# SINGLE-ELEMENT TEARING AND MODIFICATION OF SPARSE SYMMETRIC SYSTEMS

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TR 72 - 150

December 1972

Simultaneously issued as a technical report by Harvard University.

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ABSTRACT

Tearing and modification obtains the solution of a linear system synthetically by first solving a slightly different ("torn") system and then modifying that solution. We show that single-element tearing of symmetric systems is rarely advantageous when the modified system is solved by elimination, and we classify those systems for which it is advantageous.

*Department of Computer Science. The author's work was supported by ONR contract N00014-67-A-0077-0021.

†Center for Research in Computing Technology. The author's work was supported by ONR contract

## 1. INTRODUCTION

The complexity analysis of single-element tearing and modification of sparse symmetric systems is the initial step toward understanding the relation between solving large sparse systems directly by (optimally ordered) Gaussian elimination and solving by synthetic methods based on the matrix modification identity (equation (2.1)). For example, when the sparse system represents the nodal admittance matrix of a (resistive) electrical network one examines the complexity of solution by removing an element corresponding to a resistor, say, solving this "torn" system, and then recapturing the solution of the original system. In this paper we examine in detail several aspects both of modifying an already computed solution and of computing an initial solution by first tearing the original system and then using modification.

In §2 we discuss the modification of a solution by giving a specific LU interpretation of the modification identity, equation(2.1). This section is more detailed with respect to rank one modification, but less general than our discussion in [1].

In §3 we discuss only single-element tearing of symmetric systems, giving particular attention to specific comparisons of the number of arithmetic operations involved when solving the equations. Here we have in mind the model of a resistive electrical network.

## 2. MODIFICATION.

Suppose we have solved the $n\times n$ system $By=b$, and now wish to solve the system $Ax=b$, where $A\equiv B+\sigma vw^t$, where $v$, $w$ are non-zero n-vectors and $\sigma$ is a non-zero scalar.

The method of modification is based on the identity (Householder[2], p.123):

$$(2.1) \qquad A^{-1} = B^{-1}-\tau B^{-1}vw^tB^{-1} ,$$

where $\tau = 1/(w^tB^{-1}v+1/\sigma)$. Thus,

$$(2.2) \qquad x = y-\tau B^{-1}vw^ty .$$

If $B=LU$, where $L$ is unit lower triangular and $U$ is upper triangular, then backsolve

$$(2.3) \quad \begin{cases} Lc=v & \text{and} \\ Uz=c , & \text{calculate} \\ \beta=w^ty, & \\ \gamma=w^tz, & \text{and} \\ \alpha=\beta/(\gamma+1/\sigma), & \text{then} \\ x=y-\alpha z. & \end{cases}$$

If $B$ is symmetric positive definite, then $B=LDL^t$, where $L$ is unit lower triangular, (symmetric Gaussian elimination), or $B=\tilde{L}\tilde{L}^t$, where $\tilde{L}=LD^{1/2}$, (Cholesky decomposition). So backsolve

$$
(2.4)\quad \begin{cases} Lc=v\,, & \text{form} \\ Dg=c\,, & \text{backsolve} \\ L^t z=g\,, & \text{calculate} \\ \beta=w^t y, & \\ \gamma=w^t z, & \text{and} \\ \alpha=\beta/(\gamma+1/\sigma)\,, & \text{then} \\ x=y-\alpha z & \end{cases}
$$

If only one element in B is changed, say the (i,j) element, then $v=e_i$, $w=e_j$, where $e_k$ is the $k^{th}$ column of the identity matrix of order n. Then (2.3) becomes:

$$
(2.5)\quad \begin{cases} Lc=e_i \\ Uz=c \\ \beta=y_j \\ \gamma=z_j \\ \alpha=\beta/(\gamma+1/\sigma) \\ x=y-\alpha z \end{cases}
$$

When B is symmetric, $B=LDL^t$, $v=e_i$, $w=e_j$, then (2.4) becomes:

$$
(2.6)\quad \begin{cases} Lc=e_i \\ Dg=c \\ L^t z=g \\ \beta=y_j \\ \gamma=z_j \\ \alpha=\beta/(\gamma+1/\sigma) \\ x=y-\alpha z \end{cases}
$$

In (resistive) electrical network problems, B is symmetric and a change in an off-diagonal entry (i,j) (corresponding to a resistor, say) necessitates a change in two diagonal elements (i,i) and (j,j). Here one takes $v=e_i \pm e_j=w$ and $A-B=\sigma vw^t=\sigma(e_i \pm e_j)(e_i^t \pm e_j^t)=\sigma(e_i e_i^t \pm e_i e_j^t \pm e_j e_i^t + e_j e_j^t)$. For example, in order to zero out $B_{ij}$ (corresponding to the removal of a resistor), take $\sigma=B_{ij}$ and $v=e_i-e_j$ (or $v=e_j-e_i$).

If $v=e_i-e_j=w$, then $\beta=w^t y=y_i-y_j$, $\gamma=w^t z=z_i-z_j$, and (2.4) becomes:

$$(2.7)\quad \begin{cases} Lc=e_i-e_j \\ Dg=c \\ L^t z=g \\ \alpha=(y_i-y_j)/(z_i-z_j+1/\sigma) \\ x=y-\alpha z. \end{cases}$$

The method of modification is related to bordering by the following:

<u>Theorem 1</u> ([1],§3). Let x be the solution to Ax=b. Then x can be augmented to satisfy the system

$$(2.8)\quad \begin{bmatrix} B & v \\ w^t & -\sigma^{-1} \end{bmatrix} \begin{bmatrix} x \\ \rho \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} .$$

Note that Theorem 1 follows from $A=B+\sigma vw^t$ and is independent of any specific interpretation such as (2.1)-(2.7). In Section 3, we will only consider interpretation (2.7).

For the general theory of modification and the general form of Theorem 1, see Bunch and Rose [1], Sections 3 and 4.

## 3. SINGLE-ELEMENT TEARING OF SYMMETRIC SYSTEMS

Let us consider the n×n system Ax=b. Suppose that the system were easier to solve if the $a_{ij}$ element were not there. In other words, we consider the "torn" matrix:

$$B \equiv A - \sigma v w^t, \tag{3.1}$$

where $\sigma = a_{ij}$ and $v = e_i$, $w = e_j$, or as in (2.7), when A is symmetric, take $v = e_i - e_j = w$, and then B is also symmetric.

We then solve By=b and recapture the solution x by means of

$$A^{-1} = B^{-1} - \tau B^{-1} v w^t B^{-1}, \text{ where } \tau = 1/(w^t B^{-1} v + 1/\sigma), \tag{3.2}$$

yielding

$$x = y - \tau B^{-1} v w^t y. \tag{3.3}$$

Thus, single-element tearing, (3.1)-(3.3), can be considered as a special case of modification, (2.1)-(2.2), and we would solve Ax=b by:

$$(3.4)\quad \begin{cases} \text{(i)} & \text{solve} & By=b & \text{for } y, \\ \text{(ii)} & \text{solve} & Bz=v & \text{for } z, \\ \text{(iii)} & \text{calculate} & \alpha = w^t y/(w^t z + 1/\sigma), & \\ \text{(iv)} & \text{form} & x = y - \alpha z. & \end{cases}$$

Suppose $B = B^t = LDL^t$, where L is unit lower triangular and D is diagonal. Let $d_i$ be the number of non-zero elements to the right of the diagonal in the first row of the reduced matrix of order n-i+1 in the elimination process (i.e. when the variable $x_i$ is being eliminated); $(d_1,\cdots,d_{n-1})$ is called the degree sequence of B.

Rose [3] has shown that the decomposition of $B=LDL^t$ requires

$$(3.5)\quad \sum_{i=1}^{n-1} d_i(d_i+3)/2 \text{ multiplications and } \sum_{i=1}^{n-1} d_i(d_i+1)/2 \text{ additions,}$$

where an addition is counted whenever fill-in (o+a) occurs; while the backsolving of $LDL^t y=b$ requires

$$(3.6)\quad n+2\sum_{i=1}^{n-1} d_i \text{ multiplications and } 2\sum_{i=1}^{n-1} d_i \text{ additions, when b is}$$

arbitrary.

For operation counts in the unsymmetric case, B=LU, see Bunch and Rose [1], Section 2.

Single-element tearing can be advantageous as the following example shows:

Example 1. Let $T_n$ be an n×n symmetric tridiagonal matrix and $S_n=T_n+\sigma(e_1-e_n)(e_1-e_n)^t$.

$$T_n = \begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}, S_n = \begin{bmatrix} \tilde{a}_1 & b_1 & & & -\sigma \\ b_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & a_{n-1} & b_{n-1} \\ -\sigma & & & b_{n-1} & a_n \end{bmatrix}, \quad \tilde{a}_1=a_1+\sigma,\ \tilde{a}_n=a_n+\sigma.$$

If $S_n$ is solved directly, then $S=\hat{L}\hat{D}\hat{L}^t$, where $\hat{L}^t=\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

Thus $d_i=2$ for $1\leq i\leq n-2$, $d_{n-1}=1$. So the decomposition requires 5n-8 multiplications and 3n-5 additions (or 2n-2 additions if an addition is not counted when fill-in occurs), while the backsolving requires 5n-6 multiplications and 4n-6 additions.

Solving $S_n x=b$ directly by symmetric Gaussian elimination requires 10n-14 multiplications and 7n-11 additions (or 6n-8 additions if an addition is not counted whenever fill-in occurs).

Solving $S_n x=b$ by tearing, we take $A=S_n$, $B=T_n$, $v=e_1-e_n=w$, $\sigma=a_{1n}$ in (3.4). Then $T_n=LDL^t$ where

$$L^t=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \text{ so } d_i=1 \text{ for } 1\leq i\leq n-1.$$

Forming $T_n=LDL^t$ requires 2n-2 multiplications and n-1 additions. Then (i) requires 3n-2 multiplications and 2n-2 additions, and so does (ii). In (iii), $\alpha=(y_i-y_j)/(z_i-z_j+1/\sigma)$ requires 2 multiplications and 3 additions. (iv) requires n multiplications and n additions.

Thus tearing requires 9n-14 multiplications and 6n-2 additions.

Now in step (ii) above, we are solving $LDL^t z=v$, where $v=e_1-e_n$. We could take advantage of the special structure of v to save operations.

($\alpha$) Solve $Lc=v$ for v.

$$Lc=\begin{bmatrix} 1 & & & \\ L_{21} & \ddots & & \\ & \ddots & \ddots & \\ & & L_{n,n-1} & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}.$$

$c_1=1$, $c_2=-L_{21}$ } no operations

$c_3=-L_{32}c_2$, ⋮, $c_{n-1}=-L_{n-1,n-2}c_{n-2}$ } 1 multiplication each

$c_n=-L_{n,n-1}c_{n-1}-1$ } 1 multiplicat 1 addition

Thus ($\alpha$) requires n-2 multiplications and 1 addition.

($\beta$) Solve Dg=c for g. $g_i=c_i/D_{ii}$. c is a full vector, so ($\beta$) requires n multiplications and no additions.

($\gamma$) Solve $L^t z=g$ for z.

$$L^t z = \begin{bmatrix} 1 & L_{21} & & 0 \\ & \ddots & \ddots & \vdots \\ & & \ddots & L_{n,n-1} \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} g_1 \\ \vdots \\ g_n \end{bmatrix}.$$

g is full, so ($\gamma$) requires n-1 multiplications and n-1 additions.

Thus (ii) could be done in 3n-3 multiplications and n additions. So tearing could be performed in 9n-5 multiplications and 5n additions.

Example 2. Let A be given by $a_{ii}\neq 0$ for $1\leq i\leq n$, $a_{n-1,1}\neq 0$, $a_{j1}=0$ for $2\leq j\leq n-2$, $a_{n1}=-\sigma$, $a_{n,n-1}\neq 0$, $a_{nj}=0$ for $2\leq j\leq n-2$, otherwise arbitrarily sparse.

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 & a_{n-1,1} & -\sigma \\ 0 & a_{22} & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & & & \ddots & & 0 \\ a_{n-1,1} & & & & \ddots & a_{n,n-1} \\ -\sigma & 0 & \cdots & 0 & a_{n,n-1} & a_{nn} \end{bmatrix}$$

Let $B=A-\sigma vv^t$, where $v=e_1-e_n$. Let $(d_1,\ldots,d_{n-1})$ and $(\hat{d}_1,\ldots,\hat{d}_{n-1})$ be the degree sequences of B and A, respectively.

Then $d_1=1$, $\hat{d}_1=2$, $\hat{d}_i=d_i+1$ for $2\le i\le n-2$, and $\hat{d}_{n-1}=1=d_{n-1}$.

Solving Ax=b directly requires $n+\sum_{i=1}^{n-1}\hat{d}_i(\hat{d}_i+7)/2=n+4+\sum_{i=1}^{n-2}(d_i+1)(d_i+8)/2=5n-4+\sum_{i=1}^{n-2}d_i(d_i+9)/2$ multiplications and

$\sum_{i=1}^{n-1}\hat{d}_i(\hat{d}_i+5)/2=3+\sum_{i=1}^{n-2}(d_i+1)(d_i+6)/2=3n-3+\sum_{i=1}^{n-2}d_i(d_i+7)/2$ additions.

Solving Ax=b by tearing:

Decomposing $B=LDL^t$ and backsolving By=b requires:

$n+\sum_{i=1}^{n-1}d_i(d_i+7)/2=n+4+\sum_{i=1}^{n-2}d_i(d_i+7)/2$ multiplications and

$\sum_{i=1}^{n-1}d_i(d_i+5)/2=3+\sum_{i=1}^{n-2}d_i(d_i+5)/2$ additions.

If we do not take advantage of the structure of $v=e_1-e_n$, backsolving Bz=v requires $n+2\sum_{i=1}^{n-1}d_i$ multiplications and $2\sum_{i=1}^{n-1}d_i$ additions.

Taking advantage of the structure of $v=e_1-e_n$:

($\alpha$) Solve $Lc=e_1-e_n$ for c.

$$\begin{bmatrix} 1 & & & & 0 \\ 0 & 1 & & & \\ & & \ddots & & \\ 0 & & & \ddots & \\ L_{n-1,1} & & & & \\ 0 & & & L_{n,n-1} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}. \quad \left.\begin{array}{l} c_1=1 \\ c_2=0 \\ c_{n-2}=0 \\ c_{n-1}=-L_{n-1,1} \\ c_n=-L_{n,n-1}c_{n-1}-1 \end{array}\right\} \begin{array}{l}\text{1 multiplication} \\ \text{1 addition.}\end{array}$$

(β) Solving $Dg=c$ for $g$.

$$\left.\begin{aligned} g_1 &= 1/D_{11} \\ g_2 &= 0 \\ g_{n-2} &= 0 \\ g_{n-1} &= c_{n-1}/D_{n-1,n-1} \\ g_n &= c_n/D_{nn} \end{aligned}\right\} \quad \text{3 multiplications, no additions.}$$

(γ) Solving $L^t z=g$ for $z$.

$$L^t z = \begin{bmatrix} 1 & 0 - - - - 0 & L_{n-1,1} & 0 \\ & \ddots & & \\ & & \ddots & L_{n,n-1} \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} g_1 \\ 0 \\ \vdots \\ 0 \\ g_{n-1} \\ g_n \end{bmatrix}$$

$z_n = g_n$

$z_{n-1} = g_{n-1} - L_{n,n-1} z_n$ ; $d_{n-1}$ multiplications, $d_{n-1}$ additions

$z_{n-2} =$ ⋮ ; $d_{n-2}$ " , $d_{n-2}-1$ "

⋮

$z_2 =$ ⋮ ; $d_2$ " , $d_2-1$ "

$z_1 = g_1 - L_{n-1,1} z_{n-1}$ ; $d_1$ " , $d_1$ "

Solving (γ) requires $\sum_{i=1}^{n-1} d_i$ multiplications and $\sum_{i=1}^{n-1} d_i - n+3$ additions.

Thus taking advantage of $v=e_1-e_n$, we need $5+\sum_{i=1}^{n-2} d_i$ multiplications and $5-n+\sum_{i=1}^{n-2} d_i$ additions to solve $Bz=v$.

Calculating $\alpha$ requires 2 multiplications and 3 additions, and calculating $x=y-\alpha z$ requires $n$ multiplications and $n$ additions.

Tearing requires:

(a) $3n+8+\sum_{i=1}^{n-2} d_i(d_i+11)/2$ multiplications and

(b) $n+8+\sum_{i=1}^{n-2} d_i(d_i+9)/2$ additions

if the structure of $v=e_1-e_n$ is not used in solving $Bz=v$, and

(c) $2n+11+\sum_{i=1}^{n-2} d_i(d_i+9)/2$ multiplications and

(d) $11+\sum_{i=1}^{n-2} d_i(d_i+7)/2$ additions

if the structure of $v=e_1-e_n$ is used in solving $Bz=v$.

Using (a), tearing saves multiplications only if $\sum_{i=1}^{n-2} d_i<2n-12$, and if so then tearing saves $2n-12-\sum_{i=1}^{n-2} d_i \leq 2n-12-(n-2)=n-10$ multiplications ($d_i \geq 1$ for every i if B is not reducible). Using (c), tearing always saves $3n-15$ multiplications.

Using (b), tearing saves additions only if $\sum_{i=1}^{n-2} d_i<2n-11$, and if so then tearing saves $2n-11-\sum_{i=1}^{n-2} d_i<2n-11-(n-2)=n-9$ additions. Using (d), tearing always saves $3n-14$ additions.

Theorem 1 implies that, in the context of optimal ordering, tearing may be of no advantage if there exists an optimal ordering which orders $\rho$ first in the system (2.8).

Bunch and Rose ([1], Section 4) show that, without loss of generality, we may assume that the torn elemnt is in the (1,n) position, i.e. we may assume $v=e_1-e_n$.

Further, without loss of generality, we may assume that the torn matrix B is irreducible, i.e., there does not exist a permutation matrix P such that $PBP^t=\begin{bmatrix} E & O \\ O & F \end{bmatrix}$.

Let $(d_1,\ldots,d_{n-1})$ and $(\hat{d}_1,\ldots,\hat{d}_{n-1})$ be the degree sequences of B and A respectively, where $B=A-\sigma vv^t$, $v=e_1-e_n$, $\sigma=a_{1n}$.

Then $d_i\leq\hat{d}_i\leq d_i+1$ for $1\leq i\leq n-2$ and $d_{n-1}=1=\hat{d}_{n-1}$. Let $m(LDL^t)$ be the number of multiplications necessary to solve Ax=b by symmetric Gaussian elimination. Let m(tear)', and m(tear), be the number of multiplications required to solve Ax=b by tearing when one does, and does not, take advantage of the structure of $v=e_1-e_n$ in solving Bz=v. Let $a(LDL^t)$, a(tear)', and a(tear) be defined similarly for additions, where an addition is counted whenever fill-in occurs.

Theorem 2.

$$n+4+\sum_{i=1}^{n-2} d_i(d_i+7)/2\leq m(LDL^t)\leq 5n-4+\sum_{i=1}^{n-2} d_i(d_i+9)/2.$$

$$3+\sum_{i=1}^{n-2} d_i(d_i+5)/2\leq a(LDL^t)\leq 3n-3+\sum_{i=1}^{n-2} d_i(d_i+7)/2.$$

$$m(\text{tear})=3n+8+\sum_{i=1}^{n-2} d_i(d_i+11)/2.$$

$$a(\text{tear})=n+8+\sum_{i=1}^{n-2} d_i(d_i+9)/2.$$

$$m(\text{tear})'=3n+10-k+\sum_{i=1}^{n-2} d_i(d_i+9)/2+\sum_{i=k}^{n-2} d_i \quad \text{and}$$

$$a(\text{tear})'=11+\sum_{i=k}^{n-2} d_i+\sum_{i=1}^{n-2} d_i(d_i+7)/2,$$

where $k=\min\{j:A_{j1}\neq 0 \text{ for } j>1\}$, so $2\leq k\leq n-1$.

<u>Proof</u>. $m(LDL^t)=n+\sum_{i=1}^{n-1} \hat{d}_i(\hat{d}_i+7)/2 \quad n+\sum_{i=1}^{n-2} d_i(d_i+7)/2\geq n+4+\sum_{i=1}^{n-2} d_i(d_i+7)/2.$

$$m(LDL^t)\leq n+4+\sum_{i=1}^{n-2} (d_i+1)(d_i+8)/2=5n-4+\sum_{i=1}^{n-2} d_i(d_i+9)/2.$$

$$a(LDL^t)=\sum_{i=1}^{n-1} \hat{d}_i(\hat{d}_i+5)/2\leq 3+\sum_{i=1}^{n-2} (d_i+1)(d_i+6)/2=3n-3+\sum_{i=1}^{n-2} d_i(d_i+7)/2.$$

$$a(LDL^t)\geq 3+\sum_{i=1}^{n-2} d_i(d_i+5)/2.$$

$$m(\text{tear})=\sum_{i=1}^{n-1} d_i(d_i+3)/2+2\{n+2\sum_{i=1}^{n-1} d_i\}+2+n=3n+8+\sum_{i=1}^{n-2} (d_i^2+11d_i)/2.$$

$$a(\text{tear})=\sum_{i=1}^{n-1} d_i(d_i+1)/2+2\{2\sum_{i=1}^{n-1} d_i\}+3+n=n+8+\sum_{i=1}^{n-2} d_i(d_i+9)/2.$$

However, one could solve $Bz=LDL^tz=v=e_1-e_n$ by:

(α)
$$Lc=\begin{bmatrix} 1 & & & & 0 \\ 0 & \ddots & & & \\ 0 & & \ddots & & \\ L_{k1} & & 1 & & \\ \vdots & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ -1 \end{bmatrix} \qquad \left.\begin{array}{l} c_1=1 \\ c_2=0 \\ \vdots \\ c_{k-1}=0 \\ c_k=-L_{k1} \end{array}\right\}\text{no operations.}$$

$$\sum_{i=k}^{n-1} d_i \text{ multiplications,} \quad \sum_{i=k}^{n-1} d_i-n+k+1 \text{ additions.} \qquad \left\{\begin{array}{l} c_{k+1}=\cdots \\ c_n \quad =\cdots \end{array}\right.$$

(β) $Dg=c=\begin{bmatrix}1\\0\\ \vdots\\0\\c_k\\ \vdots\\c_n\end{bmatrix}$ requires n-k+2 multiplications and no additions.

(γ) $$L^t z=\begin{bmatrix}1 & 0 \cdots L_{k1} & \cdots & 0\\ & \ddots & & \\ & & \ddots & L_{n,n-1}\\ & & & 1\end{bmatrix}\begin{bmatrix}z_1\\ \vdots\\ \vdots\\ z_n\end{bmatrix}=g=\begin{bmatrix}g_1\\0\\ \vdots\\0\\g_k\\ \vdots\\g_n\end{bmatrix}.$$

$$\left.\begin{aligned}&z_n=g_n\\&z_{n-1}=g_{n-1}-L_{n,n-1}z_n\\&\vdots\\&z_k=g_k-\cdots\\&z_{k-1}=0-\cdots\\&\vdots\\&z_2=0-\cdots\\&z_1=g_1-\cdots\end{aligned}\right\}\quad \sum_{i=1}^{n-1} d_i \text{ multiplications, } \sum_{i=1}^{n-1} d_i-k+2 \text{ additions.}$$

Hence $m(\text{tear})'=\sum_{i=1}^{n-1} d_i(d_i+3)/2+n+2\sum_{i=1}^{n-1} d_i+\sum_{i=k}^{n-1} d_i+n-k+2+\sum_{i=1}^{n-1} d_i+2+n=$

$\sum_{i=1}^{n-2} d_i(d_i+9)/2+\sum_{i=k}^{n-1} d_i+3n-k+10$ multiplications, and

$a(\text{tear})'=\sum_{i=1}^{n-1} d_i(d_i+1)/2+2\sum_{i=1}^{n-1} d_i+\sum_{i=k}^{n-1} d_i+\sum_{i=1}^{n-1} d_i-n+3+3+n=$

$11+\sum_{i=k}^{n-2} d_i+\sum_{i=1}^{n-2} d_i(d_i+7)/2.$ qed.

Corollary 1. If $m(\text{tear})<m(LDL^t)$, then $\sum_{i=1}^{n-2} d_i<2n-12$.

If $m(\text{tear})'<m(LDL^t)$, then $\sum_{i=k}^{n-2} d_i<2n-14+k$.

Corollary 2. If $a(\text{tear})<a(LDL^t)$, then $\sum_{i=1}^{n-2} d_i<2n-11$.

If $a(\text{tear})'<a(LDL^t)$, then $\sum_{i=k}^{n-2} d_i<3n-14$.

Corollary 3. $m(LDL^t)-m(\text{tear})\leq 2n-12-\sum_{i=1}^{n-2} d_i\leq n-10$.

$m(LDL^t)-m(\text{tear})'\leq 2n-14+k-\sum_{i=k}^{n-2} d_i\leq n-13+2k\leq 3n-15$.

$a(LDL^t)-a(\text{tear})\leq 2n-11-\sum_{i=1}^{n-2} d_i\leq n-9$.

$a(LDL^t)-a(\text{tear})'\leq 3n-14-\sum_{i=k}^{n-2} d_i\leq 2n+k-13\leq 3n-14$.

Proof. Since B is irreducible, $d_i\geq 1$ for every i.

Compare with examples 1 and 2.

Corollary 4. If $d_i\geq 2$ for at least n-10 variables, then $m(\text{tear})\geq m(LDL^t)$.

Proof. $\sum_{i=1}^{n-2} d_i\geq 2(n-10)+8=2n-12$. Hence $m(\text{tear})\geq m(LDL^t)$.

Let us look at tearing at its worst.

Corollary 5. $m(\text{tear})-m(LDL^t)\geq\frac{1}{2}n^2-\frac{3}{2}n+13$ is possible.

$a(\text{tear})-a(LDL^t)\geq\frac{1}{2}n^2-\frac{3}{2}n+12$ is possible.

Proof. Since $b_{1n}=0$, $d_1\leq n-2$, and $d_i\leq n-i-1$ for $2\leq i\leq n-2$, $d_{n-1}=1$.

$m(\text{tear})-m(LDL^t)\geq\sum_{i=1}^{n-2} d_i-2n+12\geq\frac{1}{2}n^2-\frac{3}{2}n+13$.

$$a(\text{tear})-a(LDL^t)\geq\sum_{i=1}^{n-2} d_i-2n+11\geq\frac{1}{2}n^2-\frac{3}{2}n+12.$$

Corollary 6. $m(\text{tear})'-m(LDL^t)\geq\frac{1}{2}n^2-\frac{9}{2}n+15$ is possible.

$a(\text{tear})'-a(LDL^t)\geq\frac{1}{2}n^2-\frac{11}{2}n+13$ is possible.

Proof. Let $k=2$. Then $d_1\leq n-2$, $d_i\leq n-i-1$ for $2\leq i\leq n-2$, $d_{n-1}=1$.

$$m(\text{tear})'-m(LDL^t)\geq\sum_{i=2}^{n-2} d_i-2n+12\geq\frac{1}{2}n^2-\frac{9}{2}n+15.$$

$$a(\text{tear})'-a(LDL^t)\geq\sum_{i=2}^{n-2} d_i-3n+14\geq\frac{1}{2}n^2-\frac{11}{2}n+13.$$

Note that if an addition is not counted when fill-in occurs, then the counts for $a(LDL^t)$, $a(\text{tear})$, and $a(\text{tear})'$ will be slightly different.

## 5. REMARKS

A more general treatment of this subject can be found in Bunch and Rose [1].

A summary of this report will appear in the Proceedings of the Sixth Hawaii International Conference on System Sciences (HICSS-6), January, 1973.

## REFERENCES

[1] Bunch, J.R., and D.J. Rose, "Partitioning, tearing, and modification of sparse linear systems", Cornell University Technical Report TR72-149. Also available as a technical report from Harvard University. Submitted to *Linear Algebra and Its Applications*.

[2] Householder, A.S., *The Theory of Matrices in Numerical Analysis*, Blaisdell, New York, 1964.

[3] Rose, D.J., "A graph-theoretic study of the numerical solution of sparse positive definite systems of linear equations", *Graph Theory and Computing*, R. Read, editor, Academic Press, New York, 1972.