

where $Z = \sqrt{1+|z|^2}$, are v_i and v_k. Therefore, for $j < p$ or $j \geq q$, we have

$$\forall_{F_z \in S}\ \lambda_m^{(j)}(F_z) = \lambda_m^{(j)}(F), \tag{4.2}$$

for any $m = 1, \ldots, n-j$. Denote by $\rho(|z|) = \frac{(v_i+|z|^2 v_k)}{Z}$. While a goes to ∞, ρ

decreases its value from v_i to v_k. Let

$$i' = \min\{l\,;\ v_l \in \{F_{11}, \ldots, F_{qq}\},\ \ v_i > v_l\}.$$

This implies that $i+1 \leq i' \leq k$. Note that i' is not necessarily $i+1$, as it might happen that v_{i+1} is a diagonal entry of F that does not sit in a submatrix $(F)_q$. Lemmas 4.4.2 and 4.4.3 below show that the set $\Lambda(\{F_z \mid \rho(|z|) \in [v_{i'}, v_i]\})$ is an edge of $\mathcal{P}$ starting from V_F. Now we need to compute it's length relative to the length of the edge generator (= −isotropy weight). Notice that the projection pr (induced by inclusion $T^n \hookrightarrow T_{GT}$) maps the isotropy weights of T_{GT} action to the isotropy weights of T^n action. If $e = c_l \eta_l$ is the edge of $\mathcal{P}$, then $pr(e) = c_l pr(\eta_l)$ is the part of the corresponding edge e' of $\mathcal{Q}_1$ starting from the vertex $\mu(F)$. The edge generator in the direction $pr(\eta_l)$ is $-e_{pp} + e_{qq}$ (because the isotropy weight of the standard action of maximal torus is $e_{pp} - e_{qq}$). We will denote $\widetilde{Z} := \{F_z \,|\, \rho(|z|) = v_{i'}\}$ and $\widetilde{V} := \Lambda(\widetilde{Z})$, regardless of the fact if it is a vertex or an interior point of and edge in $\mathcal{P}$. Notice that $\widetilde{V}$, has values of Λ that are different from those of F in exactly $(q-p)$ places. Precisely, for every $p \leq j < q$, there is exactly one s such that $\lambda_s^{(j)}(F) = v_i$ while $\lambda_s^{(j)}(\widetilde{Z}) = v_{i'}$. Recall from section 3.5 that the $k-th$ coordinate of $pr(\{\lambda_*^{(*)}\})$ is given by

$$(\,pr(\{\lambda_*^{(*)}\})\,)_k = \sum_{s=1}^{k} \lambda_s^{(k)} - \sum_{s=1}^{k-1} \lambda_s^{(k-1)}$$

for $k > 1$ and is equal to $\lambda_1^{(1)}$ for $k = 1$. Therefore $\mu(F) = pr(\Lambda(F))$ and $\mu(\widetilde{Z}) = pr(\Lambda(\widetilde{Z}))$ differ only at p-th and q-th coordinates:

$$\begin{aligned}(\,pr(\Lambda(F))\,)_p &= \sum_{s=1}^{p} \lambda_s^{(p)}(F) - \sum_{s=1}^{p-1} \lambda_s^{(p-1)}(F)\\ &= \sum_{s=1}^{p} \lambda_s^{(p)}(\widetilde{Z}) + v_i - v_{i'} - \sum_{s=1}^{p-1} \lambda_s^{(p-1)}(\widetilde{Z}) = (\,pr(\Lambda(\widetilde{Z}))\,)_p + v_i - v_{i'}\end{aligned}$$

$$\begin{aligned}(\, pr(\Lambda(F))\,)_q &= \sum_{s=1}^{q} \lambda_s^{(q)}(F) - \sum_{s=1}^{q-1} \lambda_s^{(q-1)}(F) \\ &= \sum_{s=1}^{q} \lambda_s^{(q)}(\widetilde{Z}) + v_i - v_{i'} - (\sum_{s=1}^{q-1} \lambda_s^{(q-1)}(\widetilde{Z}) + v_i - v_{i'}\,) \\ &= (\, pr(\Lambda(\widetilde{Z}))\,)_q - (v_i - v_{i'})\end{aligned}$$

Thus

$$\overline{\mu(F)\,\mu(\widetilde{Z})} = (v_i - v_{i'})(-e_{pp} + e_{qq}),$$

and the edge e of $\mathcal{P}$ is at least $(v_i - v_{i'})$ multiple of the weight spanning it. Recall from definition of i' that $(v_i - v_{i'}) \geq (v_i - v_{i+1})$.

In case where $v_k > v_i$, $\rho(|z|)$ would be increasing its value from v_i to v_k and we would prove in an analogous way that the edge joining F and F' is at least $(v_{i-1} - v_i)$ multiple of the edge generator.

Notice that different pairs of p and q (such that $F_{pp} \neq F_{qq}$) give different edges. This follows, for example, from the fact that for $j < p$ or $j \geq q$, we have $\lambda_s^{(j)}(F_z) = \lambda_s^{(j)}(F)$. Therefore we found D distinct edges of $\mathcal{P}$ starting from V_F. The Proposition 4.3.9 gives that these must be all the edges.

Now suppose that m is the index such that the minimum of $\{v_i - v_{i+1}\,|\, i = 1, \ldots, s\}$ is equal to $v_m - v_{m+1}$. There are indices $p < q$ such that $F_{p,p} = v_m$ and $F_{q,q} = v_{m+1}$, or $F_{p,p} = v_{m+1}$ and $F_{q,q} = v_m$. Let F' be the diagonal matrix obtained from F by switching p-th and q-th entry. Then $\widetilde{Z} = F'$, $\widetilde{V} = \Lambda(F')$ and the edge of $\mathcal{P}$ between these two vertices is exactly $(v_m - v_{m+1})$ multiple of the edge generator. $\square$

The above proof used two lemmas that we formulate and prove below.

Lemma 4.4.2. *For z such that $v_i > \frac{(v_i+|z|^2 v_k)}{Z} = \rho(|z|) > v_{i'}$ the point $\Lambda(F_z)$ is in the interior of an edge of $\mathcal{P}$.*

Proof. Let m be such that

$$\lambda_m^{(q-1)}(F_z) = v_i > \rho(|z|) = \lambda_{m+1}^{(q-1)}(F_z).$$

We will show that for any $(j,l) \neq (q-1,m)$, $j = 1,\ldots,n-1$, $l = 1,\ldots,j$, we have that

$$\lambda_l^{(j)}(F_z) = \lambda_l^{(j+1)}(F_z) \text{ or } \lambda_l^{(j)}(F_z) = \lambda_{l+1}^{(j+1)}(F_z),$$

and use the Lemma 4.3.6. The matrix $(F_z)_q := \Phi^q(F_z)$ is diagonal, thus, repeating the proof of Lemma 4.3.5 for $(F_z)_q$, we can show that the above claim holds for $j < q-1$ and any l. Also, for $j \geq q$ the claim holds, due to equations (4.2) and Lemma 4.3.5. Thus, for $j \neq q-1$ and any l, the function $\lambda_l^{(j)}$ is equal at F_z to its lower or upper bound.

Now assume $j = q-1$ and notice that

$$spectrum((F_z)_q) = spectrum((F_z)_{q-1}) \cup \{v_i, v_k\} \setminus \{\rho(|z|)\}.$$

The Figure 4.4 presents sequences of ordered eigenvalues of $(F_z)_{q-1}$ and $(F_z)_q$. This

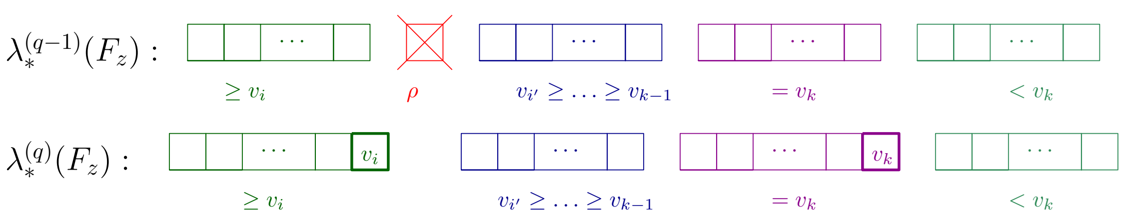


Figure 4.2: Eigenvalues of $(F_z)_{q-1}$ and $(F_z)_q$.

presentation helps to note that

$$\forall_{t \neq m,}\ \lambda_t^{(q-1)}(F_z) \geq v_k \ \Rightarrow \lambda_t^{(q-1)}(F_z) = \lambda_t^{(q)}(F_z),$$

$$\forall_{t\neq m,}\ \lambda_t^{(q-1)}(F_z) < v_k \ \Rightarrow \lambda_t^{(q-1)}(F_z) = \lambda_{t+1}^{(q)}(F_z).$$

Thus by the Lemma 4.3.6, $\Lambda(F_z)$ is on the edge of $\mathcal{P}$. All eigenvalues of $(F_z)_q$ are equal to some element of the set $\{v_1, \ldots, v_{n-s}\}$. Therefore $\lambda_m^{(q-1)}(F_z) = \rho(|z|) \in (v_{i'}, v_i)$ is not equal to $\lambda_m^{(q)}(F_z)$ nor $\lambda_{m+1}^{(q)}(F_z)$, so $\Lambda(F_z)$ is not a vertex of $\mathcal{P}$. □

Lemma 4.4.3. $\Lambda(\,\{F_z\,|\,\rho(|z|) = v_{i'}\}\,)$ *is a vertex of* $\mathcal{P}$.

Proof. Similarly to the proof of Lemma 4.4.2, we show that for $(j, l) \neq (q-1, m)$, $j = 1, \ldots, n-1$, $l = 1, \ldots, j$, the function $\lambda_l^{(j)}$ at F_z is equal to its lower or upper bound (again use Figure 4.4). However this time $\lambda_m^{(q-1)}(F_z) = \rho(|z|) = v_{i'} = \lambda_{m+1}^{(q)}(F_z)$. We use Lemma 4.3.3 to deduce that $\Lambda(\,\{F_z\,|\,\rho(|z|) = v_{i'}\}\,)$ is a vertex of $\mathcal{P}$. □

Proof. (of Theorem 4.0.2) Proposition 4.4.1 together with Proposition 3.2.6 give the proof of Theorem 4.0.2, as explained in the Example 3.2.7. □

CHAPTER 5

COADJOINT ORBITS OF THE SPECIAL ORTHOGONAL GROUP

In this chapter we consider coadjoint orbits of the special orthogonal group. Let $G = SO(2n+1)$ or $G = SO(2n)$. Then the Lie algebra $\mathfrak{g}$ is the vector space of skew symmetric matrices of appropriate size. We will identify the Lie algebra dual $\mathfrak{g}^*$ with $\mathfrak{g}$ using the G invariant pairing in $\mathfrak{g}$, $(A, B) = -\frac{1}{2}\text{trace}(AB)$. Throughout the paper we use the notation

$$R(\alpha) = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}, \quad L(a) = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$$

We make the following choices of maximal tori

$$T_{SO(2n+1)} = \left\{ \begin{pmatrix} R(\alpha_1) & & & & \\ & R(\alpha_2) & & & \\ & & \ddots & & \\ & & & R(\alpha_n) & \\ & & & & 1 \end{pmatrix} \right\}, \quad T_{SO(2n)} = \left\{ \begin{pmatrix} R(\alpha_1) & & & \\ & R(\alpha_2) & & \\ & & \ddots & \\ & & & R(\alpha_n) \end{pmatrix} \right\}$$

where $\alpha_j \in S^1$. The corresponding Lie algebra duals are

$$\mathfrak{t}^*_{SO(2n+1)} = \left\{ \begin{pmatrix} L(a_1) & & & & \\ & L(a_2) & & & \\ & & \ddots & & \\ & & & L(a_n) & \\ & & & & 0 \end{pmatrix} \right\}, \quad \mathfrak{t}^*_{SO(2n)} = \left\{ \begin{pmatrix} L(a_1) & & & \\ & L(a_2) & & \\ & & \ddots & \\ & & & L(a_n) \end{pmatrix} \right\}$$

and we choose the positive Weyl chambers to consist of matrices with $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq 0$ in the case $G = SO(2n+1)$, and $a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_{n-1} \geq |a_n|$ in the case $G = SO(2n)$. We are using the convention that the exponential map $exp : \mathfrak{t}_{SO(2)} \to T_{SO(2)}$ is given by $L(a) \to R(2\pi a)$, that is $S^1 \cong \mathbb{R}/\mathbb{Z}$. A point $\lambda \in \mathfrak{g}^*$ and a coadjoint orbit through it are called **regular** if the stabilizer of λ under coadjoint action is the maximal torus. Coadjoint orbits are in bijection with points in the positive Weyl chamber. Under this bijection, the regular points correspond to the interior of the chamber. Fix a point $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n)$, in the interior of the positive Weyl chamber, $\mathfrak{t}^*_+$,

$$\lambda = \begin{cases} \begin{pmatrix} L(\lambda_1) & & & & & \\ & L(\lambda_2) & & & & \\ & & L(\lambda_3) & & & \\ & & & \ddots & & \\ & & & & L(\lambda_n) & \\ & & & & & 0 \end{pmatrix} \in \mathfrak{t}^*_{SO(2n+1)} & \text{if } G = SO(2n+1) \\ \begin{pmatrix} L(\lambda_1) & & & & \\ & L(\lambda_2) & & & \\ & & L(\lambda_3) & & \\ & & & \ddots & \\ & & & & L(\lambda_n) \end{pmatrix} \in \mathfrak{t}^*_{SO(2n)} & \text{if } G = SO(2n) \end{cases}$$

Denote the orbit of the coadjoint action of G on λ by $\mathcal{O}_\lambda$. The orbit is also a symplectic manifold, with Kostant-Kirillov symplectic form. The dimension of $\mathcal{O}_\lambda$ is equal to

$$\dim \mathcal{O}_\lambda = \dim(\mathfrak{g}^*) - \dim T_G = \begin{cases} n(2n+1) - n = 2n^2 & \text{if } G = SO(2n+1) \\ n(2n-1) - n = 2n(n-1) & \text{if } G = SO(2n). \end{cases}$$

In this section we prove the Theorem 3.1.1 for the special orthogonal group. The analysis of the root system of the special orthogonal groups done in Subsection 5.1, and inequalities imposed on λ, imply that if $G = SO(2n+1)$ the minimum $\min\{\, |\langle \alpha^\vee, \lambda \rangle| \,;\alpha^\vee \text{ a coroot}\}$ is equal to

$$\min\{\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n,\, 2\lambda_n\},$$

while for $G = SO(2n)$ the minimum is

$$\min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}.$$

Below we give precise statement of the Theorem 3.1.1 in the case of special orthogonal group.

Theorem 5.0.4. *The Gromov width of the coadjoint orbit of the special orthogonal group passing through a point $\lambda = (\lambda_1, \ldots, \lambda_n) \in$ int $\mathfrak{t}^*_+$ in the positive Weyl chamber (chosen above) is at least*

$$\min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n,\ 2\lambda_n\}$$

if $G = SO(2n+1)$, and is at least

$$\min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}$$

if $G = SO(2n)$.

In the case of $G = SO(2n+1)$ this result can be strengthened to cover also a class of orbits that are not regular (see [33, Theorem 7.1]).

5.1 Root system of the special orthogonal group

The root system of a group G consists of vectors in $\mathfrak{t}^*$, the dual of the Lie algebra of the maximal torus of G. The coroot $\alpha^\vee$ corresponding to a root α is an element of $\mathfrak{t}$ given by the condition $x(\alpha^\vee) = 2\,\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}$ for all $x \in \mathfrak{t}^*$. Recall that $x(\alpha^\vee) = -\frac{1}{2}\mathrm{trace}(x\,\alpha^\vee)$. We will often denote this pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ by $\langle , \rangle$. We identify $\mathfrak{t}^*$ (so also $\mathfrak{t}$) with $\mathbb{R}^n$ by sending matrices

$$\begin{pmatrix} L(a_1) & & & & \\ & L(a_2) & & & \\ & & \ddots & & \\ & & & L(a_n) & \\ & & & & 0 \end{pmatrix} \in \mathfrak{t}^*_{SO(2n+1)}, \quad \begin{pmatrix} L(a_1) & & & \\ & L(a_2) & & \\ & & \ddots & \\ & & & L(a_n) \end{pmatrix} \in \mathfrak{t}^*_{SO(2n)}$$

to $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. With this identification, the pairing $\langle , \rangle$ in $\mathfrak{t}^*$ is just the standard scalar product.

The root system of the group $SO(2n+1)$ consists of vectors $= \pm e_j$, $j = 1, \ldots n$, of squared length 1, and vectors $\pm(e_j \pm e_k)$, $j \neq k$, of squared length 2 in the Lie algebra dual $\mathfrak{t}^*_{SO(2n+1)}$. Therefore this root system for $SO(n)$ is non-simply laced. Note that

$$\langle (e_j \pm e_k)^\vee, \lambda \rangle = 2 \frac{\langle e_j \pm e_k, \lambda \rangle}{\langle e_j \pm e_k, e_j \pm e_k \rangle} = \lambda_j \pm \lambda_k$$

and

$$\langle (e_j)^\vee, \lambda \rangle = 2 \frac{\langle e_j, \lambda \rangle}{\langle e_j, e_j \rangle} = 2\lambda_j.$$

Therefore for λ in our chosen positive Weyl chamber

$$\min\{|\langle \alpha^\vee, \lambda \rangle| \,; \alpha^\vee \text{ a coroot}\} = \min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \, 2\lambda_n\}.$$

The root system for $SO(2n)$ is simply laced and consists of vectors $\pm(e_j \pm e_k)$, $j \neq k$, of squared length 2. Note that

$$\langle (e_j \pm e_k)^\vee, \lambda \rangle = 2 \frac{\langle e_j \pm e_k, \lambda \rangle}{\langle e_j \pm e_k, e_j \pm e_k \rangle} = \lambda_j \pm \lambda_k.$$

Therefore for λ in a positive Weyl chamber

$$\min\{|\langle \alpha^\vee, \lambda \rangle| \,; \alpha^\vee \text{ a coroot}\} = \min\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}.$$

5.2 The Gelfand-Tsetlin system for the special orthogonal group

In this section we apply the general construction of the Gelfand-Tsetlin system to the case of $SO(n)$. Consider the following sequence of subgroups

$$G_n = SO(n) \supset G_{n-1} = SO(n-1) \supset G_{n-2} = SO(n-2) \supset \ldots \supset G_2 = SO(2).$$

For these groups we make the following choices of maximal tori.

$$T_{SO(2k+1)} = \begin{pmatrix} R(\alpha_1) & & & & \\ & R(\alpha_2) & & & \\ & & \ddots & & \\ & & & R(\alpha_k) & \\ & & & & 1 \end{pmatrix}, \quad T_{SO(2k)} = \begin{pmatrix} R(\alpha_1) & & & \\ & R(\alpha_2) & & \\ & & \ddots & \\ & & & R(\alpha_k) \end{pmatrix}.$$

The positive Weyl chambers are chosen in an analogous way to the case described in the Introduction. Take any G_k from this sequence, $k = 2, \ldots, 2n$. The group G_k injects into G by

$$G_k \ni B \mapsto \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I \end{array}\right).$$

Therefore it also act on $\mathcal{O}_\lambda$ by a subaction of the coadjoint action. This action is Hamiltonian with a momentum map $\Phi^k : \mathcal{O}_\lambda \to \mathfrak{so}(k)^*$ sending a matrix $A = [a_{ij}]$ to the $k \times k$ top left submatrix of A, which we denote by $\Phi^k(A)$ or $(A)_k$ for short. The action of the Gelfand-Tsetlin torus is defined using the following functions. Compose the map Φ^k with the map $s_k : \mathfrak{so}(k)^* \to (\mathfrak{t}_{SO(k)})^*_+$ sending $A \in \mathfrak{so}(k)^*$ to the unique point of intersection of the $SO(k)$-orbit, $SO(k) \cdot A$, with the positive Weyl chamber. Recall that we identify Lie algebra dual $(\mathfrak{t}_{SO(k)})^*$ with $\mathbb{R}^{\lfloor \frac{k}{2} \rfloor}$, as explained in the previous section. The positive Weyl chamber, $(\mathfrak{t}_{SO(k)})^*_+$, is identified with the subset of points $(x_1, \ldots, x_{\lfloor \frac{k}{2} \rfloor}) \in \mathbb{R}^{\lfloor \frac{k}{2} \rfloor}$ satisfying $x_1 \geq x_2 \geq \ldots \geq x_{\lfloor \frac{k}{2} \rfloor}$, for k odd, and $x_1 \geq x_2 \geq \ldots \geq x_{\frac{k}{2}-1} \geq |x_{\frac{k}{2}}|$, for k even.

The composition $s_k \circ \Phi^k : \mathcal{O}_\lambda \to (\mathfrak{t}_{SO(k)})^*_+$ gives us $\lfloor \frac{k}{2} \rfloor$ continuous (not everywhere smooth) functions which we denote

$$\Lambda^{(k)} := (\lambda_1^{(k)}, \dots, \lambda_{\lfloor \frac{k}{2} \rfloor}^{(k)}).$$

In this notation the superscript keeps track of the dimension of the matrices in the group (not the dimension of the maximal torus). Note that due to our choices of positive Weyl chambers, the only Gelfand-Tsetlin functions that can be negative are $\{x^{(k)}_{\frac{k}{2}}\}$, for k even.

$$\begin{array}{ccc} \mathcal{O}_\lambda & \xrightarrow{\Phi^k} & \mathfrak{so}(k)^* \\ & \searrow^{\Lambda^{(k)}} & \downarrow s_k \\ & & (\mathfrak{t}_{SO(k)})^*_+ \end{array}$$

These functions are related to the following action of $T_{SO(k)}$ denoted by $*$. An element $t \in T_{SO(k)}$ acts on a point $A \in \mathcal{O}_\lambda$ by the standard $SO(k)$ action of $B^{-1}\, t\, B$, where $B \in SO(k)$ is such that $B\, \Phi^k(A)\, B^{-1} \in (\mathfrak{t}_{SO(k)})^*_+$:

$$t * A := \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I_{n-k} \end{array} \right) A \left(\begin{array}{c|c} B^{-1}\, t\, B & \\ \hline & I_{n-k} \end{array} \right)^{-1}.$$

Note that for regular orbits, the unique wall of the positive Weyl chamber from Proposition 3.4.1 applied to $SO(k)$ action, σ_k^o, is simply the interior of the positive Weyl chamber. Therefore the Propositions 3.4.2 and 3.5.1 give:

Proposition 5.2.1. *The function $\Lambda^{(k)}$ is smooth at the preimage of the interior of the positive Weyl chamber,*

$$U_{SO(k)} := (\Lambda^{(k)})^{-1}(int\ (\mathfrak{t}_{SO(k)})^*_+).$$

Moreover, the $*$ action of the torus $T_{SO(k)}$ on $U_{SO(k)}$ is Hamiltonian and $\Lambda^{(k)}$ is a momentum map.

If $G = SO(2n+1)$, putting together these functions for $k = 1, \dots 2n$ we obtain a function, denoted by $\Lambda = \{\lambda_j^{(k)} \,|1 \leq k \leq 2n,\, 1 \leq j \leq \lfloor \frac{k}{2} \rfloor\}$, mapping $\mathcal{O}_\lambda$ to $\mathbb{R}^N$, where

$$N = n + 2(n-1) + 2(n-2) + \dots + 2 \cdot 2 = n + n(n-1) = n^2.$$

If $G = SO(2n)$, then we obtain a function $\Lambda = \{\lambda_j^{(k)} \,|1 \leq k \leq 2n-1,\, 1 \leq j \leq \lfloor \frac{k}{2} \rfloor\}$, mapping $\mathcal{O}_\lambda$ to $\mathbb{R}^N$, with

$$N = 2(n-1) + 2(n-2) + \dots + 2 \cdot 2 = n(n-1).$$

In both cases N is equal to half of the dimension of a regular coadjoint orbit of G.

Putting the actions together we obtain the Hamiltonian action of the **Gelfand-Tsetlin torus** $T = T_{GT} = T_{SO(n-1)} \oplus \dots \oplus T_{SO(2)} \cong (S^1)^N$ on the dense open subset

$$U := \cap_k U_{SO(k)}$$

of the coadjoint orbit $\mathcal{O}_\lambda$ where all functions $\Lambda^{(k)}$ are smooth. This action is called the **Gelfand-Tsetlin action** and its momentum map is Λ.

5.3 The Gelfand-Tsetlin polytope for the special orthogonal group

In this section we describe in details the image of Gelfand-Tsetlin functions, $\Lambda(\mathcal{O}_\lambda)$. The fact that the image forms a polytope seems to be well known. However we could not find a reference for this fact. Therefore we prove it below. The following lemmas are helpful in analyzing the image of Gelfand-Tsetlin functions. Their proofs are in the Appendix B.

Lemma 5.3.1. *For any real numbers*

$$b_1 \geq a_1 \geq b_2 \geq a_2 \geq \ldots \geq a_{k-1} \geq b_k \geq |a_k| \tag{5.1}$$

there exist a real vector $Y = [y_1, \ldots, y_{2k}]^T$ *in* $\mathbb{R}$ *such that the skew symmetric matrices*

$$A := \left(\begin{array}{cccc|c} L(a_1) & & & & \\ & L(a_2) & & & \\ & & \ddots & & Y \\ & & & L(a_k) & \\ \hline & -Y^T & & & 0 \end{array}\right) \text{ and } S := \left(\begin{array}{cccc|c} L(b_1) & & & & \\ & L(b_2) & & & 0 \\ & & \ddots & & \\ & & & L(b_k) & \\ \hline & 0 & & & 0 \end{array}\right).$$

are in the same $SO(2k+1)$ *orbit. Moreover,*

(1) if a_j, b_j *are not satisfying inequalities (5.1), then such* Y *does not exist,*

(2) if j *is the unique index from* $1, \ldots, k$ *such that* $a_j = b_m$ *for some* m*, then* $y_{2j-1} = y_{2j} = 0$.

Here is the even dimensional analogue.

Lemma 5.3.2. *For any real numbers*

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq b_{k-1} \geq |a_k| \tag{5.2}$$

there exist a real vector $Y = [y_1, \ldots, y_{2k-1}]^T$ *in* $\mathbb{R}$ *such that the skew symmetric matrices*

$$A := \left(\begin{array}{ccccc|c} L(b_1) & & & & & \\ & L(b_2) & & & & \\ & & \ddots & & & Y \\ & & & L(b_{k-1}) & & \\ & & & & 0 & \\ \hline & -Y^T & & & & 0 \end{array}\right) \text{ and } \left(\begin{array}{cccc} L(a_1) & & & \\ & L(a_2) & & \\ & & \ddots & \\ & & & L(a_k) \end{array}\right).$$

are in the same $SO(2k)$ orbit. Moreover,

(1) if a_j, b_j are not satisfying inequalities (5.2), then such Y does not exist,

(2) if j is the unique index from $1, \dots, k$ such that $b_j = a_m$ for some m, then $y_{2j-1} = y_{2j} = 0$.

5.4 The polytope for $SO(2n + 1)$

Now we are ready to describe the image of the Gelfand-Tsetlin functions for the case $G = SO(2n+1)$, in $\mathbb{R}^{n^2}$. Let $\{x_j^{(k)} \,|1 \le k \le 2n,\, 1 \le j \le \lfloor \frac{k}{2} \rfloor\}$ be the basis of $(\mathbb{R}^{n^2})^*$ dual to the standard basis of $\mathbb{R}^{n^2}$.

Proposition 5.4.1. *For $SO(2n+1)$ the image of the Gelfand-Tsetlin functions $\Lambda : \mathcal{O}_\lambda \to \mathbb{R}^{n^2}$ is the polytope, which we will denote by $\mathcal{P}$, defined by the following set of inequalities*

$$\begin{cases} x_1^{(2k)} \ge x_1^{(2k-1)} \ge x_2^{(2k)} \ge x_2^{(2k-1)} \ge \dots \ge x_{k-1}^{(2k)} \ge x_{k-1}^{(2k-1)} \ge |x_k^{(2k)}|, \\ x_1^{(2k+1)} \ge x_1^{(2k)} \ge x_2^{(2k+1)} \ge x_2^{(2k)} \ge \dots \ge x_k^{(2k+1)} \ge |x_k^{(2k)}|, \end{cases} \tag{5.3}$$

for all $k = 1, \dots, n$, where $x_j^{(2n+1)} = \lambda_j$.

Proof. The above proposition follows from consecutive applications of Propositions 5.3.1 and 5.3.2. We will show only the first two steps as the next ones are analogous. (Similar procedure for the unitary case is described in the proof of Proposition 4.3.1)

Take any point $a = (a_j^{(l)}) \in \mathbb{R}^{n^2}$ satisfying inequalities (5.3). Lemma 5.3.1

implies that there exist a real vector Y_1 such that the matrix

$$A_1 := \left(\begin{array}{cccc|c} L(a_1^{(2n)}) & & & & \\ & L(a_2^{(2n)}) & & & \\ & & \ddots & & Y_1 \\ & & & L(a_n^{(2n)}) & \\ \hline & & -Y_1^T & & 0 \end{array}\right)$$

is in the same $SO(2k+1)$ orbit as λ, i.e. $B_1\, A_1\, B_1^{-1} = \lambda$ for some matrix $B_1 \in SO(2n+1)$. Now we apply Lemma 5.3.2 to find a real vector Y_2 and a matrix $B_2 \in SO(2n)$ such that for the matrix

$$A_2 := \left(\begin{array}{ccccc|c} L(a_1^{(2n-1)}) & & & & & \\ & L(a_2^{(2n-1)}) & & & & \\ & & \ddots & & & Y_2 \\ & & & L(a_{n-1}^{(2n-1)}) & & \\ & & & & 0 & \\ \hline & & -Y_2^T & & & 0 \end{array}\right)$$

we have

$$B_2 A_2 B_2^{-1} = \begin{pmatrix} L(a_1^{(2n)}) & & & \\ & L(a_2^{(2n)}) & & \\ & & \ddots & \\ & & & L(a_n^{(2n)}) \end{pmatrix}.$$

Therefore the matrix

$$\left(\begin{array}{c|c} A_2 & B_2^{-1} Y_1 \\ \hline -Y_1^T B_2 & 0 \end{array}\right)$$

has desired values of the Gelfand-Tsetlin functions $a_*^{(2n)}, a_*^{(2n-1)}$ and is in $\mathcal{O}_\lambda$ as

$$B_1 \left(\begin{array}{c|c} B_2 & \\ \hline & 1 \end{array}\right) \left(\begin{array}{c|c} A_2 & B_2^{-1} Y_1 \\ \hline -Y_1^T B_2 & 0 \end{array}\right) \left(\begin{array}{c|c} B_2^{-1} & \\ \hline & 1 \end{array}\right) B_1^{-1}$$

$$= B_1 \left(\begin{array}{c|c} B_2 A_2 B_2^{-1} & Y_1 \\ \hline -Y_1^T & 0 \end{array} \right) B_1^{-1} = B_1\, A_1\, B_1^{-1} = \lambda.$$

Successively repeating similar steps, one can construct a matrix in $\mathcal{O}_\lambda$ with prescribed values of Gelfand-Tsetlin functions if only these values satisfy inequalities (5.3). $\square$

We can think of the Gelfand-Tsetlin polytope as the set of points whose coordinates fit into the following triangle of inequalities. Let the first row be given by $\lambda_1, \ldots, \lambda_n$ (or $|\lambda_n|$ in $SO(2n)$ case). Form next rows from the coordinates with the same superscript so that top left and right left neighbors of the coordinate $x_j^{(k)}$ are $x_j^{(k+1)}$ and $x_{j+1}^{(k+1)}$. The value of $x_j^{(k)}$ must be between the values of its top left and top right neighbors.

λ_1	λ_2	$\ldots$	λ_{n-1}	λ_n
$x_1^{(2n)}$	$x_2^{(2n)}$	$\ldots$	$x_{n-1}^{(2n)}$	$\lvert x_n^{(2n)} \rvert$
$x_1^{(2n-1)}$	$x_2^{(2n-1)}$	$\ldots$	$x_{n-1}^{(2n-1)}$	

$\ldots$

5.5 The polytope for $SO(2n)$

Situation for $G = SO(2n)$ is very similar. Let $\{x_j^{(k)}\,|1 \leq k \leq 2n-1,\, 1 \leq j \leq \lfloor \frac{k}{2} \rfloor\}$ be the basis of $(\mathbb{R}^N)^* = (\mathbb{R}^{n(n-1)})^*$ dual to the standard basis of $\mathbb{R}^N$.

Proposition 5.5.1. *For $SO(2n)$ the image of the Gelfand-Tsetlin functions $\Lambda : \mathcal{O}_\lambda \to \mathbb{R}^{n(n-1)}$ is the polytope, which we will denote by $\mathcal{P}$, defined by the following* *set of inequalities*

$$\begin{cases} x_1^{(2k)} \geq x_1^{(2k-1)} \geq x_2^{(2k)} \geq x_2^{(2k-1)} \geq \ldots \geq x_{k-1}^{(2k)} \geq x_{k-1}^{(2k-1)} \geq |x_k^{(2k)}|, \\ x_1^{(2k+1)} \geq x_1^{(2k)} \geq x_2^{(2k+1)} \geq x_2^{(2k)} \geq \ldots \geq x_k^{(2k+1)} \geq |x_k^{(2k)}|, \end{cases} \tag{5.4}$$

for all $k = 1, \ldots, n$, *where* $x_j^{(2n)} = \lambda_j$ *for* $j = 1, \ldots, n$.

Proof. Analogous to the proof of Proposition 5.4.1. □

Here we also can present these inequalities in the form of a triangle of inequalities similar to the $SO(2n+1)$ case above.

5.6 Isotropy weights of the Gelfand-Tsetlin action

Notice that $\Lambda(\lambda)$ is a vertex of $\mathcal{P}$. This is because at this point all the Gelfand-Tsetlin functions are equal to their upper bounds. If on the triangle of inequalities we connect by a line all coordinates of $\Lambda(\lambda)$ with the same values, then we obtain the picture in Figure 5.1.

We will analyze edges starting from $\Lambda(\lambda)$. For more details about identifying vertices and edges of the Gelfand-Tsetlin polytope, see Lemmas 4.3.3 and 4.3.6 or [39]. Basically, to obtain an edge starting from $\Lambda(\lambda)$, we pick one of the inequalities defining $\mathcal{P}$ that are equations at $\Lambda(\lambda)$, and consider the set of points in $\mathcal{P}$ satisfying all the same equations that $\Lambda(\lambda)$ satisfies, except possibly this chosen one. It is important to note that in this way we obtain ALL the edges starting from $\Lambda(\lambda)$. This procedure may not work if instead of $\Lambda(\lambda)$ we analyze a vertex V' of $\mathcal{P}$ such that $\Lambda^{-1}(V')$ is not in a subset of U.

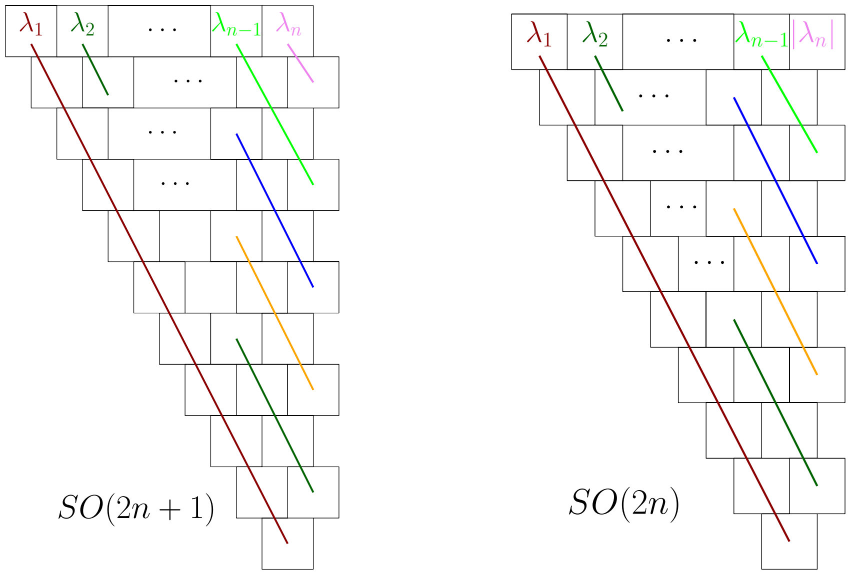


Figure 5.1: The values of Gelfand-Tsetlin functions for $\Lambda(\lambda)$ in $G = SO(2n+1)$ and $G = SO(2n)$ cases.

Pick any $k \in \{1, \dots, n\}$ for $G = SO(2n+1)$, or $k \in \{1, \dots, n-1\}$ for $G = SO(2n)$, and $j \in \{1, \dots, k\}$. Consider the set $E := E_j^{(2k)}$, that is the image of points where all the Gelfand-Tsetlin functions are equal to their upper bound, apart from the function $\lambda_j^{(2k)}$. That is, E is the line segment consisting of points $\mathbf{a} \in \mathbb{R}^N$ satisfying

$$\begin{aligned}
&a_l^{(m)} = \lambda_l \text{ for all } m \text{ and for all } l \neq j, \\
&a_j^{(m)} = \lambda_j \text{ for all } m > 2k, \\
&a_j^{(m)} = a_j^{(2k)} \text{ for all } m \leq 2k, \\
&a_j^{(2k)} \in [\lambda_{j+1}, \lambda_j] \text{ if } j < k, \\
&a_j^{(2k)} \in [-\lambda_k, \lambda_k] \text{ if } j = k.
\end{aligned} \tag{5.5}$$

The following graphical presentation (of the case $j < k$) can be helpful.

$$
\begin{array}{ccccccccccccc}
\lambda_{j-1} & & & \lambda_j & & & \lambda_{j+1} & & & & & & \\
 & = & & & = & & & = & & & & & \\
 & & a_{j-1}^{(2k+1)} & & & a_j^{(2k+1)} & & & a_{j+1}^{(2k+1)} & & & & \\
 & & & = & & & > & & & = & & & \\
 & & & & a_{j-1}^{(2k)} & & & a_j^{(2k)} & & & a_{j+1}^{(2k)} & & \\
 & & & & & = & & & = & & & = & \\
 & & & & & & a_{j-1}^{(2k-1)} & & & a_j^{(2k-1)} & & & a_{j+1}^{(2k-1)}
\end{array}
$$

The set E is an edge of $\mathcal{P}$. Proof of this fact in nearly identical as in the unitary case, described in Lemma 4.3.6. The vertex $\Lambda(\lambda)$ belongs to E. Denote by $\overset{-\circ}{E}$ the half open line segment: E minus the other endpoint, i.e. $\overset{-\circ}{E} = \Lambda(\lambda) \cup \text{ int } E$. From the definition of U it follows that if $q \in U$ and $\Lambda(q)$ belongs to a face $\mathcal{F}$ of the polytope $\mathcal{P}$, then $\Lambda^{-1}(\text{int } \mathcal{F})$ is in U. Therefore $\Lambda^{-1}(\overset{-\circ}{E})$ is also contained in U and is equipped with a smooth action of the Gelfand-Tsetlin torus. Below we analyze carefully which matrices are in $\Lambda^{-1}(\overset{-\circ}{E})$.

Lemma 5.6.1. $\Lambda^{-1}(\overset{-\circ}{E})$ *is a disc invariant under the action of the Gelfand-Tsetlin torus.*

To make the notation easier, we will write $A \sim B$ if A can be conjugated to B using a special orthogonal matrix of appropriate size. We also write $(A)_l$ for the $l \times l$ top left submatrix of A.

Proof. Applying the Propositions 5.3.2 and 5.3.1 we deduce that, in the $G =$

$SO(2n+1)$ case, $\Lambda^{-1}(\overset{\circ}{\bar{E}})$ consists of matrices M,

$$
\left(\begin{array}{c|c|c|c}
\begin{matrix} L(\lambda_1) & & & & & & \\ & \ddots & & & & & \\ & & L(\lambda_{j-1}) & & & & \\ & & & L(a_j^{(2k)}) & & & \\ & & & & L(\lambda_{j+1}) & & \\ & & & & & \ddots & \\ & & & & & & L(\lambda_k) \end{matrix} & P & & \\ \cline{1-2}
-P^T & 0 & Y & 0 \\ \cline{1-3}
-Y^T & & 0 & \\ \hline
0 & & & \begin{matrix} L(\lambda_{k+2}) & & & \\ & \ddots & & \\ & & L(\lambda_n) & \\ & & & 0 \end{matrix}
\end{array}\right)
$$

where

$$a_j^{(2k)} \in (\lambda_{j+1}, \lambda_j] \text{ if } j < k,$$

$$a_j^{(2k)} \in (-\lambda_k, \lambda_k] \text{ if } j = k,$$

and the real vectors P and Y are such that

$$(M)_{2k+1} \sim (\lambda)_{2k+1} \text{ and } (M)_{2k+2} \sim (\lambda)_{2k+2}.$$

Top right $(2k+2)\times(2n+1-2k-2)$ minor, and bottom left $(2n+1-2k-2)\times(2k+2)$ minor of M must be zero in order to have $(M)_l \sim (\lambda)_l$ for all $l > 2k+2$.

The Proposition 5.3.1 implies that the l-th coordinate of P, p_l, must be zero for all $l \neq 2j-1, 2j$. The traces of $(\,(M)_{2k+1}\,)^2$ and $(\,(\lambda)_{2k+1}\,)^2$ need to be equal, therefore $p_{2j-1}^2 + p_{2j}^2 = \lambda_j^2 - (a_j^{(2k)})^2$. This gives a circle of solutions for every choice of $a_j^{(2k)}$ in $(\lambda_{j+1}, \lambda_j)$ and the unique solution of $p_{2j-1}^2 = p_{2j}^2 = 0$ if $a_j^{(2k)} = \lambda_j$.

Now we analyze conditions on vector Y. We are to have that $(M)_{2k+2} \sim (\lambda)_{2k+2}$.

If $B \in SO(2k+1)$ is such that $B\,(M)_{2k+1}\,B^{-1} = (\lambda)_{2k+1}$, then

$$\left(\begin{array}{c|c} B & \\ \hline & 1 \end{array}\right)\left(\begin{array}{c|c} (M)_{2k+1} & Y \\ \hline -Y^T & 0 \end{array}\right)\left(\begin{array}{c|c} B^{-1} & \\ \hline & 1 \end{array}\right) = \left(\begin{array}{c|c} (\lambda)_{2k+1} & BY \\ \hline -Y^T B^{-1} & 0 \end{array}\right).$$

Therefore

$$\left(\begin{array}{c|c} (\lambda)_{2k+1} & BY \\ \hline -Y^T B^{-1} & 0 \end{array}\right) \sim \left(\begin{array}{c|c} (M)_{2k+1} & Y \\ \hline -Y^T & 0 \end{array}\right) = (M)_{2k+2} \sim (\lambda)_{2k+2}.$$

Denote the coordinates of the vector BY by $(v_1, \ldots, v_{2k+1})$. According to the Lemma 5.3.2 the condition that

$$\left(\begin{array}{c|c} (\lambda)_{2k+1} & BY \\ \hline -Y^T B^{-1} & 0 \end{array}\right) \sim (\lambda)_{2k+2}$$

implies that

$$v_1 = \ldots = v_{2k} = 0, \;\; v_{2k+1}^2 = \lambda_{k+1}^2.$$

Therefore

$$BY = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_{k+1} \end{pmatrix} \text{ or } BY = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -\lambda_{k+1} \end{pmatrix}. \tag{5.6}$$

For any choice of vector P, matrix B is uniquely defined only up to multiplication by an element of maximal torus of $SO(2k+1)$. Every element t of this torus has k (2×2) blocks of rotations on the diagonal, the last diagonal entry equal to 1, and all other entries zero. Therefore we have exactly two solutions to equation (5.6):

$$Y = B^{-1}(0, \ldots, 0, \pm\lambda_{k+1})^T.$$

For both of these solutions $(M)_{2k+2}$ has the desired characteristic polynomial $q_{2k+2}(t) = \prod_{l=1}^{k+1}(t^2 + \lambda_l^2)$. However only one of them will give us matrix in the

$SO(2k+2)$-orbit of $(\lambda)_{2k+2}$ as explained in the proof of Lemma 5.3.2 given in the Appendix B. This means that the vector Y is uniquely defined for every choice of vector P. Therefore the preimage of E is a disk.

If $G = SO(2n)$ the proof is nearly identical. Just delete last row and column in the presentation of M. Conditions on X and Y stay the same.

$\square$

Now we analyze the isotropy weights of the action.

Lemma 5.6.2. *The weight of the Gelfand-Tsetlin torus on $T_\lambda \Lambda^{-1}(\overset{-\circ}{E})$ is $-w_j^{(2k)}$, where*

$$w_j^{(2k)} := \sum_{l=2j}^{2k} x_j^{(l)}$$

and E is an edge of $\mathcal{P}$ equal to the vector

$$\begin{aligned} \langle\, (e_j - e_{j+1})^\vee, \lambda \,\rangle \; w_j^{(2k)} &= (\lambda_j - \lambda_{j+1})\, w_j^{(2k)} && \text{if } j < k, \\ \langle e_k^\vee, \lambda \rangle \, w_k^{(2k)} &= 2\,\lambda_k\, w_k^{(2k)} && \text{if } j = k \end{aligned}$$

Remark 5.6.3. *Lemmas 5.6.2 and 5.6.5 find all the isotropy weights of the Gelfand-Tsetlin torus action at λ. Consider the lattice generated by the isotropy weights. Notice that for the special orthogonal group the isotropy weights are primitive vectors in the lattice they generate. This fact has an important consequence. To apply Proposition 3.2.6 we need to find c such that the set $E_j^{(2k)}$ is equal to the $(-c)$ times the isotropy weight along E. In our case, the c we need is the same as the lattice length of E with respect to the weight lattice, exactly because all the isotropy weights are primitive. We want to point out that this is not necessarily true in general.*

Proof. To make notation easier we concentrate on the case $G = SO(2n+1)$. The proof for $G = SO(2n)$ is nearly identical.

An element $R \in T_{SO(l)}$ of maximal torus of $SO(l)$, with $l \geq 2k+2$, acts on a matrix $M \in \Lambda^{-1}(\overset{-\circ}{E})$ by conjugation with

$$\left(\begin{array}{c|c} B^{-1}RB & \\ \hline & I_{2n+1-l} \end{array}\right)$$

where $B \in SO(l)$ is such that $B(M)_l B^{-1} = (\lambda)_l \in (\mathfrak{t}_{SO(l)})^*_+$. This action is trivial. To see this denote by S the bottom left $(n+1-l) \times (n+1-l)$ submatrix of M. Then

$$\left(\begin{array}{c|c} B^{-1}RB & \\ \hline & I \end{array}\right) \left(\begin{array}{c|c} (M)_l & 0 \\ \hline 0 & S \end{array}\right) \left(\begin{array}{c|c} B^{-1}R^{-1}B & \\ \hline & I \end{array}\right) = \left(\begin{array}{c|c} (M)_l & 0 \\ \hline 0 & S \end{array}\right).$$

Therefore the functions $x_*^{(l)}$ with $l \geq 2k+2$ are constant on $\Lambda^{-1}(\overset{-\circ}{E})$.

Now consider the action of maximal torus of $SO(2k+1)$, $T_{SO(2k+1)}$. Let $B \in SO(2k+1)$ be such that $B(M)_{2k+1}B^{-1} = (\lambda)_{2k+1} \in (\mathfrak{t}_{SO(2k+1)})^*_+$. Denote by S the bottom right $(2n-2k) \times (2n-2k)$ submatrix of M. An element R of $T_{SO(2k+1)}$ has the form

$$R = \begin{pmatrix} R(\alpha_1) & & & \\ & \ddots & & \\ & & R(\alpha_k) & \\ & & & 1 \end{pmatrix}$$

and it acts on M by

$$\left(\begin{array}{c|c} B^{-1}RB & \\ \hline & I_{2n-2k} \end{array}\right) \left(\begin{array}{c|c} (M)_{2k+1} & \left(\begin{array}{c|c} Y & 0 \end{array}\right) \\ \hline \left(\begin{array}{c} -Y^T \\ \hline 0 \end{array}\right) & S \end{array}\right) \left(\begin{array}{c|c} B^{-1}R^{-1}B & \\ \hline & I_{2n-2k} \end{array}\right)$$

$$= \left(\begin{array}{c|c} (M)_{2k+1} & \left(B^{-1}R^{-1}BY \,\middle|\, 0 \right) \\ \hline \left(\begin{array}{c} -Y^T\,(BRB^{-1})^T \\ \hline 0 \end{array} \right) & S \end{array} \right).$$

Recall that

$$BY = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm\lambda_{k+1} \end{pmatrix}, \text{ so } RBY = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm\lambda_{k+1} \end{pmatrix} = BY, \text{ and } B^{-1}RBY = Y.$$

Therefore this action is also trivial.

Now let $T_{SO(l)}$ be the chosen maximal torus of $SO(l)$ for $l \leq 2k$. An element of $T_{SO(l)}$ is of the form $R = \mathrm{diag}(R(\alpha_1), \ldots, R(\alpha_{\frac{l-1}{2}}), 1)$ or $R = \mathrm{diag}(R(\alpha_1), \ldots, R(\alpha_{\frac{l}{2}}))$. Note that for $l \leq 2k$ the submatrix $(M)_l$ is in the positive Weyl chamber $(\mathfrak{t}_{SO(l)})^*_+$. Therefore an element $R \in T_{SO(l)}$ acts on M simply by conjugation. Denote by W the top right $l \times (2n+1-l)$ submatrix of M, and by S the bottom right $(2n+1-l) \times (2n+1-l)$ submatrix of M. With this notation, the action of R is the following.

$$\left(\begin{array}{c|c} R & \\ \hline & I \end{array} \right) \left(\begin{array}{c|c} (M)_l & W \\ \hline -W^T & S \end{array} \right) \left(\begin{array}{c|c} R^{-1} & \\ \hline & I \end{array} \right) = \left(\begin{array}{c|c} (M)_l & RW \\ \hline -(RW)^T & S \end{array} \right).$$

Only two of the columns of W maybe be non-zero: column $(2k+2)$-nd contains the first l coordinates of the vector Y, and column $(2k+1)$-st contains the first l coordinates of the vector P. We already showed that the only possibly non-zero entries of the vector P are p_{2j-1} and p_{2j}. Therefore the submatrix W has possibly non-zero entries in the $(2k+1)$-st column if and only if $l \geq 2j$. In this case, notice that only the j-th circle of $T_{SO(l)}$ acts on the $(2k+1)$-st column, with speed 1.

$$R \begin{pmatrix} 0 \\ \vdots \\ 0 \\ p_{2j-1} \\ p_{2j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ R(\alpha_j) \begin{pmatrix} p_{2j-1} \\ p_{2j} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Recall that the vector Y is uniquely determined by the vector P. Therefore, when we analyze the action of T on $T_\lambda \Lambda^{-1}(\overset{-\circ}{E})$, independent variables are only in W, S, P. This means that the weight of the Gelfand-Tsetlin torus on $T_\lambda \Lambda^{-1}(\overset{-\circ}{E})$ is

$$-w_j^{(2k)} := -\sum_{l=2j}^{2k} x_j^{(l)}.$$

The conditions (5.5) imply that the set E is an edge of the polytope $\mathcal{P}$ given by the vector

$$(\lambda_j - \lambda_{j+1})\, w_j^{(2k)} = \langle\, (e_j - e_{j+1})^\vee, \lambda\,\rangle\; w_j^{(2k)},$$

if $j < k$, and by the vector

$$\langle\, e_k^\vee, \lambda\,\rangle\; w_k^{(2k)} = 2\,\lambda_k\, w_k^{(2k)}$$

if $j = k$.

Recall that for $G = SO(2n+1)$ we were taking k from the set $\{1, \ldots, n\}$, and for $G = SO(2n)$ we had $k \in \{1, \ldots, n-1\}$. Therefore the collection of lattice lengths of edges $E_j^{(2k)}$ is

$$\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\lambda_1, \ldots, 2\lambda_n\} \text{ for } G = SO(2n+1)$$

$$\{\lambda_1 - \lambda_2, \dots, \lambda_{n-2} - \lambda_{n-1}, 2\lambda_1, \dots, 2\lambda_{n-1}\} \text{ for } G = SO(2n).$$

□

Now we analyze the other edges starting from $\Lambda(\lambda)$. We still think of $(\mathbb{R}^N)^*$ as having coordinates $\{x_j^{(k)}\}$, for appropriate k, j. Pick any $k < n$ and $j \leq k$. Consider the set $F := F_j^{(2k+1)}$, that is the image of points where all the Gelfand-Tsetlin functions are equal to their upper bound, apart from the function $\lambda_j^{(2k+1)}$. That is, F is the set of points satisfying

$$\begin{aligned} a_l^{(m)} &= \lambda_l \text{ for all } m \text{ and all } l \neq j, \\ a_j^{(m)} &= \lambda_j \text{ for all } m \geq 2k+2, \\ a_j^{(m)} &= a_j^{(2k+1)} \text{ for all } m \leq 2k+1, \end{aligned} \tag{5.7}$$

where $a_j^{(2k+1)} \in [\lambda_{j+1}, \lambda_j]$, unless $G = SO(2n)$ and $k = n-1, j = n-1$ when $a_{n-1}^{(2n-1)} \in [|\lambda_n|, \lambda_{n-1}]$. Here is graphical presentation

$$\begin{array}{ccccccccccccc} \lambda_{j-1} & & & & \lambda_j & & & & \lambda_{j+1} & & & & \\ & \searrow^{=} & & & & \searrow^{=} & & & & \searrow^{=} & & & \\ & & a_{j-1}^{(2k+2)} & & & & a_j^{(2k+2)} & & & & a_{j+1}^{(2k+2)} & & \\ & & & \searrow^{=} & & & & \searrow^{\geq} & & & & \searrow^{=} & \\ & & & & a_{j-1}^{(2k+1)} & & & & a_j^{(2k+1)} & & & & a_{j+1}^{(2k+1)} \\ & & & & & \searrow^{=} & & & & \searrow^{=} & & & \searrow^{=} \\ & & & & & & a_{j-1}^{(2k)} & & & & a_j^{(2k)} & & a_{j+1}^{(2k)} \end{array} \tag{5.8}$$

Again, similarly to the unitary case (Lemma 4.3.6), one can show that F is an edge of $\mathcal{P}$. Let $\overset{-\circ}{F} = \Lambda(\lambda) \cup \text{int } F$ denote the edge F without the second endpoint. From the definition of U and the fact that $\Lambda(\lambda) \in U$, it follows that the set $\Lambda^{-1}(\overset{-\circ}{F})$ is also contained in U. Therefore it is equipped with a smooth action of the Gelfand-Tsetlin torus.

Lemma 5.6.4. $\Lambda^{-1}(\overset{-\circ}{F})$ *is a disc invariant under the action of the Gelfand-Tsetlin torus.*

Proof. In this proof we again concentrate on the case $G = SO(2n+1)$ as the procedure for $G = SO(2n)$ is analogous.

$\Lambda^{-1}(\overset{-\circ}{F})$ consists of matrices M of the form

$$
\left(\begin{array}{cccccccc|c|cccc}
L(\lambda_1) & & & & & & & 0 & & & & & \\
 & \ddots & & & & & & & & & & & \\
 & & L(\lambda_{j-1}) & & & & & & & & & & \\
 & & & L(a_j^{(2k+1)}) & & & & \vdots & Y & & 0 & & \\
 & & & & L(\lambda_{j+1}) & & & & & & & & \\
 & & & & & \ddots & & & & & & & \\
 & & & & & & L(\lambda_k) & 0 & & & & & \\
0 & & & \cdots & & & 0 & 0 & & & & & \\
\hline
 & & & -Y^T & & & & & 0 & & & & \\
\hline
 & & & & & & & & & L(\lambda_{k+2}) & & & \\
 & & & 0 & & & & & & & \ddots & & \\
 & & & & & & & & & & & L(\lambda_n) & \\
 & & & & & & & & & & & & 0
\end{array}\right)
$$

where $a_j^{(2k+1)} \in (\lambda_{j+1}, \lambda_j]$, unless $G = SO(2n)$ and $k = j = n-1$ when $a_{n-1}^{(2n-1)} \in (|\lambda_n|, \lambda_{n-1}]$, and the real vector Y is such that $(M)_{2k+2} \sim (\lambda_{2k+2})$. Notice that the top right and bottom left minors have to be zero to have that $(M)_l \sim (\lambda)_l$ for any $l > 2k+2$. The Proposition 5.3.2 implies that

$$y_l = 0 \text{ for all } l \neq 2j-1,\ 2j,\ 2k+1$$

and that y_{2k+1} and $y_{2j-1}^2 + y_{2j}^2$ are uniquely defined. If $a_j^{(2k+1)} = \lambda_j$, then $y_{2j-1} = y_{2j} = 0$, and $y_{2k+1} = -\lambda_j$. For each $a_j^{(2k+1)} \in (\lambda_{j+1}, \lambda_j)$ we have a circle worth of choices for $y_{2j-1}, y_{2j} = 0$, and unique choice for y_{2k+1}. Therefore $\Lambda^{-1}(\overset{-\circ}{F})$ is a 2-dimensional disk. □

Now we analyze the isotropy weights of the action.

Lemma 5.6.5. *The weight of the Gelfand-Tsetlin torus on* $T_\lambda \Lambda^{-1}(\overset{-\circ}{F})$ *is* $-w_j^{(2k+1)}$*, where*

$$w_j^{(2k+1)} := \sum_{l=2j}^{2k+1} x_j^{(l)}$$

and F *is an edge of* $\mathcal{P}$ *equal to the vector*

$$\langle\, (e_j - e_{j+1})^\vee, \lambda \,\rangle \; w_j^{(2k+1)} = (\lambda_j - \lambda_{j+1})\, w_j^{(2k+1)},$$

unless $G = SO(2n)$ *and* $k = n-1, j = n-1$ *when* F *isan edge of* $\mathcal{P}$ *equal to the vector* $(\lambda_{n-1} - |\lambda_n|)\, w_{n-1}^{(2n-1)}$.

Proof. For simplicity of notation assume that $G = SO(2n+1)$. To obtain the proof in the case $G = SO(2n)$ one only needs to delete the last row and column of M.

First consider the action of $T_{SO(l)}$ with $l \geq 2k+2$. An element $R \in T_{SO(l)}$ of the maximal torus of $SO(l)$ acts on matrix $M \in \Lambda^{-1}(\overset{-\circ}{F})$ by conjugation with

$$\left(\begin{array}{c|c} B^{-1}RB & \\ \hline & I_{2n+1-l} \end{array}\right)$$

where $B \in SO(l)$ is such that $B(M)_l B^{-1} = (\lambda)_l \in (\mathfrak{t}_{SO(l)})^*_+$. Denote by S the bottom left $(n+1-l) \times (n+1-l)$ submatrix of M. Have

$$\left(\begin{array}{c|c} B^{-1}RB & \\ \hline & I \end{array}\right) \left(\begin{array}{c|c} (M)_l & 0 \\ \hline 0 & S \end{array}\right) \left(\begin{array}{c|c} B^{-1}R^{-1}B & \\ \hline & I \end{array}\right) = \left(\begin{array}{c|c} (M)_l & 0 \\ \hline 0 & S \end{array}\right).$$

Therefore the functions $x_*^{(l)}$ for $l \geq 2k+2$ are constant on $\Lambda^{-1}(\overset{-\circ}{F})$ and the action is trivial.

Now consider the action of $T_{SO(l)}$, for $l \leq 2k+1$. An element R of $T_{SO(l)}$ has the form

$$R = \begin{pmatrix} R(\alpha_1) & & & \\ & \ddots & & \\ & & R(\alpha_{\lfloor \frac{l}{2} \rfloor}) & \\ & & & 1 \end{pmatrix} \text{ or } R = \begin{pmatrix} R(\alpha_1) & & \\ & \ddots & \\ & & R(\alpha_{\frac{l}{2}}) \end{pmatrix}.$$

Denote by W the top right $l \times (2n+1-l)$ submatrix of M, and by S the bottom right $(2n+1-l) \times (2n+1-l)$ submatrix of M. Notice that $(M)_l \in (\mathfrak{t}_{SO(l)})^*_+$. Therefore the action of R is the following.

$$\left(\begin{array}{c|c} R & \\ \hline & I \end{array}\right) \left(\begin{array}{c|c} (M)_l & W \\ \hline -W^T & S \end{array}\right) \left(\begin{array}{c|c} R^{-1} & \\ \hline & I \end{array}\right) = \left(\begin{array}{c|c} (M)_l & RW \\ \hline -(RW)^T & S \end{array}\right).$$

Only one of the columns of W maybe be non-zero: column $(2k+2)$-nd contains the first l coordinates of the vector Y. We already showed that the only possibly non-zero entries of the vector Y are y_{2j-1}, y_{2j} and y_{2k+1}. Therefore the submatrix W has possibly non-zero entries in the $(2k+1)$-st column if and only if $l \geq 2j-1$. The action does not change the $(2k+1, 2k+1)$-th entry of M, namely y_{2k+1}. This is because this entry is a part of W only in the case $l = 2k+1$. In that case, R acts on this entry by multiplication by its $(2k+1, 2k+1)$-th entry, which is equal to 1. There is however nontrivial action on the $(2k+1, 2j-1)$-th and $(2k+1, 2j)$-th entries of M if only $l \geq 2j$. The j-th circle of $T_{SO(l)}$ acts on the $(2k+1)$-st column, rotating them with speed 1.

$$R\begin{pmatrix} 0 \\ \vdots \\ 0 \\ y_{2j-1} \\ y_{2j} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ R(\alpha_j)\begin{pmatrix} y_{2j-1} \\ y_{2j} \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This means that the weight of the Gelfand-Tsetlin torus on $T_\lambda \Lambda^{-1}(\overset{-\circ}{F})$ is

$$-w_j^{(2k+1)} := -\sum_{l=2j}^{2k+1} x_j^{(l)}.$$

The condition (5.7) implies that F is an edge of $\mathcal{P}$ equal to the vector

$$(\lambda_j - \lambda_{j+1}) w_j^{(2k+1)},$$

unless $G = SO(2n)$ and $k = n-1, j = n-1$ when F is equal to the vector $(\lambda_{n-1} - |\lambda_n|) w_{n-1}^{(2n-1)}$.

Note the collection of lattice lengths of edges $F_j^{(2k+1)}$ is

$$\{\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n\} \text{ for } G = SO(2n+1),$$

$$\{\lambda_1 - \lambda_2, \dots, \lambda_{n-2} - \lambda_{n-1}, \lambda_{n-1} - |\lambda_n|\} \text{ for } G = SO(2n).$$

□

Remark 5.6.6. *In this work we explicitly calculated the isotropy weights. The fact that they are primitive vectors proves that the action is effective.*

Note that there is also another way to obtain the isotropy weights of the action on $T_\lambda \mathcal{O}_\lambda$. We already know that on the set $U \subset \mathcal{O}_\lambda$ where the Gelfand-Tsetlin functions are smooth, they integrate to a Hamiltonian torus action with momentum map $\Lambda_{|U}$, turning U into a (non-compact) toric manifold. It is easy to see that neighborhood of λ is in U. Therefore the isotropy weights of the Gelfand-Tsetlin action on $T_\lambda \mathcal{O}_\lambda$ are (negative) multiples of the primitive generators of edges of $\mathcal{P}$ starting from $\Lambda(\lambda)$. A priori we don't know if the multiple is (-1). One could use the description of the Gelfand-Tsetlin action (see Section 3.5) to show that the action is effective, as we did in Lemma 4.2.2 for the unitary case. This would imply that the isotropy weights at the fixed point λ are $(-1)\cdot$(primitive vectors generating edges of $\mathcal{P}$ starting from $\Lambda(\lambda)$), because for the case of proper group actions on connected manifolds effectiveness implies local effectiveness (see Corollary B.42 in Appendix B of [15]).

We summarize the above section in the following corollary.

Corollary 5.6.7. *Every edge of $\mathcal{P}$ starting from $\Lambda(\lambda)$ has lattice length equal to at least* $\min\{\, |\langle \alpha^\vee, \lambda \rangle| \,;\alpha^\vee$ *a coroot*$\}$.

Proof. Direct application of Lemmas 5.6.2 and 5.6.5 would give us lower bounds for lattice lengths equal to

$$\min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\,\lambda_1, \ldots, 2\,\lambda_n\} \text{ if } G = SO(2n+1),$$

$$\min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} - |\lambda_n|, 2\,\lambda_1, \ldots, 2\,\lambda_{n-2}, 2\,\lambda_{n-1}\} \text{ if } G = SO(2n).$$

Inequalities coming from the fact that λ is in the positive Weyl chamber imply that the minimum over the first set is equal to

$$\min\{\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n, 2\,\lambda_n\},$$

while the minimum over the second set is equal to

$$\min\{\lambda_1 - \lambda_2, \dots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}.$$

For example,

$$2\,\lambda_{n-1} > \lambda_{n-1} + |\lambda_n| = \lambda_{n-1} \pm \lambda_n,$$

$$\lambda_{n-1} - |\lambda_n| = \min\{\lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n\}.$$

Analysis of root systems done in Subsection 5.1 gives that in both cases the minimum is equal to $\min\{\,|\langle\alpha^\vee, \lambda\rangle|\ ; \alpha^\vee$ a coroot$\}$. □

5.7 The proof of lower bounds for Gromov width of $SO(n)$ coadjoint orbits

.

Proof. To prove the Theorem 5.0.4, we will proceed as in the Example 3.2.7. Recall that $2N$ is the dimension of the orbit $\mathcal{O}_\lambda$, where $N = n^2$ if $G = SO(2n+1)$ and $N = n(n-1)$ if $G = SO(2n)$. The point $\lambda \in \mathcal{O}_\lambda$ is a fixed point for the action of the Gelfand-Tsetlin torus. Moreover, preimage of $\Lambda(\lambda)$ is a single fixed point, $\{\lambda\}$. From the definition of U it follows that $\lambda \in U$ and that

$$\mathcal{T} := \bigcup_{\substack{\mathcal{F} \text{ face of } \mathcal{P} \\ \Lambda(\lambda) \in \mathcal{F}}} \Lambda^{-1}(\text{rel-int } \mathcal{F}) \subset U.$$

Moreover the action of the Gelfand-Tsetlin torus on $\mathcal{T}$ is centered around $\Lambda(\lambda)$. Denote the isotropy weights of the action $T_{GT} \curvearrowright T_\lambda\mathcal{T} = T_\lambda\mathcal{O}_\lambda$ by $-\eta_1, \dots, -\eta_N$. Let $r = \min\{\,|\langle\alpha^\vee, \lambda\rangle|\ ; \alpha^\vee$ a coroot$\}$. Corollary 5.6.7 shows that lattice lengths of all edges starting from $\Lambda(\lambda)$ are at least r. Therefore

$$\Lambda(\lambda) + \pi \sum_{i=1}^{N} |z_i|^2 \eta_i \in \mathcal{T}$$

for any $z \in B_r^{2N}$, ball of capacity r. Proposition 3.2.6 gives symplectic embedding of the ball of the capacity r. Therefore r is the lower bounds for Gromov width. $\square$

APPENDIX A

CENTERED REGIONS FOR NON-SIMPLY LACED GROUPS

Let G be a compact, connected, non-simply laced Lie group, and T be a choice of maximal torus. Choose positive Weyl chamber and let $p \in (\mathfrak{t})^*_+$ be a point in the interior of this chamber. Consider the coadjoint orbit M, through p, and denote by $2N$ the dimension of M. Coadjoint action of the maximal torus T on M is Hamiltonian. Denote the momentum map for this action by $\mu : M \to \mathfrak{t}^*$. Let $\mathcal{Q}_1 = \mu(\overline{\{x \in M;\ \dim(T \cdot x) = 1\}})$ be the image of the 1-skeleton of M. Then $\mathcal{Q}_1$ is an N-valent graph contained in the polytope $\mu(M)$. (This follows from the fact that T acts on M in a GKM fashion. For more about GKM manifolds see [11], [36]). Note that $p = \mu(p)$ is the fixed point of this action. Let $\mathcal{T} \subset \mathfrak{t}^*$ be such that $\mu^{-1}(\mathcal{T})$ is centered around p. In particular, for any edge E of $\mathcal{Q}_1$, $E \cap \mathcal{T} \neq \emptyset$ if and only if $p \in E$. One could apply Proposition 3.2.6 and obtain some lower bound for Gromov width of M as explained in the Example 3.2.7. In this section we show that in the case of non-simply laced group, this lower bound is weaker (i.e. lower) then the predicted Gromov width of the coadjoint orbit,

$$\min\left\{|\langle\alpha^\vee, p\rangle|\,;\, \alpha^\vee \text{ a coroot }\right\}.$$

This observation makes our result for the $SO(2n+1)$ coadjoint orbits even more interesting, as the root system for $SO(2n+1)$ is non-simply laced.

Let $\alpha, \beta \in \mathfrak{t}^*$ be two roots of Euclidean lengths $||\alpha|| > ||\beta||$. For any root η let $\sigma_\eta : \mathfrak{t}^* \to \mathfrak{t}^*$ denote the reflection through hyperplane perpendicular to η. Then the image of α under the reflection σ_β,

$$\sigma_\beta(\alpha) := \alpha - 2\frac{\langle\beta, \alpha\rangle}{\langle\beta, \beta\rangle}\beta = \alpha - \langle\beta^\vee, \alpha\rangle\,\beta,$$

is also a root (see condition R3 in III.9.2 of [17]). What is more,

$$||\alpha|| = ||\sigma_\beta(\alpha)||.$$

For any root η, the points p and $\sigma_\eta(p)$ are connected by an edge of $\mathcal{Q}_1$. In particular there exist and edge in $\mathcal{Q}_1$ joining p with a point

$$\sigma_\alpha(p) := p - 2\frac{\langle\alpha, p\rangle}{\langle\alpha,\alpha\rangle}\alpha.$$

Call this edge E_1. Denote by E_2 the edge in $\mathcal{Q}_1$ from $\sigma_\beta(p)$ in the direction of $\sigma_\beta(\alpha)$, joining $\sigma_\beta(p)$ with a vertex $\sigma_{\sigma_\beta(\alpha)}(\sigma_\beta(p))$. The definition of centered region implies that the edge E_2 has to be disjoint from $\mathcal{T}$. We want to know how big portion of the edge E_1 is contained in $\mathcal{T}$. Definitely the intersection of edges E_1 and E_2 is not in $\mathcal{T}$. These edges intersect if there exists t, s such that

$$\sigma_\beta(p) + s\sigma_\beta(\alpha) = p + t\alpha.$$

This means:

$$\begin{aligned} p + t\alpha = \sigma_\beta(p) + s\sigma_\beta(\alpha) &= p - 2\frac{\langle\beta, p\rangle}{\langle\beta,\beta\rangle}\beta + s\left(\alpha - 2\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\beta\right), \\ t\alpha &= -2\,\frac{\langle\beta, p\rangle}{\langle\beta,\beta\rangle}\,\beta + s\alpha - 2s\,\frac{\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\,\beta, \\ (t-s)\alpha &= -\,\frac{2}{\langle\beta,\beta\rangle}\,\left(\,\langle\beta, p\rangle + s\langle\beta,\alpha\rangle\,\right)\,\beta. \end{aligned}$$

As α and β are roots of different lengths, the only solution to the above equation is when $t = s$ and $\langle\beta, p\rangle + s\langle\beta,\alpha\rangle = 0$. The point p was chosen from the interior of the positive Weyl chamber, thus $\langle\beta, p\rangle \neq 0$. The solution exists if also $\langle\beta,\alpha\rangle \neq 0$ and is

$$t = s = -\,\frac{\langle\beta, p\rangle}{\langle\beta,\alpha\rangle} = -2\,\frac{\langle\beta, p\rangle}{\langle\beta,\beta\rangle}\,\left(\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}\right)^{-1}.$$

The values of $\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle}$ can only be $0, \pm1, \pm2, \pm3$ ([17, Chapter 9]). By the above, we know it is not 0. If $\frac{2\langle\beta,\alpha\rangle}{\langle\beta,\beta\rangle} = \pm1$, then $||\alpha|| = ||\beta||$ ([17]) contrary to our assumptions.

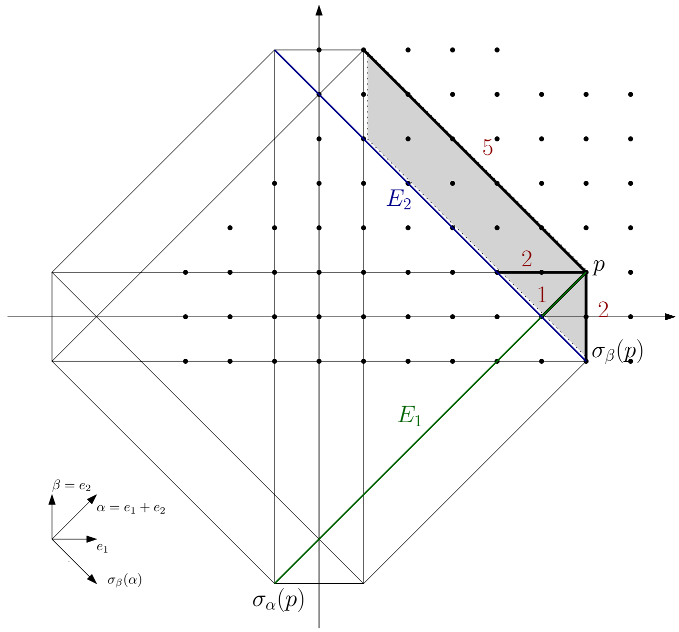


Figure A.1: One-skeleton of $SO(5)$ coadjoint orbit

Thus it has to be ± 2 or ± 3. In both cases we get that the solution

$$|t| = 2 \left| \frac{\langle \beta, p \rangle}{\langle \beta, \beta \rangle} \left(\frac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} \right)^{-1} \right| < 2 \left| \frac{\langle \beta, p \rangle}{\langle \beta, \beta \rangle} \right| = \langle \beta^\vee, p \rangle.$$

This means that the portion of the edge E_1 contained in $\mathcal{T}$ has length strictly less then $\langle \beta^\vee, p \rangle \, ||\alpha||$. Therefore the lower bound for Gromov width that we can obtain from the centered region $\mathcal{T}$ is less then $\langle \beta^\vee, p \rangle$ (the isotropy weight along the sphere $\mu^{-1}(E_1)$ is α). It may happen that the minimum $\min\{ \left| \langle \alpha_j^\vee, p \rangle \right| ; \alpha_j \text{ a coroot}\}$ is equal to $\langle \beta^\vee, p \rangle$. In this case, the predicted lower bound of Gromov width of the orbit is strictly greater then the bound one could get from the centered region for the standard action of the maximal torus.

For example, consider $SO(5)$ coadjoint orbit M through a block diagonal matrix $p = \text{diag}(L(6), L(1), 1)$ in $\mathfrak{so}(5)^*$. The momentum polytope $\mu(M)$, together with the image of 1-skeleton are presented on Figure A.1. Edge lengths are given with respect to the weight lattice. Preimage of the shaded region is the maximal subset centered around p for the standard action of maximal torus. The portion of edge E_1 contained in this region is of length

$$\left| \frac{\langle e_2, (6,1) \rangle}{\langle e_2, e_1 + e_2 \rangle} \right| = 1.$$

Therefore using this centered region, we can construct embeddings of a ball of capacity at most 1. Regions centered at the other fixed points would give the same result. The Theorem 5.0.4 provides a better lower bound, because the pairings of p with coroots $e_1^\vee, e_2^\vee, (e_1 + e_2)^\vee, (e_1 - e_2)^\vee$ give (respectively): 12, 2, 7, 5 and minimum of this set is 2.

APPENDIX B

PROOFS OF LEMMAS 5.3.1 AND 5.3.2

Proof Lemma 5.3.1. We are given real numbers satisfying inequalities (5.1):

$$b_1 \geq a_1 \geq b_2 \geq a_2 \geq \ldots \geq a_{k-1} \geq b_k \geq |a_k|$$

and we are to show that there exist a real vector $Y = [y_1, \ldots, y_{2k}]^T$ in $\mathbb{R}^{2k}$ such that the skew symmetric matrices

$$A := \left(\begin{array}{cccc|c} L(a_1) & & & & \\ & L(a_2) & & & Y \\ & & \ddots & & \\ & & & L(a_k) & \\ \hline & & -Y^T & & 0 \end{array}\right) \text{ and } S := \left(\begin{array}{cccc|c} L(b_1) & & & & \\ & L(b_2) & & & 0 \\ & & \ddots & & \\ & & & L(b_k) & \\ \hline & & 0 & & 0 \end{array}\right).$$

are in the same $SO(2k+1)$ orbit.

Proof. Two matrices in $\mathfrak{so}(2k+1)^*$ are in the same $SO(2k+1)$ orbit if and only if they have the same characteristic polynomial. The characteristic polynomial for A, $\chi_A(t)$ is

$$\chi_A(t) = \begin{vmatrix} t & a_1 & & & & -y_1 \\ -a_1 & t & & & & -y_2 \\ & & \ddots & & & \vdots \\ & & & t & a_k & -y_{2k-1} \\ & & & -a_k & t & -y_{2k} \\ y_1 & y_2 & \ldots & y_{2k-1} & y_{2k} & t \end{vmatrix}$$

$$= -y_1 \left(-a_1 \begin{vmatrix} 0 & t & a_2 & & & \\ 0 & -a_2 & t & & & \\ & & & \ddots & & \\ & & & & t & a_k \\ & & & & -a_k & t \\ y_2 & y_3 & y_4 & \dots & y_{2k-1} & y_{2k} \end{vmatrix} - y_1 \begin{vmatrix} t & 0 & 0 & & & 0 \\ 0 & t & a_2 & & & \\ 0 & -a_2 & t & & & \\ & & & \ddots & & \\ & & & & t & a_k \\ 0 & & & & -a_k & t \end{vmatrix} \right) +$$

$$+y_2 \left(-a_1 \, (y_1 \prod_{j \neq 1} (t^2 + a_j^2)) \; + y_2 \, (t \prod_{j \neq 1} (t^2 + a_j^2)) \right) + \ldots + t \prod_{j=1}^{k} (t^2 + a_j^2)$$

$$= (a_1 y_1 y_2 + y_1^2 t - a_1 y_1 y_2 + y_2^2 t) \prod_{j=2}^{k} (t^2 + a_j^2) + \ldots + (y_{2k-1}^2 + y_{2k}^2) \, t \prod_{j=1}^{k-1} (t^2 + a_j^2) + t \prod_{j=1}^{k} (t^2 + a_j^2)$$

$$= t \sum_{l=1}^{k} (y_{2l-1}^2 + y_{2l}^2) \prod_{j \neq l} (t^2 + a_j^2) + t \prod_{j=1}^{k} (t^2 + a_j^2).$$

The characteristic polynomial for S is $\chi_S(t) = t \prod_{j=1}^{k} (t^2 + b_j^2)$. Simplifying t we get the equation

$$\sum_{l=1}^{k} (y_{2l-1}^2 + y_{2l}^2) \prod_{j \neq l} (t^2 + a_j^2) + \prod_{j=1}^{k} (t^2 + a_j^2) = \prod_{j=1}^{k} (t^2 + b_j^2). \tag{B.1}$$

Case 1. Assume first that a and b are regular, that is

$$b_1 > a_1 > b_2 > a_2 > \ldots > a_{k-1} > b_k > |a_k|. \tag{B.2}$$

Then we can write the Equation B.1 as

$$\prod_{j=1}^{k} (t^2 + a_j^2) \left(1 + \sum_{l=1}^{k} \frac{y_{2l-1}^2 + y_{2l}^2}{t^2 + a_l^2} \right) = \prod_{j=1}^{k} (t^2 + b_j^2).$$

Substituting $t = \pm i b_s$ for $s = 1, \ldots, k$ we get the system of equations

$$\forall_{s=1,\ldots,k} \quad \left(1 + \sum_{l=1}^{k} \frac{y_{2l-1}^2 + y_{2l}^2}{-b_s^2 + a_l^2} \right) = 0.$$

Introduce the notation

$$w_l = y_{2l-1}^2 + y_{2l}^2.$$

Solving the Equation B.1 for regular case is equivalent to finding nonnegative solution in $w's$ to the system of linear conditions

$$\forall_{s=1,\dots,k} \quad \sum_{l=1}^{k} \frac{w_l}{b_s^2 - a_l^2} = 1. \tag{B.3}$$

Denote by $M = [m_{sl}]$, $m_{sl} = \frac{1}{b_s^2 - a_l^2}$ the matrix of this system of equations. Matrices of this type are called Cauchy matrices. In 1959 Schechter ([34]) proved that

$$\det M = \frac{\prod_{i=2}^{k} \prod_{j=1}^{i-1} (b_i^2 - b_j^2)(a_i^2 - a_j^2)}{\prod_{i=1}^{k} \prod_{j=1}^{k} (b_i^2 - a_j^2)} \neq 0.$$

Moreover, he showed that the inverse matrix $M^{-1} = [m^{ij}]$ is given by the formula

$$m^{ij} = (b_j^2 - a_i^2) B_j(a_i^2) A_i(b_j^2)$$

where $B_j(x), A_i(x)$ are the Lagrange polynomials for (b_i^2) and (a_j^2). This means that

$$A_i(x) = \frac{A(x)}{A'(a_i^2)(x - a_i^2)} \text{ and } B_i(x) = \frac{B(x)}{B'(b_i^2)(x - b_i^2)},$$

with

$$A(x) = \prod_{i=1}^{k} (x - a_i^2) \text{ and } B(x) = \prod_{i=1}^{k} (x - b_i^2).$$

Therefore, the solution to our system is given by (see also [29, Ch VIII])

$$w_l = -\frac{\prod_{j=1}^{n} (a_l^2 - b_j^2)}{\prod_{j \neq l,\, j=1}^{n} (a_l^2 - a_j^2)}.$$

Notice that, due to inequalities B.2, the numerator is positive if and only if $\sharp\{j;\ j \geq l\}$ is even, while the denominator is positive if and only if $\sharp\{j;\ j > l\}$ is even. Thus w_l is always positive, as required.

If the inequalities 5.1 are not satisfied, then some w_l is negative and therefore there is no solution in y's.

Case 2. Suppose that b is regular but a is not, that is there exists j_0 such that $a_{j_0} = b_m$ (that is $m = j_0$ or $j_0 + 1$).

Suppose for a moment that a_{j_0} is the only coordinate of a that is equal to b_m, that is, $b_m \neq a_j$ for all $j \neq j_0$. Then, substituting $t = ib_m$ in Equation (B.1), we get that

$$w_{j_0} \prod_{j \neq j_0} (a_j^2 - b_m^2) = 0,$$

thus $w_{j_0} = 0$. Therefore $y_{2j_0-1} = y_{2j_0} = 0$. This means that ever term in Equation B.1 contains a factor $(t^2 + b_m^2)$ and we can simplify this factor. Then we arrive at the equation with just $k-1$ variables $w_1, \ldots, \widehat{w_{j_0}}, \ldots w_k$ and $2k-2$ parameters which are now regular or at least less degenerate. Repeating this step if necessary, we get to the equation similar to Equation (B.1) that is regular (and has less variables and parameters).

Now suppose that a_{j_0} is not the only coordinate of a that is equal to b_m. As b is regular, this can happen if and only if $a_{m-1} = b_m = a_m$. Now every term in Equation B.1 contains a factor $(t^2 + b_m^2)$. We simplify this factor. Introducing new variables and parameters for $j = 1, \ldots, k-1$

$$\widetilde{a_j} = \begin{cases} a_j & j < m \\ a_{j+1} & j \geq m \end{cases}, \quad \widetilde{b_j} = \begin{cases} b_j & j < m \\ b_{j+1} & j \geq m \end{cases}, \quad \widetilde{w_j} = \begin{cases} w_j & j < m-1 \\ w_{m-1} + w_m & j = m-1 \\ w_{j+1} & j > m-1 \end{cases}$$

we get the equation

$$\sum_{l=1}^{k-1} (\widetilde{w_l}) \prod_{j \neq l} (t^2 + \widetilde{a_j}^2) + \prod_{j=1}^{k} (t^2 + \widetilde{a_j}^2) = \prod_{j=1}^{k} (t^2 + \widetilde{b_j}^2),$$

which is regular or at least less degenerate then the one we started with. Repeating the above steps if necessary, we obtain a regular equation and can find the solution using the inverse of appropriate Cauchy matrix.

Case 3. Now we deal with the case of b non-regular. Again we will try to reduce it, step by step, to the regular case. Suppose that $b_j = b_{j+1}$ for some index j. Then a_j is forced by the inequalities (5.3) to be also equal to b_j.

If no other a_l is equal to a_j, then substituting $t = ib_j$ we obtain that $w_j = 0$. Therefore $y_{2j-1} = y_{2j} = 0$. This means that every term in the Equation (B.1) contains the factor $(t^2 + b_j^2)$. Simplifying this factor we arrive at the equation that is one step less degenerate.

If there are other a_l also equal to a_j, then every term in the Equation (B.1) contains the factor $(t^2 + b_j^2)$. We can simplify this factor and, similarly to the case above, introduce new variables to obtain an equation that is one step less degenerate.

It is clear from the proof that if there exists unique index j such that $a_j = b_m$, then $y_{2j-1} = y_{2j} = 0$. □

Proof of Lemma 5.3.2. Now we proof the even dimensional analogue, that is Lemma 5.3.2. We are given real numbers satisfying inequalities (5.2) recalled below:

$$a_1 \geq b_1 \geq a_2 \geq b_2 \geq \ldots \geq b_{k-1} \geq |a_k|$$

and we are to find a real vector $Y = [y_1, \ldots, y_{2k-1}]^T$ in $\mathbb{R}^{2k-1}$ such that the skew symmetric matrices

$$A := \left(\begin{array}{ccccc|c} L(b_1) & & & & & \\ & L(b_2) & & & & \\ & & \ddots & & & Y \\ & & & L(b_{k-1}) & & \\ & & & & 0 & \\ \hline & & -Y^T & & & 0 \end{array}\right) \text{ and } S := \begin{pmatrix} L(a_1) & & & \\ & L(a_2) & & \\ & & \ddots & \\ & & & L(a_k) \end{pmatrix}.$$

are in the same $SO(2k)$ orbit.

If two matrices in $\mathfrak{so}(2k)^*$ are in the same $SO(2k)$ orbit, then in particular they have the same characteristic polynomial. We could proceed as in the odd dimensional case and start with comparing the characteristic polynomials of A and S. This would again involve, for regular case, solving some linear system of equations, with unknowns $\{y_{2l-1}^2 + y_{2l}^2, y_{2k-1}\}$, given by a Cauchy matrix. By the result of Schechter we know the inverse matrix, but it is still computationally challenging to show that the solution is nonnegative (except possibly at y_{2k-1}). For this reason, and to present another approach, we will proceed differently. We will transform the problem into a problem for the unitary case. In particular we use the following Lemma, which is a slight strengthening of Lemma 4.3.2 (Lemma 3.5 in [30], see also [14]).

Lemma B.0.1. *For any real numbers* $\mu_1 \geq \nu_1 \geq \mu_2 \geq \ldots \geq \mu_{2k-1} \geq \nu_{2k-1} \geq \mu_{2k}$ *there exist* $x_1, \ldots, x_{2k-1}$ *in* $\mathbb{C}$ *and* x_{2k} *in* $\mathbb{R}$ *such that the Hermitian matrix*

$$A := \begin{pmatrix} \nu_1 & & 0 & \bar{x}_1 \\ & \ddots & & \vdots \\ 0 & & \nu_{2k-1} & \bar{x}_{2k-1} \\ x_1 & \ldots & x_{2k-1} & x_{2k} \end{pmatrix},$$

has eigenvalues $\mu_1, \ldots, \mu_{2k}$. *Inequalities between* μ_j *and* ν_j *are necessary for such* $x_1, \ldots, x_{k+1}$ *to exist. Moreover*

1. The solution is not unique: gives conditions only on the values $|x_1|, \ldots, |x_{2k-1}|$ *and* x_{2k}. *The coordinate* x_{2k} *is uniquely defined by trace condition.*

If $\nu_1, \ldots, \nu_{2k-1}$ *are distinct then also* $|x_1|, \ldots, |x_{2k-1}|$ *are uniquely defined.*

If $\nu_l = \ldots = \nu_{l+s}$, *then only the value* $|x_l|^2 + \ldots + |x_{l+s}|^2$ *is uniquely defined.*

2. If m *is the unique index such that* $\mu_j = \nu_m$ *then* $x_m = 0$.

3. Suppose that $\nu_l = -\nu_{2k-l}$, $\mu_l = -\mu_{2k+1-l}$, *for* $l = 1, \ldots, k$, *(so* $\nu_k = 0$*). Then there exists a solution with* $|x_l| = |x_{2k-l}|$ *for* $l = 1, \ldots, k$ *and* $x_{2k} = 0$.

Proof. Here we only prove the additional, strengthening statements 1, 2 and 3.

1. Equation

$$\det(A - t\ Id) = \Pi_j(\mu_j - t) \tag{B.4}$$

only involves $|x_1|, \ldots, |x_{2k-1}|$ and x_{2k}. Guillemin and Sternberg while proving the existence of solution (see [14]), also showed that $|x_1|, \ldots, |x_{2k-1}|$ and x_{2k} are uniquely defined in a generic situation, that is if $\nu_1, \ldots, \nu_{2k-1}$ are distinct. If they are not distinct, then dividing equation B.4 by appropriate factors $(t - \nu_l)^s$ we reduce the problem to the generic one, with new variable $y = |x_l|^2 + \ldots + |x_{l+s}|^2$, instead of variables $|x_l|, \ldots, |x_{l+s}|$. The same reduction is explained in details in Case 2 of the proof of Lemma 5.3.1 above.

2. The characteristic polynomial of matrix A is

$$t \prod_{l=1}^{2k-1} (t - \nu_l) - \sum_{i=1}^{2k-1} |x_i|^2 \prod_{l \neq i} (t - \nu_l).$$

This must be equal to $\prod_{l=1}^{2k}(t - \mu_l)$, the characteristic polynomial of S. Therefore, substituting $t = \mu_j$ we get

$$0 = t \prod_{l=1}^{2k-1} (\mu_j - \nu_l) - \sum_{i=1}^{2k-1} |x_i|^2 \prod_{l \neq i} (\mu_j - \nu_l) = -|x_m|^2 \prod_{l \neq m} (\mu_j - \nu_l).$$

This means that $x_m = 0$, because m is the unique index such that $\mu_j = \nu_m$.

3. The trace of A is $0 = \sum_{l=1}^{2k} \mu_l = \sum_{l=1}^{2k-1} \nu_l + x_{2k}$, thus $x_{2k} = 0$. Notice that conjugating A with a matrix of permutation switching l with $2k - l$, for $l = 1, \ldots, k$, (which is in $U(n)$), will give the matrix A', with the same eigenvalues as A.

$$A' := \begin{pmatrix} \nu_{2k-1} & & 0 & \bar{x}_{2k-1} \\ & \ddots & & \vdots \\ 0 & & \nu_1 & \bar{x}_1 \\ x_{2k-1} & \ldots & x_1 & 0 \end{pmatrix} = \begin{pmatrix} -\nu_1 & & 0 & \bar{x}_{2k-1} \\ & \ddots & & \vdots \\ 0 & & -\nu_{2k-1} & \bar{x}_1 \\ x_{2k-1} & \ldots & x_1 & 0 \end{pmatrix}$$

Eigenvalues of $(-A')$ are $\{-\mu_l;\ l = 1, \ldots 2k\} = \{\mu_l;\ l = 1, \ldots 2k\}$, the same as of the matrix A. Therefore the sequence $(-x_{2k-1}, \ldots, -x_1, 0)$ is also a solution to question in the Lemma B.0.1. In case $\nu_1, \ldots, \nu_{2k-1}$ are distinct, then the absolute values of the solution are uniquely defined. Therefore $|x_l| = |-x_{2k-l}| = |x_{2k-l}|$ for $l = 1, \ldots, k$. If $\nu_l = \ldots = \nu_{l+s}$, then also $\nu_{2k-l} = \ldots = \nu_{2k-l-s}$ and equation B.4 imposes the same conditions on $|x_l|^2 + \ldots + |x_{l+s}|^2$ and on $|x_{2k-l}|^2 + \ldots + |x_{2k-l-s}|^2$. Therefore we can alter the solution to satisfy $|x_l| = |x_{2k-l}|$ for $l = 1, \ldots, k$. $\square$

Now we are ready to prove Lemma 5.3.2.

Proof. Applying the Lemma B.0.1 we get that there exists $X = (x_1, \ldots, x_{2k-1}) \in \mathbb{C}^{2k-1}$, such that the matrix

$$\left(\begin{array}{cccccccc|c}
b_1 & & & & & & & & \overline{x}_1 \\
& b_2 & & & & & & & \overline{x}_3 \\
& & \ddots & & & & & & \vdots \\
& & & b_{k-1} & & & & & \overline{x}_{2k-3} \\
& & & & 0 & & & & \overline{x}_{2k-1} \\
& & & & & -b_{k-1} & & & \overline{x}_{2k-2} \\
& & & & & & \ddots & & \vdots \\
& & & & & & & -b_2 & \overline{x}_4 \\
& & & & & & & -b_1 & \overline{x}_2 \\
\hline
x_1 & x_3 & \ldots & x_{2k-3} & x_{2k-1} & x_{2k-2} & \ldots \; x_4 & x_2 & 0
\end{array}\right)$$

has eigenvalues $(a_1, \ldots, |a_k|, -|a_k|, \ldots, -a_1)$, and $|x_{2j-1}| = |x_{2j}|$ for $j = 1, \ldots, k-1$. Conjugating with a permutation matrix (which is also in $U(2k)$) will not change the eigenvalues. Therefore there exist a matrix $B \in U(2k)$ such that

$$B \left(\begin{array}{cccccc|c}
b_1 & & & & & & \overline{x}_1 \\
& -b_1 & & & & & \overline{x}_2 \\
& & \ddots & & & & \vdots \\
& & & b_{k-1} & & & \overline{x}_{2k-3} \\
& & & & -b_{k-1} & & \overline{x}_{2k-2} \\
& & & & & 0 & \overline{x}_{2k-1} \\
\hline
x_1 & x_2 & \ldots & x_{2k-3} & x_{2k-2} & x_{2k-1} & 0
\end{array}\right) B^{-1} = \begin{pmatrix}
a_1 & & & & \\
& -a_1 & & & \\
& & \ddots & & \\
& & & a_k & \\
& & & & -a_k
\end{pmatrix} \quad \text{(B.5)}$$

Notice that

$$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = 2 \begin{pmatrix} ix & 0 \\ 0 & -ix \end{pmatrix}.$$

Define the matrices $J_m \in U(2m)$, $L_m \in U(2m+1)$ in the following way

$$J_m := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & & & \\ i & 1 & & & \\ & & \ddots & & \\ & & & 1 & i \\ & & & i & 1 \end{pmatrix}, \quad L_m := \left(\begin{array}{c|c} J_m & 0 \\ \hline 0 & 1 \end{array}\right).$$

We will suppress m from the notation when the dimension is understood. Have

$$J \begin{pmatrix} L(a_1) & & \\ & \ddots & \\ & & L(a_k) \end{pmatrix} J^{-1} = \begin{pmatrix} ia_1 & & & & \\ & -ia_1 & & & \\ & & \ddots & & \\ & & & ia_k & \\ & & & & -ia_k \end{pmatrix}.$$

Also

$$i \left(\begin{array}{cccccc|c} b_1 & & & & & & \overline{x}_1 \\ & -b_1 & & & & & \overline{x}_2 \\ & & \ddots & & & & \vdots \\ & & & b_{k-1} & & & \overline{x}_{2k-3} \\ & & & & -b_{k-1} & & \overline{x}_{2k-2} \\ & & & & & 0 & \overline{x}_{2k-1} \\ \hline x_1 & x_2 & \dots & x_{2k-3} & x_{2k-2} & x_{2k-1} & 0 \end{array}\right)$$

$$= \left(\begin{array}{c|c} L & \\ \hline & 1 \end{array}\right) \left(\begin{array}{c|c} L^{-1} & \\ \hline & 1 \end{array}\right) i \left(\begin{array}{cccc|c} b_1 & & & & \\ & \ddots & & & \\ & & -b_{k-1} & & X^* \\ & & & 0 & \\ \hline & X & & & 0 \end{array}\right) \left(\begin{array}{c|c} L & \\ \hline & 1 \end{array}\right) \left(\begin{array}{c|c} L^{-1} & \\ \hline & 1 \end{array}\right) =$$

$$= \left(\begin{array}{c|c} L & \\ \hline & 1 \end{array}\right) \left(\begin{array}{cccc|c} L(b_1) & & & & \\ & \ddots & & & \\ & & L(b_{k-1}) & & iL^{-1}\,X^* \\ & & & 0 & \\ \hline & & iX\,L & & 0 \end{array}\right) \left(\begin{array}{c|c} L^{-1} & \\ \hline & 1 \end{array}\right) =$$

$$= \left(\begin{array}{c|c} L & \\ \hline & 1 \end{array}\right) \; A \; \left(\begin{array}{c|c} L^{-1} & \\ \hline & 1 \end{array}\right)$$

where

$$A := \left(\begin{array}{cccc|c} L(b_1) & & & & \\ & \ddots & & & \\ & & L(b_{k-1}) & & iL^{-1}\,X^* \\ & & & 0 & \\ \hline & & iX\,L & & 0 \end{array}\right)$$

Together with Equation B.5 this gives that

$$S = \begin{pmatrix} L(a_1) & & \\ & \ddots & \\ & & L(a_k) \end{pmatrix} = J^{-1}\, i \begin{pmatrix} a_1 & & & & \\ & -a_1 & & & \\ & & \ddots & & \\ & & & a_k & \\ & & & & -a_k \end{pmatrix} J$$

$$= J^{-1}\, B \left(\begin{array}{c|c} L & \\ \hline & 1 \end{array}\right) \; A \; \left(\begin{array}{c|c} L^{-1} & \\ \hline & 1 \end{array}\right) B^{-1}\, J$$

Notice that we can choose X so that A is not only in $\mathfrak{u}(2k)^*$ but also in $\mathfrak{so}(2k)^*$.

If $x_j = r_j + iw_j$, then

$$Y := i\,L^{-1}\,X^* = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & & & & \\ -i & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & -i & \\ & & & -i & 1 & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} w_1 + ir_1 \\ w_2 + ir_2 \\ \cdots \\ w_{2k-3} + ir_{2k-3} \\ w_{2k-2} + ir_{2k-2} \\ w_{2k-1} + ir_{2k-1} \end{pmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} w_1 + r_2 + i(r_1 - w_2) \\ w_2 + r_1 + i(r_2 - w_1) \\ \cdots \\ w_{2k-3} + r_{2k-2} + i(r_{2k-3} - w_{2k-2}) \\ w_{2k-2} + r_{2k-3} + i(r_{2k-2} - w_{2k-3}) \\ w_{2k-1} + ir_{2k-1} \end{pmatrix}.$$

This vector is real if and only if $r_{2j-1} = w_{2j}$ and $r_{2j} = w_{2j-1}$, for $j = 1, \ldots, k-1$ and $r_{2k-1} = 0$. According to Lemma B.0.1, only the absolute values of x_j's are uniquely defined and $|x_{2j-1}| = |x_{2j}|$ for $j = 1, \ldots, k-1$. Therefore, if we take any $x_{2j-1} = r_{2j-1} + iw_{2j-1}$ with prescribed absolute value, and put $x_{2j} = w_{2j-1} + ir_{2j-1}$, $x_{2k-1} = |x_{2k-1}|$ then vectors $i\,L_k^{-1}\,X^*$ and its transpose conjugate $-i\,X\,L_k$ are real and $A \in \mathfrak{so}(2k)^*$.

Moreover, the only two matrices in the positive Weyl chamber with the same characteristic polynomial as the matrix A are

$$S = \begin{pmatrix} L(a_1) & & \\ & \ddots & \\ & & L(a_k) \end{pmatrix}, \; \widetilde{S} := \begin{pmatrix} L(a_1) & & \\ & \ddots & \\ & & L(-a_k) \end{pmatrix}.$$

These matrices are $O(2k)$ conjugate but not $SO(2k)$ conjugate. Let $R \in O(2k)$ denote the diagonal matrix with all 1's on diagonal except the last, $2k$-th, entry that is equal to -1. Then

$$\widetilde{S} = R\,S\,R^{-1}.$$

If the matrix A we have constructed is in fact in the $SO(2k)$ orbit through $\widetilde{S}$, then the matrix

$$R\,A\,R^{-1} = \left(\begin{array}{ccccc|c} L(b_1) & & & & & \\ & L(b_2) & & & & \\ = & & \ddots & & & -Y \\ & & & L(b_k) & & \\ & & & & 0 & \\ \hline & & Y^T & & & 0 \end{array}\right)$$

is in the $SO(2k)$ orbit through S. Therefore, if Y is the vector such that matrices A and S have the same characteristic polynomial, then either Y or $-Y$ is the solution we need. Again we have that $y_{2j-1}^2 + y_{2j}^2 = 2r_{2j-1}^2 + 2w_{2j}^2 = 2|x_{2j-1}|^2$ and $y_{2k-1} = \pm|x_{2k-1}|$ are uniquely defined. $\square$

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