EXTREME VALUE THEORY FOR LONG RANGE DEPENDENT STABLE RANDOM FIELDS

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Abstract. We study the extremes for a class of a symmetric stable random fields with long range dependence. We prove functional extremal theorems both in the space of sup measures and in the space of cadlag functions of several variables. The limits in both types of theorems are of a new kind, and only in a certain range of parameters these limits have the Fréchet distribution.

1. Introduction

Extreme value theorems describe the limiting behaviour of the largest values in increasingly large collections of random variables. The classical extremal theorems, beginning with Fisher and Tippett (1928) and Gnedenko (1943), deal with the extremes of i.i.d. (independent and identically distributed) random variables. The modern extreme value theory techniques allow us to study the extremes of dependent sequences; see Leadbetter et al. (1983) and the expositions in Coles (2001) and de Haan and Ferreira (2006). The effect of dependence on extreme values can be restricted to a loss in the effective sample size, through the extremal index of the sequence. When the dependence is sufficiently long, more significant changes in extreme value may occur; see e.g. Samorodnitsky (2004), Owada and Samorodnitsky (2015b). The present paper aims to contribute to our understanding of the effect of memory on extremes when the time is of dimension larger than 1, i.e. for random fields.

We consider a discrete time stationary random field $\mathbf{X} = \left(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\right)$. For $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}^d$ we would like to study the extremes of the random field over growing hypercubes of the type

$$[\mathbf{0}, \mathbf{n}] = \left\{\mathbf{0} \le \mathbf{k} \le \mathbf{n}\right\},\ \mathbf{n} \to \boldsymbol{\infty}\,,$$

where $\mathbf{0}$ is the vector with zero coordinates, the notation $\mathbf{s} \le \mathbf{t}$ for vectors $\mathbf{s} = (s_1, \ldots, s_d)$ and $\mathbf{t} = (t_1, \ldots, t_d)$ means that $s_i \le t_i$ for all $i = 1, \ldots, d$, and the notation $\mathbf{n} \to \boldsymbol{\infty}$ means that all d components of the vector $\mathbf{n}$ tend to infinity. Denote

$$M_{\mathbf{n}} = \max_{\mathbf{0} \le \mathbf{k} \le \mathbf{n}} X_{\mathbf{t}}\,.$$

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What limit theorems does the array $(M_{\mathbf{n}})$ satisfy? It was shown by Leadbetter and Rootzén (1998) that under appropriate strong mixing conditions, only the classical three types of limiting distributions (Gumbel, Fréchet and Weibull) may appear (even when forcing $\mathbf{n} \to \boldsymbol{\infty}$ along a monotone curve). In the case when the marginal distributions of the field $\mathbf{X}$ have regularly varying tails, this allows only the Fréchet distribution as a limit.

In this paper we will discuss only random fields with regularly varying tails, in which case the experience from the classical extreme value theory tells us to look for limit theorems for the type

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}} \Rightarrow Y \quad \text{as } \mathbf{n} \to \boldsymbol{\infty} \tag{1.1}$$

for some nondegenerate random variable Y. The regular variation of the marginal distributions means that

$$P(X(\mathbf{0}) > x) = x^{-\alpha} L(x),\ \alpha > 0,\ L \text{ slowly varying}, \tag{1.2}$$

see e.g. Resnick (1987). Notice that the assumption is only on the right tail of the distribution since, in most cases, one does not expect a limit theorem for the partial maxima as in (1.1) to be affected by the left tail of $X(\mathbf{0})$.

If the random field $\mathbf{X}$ consists of i.i.d. random variables satisfying the regular variation condition (1.2), then the classical extreme value theory tells us that the convergence in (1.1) holds if we choose

$$b_{\mathbf{n}} = \inf\left\{x > 0 :\ P(X(\mathbf{0}) > x) \le (n_1 \cdots n_d)^{-1}\right\}, \tag{1.3}$$

in which case the limiting random variable Y has the standard Fréchet distribution. We are interested in understanding how the spatial dependence in the random field $\mathbf{X}$ affects the scaling in and the distribution of the limit not only in (1.1), but in its functional versions, which can be stated in different spaces, for example in the space $D(\mathbb{R}^d_+)$ of right continuous, with limits along monotone paths, functions (see Straf (1972)), or in the space of random sup measures $\mathcal{M}(\mathbb{R}^d_+)$; see O'Brien et al. (1990). We will describe the relevant spaces below.

If the time is one-dimensional, and the memory in the stationary process is short, then the standard normalization (1.3) is still the appropriate one, and the limits both in (1.1) and its functional versions change only through a change in a multiplicative constant; see Samorodnitsky (2016) and references therein. However, when the memory becomes sufficiently long, both the order of magnitude of the normalization in the limit theorems changes, and the nature of the limit changes as well; see Samorodnitsky (2004) and Owada and Samorodnitsky (2015b). Furthermore, the limit may even stop having the Fréchet distribution (or Fréchet marginal distributions, in the functions limit theorems); see Samorodnitsky and Wang (2017). It is reasonable to expect that similar phenomena happen for random fields, but because it is harder to quantify how long the memory is when the time is not one-dimensional, less is known in this case.

In this paper we will concentrate on the case where the random field $\mathbf{X}$ is a symmetric α-stable (SαS) random field, $0 < \alpha < 2$. Recall that this means that every finite linear combination of the values of the values of the random field has a one-dimensional SαS distribution, i.e. has a characteristic function of the form $\exp\{-\sigma^\alpha |\theta|^\alpha\}$, $\theta \in \mathbb{R}$, where

$\sigma \in [0, \infty)$ is a scale parameter that depends on the linear combination; see Samorodnitsky and Taqqu (1994). The marginal distributions of SαS random fields satisfy the regular variation assumption (1.2) with $0 < \alpha < 2$ that coincides with the index of stability. In this case a series of results on the relation between the sizes of the extremes of stationary SαS random fiels and certain ergodic-theoretical properties of the Lévy measures of these fields is due to Parthanil Roy and his coworkers; see Roy and Samorodnitsky (2008), Chakrabarty and Roy (2013), Sarkar and Roy (2016). These results are made possible because of the connection between the structure of the SαS random fields and ergodic theory established by Rosiński (2000).

This paper contributes to understanding the extremal limit theorems for SαS random fields and their connection to the dynamics of the Lévy measures. In this sense our paper is related to the ideas of Rosiński (2000). However, we will restrict ourselves to certain Markov flows. This will allow us to avoid, to a large extent, the language of ergodic theory, and state everything in purely probabilistic terms. There is not doubt, however, that our results could be extended to more general dynamical systems acting on the Lévy measures of SαS random fields. The generality in which work is sufficient to demonstrate the new phenomena that may arise in extremal limit theorems for random fields with long range dependence. We will exhibit new types of limits, some of which will have non-Fréchet distributions, both in the space of random sup measures and in the space $D(\mathbb{R}^d_+)$.

This paper is organized as follows. In Section 2 we introduce the class of stationary symmetric α-stable random fields we will study in this paper. In Section 3 we provide some background on random closed sets and random sup measures, and describe the limiting random sup measure that appears as the weak limit the extremal theorem in Section 4. Finally, in Section 5 we prove versions of our extremal limit theorems in the space $D(\mathbb{R}^d_+)$.

Notation: For a function g on an arbitrary set with values in a linear space we denote the set of zeros of g by $\mathcal{Z}(g)$. Arithmetic operations involved vectors are performed component-wise. Thus, if $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$, then, say, $\mathbf{xy} = (x_1 y_1, \ldots, x_d y_d)$. This extends to sets: if $A = A_1 \times \cdots \times A_d$, then $\mathbf{x}A = x_1 A_1 \times \cdots \times x_d A_d$.

2. A SαS random field with long range dependence

We start with a construction of a family of stationary SαS random fields, $0 < \alpha < 2$, whose memory has a natural finite-dimensional parameterization. It is an extension to random fields of models considered before in the case of one-dimensional time; see e.g. Resnick et al. (2000), Samorodnitsky (2004), Owada and Samorodnitsky (2015a,b), Owada (2016) and Lacaux and Samorodnitsky (2016).

We start with d σ-finite, infinite measures on $\left(\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0})\right)$ defined by

$$\mu_i := \sum_{k \in \mathbb{Z}} \pi_k^{(i)} P_k^{(i)}\,, \tag{2.1}$$

where for $i = 1, \ldots, d$, $P_k^{(i)}$ is the law of an irreducible aperiodic null-recurrent Markov chain $(Y_n^{(i)})_{n \geq 0}$ on $\mathbb{Z}$ starting at $Y_0^{(i)} = k \in \mathbb{Z}$. Further, $(\pi_k^{(i)})_{k \in \mathbb{Z}}$ is its unique (infinite)

invariant measure satisfying $\pi_0^{(i)} = 1$. Given this invariant measure, we can extend the probability measures $P_k^{(i)}$ from measures on $\mathbb{Z}^{\mathbb{N}_0}$ to measures on $\mathbb{Z}^{\mathbb{Z}}$ which, in turn, allows us to extend the measure μ_i in (2.1) to $\mathbb{Z}^{\mathbb{Z}}$ as well. We will keep using the same notation as in (2.1).

We will work with the product space

$$(E, \mathcal{E}) = \Big(\mathbb{Z}^{\mathbb{Z}} \times \cdot \times \mathbb{Z}^{\mathbb{Z}},\ \mathcal{B}(\mathbb{Z}^{\mathbb{Z}}) \times \cdot \times \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\Big)$$

of d copies of $\big(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\big)$, on which we put the product, σ-finite, infinite, measure

$$\mu = \mu_1 \times \cdot \times \mu_d\,.$$

The key assumption is a regular variation assumption on the return times of the Markov chains $(Y_n^{(i)})_{n\geq 0}$, $i = 1, \ldots, d$. For $\mathbf{x} = (\ldots, x_{-1}, x_0, x_1, x_2 \ldots) \in \mathbb{Z}^{\mathbb{Z}}$ we define the first return time to the origin by $\varphi(\mathbf{x}) = \inf\{n \geq 1 :\ x_n = 0\}$. We assume that for $i = 1, \ldots, d$ we have

$$P_0^{(i)}(\varphi > n) \in RV_{-\beta_i} \tag{2.2}$$

for some $0 < \beta_i < 1$. This implies that

$$\begin{aligned} &\mu_i\left(\{\mathbf{x} :\ x_k = 0 \text{ for some } k = 0, 1, \ldots, n\}\right) \\ &\sim \sum_{k=1}^{n} P_0^{(i)}(\varphi > k) \sim (1-\beta_i)^{-1} n P_0^{(i)}(\varphi > n) \in RV_{1-\beta_i}\,. \end{aligned} \tag{2.3}$$

See Resnick et al. (2000).

On $\mathbb{Z}^{\mathbb{Z}}$ there is a natural left shift operator

$$T\big((\ldots, x_{-1}, x_0, x_1, x_2 \ldots)\big) = (\ldots, x_0, x_1, x_2, x_3 \ldots)\,.$$

It is naturally extended to a group action of $\mathbb{Z}^d$ on E as follows. Writing an element $\mathbf{x} \in E$ as $\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})$ with $\mathbf{x}^{(i)} = (\ldots, x_{-1}^{(i)}, x_0^{(i)}, x_1^{(i)}, x_2^{(i)} \ldots)\big) \in \mathbb{Z}^{\mathbb{Z}}$ for $i = 1, \ldots, d$, we set for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$,

$$T^{\mathbf{n}}\mathbf{x} = (T^{n_1}\mathbf{x}^{(1)}, \ldots, T^{n_d}\mathbf{x}^{(d)}) \in E\,. \tag{2.4}$$

Even though we are using the same notation T for operators acting on different spaces, the meaning will always be clear from the context. Note that each individual left shift T on $\big(\mathbb{Z}^{\mathbb{N}_0}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}_0}), \mu_j\big)$ is measure preserving (because each $(\pi_i^{(j)})_{i\in\mathbb{Z}}$ is an invariant measure.) It is also conservative and ergodic by Theorem 4.5.3 in Aaronson (1997). Therefore, the group action $\mathcal{T} = \{T^{\mathbf{n}} : \mathbf{n} \in \mathbb{Z}^d\}$ is conservative, ergodic and measure preserving on $(E, \mathcal{E}, \mu)$.

Equipped with a measure preserving group action on the space $(E, \mathcal{E})$ we can now define a stationary symmetric α-stable random field by

$$X_{\mathbf{n}} = \int_E f \circ T^{\mathbf{n}}(\mathbf{x})\, M(d\mathbf{x}), \quad \mathbf{n} \in \mathbb{Z}^d\,, \tag{2.5}$$

where M is a $S\alpha S$ random measure on $(E, \mathcal{E})$ with control measure μ, and

$$f(\mathbf{x}) = \mathbf{1}(\mathbf{x}^{(i)} \in A,\ i = 1, \ldots, d),\ \mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)})\,. \tag{2.6}$$

where $A = \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}} : x_0 = 0\}$. Clearly, $f \in L^\alpha(\mu)$, which guarantees that the integral in (2.5) is well defined. We refer the reader to Samorodnitsky and Taqqu (1994) for general information on stable processes and integrals with respect to stable measures, and to Rosiński (2000) on more details on stationary stable random fields and their representations.

The random field model defined by (2.5) is attractive because the key parameters involve in its definition have a clear intuitive meaning: the index of stability $0 < \alpha < 2$ is responsible for the heaviness of the tails, while $0 < \beta_i < 1$, $i = 1, \ldots, d$ (defined in (2.2)) are responsible for the "length of the memory". The latter claim is not immediately obvious, but its (informal) validity will become clearer in the sequel.

The following array of positive numbers will play the crucial role in the extremal limit theorems in this paper. Denote for $n = 1, 2, \ldots$ and $i = 1, \ldots, d$,

$$b_n^{(i)} = \left(\mu_i\left(\{\mathbf{x} : x_k = 0 \text{ for some } k = 0, 1, \ldots, n\}\right)\right)^{1/\alpha},$$

and let

$$b_{\mathbf{n}} = \prod_{i=1}^{d} b_{n_i}^{(i)}, \ \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d. \tag{2.7}$$

Then $b_{\mathbf{n}}^\alpha = \mu(B_{\mathbf{n}})$, where

$$B_{\mathbf{n}} = \{\mathbf{x} = (\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(d)}) \in E : x_{k_i}^{(i)} = 0 \ \text{ for some } \ 0 \le k_i \le n_i, \ \text{each } i = 1, \ldots, d\}.$$

Therefore, we can define, for each $\mathbf{n} \in \mathbb{N}_0^d$, a probability measure $\eta_{\mathbf{n}}$ on $(E, \mathcal{E})$ by

$$\eta_{\mathbf{n}}(\cdot) = b_{\mathbf{n}}^{-\alpha} \mu(\cdot \cap B_{\mathbf{n}}). \tag{2.8}$$

This probability measure allows us to represent the restriction of the stationary $S\alpha S$ random field $\mathbf{X}$ in (2.5) to the hypercube $[\mathbf{0}, \mathbf{n}] = \{\mathbf{0} \le \mathbf{k} \le \mathbf{n}\}$ as a series, described below, and that we will find useful in the sequel. It is useful to note also that the measure $\eta_{\mathbf{n}}$ is the product measure of d probability measures on $\left(\mathbb{Z}^{\mathbb{Z}}, \mathcal{B}(\mathbb{Z}^{\mathbb{Z}})\right)$: $\eta_{\mathbf{n}} = \eta_{n_1}^{(1)} \times \cdot \times \eta_{n_d}^{(d)}$ for $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{N}_0^d$, where for $i = 1, \ldots, d$ and $n \ge 0$,

$$\eta_n^{(i)}(\cdot) = (b_n^{(i)})^{-\alpha} \mu_i\big(\cdot \cap \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}} : x_k = 0 \text{ for some } 0 \le k \le n\}\big). \tag{2.9}$$

The restriction of the stationary $S\alpha S$ random field $\mathbf{X}$ in (2.5) to the hypercube $[\mathbf{0}, \mathbf{n}]$ admits, in law, the series representation

$$X_{\mathbf{k}} = b_{\mathbf{n}} C_\alpha^{1/\alpha} \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}), \ \mathbf{0} \le \mathbf{k} \le \mathbf{n}, \tag{2.10}$$

with $A^d = A \times \cdot \times A$ the direct product of d copies of A and A is in (2.6), where the constant C_α is the tail constant of the α-stable random variable:

$$C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} & \alpha \ne 1 \\ 2/\pi & \alpha = 1 \end{cases}.$$

Furthermore, $\{\epsilon_j\}$ is a iid sequence of Rademacher random variables, $\{\Gamma_j\}$ is the sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$, and $\{U_{j,\mathbf{n}}\}$ are iid E-valued random elements with common law $\eta_{\mathbf{n}}$. The sequence $\{\epsilon_j\}$, $\{\Gamma_j\}$ and $\{U_{j,\mathbf{n}}\}$ are independent. See Samorodnitsky and Taqqu (1994) for details.

3. Stable regenerative sets and random sup measures

In this section we describe the limiting object one obtains in an extremal limit theorem from the random field $\mathbf{X}$ of the previous section. We start with a bit of technical background information on random closed sets and random sup measures. The reader should consult Molchanov (2017) for more details.

Let $\mathbb{E}$ be a locally compact and second countable Hausdorff topological space (it will be $\mathbb{R}^d$ or $[0,1]^d$ in our case). We denote by $\mathcal{G}, \mathcal{F}, \mathcal{K}$ the families of open, closed, compact sets of $\mathbb{E}$, respectively. The Fell topology on the space $\mathcal{F}$ of closed sets has a subbasis consisting of the sets

$$\mathcal{F}_G = \{F \in \mathcal{F} : F \cap G \neq \emptyset\},\ G \in \mathcal{G}$$
$$\mathcal{F}^K = \{F \in \mathcal{F} : F \cap K = \emptyset\},\ K \in \mathcal{K}\,.$$

The Fell topology is metrizable and compact.

A random closed set is a measurable mapping from a probability space to $\mathcal{F}$ equipped with the Borel σ-field $\mathcal{B}(\mathcal{F})$ generated by the Fell topology. A specific random closed set in $\mathbb{R}$, the so-called stable regenerative set, is the key for describing the main results of this paper.

For $0 < \beta < 1$ let $(L_\beta(t),\, t \geq 0)$ be the standard β-stable subordinator. That is, it is an increasing Lévy process with Laplace transform $\mathbb{E}e^{-\theta L_\beta(t)} = e^{-t\theta^\beta}$, $\theta \geq 0$. The β-stable regenerative set is defined to be the closure of the range of the β-subordinator, viewed as a random closed set of $\mathbb{R}$:

$$R_\beta \stackrel{d}{:=} \overline{\{L_\beta(t)\,, t \geq 0\}}\,. \tag{3.1}$$

See e.g. Fitzsimmons and Taksar (1988). Products of shifted stable regenerative sets produce random closed subsets of $\mathbb{R}^d$ as follows.

For $0 < \beta_i < 1$, $i = 1, \ldots, d$, let $R^{(i)}_{\beta_i}$, $i = 1, \ldots, d$ be independent β_i-stable regenerative sets. Let $v^{(i)} > 0$, $i = 1, \ldots, d$, and denote $\tilde{R}^{(i)}_{\beta_i} = v^{(i)} + R^{(i)}_{\beta_i}$. Then

$$\tilde{R}_{\boldsymbol{\beta}} := \prod_{i=1}^{d} \tilde{R}^{(i)}_{\beta_i} \tag{3.2}$$

is a random closed subset of $\mathbb{R}^d$. Such random closed sets have interesting intersection properties. The following proposition follows from Lemma 3.1 of Samorodnitsky and Wang (2017).

Proposition 3.1. *Let $\{\tilde{R}_{\boldsymbol{\beta},j}\}_{j\geq 1}$ be independent random closed sets in $\mathbb{R}^d$ as defined by (3.2). Suppose that the corresponding shift vectors $(v^{(i)}_j,\, i = 1, \ldots, d)_{j\geq 1}$ satisfy $v^{(i)}_{j_1} \neq v^{(i)}_{j_2}$ if $j_1 \neq j_2$ for each $i = 1, \ldots, d$. Then for any $m = 1, 2, \ldots$,*

$$P(\cap_{j=1}^m \tilde{R}_{\boldsymbol{\beta},j} \neq \emptyset) = 0 \text{ or } 1\,.$$

The probability is equal to 1 if and only if $m < \min_{i=1,\ldots,d}(1-\beta_i)^{-1}$.

The next object to define is a sup measure. For simplicity we take $\mathbb{E}$ be the space $[0,1]^d$ or $\mathbb{R}^d$. The details of the presentation below can be found in O'Brien et al. (1990). A map $m : \mathcal{G} \to [0,\infty]$ a called sup measure if $m(\emptyset) = 0$ and for an arbitrary collection of open sets $\{G_\gamma\}$ we have $m(\cup_\gamma G_\gamma) = \sup_\gamma m(G_\gamma)$.

The sup derivative $d^\vee m$ of a sup measure m is defined by

$$d^\vee m(t) := \inf_{t\in G} m(G), \quad G \in \mathcal{G}\,. \tag{3.3}$$

It is automatically an upper semi-continuous function. Conversely, for any function $f : \mathbb{E} \to [0,\infty]$, its sup integral $i^\vee f$ is defined as

$$i^\vee f(G) := \sup_{t\in G} f(t), \quad G \in \mathcal{G}\,. \tag{3.4}$$

If f is upper semi-continuous, then $f = d^\vee i^\vee f$. Furthermore, $m = i^\vee d^\vee m$, and one can use (3.4) to extend the domain of a sup measures to sets that are not necessarily open, by setting

$$m(B) := \sup_{t\in B} d^\vee m(t), \quad B \subset \mathbb{E}\,.$$

On the space SM of all sup measures we introduce a topology, the so-called sup vague topology, by saying that a sequence $\{m_n\}$ of sup measures converges to a sup measure m if

$$\limsup_{n\to\infty} m_n(K) \le m(K) \text{ for all } K \in \mathcal{K} \quad \text{and} \quad \liminf_{n\to\infty} m_n(G) \ge m(G) \text{ for all } G \in \mathcal{G}.$$

The space of sup measures with sup vague topology is compact and metrizable; see Theorem 2.4. in Norberg (1990), and we will often use the notation $\mathcal{M}(\mathbb{E})$ for the space of sup measures on $\mathbb{E}$.

A random sup measure is a measurable map from a probability space into SM equipped with the Borel σ-field induced by the sup vague topology. For a random sup measure η, a continuity set is an open set G such that $\eta(G) = \eta(\bar{G})$ (the closure of G) a.s., and a useful criterion for weak convergence in the sup vague topology of random sup-measures is as follows. Let $\{\eta_n\}_{n\ge1}$ be a sequence of random sup measures, and η a random sup measure. Then $\eta_n \Rightarrow \eta$ if and only if

$$(\eta_n(B_1), \ldots, \eta_n(B_m)) \Rightarrow (\eta(B_1), \ldots, \eta(B_m)) \tag{3.5}$$

for arbitrary disjoint open rectangles $B_1, \ldots, B_m$ in $\mathbb{E}$ that are continuity sets for η.

We are now ready to construct the random sup measure that will appear as the limit in the extremal limit theorem in the space SM of the next section. We will define this measure through its sup derivative, which is a random upper semi-continuous function. Let $0 < \beta_i < 1$, $i = 1, \ldots, d$. We start with d independent families of iid β_i-stable regenerative sets $\{R^{(i)}_{\beta_i,j}\}_{j\ge1}$, $i = 1, \ldots, d$. Furthermore, let $(U_{\alpha,j}, V_{\boldsymbol{\beta},j})_{j\ge1}$ be a measurable enumeration of the points of a Poisson point process on $\mathbb{R}\times\mathbb{R}^d$, independent of the stable regenerative sets, with the mean measure

$$\alpha u^{-1-\alpha} du \prod_{i=1}^{d} (1-\beta_i) v_i^{-\beta_i} dv_i, \quad u, v_1, \ldots, v_d > 0\,.$$

Then the triples $(U_{\alpha,j}, V_{\boldsymbol{\beta},j}, R_{\boldsymbol{\beta},j})_{j\geq 1}$ form a Poisson point process on $\mathbb{R}\times\mathbb{R}^d\times\mathcal{F}(\mathbb{R}^d)$ with the mean measure

$$\alpha u^{-1-\alpha}du\left(\prod_{i=1}^d(1-\beta_i)v_i^{-\beta_i}dv_i\right)d\tilde{P}_{\boldsymbol{\beta}},\quad u,v_1,\ldots,v_d>0\,. \tag{3.6}$$

Here $\tilde{P}_{\boldsymbol{\beta}}$ is a probability measure on $\mathcal{F}(\mathbb{R})^d$ defined by

$$\tilde{P}_{\boldsymbol{\beta}}=(P_{\beta_1}\times\cdots\times P_{\beta_d})\circ H^{-1}\,,$$

with P_β being the law of the β-stable regenerative set, in (3.1), and $H:\,(\mathcal{F}(\mathbb{R}))^d\to\mathcal{F}(\mathbb{R}^d)$ is defined by

$$H(F_1,\ldots,F_d)=F_1\times\cdots\times F_d\,.$$

Let

$$\eta_{\alpha,\boldsymbol{\beta}}(\mathbf{t})=\sum_{j=1}^\infty U_{\alpha,j}\mathbf{1}_{\{\mathbf{t}\in V_{\boldsymbol{\beta},j}+R_{\boldsymbol{\beta},j}\}}\ \ \mathbf{t}\in\mathbb{R}^d\,. \tag{3.7}$$

Several observations are in order. First of all, by Proposition 3.1, on event of probability 1, for each $\mathbf{t}$ the series in (3.7) has less than

$$\ell(\boldsymbol{\beta}):=\min_{i=1,\ldots,d}(1-\beta_i)^{-1}$$

non-zero terms, so there are no convergence issues. On the same event the function defined by (3.7) is upper semi-continuous. Indeed, for any finite ℓ the function

$$\sum_{j=1}^\ell U_{\alpha,j}\mathbf{1}_{\{\mathbf{t}\in V_{\boldsymbol{\beta},j}+R_{\boldsymbol{\beta},j}\}}\ \ \mathbf{t}\in\mathbb{R}^d$$

is upper semi-continuous since each terms in this finite sum is upper semi-continuous due to the fact that each shifted product of stable regenerative sets is a closed set. Moreover, it is easy to check that, on each compact set, the uniform distance between this function and that defined in (3.7), goes to zero as $\ell\to\infty$; see p. 10 in Samorodnitsky and Wang (2017).

We now define a random sup measure as the sup integral of the random upper semi-continuous function in (3.7), and we will use the same notation, $\eta_{\alpha,\boldsymbol{\beta}}$, for this sup measure. That is,

$$\eta_{\alpha,\boldsymbol{\beta}}(B)=\sup_{t\in B}\sum_{j=1}^\infty U_{\alpha,j}\mathbf{1}_{\{\mathbf{t}\in V_{\boldsymbol{\beta},j}+R_{\boldsymbol{\beta},j}\}},\ \ B\in\mathcal{B}(\mathbb{R}^d)\,. \tag{3.8}$$

Remark 3.1. The random sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ defined by (3.8) is stationary, in the sense that for every $\mathbf{x}\geq\mathbf{0}$, $\eta_{\alpha,\boldsymbol{\beta}}(\cdot+\mathbf{x})\stackrel{d}{=}\eta_{\alpha,\boldsymbol{\beta}}$. This follows from the shift invariance of the law the random upper semi-continuous function in (3.7) as in Proposition 3.2 in Samorodnitsky and Wang (2017) dealing with the case $d=1$. The argument in that proposition also shows that the random sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ is self-similar, in the sense that for any $c_1>0,\ldots,c_d>0$,

$$\eta_{\alpha,\boldsymbol{\beta}}\circ p_{c_1,\ldots,c_d}\stackrel{d}{=}\prod_{i=1}^d c_i^{(1-\beta_i)/\alpha}\eta_{\alpha,\boldsymbol{\beta}}\,,$$

where $p_{c_1,\ldots,c_d}:\,\mathbb{R}^d\to\mathbb{R}^d$ is the multiplication functional $p_{c_1,\ldots,c_d}(t_1,\ldots,t_d)=(c_1t_1,\ldots,c_dt_d)$.

Importantly,, $\eta_{\alpha,\boldsymbol{\beta}}$ is a Fréchet random sup measure if and only if the sets $(V_{\boldsymbol{\beta},j} + R_{\boldsymbol{\beta},j})$, $j = 1, 2, \ldots$ are a.s. disjoint. According to Proposition 3.1, a necessary and sufficient condition for this is $\beta_i \leq 1/2$ for some $i = 1, \ldots, d$.

The restriction of the random sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ in (3.8) to the hypercube $[\mathbf{0}, \mathbf{1}]$ has a somewhat more convenient representation. Let $\{R^{(i)}_{\beta_i,j}\}_{j\geq 1}$, $i = 1, \ldots, d$ be as stable regenerative sets as above, and let $\{V^{(i)}_j\}_{j\geq 1}$ be d independent families of iid random variables on $[0, 1]$ with distributions given by

$$P(V^{(i)}_1 \leq x) := x^{1-\beta_i}, \quad x \in [0, 1]\,. \tag{3.9}$$

Let now $\{\Gamma_j\}$ be the sequence of the arrival times of a unit rate Poisson process on $(0, \infty)$. Assume that the families $\{V^{(1)}_j\}_{j\geq 1}, \ldots, \{V^{(d)}_j\}_{j\geq 1}$, $\{R^{(1)}_{\beta_1,j}\}_{j\geq 1}, \ldots, \{R^{(d)}_{\beta_d,j}\}_{j\geq 1}$ and the Poisson process are independent. Denoting

$$\tilde{R}^{(i)}_{\beta_i,j} = V^{(i)}_j + R^{(i)}_{\beta_i,j}, \quad 1 \leq i \leq d,\ j \geq 1$$

and

$$\tilde{R}_{\boldsymbol{\beta},j} = \prod_{i=1}^{d} \tilde{R}^{(i)}_{\beta_i,j} \subset \mathbb{R}^d\,,$$

an alternative representation for the random upper semi-continuous function in (3.7) restricted to $[\mathbf{0}, \mathbf{1}]$ is

$$\eta_{\alpha,\boldsymbol{\beta}}(\mathbf{t}) = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{\mathbf{t}\in\tilde{R}_{\boldsymbol{\beta},j}\}}, \quad \mathbf{t} \in [0, 1]^d\,, \tag{3.10}$$

with the corresponding change in (3.8).

4. Convergence of the random sup measures

In this section we establish the first functional extremal theorem for the stationary random field $\mathbf{X}$ in (2.5). The random field naturally induces a family of random sup-measures $\{\eta_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ by

$$\eta_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n}\in B} X_{\mathbf{k}}, \quad B \in \mathcal{B}([0, \infty)^d)\,. \tag{4.1}$$

In the following theorem we prove an extremal theorem in the space of the random sup measures.

Theorem 4.1. *For all $0 < \alpha < 2$ and $0 < \beta_i < 1$, $i = 1, \ldots, d$,*

$$\frac{1}{b_{\mathbf{n}}}\eta_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \eta_{\alpha,\boldsymbol{\beta}}, \quad \mathbf{n} \to \boldsymbol{\infty}\,, \tag{4.2}$$

where $\eta_{\alpha,\boldsymbol{\beta}}$ is the random sup-measure defined in (3.7)*. The weak convergence holds in the space of sup measures $\mathcal{M}(\mathbb{R}^d)$ equipped with the sup vague topology.*

To simplify the notation, we will show the weak convergence in $\mathcal{M}([\mathbf{0},\mathbf{1}])$. Note that by (2.10) we can represent, in law, the sup measure in the left hand side of (4.2) as

$$\frac{1}{b_{\mathbf{n}}}\eta_{\mathbf{n}}(B) = \max_{\mathbf{k}/\mathbf{n}\in B} C_\alpha^{1/\alpha}\sum_{j=1}^{\infty}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{A^d}\circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\ B\in\mathcal{B}([0,1]^d)\,. \tag{4.3}$$

As it is often done, we prove Theorem 4.1 via a truncation argument. We fix an $\ell\in\mathbb{N}$ and construct a truncated random sup-measure $\eta_{\mathbf{n},\ell}$ so that

$$\frac{1}{b_{\mathbf{n}}}\eta_{\mathbf{n},\ell}(B) = \max_{\mathbf{k}/\mathbf{n}\in B} C_\alpha^{1/\alpha}\sum_{j=1}^{\ell}\epsilon_j\Gamma_j^{-1/\alpha}\mathbf{1}_{A^d}\circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\ B\in\mathcal{B}([0,1]^d)\,. \tag{4.4}$$

Note that we can write

$$U_{j,\mathbf{n}}(\mathbf{k}) = \big(U^{(1)}_{j,n_1}(k_1),\dots,U^{(d)}_{j,n_d}(k_d)\big)$$

for $\mathbf{n}=(n_1,\dots,n_d)$ and $\mathbf{k}=(k_1,\dots,k_d)$, with independent components in the right hand side, where $U^{(i)}_{j,n}$ has the law $\eta^{(i)}$ given in (2.9), $i=1,\dots,d$. Therefore, the set of zeroes of $U_{j,\mathbf{n}}$ satisfies

$$\mathcal{Z}(U_{j,\mathbf{n}}) = \mathcal{Z}(U^{(1)}_{j,n_1})\times\cdots\times\mathcal{Z}(U^{(d)}_{j,n_d})\,.$$

To proceed, we need to introduce new notation. Let $S\subset\mathbb{N}$. We set

$$\hat{I}^{(i)}_{S,n} = \cap_{j\in S}\mathcal{Z}(U^{(i)}_{j,n}),\ i=1,\dots,d,\ n\geq 1,\ \hat{I}_{S,\mathbf{n}} = \cap_{j\in S}\mathcal{Z}(U_{j,\mathbf{n}}),\ \mathbf{n}\in\mathbb{N}^d,$$
$$I^{(i)}_S = \cap_{j\in S}\tilde{R}^{(i)}_{\beta_i,j},\ i=1,\dots,d,\ I_S = \cap_{j\in S}\tilde{R}_{\boldsymbol{\beta},j}\,.$$

At this stage the random objects described above do not need to be defined on the same probability space. We need the following extension of Theorem 5.4 of Samorodnitsky and Wang (2017).

Proposition 4.1.

$$\left(\frac{1}{\mathbf{n}}\hat{I}_{S,\mathbf{n}}\right)_{S\subset\{1,\dots,\ell\}} \Rightarrow (I_S)_{S\subset\{1,\dots,\ell\}},\ \mathbf{n}\to\boldsymbol{\infty}\,,$$

in $\big(\mathcal{F}([\mathbf{0},\mathbf{1}])\big)^{2^\ell}$.

Proof. By Theorem 5.4 of Samorodnitsky and Wang (2017), for each $i=1,\dots,d$ and $S\subset\{1,\dots,\ell\}$,

$$\frac{1}{n}\left[\hat{I}^{(i)}_{S,n}\cap[0,1]\right]\Rightarrow I^{(i)}_S\cap[0,1],\quad n\to\infty\,,$$

in the sense of weak convergence of random closed sets. By Corollaries 1.7.13 and 1.7.14 in Molchanov (2017) applied to rectangles of the type $\prod_{i=1}^d[a_i,b_i]$, $0\leq a_i\leq b_i\leq 1$, $i=1,\dots,d$, we conclude that for every $S\subset\{1,\dots,\ell\}$,

$$\frac{1}{\mathbf{n}}\hat{I}_{S,\mathbf{n}}\Rightarrow I_S,\ \mathbf{n}\to\boldsymbol{\infty}\,,$$

($\mathcal{F}([\mathbf{0},\mathbf{1}])$). By Theorem 2.1 (ii) in Samorodnitsky and Wang (2017), this implies the joint convergence in the proposition. □

For $S \subset \{1, \dots, \ell\}$ we define now

$$\hat{I}^*_{S,\mathbf{n}} = \hat{I}_{S,n} \cap \left(\bigcup_{j \in \{1,\dots,\ell\} \setminus S} \mathcal{Z}(U_{j,\mathbf{n}}) \right)^c , \tag{4.5}$$

the set of times where only the Markov chains corresponding to $j \in S$ reach 0. Similarly we define

$$I^*_S = I_S \cap \left(\bigcup_{j \in \{1,\dots,\ell\} \setminus S} \tilde{R}_{\boldsymbol{\beta},j} \right)^c , \tag{4.6}$$

As in the case of the one-dimensional time, for large $\mathbf{n}$ the sets $\hat{I}^*_{S,\mathbf{n}}$ and I^*_S are likely to be alike.

Lemma 4.1. *For an open rectangle $B \subset [0,1]^d$, let $H_{\mathbf{n}}(B)$ be the event*

$$H_{\mathbf{n}}(B) := \bigcup_{S \subset \{1,\dots,\ell\}} \left(\left\{ \frac{\mathbf{1}}{\mathbf{n}} \hat{I}_{S,\mathbf{n}} \cap B \neq \emptyset \right\} \cap \left\{ \frac{\mathbf{1}}{\mathbf{n}} \hat{I}^*_{S,\mathbf{n}} \cap B = \emptyset \right\} \right) \tag{4.7}$$

Then, $\lim_{\mathbf{n} \to \boldsymbol{\infty}} P(H_{\mathbf{n}}(B)) = 0$.

Proof. Write $B = B_1 \times \cdots \times B_d$, with $B_1, \dots, B_d$ open rectangles in $[0,1]$. Denoting

$$\hat{I}^{(i)*}_{S,n} := \hat{I}^{(i)}_{S,n} \cap \left(\bigcup_{j \in \{1,\dots,\ell\} \setminus S} \mathcal{Z}(U^{(i)}_{j,n}) \right)^c , \ i = 1, \dots, d, \ S \subset \{1, \dots, \ell\}, \ n = 1, 2, \dots ,$$

we have

$$H_{\mathbf{n}}(B) \subset \bigcup_{S \subset \{1,\dots,\ell\}} \bigcup_{i=1,\dots,d} \left(\left\{ \frac{1}{n_i} \hat{I}^{(i)}_{S,n_i} \cap B_i \neq \emptyset \right\} \cap \left\{ \frac{1}{n_i} \hat{I}^{(i)*}_{S,n_i} \cap B_i = \emptyset \right\} \right) .$$

The right hand side above is a finite union events, and the probability of each one is asymptoticly vanishing by Lemma 5.5 in Samorodnitsky and Wang (2017). □

Remark 4.1. The argument of Lemma 5.5 in Samorodnitsky and Wang (2017) shows also the following version of the lemma: let

$$H^*_{\mathbf{n}} = \bigcup_{a_i > 0, i=1,\dots,d} H_{\mathbf{n}} \left(\prod_{i=1}^d (0, a_i) \right) .$$

Then $\lim_{\mathbf{n} \to \boldsymbol{\infty}} P(H^*_{\mathbf{n}}) = 0$. We will find this formulation useful in the sequel.

We are now ready to prove convergence of the truncated random sup-measures.

Proposition 4.2. *Let $\ell \geq 1$, and define a random sup-measure $\eta_{\alpha,\boldsymbol{\beta},\ell}$ by*

$$\eta_{\alpha,\beta,\ell}(B) = \sup_{\mathbf{t} \in B} \sum_{j=1}^{\ell} \Gamma_j^{-1/\alpha} \mathbf{1}_{\{\mathbf{t} \in \tilde{R}_{\beta,j}\}}, \quad B \in \mathcal{B}([0,1]^d) . \tag{4.8}$$

Then

$$\frac{1}{b_{\mathbf{n}}} \eta_{\mathbf{n},\ell} \Rightarrow \left(\frac{C_\alpha}{2} \right)^{1/\alpha} \eta_{\alpha,\boldsymbol{\beta},\ell}, \quad \mathbf{n} \to \boldsymbol{\infty}$$

in the space of sup measures $\mathcal{M}([\mathbf{0},\mathbf{1}])$ equipped with the sup vague topology.

Proof. We start by observing that an alternative expression for the random sup-measure $\eta_{\alpha,\boldsymbol{\beta},\ell}$ is

$$\eta_{\alpha,\beta,\ell}(B) = \max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{I_S\cap B\neq\emptyset\}} \sum_{j\in S} \Gamma_j^{-1/\alpha}\,. \tag{4.9}$$

Since stable subordinators do not hit fixed points, by (3.5) it suffices to show that for any m disjoint open rectangles $B_r = \prod_{i=1}^d (a_i^{(r)}, b_i^{(r)})$, $r=1,\ldots,m$ in $[0,1]^d$, we have a convergence of random vectors:

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1),\ldots,\eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \left(\eta_{\alpha,\boldsymbol{\beta},\ell}(B_1),\ldots,\eta_{\alpha,\boldsymbol{\beta},\ell}(B_m)\right)\,.$$

It is clear that for any $r=1,\ldots,m$, on the compliment of the event $H_{\mathbf{n}}(B_r)$,

$$\max_{\mathbf{k}/\mathbf{n}\in B_r} \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d}\circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) = \max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \epsilon_j\Gamma_j^{-1/\alpha}\,.$$

Since $\hat{I}_{S,\mathbf{n}}$ is decreasing as the set S increases, we can choose, for a fixed S, the set $S' = \{j\in S : \epsilon_j = 1\}$ to obtain

$$\max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \epsilon_j\Gamma_j^{-1/\alpha} = \max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbf{1}(\epsilon_j=1)\Gamma_j^{-1/\alpha}\,.$$

Hence, on the compliment of the event $H_{\mathbf{n}}(B_1)\cup\cdots\cup H_{\mathbf{n}}(B_m)$,

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1),\ldots,\eta_{\mathbf{n},\ell}(B_{\mathbf{n}})\right) = C_\alpha^{1/\alpha}\left(\max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}\right)_{r=1,\ldots,m}\,.$$

By Proposition 4.1 the random vector in the right hand side converges weakly as $\mathbf{n}\to\boldsymbol{\infty}$ to the random vector

$$C_\alpha^{1/\alpha}\left(\max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{I_S\cap B_r\neq\emptyset\}} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}\right)_{r=1,\ldots,m}\,.$$

Since, by Lemma 4.1, the event $H_{\mathbf{n}}(B_1)\cup\cdots\cup H_{\mathbf{n}}(B_m)$ has an asymptotically vanishing probability, the random vector

$$\frac{1}{b_{\mathbf{n}}}\left(\eta_{\mathbf{n},\ell}(B_1),\ldots,\eta_{\mathbf{n},\ell}(B_m)\right)$$

converges weakly to the same limit. The claim of the proposition follows by noticing that the thinned Poisson random measure $(\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha})_{j\geq 1}$ has the same law as $(2^{-1/\alpha}\Gamma_j^{-1/\alpha})_{j\geq 1}$ and using (4.9). □

We now deal with the part of the random sup measure in Theorem 4.1 that is left after the truncation procedure above. The following proposition is crucial.

Proposition 4.3. *For all $\delta > 0$,*

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) = 0\,. \tag{4.10}$$

Proof. Clearly,

$$\begin{aligned} & P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) \\ \le & P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right) \\ & + P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right)\,. \end{aligned} \tag{4.11}$$

By symmetry,

$$\begin{aligned} & P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right) \\ \le & \, 2P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right)\,. \end{aligned}$$

The sum in the right hand side is a representation, in law, of the restriction to the set $\mathbf{0} \le \mathbf{k} \le \mathbf{n}$ of the random field $(b_{\mathbf{n}}^{-1} Y_{\mathbf{k}},\, \mathbf{k} \in \mathbb{Z}^d)$, where $(Y_{\mathbf{k}},\, \mathbf{k} \in \mathbb{Z}^d)$ is a stationary symmetric infinitely divisible random field defined, similarly to the original stationary symmetric α-stable random field in (2.5), by

$$Y_{\mathbf{k}} = \int_E f \circ T^{\mathbf{k}}(\mathbf{x})\, \tilde{M}(d\mathbf{x}), \quad \mathbf{k} \in \mathbb{Z}^d\,, \tag{4.12}$$

with the distinction that the local Lévy measure $\tilde{\rho}$ of the symmetric infinitely divisible random measure in (4.12) has the density $\alpha|x|^{-(\alpha+1)}$ restricted to $|x| \le 1$. See Chapter 3 in Samorodnitsky (2016). In particular, each $Y_{\mathbf{k}}$ has a Lévy measure with a bounded support and, hence, has (faster than) exponentially fast decaying tails. See e.g. Sato (1999) Chapter 5. We conclude by the regular variation (2.3) of the factors in (2.7) that for $\mathbf{n} = (n_1, \ldots, n_d)$,

$$\begin{aligned} & P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j > b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta/2 \right) \\ \le & \, 2P\big(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} |Y_{\mathbf{k}}| > b_{\mathbf{n}}(\delta/2) \big) \le 2 \prod_{i=1}^{d} (1+n_i) P\big(|Y_{\mathbf{0}}| > b_{\mathbf{n}}(\delta/2) \big) \to 0 \end{aligned}$$

as $\mathbf{n} \to \boldsymbol{\infty}$. Therefore, the claim of the proposition will follow once we prove that

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} P\left(\max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right) = 0\,. \tag{4.13}$$

To this end, let $M > 0$ and set $D_\ell^M := \{\Gamma_{\ell+1} \ge M\}$. By the Strong Law of Large Numbers, $\lim_{\ell\to\infty} P(D_\ell^M) = 1$. Therefore, we may replace the probability in (4.13) by

$$P\left(\left\{ \max_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right)\,.$$

The above quantity does not exceed

$$\begin{aligned} &\sum_{\mathbf{0}\le\mathbf{k}\le\mathbf{n}} P\left(\left\{ \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right) \qquad (4.14)\\ &= \prod_{i=1}^{d} (n_i+1) P\left(\left\{ \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right)\,, \end{aligned}$$

since the probabilities in the sum in (4.14) do not depend on $\mathbf{k}$. Using symmetry, we have

$$\begin{aligned} &P\left(\left\{ \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(\Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d}(U_{j,\mathbf{n}}) \right| > \delta \right\} \cap D_\ell^M \right)\\ \le &P\left(\left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(M \le \Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d}(U_{j,\mathbf{n}}) \right| > \delta \right)\\ \le &2P\left(\left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}(M \le \Gamma_j \le b_{\mathbf{n}}^{\alpha}) \mathbf{1}_{A^d}(U_{j,\mathbf{n}}) \right| > \delta \right)\,. \qquad (4.15) \end{aligned}$$

Notice that the Poisson point process $(b_{\mathbf{n}}^{-\alpha}\Gamma_j \mathbf{1}_{A^d}(U_{j,\mathbf{n}}))_j$ has the same law as the Poisson point proces $(\Gamma_j)_j$. Therefore, the probability in (4.15) coincides with

$$\begin{aligned} &P\left(b_{\mathbf{n}}^{-1} \left| \sum_{j=1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}} \right| > \delta \right) \le P\left(\left| b_{\mathbf{n}}^{-1} \sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}} \right| > \delta/2 \right)\\ \le &b_{\mathbf{n}}^{-p} (\delta/2)^{-p} E \left| \sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{M b_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}} \right|^p \end{aligned}$$

for any $p > 0$, where $j_M = \lceil M^{1/\alpha}\delta/2 \rceil$. If $p > 0$ is large enough, then by the regular variation (2.3),

$$\lim_{\mathbf{n}\to\boldsymbol{\infty}} \prod_{i=1}^{d} (n_i+1) b_{\mathbf{n}}^{-p} \to 0\,.$$

Therefore, (4.13) will follow once we check that for any $p > 0$ we can take $M > 0$ large enough so that

$$\limsup_{\mathbf{n}\to\boldsymbol{\infty}} E\left|\sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{Mb_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right|^p < \infty\,. \tag{4.16}$$

Let us take $p = 2k$, an even integer. By the Khintchine inequality (see e.g. (A.1) in Nualart (1995)), there is a constant $c_p \in (0,\infty)$ such that

$$\begin{aligned} E\left|\sum_{j=j_M+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{\{Mb_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\right|^p &\le c_p E\left(\sum_{j=j_M+1}^{\infty} \big(\Gamma_j^{-1/\alpha} \mathbf{1}_{\{Mb_{\mathbf{n}}^{-\alpha} \le \Gamma_j \le 1\}}\big)^2\right)^{p/2} \\ \le c_p E\left(\sum_{j=j_M+1}^{\infty} \Gamma_j^{-2/\alpha}\right)^k &\le c_p \left(\sum_{j=j_M+1}^{\infty} \big(E(\Gamma_j^{-2k/\alpha})\big)^{1/k}\right)^k\,. \end{aligned}$$

The claim (4.16) now follows since $E\big(\Gamma_j^{-2k/\alpha}\big) < \infty$ for $j > 2k/\alpha$, and

$$E\big(\Gamma_j^{-2k/\alpha}\big) \sim j^{-2k/\alpha},\ j \to \infty\,.$$

□

We are now ready to finish the proof of Theorem 4.1.

Proof of Theorem 4.1 . Once again, by (3.5) it suffices to show that for any m disjoint open rectangles $B_r = \prod_{i=1}^d (a_i^{(r)}, b_i^{(r)})$, $r = 1,\dots,m$ in $[0,1]^d$, we have

$$\frac{1}{b_{\mathbf{n}}} \left(\eta_{\mathbf{n}}(B_1),\dots,\eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \left(\eta_{\alpha,\boldsymbol{\beta}}(B_1),\dots,\eta_{\alpha,\boldsymbol{\beta},\ell}(B_m)\right)\,.$$

Since by Proposition 4.2

$$\frac{1}{b_{\mathbf{n}}} \left(\eta_{\mathbf{n},\ell}(B_1),\dots,\eta_{\mathbf{n},\ell}(B_m)\right) \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \left(\eta_{\alpha,\boldsymbol{\beta},\ell}(B_1),\dots,\eta_{\alpha,\boldsymbol{\beta},\ell}(B_m)\right)\,,$$

and $\eta_{\alpha,\boldsymbol{\beta},\ell}$ increases to $\eta_{\alpha,\boldsymbol{\beta}}$ almost surely, we can use the "convergence together" argument, as in Theorem 3.2 of Billingsley (1999). To this end we need to check that for each $r = 1,\dots,m$ and any $\epsilon > 0$,

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} P\left(\frac{1}{b_{\mathbf{n}}} \left|\eta_{\mathbf{n}}(B_r) - \eta_{\mathbf{n},\ell}(B_r)\right| > \epsilon\right) = 0\,.$$

This is, however, an immediate conclusion from Proposition 4.3. □

Remark 4.2. Note that the limiting sup measure in Theorem 4.1 is Fréchet only when $\beta_i \le 1/2$ for some $i = 1,\dots,d$. See Remark 3.1.

The stationary random field $\mathbf{X}$ in (2.5) induces another family of random sup measures $\{\tilde{\eta}_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{N}^d}$ via

$$\tilde{\eta}_{\mathbf{n}}(B) := \max_{\mathbf{k}/\mathbf{n}\in B} |X_{\mathbf{k}}|, \quad B \in \mathcal{B}([0,\infty)^d)\,. \tag{4.17}$$

This family of random sup-measures satisfies the following analogue of Theorem 4.1.

Theorem 4.2. *For all $0 < \alpha < 2$ and $0 < \beta_i < 1$, $i = 1, \dots, d$,*

$$\frac{1}{b_{\mathbf{n}}}\tilde{\eta}_{\mathbf{n}} \Rightarrow \left(\frac{C_\alpha}{2}\right)^{1/\alpha} \max\big(\eta^{(1)}_{\alpha,\boldsymbol{\beta}},\, \eta^{(2)}_{\alpha,\boldsymbol{\beta}}\big), \quad \mathbf{n} \to \boldsymbol{\infty}\,, \tag{4.18}$$

where $\eta^{(1)}_{\alpha,\boldsymbol{\beta}}$ and $\eta^{(2)}_{\alpha,\boldsymbol{\beta}}$ are two independent copies of the random sup measure defined in (3.7)*. The weak convergence holds in the space of sup measures $\mathcal{M}(\mathbb{R}^d)$ equipped with the sup vague topology.*

Proof. Once again, we will show the weak convergence in $\mathcal{M}([\mathbf{0}, \mathbf{1}])$. We continue using the notation of the proof of Theorem 4.1. The same argument as in the that proof works once we show that, in the obvious notation, for any $\ell = 1, 2, \dots$,

$$\frac{1}{b_{\mathbf{n}}}\tilde{\eta}_{\mathbf{n},\ell} \Rightarrow \frac{C_\alpha^{1/\alpha}}{2} \max\big(\eta^{(1)}_{\alpha,\boldsymbol{\beta},\ell},\, \eta^{(2)}_{\alpha,\boldsymbol{\beta},\ell}\big), \quad \mathbf{n} \to \boldsymbol{\infty}\,, \tag{4.19}$$

where

$$\frac{1}{b_{\mathbf{n}}}\tilde{\eta}_{\mathbf{n},\ell}(B) = \max_{\mathbf{k}/\mathbf{n} \in B} \left| \sum_{j=1}^{\ell} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right|, \; B \in \mathcal{B}([0,1]^d)\,. \tag{4.20}$$

Outside the vanishing event $H_n(B_1) \cup \cdots \cup H_n(B_m)$ we now have

$$\frac{1}{b_{\mathbf{n}}}\left(\tilde{\eta}_{\mathbf{n},\ell}(B_1), \dots, \tilde{\eta}_{\mathbf{n},\ell}(B_{\mathbf{n}})\right) = C_\alpha^{1/\alpha} \left(\max \left[\max_{S \subset \{1,\dots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}} \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbf{1}_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}, \right.\right.$$
$$\left.\left. \max_{S \subset \{1,\dots,\ell\}} \mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}} \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbf{1}_{\{\epsilon_j = -1\}} \Gamma_j^{-1/\alpha} \right] \right)_{r=1,\dots,m}.$$

Using again Proposition 4.1 and Lemma 4.1 we conclude that, as $\mathbf{n} \to \boldsymbol{\infty}$,

$$\frac{1}{b_{\mathbf{n}}}\left(\tilde{\eta}_{\mathbf{n},\ell}(B_1), \dots, \tilde{\eta}_{\mathbf{n},\ell}(B_{\mathbf{n}})\right) \Rightarrow C_\alpha^{1/\alpha} \left(\max \left[\max_{S \subset \{1,\dots,\ell\}} \mathbf{1}_{\{I_S \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbf{1}_{\{\epsilon_j = 1\}} \Gamma_j^{-1/\alpha}, \right.\right.$$
$$\left.\left. \max_{S \subset \{1,\dots,\ell\}} \mathbf{1}_{\{I_S \cap B_r \neq \emptyset\}} \sum_{j \in S} \mathbf{1}_{\{\epsilon_j = -1\}} \Gamma_j^{-1/\alpha} \right] \right)_{r=1,\dots,m}.$$

The statement of the theorem now follows since $(\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha})_j$ and $(\mathbf{1}_{\{\epsilon_j=-1\}}\Gamma_j^{-1/\alpha})_j$ are two independent Poisson random measures, each with the same law as $(2^{-1/\alpha}\Gamma_j^{-1/\alpha})_{j\geq 1}$, and using (4.9). □

Remark 4.3. It is interesting to observe that in the case $0 < \beta_i \leq 1/2$ for some $i = 1, \dots, d$, the sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ is a Fréchet random sup measure, and so (4.2) can be reformulated as

$$\frac{1}{b_{\mathbf{n}}}\tilde{\eta}_{\mathbf{n}} \Rightarrow C_\alpha^{1/\alpha}\eta_{\alpha,\boldsymbol{\beta}}, \quad \mathbf{n} \to \boldsymbol{\infty}\,. \tag{4.21}$$

See Remark 3.1. However, if $0 < \beta_i > 1/2$ for all $i = 1, \dots, d$, then the random sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ is not max-stable, so (4.21) is no longer a valid statement of Theorem 4.2.

5. Convergence of the partial maxima processes

In this section we prove another version of a functional extremal theorem for the stationary random field $\mathbf{X}$ in (2.5). This time we will be working in the space $D(\mathbb{R}^d_+)$, and the limit will itself be a random field. The random field $\mathbf{X}$ induces an array of partial maxima random fields $\{M_{\mathbf{n}}\}$ by

$$M_{\mathbf{n}}(\mathbf{t}) := \max_{\mathbf{0} \le \mathbf{k} \le \mathbf{nt}} X_{\mathbf{k}}, \quad \mathbf{t} \in \mathbb{R}^d_+ .$$

The random sup measure $\eta_{\alpha,\boldsymbol{\beta}}$ in (3.7) also induces a random field $W_{\alpha,\boldsymbol{\beta}}$, by

$$W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}) := \eta_{\alpha,\boldsymbol{\beta}}([\mathbf{0}, \mathbf{t}]), \quad \mathbf{t} \in \mathbb{R}^d_+ .$$

Remark 5.1. It follows immediately from Remark 3.1 that the random field $\big(W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d_+\big)$ is self-similar, in the sense that for any $c_1 > 0, \dots, c_d > 0$

$$\big(W_{\alpha,\boldsymbol{\beta}}((c_1 t_1, \dots, c_d t_d)),\, \mathbf{t} \in \mathbb{R}^d_+\big) \stackrel{d}{=} \left(\prod_{i=1}^d c_i^{(1-\beta_i)/\alpha} W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d_+ \right) .$$

This is, of course, what a multivariate version of Lamperti's theorem requires from the limit in any functional extremal theorem; see e.g. Theorem 8.1.5 in Samorodnitsky (2016).

The following functional extremal theorem is the main result of this section.

Theorem 5.1. *For all $0 < \alpha < 2$ and $0 < \beta_i < 1$, $i = 1, \dots, d$,*

$$\left(\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{t}),\, t \in \mathbb{R}^d_+ \right) \Rightarrow \left(\left(\frac{C_\alpha}{2} \right)^{1/\alpha} W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t} \in \mathbb{R}^d_+ \right) \tag{5.1}$$

in the Skorohod J_1 topology on the space $D(\mathbb{R}^d_+)$.

Proof. The usual reference for multiparameter weak convergence is Straf (1972). For our purposes there is little difference between the properties of weak convergence in $D(\mathbb{R}^d_+)$ for $d = 1$ and $d > 1$. We will show weak convergence in $D([\mathbf{0}, \mathbf{1}])$, and we use the series representation (2.10). By (4.3) we can write, in law,

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n}}(\mathbf{t}) = \max_{\mathbf{0} \le \mathbf{k}/\mathbf{n} \le \mathbf{t}} C_\alpha^{1/\alpha} \sum_{j=1}^\infty \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\, \mathbf{t} \in [\mathbf{0}, \mathbf{1}] . \tag{5.2}$$

We again use a truncation argument. For $\ell \in \mathbb{N}$, we define random fields $M_{\mathbf{n},\ell}$ by

$$\frac{1}{b_{\mathbf{n}}} M_{\mathbf{n},\ell}(\mathbf{t}) = \max_{\mathbf{0} \le \mathbf{k}/\mathbf{n} \le \mathbf{t}} C_\alpha^{1/\alpha} \sum_{j=1}^\ell \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d} \circ T^{\mathbf{k}}(U_{j,\mathbf{n}}),\, \mathbf{t} \in [\mathbf{0}, \mathbf{1}] . \tag{5.3}$$

Similarly, starting with the truncated random sup measure $\eta_{\alpha,\boldsymbol{\beta},\ell}$ we define a random field $W_{\alpha,\boldsymbol{\beta},\ell}$ by

$$W_{\alpha,\boldsymbol{\beta},\ell}(\mathbf{t}) := \eta_{\alpha,\boldsymbol{\beta},\ell}([\mathbf{0}, \mathbf{t}]), \quad \mathbf{t} \in [\mathbf{0}, \mathbf{1}] .$$

We start by proving that

$$\left(\frac{1}{b_{\mathbf{n}}}M_{\mathbf{n},\ell}(\mathbf{t}),\, t\in\mathbb{R}^d_+\right)\Rightarrow\left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha}W_{\alpha,\boldsymbol{\beta},\ell}(\mathbf{t}),\,\mathbf{t}\in[\mathbf{0},\mathbf{1}]\right). \tag{5.4}$$

Note that the representation (4.9) can be written, in law, as

$$W_{\alpha,\boldsymbol{\beta},\ell}(\mathbf{t})=\max_{S\subset\{1,\ldots,\ell\}}\mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\,2^{1/\alpha}\sum_{j\in S}\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\quad \mathbf{t}\in[0,1]^d. \tag{5.5}$$

Furthermore, from the argument in Proposition 4.2 we know that outside of an event $A_{\mathbf{n}}$ whose probability goes to zero as $\mathbf{n}\to\boldsymbol{\infty}$, the random field $\big((1/b_{\mathbf{n}})M_{\mathbf{n},\ell}(\mathbf{t}),\,\mathbf{t}\in[\mathbf{0},\mathbf{1}]\big)$ coincides with the random field

$$\max_{S\subset\{1,\ldots,\ell\}}\mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}C_\alpha^{1/\alpha}\sum_{j\in S}\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\quad \mathbf{t}\in[\mathbf{0},\mathbf{1}]\,;$$

see Remark 4.1. Therefore, (5.4) will follows once we prove that

$$\begin{aligned}&\left(\max_{S\subset\{1,\ldots,\ell\}}\mathbf{1}_{\{(\mathbf{1}/\mathbf{n})\hat{I}_{S,\mathbf{n}}\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\sum_{j\in S}\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\,\mathbf{t}\in[\mathbf{0},\mathbf{1}]\right)\\ &\Rightarrow\left(\max_{S\subset\{1,\ldots,\ell\}}\mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\sum_{j\in S}\mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha},\,\mathbf{t}\in[\mathbf{0},\mathbf{1}]\right).\end{aligned} \tag{5.6}$$

Since the Fell topology on $\mathcal{F}([0,1]^d)$ is separable and metrizable (see Salinetti and Wets (1981)), by the Skorohod representation theorem, we can find a common probability space for $\big(\hat{I}_{S,\mathbf{n}},\,S\subset\{1,\ldots,\ell\}\big)$ and $\big(I_s,\,S\subset\{1,\ldots,\ell\}\big)$ such that the convergence in Proposition 4.1 becomes the almost sure convergence. On that probability space we will prove a.s. convergence in (5.6).

For $i=1,\ldots,d$ and $S\subset\{1,\ldots,\ell\}$ denote

$$t^{(i)}_{S,\mathbf{n}}=\inf\big\{t>0:\,t\in n_i^{-1}\hat{I}^{(i)}_{S,n_i}\big\},\quad t^{(i)}_S=\inf\big\{t>0:\,t\in I^{(i)}_S\big\}.$$

Then the a.s. convergence in Proposition 4.1 implies that $t^{(i)}_{S,\mathbf{n}}\to t^{(i)}_S$ a.s. as $\mathbf{n}\to\boldsymbol{\infty}$ for every $i=1,\ldots,d$ and $S\subset\{1,\ldots,\ell\}$. If we denote $\mathbf{t}_{S,\mathbf{n}}=\big(t^{(1)}_{S,\mathbf{n}},\ldots,t^{(d)}_{S,\mathbf{n}}\big)$ and $\mathbf{t}_S=\big(t^{(1)}_S,\ldots,t^{(d)}_S\big)$ for $S\subset\{1,\ldots,\ell\}$, then $\mathbf{t}_{S,\mathbf{n}}\to\mathbf{t}_S$ a.s. as $\mathbf{n}\to\boldsymbol{\infty}$. Since stable subordinators do not hit fixed points, the 2^ℓ points $\mathbf{t}_S, S\subset\{1,\ldots,\ell\}$ are distinct. Furthermore, given $(\epsilon_j,\Gamma_j)_{j\in S}$, these points determine the realization of the random field in the right hand side of (5.6), while the 2^ℓ points $\mathbf{t}_{S,\mathbf{n}}, S\subset\{1,\ldots,\ell\}$ determine the realization of the random field in the left hand side of (5.6). Therefore, any homeomorphism of $[0,1]^d$ onto itself that fixes the origin and moves $\mathbf{t}_{S,\mathbf{n}}$ to $\mathbf{t}_S$ for each $S\subset\{1,\ldots,\ell\}$ makes the values of the field in the left hand side of (5.6) equal to the values of the field in the right hand side of (5.6). The convergence $\mathbf{t}_{S,\mathbf{n}}\to\mathbf{t}_S$ for each S guarantees that these homeomorphisms can be chosen to converge to the identity in the supremum norm. Hence a.s. convergence in (5.6).

To complete the proof we use, once again, the "convergence together" argument in Theorem 3.2 of Billingsley (1999). The first step to this end is to show that

$$\big(W_{\alpha,\boldsymbol{\beta},\ell}(\mathbf{t}),\, \mathbf{t}\in[\mathbf{0},\mathbf{1}]\big) \Rightarrow \big(W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t}\in[\mathbf{0},\mathbf{1}]\big) \tag{5.7}$$

as $\ell\to\infty$ in the Skorohod J_1 topology on the space $D([\mathbf{0},\mathbf{1}])$. Since we can represent the random field in the right hand side of (5.7), in law, as

$$W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}) = \sup_{S\subset\mathbb{N}} \mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha}, \quad \mathbf{t}\in[\mathbf{0},\mathbf{1}]\,, \tag{5.8}$$

we will use the representations in law, (5.5) and (5.8), and prove a.s. convergence of the random fields in the right hand sides of these representations. Furthermore, we will show this a.s. convergence in the uniform distance. To this end, fix $\mathbf{t}\in[\mathbf{0},\mathbf{1}]$ and note that by Proposition 3.1, with probability 1, any set $S\subset\mathbb{N}$ such that $I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset$ must be of cardinality smaller than $\min_{i=1,\ldots,d}(1-\beta_i)^{-1}$. Furthermore, for any such S, the set $S\cap\{1,\ldots,\ell\}$ contributes to the maximum in the right hand side in (5.5). Therefore,

$$\begin{aligned} 0 &\le \sup_{S\subset\mathbb{N}} \mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} - \max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} \\ &\le 2^{1/\alpha} \min_{i=1,\ldots,d}(1-\beta_i)^{-1}\Gamma_{\ell+1}^{-1/\alpha}\,. \end{aligned}$$

That is,

$$\begin{aligned} &\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \left| \sup_{S\subset\mathbb{N}} \mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} - \max_{S\subset\{1,\ldots,\ell\}} \mathbf{1}_{\{I_S\cap[\mathbf{0},\mathbf{t}]\neq\emptyset\}}\, 2^{1/\alpha} \sum_{j\in S} \mathbf{1}_{\{\epsilon_j=1\}}\Gamma_j^{-1/\alpha} \right| \\ &\le 2^{1/\alpha} \min_{i=1,\ldots,d}(1-\beta_i)^{-1}\Gamma_{\ell+1}^{-1/\alpha} \to 0 \end{aligned}$$

as $\ell\to\infty$, proving the a.s. convergence in the uniform distance.

For the second ingredient in the "convergence together" argument we use again the uniform distance and prove that for any $\epsilon>0$,

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} P\left(\frac{1}{b_{\mathbf{n}}} \sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} |M_{\mathbf{n}}(\mathbf{t}) - M_{\mathbf{n},\ell}(\mathbf{t})| > \epsilon \right) = 0,.$$

By (5.2) and (5.3) it is enough to prove that

$$\lim_{\ell\to\infty} \limsup_{\mathbf{n}\to\boldsymbol{\infty}} P\left(\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \max_{\mathbf{0}\le\mathbf{k}/\mathbf{n}\le\mathbf{t}} \left| \sum_{j=\ell+1}^{\infty} \epsilon_j \Gamma_j^{-1/\alpha} \mathbf{1}_{A^d}\circ T^{\mathbf{k}}(U_{j,\mathbf{n}}) \right| > \epsilon \right) = 0\,.$$

This is, however, an immediate consequence of Proposition 4.3. □

Theorem 5.1 has a natural counterpart for the partial maxima of the absolute values of the random field (2.5). It can be obtained along the same lines as we obtained Theorem 4.2. We omit the argument.

Theorem 5.2. *Let $0<\alpha<2$ and $0<\beta_i<1$, $i=1,\ldots,d$. Define*

$$\tilde{M}_{\mathbf{n}}(\mathbf{t}) := \max_{\mathbf{0}\le\mathbf{k}\le\mathbf{nt}} |X_{\mathbf{k}}|, \quad \mathbf{t}\in\mathbb{R}_+^d\,.$$

Then

$$\left(\frac{1}{b_{\mathbf{n}}}\tilde{M}_{\mathbf{n}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right) \Rightarrow \left(\left(\frac{C_\alpha}{2}\right)^{1/\alpha} \max\big(W^{(1)}_{\alpha,\boldsymbol{\beta}}(\mathbf{t}), W^{(2)}_{\alpha,\boldsymbol{\beta}}(\mathbf{t})\big),\, \mathbf{t}\in\mathbb{R}_+^d\right) \tag{5.9}$$

in the Skorohod J_1 topology on the space $D(\mathbb{R}_+^d)$. Here $\left(W^{(1)}_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right)$ and $\left(W^{(2)}_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right)$ are two independent copies of the limiting process in Theorem 5.1.

Remark 5.2. The structure of the limit in Theorem 5.2 together with Remark 5.1 immediately implies that the random field in the right hand side of (5.9) is self-similar, Furthermore, as in Remark 4.3, in the case $0<\beta_i\le 1/2$ for some $i=1,\ldots,d$, and only in that case, the limiting random field in Theorem 5.2 is Fréchet. In this case an alternative way of stating the theorem is

$$\left(\frac{1}{b_{\mathbf{n}}}\tilde{M}_{\mathbf{n}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right) \Rightarrow \left(C_\alpha^{1/\alpha} W_{\alpha,\boldsymbol{\beta}}(\mathbf{t}),\, \mathbf{t}\in\mathbb{R}_+^d\right).$$

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