TOWERS OF BOREL FIBRATIONS AND GENERALIZED QUASI-INVARIANTS

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In this dissertation we construct and study topological spaces that generalize spaces of quasi-invariants of finite reflection groups introduced in [BR1], [BR2]. These spaces are obtained by applying the classical fibre-cofiber construction and its generalization to classical fibrations associated to compact connected Lie groups. Our results can be viewed as a natural extension of results of [BR1, BR2] to higher rank Lie groups.

We give a number of explicit examples and computations. These examples include the classifying spaces of classical Lie groups of arbitrary rank, associated homogeneous spaces as well as classifying spaces of commutativity of classical Lie groups introduced in [AG]. As an application, we compute the equivariant K-theory of the towers of homotopy fibers for the case of classifying spaces of classical Lie groups and compare our results to those of [BR1] in the rank one case. Additionally, we explore spherical fibrations and consider conjugation action in the rank one case. We give explicit combinatorial presentations and study algebraic properties of rings of generalized quasi-invariants arising from the proposed topological construction.

BIOGRAPHICAL SKETCH

Yun was born in a small town in Hubei province, China, renowned for a series of reference books (which, in reality, do not exist) for the college entrance examination (gaokao) in mainland China. Yun spent most of her childhood in the capital city of Hubei province, which has now become well-known due to the Covid pandemic. After receiving her B.Sc. in Mathematics and Applied Mathematics from Fudan University in Shanghai, Yun started her Ph.D. journey in the gorgeous town that is home to Cornell university in 2016. Yun will continue her math career as a postdoctoral researcher at Indiana University, Bloomington in the fall of 2023.

This thesis is dedicated to my grandparents.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Compact Lie groups are fundamental objects that play a role in many areas of mathematics. In some areas[1], however, it is not a Lie group G itself but rather its classifying space BG that takes a central stage. The classifying space encodes the algebraic structure (multiplication and unit) of G directly into a topological space, allowing one to use powerful homotopy-theoretic methods to study Lie groups. A classical theorem in algebraic topology (see, e.g., [N]) asserts that the isomorphism type of a compact Lie group is uniquely determined by the homotopy type of its classifying space. This leads to the idea of developing Lie theory via classifying spaces in purely homotopy-theoretic way — a perspective that has brought some spectacular advances in algebraic topology (see, e.g., [DW1]) in recent years.

One model for constructing the classifying space BG of a compact connected Lie group G involves taking the quotient of a contractible space by a (proper free) G-action. This contractible free G-space is a model of the universal principal bundle EG for the Lie group G: using it, we obtain the natural fibration

$$p : BT = EG/T \longrightarrow BG = EG/G$$

where T is a maximal torus of G. A classical result of A. Borel (see [B3]) states that the map p induces an injective ring homomorphism $p^* : H^*(BG;\mathbb{Q}) \hookrightarrow H^*(BT;\mathbb{Q})$ on rational cohomology. Denoting $W = N_G(T)/T$ the associated Weyl group of

[1]For example, in algebraic topology.

(G, T), we have a natural W-action on BT induced by the conjugation action of W on T and, consequently, on its rational cohomology. The image of p^* turns out to be precisely the subalgebra of W-invariant polynomials:

$$p^* : H^*(BG; \mathbb{Q}) \xrightarrow{\cong} H^*(BT; \mathbb{Q})^W \hookrightarrow H^*(BT; \mathbb{Q}).$$

This map equips $H^*(BT; \mathbb{Q})$ with the structure of a $H^*(BG; \mathbb{Q})$-module and by Chevalley's Theorem (see [C2]), is free of rank $|W|$. A basic example is $G = \mathrm{U}(n)$ and $W = S_n$, in which case we can identify $H^*(BT; \mathbb{Q})$ with the polynomial algebra $\mathbb{Q}[t_1, \ldots, t_n]$ in n variables, and $H^*(BG; \mathbb{Q})$ with the subalgebra of classical symmetric polynomials in $\mathbb{Q}[t_1, \ldots, t_n]$.

In this way, polynomial algebras and their W-invariant subalgebras can be realized as the rational cohomology of classifying spaces of compact connected Lie groups. This classical observation serves as a starting point for the realization problem for algebras of W-quasi-invariants.

Quasi-invariants are natural generalizations of invariant polynomials, originally introduced by O. Chalykh and A. Veselov [CV1] in their study of commutative rings of differential operators. Given the algebra $R := \mathbb{R}[V]$ of polynomials on a geometric representation V of a finite reflection group W (i.e., a finite group generated by the reflections s_α along hyperplanes H_α in V), we write $S = R^W$ for its subalgebra consisting of W-invariant polynomials. We can characterize W-invariant polynomials as the ones satisfying the equations $f - s_\alpha f = 0$ for each reflection $s_\alpha \in W$; the W-quasi-invariant polynomials f in R are then defined by the weaker condition that the differences $f - s_\alpha f$ vanishing on the hyperplanes H_α up to a certain order specified by non-negative integers m_α assigned to H_α (see Definition 2.1). This assignment $\alpha \mapsto m_\alpha$ is referred to as a multiplicity function for W. For a fixed m, the quasi-invariant polynomials form a graded subalgebra $Q_m(W)$ of R that contains all W-invariant polynomials. When m varies, the subalgebras $Q_m(W)$ give a natural filtration on R, in which $Q_0(W) = R$ and $Q_\infty(W) = S$. A deep theorem in the theory of quasi-invariants asserts that, for all m, $Q_m(W)$ is a free module over the algebra S of invariant polynomials of rank $|W|$ (see Theorem 2.2.10). For $m = 0$, this is the classical result of Chevalley mentioned above.

In their recent work [BR1], Yu. Berest and A. C. Ramadoss formulated a topological realization problem[2] for algebras of quasi-invariants for an arbitrary compact connected Lie group G. Specifically, they asked for the existence of topological spaces $X_m = X_m(G, T)$ (one for each multiplicity m) together with natural maps $BT \to X_m(G, T) \to BG$ satisfying certain axioms that are homotopy-theoretic analogs of geometric properties of varieties of quasi-invariants studied in [BEG] (see Section 2.3). They found that — in the rank one case (for $G = \mathrm{SU}(2)$) — the solution of the realization problem for algebras of quasi-invariants is given by a classical topological construction known as the Ganea ('fiber-cofiber') construction [G1][3].

However, in the general (higher rank) cases, Berest and Ramadoss showed that the classical Ganea construction does not produce spaces of quasi-invariants: instead, they use a sophisticated gluing construction represented by homotopy colimits. This leads us to the main question that we study in the present work:

Does there exist a version (generalization) of the Ganea construction that still works for higher rank groups, producing spaces with cohomological properties similar to those of

[2] It is worth mentioning that the question of topological realization of algebras of quasi-invariants was raised in the unpublished preprint [FF], where an explicit geometric construction was proposed in the case of $G = \mathrm{U}(2)$.

[3] The Ganea construction plays an important role in abstract homotopy theory, particularly in the study of the so-called Lusternik-Schnirelmann (LS) categories of spaces [G2]. However, this construction did not seem to be used in homotopy theory of compact Lie groups.

the spaces of quasi-invariants $X_m(G,T)$ constructed in [BR1]*?*

Through an examination of the conditions under which the classical Ganea construction works in the rank one case, we find some natural topological generalizations. These generalizations allow us to construct spaces whose rational cohomology have algebraic properties similar to those of classical quasi-invariants.

Instead of applying the classical Ganea construction, which involves the homotopy cofiber of homotopy fibers associated with a fibration $p_0 : X_0 \to BG$, an alternative approach is to combine homotopy pullbacks and homotopy pushouts of two fibrations $p_0 : X_0 \to BG$ and $Y_0 \to BG$ to construct a new fibration:

$$p_1 : X_1 = X_0 *_{BG} Y_0 \longrightarrow BG$$

A similar construction has already appeared in the literature on abstract homotopy theory under the name 'join construction' (see [D1]): we use a slightly generalized version of it that we call a *generalized relative join construction*. The (generalized) relative join construction can be iterated, giving rise to a tower of fibrations:

$$X_0 \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_m \longrightarrow \cdots \longrightarrow BG.$$

together with natural maps $p_m : X_m \to BG$.

It is important to note that the classical Ganea construction corresponds to a special case of the relative join construction when the second fibration is chosen to be $EG \to BG$. By expanding the scope of the approach to include other fibrations in the relative join construction, we obtain a broader framework that extends beyond the classical Ganea construction.

In the second part of their work [BR2], Berest and Ramadoss construct topological realizations of algebras of quasi-invariants for cyclic groups (viewed as rank one complex (pseudo-)reflection groups) in terms of p-local spaces (p-compact groups) called Sullivan spheres. To this end they use a different generalization of the classical Ganea construction which they call *Ganea diagrams*. A natural question that arises in this direction is whether one can combine the two generalizations — the relative join construction and the Ganea diagrams introduced in [BR2] — to provide a solution to the realization problem for classical quasi-invariants of higher rank Weyl groups. We leave this question as a subject for future work and exploration.

1.2 Main Results

1.2.1 Main Theorem

One of the main results of this dissertation is the following theorem (Theorem 6.3.1):

Theorem 1.2.1. *Let G be a compact connected Lie group with a maximal torus T and associated Weyl group W. Consider closed subgroups $\widetilde{G}$ and H of G that satisfy the following conditions:*

(a) $T \subseteq \widetilde{G}$ *(denoting the corresponding Weyl group as $\widetilde{W} := N_{\widetilde{G}}(T)/T$).*

(b) $H \subseteq \widetilde{G}$ *and $\widetilde{G}/H \cong \mathbb{S}^{2k-1}$ for some $k \geq 1$.*

Let $\widetilde{p}_0 : X_0 \to B\widetilde{G}$ be a fibration satisfying the following properties:

(a) *The homotopy fiber* $K_0 = \mathrm{hofib}\,(X_0 \to B\widetilde{G})$ *has even-dimensional rational cohomology.*

(b) $\dim_{\mathbb{Q}} H^*(K_0) = |\widetilde{W}|$.

We can construct spaces $X_m = X_m(X_0, G, \widetilde{G}, H)$ *together with natural fibrations* $\widetilde{p}_m : X_m \to B\widetilde{G}$ *and* $p_m : X_m \to BG$ *for all* $m \geq 0$*, which factors through* $\pi_m : X_m \to X_{m+1}$ *and satisfies*

1. *The rational cohomology* $H^*(X_m;\mathbb{Q})$ *is a free module over* $H^*(BG;\mathbb{Q})$ *of rank* $|W|$*.*
2. *The natural maps on rational cohomology*

$$H^*(BG;\mathbb{Q}) \hookrightarrow H^*(B\widetilde{G};\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(X_m;\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(X_0;\mathbb{Q}) \tag{1.1}$$

   are injective.

3. *When* $\widetilde{p}_0$ *is a Borel fibration (see Proposition 4.2.4 for definition), so is* $\widetilde{p}_m$*.*

As a consequence of this theorem, we can obtain an explicit expression for the rational cohomology of X_m and a basis of $H^*(X_m)$ as a free module over $H^*(BG)$ in many cases.

Classifying Spaces of Classical Lie Groups One straightforward generalization occurs when $X_0 = BT$. In this case, $H^*(X_0;\mathbb{Q})$ is a polynomial algebra.

As an example, we will consider the case of $G = \mathrm{U}(n)$.

Example 1.2.2 (Corollary 7.1.3)**.** For $G = \widetilde{G} = \mathrm{U}(n)$ and $H = \mathrm{U}(n-1)$, if we write $H^*(BT;\mathbb{Q}) = \mathbb{Q}[t_1, \ldots, t_n]$ where $\deg(t_i) = 2$, then by Borel theorem, we know $H^*(B\,\mathrm{U}(n);\mathbb{Q}) = \mathbb{Q}[c_1, \ldots, c_n]$ where $c_i = \sigma_i(t_1, \ldots, t_n)$ are the i-th elementary symmetric polynomials. In this case, the corresponding space X_m has rational cohomology

$$H^*(X_m;\mathbb{Q}) = \mathbb{Q}[c_1,\dots,c_n] + \mathbb{Q}[t_1,\dots,t_n]c_n^m$$

or equivalently, we can express it as

$$H^*(X_m;\mathbb{Q}) = \{f \in \mathbb{Q}[t_1,\dots,t_n \mid s_{ij}f \equiv f \mod (c_n^m), 1 \le i < j \le n\}$$

where $s_{ij}(f)(t_1,\dots,t_i,\dots,t_j,\dots,t_n) = f(t_1,\dots,t_j,\dots,t_i,\dots,t_n)$.

A basis of $H^*(X_m;\mathbb{Q})$ as a free module over $H^*(B\,\mathrm{U}(n);\mathbb{Q})$ is given by

$$\{1, t_1^{i_1}\cdots, t_n^{i_n}c_n^m, 0 \le i_1 < 1, \dots, 0 \le i_n < n, (i_1,\dots,i_n) \neq (0,\dots,0)\}$$

and it Hilbert series is given by

$$p_{X_m}(t) = \frac{1-t^{2mn}}{\prod_{i=1}^n(1-t^{2i})} + \frac{t^{2mn}}{(1-t^2)^n}.$$

Additionally, we compute in this case the equivariant K-theory of the (homotopy) fiber $F_m \coloneqq \mathrm{hofib}\{X_m \to BG\}$,

$$K^*_{\mathrm{U}(n)}(F_m) = \mathbb{Z}[c_1,\dots,c_n,c_n^{-1}] + \prod_{i=1}^n (t_i-1)^m \cdot \mathbb{Z}[t_1,\dots,t_n,(t_1\cdots t_n)^{-1}],$$

which can be viewed as a multiplicative analogue of the rational cohomology.

In the case of $G = \mathrm{SU}(2)$, this construction agrees with the one presented in [BR1]. Hence, our approach can be regarded as a different generalization for the higher rank cases. There is an important distinction: the rational cohomology $H^*(X_m;\mathbb{Q})$ is not Gorenstein. Additionally, if we look at the variety $A_m = \mathrm{Spec}(H^*(X_m;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C})$, the normalization map $\pi_m : \mathbb{A}^n \twoheadrightarrow A_m$ is not injective, which differs from the cases of classical quasi-invariants.

Classifying Spaces of Commutativity Another interesting collection of examples arises when considering $X_0 = B_{comm}G$, the classifying spaces of commutativity.

For a topological group G, Adem, Cohen and Torres-Giese [ACG] showed that the space of homomorphisms $\mathrm{Hom}(\mathbb{Z}^n, G)$ can be assembled to form a simplicial space whose geometric realizations $B_{comm}G$ is called the *classifying space of commutativity* of G. Adem and Gómez studied in [AG] the homotopy-theoretic properties of $B_{comm}(G)$ and showed that for a compact connected Lie group G, the rational cohomology of (the path components of) $B_{comm}G$ is a free module over $H^*(BG; \mathbb{Q})$ of rank W. They also computed the rational cohomology of $B_{comm}G$ for $G = \mathrm{U}(n), \mathrm{SU}(n)$ and $\mathrm{Sp}(n)$.

Our construction can be applied to the fibration $B_{comm}G \to BG$, allowing us to define X_m for the classifying spaces of commutativity in a similar manner. For $G = \mathrm{U}(n), \mathrm{SU}(n)$ and $\mathrm{Sp}(n)$, explicit bases are given using combinatorial methods (see, e.g., [A2] [V] [G3] [G4]). Here, we present the result for $G = \mathrm{U}(n)$.

Example 1.2.3 (Corollary 9.3.5)**.** For $G = \widetilde{G} = \mathrm{U}(n)$ and $H = \mathrm{U}(n-1)$, if we write

$$H^*(B_{comm}G; \mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^W / (c_1(X), \ldots, c_n(X))$$

where $c_i(X) = \sigma_i(x_1, \ldots, x_n)$, then the rational cohomology of X_m is given by

$$H^*(X_m; \mathbb{Q}) = \mathbb{Q}[c_1(Y), \ldots, c_n(Y)] + H^*(B_{comm}G_1)c_n(Y)^m$$

where $c_i(Y) = \sigma_i(y_1, \ldots, y_n)$. Let $\{g_\sigma\}_{\sigma \in S_n}$ be a basis of $H^*(B_{comm}G_1; \mathbb{Q})$ (see Definition 9.3.3 for an explicit construction) where $g_e = 1$, then $H^*(X_m; \mathbb{Q})$ is a graded module over $H^*(BU(n), \mathbb{Q})$ with basis

$$\{1, g_\sigma c_n(Y)^m\}_{\sigma \in S_n \setminus \{e\}}.$$

And the Hilbert series of $H^*(X_m)$ is given by

$$p_{X_m}(t) = \frac{1 - t^{2mn}}{\prod_{i=1}^{n}(1 - t^{2i})} + \frac{t^{2mn}(\sum_{\sigma \in S_n} t^{\deg(g_\sigma)})}{\prod_{i=1}^{n}(1 - t^{2i})}.$$

1.2.2 Other Examples

The main theorem in our work arises from considering two fibrations, $X_0 \to BG$ and $Y_0 = BH \to BG$. We also discovered additional examples of X_0 and Y_0 where we can apply the relative join construction to obtain spaces X_m with nice cohomological properties.

Spherical Fibrations Instead of considering $X_0 = BT$, we can choose alternative spaces that give rise to spherical fibrations $\mathbb{S}^{2k} \to X_0 \to BG$ and $\mathbb{S}^{2k+1} \to Y_0 \to BG$. In these cases, we can construct examples whose rational cohomology satisfies the Gorenstein property.

Example 1.2.4 (Proposition 10.2.2)**.** For $G = \mathrm{SO}(2n+1)$ and $m > 1$, there is a space X_m together with a fibration $p_m : X_m \to B\,\mathrm{SO}(2n+1)$ such that the rational cohomology of X_m is given by

$$H^*(X_m) = H^*(X_m) = \mathbb{Q}[q_1, \ldots, q_n, \xi_m]/(\xi_m^2)),$$

where $\deg(\xi_m) = 4mn + 2$. It is important to note that in this case, the induced map of $\pi_0 : X_0 = BT \to X_1$ on rational cohomology is not injective.

Conjugation Action in the Rank One Case In the rank one case, we can replace the left G-action on the fiber G of the second fibration $Y_0 = EG \to BG$ with the conjugation action. This provides an alternative example where the rational cohomology differs from the classical one.

Example 1.2.5 (Proposition 11.1.3)**.** For $G = \mathrm{SU}(2)$ and $m > 1$, there is a space X_m together with a fibration $p_m : X_m \to B\,\mathrm{SU}(2)$ such that the rational cohomology of X_m is given by

$$H^*(X_m) = \mathbb{Q}[q_1, \xi_m]/(\xi_m^2),$$

where $\deg(\xi_m) = 4m + 2$. It is worth noting that in this case, we still have the map $\pi_0 : X_0 = BT \to X_1$, although the induced map on rational cohomology is not injective.

1.3 Organization of the Dissertation

The dissertation is organized as follows.

Chapter 2 This chapter provides a review of classical quasi-invariants, covering their definitions, properties, and their connections to quantum Calogero-Moser systems and rational Cherednik algebras.

Chapter 3 The focus of this chapter is on Borel's theorem and the topological realization problem of quasi-invariants.

Chapter 4 In this chapter, we review the Ganea construction and the relative join construction. These techniques play a vital role in topological realization of quasi-invariants and serve as foundation for our subsequent generalizations.

Chapter 5 This chapter centers on the rank one case and presents the solution to the realization problem as outlined in [BR1] using Ganea construction.

Chapter 6 Building upon the previous chapters, we study in Chapter 6 special cases of the relative join construction. We apply this technique to construct towers of fibrations that give rise to generalized spaces of quasi-invariants. The main theorem (Theorem 6.3.1) is also proven in this chapter.

Chapters 7-11 These chapters present various examples of the relative join construction. The case of classifying spaces of classical Lie groups are discussed in Chapter 7 and its algebraic version is in Chapter 8. Chapter 9 focuses on classifying spaces of commutativity. Both Chapter 7 and Chapter 9 are direct applications of the main theorem. In Chapter 10, we study spherical fibrations that produce Gorenstein examples. Additionally, Chapter 11 is the application of the relative join construction in the context of the conjugation action for the rank one case. While these two examples do not directly follow from the main theorem, their proofs share some similarities.

CHAPTER 2

QUASI-INVARIANTS OF FINITE REFLECTION GROUPS

Quasi-invariants of finite reflection groups have been the subject of study in various branches of mathematics. Their first appearance can be traced back to the work of O. Chalykh and A. Veselov [CV1], where they were introduced in the context of commutative Schrödinger operators. Since then they have been studied by numerous researchers in a wide range of problems related to geometric analysis and mathematical physics [CV2][VSC][B1] [CFV] [FV2][FV3] [F], noncommutative algebra [ES] [BEG] [BC1], representation theory [FV1] [EG1] [BEG] [BC1], and combinatorics [VKM][GW1] [GW2] [GW3] [BM].

In this chapter, we will provide an overview of the fundamental properties of quasi-invariants and their connections to quantum Calogero-Moser systems and rational Cherednik algebras. While our focus will be on working over the base field $k = \mathbb{C}$, it is important to note that the algebraic properties of quasi-invariants also hold over the fields $\mathbb{Q}$ and $\mathbb{R}$.

2.1 Finite Reflection Groups

2.1.1 Definitions

Let V be a real Euclidean space equipped with a positive definite symmetric bilinear form $(-,-) : V \times V \to \mathbb{R}$.

Definition 2.1.1. A **reflection** s_α is defined as a linear operator on V that maps a nonzero vector α to $-\alpha$ while leaving the hyperplane H_α orthogonal to α

unchanged. More explicitly, the reflection s_α is given by the formula:

$$s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha.$$

Here, (β, α) denotes the inner product of vectors β and α, and (α, α) represents the inner product of α with itself.

It is easy to see that a reflection is an order 2 orthogonal linear transformation.

Definition 2.1.2. Let V be a finite-dimensional Euclidean space. A **finite reflection group** W is a finite subgroup of the general linear group $\mathrm{GL}(V)$ of V that is generated by orthogonal reflections. These reflections are associated with hyperplanes passing through the origin of V.

2.1.2 Invariants of Finite Reflection Groups

Let $k[V] = \mathrm{Sym}_k(V^*)$ be the symmetric algebra of the dual space, which is the same as the algebra of polynomial functions on V. There is a natural action of W on $k[V]$, coming from the action of W on V^*:

$$(w \cdot f)(v) = f(w^{-1}v)$$

for $w \in W, f \in V^*, v \in V$. This action preserves the natural grading of the polynomial algebra, ensuring that the degrees of the polynomials remain unchanged under the action of W.

Definition 2.1.3. A polynomial $f \in k[V]$ is W**-invariant** if $w \cdot f = f$ for all $w \in W$. We denote the subalgebra of W-invariant polynomials in $k[V]$ as $k[V]^W$.

It can be shown that $k[V]^W$ is also a polynomial algebra (see, e.g. [H4] Theorem 3.5).

Theorem 2.1.4 ([C2] Theorem (A))**.** *Let W be a finite reflection group acting on an n-dimensional vector space V over k. Then $k[V]^W$ is generated as an k-algebra by n homogeneous algebraically independent polynomials of positive degrees (together with 1).*

Furthermore, we have the following result (see, e.g., [H4] Proposition 3.6).

Proposition 2.1.5. *$k[V]$ is a free module over $k[V]^W$ of rank $|W|$.*

2.2 Quasi-invariants of Finite Coxeter Groups

2.2.1 Quasi-invariants

Let $\mathcal{A}_W$ be the set of reflection hyperplanes of W. The invariant algebra $k[V]^W$ can be characterized as the subalgebra of polynomials p satisfying

$$s_\alpha(p) = p, \ \ \forall \, H_\alpha \in \mathcal{A}_W.$$

To define quasi-invariants, we modify the invariant condition in the following way.

For each reflection hyperplane $H \in \mathcal{A}_W$, we choose a linear functional $\alpha_H \in V^*$ such that $H_\alpha = \ker(\alpha_H)$, and assign a non-negative integer m_α called the **multiplicity** of α. Note W acts naturally on the set of hyperplanes, and we require the multiplicity to be W-invariant, i.e. $m_\alpha = m_{\alpha'}$ whenever α and and α' belong to the same W-orbit.

Definition 2.2.1 ([CV1])**.** A polynomial p is **W-quasi-invariant of multiplicity** m

if it satisfies the condition

$$s_\alpha(p) \equiv p \mod \langle \alpha_H \rangle^{2m_\alpha}, \ \ \forall\, H_\alpha \in \mathcal{A}_W. \tag{2.1}$$

We denote by $Q_m(W)$ the subspace of $k[V]$ generated by all such polynomials.

There is a partial order on the set $\mathcal{M}_W$ of multiplicity functions $m : \mathcal{A}_W \to \mathbb{Z}_{\geq 0}$ given by $m \leq m'$ if $m_\alpha \leq m'_\alpha$ for all $H_\alpha \in \mathcal{A}_W$. This makes $\mathcal{M}_W$ a poset category.

In the following lemma, we outline elementary properties of quasi-invariants that directly follow from the definition.

Lemma 2.2.2. *1. $Q_m(W)$ is a finitely generated subalgebra of $k[V]$ that contains the invariant subalgebra $k[V]^W$ and is stable under the action of W.*

2. *The integral closure of $Q_m(W)$ is $k[V]$.*

3. *The algebras of quasi-invariants of different multiplicities form a contravariant diagram of shape $\mathcal{M}_W$, i.e. a functor*

$$\begin{aligned} \mathcal{M}_W^{op} &\longrightarrow \mathbf{Comm}_k \\ m &\longmapsto Q_m(W) \end{aligned}$$

The diagram of quasi-invariants can be viewed as a filtration on $k[V]$ indexed by $\mathcal{M}_W^{op}$, giving rise to a filtration of $k[V]$ as follows:

$$k\,[V]^W \subset \cdots \subset Q_m(W) \subset Q_{m'}(W) \subset \cdots \subset k\,[V]$$

for $m \geq m'$, and

$$\bigcap_m Q_m(W) = k[V]^W.$$

4. *$Q_m(W)$ is a finite rank module over $k[V]^W$.*

2.2.2 Algebraic Properties

Before discussing the algebraic properties of $Q_m(W)$, let's review some relevant definitions.

Definition 2.2.3. Let R be a commutative ring, $I \subset R$ is an ideal and M is a finitely generated R-module satisfying $IM \subsetneq M$, then the ***I*-depth** of M is defined as

$$\operatorname{depth}_I(M) := \min\{i : \operatorname{Ext}^i(R/I, M) \neq 0\}.$$

The **depth** of a local (graded) ring R with maximal (homogeneous) ideal $\mathfrak{m}$ is its $\mathfrak{m}$-depth as a module over itself.

Definition 2.2.4. The **Krull dimension** of R is the supremum of the lengths of all chains of prime ideals in R.

Definition 2.2.5. For a commutative Noetherian local ring R, a finite R-module $M \neq 0$ is a **Cohen-Macaulay** module if $\operatorname{depth}(M) = \dim(M)$. If R is an arbitrary Noetherian ring, then M is a **Cohen-Macaulay** module if $M_{\mathfrak{m}}$ is Cohen-Macaulay for all maximal ideals $\mathfrak{m}$ such that $M_{\mathfrak{m}} \neq 0$. A Noetherian ring R is a **Cohen-Macaulay** ring if it is a Cohen-Macaulay module as an R-module.

Definition 2.2.6. Let R be a finitely generated graded algebra of dimension n over $R_0 = k$ where k is a field. A **homogenous system of parameters** for R is a sequence of homogeneous elements $F_1, \ldots, F_n$ of positive degree in R such that $n = \dim(R)$ and $R/(F_1, \ldots, F_n)$ has Krull dimension 0.

Note when R is a such a graded ring, a homogeneous system of parameter always exists.

Theorem 2.2.7. *Let R be a finitely generated graded algebra of dimension n over $R_0 = k$ where k is a field. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. The following conditions are equivalent.*

(1) *Some homogeneous system of parameters is a regular sequence.*

(2) *Every homogeneous system of parameters is a regular sequence.*

(3) *For some homogeneous system of parameters* $F_1, \ldots, F_n$*, R is a free-module over* $k[F_1, \ldots, F_n]$.

(4) *For every homogeneous system of parameters* $F_1, \ldots, F_n$*, R is a free-module over* $k[F_1, \ldots, F_n]$.

(5) $R_{\mathfrak{m}}$ *is Cohen-Macaulay.*

(6) *R is Cohen-Macaulay.*

In particular, if $S = k[V]^W$ is the ring of invariants of a finite reflection group W, S is a polynomial ring and therefore a regular graded ring, so if $Q_m(W)$ is a finite free graded module over $k[V]^W$, then $Q_m(W)$ is Cohen-Macaulay.

Definition 2.2.8. A **Gorenstein local ring** is a commutative Noetherian local ring R with finite injective dimension as an R-module. A **Gorenstein ring** is a commutative Noetherian ring such that each localization at a prime ideal is a Gorenstein local ring.

The following proposition provides a simple criterion for a Cohen-Macaulay integral domain k-algebra to be Gorenstein.

Proposition 2.2.9 ([S5] Theorem 4.4)**.** *For a finitely generated commutative graded algebra R over a field k such that R is a Cohen–Macaulay integral domain, R is Gorenstein if and only if its Hilbert series is symmetric in the sense that*

$$f(t^{-1}) = (-1)^n t^s f(t)$$

for some integer s*, where n is the Krull dimension of R.*

We are now ready to state the main algebraic properties of the algebra of quasi-invariants.

Theorem 2.2.10 ([FV1] [EG1] [BEG])**.** *For any multiplicity m, $Q_m(W)$ is a free module over $k[V]^W$ of rank equal to $|W|$ and thus Cohen-Macaulay. Moreover $Q_m(W)$ is a Gorenstein algebra with Gorenstein shift $a = \dim(V) - 2\sum_{\alpha\in\mathcal{A}_W} m_\alpha$.*

Remark. In [FV1], Theorem 2.2.10 is proved for dihedral groups and conjectured for general W. [EG1] and [BEG] provide two different proofs for arbitrary finite reflection group W. A generalization to complex reflection groups appears in [BC1] where the Cohen-Macaulay property still holds.

2.2.3 Example: Rank 1 Case

An illustrative example is given in the rank one case.

Let $V = \mathbb{R}$ be a one dimensional Euclidean space. $W = \mathbb{Z}/2\mathbb{Z}$ acts on V by $\tau(v) = -v$, which induces an action on $k[V] \cong k[x]$ by $\tau(x) = -x$. The algebra of invariants is given by $k[x]^{\mathbb{Z}/2\mathbb{Z}} = k[x^2]$. In this case, any multiplicity function is given by a natural number $m \geq 0$. We choose the linear functional to be x, then the algebra of quasi-invariants of multiplicity m is given by

$$Q_m(\mathbb{Z}/2\mathbb{Z}) = \{f(x) \in k[x] : x^{2m} \mid (f(x) - f(-x))\} = k[x^2, x^{2m+1}].$$

It can be seen directly in this case that

$$Q_m(\mathbb{Z}/2\mathbb{Z}) = k[x^2] \cdot 1 \oplus k[x^2] \cdot x^{2m+1}$$

is a free module over the algebra of invariants $k[x^2]$ of rank 2, with basis $\{1, x^{2m+1}\}$. Therefore $Q_m(\mathbb{Z}/2\mathbb{Z})$ is Cohen-Macaulay. The Hilbert series of $Q_m(\mathbb{Z}/2\mathbb{Z})$ is given by

$$h_{Q_m(\mathbb{Z}/2\mathbb{Z})}(t) = \frac{1 + t^{2m+1}}{1 - t^2}$$

which shows that $Q_m(\mathbb{Z}/2\mathbb{Z})$ is Gorenstein.

2.3 The Variety of Quasi-invariants

Definition 2.3.1 ([BEG]). The **variety of quasi-invariants** of multiplicity m is defined as

$$V_m(W) := \operatorname{Spec} Q_m(W).$$

These varieties are equipped with natural projections

$$p_m : V_m(W) \longrightarrow V//W$$

where $V//W = \operatorname{Spec}(\mathbb{C}[V]^W)$. They form a diagram of schemes over the poset $\mathcal{M}_W$

$$\begin{aligned} \mathcal{M}_W &\longrightarrow \mathbf{Sch}_k \\ m &\longmapsto V_{m(W)} \\ m < m' &\mapsto (V_m(W) \xrightarrow{\pi_{m/m'}} V_{m'}(W)) \end{aligned} \tag{2.2}$$

with the following formal properties (cf. [BR1] Section 2.3):

1. $V_m(W)$ is a reduced irreducible scheme of finite type over k, equipped with W-action such that the maps in the diagram (2.2) are W-equivariant.
2. The morphism $p_0 : V_0(W) \to V//W$ coincides with the canonical projection $p : V \to V//W$ and the triangles

$$\begin{array}{ccc} V_m(W) & \xrightarrow{\pi_{m,m'}} & V_{m'}(W) \\ & {\scriptstyle p_m} \searrow \quad \swarrow {\scriptstyle p_{m'}} & \\ & V//W & \end{array}$$

commute for all $m < m'$. Thus we have a diagram of W-schemes over $V//W$.

3. The diagram (2.2) over $V//W$ induces an equivalence

$$\operatorname{colim}_{m\in\mathcal{M}_W} V_m(W) \xrightarrow{\sim} V//W$$

4. Each projection p_m factors naturally through $V_m//W$, inducing an isomorphism of schemes $V_m(W)//W \cong V//W$ for any m.

The next lemma is a deeper geometric property of varieties of quasi-invariants first proved in [BEG].

Lemma 2.3.2 ([BEG], Lemma 7.3)**.** *Each map $\pi_{m,m'} : V_m(W) \to V_{m'}(W)$ is a universal homeomorphism of schemes, i.e. a finite morphism of schemes that is (set-theoretically) surjective and injective on closed points.*

2.4 Quantum Calogero-Moser System

The algebra of quasi-invariants was first introduced in the study of quantum Calogero-Moser systems in [CV1]. We will briefly review its connection here.

Let $V = \mathbb{R}^n$ be a Euclidean space equipped with the standard inner product $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Consider a finite reflection group W and a W-invariant function $k : \mathcal{M}_W \to \mathbb{C}$, where $\mathcal{M}_W$ is the set of reflection hyperplanes of W. The function k is not necessarily integer-valued. For each reflection hyperplane $H \in \mathcal{M}_W$, we choose a linear functional $\alpha \in V^*$ such that $H = \ker(\alpha_H)$ and identify H with α_H. In [BC2], this pair $(\mathcal{A}_W, k)$ is referred to as a **configuration** in $\mathbb{R}^n$.

2.4.1 Calogero-Moser Operators

Let $V^{\text{reg}} := \{x \in V : (\alpha, x) \neq 0, \forall \alpha \in \mathcal{A}_W\}$ be the complement to the union of all reflection hyperplanes of W. We denote by $\mathbb{C}[V^{\text{reg}}]$ and $\mathcal{D}(V^{\text{reg}})$ the rings of regular functions and regular differential operators on V^{reg}, respectively.

Definition 2.4.1. For a given configuration $(\mathcal{A}_W, k)$, we associate a **Calogero-Moser operator**, denoted by L_k, which takes the form :

$$L_k = \Delta - \sum_{\alpha \in \mathcal{A}_W} \frac{k_\alpha(k_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} \tag{2.3}$$

where $\Delta = \sum_i \partial_i^2$ is the Laplacian operator in $\mathbb{R}^n$.

Example 2.4.2. The standard rational Calogero-Moser operator corresponds to the root system of type A_{n-1} with all $k_\alpha = k$. In this case, the operator L_k takes the form:

$$L_k = \Delta - \sum_{1 \leq i < j \leq n} \frac{k(k+1)}{(x_i - x_j)^2}.$$

where Δ is the Laplacian operator and x_i represents the coordinates of the particles. This operator can be interpreted as a quantum Hamiltonian describing the interactions between n particles on a one-dimensional line.

Definition 2.4.3. A **quantum integral** of a differential operator D is a nonconstant differential operator D' that commutes with D, i.e., $[D, D'] = 0$. If there exist n commuting algebraically independent quantum integrals, including D, denoted as $D_1, D_2, \ldots, D_n$, then the system is called a **quantum completely integrable system**. If there is an additional operator D_0 that commutes with D_i for $1 \leq i \leq n$ and is independent of them, then the system is called **algebraically integrable**.

2.4.2 Dunkl Operators

Note the W-action on V restricts to a free action on V^{reg}, so W acts naturally on $\mathbb{C}[V^{\text{reg}}]$ and $\mathcal{D}(V^{\text{reg}})$ and we can then form the crossed product $\mathcal{D}W := \mathcal{D}(V^{\text{reg}}) * W$, which is generated by $\mathcal{D}(V^{\text{reg}})$ and W subject to the relations:

$$wDw^{-1} = w \cdot D, \ \forall w \in W.$$

Definition 2.4.4. Given $y \in V$, we associate the following **Dunkl operator** (with respect to a configuration $(\mathcal{A}_W, k)$)

$$D_y = \partial_y + \sum_{\alpha \in \mathcal{A}_W} \frac{(\alpha, y)}{(\alpha, x)} k_\alpha \cdot s_\alpha$$

These Dunkl operators have the following properties.

Proposition 2.4.5 ([D4])**.** *For any $y, z \in V$ and $w \in W$,*

1. *D_y is an operator of degree -1 with respect to the natural grading of $\mathcal{D}W$.*
2. *$D_y D_z = D_z D_y$.*
3. *$wD_y w^{-1} = D_{wy}$.*

These properties ensure that the assignment $y \mapsto D_y$ can be extended to an (injective) algebra homomorphism:

$$\theta : \mathbb{C}[V^*] \hookrightarrow \mathcal{D}W$$
$$q \longmapsto D_q$$

Definition 2.4.6 ([EG2])**.** The **rational Cherednik algebra** $H_k = H_k(W)$ is generated by V, V^* and W, subject to the defining relations:

$$w \cdot x \cdot w^{-1} = w(x) \qquad w \cdot y \cdot w^{-1} = w(y)$$
$$[x\cdot, x'] = 0 \qquad [y, y'] = 0$$
$$[y, x] = y(x) - \textstyle\sum_{\alpha \in \mathcal{A}_W} k_\alpha \alpha(y) x(\alpha^\vee) \cdot s_\alpha$$

for any $x, x' \in V^*, y, y' \in V$, and $w \in W$. Here $\alpha^\vee = \frac{2(\alpha,\cdot)}{(\alpha,\alpha)} \in V^{**} \cong V$ is the coroot of α.

The map θ can be extended to a map

$$\Theta_k : H_k \longrightarrow \mathcal{D}W$$

satisfying

$$x \in V^* \longmapsto x, \qquad y \in V \longmapsto D_y, \qquad w \in W \longmapsto w,$$

which is known as the **Dunkl representation** of H_k.

Let $\mathbf{e} = \frac{1}{|W|}\sum_{w\in W} w$ be the symmetrizer in $\mathbb{C}[W]$, then $\mathbf{e} \in H_k$ is an idemponent.

Definition 2.4.7. The algebra $\mathbf{e}H_k\mathbf{e}$ is called **spherical subalgebra** of H_k.

The restriction of the Dunkl representation to the spherical subalgebra gives an injective algebra homomorphism

$$\Theta_k^{spher} : \mathbf{e}H_k\mathbf{e} \hookrightarrow \mathbf{e}\mathcal{D}W\mathbf{e} \cong \mathcal{D}(V^{\mathrm{reg}})^W. \tag{2.4}$$

called **spherical Dunkl representation**.

Proposition 2.4.8 ([H2]). *Let $v_1, \dots, v_n$ be an orthonormal basis of V and $q = v_1^2 + \cdots + v_n^2 \in \mathbb{C}[V^*]^W$, then $\Theta_k^{spher}(\mathbf{e}D_q\mathbf{e}) = L_k$ is the Calogero-Moser operator (2.3). The image of $\mathbf{e}\mathbb{C}[V^*]^W\mathbf{e}$ under the spherical Dunkl representation (2.4) forms a commutative subalgebra in $\mathcal{D}(V^{\mathrm{reg}})^W$ and thus the operator L_k defines a quantum completely integrable system.*

When k is not an integer-valued multiplicity, the image of $\mathbf{e}\mathbb{C}[V^*]^W\mathbf{e}$ under the spherical Dunkl representation is a maximal commutative subalgebra.

When k is an integer-valued multiplicity, this algebra can be extended to a larger commutative subalgebra in $\mathcal{D}(V^{\mathrm{reg}})$.

Theorem 2.4.9 ([CV1], [VSC]). *For any finite reflection group W and integer-valued multiplicity $k = m$, there exists a an algebra embedding*

$$\begin{aligned} \theta : Q_m(W) &\longrightarrow \mathcal{D}(V^{\text{reg}}) \\ q &\longmapsto L_q \end{aligned} \tag{2.5}$$

The image $\mathcal{D}_m(W)$ of this map has the following property.

Theorem 2.4.10 ([CFV]). *The image $\mathcal{D}_m(W)$ of (2.5) is the maximal commutative subalgebra of differential operators in $\mathcal{D}(V^{\text{reg}})$ which contains $\mathcal{D}(V^{\text{reg}})^W$.*

The explicit formula of L_q is given in [B1] by

$$L_q = \frac{1}{2^d d!}(\text{ad}_{L_k})^d(\hat{q}) \tag{2.6}$$

where $d = \deg(q)$, $\text{ad}_{L_k}(A) = [L_k, A]$ and $\hat{q}$ is the operator of multiplication by q. A direct consequence of this formula is the following result.

Theorem 2.4.11 ([CFV]). *The space $Q_m(W)$ is stable under the action of all operators $L_q \in D_m(W)$.*

2.4.3 Rational Cherednik Algebra and Quasi-invariants

Let $\mathcal{D}(V^{\text{reg}})^W_-$ be the algebra spanned by homogeneous element $D \in \mathcal{D}(V^{\text{reg}})^W$ with $\text{order}(D) + \deg(D) \leq 0$, where the order of a partial differential operator D is the highest order derivative it contains, and the degree of D is the power to which the highest order derivative is raised. Let $\mathscr{C}_k$ be the centralizer of L_k in $\mathcal{D}(V^{\text{reg}})^W_-$ and $\mathscr{B}_k$ be the subalgebra in $\mathcal{D}(V^{\text{reg}})^W$ generated by $\mathscr{C}_k$ and $\mathbb{C}[V]^W \subset \mathcal{D}(V)^W$.

Theorem 2.4.12 ([EG2] Theorem 4.8, [BEG] Proposition 4.10). *For integer-valued multiplicity m, the image of Θ_m^{spher} is $\mathscr{B}_m$.*

By Theorem 2.4.11 and Theorem 2.4.10, $Q_m(W)$ is stable under the action of $\mathscr{C}_m$, thus there is an action of $\mathscr{B}_m$ on $Q_m(W)$. This action commutes with the W-action on $Q_m(W)$ so $Q_m(W)$ obtains a $\mathbb{C}[W] \otimes \mathbf{e}H_m\mathbf{e}$-module structure.

CHAPTER 3

TOPOLOGICAL REALIZATION OF QUASI-INVARIANTS

Quillen's rational homotopy theory [Q] implies that any reduced, locally finite, graded commutative $\mathbb{Q}$-algebra A is topologically realizable. This means that there exists a topological space X such that its rational cohomology $H^*(X;\mathbb{Q})$ is isomorphic to A. Consequently, for any given multiplicity m, the algebra $Q_m(W)$ is realizable.

Instead of focusing on the realization of individual algebras for each multiplicity value, Berest and Ramadoss [BR1] posed the question of topological realization for the diagram of quasi-invariants and presented a systematic approach to construct spaces X_m in the case of rank one. More specifically, they want to define a functor:

$$X : \mathcal{M}_W^{op} \longrightarrow \mathrm{Ho}(\mathbf{Top}_*)$$

$$m \longmapsto X_m$$

such that the composite $X \circ H^* = Q$ recovers the functor

$$Q : \mathcal{M}_W^{op} \longrightarrow \mathbf{Comm}_{\mathbb{Q}}$$

$$m \longmapsto Q_m(W)$$

together with additional nice homotopical properties. In this context, $H^* : \mathrm{Ho}(\mathbf{Top}*) \to \mathbf{Comm}_{\mathbb{Q}}$ represents the functor that assigns to each homotopy type of pointed topological space its rational cohomology algebra.

The construction we are discussing is rooted in a fundamental result proved by A. Borel in his work on the rational cohomology of classifying spaces for compact Lie groups [B3]. We will provide a brief overview of this result. Additional details on universal principal bundles and classifying spaces can be found in the appendix.

3.1 Borel's Theorem

Let G be a compact connected Lie group with maximal torus T and associated Weyl group $W = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. Then W acts on T by conjugation:

$$\begin{aligned} W \times T &\longrightarrow T \\ (nT, t) &\longmapsto ntn^{-1}. \end{aligned}$$

Let EG be Milnor's model of universal G-bundle and $BG = EG/G$ be a model of classifying space of G. As EG is contractible, we can restrict the right G-action on EG to obtain a T-action, making EG a model of universal T-bundle. Consequently, we can use $BT = EG/T$ as our model of classifying space for T. Moreover, the Weyl group W naturally acts on BT via:

$$\begin{aligned} W \times BT &\longrightarrow BT \\ (nT, [x]_T) &\longmapsto \left[xn^{-1}\right]_T. \end{aligned}$$

Let $p : BT \to BG$ be the natural fibration induced by the inclusion map $i : T \hookrightarrow G$.

The following Borel's theorem is crucial to our construction, and a proof can be found in [DW2] Theorem 5.14(3).

Theorem 3.1.1 (Borel)**.** *The natural inclusion $i : T \hookrightarrow G$ induces a monomorphism on rational cohomology rings:*

$$i^* : H^*(BG;\mathbb{Q}) \hookrightarrow H^*(BT;\mathbb{Q})$$

whose image is the subring of W-invariants

$$H^*(BG,;\mathbb{Q}) \cong H^*(BT;\mathbb{Q})^W.$$

Now, let $V = \pi_1(T) \otimes \mathbb{Q}$, which is a $\mathbb{Q}$-vector space of dimension n, the rank of the group G. We establish an identification between $H^*(BT;\mathbb{Q})$ and the polynomial ring $\mathbb{Q}[V]$ as follows:

The natural action of the Weyl group W on T induces an action of W on $\pi_1(T)$, which extends to a linear action of W on V:

$$\rho : W \longrightarrow GL_{\mathbb{Q}}(V)$$

This action is a monomorphism that realizes W as a reflection subgroup of $\mathrm{GL}_{\mathbb{Q}}(V)$ (see [DW2] Theorem 5.16).

On the other hand, since T is a compact connected Lie group, the natural map $T \to \Omega BT$ is a (pointed) homotopy equivalence. Consequently, we can identify:

$$\pi_2(BT) \cong \pi_1(\Omega BT) \cong \pi_1(T).$$

The second Hurewicz homomorphism $\pi_2(BT) \to H_2(BT;\mathbb{Q})$ extends to an isomorphism of $\mathbb{Q}$-vector spaces $V = \mathbb{Q} \otimes \pi_2(BT) \xrightarrow{\cong} H_2(BT;\mathbb{Q})$ which in turn dualizes to an isomorphism $V^* \cong H^2(BT;\mathbb{Q})$. This isomorphism further extends to an isomorphism of graded $\mathbb{Q}$-algebras

$$H^*(BT;\mathbb{Q}) \cong \mathrm{Sym}_{\mathbb{Q}}(V^*) = \mathbb{Q}[V].$$

Therefore, using Borel's theorem, we can identify

$$H^*(BG;\mathbb{Q}) \cong \mathbb{Q}[V]^W$$

and the inclusion $\mathbb{Q}[V]^W \hookrightarrow \mathbb{Q}[V]$ can be realized by the principal fibration $BT \twoheadrightarrow BG$.

To simplify the notation, let's denote the rational cohomology of a space X as $H^*(X) = H^*(X;\mathbb{Q})$ from now on.

We begin by considering a model $B\,\mathrm{U}(1) = \mathbb{C}P^\infty$ of classifying space for the unitary group $\mathrm{U}(1)$. In this case, the rational cohomology of $B\,\mathrm{U}(1)$ is given by $H^*(B\,\mathrm{U}(1)) = H^*(\mathbb{C}P^\infty) = \mathbb{Q}[t]$, where $\deg(t) = 2$. We can extend this result to a torus $T = U(1)^n$ of rank n by observing that classifying spaces commutes with products. This extension yields

$$H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$$

where $\deg(t_i) = 2$.

Using Borel's theorem, we can establish similar identifications for the rational cohomology of classifying spaces associated with certain classical Lie groups.

Example 3.1.2.

1. $G = U(n)$, then $W = S_n$ acts on $\mathbb{Q}[t_1, \ldots, t_n]$ by permutation of indices, thus $H^*(B\,U(n)) = \mathbb{Q}[c_1, \ldots, c_n]$ where $c_i = \sigma_i(t_1, \ldots, t_n)$ is the i-th elementary symmetric polynomial.
2. $G = SU(n)$, then $W = S_n$ acts on $\mathbb{Q}[t_1, \ldots, t_n]/(c_1)$ by permutation of indices, thus $H^*(B\,SU(n)) = \mathbb{Q}[c_2, \ldots, c_n]$.
3. $G = Sp(n)$, then $W = B_n$ acts on $\mathbb{Q}[t_1, \ldots, t_n]$ by sign changes and permutation of indices, thus $H^*(B\,Sp(n)) = \mathbb{Q}[q_1, \ldots, q_n]$ where $q_i = \sigma_i(t_1^2, \ldots, t_n^2)$.
4. $G = SO(2n + 1)$, then $W = B_n$ acts on $\mathbb{Q}[t_1, \ldots, t_n]$ by sign changes and permutation of indices, thus $H^*(B\,SO(2n + 1) = \mathbb{Q}[q_1, \ldots, q_n]$.
5. $G = SO(2n)$, then $W = D_n$ acts on $\mathbb{Q}[t_1, \ldots, t_n]$ by even number of sign changes and permutation of indices, thus $H^*(B\,SO(2n)) = \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n]$.

3.2 Topological Realization of Quasi-invariants

Berest and Ramadoss proposed in [BR1] the following realization problem which extends the classical result of Borel.

3.3 The Realization Problem

Given a compact connected Lie group G with maximal torus T and associated Weyl group W, construct a diagram of spaces $X_m = X_m(G, T)$ over the poset $\mathcal{M}_W$:

$$\begin{array}{ccccccccc} BT = X_0 & \longrightarrow & \cdots & \longrightarrow & X_m & \xrightarrow{\pi_{m,m'}} & X_{m'} & \longrightarrow & \cdots \\ \downarrow p & & \swarrow p_m & & & \swarrow p_{m'} & & & \\ BG & & & & & & & & \end{array} \tag{3.1}$$

with the following properties

(QI1) each $X_m(G, T)$ is a CW-complex equipped with W-action and all maps are W-equivariant. The map $p_0 : X_0 \to BG$ coincides with the natural fibration $p : BT \to BG$ and the above construction commutes up to homotopy.

(QI2) The diagram (3.1) converges to BG in the sense that

$$\text{hocolim } \{X_1 \to X_2 \to \cdots \to X_m \to \cdots\} \xrightarrow{\sim} BG$$

where the homotopy colimit is taken over the telescope diagram and $\sim$ is a weak homotopy equivalence.

(QI3) Each map $p_m : X_m \to BG$ factors through the space $(X_m)_{hW}$ of homotopy orbits of the action of W on X_m

$$\begin{array}{ccc} X_m & \xrightarrow{p_m} & BG \\ & {\scriptstyle q_m}\searrow \quad \nearrow{\scriptstyle \overline{p}_m} & \\ & (X_m)_{hW} & \end{array}$$

and the induced map $\overline{p}_m$ gives a rational cohomology isomorphism,

$$H^*_W(X_m) \cong H^*(BG).$$

(QI4) Each π_i induces an injective rational cohomology algebra map so that the map $p^* : H^*(BG) \to H^*(BT)$ factors into a $\mathcal{M}_W^{op}$-diagram of $\mathbb{Q}$-algebras

$$H^*(BG) \hookrightarrow \cdots \hookrightarrow H^*(X_{m'}) \xhookrightarrow{\pi_{m,m'}} H^*(X_m) \hookrightarrow \cdots \hookrightarrow H^*(BT)$$

where $m < m'$.

(QI5) With natural identification $H^*(BT) = \mathbb{Q}[V]$, the map $\pi_{0,m}^* : H^*(X_m) \to H^*(BT)$ induces isomorphisms

$$H^*(X_m) \otimes \mathbb{C} \cong Q_m(W)$$

where $Q_m(W)$ are the subalgebras of W-quasi-invariants of multiplicity m in $\mathbb{C}[V]$.

Berest and Ramadoss [BR1] discover that the realization problem can be solved in the rank one case through Ganea construction. In the upcoming chapter we will review Ganea construction and its generalizations.

CHAPTER 4

RELATIVE JOIN CONSTRUCTION

In this chapter, our focus is on (pointed) compact-generated spaces, specifically CW-complexes. From a homotopical perspective, many classical topological constructions, which are built from diagrams of spaces, are not homotopy invariant. This means that if we replace the diagram of spaces with a homotopy equivalent one (as defined in Definition 4.1.5), the resulting construction does not yield a space that is homotopy equivalent to the original one. This observation motivates algebraic topologists to develop concepts that are homotopy invariant. In particular, definitions such as the homotopy fiber and homotopy cofiber of a map between spaces, the Ganea ('fiber-cofiber') construction associated with fibrations, and the generalization to homotopy pullback and homotopy pushout, as well as the relative join construction, are examples of homotopy invariant constructions.

It is important to mention that the construction of homotopy pushouts, homotopy pullbacks, and specifically homotopy fibers and homotopy cofibers can be carried out in a pointed model category using abstract homotopy theory, as described in [D1]. However, since we are working with topological spaces, we have explicit models for these constructions. The main references of this chapter are [H1], [S1] and [D1].

4.1 Homotopy Fiber

In general, taking the fiber of a map at an arbitrary point does not preserve homotopy equivalence. However, this issue can be resolved by "replacing a map by a fibration," or more precisely, factoring a map f through a weak equivalence followed by a fibration and then taking the fiber of this fibration. Such a factorization is only unique up to homotopy, and therefore, the construction of homotopy fiber is also unique up to homotopy.

Definition 4.1.1. The **pathspace fibration** of $f : A \to B$ is defined as

$$E^f := \{(a, \gamma) \in A \times B^I \mid \gamma(0) = f(a)\}.$$

Definition 4.1.2. The **homotopy fiber** F_f or mapping cylinder of $f : A \to B$ is defined as the fiber of $E^f \to B$,

$$\begin{array}{ccc} F_f = \mathrm{hofib}(f) & \longrightarrow & E^f \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \beta^f} \\ * & \xrightarrow{b_0} & B \end{array}$$

Explicitly,

$$F_f = \{(a, \gamma) \in A \times B^I | f(a) = \gamma(0), b_0 = \gamma(1)\}.$$

In practice, we usually work with pointed spaces and based maps $f : (A, a_0) \to (B, b_0)$ so we have a canonical choice of homotopy fiber and homotopy cofiber.

Example 4.1.3. When $f : b_0 \hookrightarrow B$ is the inclusion of the basepoint, Then $E^f = \mathcal{P}B = \mathrm{Map}_*(I, B)$ is the **based path space** of B, the fiber ΩB is **(based) loop space** of B which is defined as the pullback of the following diagram:

$$\begin{array}{ccc} \Omega B & \longrightarrow & \mathcal{P}B \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & B \end{array}$$

and we get the loop space fibration

$$\Omega B \longrightarrow \mathcal{P}B \longrightarrow B.$$

When $p : E \to B$ is already a fibration with fiber F, the following proposition (see, e.g., [H1] Proposition 4.65) shows that the homotopy fiber of p is in fact homotopy equivalent to the fiber F, so up to homotopy we do not distinguish fiber and homotopy fiber of a fibration.

Proposition 4.1.4. *If $p : E \to B$ is a fibration, then the inclusion $E \to E^p$ is a fiber homotopy equivalence. In particular the homotopy fibers of p are homotopy equivalent to the fibers of p.*

In particular, we do not need to work with actual fibration if we are only interested in spaces up to homotopy.

Definition 4.1.5. The **homotopy category** of pointed spaces is the category whose objects are pointed topological spaces and morphisms are equivalence classes of pointed maps under the relation of homotopy rel $\{*\}$. Two diagrams of spaces are **equivalent up to homotopy** if there is a natural equivalence between them in the homotopy category.

Definition 4.1.6. A sequence $W \to X \to Y$ is called a **homotopy fibration** if it is equivalent up to homotopy to a fibration sequence. Explicitly, there is a homotopy commutative diagram

$$\begin{array}{ccccc} W & \longrightarrow & X & \longrightarrow & Y \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ F & \longrightarrow & E & \longrightarrow & B \end{array}$$

where the vertical maps are homotopy equivalences and the bottom row is a fibration sequence.

4.2 Borel Construction

Definition 4.2.1. Let X be a G-space, the space $X_{hG} \coloneqq X \times_G EG$ is called the **Borel Construction** on X.

Definition 4.2.2. The equivariant cohomology of X is defined as $H^*_G(X) \coloneqq H^*(X_{hG})$.

Example 4.2.3. When $X = *$, $H^*_G(*) = H^*(BG)$.

Proposition 4.2.4. *The topological Borel construction X_{hG} fits into a fibration*

$$X \longrightarrow X_{hG} \longrightarrow BG.$$

which is called **Borel fibration**.

Combing this proposition and Theorem B.4.10 we have the following result.

Proposition 4.2.5. *For a compact Lie group G and a G-space X, if there is a fibration*

$$X \longrightarrow E \xrightarrow{f} BG$$

such that there is commutative diagram

$$\begin{array}{ccccc} X & \longrightarrow & X_{hG} & \longrightarrow & BG \\ \| & & \downarrow{\scriptstyle \alpha} & & \| \\ X & \longrightarrow & E & \longrightarrow & BG \end{array}$$

then α is a homotopy equivalence, $X_{hG} \simeq E$.

Applying Corollary B.4.14, we have the following result.

Corollary 4.2.6. *Assume X and G are connected. If $X \to X/G$ is a principal G-bundle, then $X_{hG} \simeq X/G$ and thus $H^*_G(X) = H^*(X_{hG})$.*

4.3 Homotopy Cofiber

Definition 4.3.1. Let $f : A \to B$ be a map of pointed spaces, the (reduced) **mapping cylinder** M_f of f is the following pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow \\ (A \times I)/(* \times I) & \longrightarrow & M_f \end{array}$$

where $i_0(a) = (a, 0)$. Explicitly, $M_f = (A \times I)/(* \times I) \cup_A B$.

By applying this construction to the basepoint inclusion map $* \to A$, we obtain a cofibration

$$i_1 : A \longrightarrow CA$$

where $CA = (A \times I)/((A \times \{0\} \cup (* \times I))$ is the (reduced) **mapping cone** of A. The cofiber ΣA of of i_1 is called the (reduced) **suspension** of A. Note

$$CA = A \wedge I, \ \Sigma A = \mathbb{S}^1 \wedge A$$

where $X \wedge Y := (X \times Y)/(X \vee Y)$.

Remark. The unreduced mapping cylinder and mapping cone can be defined by replacing $(A \times I)/(* \times I)$ by $A \times I$ in the definition. For a pointed space with nondegenerate basepoint, the reduced and unreduced versions are homotopy equivalent.

Definition 4.3.2. The **homotopy cofiber** of $f : A \to B$ is the **mapping cone** $C_f = M_f/(A \times \{0\})$.

4.4 Homotopy Pullback and Homotopy Pushout

In general, pullback or pushout does not preserve homotopy equivalence. However, when we consider the pullback along a fibration or the pushout along a cofibration, they do preserve homotopy equivalences.

Proposition 4.4.1 ([S1] Proposition 7.6.1)**.** *Let*

$$\begin{array}{ccc} E & \xrightarrow{g'} & A \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

be a homotopy commutative square in which A, B and C have the homotopy type of connected CW complexes, then the following are equivalent:

1. *there exists an induced map of fibers $F_{g'} \to F_g$ which is a homotopy equivalence.*
2. *the diagram is equivalent in the homotopy category to a pullback in which the horizontal maps are fibrations;*
3. *the diagram is equivalent in the homotopy category to a pullback in which the vertical maps are fibrations;*
4. *there exists an induced map of homotopy fibers $F_{f'} \to F_f$ which is a homotopy equivalence.*

The conditions are satisfied when, by replacing either f or g with a fibration, there exists a homotopy equivalence between E and the pullback. This observation motivates the definition of the homotopy pullback.

Definition 4.4.2. Given maps $f : A \to B$ and $g : C \to B$, there is a unique homotopy commutative diagram (up to equivalence of diagrams in the homotopy category) in which the maps f, g form the bottom right corner and the equivalent conditions above are satisfied. This square is called the **homotopy pullback** of f and g and we write $E \simeq A \times^h_B C$. Dually we can define **homotopy pushout** for

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ C & \xrightarrow{f'} & D \end{array}$$

and write $D \simeq A \vee^h_B C$.

A standard construction of homotopy pullback of $A \xrightarrow{f} B \xleftarrow{g} C$ is

$$E_{f,g} = \{(a, \gamma, c) \in A \times B^I \times C | \, f(a) = \gamma(0), g(c) = \gamma(1).$$

A standard construction of homotopy pushout of $A \xleftarrow{f} B \xrightarrow{g} C$ is

$$M_{f,g} = (A \coprod (B \times I) \coprod C)/(f(b) \sim (b, 0), g(b) \sim (b, 1)).$$

If we work with pointed spaces, we need to replace $B \times I$ with $B \times I/(* \times I)$.

4.5 Ganea Construction

Definition 4.5.1. For pointed spaces A and B, the (reduced) **join** of A and B, denoted $A * B$ is defined by

$$A * B := (A \times I \times B)/ \sim$$

where $(a, 0, b) \sim (a', 0, b), (a, 1, b) \sim (a, 1, b')$ and $(*, t, *) \sim (*, t', *)$ for all $a, a' \in A, b, b' \in B$ and $t, t' \in I$. We usually write elements in $A * B$ as $ta + (1-t)b, 0 \leq t \leq 1$.

The join construction is associative, and the m-fold join $A_1 * A_2 * \cdots * A_m$ can be defined either inductively or simultaneously.

Note $\Sigma(A \wedge B) = (A * B)/(CA \vee CB)$ where $CA \vee CB$ is contractible, thus we have the following result (see, e.g., [S1] Proposition 7.1.1).

Proposition 4.5.2. *If the basepoints of A and B are closed subsets then the quotient map $A * B \to \Sigma(A \wedge B)$ is a homotopy equivalence.*

In particular the inclusion $A \hookrightarrow A * B$ and $B \hookrightarrow A * B$ are null homotopic.

Throughout the rest of this section we assume all spaces are CW complexes to ensure all basepoints are closed.

It turns out that the join construction is given by homotopy pushout (see, e.g., [S1] Proposition 7.7.2).

Proposition 4.5.3. *The homotopy pushout of the projection maps $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ is homotopy equivalent to $A * B$.*

Proof. The homotopy type of homotopy pushout can be seen by taking the actual pushout after replace $\pi_1 : A \times B \to A$ by the cofibration $A \times B \hookrightarrow A \times CB$. □

The Ganea construction[1] provides a method for constructing a new fibration from an existing one (see, e.g., [S1] Theorem 7.7.3).

Theorem 4.5.4 (Ganea)**.** *Let $F \xrightarrow{j} E \xrightarrow{p} B$ be a fibration and let $\widetilde{p} : E/F \to B$ be the induced map. Then there is a homotopy commutative diagram of homotopy fibrations*

$$\begin{array}{ccccc} F & \xrightarrow{j} & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle *} & & \downarrow & & \| \\ \Omega B * F & \longrightarrow & E/F & \xrightarrow{p_1} & B \end{array}$$

[1]This construction was originally introduced by Ganea in [G1] and [G2] to study the Lusternik-Schnirelmann category of a topological space B, which is the least integer $k \geq 0$ such that B can be covered by $k + 1$ contractible subsets of B.

Furthermore, up to homotopy the composition $\Omega B * F \to E/F \to \Sigma F$ is the map $(w,t,f) \mapsto (\mu(w^{-1}, f), t)$ where $\mu : \Omega B \times F \to F$ is the action map from the induced principal homotopy fibration $\Omega B \to F \to E$.

Proof. We only sketch the part proof for the homotopy type of $\mathrm{hofib}(p_1)$. Up to homotopy equivalence we may replace E/F by $E \cup_F CF$ where $i : E \cup_F CF \xrightarrow{\simeq} E/F$ is the map that collapses the cone. The homotopy fiber Q of $\widetilde{p}i$ is homotopy equivalent to the pullback Q of the fibration $\mathcal{P}B \to B$ and $E \cup_F CF \to B$. Let Q_E, Q_F and Q_{CF} be the pullback of $\mathcal{P}B \to B$ with the composition of $E \cup_F CF \to B$ of the inclusions from E, F and CF into $E \cup_F CF$. The space Q is given by $Q_E \cup_{Q_F} Q_{CF}$. Note $Q_E \simeq F$ and $Q_F \simeq \Omega B \times F$ and $Q_{CF} \simeq \Omega B \times CF$, where the last two homotopy equivalences are because the maps $F \to E \cup_F CF \to B$ and $CF \to E \cup_F CF \to B$ are trivial, thus $Q \simeq F \cup_{\Omega B \times F} (\Omega B \times CF) = \Omega B * F$. □

Remark. The Ganea construction, which involves taking homotopy fiber followed by homotopy cofiber, is also referred to as the **fiber-cofiber construction**.

The above construction can be iterated, and we get a tower of (homotopy) fibration sequence over B:

$$\begin{array}{ccccc} F & \xrightarrow{j} & X & \xrightarrow{p} & B \\ \downarrow & & \downarrow{\scriptstyle \pi_0} & & \| \\ F_1 & \xrightarrow{j_1} & X_1 & \xrightarrow{p_1} & B \\ \downarrow & & \downarrow{\scriptstyle \pi_1} & & \| \\ F_2 & \xrightarrow{j_2} & X_2 & \xrightarrow{p_2} & B \\ \downarrow & & \downarrow & & \| \\ \vdots & & \vdots & & \vdots \end{array} \tag{4.1}$$

where

$$X_m := \mathrm{hocof}(j_{m-1}), \qquad F_m := \mathrm{hofib}(p_m), \qquad m \geq 1$$

Ganea theorem implies that

$$F_m \simeq F * \Omega B * \cdots * \Omega B$$

which is compatible with the fiber inclusion $F_m \hookrightarrow F_{m+1}$. Furthermore, the whisker maps $p_m : X_m \to B$ induce a weak homotopy equivalence

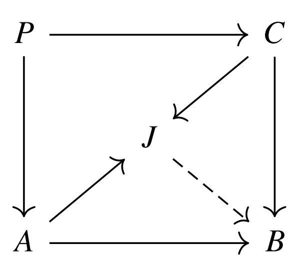
$$\text{hocolim}\{X \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} X_2 \xrightarrow{\pi_2} \cdots \to X_m \to \cdots\} \xrightarrow{\sim} B$$

where the homotopy colimit is taken over the telescope diagram in the middle of (4.1).

4.6 Relative Join Construction

Follow the convention in [D1], the (relative) join is defined as follows.

Definition 4.6.1. Let $A \to B$ and $C \to B$ be two maps and let $P \simeq A \times_B^h C$ and $J \simeq A \vee_P^h C$, then we call J (together with the whisker map $J \to B$) the **join** of A and C over B and write $J \simeq A *_B C$. When $B \simeq *$, we simply write $J \simeq A * C$.



Example 4.6.2. The usual join of two (pointed) spaces $A * B$ is the join of A and B over $*$.

To distinguish the actual join of spaces (over a point) with the join of A and C over B, we will call $A *_B C$ **relative join**.

Example 4.6.3. If $F \hookrightarrow E \xrightarrow{p} B$ is a fibration, then $E/F \simeq E *_B *$.

Proposition 4.6.4 ([D1] Proposition 4.2)**.** *Let $F \to E \to B$ and $F' \to E' \to B$ be two fibration sequence, then there is a fibration sequence*

$$F * F' \longrightarrow E *_B E' \longrightarrow B$$

Note the relative join construction is also associative.

We have the following corollary which generalizes the Ganea tower (4.1).

Corollary 4.6.5. *Let $F \to E \to B$ and $F' \to E' \to B$ be two fibration sequence, then there is a tower of fibration sequences*

$$\begin{array}{ccccc}
F & \longrightarrow & E & \longrightarrow & B \\
\downarrow & & \downarrow & & \| \\
F * F' & \longrightarrow & E *_B E' & \longrightarrow & B \\
\downarrow & & \downarrow & & \| \\
\cdots & & \cdots & & \cdots \\
\downarrow & & \downarrow & & \| \\
F *^m F' & \longrightarrow & E *_B^m E' & \longrightarrow & B \\
\downarrow & & \downarrow & & \| \\
\cdots & & \cdots & & \cdots
\end{array} \tag{4.2}$$

where

$$E *_B^m E' = (E *_B^{m-1} E') *_B E'$$

is defined inductively.

An interesting application of the relative join construction is the following result.

Proposition 4.6.6. *If $F_1 \to X_1 \to BG$ and $F_2 \to X_2 \to BG$ are two Borel (homotopy) fibrations, then $F_1 * F_2 \to X_1 *_{BG} X_2 \to BG$ is a Borel (homotopy) fibration and $X_1 *_{BG} X_2 \simeq (F_1 * F_2)_{hG}$.*

Proof. For simplicity we can assume $X_1 = (F_1)_{hG}$ and $X_2 = (F_2)_{hG}$. We have the following commutative diagram of fibrations

$$\begin{array}{ccccc}
F_1 \times F_2 & \longrightarrow & X_1 \times_B X_2 & \longrightarrow & BG \\
\| & & \downarrow & \lrcorner & \downarrow \Delta \\
F_1 \times F_2 & \longrightarrow & X_1 \times X_2 & \longrightarrow & BG \times BG
\end{array}$$

where $X_1 \times X_2 = (F_1 \times F_2)_{h(G\times G)}$. We have another fibration

$$F_1 \times F_2 \longrightarrow (F_1 \times F_2)_{hG} \longrightarrow BG$$

which factors as follows

$$\begin{array}{ccccc}
F_1 \times F_2 & & \longrightarrow & & F_1 \\
\downarrow & \searrow & & & \downarrow \\
 & & (F_1 \times F_2)_{hG} & \longrightarrow & X_1 \\
\downarrow & & \downarrow & & \downarrow \\
F_2 & \longrightarrow & X_2 & \longrightarrow & BG
\end{array}$$

thus by universal property of pullback, we know there is a map $\alpha : (F_1 \times F_2)_{hG} \to X_1 \times_B X_2$ such that the following diagram commutes

$$\begin{array}{ccccc}
F_1 \times F_2 & \longrightarrow & (F_1 \times F_2)_{hG} & \longrightarrow & BG \\
\| & & \downarrow \alpha & & \| \\
F_1 \times F_2 & \longrightarrow & X_1 \times_B X_2 & \longrightarrow & BG
\end{array}$$

thus by Proposition 4.2.5,

$$(F_1 \times F_2)_{hG} \simeq X_1 \times_B X_2.$$

Let $\widetilde{F}_1$ be a replacement of F_1 to construct the following homotopy pushout as actual pushout.

$$\begin{array}{ccccc}
F_1 \times F_2 & \longrightarrow & \widetilde{F}_1 & \overset{\sim}{\twoheadrightarrow} & F_1 \\
\downarrow & & \downarrow & & \\
F_2 & \longrightarrow & F_1 * F_2 & &
\end{array}$$

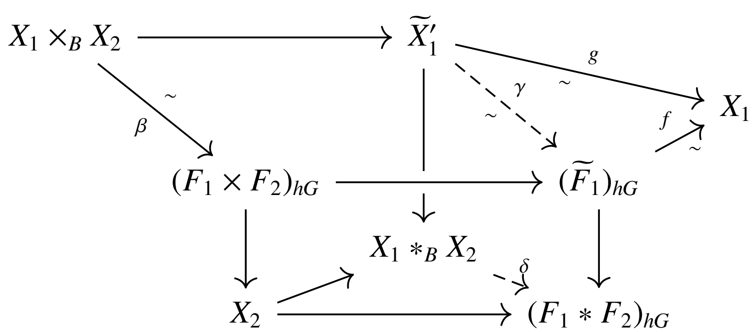
It induces a commutative diagram on the Borel construction

$$\begin{array}{ccccc} (F_1 \times F_2)_{hG} & \longrightarrow & (\widetilde{F}_1)_{hG} & \xrightarrow[\sim]{f} & X_1 \\ \downarrow & & \downarrow & & \\ X_2 & \longrightarrow & (F_1 * F_2)_{hG} & & \end{array}$$

Similarly, let $\widetilde{X}_1$ be a replacement of X_1 to construct the following homotopy pushout as actual pushout.

$$\begin{array}{ccccc} X_1 \times_B X_2 & \longrightarrow & \widetilde{X}_1 & \xrightarrow[\sim]{g} & X_1 \\ \downarrow & & \downarrow & & \\ X_2 & \longrightarrow & X_1 *_B X_2 & & \end{array}$$

Putting them together we obtain a commutative diagram



where β is the homotopy inverse of α and $\gamma = h \circ g$ where $h : X_1 \to (\widetilde{F}_1)_{hG}$ is the homotopy inverse of f. Thus by universal property of pushout there is a induced map $\delta : X_1 *_B X_2 \to (F_1 * F_2)_{hG}$, so by applying Proposition 4.2.5 again, we see that $X_1 *_B X_2 \simeq (F_1 * F_2)_{hG}$ and $F_1 * F_2 \to X_1 *_{BG} X_2 \to BG$ is a (homotopy) Borel fibration. □

CHAPTER 5

TOPOLOGICAL REALIZATION OF QUASI-INVARIANTS

In this chapter we review the solution of topological realization problem of quasi-invariants presented in Section 3.3 in the rank one case, as described in [BR1]. They also compute the equivariant K-theory of the fibers in this case. These computations offer extra information derived from the spaces of quasi-invariants. To provide necessary background, we include a brief review of equivariant K-theory for compact connected Lie groups in the appendix.

5.1 Topological Realization in Rank 1

Let $G = \mathrm{SU}(2)$ with maximal torus $T = \mathrm{U}(1)$ and Weyl group $W = \mathbb{Z}/2\mathbb{Z}$. In this case $\mathcal{M}_W = \mathbb{Z}_{\geq 0}$. Consider the following fibration

$$G/T \xrightarrow{\ j\ } BT \xrightarrow{\ p\ } BG \tag{5.1}$$

where p is the map induced by the inclusion $i : T \hookrightarrow G$.

Theorem 5.1.1 ([BR1] Theorem 3.9)**.** *The diagram of spaces obtained by applying iterated Ganea construction* (4.1) *to the fibration* (5.1) *is given as follows:*

$$BT = X_0(G,T) \longrightarrow X_1(G,T) \longrightarrow \cdots \longrightarrow X_m(G,T) \xrightarrow{\ \pi_m\ } X_{m+1}(G,T) \tag{5.2}$$

with the whisker map p_m induced by the Ganea construction. It satisfies the following properties:

(QI1) *Each X_m is a CW-complex equipped with a W-action and all the maps are W-equivariant. In particular the map $p_0 : X_0(G,T) \to BG$ coincides with the* *canonical map $p : BT \to BG$, and for all m, the following diagram commutes up to homotopy:*

$$\begin{array}{ccc} X_m(G,T) & \xrightarrow{\pi_m} & X_{m+1}(G,T) \\ & {\scriptstyle p_m}\searrow \quad \swarrow{\scriptstyle p_{m+1}} & \\ & BG & \end{array}$$

(QI2) *The diagram* (5.2) *'converges' to BG in the sense that the maps p_m induce a weak homotopy equivalence of spaces:*

$$\mathrm{hocolim}_{\mathcal{M}_W} X_m(G,T) \xrightarrow{\sim} BG.$$

(QI3) *Each map $p_m : X_m(G,T) \to BG$ factors naturally (in m) through the fiber inclusions into to the space $X_m(G,T)_{hW}$ of the homotopy orbits of the action of W on $X_m(G,T)$:*

$$\begin{array}{ccc} X_m(G,T) & \xrightarrow{p_m} & BG \\ & \searrow \quad \nearrow{\scriptstyle \overline{p}_m} & \\ & X_m(G,T)_{hW} & \end{array}$$

and the map $\overline{p}_m$ induces isomorphism on rational cohomology

$$H_W(X_m) \cong H^*(BG).$$

(QI4) *Each map π_m induces an injective map on rational cohomology so that the Borel homomorphism p^* factors into a $\mathcal{M}_W^{op}$-diagram of algebras*

$$H^*(BG) \hookrightarrow \cdots \hookrightarrow H^*(X_m(G,T)) \overset{\pi_m^*}{\hookrightarrow} H^*(X_{m+1}(G,T)) \hookrightarrow \cdots \hookrightarrow H^*(BT).$$

(QI5) *With natural identification $H^*(BT) = \mathbb{Q}[V]$, the maps $\pi_{0,m}^* : H^*(X_m) \to H^*(BT)$ given by the composite of π_j's induces isomorphisms*

$$H^*(X_m) \otimes \mathbb{C} \cong Q_m(W)$$

for all m.

The homotopy fiber F_m at stage m can be represented by the (unreduced) iterated join

$$F_m = G/T *^m \Omega BG \simeq G/T *^m G = G/T * E_{m-1}G \tag{5.3}$$

where $E_{m-1}G$ is Milnor's model for $(m-1)$-universal principal G-bundle and it carries a left holonomy action $\Omega BG \times F \to F_m$ that under the identification $\Omega BG \simeq G$, corresponds to the diagonal action of G

$$\begin{array}{ccc} G \times F_m & \longrightarrow & F_m \\ (g, (t_0 g_0 T + t_1 g_1 + \cdots + t_m g_m)) & \mapsto & (t_0 g g_0 T + t_1 g g_1 + \cdots + t_m g g_m) \end{array} \tag{5.4}$$

where $g_i \in G$ and $t_0, \cdots, t_m) \in \Delta^m$. By Proposition 4.6.6, the fibration

$$F_m \longrightarrow X_m \longrightarrow BG$$

can be identified with the Borel fibration

$$F_m \longrightarrow (F_m)_{hG} \longrightarrow BG. \tag{5.5}$$

5.2 Equivariant K-theory

Let $\hat{T} := \mathrm{Hom}(T, \mathrm{U}(1))$ be the character lattice and $R(T)$ the representation ring of T. We identify $R(T) = \mathbb{Z}[\hat{T}]$ and write e^λ the element in $R(T)$ corresponding to the characters $\lambda \in \hat{T}$.

Let Φ be the root system determine by (G, T) and choose Φ_+ a subset of positive roots in Φ. Let s_α be the reflection in W corresponding to $\alpha \in \Phi_+$, then the difference $e^\lambda - e^{s_\alpha(\lambda)}$ in $R(T)$ is uniquely divisible by $1 - e^\alpha$ for any $\lambda \in \hat{T}$.

Definition 5.2.1. An element $f \in R(T)$ is called **exponential quasi-invariant** of W of multiplicity m if

$$\frac{(1 - s_\alpha) f}{1 - e^\alpha} \equiv 0 \mod (1 - e^{\frac{\alpha}{2}})^{2m_\alpha}, \ \forall \alpha \in \Phi_+$$

We write $\mathcal{Q}_m(W)$ the set of all exponential quasi-invariants of multiplicity m.

Theorem 5.2.2 ([BR1] Theorem 5.6)**.** *There is a natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded commutative rings*

$$K_G^*(F_m) \cong \mathcal{Q}_M(W)$$

thus $K_G^0(F_m) \cong \mathcal{Q}_m(W)$ and $K_G^1(F_m) = 0$ for all $m \in \mathbb{Z}_+$.

5.3 Ganea Construction in the Higher Rank Case

The Ganea construction, when applied to the higher rank cases, fails to produce spaces with desirable algebraic properties. This limitation is shown in [BR1] Example 3.8. Specifically, when the rank of the Lie group, denoted as $\mathrm{rk}(G)$, is greater than or equal to 2, the rational cohomology algebra $H^*(X_1)$ of the constructed space X_1 is not a free graded module over the rational cohomology algebra $H^*(BG)$.

Alternatively, they proposed an modified approach, applying the relative join construction with respect to each hyperplane contained in the configuration, resulting in a diagram of spaces of (partial) quasi-invariants. These spaces are "glued" using homotopy colimit to obtain a space $\widetilde{X}_m$. The even dimensional rational cohomology of this space coincides with classical quasi-invariants, while the odd-dimensional rational cohomology remains non-vanishing.

As a result, a natural question arises: how can we generalize this construction in a way that applies to higher rank Lie groups and constructs spaces with similar properties to the spaces of classical quasi-invariants proposed in [BR1]? We present in the next chapter one such generalization.

CHAPTER 6

TOWERS OF BOREL FIBRATIONS

Motivated by the construction in the rank one case, we want to generalize the towers of (Borel) fibrations to the higher rank cases and study algebraic properties of these spaces. In this chapter, we approach one such generalization that applies the relative join construction. This construction produces spaces with rational cohomology similar to classical quasi-invariants, although they differ in the higher rank case.

6.1 Towers of Fibrations

Consider a compact connected Lie group G with a maximal torus T of rank n. Denote the corresponding Weyl group as $W = N_G(T)/T = W_G(T)$.

Let $\widetilde{G} \subseteq G$ be a closed subgroup of G that contains T. According to Theorem C.4.3, the quotient space $G/\widetilde{G}$ has only finite even-dimensional rational cohomology. We define $\widetilde{W}$ as the Weyl group of $\widetilde{G}$. The Serre spectral sequence associated to the fibration

$$\widetilde{G}/T \longrightarrow G/T \longrightarrow G/\widetilde{G}$$

degenerates and thus $\dim_{\mathbb{Q}} H^*(G/\widetilde{G}) = |W|/|\widetilde{W}|$.

We have another fibration

$$G/\widetilde{G} \hookrightarrow B\widetilde{G} \overset{r}{\twoheadrightarrow} BG. \tag{6.1}$$

The Leray-Serre spectral sequence associated with this fibration collapses since both fiber and base space have only even dimensional rational cohomology. By Theorem B.6.1 and Borel Theorem 3.1.1, we know that

- $r^* : H^*(BG) \to H^*(B\widetilde{G})$ is a monomorphism.
- $H^*(B\widetilde{G})$ is a finite generated free module over $H^*(BG)$ of rank $|W/\widetilde{W}|$.
- $H^*(G/\widetilde{G}) \cong H^*(B\widetilde{G})/(\mathrm{Im} r^+)$ where r^+ is the restriction of r^* on positive degrees.

We have a natural composite

$$BT \longrightarrow B\widetilde{G} \xrightarrow{p} BG$$

If we identify $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$, by Borel Theorem 3.1.1 the map induced on rational cohomology is given by the natural inclusion

$$\mathbb{Q}[t_1, \ldots, t_n]^W \xhookrightarrow{r^*} \mathbb{Q}[t_1, \ldots, t_n]^{\widetilde{W}} \hookrightarrow \mathbb{Q}[t_1, \ldots, t_n].$$

Thus by Chevalley's Theorem 2.1.4, we can write

$$H^*(BG) = \mathbb{Q}[t_1, \ldots, t_n]^W = \mathbb{Q}[\theta_1, \ldots, \theta_n)]$$

and

$$H^*(B\widetilde{G}) = \mathbb{Q}[t_1, \ldots, t_n]^{\widetilde{W}} = \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n)].$$

Choose a closed subgroup $H \subset \widetilde{G}$ such that $\widetilde{G}/H \cong_{\mathbb{Q}} \mathbb{S}^{2k-1}$. It follows from the classification in Theorem C.4.4, H is a closed subgroup of $\widetilde{G}$ of rank $n-1$ such that $H^*(BH) = \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{n-1}]$. By reordering indices we can assume $\deg(\widetilde{\theta}_n) = 2k$. The following Borel fibration

$$\widetilde{G}/H \hookrightarrow BH \xrightarrow{q} B\widetilde{G}. \tag{6.2}$$

gives $H^*(BH)$ a $H^*(B\widetilde{G})$-module structure given by

$$H^*(BH) = H^*(B\widetilde{G})/(\widetilde{\theta}_n) = \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{n-1}].$$

Step 0. Let

$$K_0 \hookrightarrow X_0 \xrightarrow{\widetilde{p}} B\widetilde{G} \tag{6.3}$$

be a fibration such that $H^*(K_0)$ has finite dimensional $|\widetilde{W}|$. Then the composite $p_0 : X_0 \to B\widetilde{G} \to BG$ is also a fibration with fiber F_0

$$F_0 \hookrightarrow X_0 \xrightarrow{p_0} BG. \tag{6.4}$$

For $m = 0$, we have the following fibration

$$K_0 \longrightarrow F_0 \longrightarrow G/\widetilde{G}$$

The Serre spectral sequence associated with fibration collapses since both K_0 and $G/\widetilde{G}$ has only even dimensional rational cohomology, thus

$$\dim_{\mathbb{Q}} H^*(F_0) = \dim_{\mathbb{Q}} H^*(K_0) \cdot \dim_{\mathbb{Q}} H^*(G/\widetilde{G}) = |\widetilde{W}| \cdot |W|/|\widetilde{W}| = |W|.$$

The Leray-Serre spectral sequence associated to the fibration (6.4) collapses, and by applying Theorem B.6.1, we know that

- $p_0^* : H^*(BG) \to H^*(X_0)$ is a monomorphism.
- $H^*(X_0)$ is a finite generated free module over $H^*(BG)$ of rank $|W|$.
- $H^*(F_0) \cong H^*(X_0)/(\mathrm{Im} p_0^+)$.

Analogously, for the fibration (6.3), we have the following properties:

- $\widetilde{p}^* : H^*(B\widetilde{G}) \to H^*(X_0; \mathbb{Q})$ is a monomorphism.
- $H^*(X_0)$ is a finite generated free module over $H^*(\widetilde{G})$ of rank $|\widetilde{W}|$.
- $H^*(K_0) \cong H^*(X_0)/(\mathrm{Im}\widetilde{p}^+)$.

Step 1. Let $X_0 \times_{\widetilde{BG}} BH \to BH$ be the canonical projection map. We have the following commutative diagram

$$\begin{array}{ccc} X_0 \times_{\widetilde{BG}} BH & \longrightarrow & BH \\ \big\downarrow & & \big\downarrow q \\ X_0 & \xrightarrow{p_0} \!\!\!\!\rightarrow & BG \end{array}$$

Define

$$X_1 := \text{hocolim}\{X_0 \leftarrow X_0 \times_{\widetilde{BG}} BH \to BH\}.$$

It comes with canonical maps $p_1 : X_1 \to BG$ and $BT \to X_1$ and we have $X_1 = X_0 *_{\widetilde{BG}} BH$.

We write $F_1 := \text{hofib}\{p_1 : X_1 \to BG\}$ the (homotopy) fiber of p_1.

Inductive Step. By induction, we can construct X_{m+1} and F_{m+1} as follows: Let

$$X_m \times_{\widetilde{BG}} BH \to BH$$

be the canonical projection map. We have the following commutative diagram

$$\begin{array}{ccccc} & & X_m \times_{\widetilde{BG}} BH & \longrightarrow & BH \\ & & \big\downarrow & & \big\downarrow q \\ F_m & \hookrightarrow & X_m & \xrightarrow{p_m} \!\!\!\!\rightarrow & BG \end{array}$$

Define

$$X_{m+1} = \text{hocolim} \ \{X_m \longleftarrow X_m \times_{\widetilde{BG}} BH \longrightarrow BH\} \tag{6.5}$$

which comes together with canonical maps $p_{m+1} : X_{m+1} \to BG$ and $BT \to X_{m+1}$, and define

$$F_{m+1} := \text{hofib}\{p_{m+1} : X_{m+1} \to BG\}.$$

By induction we have $X_{m+1} = X_m *^m_{\widetilde{G}} BH = X_0 *^{m+1}_{\widetilde{BG}} BH$ where the second notion means taking relative join with BH over $\widetilde{BG}$ $m+1$ times.

Next we give an alternative description of the spaces F_m and X_m.

Define $K_m = K_0 *^m \widetilde{G}/H$. By induction on m, we can apply the relative join construction to get the following fibrations

$$K_m = K_0 *^m \widetilde{G}/H \hookrightarrow X_m = BT *^m_{B\widetilde{G}} BH \longrightarrow B\widetilde{G}. \tag{6.6}$$

The fibrations (6.6) fit into the following commutative diagram

$$\begin{array}{ccccc} F_m & \longrightarrow & G/\widetilde{G} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X_m & \longrightarrow & B\widetilde{G} & \longrightarrow & BG \end{array}$$

where both outer and the right are homotopy pullbacks, implying that the left square is also a homotopy pullback. It follows that we also have the following fibrations

$$K_m = K_0 *^m \widetilde{G}/H \hookrightarrow F_m = F_0 *^m_{G/\widetilde{G}} G/H \longrightarrow G/\widetilde{G}. \tag{6.7}$$

Furthermore, by Proposition 4.6.6, we know that if $\widetilde{p}_0 : X_0 \to B\widetilde{G}$ is a Borel fibration, then $\widetilde{p}_m : X_m \to B\widetilde{G}$ is a Borel fibration. Therefore, we can provide an alternative description of F_m and X_m as follows.

Lemma 6.1.1. *$F_m = F_0 *^m_{G/\widetilde{G}} G/H$ and if $\widetilde{p}_0 : X_0 \to B\widetilde{G}$ is a Borel fibration, $X_m = (K_m)_{h\widetilde{G}}$ where $K_m = K_0 *^m \widetilde{G}/H$.*

Corollary 6.1.2. *When $\widetilde{G} = G$ and $\widetilde{p}_0 : X_0 \to B\widetilde{G}$ is a Borel fibration, $X_m = (F_m)_{hG}$.*

6.2 Rational Homotopy Type of Joins

We can describe the rational homotopy types of the spaces K_m by the following well-known result (see, e.g. [FHT], Theorem 24.5).

Theorem 6.2.1. *The following conditions on a simply connected topological space X are equivalent:*

1. *The rational Hurewicz homomorphism $h_X : \pi_* X \otimes \mathbb{Q} \to H_*(X;\mathbb{Q})$ is surjective.*
2. *There is a rational homotopy equivalence of the form $\bigvee_{\alpha \in I} \mathbb{S}^{n_\alpha} \to X, n_\alpha \geq 2$.*
3. *There is a well-based, path-connected space Y and a rational homotopy equivalence of the form $\Sigma Y \to X$.*

By Theorem 6.2.1, if X is a path-connected space with reduced rational homology $\bar{H}_*(X)$ spanned by (finitely many) homology classes $\{v_i\}_{i\in I}$ of dimensions $|v_i| = n_i \geq 1$, then $\Sigma(X)$ is rationally equivalent to the wedge of spheres $\mathbb{S}^{n_i+1}$, one for each $i \in I$.

Example 6.2.2. In the case of $X = \widetilde{G}/T$, the flag manifold $\widetilde{G}/T$ has a cell decomposition given by the Schubert cells X_w, one for each $w \in \widetilde{W}$, of dimensions $\dim(X_w) = 2l(w)$ where $l(w)$ is the length of w. Its reduced integral (and hence, rational) homology is spanned by the classes $\{[X_w], w \in \widetilde{W}\backslash e\}$. Therefore $\Sigma(\widetilde{G}/T)$ is rationally equivalent to the wedge of spheres $\mathbb{S}^{2l(w)+1}$ indexed by $w \in \widetilde{W}\backslash e$.

By applying Proposition 4.5.2 to K_0 and $\widetilde{G}/H \simeq \mathbb{S}^{2k-1}$, we can make the following observation

$$K_1 = K_0 * \widetilde{G}/H \simeq_{\mathbb{Q}} (\bigvee_{i\in I} \mathbb{S}^{2n_i+1}) \wedge \mathbb{S}^{2k-1} \simeq \bigvee_{i\in I} \mathbb{S}^{2n_i+2k}. \tag{6.8}$$

By induction, we can see that

$$K_m = \widetilde{G}/T *^m \widetilde{G}/H \simeq_{\mathbb{Q}} \bigvee_{w\in\widetilde{W}\backslash e} \mathbb{S}^{2l(w)+2mk} \tag{6.9}$$

is rationally equivalent to a wedge of even dimensional spheres and $\dim_{\mathbb{Q}} H^*(K_m) = |\widetilde{W}|$.

6.3 Main Theorem

Starting with a fibration (6.4), we obtain a sequence of natural maps

$$X_0 \xrightarrow{\pi_0} X_1 \xrightarrow{\pi_1} X_2 \longrightarrow \cdots \longrightarrow BG$$

Our objective in this section is to prove the following theorem.

Theorem 6.3.1. *Let G be a compact connected Lie group with maximal torus T and associated Weyl group W. Let $\widetilde{G}$ and H be closed subgroups of G satisfying the conditions*

(*a*) $T \subseteq \widetilde{G}$ *(we write $\widetilde{W} := N_{\widetilde{G}}(T)/T$ for the corresponding Weyl group).*

(*b*) $H \subseteq \widetilde{G}$ *and* $\widetilde{G}/H \cong \mathbb{S}^{2k-1}$ *for some* $k \geq 1$.

Let $\widetilde{p}_0 : X_0 \to B\widetilde{G}$ be a fibration satisfying

(*a*) $K_0 = \mathrm{hofib}\{X_0 \to B\widetilde{G}\}$ *has even dimensional rational cohomology.*

(*b*) $\dim_{\mathbb{Q}} H^*(K_0) = |\widetilde{W}|$.

We define $X_m := X_0 *^m_{B\widetilde{G}} BH$ which comes with natural maps $\widetilde{p}_m : X_m \to B\widetilde{G}$ and $p_m : X_m \to BG$ for all $m \geq 0$. Then

1. $H^*(X_m;\mathbb{Q})$ *is a free module over* $H^*(BG;\mathbb{Q})$ *of rank* $|W|$.
2. $H^*(X_{m+1};\mathbb{Q}) = \ker(\alpha_m : H^*(X_m;\mathbb{Q}) \oplus H^*(BH;\mathbb{Q}) \to H^*(X_m \times_{B\widetilde{G}} BH;\mathbb{Q}))$.
3. *The natural maps on rational cohomology*

$$H^*(BG,\mathbb{Q}) \hookrightarrow H^*(B\widetilde{G},\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(X_m,\mathbb{Q}) \hookrightarrow \cdots \hookrightarrow H^*(X_0,\mathbb{Q}) \tag{6.10}$$

are injective.

Remark. The above theorem holds more generally even if we do not require $\dim_{\mathbb{Q}} H^*(K_0) = |\widetilde{W}|$. In the general case, $H^*(X_m; \mathbb{Q})$ is a free module over $H^*(BG; \mathbb{Q})$ of rank $\frac{|W|}{|\widetilde{W}|} \cdot \dim_{\mathbb{Q}} H^*(K_0)$. In most of the cases we consider, where $G = \widetilde{G}$, both $H^*(X_0)$ and $H^*(X_m)$ are naturally equipped with a W-action.

6.3.1 Freeness Property

Our proof of the first part relies on the following observation.

Proposition 6.3.2. *F_m has only finite even dimensional rational cohomology and $\dim_{\mathbb{Q}} H^*(F_m) = |W|$.*

Proof. In this case, K_m consists of only even dimensional cohomology. Applying Theorem B.6.1 to the fibration (6.7), we see that

$$H^*(F_m) = H^*(K_m) \bigotimes H^*(G/\widetilde{G}) \tag{6.11}$$

so F_m consists of only even dimensional rational cohomology and $\dim_{\mathbb{Q}} H^*(F_m) = \dim_{\mathbb{Q}} H^*(K_m) \cdot \dim_{\mathbb{Q}} H^*(G/\widetilde{G}) = |\widetilde{W}| \cdot |W/\widetilde{W}| = |W|$. □

Proposition 6.3.3. *$H^*(X_m; \mathbb{Q})$ is a free module over $H^*(BG; \mathbb{Q})$ of rank $|W|$.*

Proof. Again by applying Theorem B.6.1 to the fibration

$$F_m \longrightarrow X_m \longrightarrow BG$$

we see that

$$H^*(X_m) = H^*(F_m) \bigotimes H^*(BG) \tag{6.12}$$

is a free module over $H^*(BG)$ of rank $|W|$. □

In order to prove injectivity, we need to consider the fibration (6.6). Since K_m has only finite even dimensional rational cohomology, we know that

$$H^*(X_m;\mathbb{Q}) \cong H^*(K_m;\mathbb{Q}) \bigotimes H^*(B\widetilde{G};\mathbb{Q})$$

is a finite generated free module over $H^*(B\widetilde{G})$ of rank $|\widetilde{W}|$.

6.3.2 Computation of Rational Cohomology

Lemma 6.3.4. *There is an isomorphism of algebras*

$$H^*(X_m \times_{B\widetilde{G}} BH) \cong H^*(X_m) \bigotimes_{H^*(B\widetilde{G})} H^*(BH). \tag{6.13}$$

Proof. By [S3] Theorem 3.5 (or [M3] Theorem 7.15 and Corollary 7.18), the Eilenberg-Moore spectral sequence

$$E_2^{p,*} = Tor^{-p}_{H^*(B\widetilde{G})}(H^*(X_m), H^*(BH) \Longrightarrow H^*(X_m \times_{B\widetilde{G}} BH)$$

associated to the following (homotopy) pullback

$$\begin{array}{ccc} X_m \times_{B\widetilde{G}} BH & \longrightarrow & BH \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & B\widetilde{G} \end{array}$$

converges strongly as an algebra. Since $H^*(X_m)$ is free over $H^*(B\widetilde{G})$, we get the desired isomorphism. In particular, the isomorphism respects the multiplicative structure on both sides, which gives an algebra isomorphism. □

Corollary 6.3.5. *The odd cohomology of $X_m \times_{B\widetilde{G}} BH$ vanishes for any m.*

Now since X_{m+1} is defined as a homotopy pushout in (6.5), the associated Bousfield-Kan spectral sequence induces a (Mayer-Vietoris) long exact sequence of cohomology (see, e.g., [D3] Section 18.1)

$$\cdots \to H^{r-1}(X_m \times_{B\widetilde{G}} BH) \to H^r(X_{m+1}) \to H^r(X_m) \oplus H^r(BH) \to H^r(X_m \times_{B\widetilde{G}} BH) \to \cdots \tag{6.14}$$

In particular, because of the vanishing of odd cohomology of X_m, BH and $X_m \times_{B\widetilde{G}} BH$, the long exact sequence (6.14) splits into short exact sequences

$$0 \longrightarrow H^{2t}(X_{m+1}) \longrightarrow H^{2t}(X_m) \oplus H^{2t}(BH) \xrightarrow{\alpha_m} H^{2t}(X_m \times_{B\widetilde{G}} BH) \longrightarrow 0 \tag{6.15}$$

and $H^{2t+1}(X_{m+1}) = 0$.

Thus we can compute rational cohomology of X_m inductively by the following formula.

Proposition 6.3.6. $H^*(X_{m+1}) = \ker(\alpha_m : H^*(X_m) \oplus H^*(BH) \to H^*(X_m \times_{B\widetilde{G}} BH))$.

6.3.3 Injectivity Property

Proposition 6.3.7. *The map $\pi_m^* : H^*(X_{m+1}) \hookrightarrow H^*(X_m)$ is injective for every $m \geq 0$.*

We will do induction on m to show that each $H^*(X_{m+1}) \hookrightarrow H^*(X_m)$ is an embedding.

Note we have natural maps of commutative diagrams

$$\begin{array}{ccccc} X_m & \longleftarrow & X_m \times_{B\widetilde{G}} BH & \longrightarrow & BH \\ \uparrow & & \uparrow & & \| \\ X_{m-1} & \longleftarrow & X_{m-1} \times_{B\widetilde{G}} BH & \longrightarrow & BH \end{array}$$

which induces maps on homotopy colimit $X_m \to X_{m+1}$, so we have commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^*(X_{m+1}) & \longrightarrow & H^*(X_m) \oplus H^*(BH) & \longrightarrow & H^*(X_m \times_{B\widetilde{G}} BH) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^*(X_m) & \longrightarrow & H^*(X_{m-1}) \oplus H^*(BH) & \longrightarrow & H^*(X_{m-1} \times_{B\widetilde{G}} BH) & \longrightarrow & 0 \end{array}$$

where the middle vertical map is identity on $H^*(BH)$, so by snake lemma

$$\ker(H^*(X_{m+1}) \to H^*(X_m)) \subset \ker(H^*(X_m) \to H^*(X_{m-1}))$$

and by induction it suffices to show that

$$\ker(H^*(X_1) \xrightarrow{\gamma} H^*(X_0)) = 0.$$

Consider the short exact sequence

$$0 \to H^*(X_1) \xrightarrow{\gamma} H^*(X_0) \oplus H^*(BH) \xrightarrow{\alpha} H^*(X_0 \times_{B\widetilde{G}} BH) = H^*(X_0) \bigotimes_{H^*(B\widetilde{G})} H^*(BH) \to 0$$

$$(x, y) \longmapsto p^*_{X_0}(x) \otimes 1 - 1 \otimes p^*_{BH}(y)$$

For simplicity, let's write $M = H^*(X_0), N = H^*(BH), P = H^*(X_0) \bigotimes_{H^*(B\widetilde{G})} H^*(BH)$, then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\beta) & \longrightarrow & N & \xrightarrow{\beta} & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \ker(\alpha) & \longrightarrow & M \oplus N & \xrightarrow{\alpha} & P & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & = & M & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

where all vertical and horizontal sequences are exact. Thus there is an exact sequence

$$0 \longrightarrow \ker(\beta) \longrightarrow \ker(\alpha) \xrightarrow{\gamma} M \longrightarrow \operatorname{coker}(\beta) \longrightarrow 0$$

It suffices to show $\ker(\beta) = 0$. Observe that

$$\beta = p^*_{BH} : H^*(BH)) \longrightarrow H^*(X_0) \bigotimes_{H^*(B\widetilde{G})} H^*(BH)$$

is injective because $H^*(X_0)$ is free over $H^*(B\widetilde{G})$. So we're done with our proof.

6.3.4 Rational Cohomology

If we write

$$\begin{aligned} H^*(BT) &= \mathbb{Q}[t_1, \ldots, t_n] \\ H^*(BG) &= \mathbb{Q}[\theta_1, \ldots, \theta_n] \\ H^*(B\widetilde{G}) &= \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n] \\ H^*(BH) &= \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{n-1}] \end{aligned} \tag{6.16}$$

With these identification, we can compute the rational cohomology of X_m combining Lemma 6.3.4 and Proposition 6.3.6.

Proposition 6.3.8. *The rational cohomology of X_m is of the form*

$$H^*(X_m) = \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n] + A\widetilde{\theta}_n^m \tag{6.17}$$

where $A = H^*(X_0)$ and we identify $H^*(B\widetilde{G})$ with its image in A.

Proof. The proof is based on induction by m. When $m = 0$ this is obvious.

By Lemma 6.3.4,

$$\begin{aligned} H^*(X_m \times_{B\widetilde{G}} BH) &\cong H^*(X_m) \bigotimes_{H^*(B\widetilde{G})} H^*(BH) \\ &\cong H^*(X_m) \bigotimes_{\mathbb{Q}[\widetilde{\theta}_1,\ldots,\widetilde{\theta}_n]} \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{n-1}] \\ &\cong H^*(X_m) \bigotimes_{\mathbb{Q}[\widetilde{\theta}_1,\ldots,\widetilde{\theta}_n]} \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n]/(\widetilde{\theta}_n) \\ &\cong H^*(X_m)/(\widetilde{\theta}_n) \end{aligned}$$

and by Proposition 6.3.6, we can compute

$$P_{m+1} = \{(f, \gamma) \in H^*(X_m) \oplus \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_{n-1}] | f \equiv \gamma \mod (\widetilde{\theta}_n)\}$$

It follows by the injectivity of $\pi^*_{i,i+1} : H^*(X_{m+1}) \hookrightarrow H^*(X_m)$, one can readily check

$$H^*(X_{m+1}) = \mathbb{Q}[\widetilde{\theta}_1, \dots, \widetilde{\theta}_n] + H^*(X_m)\widetilde{\theta} = \mathbb{Q}[\widetilde{\theta}_1, \dots, \widetilde{\theta}_n] + A\widetilde{\theta}_n^{m+1}$$

and the induction holds. □

6.3.5 Basis As Free Module

Now we've seen that $H^*(X_m)$ is a subalgebra of $H^*(X_0)$ and is a free module over $H^*(BG)$ of rank $|W|$, a natural question is to get a basis of $H^*(X_m)$ as a module over $H^*(BG)$. The following proposition shows that when $G = \widetilde{G}$, given a basis of $H^*(X_0)$ as a free module over $H^*(BG)$, we can get a basis of $H^*(X_m)$ as a free module over $H^*(BG)$.

Proposition 6.3.9. *Assume $G = \widetilde{G}$. Suppose we have a basis $\{e_i\}$ of $H^*(X_0)$ as a free module over $H^*(BG)$ of rank $|W|$ with $e_1 = 1$, then*

$$\{1, e_i\theta_n^m, 1 < i \leq |W|\}$$

forms a basis of $H^*(X_m)$ as a free module over $H^*(BG)$.

Proof. We can write

$$H^*(X_0) = \bigoplus_{1 \leq i \leq |W|} \mathbb{Q}[\theta_1, \dots, \theta_n]e_i$$

and thus

$$\begin{aligned} P_m &= \mathbb{Q}[\theta_1, \dots, \theta_n] + \bigoplus_{1 \leq i \leq |W|} \mathbb{Q}[\theta_1, \dots, \theta_n]e_i\theta_n^m \\ &= \mathbb{Q}[\theta_1, \dots, \theta_n] + \mathbb{Q}[\theta_1, \dots, \theta_n]\theta_n^m + \bigoplus_{2 \leq i \leq |W|} \mathbb{Q}[\theta_1, \dots, \theta_n]e_i\theta_n^m \\ &= \mathbb{Q}[\theta_1, \dots, \theta_n] + \bigoplus_{2 \leq i \leq |W|} \mathbb{Q}[\theta_1, \dots, \theta_n]e_i\theta_n^m \end{aligned}$$

Note in the above decomposition, $\mathbb{Q}[\theta_1, \ldots, \theta_n]e_i \cap \mathbb{Q}[\theta_1, \ldots, \theta_n]e_i = \emptyset$ whenever $i \neq j$, therefore $\mathbb{Q}[\theta_1, \ldots, \theta_n] \cap \mathbb{Q}[\theta_1, \ldots, \theta_n]e_i\theta_n^m = \emptyset$ for $i > 1$, so we can write

$$P_m = \mathbb{Q}[\theta_1, \ldots, \theta_n] \bigoplus \bigoplus_{2 \leq i \leq |W|} \mathbb{Q}[\theta_1, \ldots, \theta_n]e_i\theta_n^m$$

which shows P_m is a free $\mathbb{Q}[\theta_1, \ldots, \theta_n]$-module of rank $|W|$ with desired basis. □

CHAPTER 7

CLASSIFYING SPACES OF CLASSICAL LIE GROUPS

In this chapter, we focus on the cases when $X_0 = BT$ is the classifying space of a maximal torus T in a compact connected Lie group G. We will compute the rational cohomology of the generalized spaces of quasi-invariants X_m as defined in Theorem 6.3.1 and its Hilbert series for the cases $G = \mathrm{U}(n), \mathrm{SU}(n-1), \mathrm{Sp}(n)$, and $\mathrm{SO}(n)$. we also compute the equivariant K-theory of the homotopy fibers $F_m = \mathrm{hofib}\{p_m : X_m \to BG\}$ in the cases of $G = \mathrm{U}(n), \mathrm{SU}(n-1)$, and $\mathrm{Sp}(n)$.

7.1 Relative Join Construction

Let G be a compact connected Lie group with maximal torus T and let W be the Weyl group of G, we first consider the case when $\widetilde{G} = G$ and H is a closed subgroup of G such that $G/H \simeq \mathbb{S}^{2k-1}$ is (rationally) an odd dimensional sphere. By Theorem C.4.4, it suffices to focus mostly on the case $(G, H) = (\mathrm{U}(n), H = \mathrm{U}(n-1), (\mathrm{SU}(n), \mathrm{SU}(n-1)), (\mathrm{Sp}(n), \mathrm{Sp}(n-1)), (\mathrm{SO}(2n), \mathrm{SO}(2n-1))$ and $(\mathrm{SO}(2n+1), \mathrm{SO}(2n-1))$.

Consider the classical fibration

$$G/T \overset{i}{\hookrightarrow} BT \xrightarrow{p} BG \tag{7.1}$$

and the fibration

$$G/H \hookrightarrow BH \longrightarrow BG. \tag{7.2}$$

Applying the (relative) join construction in Proposition 4.6.4, we may form a new fibration

$$F_1 = G/T * G/H \xrightarrow{i_1} X_1 = BT *_{BG} BH \xrightarrow{p_1} BG. \tag{7.3}$$

and in fact we can repeat this procedure to produce fibrations

$$F_m = G/T *^m G/H \xrightarrow{i_m} X_m = BT *^m_{BG} BH \xrightarrow{p_m} BG. \tag{7.4}$$

Note that G/T is a flag manifold with only even dimensional cohomology, so the Leray-Serre spectral sequence associated to the fibration (7.1) collapses and by applying Theorem B.6.1 we know that

(a) $p^* : H^*(BG) \to H^*(BT)$ is a monomorphism.

(b) $H^*(BT)$ is a finitely generated free $\mathrm{Im}(p^*)$-module.

7.1.1 Hilbert series

It follows from the isomorphism (6.12) that the Hilbert series of X_m is given by

$$p_{X_m}(t) = p_{F_m}(t) p_{BG}(t).$$

For classical Lie groups G,

$$p_{BG}(t) = \prod_{i=1}^{n} \frac{1}{1 - t^{2d_i}}$$

where d_i are the degrees of the homogeneous algebraically independent generators of $H^*(BG) \cong \mathbb{Q}[V]^W$, so it suffices for us to compute $p_{F_m}(t)$.

Note the Hilbert series of F_m is given by

$$q_{F_m}(t) = 1 + \sum_{w \in W \setminus \{e\}} t^{2l(w)+2mk} = 1 - t^{2mk} + t^{2mk} \sum_{w \in W} (t^2)^{l(w)}.$$

By [H4] Theorem 3.15,

$$W(t) = \sum_{w \in W} t^{l(w)} = \prod_{i=1}^{n} \frac{t^{d_i} - 1}{t - 1}.$$

Replacing t by t^{-2} and plugging $W(t^2)$ in, we get

$$p_{F_m}(t) = 1 - t^{2mk} + t^{2mk}W(t^2) = 1 - t^{2mk} + t^{2mk}\prod_{i=1}^{n}\frac{1-t^{2d_i}}{1-t^2}$$

and

$$p_{X_m}(t) = \frac{1-t^{2mk}}{\prod_{i=1}^{n}(1-t^{2d_i})} + \frac{t^{2mk}}{(1-t^2)^n}.$$

7.1.2 Rational Cohomology

Let G be one of the classical Lie groups $\mathrm{U}(n), \mathrm{SU}(n+1)$ or $\mathrm{Sp}(n)$. Let $T \subset G$ be its maximal torus and $W = N(T)/T$ be the associated Weyl group. Let $H \subset G$ be $\mathrm{U}(n-1), \mathrm{SU}(n)$ or $\mathrm{Sp}(n-1)$ respectively, then $G/H \cong \mathbb{S}^{2d_n-1}$ where d_n is the highest degree of W.

In the real case when $G = \mathrm{SO}(n)$, the choices of H depends on the parity of n. When $G = \mathrm{SO}(2n)$ we choose $H = \mathrm{SO}(2n-1)$ and when $G = \mathrm{SO}(2n+1)$ we choose $H = \mathrm{SO}(2n-1)$.

By Borel's Theorem 3.1.1, we know $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$ and $H^*(BG) = \mathbb{Q}[\theta_1, \ldots, \theta_n]$, where $\deg(\theta_i) = 2d_i$ are numbered in a non-decreasing order so that θ_n is the highest degree generator (except for the case $G = \mathrm{SO}(2n), H = \mathrm{SO}(2n-1)$). In the respective cases, $H^*(BH) = \mathbb{Q}[\theta_1, \ldots, \theta_{n-1}]$.

Applying Proposition 6.3.8 we obtain the following theorem.

Theorem 7.1.1. *The rational cohomology of X_m is given by*

$$P_m(G,T,H) := H^*(X_m) \cong \mathbb{Q}[\theta_1, \ldots, \theta_n] + \mathbb{Q}[t_1, \ldots, t_n]\theta_n^m \tag{7.5}$$

and the canonical composite of fibrations $BT \to X_m \to BG$ gives the inclusions in *rational cohomology*

$$\mathbb{Q}[\theta_1,\ldots,\theta_n] \hookrightarrow \mathbb{Q}[\theta_1,\ldots,\theta_n] + \mathbb{Q}[t_1,\ldots,t_n]\theta_n^m \hookrightarrow \mathbb{Q}[t_1,\ldots,t_n].$$

Furthermore, $H^*(X_m)$ is a free module over $H^*(BG) = \mathbb{Q}[\theta_1,\ldots,\theta_n]$ of rank $|W|$.

We will denote $P_m = P_m(G,T,H)$ when the underlying (G,T,H) is clear.

Corollary 7.1.2. *There is another description of $H^*(X_m)$ as*

$$\mathbb{Q}[\theta_1,\ldots,\theta_n] + \mathbb{Q}[t_1,\ldots,t_n]\theta_n^m = \{f \in \mathbb{Q}[t_1,\ldots,t_n] | s_\alpha(f) - f \in (\theta_n^m), \forall \alpha \in \mathcal{A}_W\}.$$

Proof. Let

$$Q_m = \{f \in \mathbb{Q}[t_1,\ldots,t_n] | s_\alpha(f) \equiv f \mod (\theta_n)^m, \alpha \in \mathcal{A}_W\}.$$

It is easy to see that $P_m \subseteq Q_m$, so we only need to show that $Q_m \subseteq P_m$.

Since θ_n is W_n-invariant, the ideal $(\theta_n)^m \subseteq \mathbb{Q}[t_1,\ldots,t_n]$ is W_n-invariant, thus Q_m can be described as

$$Q_m = \{f \in \mathbb{Q}[t_1,\ldots,t_n] | s_\alpha(f) \equiv f \mod (\theta_n)^m, s_\alpha \in W_n\}.$$

Given any $f \in Q_m$, we can decompose f as

$$f = \frac{1}{|W_n|}\sum_{s_\alpha \in W_n} s_\alpha(f) + \frac{1}{|W_n|}\sum_{s_\alpha \in W_n}(1 - s_\alpha)(f)$$

where the first component is W_n-invariant and the latter component is a sum of polynomials where each $(1 - s_\alpha)(f) \in (\theta_n)^m$, so $f \in P_m$. Thus we proved $P_m = Q_m$ as desired. □

Remark. From the expression of P_m, it's straightforward to see there is a W-action on P_m such that $P_m^W = \mathbb{Q}[\theta_1,\ldots,\theta_n]$. This W-action in fact is inherited by the W-action on $H^*(BT)$ and trivial W-action on $H^*(BH)$.

The Case of $\mathrm{U}(n)$

In this case, $H^*(BT) = \mathbb{Q}[t_1,\ldots,t_n]$ and $H^*(BG) = \mathbb{Q}[c_1,\ldots,c_n]$ where $c_i = \sigma_i(t_1,\ldots,t_n)$ and σ_i is the i-th elementary symmetric polynomial. By Proposition 6.3.8 and 6.3.9, we have the following result.

Corollary 7.1.3. *Let $X_m = BT *^m_{BG} BH$, then the rational cohomology of X_m is given by*

$$H^*(X_m) = \mathbb{Q}[c_1,\ldots,c_n] + \mathbb{Q}[t_1,\ldots,t_n]c_n^m. \tag{7.6}$$

It is a free module over $H^*(B\,\mathrm{U}(n)) = \mathbb{Q}[c_1,\ldots,c_n]$ with basis

$$\{1, t_1^{i_1}\cdots, t_n^{i_n}c_n^m, 0 \le i_1 < 1, \ldots, 0 \le i_n < n, (i_1,\ldots,i_n) \neq (0,\ldots,0)\}. \tag{7.7}$$

This follows from the following classical result (see, e.g., [AM] II.G). ?an alternative proof can be found in [LLPT] DIFF 1.3.

Lemma 7.1.4. *As a free $\mathbb{Q}[c_1,\ldots,c_n]$-module , $\mathbb{Q}[t_1,\ldots,t_n]$ has a basis $\{t_1^{i_1}\cdots t_n^{i_n}\}_{0\le i_j<j}$ called* **Artin basis**.

Proof. We need to show that the natural map

$$\Theta : \bigoplus_{0\le i_j<j, 1\le j\le n} \mathbb{Q}\left[c_1,\cdots,c_n\right] t_1^{i_1}\ldots t_n^{i_n} \longrightarrow \mathbb{Q}\left[t_1,\cdots,t_n\right]$$

$$\sum f(c_1,\ldots,c_n)t_1^{i_1}\ldots t_n^{i_n} \longmapsto \sum f(c_1,\ldots,c_n)t_1^{i_1}\ldots t_n^{i_n}$$

is an isomorphism of $\mathbb{Q}[c_1,\ldots,c_n]$-modules.

First, we need to show this map is surjective. Consider the following n polynomials defined inductively,

$$F_n(x) = (x-t_1)(x-t_2)\cdots(x-t_n) = x^n - c_1x^{n-1} + c_2x^{n-2} + \cdots + (-1)^n c_{n-1}x + (-1)^n c_n$$
$$F_i(x) := \frac{F_{i+1}(x)}{x - t_{i+1}} = \frac{F_n(x)}{(x-t_{i+1})(x-t_{i+2})\cdots(x-t_n)}, 1 \le i \le n-1$$

Applying division, we see that $F_i(x)$ is a polynomial in x of degree i with leading coefficient 1 and lower coefficients polynomials in $c_1, \dots, c_n$ and $t_{i+1}, \dots, t_n$ with integer coefficients, i.e. we can write

$$F_i(x) = x^i + a_{i,1}(c_1, \dots, c_n, t_{i+1}, \dots, t_n)x^{i-1} + \dots + a_{i,i}(c_1, \dots, c_n, t_{i+1}, \dots, t_n)$$

where $a_{i,j} \in \mathbb{Z}[c_1, \dots, c_n, t_{i+1}, \dots, t_n]$.

Now given any polynomial $g \in \mathbb{Q}[t_1, \dots, t_n]$. Since $F_1(t_1) = t_1 + a_{1,1} = 0$ and $F_1(x)$ is of degree 1, we can express

$$t_1 = -a_{1,1}(c_1, \dots, c_n, t_2, \dots, t_n)$$

so we can replace all $t_1^i, i \geq 1$ with polynomials of the form $(-a_{1,1})^i$. Since $F_2(t_2) = 0$ and $F_2(x)$ is of degree 2, we can express

$$t_2^2 = -a_{2,1}(c_1, \dots, c_n, t_3, \dots, t_n)t_2 - a_{2,2}(c_1, \dots, c_n, t_3, \dots, t_n)$$

so we can replace all $t_2^i, i > 2$ with polynomials of the form

$$r(c_1, \dots, c_n, t_{i+1}, \dots, t_n)t_2 + s(c_1, \dots, c_n, t_{i+1}, \dots, t_n).$$

Introducing these expressions gives us a way to expression $t_j^i, i \geq j$ as sums of lower degree terms with coefficients polynomials in $_1, \dots, c_n, t_{i+1}, \dots, t_n$, i.e.

$$t_j^i = r_1(c_1, \dots, c_n, t_{i+1}, \dots, t_n)t_j^{j-1} + \dots + r_j(c_1, \dots, c_n, t_{i+1}, \dots, t_n).$$

Now plugging in these expression in g from t_1 to t_n, we see g can be expressed as

$$g = \sum_{0 \leq i_j < j, 1 \leq j \leq n} r(c_1, \dots, c_n)t_1^{i_1} \cdots t_n^{i_n}$$

which show surjectivity of Θ.

Now it suffices to check that Hilbert series of both sides (with standard grading $|t_i| = 1$ for all i) are the same. For the right hand side

$$p_{RHS}(t) = \frac{1}{(1-t)^n}.$$

For the left hand side,

$$\begin{aligned}
p_{LHS}(t) &= \sum_{p=0}^{\infty}(\sum_{0\le i_j<j, 1\le j\le n} \dim_{\mathbb{Q}}(\mathbb{Q}[c_1,\dots,c_n])_{p-\sum_{j=1}^{n} i_j} t^p \\
&= \sum_{q=0}^{\infty}(\dim_{\mathbb{Q}}(\mathbb{Q}[c_1,\dots,c_n])_q(\sum_{0\le i_j<j,1\le j\le n} t^{\sum_{j=1}^{n} i_j})) \\
&= (\prod_{l=1}^{n} \frac{1}{(1-t^l)})(\prod_{j=1}^{n}(1+t+\cdots+t^{j-1})) \\
&= \frac{1}{(1-t)^n}
\end{aligned}$$

thus the Hilbert series of both sides agree. □

It follows that we can write

$$\mathbb{Q}[t_1,\dots,t_n] = \bigoplus_{0\le i_k<k, 1\le k\le n} \mathbb{Q}[c_1,\dots,c_n] t_1^{i_1}\cdots t_n^{i_n}$$

and thus by Proposition 6.3.9 $H^*(X_m)$ is a free $\mathbb{Q}[c_1,\dots,c_n]$-module of rank $|S_n| = n!$ with desired basis.

Corollary 7.1.5. *The Hilbert series of X_m is*

$$p_{X_m}(t) = \frac{1-t^{2mn}}{\prod_{i=1}^{n}(1-t^{2i})} + \frac{t^{2mn}}{(1-t^2)^n}.$$

The Case of SU(n)

The case of SU$(n+1)$ is very similar U(n) to except we need to replace $\mathbb{Q}[t_1,\dots,t_n]$ with $\mathbb{Q}[t_0,t_1,\dots,t_n]/(\sigma_1 = t_0+\cdots+t_n)$. In this case $\theta_i = \sigma_{i+1}(t_0,t_1,\dots,t_n)$ for $1\le i\le n$. By Proposition 6.3.8 and 6.3.9, we have the following result.

Corollary 7.1.6. *Let $X_m = BT *_{BG}^m BH$, then the rational cohomology of X_m is given by*

$$H^*(X_m) = \mathbb{Q}[\sigma_2,\dots,\sigma_{n+1}] + (\mathbb{Q}[t_0,t_1,\dots,t_n]/(\sigma_1))\cdot\sigma_{n+1}^m. \tag{7.8}$$

It is a free module over $H^*(B\,\mathrm{SU}(n)) = \mathbb{Q}[\sigma_2, \ldots, \sigma_n]$ *with basis*

$$\{1, t_1^{i_1} \cdots, t_n^{i_n}\sigma_n^m, 0 \le i_1 < 1, \ldots, 0 \le i_n < n, (i_1, \ldots, i_n) \ne (0, \ldots, 0)\}. \tag{7.9}$$

Corollary 7.1.7. *The Hilbert series of* $H^*(X_m)$ *is*

$$p_{X_m}(t) = \frac{1 - t^{2m(n+1)}}{\prod_{i=1}^n (1 - t^{2i+2})} + \frac{t^{2m(n+1)}}{(1 - t^2)^n}.$$

The Case of $\mathrm{Sp}(n)$

In this case, $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$ and $H^*(BG) = \mathbb{Q}[q_1, \ldots, q_n]$ where $q_i = \sigma_i(t_1^2, \ldots, t_n^2)$ is the i-th elementary signed-symmetric polynomial. Thus by similar argument, we have the following result.

Corollary 7.1.8. *Let* $X_m = BT *_{BG}^m BH$*, then the rational cohomology of* X_m *is given by*

$$H^*(X_m) = \mathbb{Q}[q_1, \ldots, q_n] + \mathbb{Q}[t_1, \ldots, t_n]q_n^m. \tag{7.10}$$

It is a free module over $\mathbb{Q}[q_1, \ldots, q_n]$ *with basis*

$$\{1, t_1^{i_1} \cdots, t_n^{i_n} q_n^m, 0 \le i_1 < 2, \ldots, 0 \le i_n < 2n, (i_1, \ldots, i_n) \ne (0, \ldots, 0)\}. \tag{7.11}$$

Corollary 7.1.9. *The Hilbert series of* $H^*(X_m)$ *is*

$$p_{X_m}(t) = \frac{1 - t^{4mn}}{\prod_{i=1}^n (1 - t^{4i})} + \frac{t^{4mn}}{(1 - t^2)^n}$$

The Case of $\mathrm{SO}(2n)$

When $G = \mathrm{SO}(2n)$ and $H = \mathrm{SO}(2n-1)$, we have

$$H^*(BG) = \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n]$$

$$H^*(BH) = \mathbb{Q}[q_1, \ldots, q_{n-1}]$$

where the inclusion $H \hookrightarrow G$ induces a $H^*(BG)$-module structure on $H^*(BH)$ given by

$$\begin{aligned}\mathbb{Q}[q_1, \ldots, q_{n-1}, c_n] &\to \mathbb{Q}[q_1, \ldots, q_{n-1}] \\ q_i &\longmapsto q_i \\ c_n &\longmapsto 0\end{aligned}$$

Corollary 7.1.10. *Let $X_m = BT *_{BG}^m BH$, then the rational cohomology of X_m is given by*

$$H^*(X_m) = \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n] + \mathbb{Q}[t_1, \ldots, t_n]c_n^m$$

and its Hilbert series is

$$p_{X_m}(t) = \frac{1 - t^{4mn}}{(1 - t^{2n}) \prod_{i=1}^{n-1}(1 - t^{4i})} + \frac{t^{4mn}}{(1 - t^2)^n}.$$

The Case of $\mathrm{SO}(2n + 1)$

When $G = \mathrm{SO}(2n + 1)$ and $H = \mathrm{SO}(2n - 1)$, the result is the same as the case $\mathrm{Sp}(n)$.

7.1.3 Equivariant K-Theory

It turns out in this case the equivariant K-theory of these spaces exhibits similar algebraic properties as the rank one case in [BR1].

Our computation is based the following two results, which only work for G with $\pi_1(G) = 1$, so we only focus on the complex and quaternion cases.

The first result is a well-known Künneth type of formula for equivariant K-theory first studied by Hodgkin (see, e.g., [BZ], Theorem 2.3).

Theorem 7.1.11. *For a compact connected Lie group G such that $\pi_1(G)$ is torsion free, for any G-spaces X, Y, there is a spectral sequence $E_r \Rightarrow K^*_G(X \times Y)$ with E_2-term*

$$E_2^{*,*} = \mathrm{Tor}^{*,*}_{R(G)}(K^*_G(X), K^*_G(Y))$$

that converges to $K^*_G(X \times Y)$ where $X \times Y$ is viewed as a G-space with diagonal action.

The second result id a Mayer-Vietoris type of formula (see, e.g., [JO]).

Lemma 7.1.12. *Let $f : U \to X$ and $g : U \to Y$ be proper equivariant maps of G-spaces. Let $Z = \mathrm{hocolim}\{X \overset{f}{\leftarrow} U \overset{g}{\rightarrow} Y\}$ where the homotopy colimit is taken in the category of G-spaces. Then the abelian groups $K^*_G(X), K^*_G(Y)$ and $K^*_G(Z)$ are related by the following six-term exact sequence.*

$$\begin{array}{ccccc} K^0_G(Z) & \longrightarrow & K^0_G(X) \oplus K^0_G(Y) & \xrightarrow{f^*-g^*} & K^0_G(U) \\ \partial \uparrow & & & & \downarrow \partial \\ K^1_G(U) & \xleftarrow{f^*-g^*} & K^1_G(X) \oplus K^1_G(Y) & \longleftarrow & K^1_G(Z) \end{array}$$

The Case of $\mathrm{U}(n)$

Recall in the construction of spaces of generalized quasi-invariants,

$$F_m = \mathrm{U}(n)/T^n *^m \mathrm{U}(n)/\mathrm{U}(n-1).$$

Theorem 7.1.13. *The equivariant K-theory of F_m is given (as abelian group) by*

$$K^*_{\mathrm{U}(n)}(F_m) = \mathcal{P}_m := \mathbb{Z}[c_1, \dots, c_n, c_n^{-1}] + \prod_{i=1}^{n}(t_i - 1)^m \cdot \mathbb{Z}[t_1, \dots, t_n, (t_1 \cdots t_n)^{-1}]$$

where $c_i = \sigma_i(t_1, \dots, t_n)$.

Proof. 1. When $m = 0$, we have

$$K^*_{\mathrm{U}(n)}(\mathrm{U}(n)/T^n) \cong K^*_{T^n}(\mathrm{pt}) = R(T^n) = \mathbb{Z}[t_1, \dots, t_n, (t_1 \cdots t_n)^{-1}]$$

$$K^*_{\mathrm{U}(n)}(\mathrm{pt}) = R(U(n)) = \mathbb{Z}[c_1, \dots, c_n, c_n^{-1}]$$

$$K^*_{\mathrm{U}(n)}(U(n)/U(n-1)) \cong K^*_{\mathrm{U}(n-1)}(\mathrm{pt}) = R(\mathrm{U}(n-1)) = \mathbb{Z}[d_1, \dots, d_{n-1}, d_{n-1}^{-1}]$$

so we have

$$K^*_{\mathrm{U}(n)}(F_0) = K^*_{\mathrm{U}(n)}(\mathrm{U}(n)/T^n) = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}].$$

The commutative diagram

$$\begin{array}{ccc} R(\mathrm{U}(n)) & \longrightarrow & R(T^n) \\ \downarrow & & \downarrow \\ R(\mathrm{U}(n-1) & \longrightarrow & R(T^{n-1}) \end{array}$$

induced on $\mathrm{U}(n)$-equivariant K-theory by

$$\begin{array}{ccc} \mathrm{U}(n)/T^{n-1} & \longrightarrow & \mathrm{U}(n)/\mathrm{U}(n-1) \\ \downarrow & & \downarrow \\ \mathrm{U}(n)/T^n & \longrightarrow & \mathrm{pt} \end{array}$$

is identified with

$$\begin{array}{ccccc} \mathbb{Z}[c_1, \ldots, c_n, c_n^{-1}] & \xhookrightarrow{i_n} & \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}] & \qquad & c_i \longmapsto \sigma_i(t_1, \ldots, t_n) \\ \downarrow\!\!\downarrow \varphi & & \downarrow\!\!\downarrow & & \\ \mathbb{Z}[d_1, \ldots, d_{n-1}, d_{n-1}^{-1}] & \xhookrightarrow{i_{n-1}} & \mathbb{Z}[t_1, \ldots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}] & \qquad & d_i \longmapsto \sigma_i(t_1, \ldots, t_{n-1}) \end{array}$$

where the right vertical map is given by the projection taking $t_n \mapsto 1$. It follows that the left vertical map is given by

$$c_1 \mapsto d_1 + 1$$

$$c_i \mapsto d_i + d_{i-1}, 1 < i < n$$

$$c_n \mapsto d_{n-1}.$$

Lemma 7.1.14. *There is an isomorphism of rings*

$$\mathbb{Z}[d_1, \ldots, d_{n-1}, d_{n-1}^{-1}] \cong \mathbb{Z}[c_1, \ldots, c_n, c_n^{-1}] / \left(\sum_{i=0}^{n} (-1)^{n-i} c_i \right)$$

where $c_0 = 1$.

Proof. Consider the following map

$$\phi : \mathbb{Z}[c_1, \ldots, c_n] \to \mathbb{Z}[d_1, \ldots, d_{n-1}]$$

$$c_i \longmapsto d_i + d_{i-1}, \qquad 1 \leq i < n$$

$$c_n \longmapsto d_{n-1}$$

where $d_0 = 1$. Note $\ker(\phi) = (\sum_{i=0}^{n}(-1)^{n-i}c_i)$, thus we have an isomorphism

$$\mathbb{Z}[d_1, \ldots, d_{n-1}] \cong \mathbb{Z}[c_1, \ldots, c_n]/\left(\sum_{i=0}^{n}(-1)^{n-i}c_i\right)$$

and localizing at the ideal (c_n) gives the desired isomorphism. □

2. When $m \geq 1$, we have

$$F_{m+1} = \operatorname{hocolim}\{F_m \leftarrow F_m \times \mathrm{U}(n)/\,\mathrm{U}(n-1) \rightarrow \mathrm{U}(n)/\,\mathrm{U}(n-1)\}$$

First we need to compute $K_{\mathrm{U}(n)}(F_m \times \mathrm{U}(n)/\,\mathrm{U}(n-1))$.

By induction on m and Lemma 7.1.14, the Tor group

$$\mathrm{Tor}^*_{R(\mathrm{U}(n))}(K^*_{\mathrm{U}(n)}(F_m), K^*_{U(n)}(\mathrm{U}(n)/\,\mathrm{U}(n-1))) = \mathrm{Tor}^*_{R(\mathrm{U}(n))}(K_{\mathrm{U}(n)}(F_m), R(\mathrm{U}(n-1)))$$

can be identified with

$$\mathrm{Tor}_{\mathbb{Z}[c_1,\ldots,c_n,c_n^{-1}]}\left(\mathcal{P}_m, \mathbb{Z}[d_1, \ldots, d_{n-1}, d_{n-1}^{-1}]\right)$$

which is given by the homology of the following complex

$$0 \longrightarrow \mathcal{P}_m \xrightarrow{\cdot a} \mathcal{P}_m \longrightarrow 0$$

where $a = \sum_{i=0}^{m}(-1)^{n-i}c_i = \prod_{i=1}^{n}(t_i - 1)$. Since $\mathcal{P}_m$ is an integral domain the first homology of the above complex vanishes. Thus by Theorem 7.1.11, the Hodgkin's spectral sequence collapses on the E_2-page and we have

$$K^*_{\mathrm{U}(n)}(F_m \times \mathrm{U}(n)/\,\mathrm{U}(n-1)) \cong \mathcal{P}_m/\left(\prod_{i=1}^{n}(t_i - 1)\right).$$

Furthermore, the projection $F_m \times \mathrm{U}(n)/\,\mathrm{U}(n-1) \rightarrow F_m$ induces the quotient map

$$\pi : \mathcal{P}_m \longrightarrow \mathcal{P}_m/\left(\prod_{i=1}^{n}(t_i - 1)\right)$$

on equivariant K-theory, and $F_m \times \mathrm{U}(n)/\,\mathrm{U}(n-1) \to \mathrm{U}(n)/\,\mathrm{U}(n-1)$ induces an injective map

$$i : \mathbb{Z}[c_1, \dots, c_n, c_n^{-1}]/\left(\sum_{i=0}^{n} (-1)^{n-i} c_i\right) \longrightarrow \mathcal{P}_m/\left(\prod_{i=1}^{n} (t_i - 1)\right)$$

by the isomorphism (1).

Now consider the homotopy pushout (2), by Lemma 7.1.12, we have the following six term exact sequence

$$\begin{array}{ccc}
K^0_{\mathrm{U}(n)}(F_{m+1}) & \xrightarrow{(i_{m,m+1}, f_{m+1})} & \mathcal{P}_m \oplus \mathbb{Z}[c_1, \dots, c_n, c_n^{-1}]/\left(\sum_{i=0}^{n} (-1)^{n-i} c_i\right) \\
\uparrow & & \downarrow{\scriptstyle \pi - i} \\
0 & & \mathcal{P}_m/\left(\prod_{i=1}^{n} (t_i - 1)\right) \\
\uparrow & & \downarrow{\scriptstyle \partial} \\
0 \oplus 0 & \longleftarrow & K^1_{\mathrm{U}(n)}(F_{m+1})
\end{array}$$

It can be seen that $\pi - i$ is surjective and thus $K^1_{\mathrm{U}(n)}(F_{m+1}) = 0$, and

$$K^0_{U(n)}(F_{m+1}) = \ker(\pi - i) = \mathbb{Z}[c_1, \dots, c_n, c_n^{-1}] + \prod_{i=0}^{n} (t_i - 1) \cdot \mathcal{P}_m = \mathcal{P}_{m+1}.$$

□

The Case of $\mathrm{SU}(n)$

In this case,

$$F_m = \mathrm{SU}(n)/T^{n-1} *^m \mathrm{SU}(n)/\,\mathrm{SU}(n-1).$$

Theorem 7.1.15. *The equivariant K-theory of F_m is given (as abelian group) by*

$$\mathcal{P}_m := \mathbb{Z}[c_1, \dots, c_{n-1}] + a^m \cdot \mathbb{Z}[t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}]$$

where

$$c_i = \sigma_i(t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1})$$

and

$$a = \sum_{i=0}^{n} (-1)^{n-i} c_i = ((t_1 \cdots t_{n-1})^{-1} - 1) \prod_{i=1}^{n-1} (t_i - 1).$$

Proof. 1. When $m = 0$, we have

$$\begin{aligned} &K^*_{\mathrm{SU}(n)}(\mathrm{SU}(n)/T^{n-1}) \cong K^*_{T^{n-1}}(\mathrm{pt}) = R(T^{n-1}) = \mathbb{Z}[t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}] \\ &K^*_{\mathrm{SU}(n)}(\mathrm{pt}) = R(\mathrm{SU}(n)) = \mathbb{Z}[c_1, \dots, c_{n-1}] \\ &K^*_{\mathrm{SU}(n)}(\mathrm{SU}(n)/\mathrm{SU}(n-1)) \cong K^*_{\mathrm{SU}(n-1)}(\mathrm{pt}) = R(\mathrm{SU}(n-1)) = \mathbb{Z}[d_1, \dots, d_{n-2}] \end{aligned}$$

so we have

$$K^*_{\mathrm{SU}(n)}(F_0) = K^*_{\mathrm{SU}(n)}(\mathrm{SU}(n)/T^{n-1}) = \mathbb{Z}[t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}].$$

The commutative diagram

$$\begin{array}{ccc} R(\mathrm{SU}(n)) & \longrightarrow & R(T^{n-1}) \\ \downarrow & & \downarrow \\ R(\mathrm{SU}(n-1) & \longrightarrow & R(T^{n-2}) \end{array}$$

induced on $\mathrm{SU}(n)$-equivariant K-theory by

$$\begin{array}{ccc} \mathrm{SU}(n)/T^{n-2} & \longrightarrow & \mathrm{SU}(n)/\mathrm{SU}(n-1) \\ \downarrow & & \downarrow \\ \mathrm{SU}(n)/T^{n-1} & \longrightarrow & \mathrm{pt} \end{array}$$

is identified with

$$\begin{array}{ccccccc} \mathbb{Z}[c_1, \dots, c_{n-1}] & \xhookrightarrow{i_n} & \mathbb{Z}[t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}] & \qquad & c_i & \longmapsto & \sigma_i(t_1, \dots, t_{n-1}, (t_1 \cdots t_{n-1})^{-1}) \\ \downarrow\!\!\downarrow \varphi & & \downarrow\!\!\downarrow & & & & \\ \mathbb{Z}[d_1, \dots, d_{n-2}] & \xhookrightarrow{i_{n-1}} & \mathbb{Z}[t_1, \dots, t_{n-2}, (t_1 \cdots t_{n-2})^{-1}] & \qquad & d_i & \longmapsto & \sigma_i(t_1, \dots, t_{n-2}, (t_1 \cdots t_{n-2})^{-1}) \end{array}$$

where the right vertical map is given by the projection taking $t_{n-1} \mapsto 1$. It follows that the left vertical map is given by

$$c_1 \mapsto d_1 + 1$$
$$c_i \mapsto d_i + d_{i-1}, 1 < i < n-1$$
$$c_{n-1} \mapsto d_{n-2} + 1.$$

Lemma 7.1.16. *There is an isomorphism of rings*

$$\mathbb{Z}[d_1, \ldots, d_{n-2}] \cong \mathbb{Z}[c_1, \ldots, c_{n-1}] / \left(\sum_{i=0}^{n} (-1)^{n-i} c_i \right)$$

where $c_0 = c_n = 1$.

Proof. Consider the following map

$$\phi : \mathbb{Z}[c_1, \ldots, c_{n-1}] \to \mathbb{Z}[d_1, \ldots, d_{n-2}]$$
$$c_i \longmapsto d_i + d_{i-1}$$

where $d_0 = d_{n-1} = 1$. Then $\ker(\phi) = (\sum_{i=0}^{n} (-1)^{n-i} c_i)$, thus we have an isomorphism

$$\mathbb{Z}[d_1, \ldots, d_{n-2}] \cong \mathbb{Z}[c_1, \ldots, c_{n-1}] / \left(\sum_{i=0}^{n} (-1)^{n-i} c_i \right).$$

□

2. When $m \geq 1$, we have

$$F_{m+1} = \operatorname{hocolim}\{F_m \leftarrow F_m \times \mathrm{SU}(n)/\mathrm{SU}(n-1) \to \mathrm{SU}(n)/\mathrm{SU}(n-1)\}$$

So first we need to compute $K_{\mathrm{SU}(n)}(F_m \times \mathrm{SU}(n)/\mathrm{SU}(n-1))$.

By induction on m and Lemma7.1.16, the Tor group

$$\mathrm{Tor}^*_{R(\mathrm{SU}(n))}(K^*_{\mathrm{SU}(n)}(F_m), K^*_{\mathrm{SU}(n)}(\mathrm{SU}(n)/\mathrm{SU}(n-1))) = \mathrm{Tor}^*_{R(\mathrm{SU}(n))}(K_{\mathrm{SU}(n)}(F_m), R(\mathrm{SU}(n-1)))$$

can be identified with

$$\mathrm{Tor}_{\mathbb{Z}[c_1,\ldots,c_{n-1}]}(\mathcal{P}_m, \mathbb{Z}[d_1, \ldots, d_{n-2}])$$

which is given by the homology of the following complex

$$0 \longrightarrow \mathcal{P}_m \xrightarrow{\cdot a} \mathcal{P}_m \longrightarrow 0$$

where

$$a := \sum_{i=0}^{n} (-1)^{n-i} c_i = ((t_1 \cdots t_{n-1})^{-1} - 1) \prod_{i=1}^{n-1} (t_i - 1).$$

Since $\mathcal{P}_m$ is an integral domain the first homology of the above complex vanishes. Thus by Theorem 7.1.11, the Hodgkin's spectral sequence collapses on the E_2-page and we have

$$K^*_{\mathrm{SU}(n)}(F_m \times \mathrm{SU}(n)/\,\mathrm{SU}(n-1)) \cong \mathcal{P}_m/\left((t_1 \cdots t_{n-1})^{-1} - 1) \prod_{i=1}^{n-1} (t_i - 1)\right).$$

Furthermore, the projection $F_m \times \mathrm{SU}(n)/\,\mathrm{SU}(n-1) \to F_m$ induces the quotient map

$$\pi : \mathcal{P}_m \longrightarrow \mathcal{P}_m/\,(a)$$

on equivariant K-theory, and $F_m \times \mathrm{SU}(n)/\,\mathrm{SU}(n-1) \to \mathrm{SU}(n)/\,\mathrm{SU}(n-1)$ induces an injective map

$$i : \mathbb{Z}[c_1, \ldots, c_{n-1}]/\,(a) \longrightarrow \mathcal{P}_m/\,(a)$$

by the isomorphism (1).

Now consider the homotopy pushout (2), by Lemma 7.1.12, we have the following six term exact sequence

$$\begin{CD}
K^0_{\mathrm{SU}(n)}(F_{m+1}) @>{(i_{m,m+1}, f_{m+1})}>> \mathcal{P}_m \oplus \mathbb{Z}[c_1, \ldots, c_{n-1}]/\,(a) \\
@AAA @VV{\pi - i}V \\
0 @. \mathcal{P}_m/\,(a) \\
@AAA @VV{\partial}V \\
0 \oplus 0 @<<< K^1_{\mathrm{SU}(n)}(F_{m+1})
\end{CD}$$

It can be seen that $\pi - i$ is surjective and thus $K^1_{SU(n)}(F_{m+1}) = 0$, and

$$K^0_{SU(n)}(F_{m+1}) = \ker(\pi - i) \cong \mathbb{Z}[c_1, \ldots, c_{n-1}] + a \cdot \mathcal{P}_m = \mathcal{P}_{m+1}.$$

□

The Case of $\mathrm{Sp}(n)$

Recall in the construction of generalized spaces of quasiinvariants,

$$F_m = \mathrm{Sp}(n)/T^n *^m \mathrm{Sp}(n)/\mathrm{Sp}(n-1).$$

Theorem 7.1.17. *The equivariant K-theory of F_m is given (as abelian group) by*

$$K^*_{\mathrm{Sp}(n)}(F_m) = \mathcal{P}_m := \mathbb{Z}[q_1, \ldots, q_n] + \prod_{i=1}^{n}(t_i + t_i^{-1} - 2)^m \cdot \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}]$$

where $q_i = \sigma_i(t_1 + t_1^{-1}, \ldots, t_n + t_n^{-1})$.

Proof. 1. When $m = 0$, we have

$$K^*_{\mathrm{Sp}(n)}(\mathrm{Sp}(n)/T^n) \cong K^*_{T^n}(\mathrm{pt}) = R(T^n) = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}]$$
$$K^*_{\mathrm{Sp}(n)}(\mathrm{pt}) = R(\mathrm{Sp}(n)) = \mathbb{Z}[q_1, \ldots, q_n]$$
$$K^*_{\mathrm{Sp}(n)}(\mathrm{Sp}(n)/\mathrm{Sp}(n-1)) \cong K^*_{\mathrm{Sp}(n-1)}(\mathrm{pt}) = R(\mathrm{Sp}(n-1)) = \mathbb{Z}[p_1, \ldots, p_{n-1}]$$

so we have

$$K^*_{\mathrm{Sp}(n)}(F_0) = K^*_{\mathrm{Sp}(n)}(\mathrm{Sp}(n)/T^n) = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}].$$

The commutative diagram

$$\begin{array}{ccc} R(\mathrm{Sp}(n)) & \longrightarrow & R(T^n) \\ \downarrow & & \downarrow \\ R(\mathrm{Sp}(n-1) & \longrightarrow & R(T^{n-1}) \end{array}$$

induced on Sp(n)-equivariant K-theory by

$$\begin{array}{ccc} \mathrm{Sp}(n)/T^{n-1} & \longrightarrow & \mathrm{Sp}(n)/\mathrm{Sp}(n-1) \\ \downarrow & & \downarrow \\ \mathrm{Sp}(n)/T^n & \longrightarrow & \mathrm{pt} \end{array}$$

is identified with

$$\begin{array}{ccccc} \mathbb{Z}[q_1,\dots,q_n] & \xhookrightarrow{i_n} & \mathbb{Z}[t_1,\dots,t_n,(t_1\cdots t_n)^{-1}] & & q_i \mapsto \sigma_i(t_1+t_1^{-1},\dots,t_n+t_n^{-1}) \\ \downarrow & & \downarrow & & \\ \mathbb{Z}[p_1,\dots,p_{n-1}] & \xhookrightarrow{i_{n-1}} & \mathbb{Z}[t_1,\dots,t_{n-1},(t_1\cdots t_{n-1})^{-1}] & & p_i \mapsto \sigma_i(t_1+t_1^{-1},\dots,t_{n-1}+t_{n-1}^{-1}) \end{array}$$

where the right vertical map is given by the projection taking $t_n \mapsto 1$. It follows that the left vertical map is given by

$$q_1 \mapsto p_1 + 2$$

$$q_i \mapsto p_i + 2p_{i-1},\ 1 < i < n$$

$$q_n \mapsto 2p_{n-1}.$$

Lemma 7.1.18. *There is an isomorphism of rings*

$$\mathbb{Z}[p_1,\dots,p_{n-1}] \cong \mathbb{Z}[q_1,\dots,q_n]/\left(\sum_{i=0}^{n}(-2)^{n-i}c_i\right)$$

where $c_0 = 2$*.*

Proof. Consider the following map

$$\begin{aligned} \phi : \mathbb{Z}[q_1,\dots,q_n] &\to \mathbb{Z}[p_1,\dots,p_{n-1}] \\ q_i &\longmapsto p_i + 2p_{i-1}, \qquad 1 \le i < n \\ q_n &\longmapsto 2p_{n-1} \end{aligned}$$

where $p_0 = 2$. Note $\ker(\phi) = (\sum_{i=0}^{n}(-2)^{n-i}q_i)$, thus we have an isomorphism

$$\mathbb{Z}[p_1,\dots,p_{n-1}] \cong \mathbb{Z}[q_1,\dots,q_n]/\left(\sum_{i=0}^{n}(-2)^{n-i}q_i\right).$$

□

2. When $m \geq 1$, we have

$$F_{m+1} = \text{hocolim}\{F_m \leftarrow F_m \times \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1) \rightarrow \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1)\}$$

So first we need to compute $K_{\mathrm{Sp}(n)}(F_m \times \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1))$.

By induction on m and Lemma 7.1.18, the Tor group

$$\mathrm{Tor}^*_{R(\mathrm{Sp}(n))}(K^*_{\mathrm{Sp}(n)}(F_m), K^*_{\mathrm{Sp}(n)}(\mathrm{Sp}(n)/\,\mathrm{Sp}(n-1))) = \mathrm{Tor}^*_{R(\mathrm{Sp}(n))}(K_{\mathrm{Sp}(n)}(F_m), R(\mathrm{Sp}(n-1)))$$

can be identified with

$$\mathrm{Tor}_{\mathbb{Z}[q_1,\ldots,q_n]}(\mathcal{P}_m, \mathbb{Z}[p_1,\ldots,p_{n-1}])$$

which is given by the homology of the following complex

$$0 \longrightarrow \mathcal{P}_m \xrightarrow{\cdot a} \mathcal{P}_m \longrightarrow 0$$

where $a = \sum_{i=0}^{m}(-2)^{n-i}q_i = \prod_{i=1}^{n}(t_i + t_i^{-1} - 2)$. Since $\mathcal{P}_m$ is an integral domain the first homology of the above complex vanishes. Thus by Theorem 7.1.11, the Hodgkin's spectral sequence collapses on the E_2-page and we have

$$K^*_{\mathrm{Sp}(n)}(F_m \times \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1)) \cong \mathcal{P}_m/\left(\prod_{i=1}^{n}(t_i + t_i^{-1} - 2)\right).$$

Furthermore, the projection $F_m \times \mathrm{Sp}(n)/\,\mathrm{Sp}(n-1) \rightarrow F_m$ induces the quotient map

$$\pi : \mathcal{P}_m \longrightarrow \mathcal{P}_m/\left(\prod_{i=1}^{n}(t_i + t_i^{-1} - 2)\right)$$

on equivariant K-theory, and $F_m \times Sp(n)/Sp(n-1) \rightarrow Sp(n)/Sp(n-1)$ induces an injective map

$$i : \mathbb{Z}[q_1,\ldots,q_n]/\left(\sum_{i=0}^{n}(-2)^{n-i}q_i\right) \longrightarrow \mathcal{P}_m/\left(\prod_{i=1}^{n}(t_i + t_i^{-1} - 2)\right)$$

by the isomorphism (1).

Now consider the homotopy pushout (2), by Lemma 7.1.12, we have the following six term exact sequence

$$
\begin{array}{ccc}
K^0_{\mathrm{Sp}(n)}(F_{m+1}) & \xrightarrow{(i_{m,m+1}, f_{m+1})} & \mathcal{P}_m \oplus \mathbb{Z}[q_1, \ldots, q_n] / \left(\sum_{i=0}^{n} (-2)^{n-i} q_i \right) \\
\uparrow & & \downarrow{\scriptstyle \pi - i} \\
0 & & \mathcal{P}_m / \left(\prod_{i=1}^{n} (t_i + t_i^{-1} - 2) \right) \\
\uparrow & & \downarrow{\scriptstyle \partial} \\
0 \oplus 0 & \longleftarrow & K^1_{\mathrm{Sp}(n)}(F_{m+1})
\end{array}
$$

It can be seen that $\pi - i$ is surjective and thus $K^1_{Sp(n)}(F_{m+1}) = 0$, and

$$
K^0_{\mathrm{Sp}(n)}(F_{m+1}) = \ker(\pi - i) \cong \mathbb{Z}[q_1, \ldots, q_n] + \prod_{i=0}^{n} (t_i + t_i^{-1} - 2) \cdot \mathcal{P}_m = \mathcal{P}_{m+1}.
$$

□

7.2 Generalized Relative Join Construction

Instead of taking $\widetilde{G} = G$ in the construction, we can choose $\widetilde{G} \subset G$ such that $T \subset \widetilde{G}$ to generalize the above construction for some classical Lie groups.

7.2.1 Hilbert Series

By Borel's theorem,

$$
H^*(B\widetilde{G}) = \mathbb{Q}[V]^{\widetilde{W}} \cong \mathbb{Q}[\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n]
$$

Let $\widetilde{d_i} = \deg(\widetilde{\theta}_i), 1 \le i \le n$ and assume $\widetilde{G}/H \cong \mathbb{S}^{2\widetilde{d_i}-1}$. thus the Hilbert polynomial of $\widetilde{G}/T$ is

$$p_{\widetilde{G}/T}(t) = \prod_{i=1}^{n} \frac{1 - t^{2\widetilde{d_i}}}{1 - t^2}.$$

From (6.9) we know the Hilbert polynomial of K_m is

$$p_{K_m}(t) = 1 - t^{2m\widetilde{d_i}} + \frac{t^{2m\widetilde{d_i}} \prod_{i=1}^{n}(1 - t^{2\widetilde{d_i}})}{(1 - t^2)^n}$$

Combining this with the fibration (6.6), we know that the Hilbert series of X_m is

$$p_{X_m}(t) = p_{K_m}(t) p_{B\widetilde{G}}(t) = \frac{1 - t^{2m\widetilde{d_i}}}{\prod_{i=1}^{n}(1 - t^{2\widetilde{d_i}})} + \frac{t^{2m\widetilde{d_i}}}{(1 - t^2)^n}.$$

7.2.2 Rational Cohomology

The rational cohomology of $X_{m,k}$ can be computed in the same way as in the join construction if we replace $H^*(BG)$ with $H^*(B\widetilde{G})$ in the computation, thus Theorem 7.1.1 still holds if we replace $\theta_1, \ldots, \theta_n$ with $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_n$.

The Case of $\mathrm{U}(n)$

Let $G = \mathrm{U}(n)$ and $\widetilde{G} = G_k = \mathrm{U}(n-k) \times \mathrm{U}(k)$. Let $H = H_k = \mathrm{U}(n-k) \times \mathrm{U}(k-1)$. The Weyl group of G_k is $S_{n-k} \times S_k$.

By Borel's Theorem,

$$H^*(BG_k) \cong \mathbb{Q}[V]^{W_k} = \mathbb{Q}[t_1 \cdots, t_n]^{S_{n-k} \times S_k} = \mathbb{Q}[b_1, \ldots, b_{n-k}, c_1, \ldots, c_k]$$

where $b_i = \sigma_i(t_1, \ldots, t_{n-k})$ and $c_i = \sigma_i(t_{n-k+1}, \ldots, t_n)$. Let d_i be the degrees of the homogeneous algebraically independent generators of $H^*(BG_k)$ in the order we present here, i.e.

$$d_i = \begin{cases} \deg(b_i) = i, & 1 \le i \le n-k, \\ \deg(c_{i-n+k}) = i-n+k, & n-k+1 \le i \le n. \end{cases}$$

In this case, the fibration (6.6) is given by

$$F_{m,k} = G/T *^m_{G/G_k} G/H_k \hookrightarrow X_{m,k} = BT *_{BG_k} BH_k \longrightarrow BG. \tag{7.12}$$

The Hilbert series of $K_{m,k}$ is given by

$$p_{K_{m,k}}(t) = 1 - t^{2mk} + \frac{t^{2mk}\prod_{i=1}^{k}(1-t^{2i})\prod_{i=1}^{n-k}(1-t^{2i})}{(1-t^2)^n}$$

and Hilbert series of $X_{m,k}$ is given by

$$p_{X_{m,k}}(t) = \frac{1-t^{2mk}}{\prod_{i=1}^{k}(1-t^{2i})\prod_{i=1}^{n-k}(1-t^{2i})} + \frac{t^{2mk}}{(1-t^2)^n}.$$

Corollary 7.2.1. *The rational cohomology of $X_{m,k}$ is given by*

$$H^*(X_{m,k}) = \mathbb{Q}[b_1, \ldots, b_{n-k}, c_1, \ldots, c_k] + \mathbb{Q}[t_1, \ldots, t_n]c_k^m$$

where $b_i = \sigma_i(t_1, \ldots, t_{n-k})$ and $c_i = \sigma_i(t_{n-k+1}, \ldots, t_n)$.

The Case of $\mathrm{Sp}(n)$

Let $G = \mathrm{Sp}(n), G_k = \mathrm{Sp}(n-k) \times \mathrm{Sp}(k)$ and $H_k = \mathrm{Sp}(n-k) \times \mathrm{Sp}(k-1)$. Then $G_k/H_k \cong \mathbb{S}^{4k-1}$. In this case, $H^*(BT) = \mathbb{Q}[t_1, \ldots, t_n]$ and $H^*(BG_k) = \mathbb{Q}[p_1, \ldots, p_{n-k}, q_1, \ldots, q_k]$ where $p_i = \sigma_i(t_1^2, \ldots, t_{n-k}^2), q_i = \sigma_i(t_{n-k+1}^2, \ldots, t_n^2)$.

The Hilbert series of $K_{m,k}$ is given by

$$p_{K_{m,k}}(t) = 1 - t^{4mk} + \frac{t^{4mk}\prod_{i=1}^{k}(1-t^{4i})\prod_{i=1}^{n-k}(1-t^{4i})}{(1-t^2)^n}$$

and Hilbert series of $X_{m,k}$ is given by

$$p_{X_{m,k}}(t) = \frac{1 - t^{4mk}}{\prod_{i=1}^{k}(1 - t^{4i})\prod_{i=1}^{n-k}(1 - t^{4i})} + \frac{t^{4mk}}{(1 - t^2)^n}.$$

Corollary 7.2.2. *The rational cohomology of $X_{m,k}$ is given by*

$$H^*(X_{m,k}) = \mathbb{Q}[p_1, \ldots, p_{n-k}, q_1, \ldots, q_k] + \mathbb{Q}[t_1, \ldots, t_n]q_k^m$$

where $p_i = \sigma_i(t_1, \ldots, t_{n-k})$ and $q_i = \sigma_i(t_{n-k+1}, \ldots, t_n)$.

CHAPTER 8

VARIETIES OF GENERALIZED QUASI-INVARIANTS

In this chapter we describe an algebraic version of the construction we introduced in the previous chapter. We apply Ganea construction in the category of derived affine schemes, as described in [BR1] Section 3.2. This allows us to generalize the construction to the realm of algebraic geometry.

8.1 Varieties of Generalized Quasi-invariants

Let $W_n \subset GL_n(\mathbb{C})$ be a finite reflection group. There is a canonical fibration

$$\mathbb{A}^n \twoheadrightarrow \mathbb{A}^n//W_n \tag{8.1}$$

Furthermore, if $W_{n-1} \subset W_n$, we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{A}^{n-1} & \hookrightarrow & \mathbb{A}^n \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \mathbb{A}^{n-1}//W_{n-1} & \hookrightarrow & \mathbb{A}^n//W_{n-1} \\ & \searrow & \downarrow\!\!\downarrow \\ & & \mathbb{A}^n//W_n \end{array}$$

which gives us a map (in fact a cofibration in our examples)

$$\mathbb{A}^{n-1}//W_{n-1} \longrightarrow \mathbb{A}^n//W_n. \tag{8.2}$$

Definition 8.1.1. Given the two maps above, we can define

$$A_1 = \text{hocolim}\{\mathbb{A}^n \leftarrow \mathbb{A}^n \times_{\mathbb{A}^n//W_n} (\mathbb{A}^{n-1}//W_{n-1}) \rightarrow \mathbb{A}^{n-1}//W_{n-1}\}$$

and inductively

$$A_{m+1} = \text{hocolim}\{A_m \leftarrow A_m \times_{\mathbb{A}^n//W_n} (\mathbb{A}^{n-1}//W_{n-1}) \rightarrow \mathbb{A}^{n-1}//W_{n-1}\}.$$

Remark. In our cases, the cofibration (8.2) is given by the following composite

$$\begin{array}{ccccc} \mathbb{C}[\theta_1, \ldots, \theta_n] & \hookrightarrow & \mathbb{C}[\theta_1, \ldots, \theta_{n-1}, t_n] & \twoheadrightarrow & \mathbb{C}[\theta_1, \ldots, \theta_{n-1}] \\ \theta_i & \longmapsto & & & \theta_i, i < n \\ \theta_n & \longmapsto & & & 0 \end{array}$$

thus our machinery would work for all the cases and we can get a uniformed explicit description of A_m.

Theorem 8.1.2. *Let $W_n = S_n$ or B_n. The variety of generalized quasi-invariants $A_m =$ Spec P_m is given explicitly by*

$$P_m = \mathbb{C}[\theta_1, \ldots, \theta_n] + \mathbb{C}[t_1, \ldots, t_n]\theta_n^m \tag{8.3}$$

where $\theta_i = \sigma_i(t_1, \ldots, t_n)$ when $W_n = S_n$ and $\theta_i = \sigma_i(t_1^2, \ldots, t_n^2)$ when $W_n = B_n$. Furthermore, P_m is a free module over $\mathbb{C}[\theta_1, \ldots, \theta_n]$ of rank $|W_n|$.

The second part is already proved in the topological case, so we only need to prove the first part.

We will given the proof for the case S_n, and the proof for B_n is similar.

8.1.1 The Case of S_n

Let $W_n = S_n$ and view $S_{n-1} \subset S_n$ as the permutation of the first $n-1$ elements. In this case, $\sigma \in S_n$ acts on $\mathbb{A}^n = \operatorname{Spec} \mathbb{C}[t_1, \ldots, t_n]$ by $\sigma \cdot t_i = t_{\sigma(i)}$, and

$$\mathbb{A}^n // W_n = \operatorname{Spec} \mathbb{C}[c_1, \ldots, c_n]$$

where $c_i = \sigma_i(t_1, \ldots, t_n]$ and σ_i is the i-th elementary symmetric polynomial.

The second cofibration corresponds to the following $\mathbb{C}$-algebra homomorphism

$$\begin{aligned} \mathbb{C}[c_1,\dots,c_n] &\twoheadrightarrow \mathbb{C}[c_1,\dots,c_{n-1}] \\ c_i &\longmapsto c_i,\, i<n \\ c_n &\longmapsto 0 \end{aligned}$$

We will prove this case by induction on m.

Case $m=1$ We have

$$A_1 = \operatorname{hocolim}\{\mathbb{A}^n \leftarrow \mathbb{A}^n \times_{\mathbb{A}^n//S_n} \mathbb{A}^{n-1}//S_{n-1} \to \mathbb{A}^{n-1}//S_{n-1}\}$$

which can be identified with

$$\operatorname{Spec}\left(\operatorname{holim}\{\mathbb{C}[t_1,\dots,t_n] \to \mathbb{C}[t_1,\dots,t_n] \underset{\mathbb{C}[c_1,\dots,c_n]}{\otimes} \mathbb{C}[c_1,\dots,c_{n-1}] \leftarrow \mathbb{C}[c_1,\dots,c_{n-1}]\}\right)$$

thus

$$\begin{aligned} A_1 &= \operatorname{Spec}\left(\operatorname{holim}\{\mathbb{C}[t_1,\dots,t_n] \to \mathbb{C}[t_1,\dots,t_n]/(c_n) \leftarrow \mathbb{C}[c_1,\dots,c_{n-1}]\}\right) \\ &= \operatorname{Spec}\left(\mathbb{C}[t_1,\dots,t_n] \times_{\mathbb{C}[t_1,\dots,t_n]/(c_n)} \mathbb{C}[c_1,\dots,c_{n-1}]\right) \end{aligned}$$

Let

$$P_1 = \mathbb{C}[t_1,\dots,t_n] \underset{\mathbb{C}[t_1,\dots,t_n]/(c_n)}{\times} \mathbb{C}[c_1,\dots,c_{n-1}]$$

and consider the following composite

$$\begin{aligned} \pi_1^* : P_1 &\hookrightarrow \mathbb{C}[t_1,\dots,t_n] \times \mathbb{C}[c_1,\dots,c_{n-1}] \xrightarrow{pr_1} \mathbb{C}[t_1,\dots,t_n] \\ (f,\alpha) &\longmapsto (f,\alpha) \longmapsto f \end{aligned} \tag{8.4}$$

we claim this is an injective map. [1]

Given $(f,\alpha),(g,\beta) \in P_1$, such that $\pi_1^*(f,\alpha) = \pi_1^*(g,\beta)$, i.e. $f=g$, we have

$$\alpha \equiv f \mod c_n \quad \text{and} \quad \beta \equiv f \mod c_n$$

[1] This is already proved in the topological case in Proposition 6.3.7, we present here a slightly different proof.

in $\mathbb{C}[t_1, \dots, t_n]$, i.e.

$$f(t_1, \dots, t_n) = \alpha(c_1, \dots, c_{n-1}) + g(t_1, \dots, t_n)c_n = \beta(c_1, \dots, c_{n-1}) + h(t_1, \dots, t_n)c_n$$

if we identify $\mathbb{C}[c_1, \dots, c_{n-1}] \subseteq \mathbb{C}[c_1, \dots, c_n] \subseteq \mathbb{C}[t_1, \dots, t_n]$ as a subalgebra. By Theorem 2.2.7, we know that $c_1, \cdots c_n$ form a regular sequence in $\mathbb{C}[t_1, \dots, t_n]$, we have $g = h$ and $\alpha = \beta$, which proves injectivity.

In this case, we may identify P_1 as a subalgebra of $\mathbb{C}[t_1, \dots, t_n]$, which consists on polynomials of the form

$$f(t_1, \dots, t_n) = \alpha(c_1, \dots, c_{n-1}) + g(t_1, \dots, t_n)c_n$$

where $\alpha \in \mathbb{C}[c_1, \dots, c_{n-1}] \subseteq \mathbb{C}[c_1, \dots, c_n] \subseteq \mathbb{C}[t_1, \dots, t_n]$ and $g \in \mathbb{C}[t_1, \dots, t_n]$. Or equivalently, we may write

$$P_1 = \mathbb{C}[c_1, \dots, c_n] + \mathbb{C}[t_1, \dots, t_n]c_n.$$

Case $m > 1$ In the inductive steps, if we write $A_m = \mathrm{Spec}(P_m)$, we have

$$\begin{aligned} A_{m+1} &= \mathrm{hocolim}\{A_m \leftarrow A_m \times_{\mathbb{A}^n//S_n} \mathbb{A}^{n-1}//S_{n-1} \to \mathbb{A}^{n-1}//S_{n-1}\} \\ &= \mathrm{Spec}\left(P_m \times_{P_m/(c_n)} \mathbb{C}\left[c_1, \dots, c_{n-1}\right]\right) \end{aligned} \tag{8.5}$$

and we may write $P_{m+1} := P_m \times_{P_m/(c_n)} \mathbb{C}\left[c_1, \dots, c_{n-1}\right]$

Using a similar composite as ((8.4)), we get

$$\begin{array}{rcccccc} \pi^*_{m+1} : P_{m+1} & \hookrightarrow & P_m \times \mathbb{C}\left[c_1, \dots, c_{n-1}\right] & \xrightarrow{pr_1} & P_m & \hookrightarrow & \mathbb{C}\left[t_1, \dots, t_n\right] \\ (f, \alpha) & \longmapsto & (f, \alpha) & \longmapsto & f & \longmapsto & f \end{array} \tag{8.6}$$

we claim this is an injective map. The proof is almost identical as the base case.

Given $(f, \alpha), (g, \beta) \in P_{m+1}$, such that $\pi^*_{m+1}(f, \alpha) = \pi^*_{m+1}(g, \beta)$, i.e. $f = g$, we have

$$\alpha \equiv f \mod c_n \quad \text{and} \quad \beta \equiv f \mod c_n$$

in P_m, i.e.

$$f(t_1,\ldots,t_n) = \alpha(c_1,\ldots,c_{n-1}) + g(t_1,\ldots,t_n)c_n = \beta(c_1,\ldots,c_{n-1}) + h(t_1,\ldots,t_n)c_n$$

where $g, h \in P_m$. By Theorem 2.2.7, $c_1, \cdots c_n$ are homogeneous (symmetric) polynomials which form a regular sequence in P_m. Therefore $g = h$ and $\alpha = \beta$, which proves injectivity.

By induction on m together with the discussion above, we get the desired result.

8.2 Varieties of Relative Generalized Quasi-invariants

Given a subgroup $W_{n,k} \subset W_n$ such that $W_{n-1,k-1} \subseteq W_{n,k} \cap W_{n-1}$, instead of considering the fibration (8.1) and (cofibration) map (8.2), we consider the following fibration

$$\mathbb{A}^n \longrightarrow\!\!\!\!\rightarrow \mathbb{A}^n//W_{n,k} \tag{8.7}$$

and another (cofibration) map

$$\mathbb{A}^{n-1}//W_{n-1,k-1} \hookrightarrow \mathbb{A}^n//W_{n,k}. \tag{8.8}$$

Definition 8.2.1. Given the two maps above, we can define

$$A_{k,1} = \operatorname{hocolim}\{\mathbb{A}^n \leftarrow \mathbb{A}^n \times_{\mathbb{A}^n//W_{n,k}} \mathbb{A}^{n-1}//V_k \to \mathbb{A}^{n-1}//W_{n-1,k-1}\}$$

and inductively

$$A_{k,m+1} = \operatorname{hocolim}\{A_{k,m} \leftarrow A_{k,m} \times_{\mathbb{A}^n//W_{n,k}} \mathbb{A}^{n-1}//W_{n-1,k-1} \to \mathbb{A}^{n-1}//W_{n-1,k-1}\}.$$

Applying Borel Theorem 3.1.1 to $W_{n,k} \subset W_n \subset GL_n(\mathbb{C})$, we have inclusions of invariant subrings

$$\mathbb{C}[t_1,\ldots,t_n]^{W_n} = \mathbb{C}[\theta_1,\ldots,\theta_n] \subset \mathbb{C}[t_1,\ldots,t_n]^{W_{n,k}} = \mathbb{C}[\alpha_1,\ldots,\alpha_{n-k},\beta_1,\ldots,\beta_k] \subset \mathbb{C}[t_1,\ldots,t_n].$$

By similar argument as before we have the following theorem.

Theorem 8.2.2. *Let $W_n = S_n$ or B_n and let $W_{n,k} = S_{n-k} \times S_k$ or $B_{n-k} \times B_k$ respectively. The variety of relative generalized quasi-invariants $A_{k,m} = \operatorname{Spec} P_{k,m}$ is given explicitly by*

$$P_{k,m} = \mathbb{C}[\alpha_1, \ldots, \alpha_{n-k}, \beta_1, \ldots, \beta_k] + \mathbb{C}[t_1, \ldots, t_n]\beta_k^m$$

where $\alpha_i = \sigma_i(t_1, \ldots, t_{n-k})$ and $\beta_i = \sigma_i(t_{n-k+1}, \ldots, t_{n-1})$ when $W_{n,k} = S_{n-k} \times S_k$, and $\alpha_i = \sigma_i(t_1^2, \ldots, t_{n-k}^2)$ and $\beta_i = \sigma_i(t_{n-k+1}^2, \ldots, t_{n-1}^2)$ when $W_{n,k} = B_{n-k} \times B_k$. Furthermore, $A_{k,m}$ is a free module over $\mathbb{C}[t_1, \ldots, t_n]^{W_n}$ of rank $|W_n|$.

8.3 Normalization

For the varieties of classical quasi-invariants, the normalization map $p_m : \mathbb{A}^n \twoheadrightarrow \operatorname{Spec} Q_m(W)$ is injective (in fact, bijective, see [BEG] Lemma 7.3(ii)). This is a crucial component of their main result: the the algebra $\mathcal{D}(A_m)$ of differential operators on A_m is Morita equivalent to the algebra $\mathcal{D}(\mathbb{A}^n)$ of differential operators on $\mathbb{A}^n$. A direct consequence of this result is that $\mathcal{D}(A_m)$ is a simple algebra, and it follows by [VdB] Theorem 6.2.5 that A_m is Cohen-Macaulay.

This motivates us to ask if the normalization map

$$\pi : \mathbb{A}^n \longrightarrow\!\!\!\!\!\!\rightarrow A_{k,m}$$

is bijective, and it suffices to check if π is injective, i.e.

Question 8.3.1. *Given two maximal ideals $\mathfrak{m}, \mathfrak{m}' \subseteq \mathbb{C}[t_1, \ldots, t_n]$, if $\mathfrak{m} \cap P_{k,m} = \mathfrak{m}' \cap P_{k,m}$, is $\mathfrak{m} = \mathfrak{m}'$?*

Unfortunately the answer to this question is negative for $n > 1$. In this last part of this chapter, we will construct the following counterexample for $n = k > 1$

when $W_n = S_n$. The counterexamples for the cases of B_n and general (n,k) are analogous.

As before, we write $c_i = \sigma_i(t_1, \ldots, t_n)$ for the i-th elementary symmetric polynomial of n variables. Then in this case $P_m = \mathbb{C}[c_1, \ldots, c_n] + \mathbb{C}[t_1, \ldots, t_n] \cdot c_n^m$.

Claim 8.3.2. *There are two maximal ideals* $\mathfrak{m}_1 \neq \mathfrak{m}_2 \subseteq \mathbb{C}[t_1, \ldots, t_n]$ *such that* $\mathfrak{m}_1 \cap P_m = \mathfrak{m}_2 \cap P_m$.

Consider the following two maximal ideals in $\mathbb{C}[t_1, \ldots, t_n]$,

$$\begin{aligned} \mathfrak{m}_1 &= (t_1 - 1, t_2, t_3, \ldots, t_n) \\ \mathfrak{m}_2 &= (t_1, t_2 - 1, t_3, \ldots, t_n) \end{aligned}$$

and we have

$$c_1 - 1, c_2, \ldots, c_n \in \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap P_m.$$

This implies the ideal

$$(c_1 - 1, c_2, \ldots, c_n) \cdot P_m \subseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap P_m.$$

Note $t_i^m \in \mathfrak{m}_1$ for $2 \leq i \leq n$, so $t_2^m \cdots t_n^m \in \mathfrak{m}_1$. Furthermore, $t_1^m - 1 \in \mathfrak{m}_1$, so

$$t_1^m t_2^m t_3^m \cdots t_n^m = (t_1^m - 1) t_2^m \cdots t_n^m + t_2^m \cdots t_n^m \in \mathfrak{m}_1$$

and thus $\mathbb{C}[t_1, \ldots, t_n] t_1^m \cdots t_n^m \subseteq \mathfrak{m}_1$.

Similarly, $t_i^m \in \mathfrak{m}_2$ for $1 \leq i \leq n, i \neq 2$, so $t_1^m t_3^m \cdots t_n^m \in \mathfrak{m}_2$. Furthermore, $t_2^m - 1 \in \mathfrak{m}_2$, so

$$t_1^m t_2^m t_3^m \cdots t_n^m = (t_2^m - 1) t_1^m t_3^m \cdots t_n^m + t_1^m t_3^m \cdots t_n^m \in \mathfrak{m}_2$$

and thus $\mathbb{C}[t_1, \ldots, t_n] t_1^m \cdots t_n^m \subseteq \mathfrak{m}_2$.

Let

$$\mathfrak{n} = (c_1 - 1, c_2, \dots, c_n) \cdot P_m + \mathbb{C}[t_1, \cdot, t_n] t_1^m \cdots t_n^m,$$

from above discussion, we know $\mathfrak{n} \subset \mathfrak{m}_1 \cap P_m$ and $\mathfrak{n} \subseteq \mathfrak{m}_2 \cap P_m$. We claim $\mathfrak{n}$ is a maximal ideal in P_m, then it follows that $\mathfrak{n} = \mathfrak{m}_1 \cap P_m = \mathfrak{m}_2 \cap P_m$.

Choose any element $f \in P_m \backslash \mathfrak{n}$, we may decompose $f = g + h$ where $g \in \mathbb{C}[c_1, \dots, c_n]$ and $h \in \mathbb{C}[t_1, \cdot, t_n] t_1^m \cdots t_n^m$. Note $g \notin \mathfrak{n}$ implies $g \notin (c_1 - 1, c_2, \dots, c_n) \cdot P_m$ and therefore $g \notin (c_1 - 1, c_2, \dots, c_n) \cdot \mathbb{C}[c_1, \dots, c_n]$ which is a maximal ideal in the ring of symmetric polynomials, so $g = g_1 + g_2$ where $g_1 \in \mathbb{C}^\times$ is a unit and $g_2 \in (c_1 - 1, c_2, \dots, c_n) \cdot \mathbb{C}[c_1, \dots, c_n]$. Thus $(f) \cdot P_m + \mathfrak{n} = (g_1) \cdot P_m + \mathfrak{n} = P_m$, which implies $\mathfrak{n}$ is maximal.

CHAPTER 9

CLASSIFYING SPACE OF COMMUTATIVITY

In this chapter, we apply of our main theorem (Theorem 6.3.1) to a different choice of the initial space X_0. Instead of considering BT, we are interested in other spaces X_0 equipped with canonical map $X_0 \to BG$ such that $H^*(X)$ is a free module over $H^*(BG)$ of rank $|W|$. One natural candidate is $B_{comm}G$, the classifying space of commuting elements in a Lie group G introduced in [ACG]. In this case, we can compute the rational cohomology of the spaces X_m. Furthermore, we give an explicit description of the basis of $H^*(X_m)$ as a free module over $H^*(BG)$ in cases where $G = \mathrm{U}(n), \mathrm{SU}(n)$, and $\mathrm{Sp}(n)$ by combinatorial methods (see, e.g., [A2] [V] [G3] [G4]).

9.1 Classifying Space of Commutativity

Definition 9.1.1 ([ACG]). Let G be a topological group and consider the simplicial space $B_{comm}(G)_* = \{\mathrm{Hom}(\mathbb{Z}^n, G)\}_{n\geq 0}$ with face maps

$$d_i : \mathrm{Hom}(\mathbb{Z}^n, G) \longrightarrow \mathrm{Hom}(\mathbb{Z}^{n-1}, G)$$

given by

$$d_i(g_1, \cdots, g_n) = \begin{cases} (g_2, \cdots, g_n), & i = 0 \\ (g_1, \cdots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \cdots, g_n), & 1 \leq i < n, \\ (g_1, \ldots, g_{n-1}), & i = n, \end{cases}$$

and degeneracy maps

$$s_j : \mathrm{Hom}(\mathbb{Z}^n, G) \longrightarrow \mathrm{Hom}(\mathbb{Z}^{n+1}, G)$$
$$(g_1, \ldots, g_n) \longmapsto (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n).$$

We denote by $B_{comm}G$ the geometric realization of the simplicial space $B_{comm}(G)_*$.

Definition 9.1.2 ([ACG]). The simplicial space $E_{comm}G_* = \{\mathrm{Hom}(\mathbb{Z}^n, G) \times G \subset G^{n+1}\}_{n\geq 0}$ can be defined analogously with face and degeneracy maps similar to EG_*, its geometric realization is dennoted by $E_{comm}G$.

The projection on the first n-coordinates defines a simplicial map

$$E_{comm}(G)_* \longrightarrow B_{comm}(G)_*$$

which induces a continuous map on geometric realizations

$$p_{comm} : E_{comm}G \longrightarrow B_{comm}G$$

that fits in the following diagram of morphisms of principal G-bundles

$$\begin{array}{ccc} E_{comm}G & \longrightarrow & EG \\ {\scriptstyle p_{comm}}\downarrow & & \downarrow{\scriptstyle p} \\ B_{comm}G & \xrightarrow{i} & BG \end{array}$$

thus up to homotopy it gives rise to a fibration sequence

$$E_{comm}G_1 \longrightarrow B_{comm}G_1 \longrightarrow BG. \tag{9.1}$$

Remark. In general, $\mathrm{Hom}(\mathbb{Z}^n, G)$ may fail to be path connected even if G is path-connected or simply-connected, so we may want to work with the identity component of $\mathrm{Hom}(\mathbb{Z}^n, G)$. When G is a classical Lie group, i.e. $\mathrm{SU}(n), \mathrm{U}(n)$ or $\mathrm{Sp}(n)$ or their finite products, $\mathrm{Hom}(\mathbb{Z}^n, G)$ is always path-connected. Furthermore, if G is connected, $B_{comm}G$ is simply-connected, and if G is simply-connected, $B_{comm}G$ is 3-connected.

The rational cohomology of $B_{comm}G_1$ is given in the following result.

Proposition 9.1.3 ([AG] Proposition 7.1)**.** *Suppose that G is a compact connected Lie group with $T \subset G$ a maximal torus and associated Weyl group W, then there is a natural isomorphism of rings*

$$\alpha_G : H^*(B_{comm}G_1) \xrightarrow{\cong} (H^*(BT) \otimes H^*(BT))^W / \mathfrak{J}_G$$

where W acts diagonally on $H^*(BT) \otimes H^*(BT)$ and $\mathfrak{J}_G$ is the ideal generated by elements of positive degrees in the image of

$$\begin{aligned} i_1 : H^*(BG) &\to (H^*(BT) \otimes H^*(BT))^W \\ x &\longmapsto x \otimes 1 \end{aligned}$$

Under this identification, the $H^*(BG)$-module structure on the algebra $(H^*(BT) \otimes H^*(BT))^W / \mathfrak{J}_G$ is given by

$$f \cdot [x \otimes y] = [x \otimes fy].$$

In particular we can identify $H^*(BG)$ as a subalgebra in $H^*(B_{comm}G_1)$.

As a consequence we have the following theorem.

Theorem 9.1.4 ([AG] Theorem 7.2)**.** *Suppose G is a compact connected Lie group, then $H^*(B_{comm}G_1)$ is a free module over $H^*(BG)$ of rank $|W|$.*

The rational cohomology of the fiber $E_{comm}G_1$ is given as follows.

Corollary 9.1.5 ([AG] Corollary 7.4)**.** *Suppose that G is a compact connected Lie group with $T \subset G$ a maximal torus and associated Weyl group W, then there is a natural isomorphism of rings*

$$\widetilde{\alpha}_G : H^*(E_{comm}G_1) \xrightarrow{\cong} (H^*(G/T) \otimes H^*(G/T))^W.$$

9.2 Relative Join Construction

Let G be a compact connected Lie group with maximal torus T and associated Weyl group W, and let H be a subgroup of $G = \widetilde{G}$ such that $G/H \cong \mathbb{S}^{2k-1}$. We may apply the relative join construction to the two fibrations (9.1) and (7.2) to get a tower of fibrations:

$$F_m = E_{comm}G_1 *^m G/H \longrightarrow X_m = B_{comm}G_1 *^m_{BG} BH \longrightarrow BG \qquad (9.2)$$

for $m \geq 0$. In this case,

1. BG has only even rational cohomology.
2. $E_{comm}G_1$ has only even rational cohomology.
3. $G/H \cong \mathbb{S}^{2k-1}$ is an odd dimensional sphere.
4. $H^*(B_{comm}G_1)$ is a free module over $H^*(BG)$ of rank $|W|$ by Theorem 9.1.4.

so by Proposition 6.3.3, $H^*(X_m)$ is a free module over $H^*(BG)$ of rank $|W|$.

9.3 Rational Cohomology

We will now describe the rational cohomology of X_m explicitly. Write

$$H^*(B_{comm}G_1) = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^W / (\theta_1(X), \ldots, \theta_n(X))$$

where $\theta_i(X)$ are the images of generators θ_i of $H^*(BG)$ in

$$\begin{aligned} i_1 : H^*(BG) &\to (H^*(BT) \otimes H^*(BT))^W \\ x &\longmapsto x \otimes 1 \end{aligned}$$

Using the same proof as in Theorem 7.1.1, we can get the following result when $X_0 = B_{comm}G_1$.

Theorem 9.3.1. *Let (G, T, H) be the same as in Theorem 7.1.1. Let $X_m = B_{comm}G_1 *^m_{BG} BH$. The rational cohomology of X_m is given by*

$$H^*(X_m) \cong \mathbb{Q}[\theta_1, \dots, \theta_n] + H^*(B_{comm}G_1) \cdot \theta_n^m$$

where $\theta_i = \theta_i(Y)$ are the images of $\theta_i \in H^*(BG)$ in $H^*(B_{comm}G_1)$, and the canonical composite of fibrations $B_{comm}G_1 \to X_m \to BG$ induces the inclusions in rational cohomology

$$\mathbb{Q}[\theta_1, \dots, \theta_n] \hookrightarrow \mathbb{Q}[\theta_1, \dots, \theta_n] + H^*(B_{comm}G_1)\theta_n^m \hookrightarrow H^*(B_{comm}G_1).$$

Furthermore, $H^*(X_m)$ is a free module over $H^*(BG) = \mathbb{Q}[\theta_1, \dots, \theta_n]$ of rank $|W|$.

9.3.1 The Case of $\mathrm{U}(n)$

For $G = \mathrm{U}(n)$ and $W = S_n$, we have

$$H^*(B_{comm}\,\mathrm{U}(n)) \cong \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}/(c_1(X), \dots, c_n(X))$$

$$H^*(E_{comm}\,\mathrm{U}(n)) \cong \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]^{S_n}/(c_1(X), \dots, c_n(X), c_1(Y), \dots, c_n(Y)).$$

Definition 9.3.2. A polynomials $p(X, Y)$ is **double symmetric** if for any $(\alpha, \beta) \in S_n \times S_n$, $(\alpha, \beta)p(X, Y) = p(X, Y)$ where $(\alpha, \beta) \cdot p(X, Y) = p(x_{\alpha_1}, \dots, x_{\alpha_n}, y_{\beta_1}, \dots, y_{\beta_n})$. We will denote $\mathbb{Q}[X, Y]^{S_n \times S_n}$ the ring of doubly symmetric polynomials. Note $\mathbb{Q}[X, Y]^{S_n \times S_n} = \mathbb{Q}[c_1(X), \dots, c_n(X), c_1(Y), \dots, c_n(Y)]$ where c_i is the i-th elementary symmetric polynomial in the corresponding variables.

Definition 9.3.3. The **descent monomial** associated with $\sigma \in S_n$ is defined to be

$$h_\sigma = \prod_{\sigma^{-1}(i) > \sigma^{-1}(i+1)} (x_1 \cdots x_i) \prod_{\sigma(j) > \sigma(j+1)} (y_{\sigma(1)} \cdots y_{\sigma(j)})$$

Consider the averaging operator

$$\rho : \mathbb{Q}[X, Y] \longrightarrow \mathbb{Q}[X, Y]^{S_n}$$
$$f \longmapsto \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma \cdot f$$

and let $g_\sigma = \rho(h_\sigma) \in \mathbb{Q}[X, Y]^{S_n}$.

Theorem 9.3.4 ([A2], Theorem 1.3)**.** *The collection $\mathfrak{G}_n = \{g_\sigma\}_{\sigma \in S_n}$ forms a basis of $\mathbb{Q}[X, Y]^{S_n}$ as free module over $\mathbb{Q}[X, Y]^{S_n \times S_n}$.*

Note we have the following isomorphism of $\mathbb{Q}[X, Y]^{S_n \times S_n}$-modules

$$\mathbb{Q}[X, Y]^{S_n} \cong \mathbb{Q}[X]^{S_n} \otimes \mathbb{Q}[Y]^{S_n} \otimes \mathfrak{G}_n$$

therefore tensoring on both sides with $\mathbb{Q}[Y]^{S_n}$ over $\mathbb{Q}[X, Y]^{S_n \times S_n}$, we get an isomorphism of $\mathbb{Q}[Y]^{S_n}$-modules

$$\mathbb{Q}[X, Y]^{S_n}/\mathfrak{I}_n = \mathbb{Q}[Y]^{S_n} \otimes_{\mathbb{Q}[X,Y]^{S_n}} \mathbb{Q}[X, Y]^{S_n} \cong \mathbb{Q}[Y]^{S_n} \otimes \mathfrak{G}_n$$

so the set $\mathfrak{G}_n$ of descent polynomials form a basis of $H^*(B_{comm}\,\mathrm{U}(n))$ as a module over $H^*(B\,\mathrm{U}(n))$. Thus by Proposition 6.3.8 and 6.3.9, we have the following result.

Corollary 9.3.5. *The rational cohomology of X_m is*

$$H^*(X_m) = \mathbb{Q}[c_1(Y), \ldots, c_n(Y)] + (\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}/(c_1(X), \ldots, c_n(X))) \cdot c_n(Y)^m$$

and it decomposes as a graded module

$$H^*(X_m) \cong \mathbb{Q}[c_1(Y), \ldots, c_n(Y)] \oplus \bigoplus_{\sigma \in S_n \setminus \{e\}} \mathbb{Q}[c_1(Y), \ldots, c_n(Y)] \cdot g_\sigma c_n(Y)^m$$

over $H^*(B\,\mathrm{U}(n))$ with basis

$$\{1, g_\sigma c_n(Y)^m\}_{\sigma \in S_n \setminus \{e\}}.$$

Definition 9.3.6. The **major index** $\mathrm{maj}(\sigma)$ of $\sigma \in S_n$ is defined to be

$$\mathrm{maj}(\sigma) := \sum_{\sigma(i)>\sigma(i+1)} i.$$

It can be seen that $\deg(g_\sigma) = 2(\mathrm{maj}(\sigma) + \mathrm{maj}(\sigma^{-1}))$.

Corollary 9.3.7. *The Hilbert series of $H^*(X_m)$ is*

$$p_{X_m}(t) = \frac{1 - t^{2mn}}{\prod_{i=1}^{n}(1 - t^{2i})} + \frac{t^{2mn}(\sum_{\sigma\in S_n} t^{2(maj(\sigma)+maj(\sigma^{-1}))})}{\prod_{i=1}^{n}(1 - t^{2i})}.$$

9.3.2 The Case of $\mathrm{SU}(n)$

For $G = \mathrm{SU}(n)$ and $W = S_n$, we have

$$H^*(B_{comm}G) \cong \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}/(c_1(X), \ldots, c_n(X), c_1(Y)).$$

In this case the set $\mathfrak{G}_n$ of descent polynomials forms a basis of $H^*(B_{comm}\,\mathrm{SU}(n))$ as a module over $H^*(B\,\mathrm{SU}(n))$. Thus by Proposition 6.3.8 and 6.3.9 we have the following result.

Corollary 9.3.8. *The rational cohomology of X_m is*

$$\mathbb{Q}[c_2(Y), \ldots, c_n(Y)] + (\mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{S_n}/(c_1(X), \ldots, c_n(X), c_1(Y))) \cdot c_n(Y)^m$$

and it decomposes as a graded module

$$H^*(X_m) \cong \mathbb{Q}[c_2(Y), \ldots, c_n(Y)] \oplus \bigoplus_{\sigma\in S_n\setminus\{e\}} \mathbb{Q}[c_2(Y), \ldots, c_n(Y)] \cdot g_\sigma c_n(Y)^m$$

over $H^*(B\,\mathrm{SU}(n))$ with basis

$$\{1, g_\sigma c_n(Y)^m\}_{\sigma\in S_n\setminus\{e\}}.$$

The Hilbert series of $H^*(X_m)$ is

$$p_{X_m}(t) = \frac{1 - t^{2mn}}{\prod_{i=2}^{n}(1 - t^{2i})} + \frac{t^{2mn}(\sum_{\sigma\in S_n} t^{2(maj(\sigma)+maj(\sigma^{-1}))})}{\prod_{i=2}^{n}(1 - t^{2i})}.$$

9.3.3 The Case of $\mathrm{Sp}(n)$

For $G = \mathrm{Sp}(n)$ and $W = B_n$, we have

$$H^*(B_{comm}G) \cong \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{B_n}/(q_1(X), \ldots, q_n(X))$$

where $q_i(X) = \sigma_i(x_1^2, \ldots, x_n^2)$ is the i-th signed symmetric polynomial, and

$$H^*(E_{comm}G) \cong \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]^{B_n}/(q_1(X), \ldots, q_n(X), q_1(Y), \ldots, q_n(Y)).$$

Given $\sigma \in B_n$, let

$$\begin{aligned} d_i(\sigma) &:= |\{i \le j \le n-1 | \sigma(j) > \sigma(j+1)\}| \\ \varepsilon_i(\sigma) &:= \begin{cases} 0, & \sigma(i) > 0, \\ 1, & \sigma(i) < 0. \end{cases} \\ f_i(\sigma) &:= 2d_i(\sigma) + \varepsilon_i(\sigma). \end{aligned}$$

Definition 9.3.9. The **diagonal signed descent monomial** associated to $\sigma \in B_n$ is defined to be

$$b_\sigma = \prod_{i=1}^n x_i^{f_i(\sigma^{-1})} y_i^{f_{|\sigma^{-1}(i)|}(\sigma)}.$$

Consider the averaging operator

$$\begin{aligned} \rho : \mathbb{Q}[X, Y] &\longrightarrow \mathbb{Q}[X, Y]^{B_n} \\ f &\longmapsto \frac{1}{|B_n|} \sum_{\sigma \in B_n} \sigma \cdot f \end{aligned}$$

and let $c_\sigma = \rho(b_\sigma) \in \mathbb{Q}[X, Y]^{B_n}$.

Theorem 9.3.10 ([G3] Theorem 1.1)**.** *the collection* $\mathfrak{C}_n = \{c_\sigma\}_{\sigma \in B_n}$ *forms a basis of* $\mathbb{Q}[X, Y]^{B_n}$ *as a free module over* $\mathbb{Q}[X, Y]^{B_n \times B_n}$.

Therefore by analogous argument for the cases of S_n, we know $\mathfrak{C}_n$ is a basis of $H^*(B_{comm}\mathrm{Sp}(n))$ over $H^*(B\,\mathrm{Sp}(n))$. Thus by Proposition 6.3.8 and 6.3.9, we have the following result.

Corollary 9.3.11. *The rational cohomology of X_m is*

$$H^*(X_m) = \mathbb{Q}[q_1(Y), \dots, q_n(Y)] + (\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]^{B_n}/(q_1(X), \dots, q_n(X))) \cdot q_n(Y)^m$$

and it decomposes as a graded module

$$H^*(X_m) \cong \mathbb{Q}[q_1(Y), \dots, q_n(Y)] \oplus \bigoplus_{\sigma \in B_n \setminus \{e\}} \mathbb{Q}[q_1(Y), \dots, q_n(Y)] \cdot c_\sigma q_n(Y)^m$$

over $H^*(B\,\mathrm{Sp}(n))$ with basis

$$\{1, c_\sigma q_n(Y)^m\}_{\sigma \in B_n \setminus \{e\}}.$$

Definition 9.3.12 ([AR])**.** The **flag major index** $\mathrm{fmaj}(\sigma)$ of a signed permutation is defined as

$$\mathrm{fmaj}(\sigma) := 2(\mathrm{maj}(\sigma) + \mathrm{neg}(\sigma))$$

where $\mathrm{neg}(\sigma) := |\{1 \leq i \leq n \mid \sigma(i) < 0\}|$ if we identify B_n with the group of signed permutation on $\mathbb{I}_n = \{-n, -n+1, \dots, -1, 1, \dots, n\}$.

It can be seen that

$$\deg(c_\sigma) = 2(\mathrm{fmaj}(\sigma) + \mathrm{fmaj}(\sigma^{-1})).$$

Corollary 9.3.13. *The Hilbert series of $H^*(X_m)$ is*

$$p_{X_m}(t) = \frac{1 - t^{4mn}}{\prod_{i=1}^n (1 - t^{4i})} + \frac{t^{4mn}(\sum_{\sigma \in B_n} t^{2(fmaj(\sigma) + fmaj(\sigma^{-1}))})}{\prod_{i=1}^n (1 - t^{4i})}.$$

CHAPTER 10

SPHERICAL FIBRATIONS

We give a further generalization that applies to some classical spherical fibrations. The general scheme is as follows.

Consider the following two triples:

- $(X, G \supseteq H)$ where X is a G-space and $H \subseteq G$ is a closed subgroup,
- $(X', G' \supseteq H')$ where X' is a G'-space and $H' \subseteq G'$ is a closed subgroup

such that

$$X_G \cong X'_{G'}. \tag{10.1}$$

Denote $B := X_G \cong X'_{G'}$. Then we have two fibrations

$$G/H \longrightarrow X_H \longrightarrow X_G = B$$

$$G'/H' \longrightarrow X'_{H'} \longrightarrow X'_{G'} = B$$

that produce a fibration

$$G/H *^m G'/H' \longrightarrow X_H *^m_B X'_{H'} \longrightarrow B$$

To ensure (10.1) is satisfied, we can always make the following choices:

$$X = \mathrm{pt}$$
$$H \subseteq G = H' \subseteq G'$$
$$X' = G'/H' = G'/G$$

then $X_G = X'_{G'} = BG$. More generally, we can take $P \subset G \cap G'$ and $X = G/P$, $X' = G'/P$, then $X_G = X'_{G'} = BP$.

10.1 Homotopy Quotient of Homogeneous Spaces

Let G be a compact connected Lie group and H a closed connected subgroup of G. Let $G \to G/H$ be the natural projection. Let $N_G(H)$ be the normalizer of H in G.

Theorem 10.1.1 ([S2]). *If the following two conditions are satisfied:*

P1 The rational homotopy type of G/H is formal.

P2 $\pi^* : H^*(G/H;\mathbb{R})^{N_G(H)} \to H^*(G;\mathbb{R})$ is injective.

then the following fibration

$$G/H \longrightarrow EG \times_H G/H \longrightarrow BH$$

induces a surjective map

$$j^* : H^*(EH \times_H G/H) = H^*_H(G/H) \longrightarrow H^*(G/H)$$

of cohomology with coefficients in $\mathbb{R}$.

Example 10.1.2. We have the following two type of examples:

1. $\mathrm{SO}(2n+1) \subset \mathrm{SO}(2n+2)$.
2. $G_2 \subset \mathrm{Spin}(7)$.

Corollary 10.1.3. *Given a pair (G, H) satisfying the above conditions (P1) and (P2), the rational equivariant cohomology of the homogeneous space G/H is*

$$H^*_H(G/H) \cong H^*(G/H) \otimes H^*(BH)$$

as a graded $\mathbb{Q}$-vector space.

The corollary above shows that the rational cohomology of $(G/H)_{hH}$ is a free module over $H^*(BH)$.

In fact, we can know more about the rational cohomology of $(G/H)_{hH}$ including the multiplicative structure (see, e.g. [C1] Proposition 6.2.4).

Proposition 10.1.4. *Let G be a Lie group and H, K be closed subgroups. Then there is a homeomorphism of G/H-bundles over BK*

$$\varphi : EG \times_K G/H \rightleftarrows BK \times_{BG} BH : \phi$$

$$[e, gH]_K \longmapsto (eK, egH)$$

$$[e, gH]_K \longleftarrow (eK, egH)$$

Proof. The composite $\varphi \circ \phi = \mathrm{Id}$ and $\phi \circ \varphi = \mathrm{Id}$. It suffices to show these two maps are well-defined.

The map φ is well-defined since given another representative $(ek, k^{-1}gh)$, it maps to the same element

$$\varphi([ek, k^{-1}ghH]_k) = (ekK, eghH) = (eK, egH).$$

For the map ϕ, note any element $(eK, fH) \in BK \times_{BG} BH$ satisfies $eG = fG$, i.e. there is a unique $g \in G$ such that $f = eg$, so any element in $BK \times_{BG} BH$ is of the form (eK, egH). To check ϕ is well-defined, note for another representative $(ek, egh) = (ek, ekk^{-1}gh)$, it maps to

$$\phi(ekK, eghH) = [ek, k^{-1}ghH]_K = [e, gH]_K$$

so ϕ is also well-defined. □

Corollary 10.1.5. *The rational cohomology of the homotopy quotient $(G/H)_{hK}$ is given by*

$$H^*((G/H)_{hK}) \cong H^*(BK \times_{BG} BH) \cong \mathrm{Tor}^*_{H^*(BG)}(H^*(BK), H^*(BH))$$

as an algebra.

Proof. The previous proposition gives the first isomorphism. The second isomorphism is the result of the Eilenberg-Moore spectral sequence. □

10.2 Gorenstein Examples

We can use the examples in Example 10.1.2 to produce examples of topological spaces satisfying Gorenstein duality.

For this, we need to choose our data such that

$$G/H \cong \mathbb{S}^{2n} \quad G'/H' \cong \mathbb{S}^{2k+1}$$

Theorem 10.2.1 (Montgomery-Samelson, Borel)**.** *If $G/H \cong \mathbb{S}^{2n}$ is an even dimensional sphere, where $H \subseteq G$ is a closed subgroup of a compact connected Lie group such that G acts effectively on G/H, then (up to equivalence) there are only two possibilities:*

1. $G = \mathrm{SO}(2n+1), H = \mathrm{SO}(2n)$, $G/H \cong \mathbb{S}^{2n}, n \geq 1$.
2. $G = G_2, H = \mathrm{SU}(2)$, *and* $G/H \cong \mathbb{S}^6$.

Remark. SU(2) does not act effectively on $\mathbb{S}^2$, we have

$$\begin{array}{ccccc} \{\pm 1\} & \hookrightarrow & \mathrm{SU}(2) & \twoheadrightarrow & \mathrm{SO}(3) \\ & & \downarrow & & \downarrow \\ & & \mathrm{SU}(2)/\mathrm{SU}(1) & \xrightarrow{\cong} & \mathrm{SO}(3)/SO(2) \end{array}$$

Let $H = \mathrm{SO}(2n), G = H' = \mathrm{SO}(2n+1), G' = \mathrm{SO}(2n+2)$. Write

$$Y = E\,\mathrm{SO}(2n+2) \times_{\mathrm{SO}(2n+1)} \mathbb{S}^{2n+1},$$

then we have the following two fibrations

$$\mathbb{S}^{2n} \cong \mathrm{SO}(2n+1)/\,\mathrm{SO}(2n) \longrightarrow B\,\mathrm{SO}(2n) \to B\,\mathrm{SO}(2n+1)$$

$$\mathbb{S}^{2n+1} \cong \mathrm{SO}(2n+2)/\,\mathrm{SO}(2n+1) \longrightarrow Y \longrightarrow B\,\mathrm{SO}(2n+1)$$

Apply the relative join construction, we can get a new fibration

$$\mathbb{S}^{4n+2} = \mathbb{S}^{2n} * \mathbb{S}^{2n+1} \longrightarrow X_1 = B\,\mathrm{SO}(2n) *_{B\,\mathrm{SO}(2n+1)} Y \longrightarrow B\,\mathrm{SO}(2n+1).$$

Through iterative application of this construction, we obtain fibrations

$$\mathbb{S}^{4mn+2} = \mathbb{S}^{2n} *^{m} \mathbb{S}^{2n+1} \longrightarrow X_m = B\,\mathrm{SO}(2n) *^{m}_{B\,\mathrm{SO}(2n+1)} Y \longrightarrow B\,\mathrm{SO}(2n+1). \tag{10.2}$$

Proposition 10.2.2. *The ring structure on $H^*(X_m)$ is given by $\mathbb{Q}[q_1, \cdots, q_n, \xi_m]/(\xi_m^2)$ for $m \geq 1$.*

Before proving this proposition in the rank one and higher rank cases, we begin with some analysis.

The rational cohomology $H^*(X_1)$ fits into a (Mayer-Vietoris) long exact sequence

$$\begin{aligned} H^{r-1}(Y_1) \to H^{r-1}(Y) \oplus H^{r-1}(B\,\mathrm{SO}(2n)) &\overset{(\alpha,-\beta)}{\longrightarrow} H^{r-1}(B\,\mathrm{SO}(2n) *_{B\,\mathrm{SO}(2n+1)} Y) \\ \longrightarrow H^{r}(Y_1) \longrightarrow H^{r}(Y) \oplus H^{r}(B\,\mathrm{SO}(2n)) &\xrightarrow{(\alpha,-\beta)} H^{r}(B\,\mathrm{SO}(2n) *_{B\,\mathrm{SO}(2n+1)} Y) \end{aligned} \tag{10.3}$$

Note in this case, the rational cohomology of Y is

$$H^*(Y) = \mathrm{Tor}^*_{H^*(B\,\mathrm{SO}(2n+2))}(H^*(B\,\mathrm{SO}(2n+1)), H^*(B\,\mathrm{SO}(2n+1))) = \mathbb{Q}[q_1, \ldots, q_n, e_{2n+1}]$$

where $q_i = \sigma_i(t_1^2, \ldots, t_n^2)$ is the i-th signed symmetric polynomial and $\deg(e_{2n+1}) = 0$.

Thus $H^*(Y)$ is a free module over $H^*(B\,\mathrm{SO}(2n+1))$ of rank 2. And it follows that

$$\begin{aligned} H^*(B\,\mathrm{SO}(2n) \times_{B\,\mathrm{SO}(2n+1)} Y) &= \mathrm{Tor}^*_{H^*(B\,\mathrm{SO}(2n+1))}(H^*(Y), H^*(B\,\mathrm{SO}(2n))) \\ &= H^*(Y)) \otimes_{H^*(B\,\mathrm{SO}(2n+1))} H^*(B\,\mathrm{SO}(2n)) \\ &= \mathbb{Q}[q_1, \ldots, q_n, e_{2n+1}] \otimes_{\mathbb{Q}[q_1,\ldots,q_n]} \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n] \\ &= \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n, e_{2n+1}] \end{aligned}$$

where $c_n = \sigma_n(t_1, \ldots, t_n)$ satisfies $c_n^2 = q_n$.

10.2.1 Rank 1 Case

When $n = 1$, the two fibrations are given by

$$\begin{aligned} &\mathbb{S}^2 \longrightarrow X_0 = B\,\mathrm{SO}(2) \longrightarrow B = B\,\mathrm{SO}(3) \\ &\mathbb{S}^3 \to Y = (B\,\mathrm{SO}(4)/B\,\mathrm{SO}(3))_{hB\,\mathrm{SO}(3)} \to B = B\,\mathrm{SO}(3) \end{aligned}$$

and the m-th step relative join gives

$$F_m = \mathbb{S}^{2+4m} = \mathbb{S}^2 *^m \mathbb{S}^3 \longrightarrow X_m = X_0 *_B^m Y \longrightarrow B$$

The Serre spectral sequence in this case collapses at E_2-page since the base and fiber have only even dimensional rational cohomology, which implies that

$$H^*(X_m) = \mathbb{Q}[q_1] \oplus \mathbb{Q}[q_1]\xi_m$$

is a free module over $H^*(B) = \mathbb{Q}[q_1]$ with basis $\{1, \xi_m\}$ where $\deg(\xi_m) = 4m + 2$.

For degree reasons we have the following result.

Claim 10.2.3. *For a rank 2 (graded) free module $P_m = \mathbb{Q}[q_1] \oplus \mathbb{Q}[q_1]\xi_m$ over $\mathbb{Q}[q_1], \deg(q_1) = 4$ with basis $\{1, \xi_m\}$ where $\deg(\xi_m) = 4m+2$, there are only two potential ring structures on M. One is $\mathbb{Q}[q_1, \xi_m]/(\xi_m^2 - q_1^{2m+1})$, and the other is $\mathbb{Q}[q_1, \xi_m]/(\xi_m^2)$.*

An immediate consequence of the previous claim is the following.

Proposition 10.2.4. *The ring structure on $H^*(X_m)$ is given by $\mathbb{Q}[q_1, \xi_m]/(\xi_m^2)$ for $m \geq 1$.*

Let first look at the case when $m = 1$.

Consider the Mayer-Vietoris sequence (10.3) in this case,

$$\cdots \to H^{r-1}(Y \times_{B\,\mathrm{SO}(3)} B\,\mathrm{SO}(2)) \longrightarrow H^r(X_1) \xrightarrow{(f,g)} H^r(Y) \oplus H^r(B\,\mathrm{SO}(2)) \\ \xrightarrow{(\alpha,\beta)} H^r(Y \times_{B\,\mathrm{SO}(3)} B\,\mathrm{SO}(2)) \longrightarrow \cdots \tag{10.4}$$

Since $H^*(X_0)$ and $H^*(X_1)$ has only even degree elements and $H^*(Y)$ is only nonzero in degree $4i$ and degree $4i+3$, the long exact sequence splits into

$$0 \longrightarrow H^{4i+1}(X_0 \times_B Y) \longrightarrow H^{4i+2}(X_1) \longrightarrow H^{4i+2}(X_0) \oplus 0 \longrightarrow H^{4i+2}(X_0 \times_B Y) \longrightarrow 0$$

and

$$0 \to 0 \oplus H^{4i+3}(Y) \longrightarrow H^{4i+3}(X_0 \times_B Y) \longrightarrow H^{4i+4}(X_1) \\ \longrightarrow H^{4i+4}(X_0) \oplus H^{4i+4}(Y) \to H^{4i+4}(X_0 \times_B Y) \to 0$$

which are given explicitly by

$$0 \longrightarrow \mathbb{Q} \cdot c_1^{2i-1} e_3 \longrightarrow H^{4i+2}(X_1) \xrightarrow{(f^{4i+2}, g^{4i+2})} \mathbb{Q} \cdot c_1^{2i+1} \oplus 0 \xrightarrow{(\alpha^{4i+2}, \beta^{4i+2})} \mathbb{Q} \cdot c_1^{2i+1} \longrightarrow 0$$

and

$$0 \to 0 \oplus \mathbb{Q} \cdot q_1^i e_3 \xrightarrow{(\alpha^{4i+3}, \beta^{4i+3})} \mathbb{Q} \cdot c_1^{2i} e_3 \to H^{4i+4}(X_1) \xrightarrow{(f^{4i+4}, g^{4i+4})} \mathbb{Q} \cdot c_1^{2i+2} \oplus \mathbb{Q} \cdot q_1^{i+1} \xrightarrow{(\alpha^{4i+4}, \beta^{4i+4})} \mathbb{Q} \cdot c_1^{2i+2} \to 0$$

where

$$(\alpha^{4i+2}, \beta^{4i+2})(c_i^{2i+1}, 0) = c_i^{2i+1}$$
$$(\alpha^{4i+3}, \beta^{4i+3})(0, q_i^i e_3) = c_i^{2i} e_3$$
$$(\alpha^{4i+4}, \beta^{4i+4})(a c_i^{2i+2}, b q_i^{i+1}) = (a-b) c_i^{2i+2}$$

thus we can see

$$H^{4i+2}(X_1) \cong H^{4i+1}(X_0 \times_B Y) \cong \mathbb{Q}$$
$$H^{4i+4}(X_1) \cong \ker\Big((\alpha^{4i+4}, \beta^{4i+4}) : H^{4i+4}(X_0) \oplus H^{4i+4}(Y) \to H^{4i+4}(X_0 \times_B Y)\Big) \cong \mathbb{Q}$$

In particular, we notice that for the composite map

$$f^* : H^*(X_1) \xrightarrow{(f^*,g^*)} H^*(X_0) \oplus H^*(Y) \xrightarrow{pr_1} H^*(X_0)$$

we have $f^{4i}(q_1^i) = q_1^i$ while $f^6(\xi_1) = 0$. Since f^* is a map of $\mathbb{Q}$-algebras, the multiplicative structure on $H^*(Y_1)$ cannot be defined by $\mathbb{Q}[q_1, \xi_1]/(\xi_1^2 - q_1^3)$, which means that the only possibility is

$$H^*(X_1) \cong \mathbb{Q}[q_1, \xi_1]/(\xi_1^2)$$

with $\deg(\xi_1) = 6$.

For $m > 1$, we can get the algebra structure of X_m by the following observation. We have natural morphisms

$$H^*(X_m) \xrightarrow{f_m} H^*(X_{m-1}) \longrightarrow \cdots \longrightarrow H^*(X_1) \xrightarrow{f_1} H^*(X_0)$$

and by comparing degrees we know that in

$$f_m : H^*(X_m) \longrightarrow H^*(X_{m-1})$$

$f_m(\xi_m) = q_1\xi_{m-1}$ or $f_m(\xi_m) = 0$. Inductively we see the map

$$f_m \circ \cdots \circ f_2 : H^*(X_m) \longrightarrow H^*(X_1)$$

is determined by where ξ_m maps to, and it's either 0 or $q_1^{m-1}\xi_1$. In either case, we know the composite

$$f_m \circ \cdots \circ f_1 : H^*(X_m) \longrightarrow H^*(BT)$$

maps ξ_m to 0, so the algebra structure on $H^*(X_m)$ can only be

$$H^*(X_m) = \mathbb{Q}[q_1, \xi_m]/(\xi_m^2).$$

10.2.2 Higher Rank Cases

When $n > 1$ and $m = 1$, by similar analysis of the Mayer-Vietoris sequence, we can show that the composite map

$$f^* : H^*(X_1) \xrightarrow{(f^*, g^*)} H^*(X_0) \oplus H^*(Y) \xrightarrow{pr_1} H^*(X_0)$$

satisfies $f^*(\xi_1) = 0$. Thus the $\mathbb{Q}$-algebra structure on $H^*(X_1) = \mathbb{Q}[q_1, \ldots, q_n] \oplus \mathbb{Q}[q_1, \ldots, q_n]\xi_1$ can only be

$$H^*(X_1) \cong \mathbb{Q}[q_1, \ldots, q_n, \xi_1]/(\xi_1^2).$$

To see this, let's consider how the Mayer-Vietoris sequence splits in the higher rank case.

$$0 \to H^{4i+1}(X_0 \times_B Y) \to H^{4i+2}(X_1) \xrightarrow{(f^{4i+2}, g^{4i+2})} H^{4i+2}(X_0) \oplus 0 \to H^{4i+2}(X_0 \times_B Y) \to 0$$

$$0 \to 0 \oplus H^{4i+3}(Y) \longrightarrow H^{4i+3}(X_0 \times_B Y) \longrightarrow H^{4i+4}(X_1)$$

$$\longrightarrow H^{4i+4}(X_0) \oplus H^{4i+4}(Y) \to H^{4i+4}(X_0 \times_B Y) \to 0$$

Note we have

$$H^*(X_0) = \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n]$$
$$H^*(X_0 \times_B Y) = \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n, e_{2n+1}]$$

thus

$$H^{even}(X_0) \cong H^{even}(X_0 \times_B Y)$$

and therefore

$$H^{4i+1}(X_0 \times_B Y) \cong H^{4i+2}(X_1).$$

In particular, we know that $\deg(\xi_1) = 4n + 2$ and therefore the map

$$(f^{4n+2}, g^{4n+2}) : H^{4n+2}(X_1) \to H^{4i+2}(X_0) \oplus 0$$
$$\xi_1 \longmapsto 0$$

is the trivial map, therefore the $\mathbb{Q}$-algebra structure on $H^*(X_1)$ can only be

$$H^*(X_1) \cong \mathbb{Q}[q_1, \ldots, q_n, \xi_1]/(\xi_1^2).$$

For $m > 1$, the Mayer-Vietoris sequence splits as follows

$$0 \to H^{4i+1}(X_{m-1} \times_B Y) \to H^{4i+2}(X_m) \xrightarrow{(f_m^{4i+2}, g_m^{4i+2})} H^{4i+2}(X_{m-1}) \oplus 0 \to H^{4i+2}(X_{m-1} \times_B Y) \to 0$$

$$0 \to 0 \oplus H^{4i+3}(Y) \longrightarrow H^{4i+3}(X_{m-1} \times_B Y) \longrightarrow H^{4i+4}(X_m)$$
$$\longrightarrow H^{4i+4}(X_{m-1}) \oplus H^{4i+4}(Y) \to H^{4i+4}(X_{m-1} \times_B Y) \to 0$$

Suppose by induction we have

$$H^*(X_{m-1}) = \mathbb{Q}[q_1, \ldots, q_{n-1}, \xi_{m-1}]/(\xi_{m-1}^2)$$
$$H^*(X_{m-1} \times_B Y) = \mathbb{Q}[q_1, \ldots, q_{n-1}, \xi_{m-1}, e_{2n+1}]/(\xi_{m-1}^2)$$

then

$$H^{even}(X_{m-1}) \cong H^{even}(X_{m-1} \times_B Y).$$

If $\deg(\xi_m) = 2mn + 2n + 2m \equiv 2 \mod 4$, then we may write $2mn + 2n + 2m = 4i + 2$ and thus

$$H^{4i+2}(X_m) = \mathbb{Q} \cdot \xi_m$$

and the first split exact sequence reads as

$$0 \to H^{4i+1}(X_{m-1} \times_B Y) \xrightarrow{\cong} H^{4i+2}(X_m) \xrightarrow{0} H^{4i+2}(X_{m-1}) \oplus 0 \xrightarrow{\cong} H^{4i+2}(X_{m-1} \times_B Y) \to 0$$

and we see $f_m^{4i+2} = 0$, i.e. $f_m(\xi_m) = 0$.

If $\deg(\xi_m) = 2mn + 2n + 2m \equiv 0 \mod 4$, then we may write $2mn + 2n + 2m = 4i$ and thus

$$H^{4i}(X_m) = H^{4i}(Y) = H^{4i}(X_{m-1} \times_B Y) = \mathbb{Q} \cdot \xi_m \oplus \mathbb{Q}[q_1, \ldots, q_n]_{(4i)}.$$

Therefore the last three terms in the second exact sequence

$$H^{4i}(X_m) \xrightarrow{(f_m^{4i}, g_m^{4i})} H^{4i}(X_{m-1}) \oplus H^{4i}(Y) \xrightarrow{(\alpha^{4i}, \beta^{4i})} H^{4i}(X_{m-1} \times_B Y) \longrightarrow 0$$

reads as

$$\mathbb{Q}\xi_m \oplus \mathbb{Q}[q_1, \ldots, q_n]_{(4i)} \to \mathbb{Q}[q_1, \ldots, q_n]_{(4i)} \oplus \mathbb{Q}[q_1, \ldots, q_n]_{(4i)} \to \mathbb{Q}[q_1, \ldots, q_n]_{(4i)}$$

Note

$$(f_m^{4i}, g_m^{4i}) : H^{4i}(X_m) \longrightarrow H^{4i}(X_{m-1}) \oplus H^{4i}(Y)$$

when restricted to $\mathbb{Q}[q_1, \ldots, q_n]_{(4i)}$ reads as

$$\begin{aligned} (f_m^{4i}, g_m^{4i}) : \mathbb{Q}[q_1, \ldots, q_n]_{(4i)} &\to \mathbb{Q}[q_1, \ldots, q_n]_{(4i)} \oplus \mathbb{Q}[q_1, \ldots, q_{n-1}, c_n]_{(4i)} \\ h &\longmapsto (h, -h) \end{aligned}$$

thus $\ker(\alpha^{4i}, \beta^{4i}) = \mathbb{Q}[q_1, \ldots, q_n]_{(4i)}$ and $f_m^{4i}(\xi_m) = 0$.

We've shown that the morphisms

$$f_m : H^*(X_m) \longrightarrow H^*(X_{m-1})$$

maps ξ_m to 0 and inductively we see the map

$$f_m \circ \cdots \circ f_1 : H^*(X_m) \longrightarrow H^*(BT)$$

maps ξ_m to 0, so the algebra structure on $H^*(X_m)$ can only be

$$H^*(X_m) = \mathbb{Q}[q_1, \ldots, q_n, \xi_m]/(\xi_m^2).$$

10.3 Another example

There a different example that would make our construction work. Consider the following two fibrations:

$$\mathbb{S}^{2n+1} \to (\mathrm{SO}(2n+2)/\mathrm{SO}(2n+1))_{h\,\mathrm{SO}(n+1)} \to B\,\mathrm{SO}(2n+1)$$

$$\mathbb{S}^{4n-1} \longrightarrow B\,\mathrm{SO}(2n-1) \longrightarrow B\,\mathrm{SO}(2n+1)$$

For simplicity we write $X = X_0 = (\mathrm{SO}(2n+2)/\,\mathrm{SO}(2n+1))_{h\,\mathrm{SO}(n+1)}$, $Y = B\,\mathrm{SO}(2n-1)$ and $B = B\,\mathrm{SO}(2n+1)$. Consider the relative join construction

$$\mathbb{S}^{4nm+2n+1} = \mathbb{S}^{2n+1} *^m \mathbb{S}^{4n-1} \longrightarrow X_m = X *_B^m Y \longrightarrow B$$

If we write $Z_m = X_m \times_B Y$, then

$$X_{m+1} = \mathrm{hocolim}\{X_m \leftarrow Z_m \rightarrow Y\}.$$

We have

$$H^*(B\,\mathrm{SO}(2n-1)) = \mathbb{Q}[q_1, \ldots, q_{n-1}]$$
$$H^*(B\,\mathrm{SO}(2n+1)) = \mathbb{Q}[q_1, \ldots, q_n]$$
$$H^*((\mathrm{SO}(2n+2)/\,\mathrm{SO}(2n+1))_{h\,\mathrm{SO}(n+1)}) = \mathbb{Q}[q_1, \ldots, q_n, e_{2n+1}]$$

with $\deg(q_i) = 4i$ and $\deg(e_{2n+1}) = 2n+1$.

Proposition 10.3.1. *The rational cohomology of X_m is given by*

$$H^*(X_m) \cong \mathbb{Q}[q_1, \ldots, q_n] + \mathbb{Q}[q_1, \ldots, q_n] \cdot q_n^m e_{2n+1}.$$

Proof. When $m = 0$, we have

$$H^*(Z_0) = H^*(X \times_B Y) \cong \mathrm{Tor}^*_{H^*(B)}(H^*(X), H^*(Y)) \cong \mathbb{Q}[q_1, \ldots, q_{n-1}, e_{2n+1}]$$

where the first isomorphism is given by the Eilenberg-Moore spectral sequence, and the second isomorphism is because $H^*(X)$ is free over $H^*(B)$. Therefore in the corresponding Mayer-Vietoris sequence, we have a surjective map

$$H^i(X_0) \oplus H^i(Y) \longrightarrow H^i(Z_0)$$

and thus the long exact sequence splits into short exact sequences

$$0 \longrightarrow H^i(X_1) \xrightarrow{(f,g)} H^i(X_0) \oplus H^i(Y) \xrightarrow{(\alpha,\beta)} H^i(Z_0) \longrightarrow 0$$

We can show that the composite

$$f : H^i(X_1) \xrightarrow{(f,g)} H^i(X_0) \oplus H^i(Y) \xrightarrow{pr_1} H^i(X_0)$$

is injective. Note

$$H^*(Y_1) = \ker(f, g) = \{(h, l) \in \mathbb{Q}[q_1, \dots, q_n, e_{2n+1}] \oplus \mathbb{Q}[q_1, \dots, q_{n-1}] \mid \\ h \equiv l \mod (q_n) \text{ in } \mathbb{Q}[q_1, \dots, q_n, e_{2n+1}]\}$$

If we write $h = h_1(q_1, \dots, q_n) + h_2(q_1, \dots, q_n)e_{2n+1}$ and $l = l(q_1, \dots, q_{n-1})$, then the above condition is equivalent to that

$$h_2 = k(q_1, \dots, q_n) \cdot q_n.$$

Consider if $h = pr_2(h, l) = pr_2(h', l') = h'$, then

$$l = h(q_1, \dots, q_{n-1}, 0) = h'(q_1, \dots, q_{n-1}, 0) = l'$$

thus the map $f : H^*(X_1) \hookrightarrow H^*(X_0)$ is injective and one can check that

$$H^*(X_1) \cong \mathbb{Q}[q_1, \dots, q_n] + \mathbb{Q}[q_1, \dots, q_n] \cdot q_n e_{2n+1}.$$

And inductively,

$$H^*(X_m) \cong \mathbb{Q}[q_1, \dots, q_n] + \mathbb{Q}[q_1, \dots, q_n] \cdot q_n^m e_{2n+1}.$$

□

CHAPTER 11

CONJUGATION ACTION

In the rank one case, the G-action on the second fiber G is the left G-action. A natural question is what happens if we replace the translation action by the conjugation G-action. In this chapter, we answer this question by constructing the corresponding Ganea tower (of fibrations) and computing its rational cohomology. Additionally, we consider a mixture of the translation and conjugation actions, and compute the rational cohomology.

11.1 Conjugation Action

Consider the following two fibrations:

$$\mathbb{S}^2 \cong \mathrm{SU}(2)/T \longrightarrow BT \longrightarrow B\,\mathrm{SU}(2)$$

$$\mathbb{S}^3 \cong \mathrm{SU}(2) \to \mathrm{SU}(2)^{ad}_{h\,\mathrm{SU}(2)} \to B\,\mathrm{SU}(2)$$

Remark. Note if we replace the second total space with the homotopy quotient of SU(2) by left translation on itself, the resulting action is free. Therefore, the homotopy quotient is homotopy equivalent to a point, or equivalently, $E\,\mathrm{SU}(2)$. By making this substitution, we recover the original rank one construction.

For the second fibration, the total space $Y = \mathrm{SU}(2)^{ad}_{h\,\mathrm{SU}(2)}$ is the homotopy quotient space of the adjoint action, which homotopy equivalent to the free loop space $\mathcal{L}B\,\mathrm{SU}(2)$ by the following lemma, (see, e.g., [G5], Appendix A, or [KSS] Lemma 9.1).

Lemma 11.1.1. *Let G be any topological group of CW type. Then there is a fiberwise homotopy equivalence*

$$\mathcal{L}BG \simeq EG \times_G G^{ad}$$

of fiberwise G-spaces over BG.

Theorem 11.1.2 ([S4] Main Theorem). *Let M be a 1-connected compact Riemannian symmetric space or a Kähler manifold, then there is an isomorphism*

$$H^*(\mathcal{L}M;\mathbb{R}) \cong \mathrm{Tor}_{**}^{H^*(M)\otimes H^*(M)}(H^*(M), H^*(M))$$

of algebras over $\mathbb{R}$.

It follows from Theorem 11.1.2[1] and [S4] Corollary 3.6, we have

$$H^*(\mathrm{SU}(2)^{ad}_{h\,\mathrm{SU}(2)}) = H^*(\mathcal{L}B\,\mathrm{SU}(2)) = H^*(\mathrm{SU}(2)) \otimes H^*(B\,\mathrm{SU}(2)) = \mathbb{Q}[e_3, q_1]$$

where $\deg(e_3) = 3, \deg(q_1) = 4$.

Thus, in this case, the cohomology coincides with the one obtained for $B\,\mathrm{SO}(3)$ in the previous chapter, giving us the same result.

Proposition 11.1.3. *In the following fibration*

$$F_m = \mathbb{S}^{4m+2} = \mathbb{S}^2 *^m \mathbb{S}^3 \longrightarrow X_m = BT *^m_{B\,\mathrm{SU}(2)} \mathrm{SU}(2)^{ad}_{h\,\mathrm{SU}(2)} \longrightarrow B\,\mathrm{SU}(2)$$

the rational cohomology of X_m is

$$H^*(X_m) = \mathbb{Q}[q_1, \xi_m]/(\xi_m^2), \;\; \deg(\xi_m) = 4m + 2.$$

11.2 Mixture of Two Actions

Having observed the two distinct cases of G-actions on G resulting in different fibrations for Ganea construction, it is natural to consider combining both actions.

[1]A Lie group G is a symmetric space if and only if the left-invariant metric is bi-invariant. In particular, such a metric exists if G is compact.

Since the order of taking relative joins does not matter, it is sufficient for us to perform m_1 steps of the relative join construction associated with the left G-action fibration, followed by m_2 steps of the relative join construction associated with the conjugation action fibration.

Proposition 11.2.1. *Let* $X_{m_1,m_2} = BT *_{BG}^{m_1} \mathrm{pt} *_{BG}^{m_2} G_{hG}^{ad}$ *for* $m_2 > 0$*, then its rational cohomology is given by*

$$H^*(X_{m_1,m_2}) = \mathbb{Q}[t^2, \xi_{m_1,m_2}]/(\xi_{m_1,m_2}^2)$$

where $\deg(\xi_{m_1,m_2}) = 4m_1 + 4m_2 + 2$.

After the first m_1 steps, we have the following two fibrations

$$\mathbb{S}^{4m_1+2} \to X_{m_1} \to BG$$

$$\mathbb{S}^3 \longrightarrow G_{hG}^{ad} \to BG$$

In this case, we have

$$H^*(X_{m_1} \times_{BG} G_{hG}^{ad}) = \mathbb{Q}[t^2, t^{2m_1+1}] \otimes_{\mathbb{Q}[t^2]} \mathbb{Q}[t^2, e_3]$$

and the following long exact sequence

$$\cdots \to H^k(X_{m_1,1}) \to H^k(X_{m_1}) \oplus H^k(G_{hG}^{ad}) \to H^k(X_{m_1} \times_{BG} G_{hG}^{ad}) \to H^{k+1}(X_{m_1,1}) \to \cdots$$

which is very similar to the rank 1 case (10.4) and splits in a similar way except that for $i < m_1$, in the exact sequence

$$0 \to H^{4i+1}(X_{m_1} \times_B Y) \to H^{4i+2}(X_{m_1,1}) \to H^{4i+2}(X_{m_1}) \oplus 0 \to H^{4i+2}(X_{m_1} \times_B Y) \to 0$$

we have

$$H^{4i+1}(X_{m_1} \times_B Y) = H^{4i+2}(X_{m_1}) = H^{4i+2}(X_{m_1} \times_B Y) = 0,$$

and for $i = m_1$ we have $H^{4i+1}(X_{m_1} \times_B Y) = 0$, thus $H^{4m_1+2}(X_{m_1,1}) = 0$.

When $i = m_1 + 1$, we have

$$H^{4m_1+6}(X_{m_1,1}) \cong H^{4m_1+5}(X_{m_1} \times_B G^{ad}_{hG}).$$

As an $\mathbb{Q}[t^2]$-module, $H^*(X_{m_1,1}) = \mathbb{Q}[t^2] \oplus \mathbb{Q}[t^2]\xi_{m_1,1}$ where $\deg(\xi_{m_1,1}) = 4m_1 + 6$.

Using a similar argument as in the case of the conjugation action, we can deduce

$$H^*(X_{m_1,1}) = \mathbb{Q}[t^2, \xi_{m_1,1}]/(\xi^2_{m_1,1}).$$

Now, considering a pair (m_1, m_2) with $m_2 > 1$, we can prove the desired result by induction on m_2. When $m_2 = 1$, the cohomology already coincides with the conjugation case of multiplicity $m_1 + 1$, thus the same induction shows that for the following fibration

$$\mathbb{S}^{4m_1+4m_2+2} \longrightarrow X_{m_1,m_2} = BT *^{m_1}_{BG} \mathrm{pt} *^{m_2}_{BG} G^{ad}_{hG} \longrightarrow BG \tag{11.1}$$

we have

$$H^*(X_{m_1,m_2}) = \mathbb{Q}[t^2, \xi_{m_1,m_2}]/(\xi^2_{m_1,m_2})$$

where $\deg(\xi_{m_1,m_2}) = 4m_1 + 4m_2 + 2$.

APPENDIX A

FINITE REFLECTION GROUPS

In this chapter, we briefly review properties and classifications of finite reflection groups and invariants of finite reflection groups. All the results are classical, thus we omit the proof here. The main reference for this chapter is [H4].

A.1 Root Systems

Let W be a finite reflection group.

Proposition A.1.1 ([H4], Proposition 1.2). *Let $t \in O(V)$ be an orthogonal transformation and α some nonzero vector in V, then $ts_\alpha t^{-1} = s_{t\alpha}$. In particular, for $w \in W$, $s_{w\alpha} \in W$ if $s_\alpha \in W$.*

This proposition implies that W acts on its set of reflecting hyperplanes by permutation $w \cdot H_\alpha = H_{w\alpha}$.

Note although the hyperplanes H_α are determined by W, the vectors α are not determined by W. To emphasize on such geometric configuration, the notion of root systems is introduced.

Definition A.1.2. Let Φ be a finite set of nonzero vectors in V satisfying:

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Define W to be the group generated by all reflections $\{s_\alpha\}_{\alpha\in\Phi}$. We call the pair (Φ, W) a **root system**. The vectors α are called the **roots**.

Positive and Simple Systems

Definition A.1.3. A **total ordering** of a real vector space V is a transitive relation on V such that

1. For any $u, v \in V$, exactly one of $u < v, u = v, v < u$ holds.
2. For any $u, v, w \in V$, if $u < v$ then $u + w < v + w$.
3. If $u < v$ and $c \in \mathbb{R}\backslash\{0\}$, then $cu < cv$ if $c > 0$ and $cv < cu$ if $c < 0$.

Given such an ordering , we say $v \in V$ is **positive** if $0 < v$, and **negative** if $v < 0$.

It follows from this definition that the sum of positive vectors is positive and the scalar multiple of a positive vector by a positive real number is positive.

Definition A.1.4. A subset Φ_+ of a root system Φ is a positive system if it consists of all the roots which are positive relative to a total ordering of V. An element $\alpha \in \Phi_+$ is called a **positive root**.

It can be seen that positive system exists, and since roots come in pairs $\{\alpha, -\alpha\}$, we can write Φ as disjoint union $\Phi = \Phi_+ \coprod \Phi_-$ where Φ_- is called a **negative system**, and it consists of roots which are negative relative to the same total ordering.

Definition A.1.5. A subset $\Delta \subset \Phi$ is called a **simple system** if Δ forms a basis for the $\mathbb{R}$-span of Φ in V and each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign, i.e. either all non-negative or non-positive. Elements in Δ are called **simple roots**.

The next theorem shows that simple system exists.

Theorem A.1.6 ([H4] Theorem 1.3)**.** *1. If Δ is a simple system in Φ, then there is a unique positive system containing Δ.*

2. Every positive system Φ_+ in Φ contains a unique simple system Δ. In particular, simple system exists.

Given any simple system Δ and corresponding positive system Φ_+, $w\Delta$ is another simple system with corresponding positive system $w\Phi_+$ for $w \in W$. It turns any two systems differ by an action of W.

Theorem A.1.7 ([H4], Theorem 1.4)**.** *Any two positive (resp. simple) systems in Φ are conjugate under W.*

The cardinality of any simple system is an invariant of Φ as it is the dimension of the subspace spanned by Φ in V, and it is called the **rank** of W.

A simple system Δ not only generates Φ, but also determines W.

Theorem A.1.8 ([H4], Theorem 1.5)**.** *For a fixed simple system Δ, W is generated by the reflections $s_\alpha, \alpha \in \Delta$.*

Now we've seen that W can be generated by relatively small number of reflections, we are interested in finding presentations of W as an abstract group using these generator with suitable relations.

Theorem A.1.9 ([H4], Theorem 1.9)**.** *W is generated by the simple reflections $\{s_\alpha, \alpha \in \Delta\}$ subject to the relations*

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \alpha, \beta \in \Delta.$$

A.2 Coxeter Group

The above abstract presentation leads us to following definition of abstract groups.

Definition A.2.1 ([C3]). A **finite Coxeter group** W is a finite abstract group defined with the presentation

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ and $m_{ij} \geq 2, i \neq j$. When $m_{ij} = \infty$, there is no relation of the form $(s_i s_j)^m = 1$ imposed. Let $S = \{s_1, \ldots, s_n\}$ be the generating set, then the pair (W, S) is called a **Coxeter system**.

In the finite case, there is an one-to-one correspondence between finite reflection groups and finite Coxeter groups: every finite Coxter group admits a faithful presentation as a finite reflection group of some Euclidean space and vice versa. As our focus is finite reflection groups, we will not distinguish between finite reflection groups and finite Coxeter groups.

A.3 Classifications

A.3.1 Coxeter Graph

The presentation of W in Theorem A.1.9 shows that as an abstract group, W is determined up to isomorphism by the set of integers $m(\alpha, \beta), \alpha, \beta \in \Delta$. One way to encode this information is to construct in a graph as follows.

Definition A.3.1. The **Coxeter graph** of W consists of

- vertex set in one-to-one correspondence to Δ, and
- labeled edges between (α, β) whenever $m(\alpha, \beta) \geq 3$ and with label $m(\alpha, \beta)$.

Note for any distinguished pair (α, β) not joined by an edge, we have $m(\alpha, \beta) = 2$.

Remark. More generally, we can define Coxeter graph for infinite Coxeter groups where we allow label ∞ for an edge (i, j) if no relation between s_i and s_j is imposed.

Irreducible Components

Definition A.3.2. A Coxeter system (W, S) is **irreducible** if the Coxeter graph Γ is connected.

Definition A.3.3. For any subset $I \subset S$, define W_I be the subgroup of W generated by all $s_i \in I$. All subgroups of W obtained this way are called **parabolic subgroups**.

Proposition A.3.4 ([H4] Proposition 2.2)**.** *Let (W, S) be a Coxeter system with Coxeter graph Γ. Let $\Gamma_1, \cdots, \Gamma_r$ be irreducible components of Γ, and let $S_1, \ldots, S_r$ be the corresponding subsets of S. Then W is the direct product of the parabolic subgroups $W_i = W_{S_i}$ and each (W_i, S_i) is irreducible.*

It follows from this Proposition that it suffices for us to classify all irreducible Coxeter systems.

A.3.2 Classification

To list all potential connected Coxeter graphs, we need to consider the following bilinear form associated to each Coxeter graph.

Bilinear Form Associated to Coxeter Graph

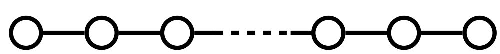
We associate to a Coxeter graph Γ with vertex set S of cardinality n a symmetric $n \times n$ matrix A by assigning

$$a(i, j) := -\cos \frac{\pi}{m(i, j)}.$$

When Γ comes from a finite reflection group, the matrix A is always positive definite as it represents the standard Euclidean inner product relative to the basis Δ of V if we choose unit simple roots. By abuse of language we will say that the Coxeter graph Γ is **positive definite** if the associated matrix is positive definite.

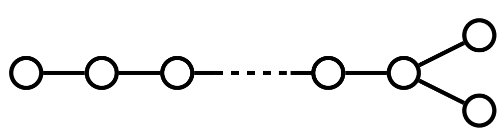
Theorem A.3.5 ([H4] Theorem 2.7)**.** *The only connected positive definite Coxeter graphs are given as follows.*

- $A_n, n \geq 1$. *The Coxeter graph is given by*

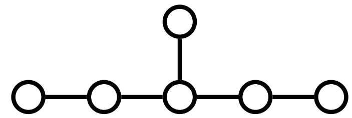


- $B_n, n \geq 2$. *The Coxeter graph is given by*

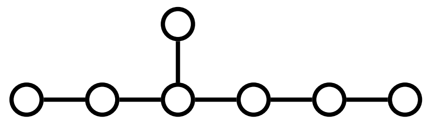
- $D_n, n \geq 4$. *The Coxeter graph is given by*



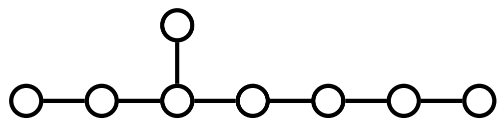
- E_6. *The Coxeter graph is given by*



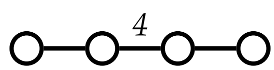
- E_7. *The Coxeter graph is given by*



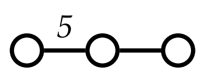
- E_8. *The Coxeter graph is given by*



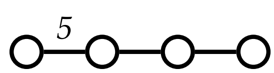
- F_4. *The Coxeter graph is given by*



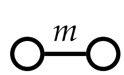
- H_3. *The Coxeter graph is given by*



- H_4. *The Coxeter graph is given by*



- $I_2(m)$. *The Coxeter graph is given by*



It turns out that for all the possible graphs above, there exists finite reflection groups with given Coxeter graph. In particular, we'd like to consider one special type of finite reflection groups associated to Lie groups.

A.4 Weyl Groups

A.4.1 Crystallographic Groups

Definition A.4.1. A **lattice** L in V is the $\mathbb{Z}$-span of a basis in V. A subgroup G of $\mathrm{GL}(V)$ is called **crystallographic** if it stabilizes a lattice L in V.

It turns out that many finite reflection are crystallographic.

Proposition A.4.2 ([H4], Proposition 2.8)**.** *If W is crystallographic, then each integer $m(\alpha, \beta)$ must be* $2, 3, 4$ *or* 6 *for $\alpha \neq \beta$ in Δ.*

A.4.2 Crystallographic Root System

Definition A.4.3. A root system (Φ, W) is called **crystallographic** if

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} \in \mathbb{Z}, \forall \alpha, \beta \in \Phi.$$

In this case the group W is called the **Weyl group** of Φ.

The integer condition ensures that $s_\alpha \beta$ is obtained from β by adding integer multiple of α, which further implies that all roots are $\mathbb{Z}$-linear combinations of Δ and the $\mathbb{Z}$-span of Δ in V is a W-stable lattice, so W is crystallographic.

The construction of crystallographic root systems are given as follows.

- $(A_n, n \geq 1)$: $V = \{v \in \mathbb{R}^{n+1} | \sum v_i = 0\}$
  $\Phi = \{e_i - e_j, 1 \leq i \neq j \leq n + 1\}$ where e_i are the standard basis vector in $\mathbb{R}^{n+1}$.

$\Delta = \{\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n\}$

$W = S_{n+1}$ acts on V by permuting the standard basis vector.

- $(B_n, n \geq 2)$: $V = \mathbb{R}^n$

  $\Phi = \{e_i, 1 \leq i \leq n\} \coprod \{\pm e_i \pm e_j, 1 \leq i < j \leq n\}$ with $2n$ short roots and $2n(n-1)$ long roots.

  $\Delta = \{\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1\} \coprod \{\alpha_n = e_n\}$

  $W = S_n \int \mathbb{Z}_2 = S_n \ltimes (\mathbb{Z}_2)^n$ where S_n acts by permuting indices and $(\mathbb{Z}_2)^n$ acts by sign changes.

- $(C_n, n \geq 2)$: $V = \mathbb{R}^n$

  $\Phi = \{2e_i, 1 \leq i \leq n\} \coprod \{\pm e_i \pm e_j, 1 \leq i < j \leq n\}$ with $2n$ long roots and $2n(n-1)$ short roots.

  $\Delta = \{\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1\} \coprod \{\alpha_n = 2e_n\}$

  $W = S_n \int \mathbb{Z}_2 = S_n \ltimes (\mathbb{Z}_2)^n$ where S_n acts by permuting indices and $(\mathbb{Z}_2)^n$ acts by sign changes.

- $(D_n, n \geq 4)$: $V = \mathbb{R}^n$

  $\Phi = \{\pm e_i \pm e_j, 1 \leq i < j \leq n\}$

  $\Delta = \{\alpha_i = e_i - e_{i+1}, 1 \leq i \leq n-1\} \coprod \{\alpha_n = e_{n-1} + e_n\}$

  $W = S_n \ltimes (\mathbb{Z}_2)^{n-1}$ where S_n acts by permuting indices and $(\mathbb{Z}_2)^{n-1}$ acts by an even number of sign changes.

- (G_2): $V = \{V = \{v \in \mathbb{R}^3 | \sum v_i = 0\}$

  $\Phi = \{\pm(e_i - e_j), 1 \leq i < j \leq n\} \coprod \{\pm(2e_i - e_j - e_k), i \neq j \neq k, i \neq k\}$ with 6 short roots and 6 long roots.

  $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = -2e_1 + e_2 + e_3\}$

  $W = D_6$ is the dihedral group of order 12.

- F_4: $V = \mathbb{R}^4$

  $\Phi = \{\pm e_i \pm e_j (i < j), \pm e_i, \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4\}$ with 24 long roots and 24 short roots.

$\Delta = \{\alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4\}$

W is the group of of symmetries of a regular solid in $\mathbb{R}^4$ having 24 faces which are octahedra.

- E_8: $V = \mathbb{R}^8$

  $\Phi = \{\pm e_i \pm e_j (i < j), \frac{1}{2}\sum_{i=1}^{8} \pm e_i$(even number of positive sign)$\}$ with 240 roots.

  $\Delta = \{\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8, \alpha_2 = e_1 + e_2, \alpha_i = e_{i-1} - e_{i-2} (3 \leq i \leq 8)\}$

- E_7: $V = \text{span}\{\alpha_i, 1 \leq i \leq 7\}$ where α_i are simple roots in type E_8.

  $\Phi = \{\pm e_i \pm e_j (1 \leq i < j \leq 6), \pm(e_7 - e_8), \pm\frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} \pm e_i\}$ (where the number of minus signs in the sum is odd) with 126 roots.

  $\Delta = \{\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8, \alpha_2 = e_1 + e_2, \alpha_i = e_{i-1} - e_{i-2} (3 \leq i \leq 7)\}$

- E_6: $V = \text{span}\{\alpha_i, 1 \leq i \leq 6\}$ where α_i are simple roots in type E_8.

  $\Phi = \{\pm e_i \pm e_j (1 \leq i < j \leq 5), \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} \pm e_i\}$ (where the number of minus signs in the sum is odd) with 126 roots.

  $\Delta = \{\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8, \alpha_2 = e_1 + e_2, \alpha_i = e_{i-1} - e_{i-2} (3 \leq i \leq 6)\}$

Remark. B_n and C_n are dual to each other in the following sense: given a (crystallographic) root system (Φ, W) with simple system Δ, define coroot $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$, then $\Phi^\vee = \{\alpha^\vee, \alpha \in \Phi\}$ is also a (crystallographic) root system with simple system $\Delta^\vee = \{\alpha^\vee, \alpha \in \Delta\}$.

A.5 Polynomial Invariants of Finite Reflection Groups

A.5.1 Degrees

The algebraically independent generators of $k[x]^W$ are not uniquely determined, although the degrees are independent of the choice of generators.

Proposition A.5.1 ([H4] Proposition 3.7)**.** *Let $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ be two sets of homogeneous, algebraically independent generators of the algebra of W-invariants. Let $d_i = \deg(f_i), e_i = \deg(g_i)$, then $d_i = e_i (1 \le i \le n)$ after renumbering if necessary.*

The numbers $d_1 \le \cdots \le d_n$ are called the **degrees** of W. The degrees of W for each type are in Table A.1.

Type	Degrees
A_n	$2, 3, 4, \ldots, n+1$
B_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2, n$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$
F_4	$2, 6, 8, 12$
G_2	$2, 6$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
$I_2(m)$	$2, m$

Table A.1: Degrees of W

A.5.2 Examples

Example A.5.2 (Symmetric Group)**.** $W = S_{n+1}$ acts by permuting indices of x_i subject to the relation $x_1 + \cdots + x_{n+1} = 0$. There are two choices of set of basic invariants, the elementary symmetric polynomials or the following

$$f_i = x_1^{i+1} + \cdots + x_{n+1}^{i+1} \ (1 \leq i \leq n).$$

Example A.5.3 (Signed Symmetric Group)**.** W acts by permuting indices and sign changes, two choices of set of basic invariants are signed symmetric polynomials $\sigma_i(x_1^2, \ldots, x_n^2)$ or the following

$$f_i = x_1^{2i} + \cdots + x_n^{2i} \ (1 \leq i \leq n).$$

Example A.5.4 (Even Signed Symmetric Group)**.** W acts by permuting indices and even sign changes. The choices of f_i are the same as type B_n for $1 \leq i \leq n-1$, and $f_n = x_1 \cdots x_n$.

APPENDIX B

PRINCIPAL BUNDLES AND CLASSIFYING SPACES

The classical results we review here work in the general setting of numerable principal bundles over any topological spaces, although we are interested in the case of locally trivial principal bundles over CW-complexes. The main references of this chapter are [M4], [H+], [M1], [M2] and [H1].

B.1 Principal Bundles

Definition B.1.1. Let B be a topological space, and X a right G-space equipped with a G-map $p : X \to B$ where G acts trivially on B. (X, p) is called a (localy trivial) **principal G-bundle** if there exists an open covering $\{U_i\}$ of B such that there are G-equivariant homeomorphisms $\phi_i : p^{-1}(U_i) \to U_i \times G$ such that the following diagram commutes

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\phi_U} & U \times G \\ & \searrow^{p|_U} & \downarrow \mathrm{pr}_1 \\ & & U \end{array}$$

where G acts on $U \times G$ by $(u, g) \cdot h = (u, gh)$. The maps $\{\phi_i\}$ is called a **local trivialization** of $p : E \to B$.

A **morphism** of principal G-bundles over B is a G-equivariant map $f : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow^{p_X} & \downarrow p_Y \\ & & B \end{array}$$

commutes as G-equivariant maps. The collection of all principal G-bundles over B form a category.

A principal G-bundle $p : X \to B$ is called **trivial** if there exists a homeomorphism

$f : X \to B \times G$ which gives an isomorphism of principal G-bundles, where the G-action on $B \times G$ is $(b, g) \cdot h = (b, gh)$.

In particular, G acts on P freely by definition.

Remark. Our definition of principal G-bundle is not the same as in [H+], as we require a stronger condition of being locally trivializable. Note however all the properties that we we present here are still valid for the definition in [H+].

Proposition B.1.2. *Any morphism of principal G-bundles is an isomorphism.*

The pullback of principal G-bundle $E \to B$ along a map $f : X \to B$ is a topological space

$$f^*E := \{(x, e) \in X \times E | f(x) = p(e)\}$$

which is the pullback along (f, p) in the category of topological spaces equipped with $f^*p : F^*E \to X$. It can be checked that (f^*E, f^*p) is also a principal G-fibration.

Proposition B.1.3. *Let H be subgroup of G, and $X \to X/G$ a principal G-bundle. If the quotient map $G \to G/H$ is a principal H-bundle, then $X \to X/H$ is a principal H-bundle.*

B.2 Numerable Principal Bundle

Definition B.2.1. Let X be a topological space with open cover $\{U_i\}$, a **partition of unity** relative to the cover $\{U_i\}$ is a set of functions $f_i : U_i \to [0, 1]$ such that

- $\overline{f_i^{-1}((0, 1])} \subset U_i$ for all j.

- $\{\overline{f_i^{-1}((0,1])}\}$ is locally finite.
- $\sum_i f_i(x) = 1, \forall x \in X$.

Definition B.2.2. A **numerable** open cover (alias normal cover) is an open cover of a topological space that admits a subordinate partition of unity.

Definition B.2.3. A principal G-bundle is **numerable** if there is a numerable open cover of trivialization.

Theorem B.2.4 ([H+] Theorem 9.9)**.** *Let G be a topological group and let $p : X \to B$ be a numerable principal G-bundle over a space B. Suppose $f \simeq g : B' \to B$ are homotopic, then $f^*E \cong g^*E$.*

B.3 Classifying Spaces and Universal Principal Bundles

Definition B.3.1. A numerable principal G-bundle $p : EG \to BG$ over a pointed space BG is **universal** if

1. for any numerable principal G-bundle $P \to X$ there exists a map $f : X \to BG$ such that $f^*EG = P$, and
2. whenever two pointed maps $f, g : X \to BG$ satisfies $f^*E \cong g^*E$, then $f \simeq g$.

Equivalently, the map

$$\begin{aligned} \Phi : [X, BG] &\longrightarrow \mathbf{Bun}_G(X) \\ [f] &\longmapsto f^*(EG) \end{aligned}$$

from the homotopy classes of continuous maps $X \to BG$ to the isomorphism classes of principal G-bundles is a bijection. BG is called the **classifying space** of G.

If we write $P_G(B)$ the set of isomorphism classes of principal G-bundles over B, then we have a functor

$$\begin{aligned} P_G(-) : \mathrm{Ho}(\mathbf{Top}) &\longrightarrow \mathbf{Set} \\ B &\longmapsto P_G(B) \end{aligned}$$

where Ho(**Top**) is the category of spaces with homotopy classes of maps. By definition, universal principal G-bundle corepresents this functor.

B.3.1 Milnor's Model

For any topological group G, consider the infinite join (as a set)

$$EG := G * G * G * \cdots$$

where an element of EG is denoted as

$$(x, t) = (g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \cdots)$$

where $x_i \in G$ and $t_i \in [0, 1]$ such that only finite $t_i \neq 0$ and $\sum_i t_i = 1$. In EG, we identify $(x, t) = (x', t')$ provided $t_i = t'_i$ and $x_i = x'_i$ if $t_i = t'_i > 0$. We define a right G-action on EG by $(x, t)g = (xg, t)$, or explicitly

$$(g_0, t_0, g_1, t_1, \ldots, g_n, t_n, \cdots)g = (g_0 g, t_0, g_1 g, t_1, \ldots, g_n g, t_n, \cdots)).$$

The topology on EG is defines as follows. Consider two families of maps

$$\begin{aligned} t_i : EG &\longrightarrow [0, 1] \\ (x, t) &\longmapsto t_i \\ g_i : t_i^{-1}(0, 1] &\longrightarrow G \\ (x, t) &\longmapsto g_i \end{aligned}$$

for $i \geq 0$. The topology on EG is the strongest topology such that the functions t_i and g_i are continuous.

It can be seen that the right G-action on EG is free, and we write $BG = EG/G$ the orbit space.

Theorem B.3.2 ([M4], [H+] Chapter 12)**.** *The quotient map $p : EG \to BG$ is a numerable principal G-bundle and it is a universal.*

Remark. When G is a compact Lie group, the strong topology on the (infinite) join agrees with the usual topology on the (infinite) join, and since we focus only on classifying space of compact connected Lie groups, we can take the usual infinite join as our model for EG and BG.

Example B.3.3.

1. Let $G = \mathbb{Z}_2 \cong S^0$ with discrete topology, then $EG \cong S^\infty$ and $BG = \mathbb{R}P^\infty$.
2. Let $G = \mathrm{U}(1) \cong S^1$, then $EG = S^\infty$ and $BG = \mathbb{C}P^\infty$.
3. Let $G = \mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong S^3$, then $EG = S^\infty$ and $BG = \mathbb{H}P^\infty$.

Proposition B.3.4 ([D2], Theorem 7.5)**.** *A numerable principal G-bundle $E \to B$ is universal if and only if E is contractible.*

Proposition B.3.5. *Let G, H be topological groups. Then the natural homotopy class*

$$B(G \times H) \longrightarrow BG \times_k BH$$

is a homotopy equivalence, where $\times_k$ means the compactly generated topology.

Remark. In general the compactly generated topology is strictly finer than the product topology, although for any spaces X and Y, $X \times_k Y \to X \times Y$ is a weak equivalence. When both BG and BH has countably many cells, the two topologies agree. For classifying space of compact Lie groups, $B(G \times H) = BG \times BH$.

B.4 Fibrations

Definition B.4.1. A map $p : E \to B$ satisfies the **homotopy lifting property** with respect to a space X if for every commutative diagram (without the dashed arrow)

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\tilde{h}_0} & E \\ \big\downarrow & \overset{\tilde{h}}{\nearrow} & \big\downarrow p \\ X \times I & \xrightarrow{h} & B \end{array}$$

there exists a lifting $\tilde{h} : X \times I \to E$ that make the diagram (with the dashed arrow) commute.

Definition B.4.2. A map $p : E \to B$ is a **fibration** or **Hurewicz fibration** if it satisfies the homotopy lifting property for all spaces X.

Definition B.4.3. A map $p : E \to B$ is a **Serre fibration** if it satisfies the homotopy lifting property for all CW-complexes.

Example B.4.4.

1. Fiber bundles are Serre fibrations.
2. Trivial fiber bundles are fibrations.
3. Covering maps are fibrations.
4. A numerable fiber bundle is a fibration. In particular a fiber bundle with a paracompact and Hausdorff base space is a fibration.

Theorem B.4.5 ([H1] Theorem 4.41)**.** *Let $p : E \to B$ is a (based) Serre fibration with based points $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. If B is path-connected, there is a long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0.$$

Proposition B.4.6. *Fibrations are stable under composition and pullback.*

Proposition B.4.7 ([H1] Proposition 4.61)**.** *Let $p : E \to B$ be a fibration. If b and b' are in the same path component of B, then $p^{-1}(b) \simeq p^{-1}(b')$ are homotopy equivalent.*

Thus up to homotopy the choice of fibers does not matter and it's convenient to work with based fibrations.

Definition B.4.8. Let $p_0 : E_0 \to B$ and $p_1 : E_1 \to B$ be fibrations. a map $f : E_0 \to E_1$ is called **fiber-preserving** if $p_0 = p_1 f$.
A fiber-preserving map $f : E_0 \to E_1$ is a **fiber homotopy equivalence** if there is a fiber-preserving map $g : E_1 \to E_0$ such that both compositions fg and gf are homotopic to the identity through fiber-preserving maps.

Proposition B.4.9 ([M2] Proposition 7.5)**.** *Let $p : D \to B$ and $q : E \to B$ be fibrations and left $f : D \to E$ be a map such that $qf = p$. Suppose that f is a homotopy equivalence, then f is a fiber homotopy equivalence.*

In fact the reverse statement is also true if B has a numerable cover.

Theorem B.4.10 ([M1]Theorem 2.6.)**.** *Assume B has a numerable cover. Let $p : D \to B$ and $q : E \to B$ be fibrations and let $f : D \to E$ be a map such that $qf = p$. Then f is a homotopy equivalence if $f_b : F_b \to F'_b$ is a homotopy equivalence for any $b \in B$.*

Proposition B.4.11 ([H1] Proposition 4.62)**.** *Let $p : E \to B$ be a fibration and let $f_0, f_1 : X \to B$ be homotopic, then the pullback fibrations $\tilde{f}_0 : X \times_B E \to X$ and $\tilde{f}_1 : X \times_B E \to X$ are fiber homotopy equivalent.*

This proposition implies that pullback along fibrations preserve homotopy equivalence.

Pathspace Fibration It turns out any map $f : A \to B$ can be factored as a homotopy equivalence followed by a fibration.

Definition B.4.12. The **pathspace fibration** of f is defined as

$$E^f := \{(x, \gamma) \in A \times B^I | \, \gamma(0) = f(x)\}.$$

We can view A as a subspace of E^f via

$$\begin{aligned} \alpha^f : A &\longrightarrow E^f \\ x &\longmapsto (x, c_{f(x)}) \end{aligned}$$

where $c_{f(x)}$ is the constant path at $f(x) \in B$, and E^f deformation retracts to A, thus α^f is a homotopy equivalence.

There is another map

$$\begin{aligned} \beta^f : E^f &\longrightarrow B \\ (x, \gamma) &\longmapsto \gamma(1). \end{aligned}$$

Thus we have the following factorization of any map.

Proposition B.4.13 ([M2] Chapter 7.3)**.** *$f : A \to B$ can be factored as*

$$A \xrightarrow[\sim]{\alpha^f} E^f \overset{\beta^f}{\twoheadrightarrow} B.$$

where α^f is a homotopy equivalence and β^f is a fibration.

Such a decomposition is not unique up to isomorphism, but unique up to homotopy in the following sense. If there is another another factorization

$$A \xrightarrow[\sim]{\alpha} E \overset{\beta}{\twoheadrightarrow} B.$$

such that α is a homotopy equivalence and β is a fibration. If we write λ the homotopy inverse of α, then $\alpha^f \circ \lambda : E \to E^f$ is a fiber homotopy equivalence. This is a result of Proposition B.4.9.

Using this factorization we can get the following corollary of Theorem B.4.10. A proof can be found here, Corollary 3.1.19, p34.

Corollary B.4.14. *Let B and B' be path connected and admits numerable open covers. Let (α, β, γ)*

$$\begin{array}{ccccc} F & \hookrightarrow & E & \xrightarrow{p} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} \\ F' & \hookrightarrow & E' & \xrightarrow{p'} & B' \end{array}$$

be a map between fibration sequences in which α and β are homotopy equivalences, then γ is a homotopy equivalence.

From this corollary we can get the uniqueness result of universal principal G-bundle and classifying space.

Proposition B.4.15. *Let $p' : E'G \to B'G$ be another numerable principal G-bundle such that $E'G$ is contractible, then p is a universal principal G-bundle.*

Corollary B.4.16. *Classifying spaces of any two universal principal G-bundles are homotopy equivalent.*

Loop Space The long exact sequence associated to a fibration gives the following result.

Proposition B.4.17 ([H1] Proposition 4.66)**.** *If $F \hookrightarrow E \to B$ is a fibration or fiber bundle with E contractible, then there is a weak homotopy equivalence $F \simeq \Omega B$.*

Corollary B.4.18. *For any topological group, ΩBG is weak homotopy equivalent to G, thus for any $n \geq 1$, $\pi_n BG \cong \pi_{n-1} G$.*

In particular, a path-connected topological group has a simply-connected classifying space. In fact we have the following stronger result.

Proposition B.4.19. *For any topological group G, G and ΩBG are homotopy equivalent if BG has a numerable open cover.*

This follows from Theorem B.4.10 and the discussion after Proposition B.4.13 when we consider the following two factorizations of the inclusion $* \to B$

$$* \longrightarrow EG \longrightarrow\!\!\!\!\!\rightarrow BG$$

$$* \longrightarrow \mathcal{P}BG \longrightarrow\!\!\!\!\!\rightarrow BG.$$

Example B.4.20. When G is a compact Lie group, $G \simeq \Omega BG$.

B.5 Cofibrations

Dualizing the definition of fibration gives the following definition of cofibration.

Definition B.5.1. A map $i : A \to B$ is a **cofibration** if for any map $f : A \to X$ that factors though i, i.e. there exists $\tilde{f} : B \to X$ such that $\tilde{f}i = f$, we can extend a homotopy $H : A \times I \to X$ where $H_0 = i$ to a homotopy $\tilde{H} : B \times I \to X$ such that $\tilde{H} \circ (f \times 1) = H$ and $\tilde{H}_0 = \tilde{f}$.

If $i : A \to B$ is a cofibration, B/A is called the **cofiber** of i.

Definition B.5.2. If (A, a_0) is a pointed space such that $\{a_0\} \hookrightarrow A$ is a cofibration, then a_0 is called a **nondegenerate basepoint**.

Any pointed space X can be replaced by one with a nondegenerate basepoint. Explicitly, let $X' = X \vee I$ with the basepoint in X' chosen to be the end of I which is not attached to X. The space X' is homotopy equivalent to X and has a nondegenerate basepoint.

Proposition B.5.3 ([S1] Proposition 7.1.9)**.** *Let $A \hookrightarrow X$ be a cofibration where A is contractible and closed in X, then the quotient map $X \to X/A$ is a homotopy equivalence.*

B.6 Spectral Sequences Associated to Fibrations

B.6.1 Leray-Hirsch Theorem

Let R be a unital commutative ring.

Theorem B.6.1 ([MT] Chapter III Theorem 4.2)**.** *Let B and F be 0-connected. Suppose that the Serre spectral sequence of a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ with trivial local coefficient ring $H^*(F;R)$ collapses, then*

1. $p^* : H^*(B;R) \to H^*(E;R)$ *is a monomorphism.*
2. $i^* : H^*(E;R) \to H^*(F;R)$ *is an epimorphism and the ideal* $\mathrm{Im}p^+ \subset \ker(i^*)$ *where* p^+ *is restriction of* p *on positive degrees.*

   Moreover if $H^i(F;R)$ is a finite generated R-free module for every i, then
3. $H^*(E;R)$ *is free as an* $\mathrm{Im}p^*$*-module with a basis* $\{e_\alpha\}$ *such that* $\{i^*e_\alpha\}$ *is a homogeneous basis of* $H^*(F;R)$ *as an R-module.*
4. $\mathrm{Im}p^*$ *is a direct summand of* $H^*(E;R)$ *and* $H^*(B;R) \cong H^*(F;R)/(\mathrm{Im}p^+)$.

Corollary B.6.2 ([MT] Chapter III Corollary 4.3)**.** *If $H^i(B;R)$ and $H^j(F;R)$ are finite generated free R-modules for every i and j in Theorem B.6.1, then so is $H^k(E;R)$ for every k, and the Hilbert polynomials satisfy*

$$p_E(t;R) = p_F(t;R) \cdot p_B(t;R).$$

Theorem B.6.3 ([MT] Chapter III Theorem 4.4)**.** *If $i^* : H^*(E;R) \to H^*(F;R)$ is an epimorphism in a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ with B 0-connected, then the system of local coefficient rings $\mathcal{H}^*(F;R)$ is trivial. In addition, if either $H^*(B;R)$ or $H^*(F;R)$ is a finite generated free R-module in each dimension, then the Serre spectral sequence collapses and Theorem B.6.1 is applicable.*

In particular, the following lemma provides special cases when Serre spectral sequence collapses.

Lemma B.6.4. *The Serre spectral sequence collapses in the following cases:*

1. $H^i(R;R) = H^i(F;R) = 0$ *for i odd.*
2. *R is a field or $\mathbb{Z}$, and $H^*(B;R), H^*(F;R)$ and $H^*(E;R)$ are finite generated free R-modules for each dimension such that*

$$p_E(t;R) = p_F(t;R) \cdot p_B(t;R).$$

B.6.2 Eilenberg-Moore Spectral Sequence

Theorem B.6.5 (Eilenberg-Moore, [S3] Theorem 3.5)**.** *Suppose given a diagram*

$$\begin{array}{ccc} F & = & F \\ \downarrow & & \downarrow \\ X \times_B Y & \longrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & B \end{array}$$

where

1. *$F \to Y \to B$ is a Serre fibration.*
2. *$F \to X \times_B Y \to X$ is pullback fibration along $f : X \to B$.*
3. *B is simply connected.*

Then there exists a spectral sequence of commutative algebras $\{E_r, d_r\}$ with

1. *$E_r \Rightarrow H^*(X \times_B Y)$,*
2. *$E_2 = \mathrm{Tor}_{H^*(B)}(H^*(X), H^*(Y))$.*

APPENDIX C

COMPACT LIE GROUPS

In this section, we will provide a brief summary of useful topological properties of compact Lie groups. The main references are [H+], [DW2] and [H3] Chapter III.

C.1 Definitions

Definition C.1.1. A **compact Lie group** is a compact differential manifold equipped with a group structure such that the multiplication and inverse maps are smooth.

Example C.1.2. Let $\mathrm{U}(n)$ be the set of matrices in $M_{n\times n}(\mathbb{C})$ which preserve the standard inner product of $\mathbb{C}^n$. Equivalently $\mathrm{U}(n) = \{M \in M_{n\times n}(\mathbb{C}) \mid MM^* = M^*M = I\}$ where M^* is s the conjugate transpose of M and I is the identity matrix. $\mathrm{U}(n)$ is called the **unitary group** of degree n.

Proposition C.1.3 ([A1] Theorem 2.27)**.** *Any closed subgroup of a compact Lie group is a compact Lie group.*

Definition C.1.4. A **torus** is a compact Lie group which is isomorphic to $U(1)^n$ for some $n \geq 0$. The number n is the **rank** of the torus.

Theorem C.1.5 ([DW2] Theorem 2.2)**.** *Any connected compact abelian Lie group is isomorphic to a torus.*

Definition C.1.6. An abelian subgroup A of a compact Lie group G is **self-centralizing** in G if $C_G(A) = A$. A is **almost self-centralizing** in G if A has finite index in $C_G(A)$.

Definition C.1.7. A **maximal torus** for a compact Lie group G is a closed subgroup T of G, isomorphic to a torus such that T is almost self-centralizing.

Theorem C.1.8 ([DW2] Theorem 4.3)**.** *Any two maximal tori T and T' in a compact Lie group G are conjugate, i.e. there exists $g \in G$ such that $T' = gTg^{-1}$.*

Definition C.1.9. Let G be a compact Lie group with maximal torus T. The **Weyl group** of T in G is the quotient $W = N_G(T)/T$.

By Theorem C.1.8, up to isomorphism (by an inner automorphism of G), W depends only on G. Furthermore, since T is abelian, the conjugation action of $N_G(T)$ on T induces an action of W on T.

C.2 Principal Bundles

Proposition C.2.1 ([B4] Lecture 3 Theorem 2)**.** *Let G be a compact Lie group and H a compact subgroup, then G/H is a manifold such that $G \to G/H$ is a principal H-bundle.*

Example C.2.2. Define $p : \mathrm{U}(n) \to S^{2n-1}$ to be the map which sends each matrix to its first column. For $x \in S^{2n-1}$, $p^{-1}(x) \cong \mathrm{U}(n-1)$) is homeomorphic to the set of orthonormal bases for the orthogonal complement of x. Then

$$\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n) \hookrightarrow S^{2n-1}$$

is a principal $\mathrm{U}(n-1)$-bundle.

Example C.2.3. Let $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}_n(\mathbb{C})$ be the **special unitary group** of degree n. Then

$$\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \hookrightarrow S^{2n-1}$$

is a principal $\mathrm{SU}(n-1)$-bundle.

Example C.2.4. If we replace $\mathbb{C}$ by $\mathbb{H}$, and define $\mathrm{Sp}(n) := \{M \in M_{n\times n}(\mathbb{H}) | \, MM^* = M^*M = I\}$, then we have a principal $\mathrm{Sp}(n-1)$-bundle

$$\mathrm{Sp}(n-1) \hookrightarrow \mathrm{Sp}(n) \hookrightarrow S^{4n-1}.$$

C.2.1 Stiefel Manifolds and Grassmann Manifolds

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $\mathbb{F}^n$ is equipped with the usual inner product in these spaces. Let $\mathrm{U}_{\mathbb{F}}(n)$ be the group of $n \times n$ matrices whose columns form orthonormal bases for $\mathbb{F}^n$. The following constructions are called real, complex, and quaternionic respectively for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The **Stiefel manifold** $V_n(\mathbb{F}^k)$ is the space of n-frames in $\mathbb{F}^k$, i.e. n-tuples of orthonormal vectors in $\mathbb{F}^k$, with subspace topology of the product of n copies of unit sphere in $\mathbb{F}^k$. The **Grassmann manifold** is the space of n-dimensional vector subspaces of $\mathbb{F}^k$. The natural projection $p : V_n(\mathbb{F}^k) \to G_n(\mathbb{F}^k)$ sending an n-frame to the subspace it spans gives $G_n(\mathbb{F}^k)$ the quotient space topology and the fiber are the spaces of n-frames in a fixed n-plane in $\mathbb{F}^k$ and thus homeomorphic to $V_n(\mathbb{F}^n)$. There is a free and transitive $\mathrm{U}_{\mathbb{F}}(n)$ action on the fiber $V_n(\mathbb{F}^n)$, which makes this quotient map a principal $\mathrm{U}_{\mathbb{F}}(n)$-bundle. This construction can be generalized to the case when $k = \infty$ where $V_n(\mathbb{F}^\infty) = \cup_k V_n(\mathbb{F}^k)$ and $G_n(\mathbb{F}^\infty) = \cup_k G_n(\mathbb{F}^k)$.

$$\begin{array}{ll}
\mathrm{O}(n) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) & \mathrm{O}(n) \longrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty) \\
\mathrm{U}(n) \longrightarrow V_n(\mathbb{C}^k) \longrightarrow G_n(\mathbb{C}^k) & \mathrm{U}(n) \longrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty) \\
\mathrm{Sp}(n) \longrightarrow V_n(\mathbb{H}^k) \longrightarrow G_n(\mathbb{H}^k) & \mathrm{Sp}(n) \longrightarrow V_n(\mathbb{H}^\infty) \longrightarrow G_n(\mathbb{H}^\infty)
\end{array}$$

The Stiefel manifolds satisfies $V_n(\mathbb{F}^k)$ is $j(k-n+1)-2$-connected where $j = \dim_{\mathbb{R}}(\mathbb{F})$ and $V_n(\mathbb{F}^\infty)$ are contractible.

C.3 Classifying Spaces of Compact Lie Groups

The classifying space BG of a compact Lie group determines G in the following sense.

Theorem C.3.1 ([N] Theorem 1.1)**.** *Two compact Lie groups G and H are isomorphic as Lie groups if and only if BG and BH are homotopy equivalent.*

We have seen that any numerable principal G-bundle $E \to B$ with contractible E is a model for EG. Thus for classical Lie groups $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$, apart from Milnor's model of infinite joins, the principal bundles of the infinite Stiefel and Grassmann manifolds in C.2.1 provide alternative models of universal principal bundles and classifying spaces.

C.4 Homogeneous Spaces

Let G be a compact connected Lie group.

Definition C.4.1. A **homogeneous space** is a G-space on which G acts transitively.

Definition C.4.2. The **Euler characteristic** of a finite CW-complex X is the sum $\chi(X) = \sum_i (-1)^i \dim_{\mathbb{Q}} H^i(X; \mathbb{Q})$.

For a homogeneous space M of the form G/K where K is a closed subgroup of G, we have the following criterion for $H^*(G/K)$ to be evenly graded.

Theorem C.4.3 ([GHV] Chapter XI, Section 2, Theorem VII, p467)**.** *Let K be a compact connected subgroup of a compact connected Lie group G. Then the following conditions are equivalent:*

1. *G and K have the same rank.*
2. *$H^*(G/K;\mathbb{Q})$ is evenly graded.*
3. *The Euler characteristic of $H^*(G/K)$ is nonzero.*

C.4.1 Homogeneous Cohomology Spheres

A **homogeneous cohomology sphere** is a homogeneous space of a compact connected Lie group with the same cohomology as a sphere. The following classification theorem of homogeneous cohomology spheres can be found in [B2] Theorem 3.1.1. or [GKK] Proposition 1.1.

Theorem C.4.4. *Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively on G/H and such that $H^*(G/H;\mathbb{Z}) \cong H^*(\mathbb{S}^n,\mathbb{Z})$ for some $n \in N$. If G/H is not a sphere, then $G = \mathrm{SO}(3)$ and H is an icosahedral group of $\mathrm{SO}(3)$/ Otherwise the action of G is equivalent to the action of a subgroup of $\mathrm{SO}(n+1)$ on the standard spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$.*
If G/H is a sphere and if G acts irreducibly (i.e. there is no proper transitive normal subgroup), then there are (up to equivalence) only the following possibilities for G, H, n):

1. $(\mathrm{SO}(n+1), \mathrm{SO}(n), n), n \in \mathbb{N}$.
2. $(\mathrm{SU}(k), \mathrm{SU}(k-1), 2k-1), k \geq 2$.
3. $(\mathrm{Sp}(k), \mathrm{Sp}(k-1), 4k-1), k \geq 2$.
4. $(G_2, \mathrm{SU}(3), 6)$.
5. $(Spin(7), G_2, 7)$.
6. $(Spin(9), Spin(7), 15)$.

If G/H is a sphere and if G has a proper transitive normal subgroup, then the possibilities for $G, H, n)$ are (up to equivalence) the following:

1. $(\mathrm{U}(k), \mathrm{U}(k-1), 2k-1), k \geq 2.$
2. $(\mathrm{U}(1) \cdot \mathrm{Sp}(k), \mathrm{U}(1) \cdot \mathrm{Sp}(k-1), 4k-1), k \geq 2.$
3. $(\mathrm{Sp}(1) \cdot \mathrm{Sp}(k), \mathrm{Sp}(1) \cdot \mathrm{Sp}(k-1), 4k-1), k \geq 2.$

C.5 Topological Complex K-theory

Let X be a compact space. Let $\mathrm{Vect}_{\mathbb{C}}(X)$ be the set of isomorphism classes of (finite dimensional) complex vector bundles over X. Whitney sum (fiber-wise direct sum) of vector bundles gives $\mathrm{Vect}_{\mathbb{C}}(X)$ a monoid structure with the 0-dimensional bundle to be the zero element. Tensor product of bundles gives $\mathrm{Vect}_{\mathbb{C}}(X)$ a semi-ring structure, with the trivial 1-dimensional vector bundle to be the multiplicative identity.

Definition C.5.1. If X is compact, $K^0(X)$ is the **Grothendieck group** of finite dimensional (complex) vector bundles over X, i.e. the ring completion of $\mathrm{Vect}_{\mathbb{C}}(X)$.

C.5.1 Equivariant Topological Complex K-theory

Let G be a compact connected Lie group.

Definition C.5.2. If X is compact, $K_G(X)$ is defined to be the Grothendieck group of finite dimensional (complex) G-vector bundles over X.

By Bott periodicity, this construction can be extended to a $\mathbb{Z}/2$-graded multiplicative generalized equivariant cohomology theory K_G^* on the spaces of (locally compact) G-spaces called ***G*-equivariant *K*-theory**.

We write $K_G^*(X) = K_G^0(X) \oplus K_G^1(X)$ for all values of K_G^* on a G-space X with understanding that $K_G^0(X) \cong K_G^{2n}(X)$ and $K_G^1(X) \cong K_G^{2n+1}(X)$ for all $n \in \mathbb{Z}$.

When X is connected, $K_G^0(X) = [X, BU]$ and $K_G^1(X) = [X, U]$ where $U = \cup_{n=1}^{\infty} U(n)$ is the infinite unitary group and a model of BU is the infinite Grassmannian $G_1(\mathbb{C}^\infty)$.

When G is trivial, $K_G^*(X)$ coincides with the ordinary K-theory $K^*(X)$ of X, while when $X = \mathrm{pt}$ is trivial, equivariant K-theory can be identified with the complex representation ring $R(G)$. In particular, we have $K_G^1(\mathrm{pt}) = 0$. In general, by functorality of K_G^*, the canonical map $X \to \mathrm{pt}$ gives a canonical $R(G)$-module structure on $K_G^*(X)$ for any G-space X.

C.5.2 Representation Ring of Classical Lie Groups

The Representation Ring of a Torus

Let $M(k_1, \ldots, k_n)$ be a one-dimensional representation of T^n where the action of T^n is given by

$$(\theta_1, \ldots, \theta_n) \cdot z = \exp[2\pi i(k_1\theta_1 + \cdots + k_n\theta_n)]z.$$

Theorem C.5.3 ([H+], Chapter 13, Theorem 9.3)**.** *The simple T^n-modules are the one-dimensional representations of T^n which are isomorphic to modules of the form $M(k_1, \ldots, k_n)$. The ring $R(T^n)$ is the polynomial ring $\mathbb{Z}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ where t_i is the* *class of* $M(0, \ldots, 0, \underset{(i)}{1}, 0, \ldots, 0)$.

Equivalently, we can write $R(T^n) = \mathbb{Z}[t_1, \ldots, t_n, (t_1 \cdots t_n)^{-1}]$.

The Representation Ring of $\mathrm{U}(n)$ and $\mathrm{SU}(n)$

Let c_i denote the class in $R(\mathrm{U}(n))$ or $R(\mathrm{SU}(n))$ of the i-th exterior power of $\Lambda^i(\mathbb{C}^n)$, where $\mathrm{U}(n)$ or $\mathrm{SU}(n)$ acts on $\mathbb{C}^n$ in the canonical way.

Theorem C.5.4 ([H⁺] Chapter 14 Theorem 3.1)**.** *Let $T^n \subseteq \mathrm{U}(n)$ be the diagonal maximal torus. The ring $R(\mathrm{U}(n)) = \mathbb{Z}[c_1, \ldots, c_n, c_n^{-1}]$, where there are no polynomial relations between $c_1, \ldots, c_n$. As a subring of $R(T^n)$, $c_i = \sigma_i(t_1, \ldots, t_n)$ where σ_i is the i-th elementary symmetric polynomial in the n variables $t_1, \ldots, t_n$. The ring $R(\mathrm{SU}(n)) = \mathbb{Z}[c_1, \ldots, c_{n-1}]$.*

The Representation Ring of $\mathrm{Sp}(n)$

Let q_i denote the class in $R(\mathrm{Sp}(n))$ of the i-th exterior power of $\Lambda^i(\mathbb{C}^{2n})$, where $\mathrm{Sp}(n)$ acts on $\mathbb{C}^{2n} = \mathbb{H}^n$ in the canonical way.

Theorem C.5.5 ([H⁺] Chapter 14 Theorem 3.1)**.** *Let $T^n \subseteq \mathrm{Sp}(n)$ be the diagonal maximal torus. The ring $R(\mathrm{Sp}(n)) = \mathbb{Z}[q_1, \ldots, q_n]$, where there are no polynomial relations between $q_1, \ldots, q_n$. As a subring of $R(T^n)$, $q_i = \sigma_i(t_1, t_1^{-1}, \ldots, t_n, t_n^{-1})$ where σ_i is the i-th elementary symmetric polynomial in the $2n$-variables $t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}$.*

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