SMOOTHNESS PROPERTIES OF SYMBOLS, CALDERÓN COMMUTATORS AND GENERALIZATIONS

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Let $m : (\mathbb{R}^n)^k \to \mathbb{C}$ be a bounded function. With this m, which we call it a symbol, we can define a multilinear operator T_m in the form

$$T_m(f_1, \cdots, f_k)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} m(\xi_1, \cdots, \xi_k)\widehat{f_1}(\xi_1) \cdots \widehat{f_k}(\xi_k)e^{2\pi i x\cdot(\xi_1+\cdots+\xi_k)}d\xi_1 \cdots d\xi_k.$$

When m is smooth enough, T_m is called a classical paraproduct and the boundedness of T_m was proved by the classical Coifman-Meyer Theorem.

We will give a weaker criteria on the smoothness of m to ensure the boundedness of T_m. After that, we will talk about its applications to classical examples, the Calderón Commutators as well as some generalizations.

BIOGRAPHICAL SKETCH

During his childhood in Hong Kong, Pok Wai Fong has already showed his deep interest towards mathematics. By participating in a number of specialized trainings and mathematical competitions during his high school studies, including the International Mathematical Olympaid, Pok Wai has acquired the problem solving skills and mathematical knowledge which prepared him well for his undergraduate studies in mathematics at The Chinese University of Hong Kong from 2007 to 2010. In the summer of 2009, he visited University of Waterloo and conducted his first mathematical research in fractal analysis. This experience further consolidated his determination to pursue higher studies in Mathematics. In 2010, He started his doctoral studies in Mathematics at Cornell University, where he conducted research in multilinear harmonic analysis under the supervision of Prof. Camil Muscalu. He will join Two Sigma Investment as a quantitative research analyst after completing his PhD program in August 2016.

To my parents Chun Chung Fong and Wai Ming Tam

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CHAPTER 1

INTRODUCTION

1.1 Historical Background

In this paper, we study the Holder type boundedness of multilinear operators of the following kind:

Definition 1.1.1. *Let* $m : (\mathbb{R}^n)^k \to \mathbb{C}$ *be a bounded function. For a multilinear operator* T_m *in the form*

$$T_m(f_1, \cdots, f_k)(x) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} m(\xi_1, \cdots, \xi_k)\widehat{f_1}(\xi_1) \cdots \widehat{f_k}(\xi_k) e^{2\pi i x\cdot(\xi_1+\cdots+\xi_k)} d\xi_1 \cdots d\xi_k$$

defined for $f_1, f_2, \cdots, f_k$ *lying in* $\mathcal{S}(\mathbb{R}^n)$*, the space of Schwartz functions on* $\mathbb{R}^n$*,* m *is called the symbol of* T_m*.*

We have the classical theorem of Coifman and Meyer [7]:

Theorem 1.1.1 (Coifman-Meyer, 1957)**.** *If* m *is a classical Marcinkiewicz-Mikhlin-Hormander symbol, that is,*

$$|\partial^\alpha m(\xi)| \lesssim \frac{1}{|\xi|^{|\alpha|}} \tag{1.1}$$

for sufficiently many multi-indices α*, then* T_m *can be extended naturally to*

$$T_m : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \cdots \times L^{p_k}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$$

as a bounded k-linear operator for any $1 < p_1, \cdots, p_k \leq \infty$*,* $0 < r < \infty$ *and* $\dfrac{1}{p_1} + \cdots + \dfrac{1}{p_k} = \dfrac{1}{r}$*.*

Here "$A \lesssim B$" means $A \leq CB$ for some universal constant C.

In the case m being a classical Marcinkiewicz-Mikhlin-Hormander symbol, we call T_m a paraproduct. A paraproduct can also be seen as a multilinear singular integral. If we let K be the inverse Fourier transform of m, then $T_m(f_1, \cdots, f_k)(x)$ can also be written formally as

$$\text{p.v.} \int_{\mathbb{R}^{nk}} f_1(x - y_1) \cdots f_k(x - y_k) K(y_1, \cdots, y_k) dy.$$

Note that when $m \equiv 1$, $K = \delta_0$ and $T_m(f_1, \cdots, f_k) = f_1 \cdots f_k$, the product of the k functions.

The Coifman-Meyer Theorem requires a very strong smoothness condition of the symbol m away from the origin. But historically we have encountered singular integrals with symbols that do not have such strong smoothness property. For example, Calderón [1] has introduced the Calderón Commutators, which arises from a number of problems including the Cauchy integral on Lipschitz curves:

Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function with $\|A'\|_\infty < 1$. It describes a Lipschitz curve Γ in the complex plane parametrized by $t \mapsto t + iA(t)$. Let f be a Schwartz function on $\mathbb{R}$. The Cauchy integral associated with Γ is

$$C_\Gamma(f)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{(x - y) + i(A(x) - A(y))} dy.$$

By expanding the power series of $(1 + iz)^{-1}$ near the origin, we can write

$$\begin{aligned} C_\Gamma(f)(x) &= \sum_{d=0}^{\infty} (-i)^d \text{p.v.} \int_{\mathbb{R}} f(y) \left(\frac{A(x) - A(y)}{x - y} \right)^d \frac{dy}{x - y} \\ &= \sum_{d=0}^{\infty} (-i)^d C_d(f, A', \cdots, A')(x), \end{aligned}$$

where C_d is defined as follows:

Definition 1.1.2. *The dth Calderón Commutator on $\mathbb{R}$ is the $(d+1)$-linear operator*

$$C_d(f, a_1, \cdots, a_d)(x) = p.v. \int_{\mathbb{R}} f(y) \prod_{j=1}^{d} \frac{A_j(x) - A_j(y)}{x - y} \frac{dy}{x - y},$$

where A_j is an antiderivative of a_j.

It turns out that the symbol associated to C_d is

$$m_d(\xi_0, \xi_1, \cdots, \xi_d) = \int_0^1 \cdots \int_0^1 \operatorname{sgn}(\xi_0 + \alpha_1 \xi_1 + \cdots + \alpha_d \xi_d) d\alpha_1 \cdots d\alpha_d.$$

For example when $d = 1$, the symbol m_1 is given by

$$m_1(\xi_0, \xi_1) = \int_0^1 \operatorname{sgn}(\xi_0 + \alpha \xi_1) d\alpha = \frac{|\xi_0 + \xi_1| - |\xi_0|}{\xi_1},$$

which is only continuous but not differentiable along the lines $\{\xi_0 + \xi_1 = 0\}$ and $\{\xi_0 = 0\}$. Therefore, the boundedness of the first Calderón Commutator already needs technology beyond the Coifman-Meyer Theorem.

Furthermore, To prove the L^2 boundedness of the Cauchy integral on Lipschitz curves, one of the approaches is to establish the boundedness of

$$C_d : L^2(\mathbb{R}) \times L^\infty(\mathbb{R}) \times \cdots \times L^\infty(\mathbb{R}) \to L^2(\mathbb{R})$$

with boundedness constant B_d and

$$\sum_{d=1}^{\infty} \|A'\|_\infty^d B_d < \infty.$$

This was first established by Coifman, McIntosh and Meyer [5], proving a polynomial growth estimate of B_d. A new proof was given by Muscalu [17] using time-frequency analysis.

In Muscalu [16], the following Holder type boundedness of the 1st Calderón Commutator was reestablished by time-frequency analysis:

Theorem 1.1.2 ($r > 1$: Calderón, 1965; $r > \frac{1}{2}$: Coifman and Meyer, 1975)**.**

$$\|C_1(f,a)\|_r \lesssim \|f\|_{p_1} \|a\|_{p_2}$$

for all $1 < p_1, p_2 \leq \infty$, $\dfrac{1}{2} < r < \infty$ *and* $\dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{r}$.

One essential property of the 1st Calderón Commutator that makes the argument work is the decay of localized fourier coefficients of its symbol. When $\widehat{\phi_0}$ and $\widehat{\phi_1}$ are bump functions such that $\widehat{\phi_0}(\xi_0)\widehat{\phi_1}(\xi_1)$ is a smooth cut-off function of a Whitney box (i.e. a box with sides parallel to the coordinate axis such that its side length is comparable to its distance to the origin) in the $\xi_0\xi_1$-plane, the Fourier transform

$$c_{n_0,n_1} := \int_{\mathbb{R}}\int_{\mathbb{R}} m_1(\xi_0,\xi_1)e^{-2\pi i(n_0\xi_0+n_1\xi_1)}\widehat{\phi_0}(\xi_0)\widehat{\phi_1}(\xi_1)d\xi_0 d\xi_1$$

satisfies the estimate

$$\left|c_{n_0,n_1}\right| \lesssim \langle n_0\rangle^{-2}\langle n_1\rangle^{-M} + \langle n_0\rangle^{-2}\langle n_0 - n_1\rangle^{-M} \tag{1.2}$$

for any $M > 0$.

Intuitively, the decay of the localized fourier coefficients describes a certain degree of smoothness of the symbol m_1. The smoothness is not as strong as in (1.1) but turned out it is enough to ensure the Holder type bound in Theorem 1.1.2.

Actually, the method in [16] yields the following, stronger form, of theorem 1.1.1:

Theorem 1.1.3. *Let $s < 1$. Let m be a symbol on $\mathbb{R}^{d+1}$ and suppose in any whitney box, the localized fourier coefficients $c_{n_0,n_1,\ldots,n_d}$ of m satisfies*

$$\left|c_{n_0,n_1,\ldots,n_d}\right| \leq \widetilde{C}_{n_0,n_1,\ldots,n_d}$$

for some positive numbers $\widetilde{C}_{n_0,n_1,\ldots,n_d}$, $n_j \in \mathbb{Z}$ *for each* j *and*

$$\sum_{n_0,\ldots,n_d} \left(\prod_{j=0}^{d} \log \left\langle n_j \right\rangle \widetilde{C}_{n_0,n_1,\ldots,n_d} \right)^{\frac{1}{s}} < \infty \tag{1.3}$$

Then

$$\|T_m(f, a_1, \ldots, a_d)\|_r \lesssim \|f\|_{p_0} \prod_{j=1}^{d} \left\|a_j\right\|_{p_j}$$

for all $1 < p_0, p_1, \ldots, p_d < \infty$, $\frac{1}{d+1} < r < \infty$, $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$ *and* $r > s$.

It was believed that the method in [16] could not be extended to prove the known quasi-Banach boundedness for the dth Calderón Coummutator, C_d, for $d \geq 2$ and the full range of exponents:

Theorem 1.1.4 ($d = 2$: Coifman and Meyer, 1975; $d \geq 3$: Duong, Grafakos and Yan, 2010)**.** *C_d satisfies the bound*

$$\|C_d(f, a_1, \ldots, a_d)\|_r \lesssim \|f\|_{p_0} \prod_{k=1}^{d} \|a_k\|_{p_k}$$

for all $1 < p_0, p_1, \ldots, p_d < \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$.

That is because, for $d \geq 2$, m_d does not behave as nicely as m_1, in the sense that the corresponding localized fourier coefficients $c_{n_0,n_1,\ldots,n_d}$ decays much slower than the localized fourier coefficients c_{n_0,n_1} of C_1.

More precisely, the decay in (1.2) means localized fourier coefficients c_{n_0,n_1} of m_1 satisfies (1.3) for $s = \frac{1}{2}$. But it seems that the localized fourier coefficients $c_{n_0,n_1,\ldots,n_d}$ of m_d, for $d \geq 2$, do not decay fast enough the satisfy (1.3) for $s = \frac{1}{d+1}$.

(1.3) roughly means

$$\widetilde{C}_{n_0,n_1,\ldots,n_d} \in L^{s+}(\mathbb{Z}^{d+1}).$$

But in fact, We only have the weaker decay

$$\widetilde{C}_{n_0,n_1,\ldots,n_d} \in L^{\frac{d}{d+1}+}(\mathbb{Z}^{d+1}),$$

guaranteed by Proposition 3.1.2. But with this weaker decay and Theorem 2.3.1, a result about the boundedness of a more complicated type of model operators, I am able to give a new prove of Theorem 1.1.4 via time-frequency analysis.

1.2 Main Results

We have the following boundedness of a new type of model operator:

Theorem 2.3.1 (with Muscalu, 2015). *Suppose at least two of the families in* $\left\{\phi^0_{I_n}\right\}_I, \ldots, \left\{\phi^{d+1}_{I_n}\right\}_I$ *are lacunary. For all* $1 < p_0, p_1, \ldots, p_d \leq \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$. *For* $\epsilon' > 1 - \frac{1}{p_1'}$ *and any* $\epsilon > 0$, *if* $K(n_0, \ldots, n_d)$ *is a function on* $\mathbb{Z}^{d+1}$ *satisfying*

$$|K(n_0, \ldots, n_d)| \lesssim \langle n_1 \rangle^{-(2-\epsilon')} \langle n_0 - n_1 \rangle^{-(1+\epsilon)} \prod_{j=2}^{d} \left\langle n_j \right\rangle^{-(1+\epsilon)},$$

we have the following bound for the following generalized form of discretized paraproduct:

$$\left\| \sum_I \sum_{n_0,n_1,\ldots,n_d \in \mathbb{Z}} \frac{K(n_0, \ldots, n_d)}{|I|^{d/2}} \prod_{j=0}^{d} \left\langle a_j, \phi^j_{I_{n_j}} \right\rangle \phi^{d+1}_I \right\|_r \lesssim \prod_{j=0}^{d} \left\|a_j\right\|_{p_j}$$

With the above theorem, we can have the following criteria of the boundedness of T_m, described by the decay of the localized fourier coefficients of m:

Theorem 2.3.3. *Let* m *be a symbol on* $\mathbb{R}^{d+1}$. *Suppose there exist* $\epsilon_0', \ldots, \epsilon_d' > 0$ *and* $\epsilon > 0$ *such that in any whitney box* B, *there exists* $k \in \{0, \ldots, d\}$ *such that* ξ_k *is never 0 in* B *and there exists another index* k' *such that the localized fourier coefficients* $c_{n_0,n_1,\ldots,n_d}$

of m satisfies

$$\left|c_{n_0,n_1,\ldots,n_d}\right| \lesssim \langle n_k\rangle^{-(2-\epsilon'_k)} \langle n_{k'}-n_k\rangle^{-(1+\epsilon)} \prod_{j\neq k,k'} \left\langle n_j\right\rangle^{-(1+\epsilon)}. \tag{1.4}$$

Then

$$\|T_m(f,a_1,\ldots,a_d)\|_r \lesssim \|f\|_{p_0} \prod_{j=1}^{d} \left\|a_j\right\|_{p_j}$$

for all $\frac{1}{1-\epsilon'_j} < p_j < \infty$ *for* $0 \le j \le d$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0}+\cdots+\frac{1}{p_d}=\frac{1}{r}$.

Combining the above theorem with the smoothness estimate in Proposition 3.1.2, the quasi-Banach boundedness for the dth Calderón Coummutator, C_d, for $d \ge 2$ and the full range of exponents will be automatic.

Note that the condition (2.9) is weaker than (1.3). In fact, the decay in Proposition 3.1.2 only implies that for the localized fourier coefficients $c_{n_0,n_1,\ldots,n_d}$ of m_d,

$$\widetilde{C}_{n_0,n_1,\ldots,n_d} \in L^{\frac{d}{d+1}+}(\mathbb{Z}^{d+1}),$$

which is weaker than

$$\widetilde{C}_{n_0,n_1,\ldots,n_d} \in L^{\frac{1}{d+1}+}(\mathbb{Z}^{d+1}),$$

the decay needed to satisfy (1.3).

Theorem 2.3.3 can also be used to prove the boundedness of a wide range of operators. For example, the following generalization of the dth Calderón Commutator could be obtained from it:

Theorem 3.3.1. *Let*

$$\widetilde{C}_d(a_0,a_1,\cdots,a_d)(x) = p.v. \int_{\mathbb{R}}\int_{\mathbb{R}} a_0(x+t+s) \prod_{j=1}^{d} \frac{\Delta_s}{s} \circ \frac{\Delta_t}{t} A_j(x) \frac{dt}{t}\frac{ds}{s},$$

where

$$A''_j = a_j$$

and

$$\frac{\Delta_t}{t} g(x) := \frac{g(x+t) - g(x)}{t}.$$

Then we have

$$\left\| \widetilde{C}_d(f, a_1, \ldots, a_d) \right\|_r \lesssim \|f\|_{p_0} \prod_{k=1}^{d} \|a_k\|_{p_k}$$

for all $1 < p_0, p_1, \ldots, p_d < \infty$, $\dfrac{1}{d+1} < r < \infty$ *and* $\dfrac{1}{p_0} + \cdots + \dfrac{1}{p_d} = \dfrac{1}{r}$.

By the same type of argument, we can also prove the boundedness of an analogue of the 1st Calderón Commutator in higher dimensions:

Theorem 3.3.2. *For* $f, A \in \mathcal{S}(\mathbb{R}^n)$, *let*

$$C_1^n(f, A) = p.v. \int_{\mathbb{R}^n} f(y) \frac{A(x) - A(y)}{|x-y|} \frac{dy}{|x-y|^n}.$$

Then

$$\left\| C_1^n(f, A) \right\|_r \lesssim \|f\|_p \, \|\nabla A\|_{p_1}$$

for any $1 < p, p_1 < \infty$, $\frac{n}{n+1} < r < \infty$ *and* $\frac{1}{p} + \frac{1}{p_1} = \frac{1}{r}$.

CHAPTER 2

BOUNDEDNESS OF T_M AND DECAY OF LOCALIZED FOURIER COEFFICIENTS OF M

In this chapter, we study the relationship between the boundedness of T_m and the decay of the localized Fourier coefficients of m.

2.1 Discretization

Let m be a symbol on $\mathbb{R}^{d+1}$. We want to describe the way to obtain the boundedness of the corresponding $(d+1)$-linear operator

$$T_m(a_0, a_1, \ldots, a_d)(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} m(\xi_0, \ldots, \xi_d) \prod_{j=0}^{d} \widehat{a_j}(\xi_j) e^{2\pi i (\sum_{j=0}^{d} \xi_j) x} d\xi_0 \cdots d\xi_d$$

via time-frequency analysis and how this is related to the decay of the localized Fourier coefficients of m.

For simplicity, we describe the discretization process for the case $d = 1$ (where T_m is a bilinear operator). The cases for larger d's are completely analogues.

Let $\widehat{\phi}$ be a bump function supported in $[-1,1]$ such that $\widehat{\phi} = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $\widehat{\psi}(\xi) = \widehat{\phi}(\xi) - \widehat{\phi}(2\xi)$. For $k \in \mathbb{Z}$, let

$$\widehat{\psi_k}(\xi) = \widehat{\psi}\left(\frac{\xi}{2^k}\right)$$

and

$$\widehat{\phi_k}(\xi) = \widehat{\phi}\left(\frac{\xi}{2^k}\right).$$

Therefore, we have

$$\sum_{l\le k}\widehat{\psi_l}(\xi)=\widehat{\phi_k}(\xi)$$

and

$$1=\sum_{k=-\infty}^{\infty}\widehat{\psi_k}(\xi)\text{ for all }\xi\neq 0.$$

As a function on $\mathbb{R}^2$,

$$\begin{aligned}1&=\left(\sum_{k_0\in\mathbb{Z}}\widehat{\psi_{k_0}}(\xi_0)\right)\left(\sum_{k_1\in\mathbb{Z}}\widehat{\psi_{k_1}}(\xi_1)\right)\\&=\sum_{k_0\ll k_1}\widehat{\psi_{k_0}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)+\sum_{k_0\sim k_1}\widehat{\psi_{k_0}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)+\sum_{k_0\gg k_1}\widehat{\psi_{k_0}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)\\&=\sum_{k_1}\widehat{\phi_{k_1-5}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)+\sum_{k_0\sim k_1}\widehat{\psi_{k_0}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)+\sum_{k_0}\widehat{\psi_{k_0}}(\xi_0)\widehat{\phi_{k_0-5}}(\xi_1)\end{aligned}\tag{2.1}$$

Here $k_0\sim k_1$ means $|k_0-k_1|<K=K(d)$. In this calculate, $d=1$ and $K(1)=5$. We need to choose $K(d)$ such that

$$d2^{-K(d)}\le 2^{-5}.\tag{2.2}$$

This makes sure that in the cases for $d>1$, functions of type $\psi_k\phi_k^d$ still have fourier transform supported in $\left[-\widetilde{c_2}2^k,-\widetilde{c_1}2^k\right]\cup\left[\widetilde{c_1}2^k,\widetilde{c_2}2^k\right]$ for some constants $\widetilde{c_1},\widetilde{c_2}>0$.

Now,

$$\begin{aligned}T_m(a_0,a_1)(x)&=\int_{\mathbb{R}}\int_{\mathbb{R}}m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0d\xi_1\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k_1}\widehat{\phi_{k_1-K}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0d\xi_1\\&\quad+\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k_0\sim k_1}\widehat{\psi_{k_0}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0d\xi_1\\&\quad+\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k_0}\widehat{\psi_{k_0}}(\xi_0)\widehat{\phi_{k_0-K}}(\xi_1)m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0d\xi_1\end{aligned}$$

We will analyze the boundedness of the new operator

$$\widetilde{T_m}(a_0,a_1)(x) := \int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{k_1}\widehat{\phi_{k_1-K}}(\xi_0)\widehat{\psi_{k_1}}(\xi_1)m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0 d\xi_1$$
$$= \sum_k \int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{\phi_{k-K}}(\xi_0)\widehat{\psi_k}(\xi_1)m(\xi_0,\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1)e^{2\pi i(\xi_0+\xi_1)x}d\xi_0 d\xi_1.$$

The other two can be analyzed in the same way.

To simplify notations, we change our notations:

- we will write $\widehat{\phi_k}(\xi_0)$ instead of $\widehat{\phi_{k-K}}(\xi_0)$.
- Note that $\widehat{\psi_k}$ is supported in $[-2^k,-2^{k-2}]\cup[2^{k-2},2^k]$.

Therefore, now we have that $\widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1)$ is a smooth cutoff function for two whitney boxes of size 2^k in the $\xi_0\xi_1$ space. In these two whitney boxes, we can decompose our symbol m into fourier series

$$\begin{aligned}\widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1)m(\xi_0,\xi_1) =& \widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1)\widehat{\phi_k^*}(\xi_0)\widehat{\psi_k^*}(\xi_1)m(\xi_0,\xi_1)\\ =& \widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1)\sum_{n_0,n_1\in\mathbb{Z}}C^k_{n_0,n_1}e^{2\pi i\frac{n_0}{2^{k+1}}\xi_0}e^{2\pi i\frac{n_1}{2^{k+1}}\xi_1}\end{aligned} \tag{2.3}$$

for some functions ϕ_k^*, ψ_k^* such that $\widehat{\phi_k^*}$, $\widehat{\psi_k^*}$ are bump functions which equal to 1 on the supports of $\widehat{\phi_k}$ and $\widehat{\psi_k}$ respectively and the supports of $\widehat{\phi_k^*}$ and $\widehat{\psi_k^*}$ are just slightly larger that those of $\widehat{\phi_k}$ and $\widehat{\psi_k}$.

For simplicity, we will just write the last line of (2.3) as

$$\widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1)\sum_{n_0,n_1\in\mathbb{Z}}C^k_{n_0,n_1}e^{2\pi i\frac{n_0}{2^k}\xi_0}e^{2\pi i\frac{n_1}{2^k}\xi_1}.$$

This simplifies our notation without affect the method of calculation below.

Hence, we can write

$$\begin{aligned}
&\widetilde{T_m}(a_0, a_1)(x)\\
&= \sum_k \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\phi_k}(\xi_0)\widehat{\psi_k}(\xi_1) m(\xi_0, \xi_1) \widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1) e^{2\pi i(\xi_0+\xi_1)x} d\xi_0 d\xi_1\\
&= \sum_{n_0,n_1\in\mathbb{Z}} \sum_k C^k_{n_0,n_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi_k}(\xi_0) e^{2\pi i n_0 \xi_0/2^k} \widehat{\phi_k}(\xi_1) e^{2\pi i n_1 \xi_1/2^k} \widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1) e^{2\pi i(\xi_0+\xi_1)x} d\xi_0 d\xi_1\\
&:= \sum_{n_0,n_1\in\mathbb{Z}} \widetilde{T}_{m,n_0,n_1}(a_0, a_1)(x),
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\widetilde{T}_{m,n_0,n_1}(a_0, a_1)(x)\\
&= \sum_k C^k_{n_0,n_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi_k}(\xi_0) e^{2\pi i n_0 \xi_0/2^k} \widehat{\phi_k}(\xi_1) e^{2\pi i n_1 \xi_1/2^k} \widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1) e^{2\pi i(\xi_0+\xi_1)x} d\xi_0 d\xi_1
\end{aligned}$$

Let us consider

$$\begin{aligned}
\widetilde{T}_{m,0,0}(a_0, a_1)(x) &= \sum_k C^k_{0,0} \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi_k}(\xi_0)\widehat{\phi_k}(\xi_1)\widehat{a_0}(\xi_0)\widehat{a_1}(\xi_1) e^{2\pi i(\xi_0+\xi_1)x} d\xi_0 d\xi_1\\
&= \sum_k C^k_{0,0} (a_0 * \psi_k)(x)(a_1 * \phi_k)(x) \qquad (2.4)\\
&:= \sum_k C^k_{0,0} \left[((a_0 * \psi_k)(x)(a_1 * \phi_k)) * \widetilde{\psi_k}\right](x). \qquad (2.5)
\end{aligned}$$

Here, note that $\widehat{\psi_k}$ is supported in $[2^{k-2}, 2^k]$ and $\widehat{\phi_k}$ is supported in $[-2^{k-5}, 2^{k-5}]$. Hence the fourier transform of $(a_0 * \psi_k)(x)(a_1 * \phi_k)$ is supported in $[c_1 2^k, c_2 2^k]$ for some $c_1, c_2 > 0$. Therefore, we can find some $\widetilde{\psi_k}$ such that $\widehat{\widetilde{\psi_k}}$ is a bump function supported in $[c_1' 2^k, c_2' 2^k]$ for some $c_1', c_2' > 0$ and $\widehat{\widetilde{\psi_k}} \equiv 1$ on $[c_1 2^k, c_2 2^k]$, so that we can complete the product in (2.4) as in (2.5). This is why we had to introduce $K(d)$ in (2.2).

Now we can estimate the L^p norm of $\widetilde{T_{m,0,0}}(a_0, a_1)$ by duality:

$$\begin{aligned}
&\int_{\mathbb{R}} \widetilde{T_{m,0,0}}(a_0, a_1)(x)h(x)\,dx \\
&= \int_{\mathbb{R}} \sum_k C_{0,0}^k \left[((a_0 * \psi_k)(x)(a_1 * \phi_k)) * \widetilde{\psi_k}\right](x)h(x)\,dx \\
&= \sum_k C_{0,0}^k \int_{\mathbb{R}} (a_0 * \psi_k)(x)(a_1 * \phi_k)(x)(h * \widetilde{\widetilde{\psi_k}})(x)\,dx \\
&\text{(where } \widetilde{\widetilde{\psi_k}}(x) = \widetilde{\psi_k}(-x)) \\
&= \sum_k C_{0,0}^k 2^{-k} \int_{\mathbb{R}} (a_0 * \psi_k)(2^{-k}y)(a_1 * \phi_k)(2^{-k}y)(h * \widetilde{\widetilde{\psi_k}}))(2^{-k}y)\,dy \\
&= \sum_k C_{0,0}^k 2^{-k} \sum_{n\in\mathbb{Z}} \int_0^1 (a_0 * \psi_k)(2^{-k}(n+\alpha))(a_1 * \phi_k)(2^{-k}(n+\alpha))(h * \widetilde{\widetilde{\psi_k}}))(2^{-k}(n+\alpha))\,d\alpha
\end{aligned}$$

Now we can write, for functions a and ψ,

$$\begin{aligned}
(a * \psi)(2^{-k}(n+\alpha)) &= \int_{\mathbb{R}} a(y)\psi(2^{-k}(n+\alpha) - y)\,dy \\
&= 2^{\frac{k}{2}} \int_{\mathbb{R}} a(y) 2^{-\frac{k}{2}} \psi(2^{-k}(n+\alpha) - y)\,dy \\
&:= 2^{\frac{k}{2}} \left\langle a, \psi_I^\alpha \right\rangle,
\end{aligned}$$

where ψ_I^α is the L^2-normalized (with respect to k) function

$$\psi_I^\alpha(y) = 2^{-\frac{k}{2}} \overline{\psi(2^{-k}(n+\alpha) - y)}$$

and I is the dyadic interval $[2^{-k}n, 2^{-k}(n+1)]$.

Hence,

$$\begin{aligned}
&\int_{\mathbb{R}} \widetilde{T}_{0,0}(a_0, a_1)(x)h(x)dx \\
=&\Lambda(a_0, a_1, h) \\
=&\sum_k C_{0,0}^k 2^{-k} \int_{\mathbb{R}} (a_0 * \psi_k)(2^{-k}y)(a_1 * \phi_k)(2^{-k}y)(h * \widetilde{\widetilde{\psi_k}})(2^{-k}y)\, dy \\
=&\sum_k C_{0,0}^k 2^{-k} \sum_{n\in\mathbb{Z}} \int_0^1 (a_0 * \psi_k)(2^{-k}(n+\alpha))(a_1 * \phi_k)(2^{-k}(n+\alpha))(h * \widetilde{\widetilde{\psi_k}})(2^{-k}(n+\alpha))\, d\alpha \\
=&\sum_I C_{0,0}^k 2^{-k} \int_0^1 2^{\frac{k}{2}} \left\langle a_0, \psi_I^{0,\alpha} \right\rangle 2^{\frac{k}{2}} \left\langle a_1, \phi_I^{1,\alpha} \right\rangle 2^{\frac{k}{2}} \left\langle h, \psi_I^{2,\alpha} \right\rangle\, d\alpha \\
=&\int_0^1 \sum_I C_{0,0}^k \frac{1}{|I|^{1/2}} \left\langle a_0, \psi_I^{0,\alpha} \right\rangle \left\langle a_1, \phi_I^{1,\alpha} \right\rangle \left\langle h, \psi_I^{2,\alpha} \right\rangle\, d\alpha \\
=&\int_0^1 \int_{\mathbb{R}} \sum_I C_{0,0}^k \frac{1}{|I|^{1/2}} \left\langle a_0, \psi_I^{0,\alpha} \right\rangle \left\langle a_1, \phi_I^{1,\alpha} \right\rangle \overline{\psi_I^{2,\alpha}(x)} h(x) dx d\alpha
\end{aligned}$$

I in the above summation runs through all the dyadic intervals in $\mathbb{R}$. i.e. all intervals in the form $\left[2^k n, 2^k(n+1)\right]$, $k, n \in \mathbb{Z}$. And $|I| = 2^{-k}$.

Hence, we see that

$$\begin{aligned}
\widetilde{T}_{0,0}(a_0, a_1)(x) &= \sum_k C_{0,0}^k \int_{\mathbb{R}} \int_{\mathbb{R}} \widehat{\psi_k}(\xi_1)\widehat{\phi_k}(\xi_2)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2) e^{2\pi i(\xi_1+\xi_2)x} d\xi_1 d\xi_2 \\
&= \int_0^1 \sum_I C_{0,0}^k \frac{1}{|I|^{1/2}} \left\langle a_0, \psi_I^{0,\alpha} \right\rangle \left\langle a_1, \phi_I^{1,\alpha} \right\rangle \overline{\psi_I^{2,\alpha}(x)} d\alpha.
\end{aligned}$$

Now we make some definitions that will be used for a lot of time throughout our discussion:

Definition 2.1.1. *For a dyadic interval I, and $x \in \mathbb{R}$, let*

$$\widetilde{d_I}(x) := \left\langle \frac{d(x, I)}{|I|} \right\rangle$$

and for a set $\Omega \subseteq R$,

$$\widetilde{d_I}(\Omega) := \left\langle \frac{d(\Omega, I)}{|I|} \right\rangle$$

Definition 2.1.2. *Let $\mathbb{I}$ be a family of dyadic intervals. $\{\phi_I\}_{I\in\mathbb{I}}$ is an L^2-normalized family of bump functions strongly adapted to $\mathbb{I}$ if*

$$|I|^{1/2}\left|\frac{d^\beta}{dx^\beta}\phi_I(x)\right| \lesssim \frac{1}{|I|^\beta}\widetilde{d_I}(x)^{-100}$$

for all $I \in \mathbb{I}$ for $\beta = 0, 1$ and

$$supp\ \widehat{\phi_I} \subset \left[c_1\,|I|^{-1}, c_2\,|I|^{-1}\right]$$

for some constants $c_1, c_2 \in \mathbb{R}$.

Definition 2.1.3. *Let $\{\phi_I\}_{I\in\mathbb{I}}$ be an L^2-normalized family of bump functions strongly adapted to $\mathbb{I}$. Then $\{\phi_I\}_{I\in\mathbb{I}}$ is lacunary if*

$$supp\ \widehat{\phi_I} \subset \left[-\widetilde{c_2}\,|I|^{-1}, -\widetilde{c_1}\,|I|^{-1}\right] \cup \left[\widetilde{c_1}\,|I|^{-1}, \widetilde{c_2}\,|I|^{-1}\right]$$

for some constants $\widetilde{c_1}, \widetilde{c_2} > 0$.

Definition 2.1.4. *A $(d+1)$-linear discretized paraproduct is a d-linear operator*

$$\Pi(a_0, \ldots, a_d)(x) = \sum_{I\in\mathbb{I}} C_I \frac{1}{|I|^{\frac{d}{2}}} \prod_{j=0}^{d} \left\langle a_j, \phi_I^j\right\rangle \phi_I^{d+1}(x),$$

where for each $1 \leq j \leq d+1$, $\left\{\phi_I^j\right\}_{I\in\mathbb{I}}$ is an L^2-normalized family of bump functions adapted to $\mathbb{I}$ and at least two of the families in $\left\{\phi_I^1\right\}_{I\in\mathbb{I}}, \ldots, \left\{\phi_I^{d+1}\right\}_{I\in\mathbb{I}}$ are lacunary, whose adaptedness satisfy Defintion 2.1.2 uniformly in α .

Therefore, the expression

$$\Pi^\alpha(a_0, a_1) := \sum_I C_I \frac{1}{|I|^{1/2}} \left\langle a_0, \psi_I^{0,\alpha}\right\rangle \left\langle a_1, \phi_I^{1,\alpha}\right\rangle \psi_I^{2,\alpha}$$

is a discretized paraproduct, with $\left\{\psi_I^{0,\alpha}\right\}_I, \left\{\psi_I^{2,\alpha}\right\}_I$ being the two lacunary families.

Definition 2.1.5. *For a dyadic interval I and $n \in \mathbb{Z}$, let*

$$I_n = I + n|I|$$

be the dyadic interval shifted n steps away from I.

Definition 2.1.6. *Let $n_1, \ldots, n_d \in \mathbb{Z}$. A $(d+1)$-linear shifted discretized paraproduct is a $(d+1)$-linear operator*

$$\Pi(a_0, \ldots, a_d)(x) = \sum_{I \in \mathbb{I}} C_I \frac{1}{|I|^{\frac{d}{2}}} \prod_{j=0}^{d} \left\langle a_j, \phi^j_{I_{n_j}} \right\rangle \phi^{d+1}_I(x),$$

where for each $1 \leq j \leq d+1$, $\left\{\phi^j_I\right\}_{I \in \mathbb{I}}$ is an L^2-normalized family of bump functions adapted to $\mathbb{I}$ and at least two of the families in $\left\{\phi^0_I\right\}_{I \in \mathbb{I}}, \ldots, \left\{\phi^{d+1}_I\right\}_{I \in \mathbb{I}}$ are lacunary.

Similar to the above calculation, we can write

$$\widetilde{T}_{n_0,n_1}(f_1, f_2)(x) = \int_0^1 \sum_I C^k_{n_0,n_1} \frac{1}{|I|^{1/2}} \left\langle f_1, \psi^{1,\alpha}_{I_{n_0}} \right\rangle \left\langle f_2, \phi^{2,\alpha}_{I_{n_1}} \right\rangle \overline{\psi^{3,\alpha}_I(x)} d\alpha$$

Or more generally, for a symbol m on $\mathbb{R}^{d+1}$, we can write T_m into a finite sum of operators of type

$$\widetilde{T} = \sum_{n_0,\ldots,n_d \in \mathbb{Z}} \widetilde{T}_{n_0,\ldots,n_d} \tag{2.6}$$

where

$$\widetilde{T}_{n_0,\ldots,n_d}(a_0, \ldots, a_d)(x) = \int_0^1 \sum_I C^k_{n_0,\ldots,n_d} \frac{1}{|I|^{d/2}} \prod_{j=0}^{d} \left\langle a_j, \psi^{1,\alpha}_{I_{n_0}} \right\rangle \psi^{d+1,\alpha}_I(x) d\alpha \tag{2.7}$$

is an average of $(d+1)$-linear shifted discretized paraproducts with respect to $\alpha \in [0, 1]$. Geometrically , the number

$$C^k_{n_0,\ldots,n_d}$$

is the $(n_0, \ldots, n_d)$-th localized fourier coefficient of m in the kth whitney box in the cone corresponding to $\widetilde{T}$ and k is the number such that $|I| = 2^{-k}$.

2.2 Model Operators

We have the estimate from [16]:

Theorem 2.2.1 (Muscalu, 2012)**.** *If the numbers $|C_I|$ are bounded above by M, then we have the following bound for the Shifted Discretized Paraproduct:*

$$\left\| \sum_{I \in \mathbb{I}} C_I \frac{1}{|I|^{\frac{d}{2}}} \prod_{j=0}^{d} \left\langle a_j, \phi_{I_{n_j}}^j \right\rangle \phi_I^{d+1}(x) \right\|_r \lesssim M \left[\Pi_{j=0}^d \log \left\langle n_j \right\rangle \right] \left[\Pi_{j=0}^d \left\| a_j \right\|_{p_j} \right]$$

for all $1 < p_0, \ldots, p_d \leq \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\sum_{j=0}^{d} \frac{1}{p_j} = \frac{1}{r}$.

Note that the case $n_j = 0$ for all j gives the $(d+1)$-linear boundedness of a $(d+1)$-linear discretized paraproduct.

By the above theorem, (2.6) and (2.7), for a general multiplier m on $\mathbb{R}^{d+1}$, we can write for $1 < p_0, \ldots, p_d \leq \infty$, $\sum_{j=0}^{d} \frac{1}{p_j} = \frac{1}{r}$ and $r \geq 1$,

$$\begin{aligned} \left\| \widetilde{T_m}(a_0, \ldots, a_d) \right\|_r &= \left\| \sum_{n_0, \ldots, n_d} \widetilde{T}_{n_0, \ldots, n_d}(a_1, \ldots, a_d) \right\|_r \\ &\leq \sum_{n_0, \ldots, n_d} \left\| \widetilde{T}_{n_0, \ldots, n_d}(a_1, \ldots, a_d) \right\|_r \\ &\lesssim \sum_{n_0, \ldots, n_d} \sup_k C_{n_0, \ldots, n_d}^k \left[\Pi_{j=0}^d \log \left\langle n_j \right\rangle \right] \left[\Pi_{j=0}^d \left\| a_j \right\|_{p_j} \right] \end{aligned}$$

Again, $C_{n_0, \ldots, n_d}^k$ is the $(n_0, \ldots, n_d)$-th fourier coefficient in the kth whitney box in $\mathbb{R}^{d+1}$ along one direction. In particular, if m is a homogenous of degree 0 (In fact this happens in most natural examples, including the Calderón Commutators), $C_{n_0, \ldots, n_d}^k$ is independent of k.

By the above estimate, we see that for $r \geq 1$ (Since L^r is this case is a normed space), if

$$\sum_{n_0,\ldots,n_d} \sup_k C^k_{n_0,\ldots,n_d} \left[\Pi_{j=0}^d \log \left\langle n_j \right\rangle\right] < \infty, \tag{2.8}$$

then we have

$$\left\|\widetilde{T_m}(a_0,\ldots,a_d)\right\|_r \lesssim \Pi_{j=0}^d \left\|a_j\right\|_{p_j},$$

which is our desired Banach estimate for $\widetilde{T_m}$.

There is a lot more detail for the case $\frac{1}{d+1} < r < 1$ and it will be covered in the next section.

2.3 The New Discretized Paraproduct

We will need to first prove the weak $(p_0,\ldots,p_d)$ bound for the discretized paraproduct and apply multililnear interpolation. We will see the detail below. And we will need the following duality lemma from [23] about estimating weak L^r norms:

Lemma 2.3.1. *Let $1 \geq r > 0$, $K > 0$ and f be a measurable function on $\mathbb{R}$. Then the following are equivalent:*

1. $\|f\|_{r,\infty} \leq K$;
2. *For every measurable set $E \subset \mathbb{R}$, there is a subset $E' \subset E$ such that $|E'| \geq \frac{1}{2}|E|$ and $\int_{E'} |f| \lesssim K|E|^{\frac{1}{r'}}$.*

In this chapter we will prove the following bound of a new type of model operators:

Theorem 2.3.1. *(with Muscalu, 2015) Suppose at least two of the families in* $\left\{\phi^0_{I_n}\right\}_I, \ldots, \left\{\phi^{d+1}_{I_n}\right\}_I$ *are lacunary. For all* $1 < p_0, p_1, \ldots, p_d \le \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$. *For* $\epsilon' > 1 - \frac{1}{p_1'}$ *and any* $\epsilon > 0$, *if* $K(n_0, \ldots, n_d)$ *is a function on* $\mathbb{Z}^{d+1}$ *satisfying*

$$|K(n_0, \ldots, n_d)| \lesssim \langle n_1 \rangle^{-(2-\epsilon')} \langle n_0 - n_1 \rangle^{-(1+\epsilon)} \prod_{j=2}^{d} \left\langle n_j \right\rangle^{-(1+\epsilon)},$$

we have the following bound for the following generalized form of discretized paraproduct:

$$\left\| \sum_I \sum_{n_0, n_1, \ldots, n_d \in \mathbb{Z}} \frac{K(n_0, \ldots, n_d)}{|I|^{d/2}} \prod_{j=0}^{d} \left\langle a_j, \phi^j_{I_{n_j}} \right\rangle \phi^{d+1}_I \right\|_r \lesssim \prod_{j=0}^{d} \left\| a_j \right\|_{p_j}$$

Fix the functions a_j, $0 \le j \le d+1$. for $n \in \mathbb{Z}$, $0 \le j \le d+1$, choose $\gamma^j_n \in \mathbb{C}$ such that

$$\left\langle a_j, \gamma^j_n \phi^j_{I_n} \right\rangle = \langle n \rangle^{-(1+\epsilon)} \left| \left\langle a_j, \phi^j_{I_n} \right\rangle \right|$$

and let

$$\widetilde{\phi}^j_I = \sum_{n \in \mathbb{Z}} \gamma^j_n \phi^j_{I_n}.$$

We know the support of fourier transform of $\widetilde{\phi}^j_I$ is contained in that of ϕ^j and

$$|I|^{\frac{1}{2}} \left| \widetilde{\phi}^j_I(x) \right| \lesssim \widetilde{d_I}(x)^{-(1+\epsilon)}.$$

Therefore, we have:

$$\begin{aligned} &\sum_I \sum_{n_0, n_1, \ldots, n_d \in \mathbb{Z}} \frac{K(n_0, \ldots, n_d)}{|I|^{d/2}} \prod_{j=0}^{d} \left\langle a_j, \phi^j_{I_{n_j}} \right\rangle \phi^{d+1}_I \\ \le &\sum_I \sum_{n_1 \in \mathbb{Z}} \frac{\langle n_1 \rangle^{-(2-\epsilon')}}{|I|^{d/2}} \left| \left\langle a_0, \widetilde{\phi}^0_{I_{n_1}} \right\rangle \right| \left| \left\langle a_1, \phi^1_{I_{n_1}} \right\rangle \right| \prod_{j=2}^{d} \left| \left\langle a_j, \widetilde{\phi}^j_I \right\rangle \right| \left| \left\langle a_{d+1}, \phi^{d+1}_I \right\rangle \right| \\ := &\Lambda(a_0, \ldots, a_{d+1}) \end{aligned}$$

And we have, for any $\epsilon'' > 0$,

$$
\begin{aligned}
&\sum_{n_1\in\mathbb{Z}} \langle n_1\rangle^{-(2-\epsilon')} \left|\left\langle a_0, \widetilde{\phi}^0_{I_{n_1}}\right\rangle\right| \left|\left\langle a_1, \phi^1_{I_{n_1}}\right\rangle\right| \\
&= \sum_{n_1\in\mathbb{Z}} \langle n_1\rangle^{-(1-\epsilon'-\epsilon'')} \left|\left\langle a_0, \langle n_1\rangle^{-(1+\epsilon'')}\widetilde{\phi}^0_{I_{n_1}}\right\rangle\right| \left|\left\langle a_1, \phi^1_{I_n}\right\rangle\right| \\
&= \sum_{k\geq 0}\sum_{|n|\sim 2^k} 2^{-(1-\epsilon'-\epsilon'')} \left|\left\langle a_0, \langle n\rangle^{-(1+\epsilon'')}\widetilde{\phi}^0_{I_n}\right\rangle\right| \left|\left\langle a_1, \phi^1_{I_n}\right\rangle\right| \\
&\leq \sum_{k\geq 0}\sum_{|n|\sim 2^k} 2^{-(1-\epsilon'-\epsilon'')} \left|\left\langle a_0, \langle n\rangle^{-(1+\epsilon'')}\widetilde{\phi}^0_{I_n}\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi^1_{I_n}\right\rangle\right|\right) \\
&:= \sum_{k\geq 0} 2^{-(1-\epsilon'-\epsilon'')k} \left|\left\langle a_0, \widetilde{\widetilde{\phi}}{}^0_I\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi^1_{I_n}\right\rangle\right|\right)
\end{aligned}
$$

In the above calculation, $|n| \sim 2^k$ means $2^{k-1} < |n| \leq 2^k$ for $k \geq 1$ and $|n| = 0$ or 1 for $k = 0$.

Therefore, we can write

$$
\begin{aligned}
&\Lambda(a_0,\ldots,a_{d+1}) \\
&\lesssim \sum_{k\geq 0} 2^{-(1-\epsilon'-\epsilon'')k} \sum_I \frac{1}{|I|^{d/2}} \left|\left\langle a_0, \widetilde{\widetilde{\phi}}{}^0_I\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi^1_{I_n}\right\rangle\right|\right) \prod_{j=2}^{d} \left|\left\langle a_j, \widetilde{\phi}^j_I\right\rangle\right| \left|\left\langle a_{d+1}, \phi^{d+1}_I\right\rangle\right| \\
&:= \sum_{k\geq 0} 2^{-(1-\epsilon'-\epsilon'')k} \Lambda^k(a_0,\ldots,a_{d+1})
\end{aligned}
$$

We will prove the boundedness of Λ^k:

Theorem 2.3.2. *Let $2 > p_0,\ldots,p_d > 1$ and close to 1. and $\frac{1}{r} = \sum_{j=0}^{d}\frac{1}{p_j}$. Then for $\epsilon' < 1 - \frac{1}{p_1}$, (recall that Λ depends on ϵ and ϵ' implicitly), $s > \frac{1}{p_1}$, functions $a_i \in L^{p_i}(\mathbb{R})$ for each $0 \leq i \leq d$ and any measurable subset $E \subseteq \mathbb{R}$, there is a subset $E' \subseteq E$ such that $|E'| \geq \dfrac{|E|}{2}$ and*

$$
\Lambda^k(a_0,\ldots,a_d,\chi_{E'}) \lesssim 2^{ks}\left[\prod_{j=0}^{d} \|a_i\|_{p_i}\right]|E'|^{\frac{1}{r'}}.
$$

With Theorem 2.3.2, we immediately get this new, weaker criteria of the local smoothness of a symbol m to ensure T_m is a bounded multilinear operator:

Theorem 2.3.3. *Let m be a symbol on $\mathbb{R}^{d+1}$. Suppose there exist $\frac{1}{2} > \epsilon_0', \ldots, \epsilon_d' > 0$ and $\epsilon > 0$ such that in any whitney box B, there exists $k \in \{0, \ldots, d\}$ such that ξ_k is never 0 in B and there exists another index k' such that the localized fourier coefficients $c_{n_0,n_1,\ldots,n_d}$ of m satisfies*

$$\left|c_{n_0,n_1,\ldots,n_d}\right| \lesssim \langle n_k \rangle^{-(2-\epsilon_k')} \langle n_{k'} - n_k \rangle^{-(1+\epsilon)} \prod_{j \neq k,k'} \left\langle n_j \right\rangle^{-(1+\epsilon)}. \tag{2.9}$$

Then

$$\|T_m(f, a_1, \ldots, a_d)\|_r \lesssim \|f\|_{p_0} \prod_{j=1}^{d} \left\|a_j\right\|_{p_j}$$

for all $\dfrac{1}{1-\epsilon_j'} < p_j < \infty$ *for* $0 \leq j \leq d$, $\dfrac{1}{d+1} < r < \infty$ *and* $\dfrac{1}{p_0} + \cdots + \dfrac{1}{p_d} = \dfrac{1}{r}$.

Proof. First, for the case $r \geq 1$, since the upper bound in (2.9) is summable in $\mathbb{Z}^{d+1}$, when T_m is decompose into shifted discretized paraproduct, we see that each piece is bounded on $L^{p_0} \times \cdots \times L^{p_d} \to L^r$, using the same argument in (2.8).

For the case $r < 1$, with Theorem 2.3.2, Lemma duality lemma, (2.6), (2.7) and the fact that the exceptional set found in Theorem 2.3.2 depends only on the functions a_j but not the parameter α in (2.7), we get the weak boundedness of T_m arbitrary close to the endpoint estimate $L^{\frac{1}{1-\epsilon_0'}} \times \cdots \times L^{\frac{1}{1-\epsilon_d'}} \to L^t$, where $t^{-1} = \sum_{j=0}^{d} \left(\frac{1}{1-\epsilon_j'}\right)^{-1}$.

Finally, we interpolate between the cases $r \geq 1$ and $r < 1$ and get the desired result. □

In the remaining of this chapter, we prove Theorem 2.3.2.

Note that the families $\{\widetilde{\widetilde{\phi}}_I^0\}_I$, $\left\{\phi_I^1\right\}_I$ and $\left\{\widetilde{\phi}_I^j\right\}_I$ for $2 \leq j \leq d$ are weakly adapted to $\mathbb{I}$ in the following sense:

Definition 2.3.1. *Let $\mathbb{I}$ be a family of dyadic intervals. Then a family of (L^2-normalized) smooth functions $\{\phi_I\}_{I\in\mathbb{I}}$ is weakly adapted to $\mathbb{I}$ if*

$$|I|^{1/2}\left|\frac{d^\beta}{dx^\beta}\phi_I(x)\right| \lesssim \frac{1}{|I|^\beta}\widetilde{d}_I(x)^{-(1+\epsilon)}$$

for some $\epsilon > 0$, for $\beta = 0, 1$.

From now on, we fix $\epsilon > 0$. All the definitions below depends on ϵ:

Here we define the shifted maximal function and shifted square function in [16]:

Definition 2.3.2. *For a dyadic interval I, let*

$$\widetilde{\chi}_I(x) = \widetilde{d}_I(x)^{-(1+\epsilon)} = \left\langle\frac{d(x,I)}{|I|}\right\rangle^{-(1+\epsilon)}.$$

Definition 2.3.3. *For $n \in \mathbb{Z}$, the shifted maximal function M_n is defined by*

$$M_n f(x) := \sup_{x\in I}\frac{1}{|I|}\int_{\mathbb{R}}|f(y)|\widetilde{\chi}_{I_n}(y)dy,$$

where the supremum is taken over all dyadic intervals I containing x.

Definition 2.3.4. *Let $n \in \mathbb{Z}$. Let$\{\phi_I\}_{I\in\mathbb{I}}$ be a family of bump functions weakly adapted to $\mathbb{I}$ and*

$$supp\ \widehat{\phi_I} \subset \left[-\widetilde{c_2}\,|I|^{-1}, -\widetilde{c_1}\,|I|^{-1}\right] \cup \left[\widetilde{c_1}\,|I|^{-1}, \widetilde{c_2}\,|I|^{-1}\right]$$

for some constants $\widetilde{c_1}, \widetilde{c_2} > 0$ (i.e. $\{\phi_I\}_{I\in\mathbb{I}}$ is a lacunary family). Then the shifted square function S^n is given by

$$S^n(f)(x) = \left[\sum_I\frac{\left|\langle f, \phi_{I_n}\rangle\right|^2}{|I|}\chi_I(x)\right]^{\frac{1}{2}}.$$

In [16], we have the following two theorems about the boundedness on $L^p(\mathbb{R})$ of the above shifted maximal function and shifted square function, with the boundedness constants being dependent on n:

Theorem 2.3.4. *For $1 < p \leq \infty$, M^n satisfies the bound*

$$\|M^n f\|_p \lesssim \log \langle n \rangle \|f\|_p .$$

Theorem 2.3.5. *S^n satisfies the weak L^1 bound*

$$\|S^n f\|_{1,\infty} \lesssim \log \langle n \rangle \|f\|_1$$

as well as the L^p bound

$$\|S^n f\|_p \lesssim \log \langle n \rangle \|f\|_p$$

for all $1 < p < \infty$.

We now define naturally a new discretized maximal function :

Definition 2.3.5. *Let*

$$\widetilde{M}^k f(x) := \sup_{|n| \leq 2^k} \sup_{x \in I \in \mathbb{I}} \frac{1}{|I|} \int_{\mathbb{R}} |f| \widetilde{\chi_{I_n}}(x) dx$$

Theorem 2.3.6. *$\widetilde{M}^k$ satisfies the bound*

$$\left\| \widetilde{M}^k f \right\|_p \lesssim 2^{ks} \|f\|_p$$

for any $1 < p \leq \infty$ and $s > \dfrac{1}{p}$.

Now, we define the new discretized square function:

Definition 2.3.6. *Let $\{\phi_I\}_{I \in \mathbb{I}}$ be a family of bump functions weakly adapted to $\mathbb{I}$ and*

$$supp\ \widehat{\phi_I} \subset \left[-\widetilde{c_2} |I|^{-1}, -\widetilde{c_1} |I|^{-1} \right] \cup \left[\widetilde{c_1} |I|^{-1}, \widetilde{c_2} |I|^{-1} \right]$$

for some constants $\widetilde{c_1}, \widetilde{c_2} > 0$.

Let

$$\widetilde{S}^k(f)(x) = \left[\sum_I \frac{\sup_{|n|\sim 2^k} \left|\langle f, \phi_{I_n}\rangle\right|^2}{|I|} \chi_I(x)\right]^{\frac{1}{2}}$$

Theorem 2.3.7. $\widetilde{S}^k$ *satisfies the bounds*

$$\left\|\widetilde{S}^k f\right\|_{1,\infty} \lesssim 2^k \|f\|_1$$

and

$$\left\|\widetilde{S}^k f\right\|_p \lesssim 2^{\frac{k}{p}} \|f\|_p$$

for any $1 < p \leq 2$.

2.3.1 Boundedness of the Maximal Function

In this subsection, we prove Theorem 2.3.6.

From [16], we know that the shifted maximal operator M_n satisfies the bound

$$\|M_n f\|_p \lesssim \log\langle n\rangle \|f\|_p$$

for $1 < p < \infty$.

Back to the original maximal operator, we let $p+ > 1$ be close to 1. Then we have

$$\begin{aligned}\left\|\widetilde{M}^k f(x)\right\|_{p+} &= \left\|\sup_{|n|\le 2^k}\sup_{x\in I\in\mathbb{I}}\frac{1}{|I|}\int_{\mathbb{R}}|f(y)|\widetilde{\chi_{I_n}}(y)dy\right\|_{p+}\\ &\le \sum_{n\le 2^k}\left\|\sup_{x\in I\in\mathbb{I}}\frac{1}{|I|}\int_{\mathbb{R}}|f(y)|\widetilde{\chi_{I_n}}(y)dy\right\|_{p+}\\ &\le \sum_{n\le 2^k}\|M_n f(x)\|_{p+}\\ &\lesssim 2^k\log\left\langle 2^k\right\rangle\|f\|_{p+}\\ &\lesssim 2^{(1+)k}\|f\|_{p+}.\end{aligned}$$

By interpolation with the trivial L^∞ bound

$$\left\|\widetilde{M}^k f\right\|_\infty \lesssim \|f\|_\infty,$$

we get

$$\left\|M^k f\right\|_q \lesssim 2^{\frac{(p+)(1+)k}{q}}\|f\|_q,$$

for $p+ < q \le \infty$.

Therefore, letting $p+$, $1+$ be close to 1, we get the desired bound in Theorem 2.3.6.

2.3.2 Boundedness of the Square Function

In this subsection, we prove Theorem 2.3.7.

we first prove the bound for $p = 2$.

$$\left\|\widetilde{S}^k(f)\right\|_2^2 = \sum_{I\in\mathbb{I}} \sup_{|n|\sim 2^k} \left|\langle f, \phi_{I_n}\rangle\right|^2$$
$$\leq \sum_{I\in\mathbb{D}} \sum_{|n|\sim 2^k} \left|\langle f, \phi_{I_n}\rangle\right|^2$$
$$\lesssim 2^k \sum_{I\in\mathbb{D}} |\langle f, \phi_I\rangle|^2 ,$$

where $\mathbb{D}$ is the set of all dyadic intervals in $\mathbb{R}$.

Now, it suffices to show that

$$\sum_{I\in\mathbb{D}} |\langle f, \phi_I\rangle|^2 \lesssim \|f\|_2^2$$

for any $f \in L^2(\mathbb{R})$.

Since the family $\{\phi_I\}_{I\in\mathbb{I}}$ is lacunary, we can assume $\widehat{f}$ is supported in $[-\widetilde{c_2}2^{-t}, -\widetilde{c_1}2^{-t}] \cup [\widetilde{c_1}2^{-t}, \widetilde{c_2}2^{-t}]$ for some $t \in \mathbb{Z}$ and prove the inequality

$$\sum_{|I|=2^t} |\langle f, \phi_I\rangle|^2 \lesssim \|f\|_2^2 \tag{2.10}$$

The left hand side of (2.10) is

$$\sum_{|I|=2^t} |\langle f, \phi_I\rangle|^2 = \sum_{|I|=2^t} \langle f, \phi_I\rangle \overline{\langle f, \phi_I\rangle}$$
$$= \left\langle f, \sum_{|I|=2^t} \langle f, \phi_I\rangle \phi_I \right\rangle$$
$$\leq \|f\|_2 \left\| \sum_{|I|=2^t} \langle f, \phi_I\rangle \phi_I \right\|_2$$

Therefore, to show (2.10), it suffices to show

$$\left\| \sum_{|I|=2^t} \langle f, \phi_I\rangle \phi_I \right\|_2 \lesssim \left[\sum_{|I|=2^t} |\langle f, \phi_I\rangle|^2 \right]^{\frac{1}{2}} , \tag{2.11}$$

which is equivalent to

$$\sum_{|I|,|J|=2^t} \langle f, \phi_I \rangle \overline{\langle f, \phi_J \rangle} \langle \phi_I, \phi_J \rangle \lesssim \sum_{|I|=2^t} |\langle f, \phi_I \rangle|^2 . \tag{2.12}$$

Now, we have

$$\begin{aligned}
\sum_{|I|,|J|=2^t} \langle f, \phi_I \rangle \overline{\langle f, \phi_J \rangle} \langle \phi_I, \phi_J \rangle &= \sum_{n\in\mathbb{Z}} \sum_{|I|=2^t} \langle f, \phi_I \rangle \overline{\langle f, \phi_{I_n} \rangle} \langle \phi_I, \phi_{I_n} \rangle \\
&\lesssim \sum_{n\in\mathbb{Z}} \sum_{|I|=2^t} \left| \langle f, \phi_I \rangle \overline{\langle f, \phi_{I_n} \rangle} \right| \langle n \rangle^{-(1+\epsilon)} \\
&\lesssim \sum_{n\in\mathbb{Z}} \sum_{|I|=2^t} \left(|\langle f, \phi_I \rangle|^2 + \left| \langle f, \phi_{I_n} \rangle \right|^2 \right) \langle n \rangle^{-(1+\epsilon)} \\
&\lesssim \sum_{|I|=2^t} |\langle f, \phi_I \rangle|^2
\end{aligned}$$

and this ends the proof of the bound for the case $p = 2$.

Now, we will prove the weak L^1 bound of $\widetilde{S}^k$. Fix $f \in L^1(\mathbb{R})$ and $\lambda > 0$. We want to estimate the size of the set

$$\left\{ x \in \mathbb{R} : \left| \widetilde{S}^k(f)(x) \right| > \lambda \right\}.$$

We apply a standard Calderón-Zygmund decomposition on f:

Let $\mathbb{J}$ be the family of maximal dyadic intervals J with respect to inclusion such that

$$\frac{1}{|J|} \int_J |f(x)|\, dx > \lambda$$

and

$$\Omega = \bigcup_{J\in\mathbb{J}} J = \{ x \in \mathbb{R} : Mf(x) > \lambda \} .$$

Then we have

$$|\Omega| = \sum_{J\in\mathbb{J}} |J| < \sum_{J\in\mathbb{J}} \frac{1}{\lambda} \int_J |f(x)|\, dx \leq \frac{1}{\lambda} \|f\|_1 .$$

Let

$$g = f\chi_{\Omega^c} + \sum_{J\in\mathbb{J}} \left(\frac{1}{|J|}\int_J f(x)dx\right)\chi_J$$

be the "good part" of f.

For $J \in \mathbb{J}$,

$$b_J = \left(f - \frac{1}{|J|}\int_J f(x)dx\right)\chi_J.$$

Finally, let

$$b = \sum_{J\in\mathbb{J}} b_J$$

be the "bad part" of f.

we have the following properties for the Calderón-Zygmund decomposition:

- $f = g + b$;
- $\|g\|_\infty \lesssim \lambda$ (because of the maximal property of each J);
- $\|g\|_1 \lesssim \|f\|_1$;
- supp $b_J \subseteq J$;
- $\int_J b_J(x)dx = 0$ for each $J \in \mathbb{J}$;
- $\|b_J\|_1 \lesssim \lambda\,|J|$.

Since $f = g + b$, we have

$$\widetilde{S}^k(f)(x) \leq \widetilde{S}^k(g)(x) + \widetilde{S}^k(b)(x)$$

and hence

$$\begin{aligned}\left\{x\in\mathbb{R} : \left|\widetilde{S}^k(f)(x)\right| > \lambda\right\} &\subseteq \left\{x\in\mathbb{R} : \left|\widetilde{S}^k(g)(x)\right| > \frac{\lambda}{2}\right\} \cup \left\{x\in\mathbb{R} : \left|\widetilde{S}^k(b)(x)\right| > \frac{\lambda}{2}\right\}\\ &:= E^g_\lambda \cup E^b_\lambda.\end{aligned}$$

$\left|E_\lambda^g\right|$ is easy to estimate since

$$\left(\frac{\lambda}{2}\right)^2 \left|E_\lambda^g\right| \leq \left\|\widetilde{S}^k(g)\right\|_2^2 \lesssim 2^k \|g\|_2^2 \lesssim 2^k \|g\|_1 \|g\|_\infty \lesssim 2^k \lambda \|g\|_1$$

and hence

$$\left|E_\lambda^g\right| \lesssim \frac{2^k}{\lambda} \|g\|_1 \lesssim \frac{2^k}{\lambda} \|f\|_1 .$$

Therefore, to prove the weak L^1 bound for $\widetilde{S}^k$, it now suffices to show

$$\left|E_\lambda^b\right| \lesssim \frac{2^k}{\lambda} \|f\|_1 .$$

For each $J \in \mathbb{J}$, let

$$\widetilde{J} = (2^{k+2} + 10)J,$$

the interval with the same center as J whose length is $(2^{k+2} + 10)$ times of that of J and

$$\widetilde{\Omega} = \bigcup_{J \in \mathbb{J}} \widetilde{J}.$$

So we have

$$\left|\widetilde{\Omega}\right| \lesssim 2^k |\Omega| \lesssim \frac{2^k}{\lambda} \|f\|_1 .$$

Hence, now it remains to show

$$\left|E_\lambda^b \setminus \widetilde{\Omega}\right| \lesssim \frac{2^k}{\lambda} \|f\|_1 . \tag{2.13}$$

Note the for any function h, we have

$$\widetilde{S}^k(h)(x) \leq \left[\sum_I \frac{\sum_{|n|\sim 2^k} \left|\langle h, \phi_{I_n}\rangle\right|^2}{|I|} \chi_I(x)\right]^{\frac{1}{2}} \leq \sum_I \sum_{|n|\sim 2^k} \frac{\left|\langle h, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)$$

and we can write

$$\begin{aligned}\int_{E_\lambda^b \setminus \widetilde{\Omega}} \widetilde{S}^k(b)(x)dx &\leq \sum_{J\in\mathbb{J}} \int_{E_\lambda^b \setminus \widetilde{\Omega}} \widetilde{S}^k(b_J)(x)dx \\ &\lesssim \sum_{J\in\mathbb{J}} \int_{\mathbb{R}\setminus\widetilde{J}} \widetilde{S}^k(b_J)(x)dx \\ &\lesssim \sum_{J\in\mathbb{J}} \sum_{I} \sum_{|n|\sim 2^k} \int_{\mathbb{R}\setminus\widetilde{J}} \frac{\left|\langle b_J, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)dx. \end{aligned} \tag{2.14}$$

For each $J \in \mathbb{J}$, we consider the corresponding term in (2.14). Here we have two cases:

Case 1: $|I| \leq |J|$. For I so that

$$\int_{\mathbb{R}\setminus\widetilde{J}} \frac{\left|\langle b_J, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)dx \tag{2.15}$$

is non-trivial, we must have $I \cap \left(\mathbb{R} \setminus \widetilde{J}\right) \neq \emptyset$. For the definition of $\widetilde{J}$ and $|I| \leq |J|$, we know that for $|n| \sim 2^k$,

$$\widetilde{d}_J(I_n) \geq 5.$$

We can estimate (2.15) by

$$\begin{aligned}\int_{\mathbb{R}\setminus\widetilde{J}} \frac{\left|\langle b_J, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)dx &\lesssim \left|\left\langle b_J, |I|^{\frac{1}{2}}\, \phi_{I_n}\right\rangle\right| \\ &\lesssim \int_{\mathbb{R}} |b|_J\, \widetilde{d}_{I_n}(x)^{-(1+\epsilon)}dx \\ &\lesssim \|b_J\|_1\, \widetilde{d}_{I_n}(J)^{-(1+\epsilon)}\end{aligned}$$

and note that when we sum over all dyadic intervals L such that $|L| \leq |J|$ and $\widetilde{d}_J(L) \geq 5$, we have

$$\sum_L \widetilde{d}_L(J)^{-(1+\epsilon)} \lesssim 1.$$

Since each such L can be represented as I_n in the summation $\sum_I \sum_{|n|\sim 2^k}$ for at most $\lesssim 2^k$ times, we see

$$\sum_{J\in\mathbb{J}}\sum_{|I|\le|J|}\sum_{|n|\sim 2^k}\int_{\mathbb{R}\setminus\widetilde{J}}\frac{\left|\left\langle b_J,\phi_{I_n}\right\rangle\right|}{|I|^{\frac{1}{2}}}\chi_I(x)dx \lesssim \sum_{J\in\mathbb{J}} 2^k\,\|b_J\|_1$$
$$\lesssim \sum_{J\in\mathbb{J}} 2^k\lambda\,|J|$$
$$\lesssim 2^k\lambda\,|\Omega|$$
$$\lesssim 2^k\,\|f\|_1\,.$$

Case 2: $|I| > |J|$. Again, we can write

$$\int_{\mathbb{R}\setminus\widetilde{J}}\frac{\left|\left\langle b_J,\phi_{I_n}\right\rangle\right|}{|I|^{\frac{1}{2}}}\chi_I(x)dx \lesssim \left|\left\langle b_J,|I|^{\frac{1}{2}}\,\phi_{I_n}\right\rangle\right|$$

Using the face that

$$\int_{\mathbb{R}} b_J(x)dx = 0$$

and denoting the central point of J by c_J, we see

$$\left|\left\langle b_J,|I|^{\frac{1}{2}}\,\phi_{I_n}\right\rangle\right| = \left|\int_{\mathbb{R}} b_J(x)\overline{|I|^{\frac{1}{2}}\,\phi_{I_n}(x)}dx\right|$$
$$= \left|\int_{\mathbb{R}} b_J(x)\overline{|I|^{\frac{1}{2}}\,[\phi_{I_n}(x)-\phi_{I_n}(c_J)]}dx\right|$$

Using mean-value theorem and the weak decay of ϕ'_{I_n}, we have

$$\left||I|^{\frac{1}{2}}\,[\phi_{I_n}(x)-\phi_{I_n}(c_J)]\right| \lesssim \frac{|x-c_J|}{|I_n|}\widetilde{d}_{I_n}(J)^{-(1+\epsilon)}$$

But since b_J is supported in J, we know $x\in J$ in order for the integrand to be non-zero. Therefore, we have

$$|x-c_J| \le \frac{|J|}{2}$$

and hence

$$\left|\left\langle b_J,|I|^{\frac{1}{2}}\,\phi_{I_n}\right\rangle\right| \lesssim \left||I|^{\frac{1}{2}}\,[\phi_{I_n}(x)-\phi_{I_n}(c_J)]\right|\|b_J\|_1 \lesssim \frac{|J|}{|I_n|}\widetilde{d}_{I_n}(J)^{-(1+\epsilon)}\,\|b_J\|_1\,.$$

Again, for each dyadic L, L can be represented as I_n in the summation $\sum_I \sum_{|n|\sim 2^k}$ for at most $\lesssim 2^k$ times. Hence, we have

$$\sum_{J\in\mathbb{J}} \sum_{|I|>|J|} \sum_{|n|\sim 2^k} \int_{\mathbb{R}\setminus\widetilde{J}} \frac{\left|\langle b_J, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)dx \lesssim \sum_{J\in\mathbb{J}} 2^k \|b_J\|_1 \sum_{|L|>|J|} \frac{|J|}{|L|} \widetilde{d}_L(J)^{-(1+\epsilon)}$$

Now observe that

$$\sum_{|L|>|J|} \frac{|J|}{|L|} \widetilde{d}_L(J)^{-(1+\epsilon)} \lesssim 1.$$

Therefore, we have

$$\begin{aligned}
\sum_{J\in\mathbb{J}} \sum_{|I|>|J|} \sum_{|n|\sim 2^k} \int_{\mathbb{R}\setminus\widetilde{J}} \frac{\left|\langle b_J, \phi_{I_n}\rangle\right|}{|I|^{\frac{1}{2}}} \chi_I(x)dx &\lesssim \sum_{J\in\mathbb{J}} 2^k \|b_J\|_1 \\
&\lesssim \sum_{J\in\mathbb{J}} 2^k \lambda |J| \\
&= 2^k \lambda |\Omega| \\
&\lesssim 2^k \|f\|_1
\end{aligned}$$

Combining the above two cases, we see, from (2.14), that

$$\int_{E^b_\lambda\setminus\widetilde{\Omega}} \widetilde{S}^k(b)(x)dx \lesssim 2^k \|f\|_1 .$$

Hence, we have

$$\left|E^b_\lambda \setminus \widetilde{\Omega}\right| \lesssim \frac{2^k}{\lambda} \|f\|_1$$

and this ends our proof of the weak L^1 bound of $\widetilde{S}^k$:

$$\left\|\widetilde{S}^k(f)\right\|_{1,\infty} \lesssim 2^k \|f\|_1 .$$

By interpolating between the L^2 bound and weak L^1 bound of $\widetilde{S}^k$, we get the conclusion of Theorem 2.3.7.

2.3.3 Boundedness of the New Discretized Paraproduct

In this subsection, using the boundedness on the maximal and square functions established in the previous two subsections, we prove Theorem 2.3.2, the weak boundedness of

$$\Lambda^k(a_0,\ldots,a_{d+1}) = \sum_{I\in\mathbb{I}} \frac{1}{|I|^{d/2}} \left|\left\langle a_0,\phi_I^0\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1,\phi_{I_n}^1\right\rangle\right|\right) \prod_{j=2}^{d} \left|\left\langle a_j,\phi_I^j\right\rangle\right| \left|\left\langle a_{d+1},\phi_I^{d+1}\right\rangle\right|,$$

where we have assumed $\mathbb{I}$ is a family of dyadic intervals, the families $\left\{\phi_I^j\right\}_I$ for $0\le j\le n$ are weakly adapted to $\mathbb{I}$ and $\left\{\phi_I^{d+1}\right\}_I$ is strong adapted to $\mathbb{I}$, and the two families $\left\{\phi_I^2\right\}_I$ and $\left\{\phi_I^d\right\}_I$ are lacunary:

Theorem 2.3.2. *Let $2 > p_0,\ldots,p_d > 1$ and close to 1. and $\frac{1}{r} = \sum_{j=0}^d \frac{1}{p_j}$. Then for $\epsilon' < 1-\frac{1}{p_1}$, (recall that Λ depends on ϵ and ϵ' implicitly), $s > \frac{1}{p_1}$, functions $a_i \in L^{p_i}(\mathbb{R})$ for each $0\le i\le d$ and any measurable subset $E\subseteq\mathbb{R}$, there is a subset $E'\subseteq E$ such that $|E'| \ge \dfrac{|E|}{2}$ and*

$$\Lambda^k(a_0,\ldots,a_d,\chi_{E'}) \lesssim 2^{ks}\left[\prod_{j=0}^{d}\|a_i\|_{p_i}\right]|E'|^{\frac{1}{r'}}.$$

Proof. We will prove the theorem for the case $d=2$ and it will be clear that the same proof works for all the cases $d\ge 2$.

Because the terms in the expression are all positive, we can assume our family $\mathbb{I}$ is finite. We will prove the bound in this case and it will be clear that the implicit constants do not depend on the cardinality of $\mathbb{I}$. Therefore, the result for the case $\mathbb{I}$ being finite implies the original case where $\mathbb{I}$ is the family of all dyadic intervals.

Because the expression is dilation invariant, we can without loss of generality assume that $|E| = 1$. Hence, we need to find $E' \subseteq E$, $|E'| \geq \frac{1}{2}$ such that

$$\sum_I \frac{1}{|I|} \left|\left\langle a_0, \phi_I^0 \right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1 \right\rangle\right| \right) \left|\left\langle a_2, \phi_I^2 \right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3 \right\rangle\right| \lesssim 2^{ks}.$$

Let

$$\Omega' := \{\, x \in \mathbb{R} : Ma_0(x) > C \,\} \cup \left\{\, x \in \mathbb{R} : \widetilde{S}^k a_1(x) > 2^{ks} C \,\right\} \cup \{\, x \in \mathbb{R} : S\, a_2(x) > C \,\},$$

for some $C > 0$, where M is the Hardy-Littlewood maximal function and $S = S^0$ is the square function S^n with $n = 0$ defined in Definition 2.3.4.

For $d \in \mathbb{N}$, let

$$\Omega_0^d := \left\{\, x \in \mathbb{R} : \exists\, 2^d \leq |m| \leq 2^{d+1}, M^m(a_0)(x) > C 2^{5d} \log \langle m \rangle \,\right\}$$

and

$$\Omega_0 := \bigcup_{d \geq 0} \Omega_0^d.$$

Similarly, let

$$\Omega_2^d := \left\{\, x \in \mathbb{R} : \exists\, 2^d \leq |m| \leq 2^{d+1}, M^m(a_2)(x) > C 2^{5d} \log \langle m \rangle \,\right\}$$

and

$$\Omega_2 := \bigcup_{d \geq 0} \Omega_2^d.$$

We also need to define a similar exceptional set for a_1.

Let

$$\Omega_1^d := \left\{\, x \in \mathbb{R} : \widetilde{M}^{\max(k,d)+1}(a_1)(x) > C 2^{(\max(k,d)+1)s} 2^{5d} \,\right\}$$

and

$$\Omega_1 := \bigcup_{d \geq 0} \Omega_1^d.$$

The purpose of defining these exceptional sets will be clear later in the proof.

Now, define

$$\widetilde{\Omega} = \Omega' \cup \Omega_0 \cup \Omega_1 \cup \Omega_2.$$

and

$$\Omega := \left\{ x \in \mathbb{R} : M(\chi_{\widetilde{\Omega}})(x) > \frac{1}{100} \right\}.$$

By the boundedness of M, M^m, $\widetilde{M}^k$ and $\widetilde{S}^k$, when C is chosen to be large enough, $|\Omega| < \frac{1}{2}$. Let

$$E' := E \backslash \Omega.$$

Hence, we have

$$|E'| \geq \frac{1}{2}.$$

Now we break the set of dyadic intervals $\mathbb{I}$ into

$$\mathbb{I} = \mathbb{I}_1 \cup \mathbb{I}_2,$$

where

$$\mathbb{I}_1 = \left\{ I \in \mathbb{I} : I \nsubseteq \Omega \right\}$$

and

$$\mathbb{I}_2 = \{ I \in \mathbb{I} : I \subseteq \Omega \}$$

Now, we break the sum

$$\sum_{I \in \mathbb{I}} \frac{1}{|I|} \left| \left\langle a_0, \phi_I^0 \right\rangle \right| \left(\sup_{|n| \sim 2^k} \left| \left\langle a_1, \phi_{I_n}^1 \right\rangle \right| \right) \left| \left\langle a_2, \phi_I^2 \right\rangle \right| \left| \left\langle \chi_{E'}, \phi_I^3 \right\rangle \right| = \sum_{\mathbb{I}_1} + \sum_{\mathbb{I}_2}$$

First, we estimate $\sum_{\mathbb{I}_1}$. Observe that $I \nsubseteq \Omega$ means there exists $x \in I$ such that $x \notin \Omega$, which means

$$\left| I \cap \widetilde{\Omega} \right| \leq \frac{|I|}{100}.$$

Now, let

$$\Gamma_0^1 = \left\{ x \in \mathbb{R} : Ma_0(x) > \frac{C}{2^1} \right\}$$

and

$$\mathbb{J}_0^1 = \left\{ I \in \mathbb{I}_1 : \left| I \cap \Gamma_0^1 \right| > \frac{|I|}{100} \right\},$$

and then

$$\Gamma_0^2 = \left\{ x \in \mathbb{R} : Ma_0(x) > \frac{C}{2^2} \right\}$$

and

$$\mathbb{J}_0^2 = \left\{ I \in \mathbb{I}_1 \backslash \mathbb{J}_0^1 : \left| I \cap \Gamma_0^2 \right| > \frac{|I|}{100} \right\}.$$

We repeat this process so that for each $m > 2$,

$$\Gamma_0^m = \left\{ x \in \mathbb{R} : Ma_0(x) > \frac{C}{2^m} \right\}$$

and

$$\mathbb{J}_0^m = \left\{ I \in \mathbb{I}_1 \backslash (\mathbb{J}_0^1 \cup \cdots \cup \mathbb{J}_0^{m-1}) : \left| I \cap \Gamma_0^m \right| > \frac{|I|}{100} \right\},$$

until we run out of intervals in $\mathbb{I}_1$.

We define a similar partition of $\mathbb{I}_1$ for a_1, described by $\widetilde{S}^k a_1$:

let

$$\Gamma_1^1 = \left\{ x \in \mathbb{R} : \widetilde{S}^k a_1(x) > \frac{2^{ks} C}{2^1} \right\}$$

and

$$\mathbb{J}_1^1 = \left\{ I \in \mathbb{I}_1 : \left| I \cap \Gamma_1^1 \right| > \frac{|I|}{100} \right\},$$

and then

$$\Gamma_1^2 = \left\{ x \in \mathbb{R} : \widetilde{S}^k a_1(x) > \frac{2^{ks} C}{2^2} \right\}$$

and

$$\mathbb{J}_1^2 = \left\{ I \in \mathbb{I}_1 \backslash \mathbb{J}_1^1 : \left| I \cap \Gamma_1^2 \right| > \frac{|I|}{100} \right\}.$$

We repeat this process so that for each $m > 2$,

$$\Gamma_1^m = \left\{ x \in \mathbb{R} : \widetilde{S}^k a_1(x) > \frac{2^{ks} C}{2^m} \right\}$$

and

$$\mathbb{J}_1^m = \left\{ I \in \mathbb{I}_1 \backslash (\mathbb{J}_1^1 \cup \cdots \cup \mathbb{J}_1^{m-1}) : \left| I \cap \Gamma_1^m \right| > \frac{|I|}{100} \right\},$$

again, until we run out of intervals in $\mathbb{I}_1$.

We further define another similar partition of $\mathbb{I}_1$ for a_2, described by $S a_2$ (We are considering $S a_2$ instead of $M a_2$ because the family $\left\{ \phi_I^2 \right\}_I$ is assumed to be lacunary):

$$\Gamma_2^1 = \left\{ x \in \mathbb{R} : S a_2(x) > \frac{C}{2^1} \right\};$$

$$\mathbb{J}_2^1 = \left\{ I \in \mathbb{I}_1 : \left| I \cap \Gamma_2^1 \right| > \frac{|I|}{100} \right\};$$

$$\Gamma_2^2 = \left\{ x \in \mathbb{R} : S a_2(x) > \frac{C}{2^2} \right\};$$

$$\mathbb{J}_2^2 = \left\{ I \in \mathbb{I}_1 \backslash \mathbb{J}_2^1 : \left| I \cap \Gamma_1^2 \right| > \frac{|I|}{100} \right\}.$$

For each $m > 2$,

$$\Gamma_2^m = \left\{ x \in \mathbb{R} : S a_2(x) > \frac{C}{2^m} \right\}$$

and

$$\mathbb{J}_2^m = \left\{ I \in \mathbb{I}_1 \backslash (\mathbb{J}_2^1 \cup \cdots \cup \mathbb{J}_2^{m-1}) : \left| I \cap \Gamma_2^m \right| > \frac{|I|}{100} \right\}.$$

Finally, we also need a partition of $\mathbb{I}_1$ described by $M\chi_{E'}$.

For $n \in \mathbb{Z}$, let

$$\Gamma_3^n = \left\{ x \in \mathbb{R} : M\chi_{E'}(x) > \frac{C}{2^n} \right\}.$$

Since the family $\mathbb{I}$ is finite, we can find N small enough such that for all $I \in \mathbb{I}_1$,

$$\left| I \cap \Gamma_3^N \right| \leq \frac{|I|}{100}.$$

Then we can use the same method to define

$$\Gamma_3^{N-1} = \left\{ x \in \mathbb{R} : M\chi_{E'}(x) > \frac{C}{2^{N-1}} \right\};$$

$$\mathbb{J}_3^{N-1} = \left\{ I \in \mathbb{I}_1 : \left| I \cap \Gamma_3^{N-1} \right| > \frac{|I|}{100} \right\}$$

and for all $m > N-1$,

$$\Gamma_3^m = \left\{ x \in \mathbb{R} : M\chi_{E'}(x) > \frac{C}{2^m} \right\}$$

and

$$\mathbb{J}_3^m = \left\{ I \in \mathbb{I}_1 \backslash (\mathbb{J}_3^1 \cup \cdots \cup \mathbb{J}_3^{m-1}) : \left| I \cap \Gamma_3^m \right| > \frac{|I|}{100} \right\}.$$

Let

$$\mathbb{J}_{m_0,m_1,m_2,m_3} = \mathbb{J}_0^{m_0} \cap \mathbb{J}_1^{m_1} \cap \mathbb{J}_2^{m_2} \cap \mathbb{J}_3^{m_3}.$$

Now, we can write

$$\sum_{I \in \mathbb{I}_1} \frac{1}{|I|} \left| \left\langle a_0, \phi_I^0 \right\rangle \right| \left(\sup_{|n| \sim 2^k} \left| \left\langle a_1, \phi_{I_n}^1 \right\rangle \right| \right) \left| \left\langle a_2, \phi_I^2 \right\rangle \right| \left| \left\langle \chi_{E'}, \phi_I^3 \right\rangle \right|$$

$$= \sum_{m_0, m_1, m_2 > 0, m_3 > N} \sum_{I \in \mathbb{J}_{m_1, m_1, m_2, m_3}} \frac{1}{|I|} \left| \left\langle a_0, \phi_I^0 \right\rangle \right| \left(\sup_{|n| \sim 2^k} \left| \left\langle a_1, \phi_{I_n}^1 \right\rangle \right| \right) \left| \left\langle a_2, \phi_I^2 \right\rangle \right| \left| \left\langle \chi_{E'}, \phi_I^3 \right\rangle \right|.$$

Now, observe that for $I \in \mathbb{J}_j^m$, we have

$$\left| I \cap \Gamma_j^{m-1} \right| \leq \frac{|I|}{100}$$

(otherwise I would be chosen in $I \in \mathbb{J}_j^{n'}$ for some $m' < m$) and therefore

$$\left| I \cap \left(\Gamma_j^{m-1} \right)^c \right| \geq \frac{99}{100} |I|.$$

So, we see that for $I \in \mathbb{J}_{m_0,m_1,m_2,m_3}$,

$$\left| I \cap \left(\Gamma_{m_0-1,m_1-1,m_2-1,m_3-1}^c \right)^c \right| \geq \frac{96}{100} |I|,$$

where

$$\Gamma^c_{m_0-1,m_1-1,m_2-1,m_3-1} = (\Gamma_1^{m_1-1})^c \cap (\Gamma_2^{m_2-1})^c \cap (\Gamma_3^{m_3-1})^c \cap (\Gamma_4^{m_4-1})^c.$$

Hence, we have

$$\begin{aligned}
&\sum_{\mathbb{I}_1} \\
&= \sum_{m_0,m_1,m_2>0,m_3>N} \sum_{I\in\mathbb{J}_{m_0,m_1,m_2,m_3}} \frac{1}{|I|} \left|\left\langle a_0, \phi_I^0\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1\right\rangle\right|\right) \left|\left\langle a_2, \phi_I^2\right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3\right\rangle\right| \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} \sum_{I\in\mathbb{J}_{m_0,m_1,m_2,m_3}} \frac{1}{|I|^2} \left|\left\langle a_0, \phi_I^0\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1\right\rangle\right|\right) \left|\left\langle a_2, \phi_I^2\right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3\right\rangle\right| |I| \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} \sum_{I\in\mathbb{J}_{m_0,m_1,m_2,m_3}} \frac{1}{|I|^2} \left|\left\langle a_0, \phi_I^0\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1\right\rangle\right|\right) \\
&\quad \left|\left\langle a_2, \phi_I^2\right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3\right\rangle\right| \left|I \cap \Gamma_{m_0-1,m_1-1,m_2-1,m_3-1}\right| \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} \int_{\Gamma^c_{m_0-1,m_1-1,m_2-1,m_3-1}} \sum_{I\in\mathbb{J}_{m_0,m_1,m_2,m_3}} \frac{1}{|I|^2} \left|\left\langle a_0, \phi_I^0\right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1\right\rangle\right|\right) \\
&\quad \left|\left\langle a_2, \phi_I^2\right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3\right\rangle\right| dx \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} \int_{\Gamma^c_{m_0-1,m_1-1,m_2-1,m_3-1}} \sum_{I\in\mathbb{J}_{m_0,m_1,m_2,m_3}} \frac{\left|\left\langle a_0, \phi_I^0\right\rangle\right|}{|I|^{1/2}} \frac{\left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1\right\rangle\right|\right)}{|I|^{1/2}} \\
&\quad \frac{\left|\left\langle a_2, \phi_I^2\right\rangle\right|}{|I|^{1/2}} \frac{\left|\left\langle \chi_{E'}, \phi_I^3\right\rangle\right|}{|I|^{1/2}} \chi_I(x) dx \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} \int_{\Gamma^c_{m_0-1,m_1-1,m_2-1,m_3-1} \cap \bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}} Ma_0(x) \widetilde{S}^k a_1(x) S a_2(x) M\chi_{E'}(x) dx \\
&\lesssim \sum_{m_0,m_1,m_2>0,m_3>N} 2^{-m_0} 2^{ks} 2^{-m_1} 2^{-m_2} 2^{-m_3} \left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right|.
\end{aligned}$$

Now,

$$\left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right| \leq \left|\bigcup \mathbb{J}_{m_0}\right| \leq \left|\left\{ x \in \mathbb{R} : M(\chi_{\Gamma_0^{m_0}})(x) > \frac{1}{100} \right\}\right| \lesssim \left|\Gamma_0^{m_0}\right| \lesssim 2^{p_0 m_0}.$$

Similarly, we have

$$\left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right| \leq \left|\bigcup \mathbb{J}_{m_1}\right| \lesssim 2^{p_1 m_1},$$

$$\left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right| \le \left|\bigcup \mathbb{J}_{m_2}\right| \lesssim 2^{p_2 m_2}$$

and

$$\left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right| \le \left|\bigcup \mathbb{J}_{m_3}\right| \lesssim 2^{\alpha m_2}$$

for every $\alpha > 1$, because $\|\chi_{E'}\|_\alpha \le 1$ for all $\alpha > 1$.

Hence, we see that for every $\theta_j \ge 0$, $\sum_{j=0}^{3} \theta_j = 1$, we have

$$\left|\bigcup \mathbb{J}_{m_0,m_1,m_2,m_3}\right| \lesssim 2^{\theta_0 p_0 m_0} 2^{\theta_1 p_1 m_1} 2^{\theta_2 p_2 m_2} 2^{\theta_3 \alpha m_3}$$

and so

$$\begin{aligned}\sum_{\mathbb{I}_1} \lesssim & \sum_{m_0,m_1,m_2>0,m_3>N} 2^{-m_0} 2^{ks} 2^{-m_1} 2^{-m_2} 2^{-m_3} 2^{\theta_0 p_0 m_0} 2^{\theta_1 p_1 m_1} 2^{\theta_2 p_2 m_2} 2^{\theta_3 \alpha m_3} \\ \lesssim & 2^{ks} \left(\sum_{m_0,m_1,m_2,m_3>0,0\ge m_3>N} + \sum_{m_0,m_1,m_2,m_3>0} \right) 2^{(\theta_0 p_0 -1)m_0} 2^{(\theta_1 p_1 -1)m_1} 2^{(\theta_2 p_2 -1)m_2} 2^{(\theta_3 \alpha -1)m_3}.\end{aligned}$$

For the first summation, we can choose θ_j and α so that $(\theta_0 p_0 - 1), (\theta_1 p_1 - 1), (\theta_2 p_2 - 1) < 0$ and $(\theta_3 \alpha - 1) > 0$.

For the second summation, we can make $(\theta_0 p_0 - 1), (\theta_1 p_1 - 1), (\theta_2 p_2 - 1), (\theta_3 \alpha - 1) < 0$.

Hence, both of the above sums converge, and we can conclude that

$$\sum_{\mathbb{I}_1} \lesssim 2^{ks}.$$

This ends the estimate for $\sum_{\mathbb{I}_1}$.

The estimate for $\sum_{\mathbb{I}_2}$ is actually much simpler, thanks to the fact that the family $\left\{ \phi_I^j \right\}_I$ is strongly adapted to $\mathbb{I}$.

Let

$$\mathbb{J}_0 = \left\{ I \in \mathbb{I}_2 : \widetilde{d}_I(\Omega) < 2 \right\}$$

and for $d > 0$, let

$$\mathbb{J}_d = \left\{ I \in \mathbb{I}_2 : 2^d \leq \widetilde{d}_I(\Omega) < 2^{d+1} \right\}$$

Recall for every $I \in \mathbb{I}_2$, $I \subseteq \Omega$ and observe that for each d, intervals in $\mathbb{J}_d$ have finite overlap and hence

$$\sum_{I \in \mathbb{J}_d} |I| \lesssim |\Omega| \lesssim 1.$$

For $I \in \mathbb{J}_d$, we know that there exists another dyadic interval $\widetilde{I}$, with the same length as I and lying m steps away from I, for some $2^d < |m| \leq 2^{d+1}$, such that $\widetilde{I} \nsubseteq \Omega$. Consider $x \in \widetilde{I} \cap \Omega^c$. For $0 \leq j \leq 2$, we have

$$x \in \widetilde{I} \cap \Omega_j^c$$

and hence

$$\frac{\left|\left\langle a_0, \phi_I^0 \right\rangle\right|}{|I|^{1/2}} \leq M^m a_0(x) \lesssim 2^{5d} \log \left\langle 2^d \right\rangle \lesssim 2^{5d} d,$$

$$\frac{\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1 \right\rangle\right|}{|I|^{1/2}} \leq \widetilde{M}^{\max(k,d)+1} a_1(x) \lesssim 2^{5d} 2^{(\max(k,d)+1)s}$$

and

$$\frac{\left|\left\langle a_2, \phi_I^2 \right\rangle\right|}{|I|^{1/2}} \leq M^m a_2(x) \lesssim 2^{5d} \log \left\langle 2^d \right\rangle \lesssim 2^{5d} d.$$

Recall that $E' = E \backslash \Omega$ and the family $\left\{ \phi_I^j \right\}_I$ is strongly adapted to $\mathbb{I}$ and therefore for any large $M > 0$, we have

$$\begin{aligned}
&\sum_{I \in \mathbb{I}_2} \frac{1}{|I|} \left|\left\langle a_0, \phi_I^0 \right\rangle\right| \left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1 \right\rangle\right| \right) \left|\left\langle a_2, \phi_I^2 \right\rangle\right| \left|\left\langle \chi_{E'}, \phi_I^3 \right\rangle\right| \\
&= \sum_{d \geq 0} \sum_{I \in \mathbb{J}_d} \frac{\left|\left\langle a_0, \phi_I^0 \right\rangle\right|}{|I|^{1/2}} \frac{\left(\sup_{|n|\sim 2^k} \left|\left\langle a_1, \phi_{I_n}^1 \right\rangle\right| \right)}{|I|^{1/2}} \frac{\left|\left\langle a_2, \phi_I^2 \right\rangle\right|}{|I|^{1/2}} \frac{\left|\left\langle \chi_{E'}, \phi_I^3 \right\rangle\right|}{|I|^{1/2}} |I| \\
&\lesssim \sum_{d \geq 0} 2^{5d} d 2^{5d} 2^{(\max(k,d)+1)s} 2^{5d} d 2^{-Md} \sum_{I \in \mathbb{J}_d} |I| \\
&\lesssim 2^{ks}
\end{aligned}$$

This concludes the proof of the theorem.

□

CHAPTER 3

THE CALDERÓN COMMUTATORS

In this chapter, we study the boundedness of the Calderón Commutators. Recall that for $d \geq 1$, the d-th Calderón Commutator is given by

$$C_d(f, a_1, \cdots, a_d)(x) = \text{p.v.} \int_{\mathbb{R}} f(y) \prod_{j=1}^{d} \frac{A_j(x) - A_j(y)}{x - y} \frac{dy}{x - y},$$

where A_j is an antiderivative of a_j. The symbol of C_d is given by

$$m_d(\xi_0, \xi_1, \cdots, \xi_d) = \int_0^1 \cdots \int_0^1 \text{sgn}(\xi_0 + \alpha_1 \xi_1 + \cdots + \alpha_d \xi_d) d\alpha_1 \cdots d\alpha_d.$$

We will prove Theorem 1.1.4 via time-frequency analysis:

Theorem 1.1.4. *($d = 2$: Coifman and Meyer, 1975; $d \geq 3$: Duong, Grafakos and Yan, 2010) C_d satisfies the bound*

$$\|C_d(f, a_1, \ldots, a_d)\|_r \lesssim \|f\|_{p_0} \prod_{k=1}^{d} \|a_k\|_{p_k}$$

for all $1 < p_0, p_1, \ldots, p_d < \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$.

Our method highly depends on the local smoothness of the symbol m_d. Hence, we will start by studying it.

3.1 Decay of localized Fourier Coefficients of C_d

In this section, we give a bound for the localized Fourier coefficients of C_d, for $d \geq 1$. The result for $d = 1$ is known in [16] and we establish the bounds for $d \geq 2$. Since the method for every $d \geq 2$ is the same, we will do the calculate for the case $d = 2$ for simplicity of notations.

Lemma 3.1.1.
$$\mathcal{F}\left(p.v.\frac{1}{x_0^2}\chi_{[0,x_0]}(x_1)\right)(\xi_0,\xi_1) = Cm_1(\xi_0,\xi_1),$$
for some absolute constant C, where $(x_0,x_1)\in\mathbb{R}^2$. (Here for $x<0$, denote $\chi_{[0,x]} = -\chi_{[x,0]}$)

Proof. In this proof, every equal sign is up to an absolute constant multiple.

We know
$$\xi \mapsto \text{sgn}(\vec{\xi}\cdot\vec{v}) = \mathcal{F}(x \mapsto (\vec{x}\cdot\vec{v})^{-1}\delta_{\mathbb{R}v}),$$
for any fixed vector $\vec{v}\in\mathbb{R}^2$. Therefore, for any test function Φ on $\mathbb{R}^2$,
$$\int_{\mathbb{R}^2}\text{sgn}(\xi_0+\alpha\xi_1)\widehat{\Phi}(\xi_0,\xi_1)d\xi_0 d\xi_1 = \text{p.v.}\int_{\mathbb{R}}\frac{1}{x}\Phi(x,\alpha x)dx.$$

Since
$$m_1(\xi_0,\xi_1) = \int_0^1 \text{sgn}(\xi_0+\alpha\xi_1)d\alpha,$$
we have
$$\begin{aligned}\int_{\mathbb{R}^2} m_1(\xi_0,\xi_1)\widehat{\Phi}(\xi_0,\xi_1)d\xi_0 d\xi_1 &= \int_0^1 \text{p.v.}\int_{\mathbb{R}}\frac{1}{x}\Phi(x,\alpha x)dxd\alpha\\ &= \text{p.v.}\int_{\mathbb{R}}\frac{1}{x_0^2}\int_0^{x_0}\Phi(x_0,x_1)dx_1dx_0\\ &= \text{p.v.}\int_{\mathbb{R}^2}\frac{1}{x_0^2}\chi_{[0,x_0]}(x_1)\Phi(x_0,x_1)dx_0dx_1\end{aligned}$$

□

Now we can write
$$\mathcal{F}^{-1}(m_1)(x_0,x_1) = \text{p.v.}\frac{1}{x_0^2}\chi_{[0,x_0]}(x_1).$$

Let b be a bump function on $\mathbb{R}$ with support in $[-1,1]$ and $b=1$ on $[-\frac{1}{2},\frac{1}{2}]$.

Let

$$k_1(x_0, x_1) = b(x_0)\mathcal{F}^{-1}(m_1)(x_0, x_1)$$

and

$$h_1(x_0, x_1) = \mathcal{F}^{-1}(m_1)(x_0, x_1) - k_1(x_0, x_1).$$

Note that:

- $\mathcal{F}^{-1}(m_1) = k_1 + h_1$.
- k_1 is supported in a ball around the origin.
- $|h_1(x_0, x_1)| \lesssim \langle (x_0, x_1) \rangle^{-2}$.

Lemma 3.1.2. *Let Φ be smooth on $\mathbb{R}^3$ such that*

$$\left| D\Phi(\vec{x}) \right| \lesssim \langle \vec{x} \rangle^{-M}$$

for some $M > 0$, then

$$\left| p.v. \int_{\mathbb{R}^2} k_1(y_1, y_2)\Phi(x_0, x_1 - y_1, x_2 - y_2) dy_1 dy_2 \right| \lesssim \langle (x_0, x_1, x_2) \rangle^{-M}.$$

Proof.

$$\begin{aligned}
&\left| \text{p.v.} \int_{\mathbb{R}^2} k_1(y_1, y_2)\Phi(x_0, x_1 - y_1, x_2 - y_2) dy_1 dy_2 \right| \\
&= \left| \text{p.v.} \int_{\mathbb{R}} \frac{b(y_1)}{y_1^2} \int_0^{y_1} \Phi(x_0, x_1 - y_1, x_2 - y_2) dy_1 dy_2 \right| \\
&= \left| \int_0^1 \frac{b(y_1)}{y_1^2} \int_0^{y_1} \Phi(x_0, x_1 - y_1, x_2 - y_2) - \Phi(x_0, x_1 + y_1, x_2 + y_2) dy_1 dy_2 \right| \\
&\leq \int_0^1 \frac{b(y_1)}{y_1^2} \int_0^{y_1} \left| \Phi(x_0, x_1 - y_1, x_2 - y_2) - \Phi(x_0, x_1 + y_1, x_2 + y_2) \right| dy_1 dy_2 \\
&\lesssim \int_0^1 \frac{b(y_1)}{y_1^2} \int_0^{y_1} 2|(y_1, y_2)| \langle (x_0, x_1, x_2) \rangle^{-M} dy_2 dy_1 \qquad \text{(by Mean Value Theorem)} \\
&\lesssim \langle (x_0, x_1, x_2) \rangle^{-M} \int_0^1 \frac{b(y_1)}{y_1^2} \int_0^{y_1} |y_1| dy_2 dy_1 \\
&\lesssim \langle (x_0, x_1, x_2) \rangle^{-M}
\end{aligned}$$

□

Now, we want to estimate the localized fourier coefficient

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} m_2(\xi,\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2,$$

where $\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\eta_2)$ is a smooth cut off function for a whitney box in $\mathbb{R}^3$.

If ϕ is the only lacunary function among ϕ, ϕ_1 and ϕ_2, then m_2 is smooth in the whitney box and we have arbitrarily fast polynomial decay with respect to (y, z_1, z_2). So it suffices to consider without loss of generality the case where ϕ_1 is a lacunary function.

In this case, η_1 is away from 0. Hence

$$\begin{aligned} m_2(\xi,\eta_1,\eta_2) &= \frac{1}{\eta_1}\int_0^1\int_0^{\eta_1}\operatorname{sgn}(\xi+t+\alpha_2\eta_2)dtd\alpha_2 \\ &\approx \int_0^1\int_0^{\eta_1}\operatorname{sgn}(\xi+t+\alpha_2\eta_2)dtd\alpha_2 := \widetilde{m_2} \end{aligned}$$

up to a multiple of smooth function. Therefore, we can instead analyze

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{m_2}(\xi,\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2$$

without affecting the rate of decay of the localized fourier coefficients.

Now we can prove the mean result about the decay of localized fourier coefficients of m_2:

Proposition 3.1.1. *For any large $M > 0$, and small $\gamma > 0$,*

$$\begin{aligned} &\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{m_2}(\xi,\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2\right| \\ \lesssim&\langle z_1\rangle^{-1}\langle y-z_1\rangle^{-M}\langle (y,z_2)\rangle^{-2}+\langle z_1\rangle^{-M}\langle (y,z_1,z_2)\rangle^{-3+\gamma}. \end{aligned}$$

Proof. We consider two ranges of the point (y, z_1, z_2) separately. Fix some large integer K. The first range is where $|z_1|^K \geq |(y, z_1, z_2)| > 1$ and the second range is $|z_1|^K < |(y, z_1, z_2)|$ and $|(y, z_1, z_2)| > 1$. The range $|(y, z_1, z_2)| \leq 1$ is compact and hence the estimate is automatic. Note that we have

$$\frac{\partial \widetilde{m_2}}{\partial \eta_1} = m_1(\xi + \eta_1, \eta_2).$$

Hence for the first range, up to some constant multiples in each term,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widetilde{m_2}(\xi, \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\phi_1}(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2 \\
=&\frac{1}{z_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widetilde{m_2}(\xi, \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\phi_1}'(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2 \\
&+ \frac{1}{z_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} m_1(\xi + \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\phi_1}(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2
\end{aligned}$$

The first term is like the original expression but with one more factor of z_1 on the denominator. The second term is of the type we will continue to analyze below. If we continue doing integration by parts on the first term, we will get an expression of the same type of the first term as well as another expression of the same type of the second term. Repeat this process again for $(M + 3)K$ times, we see that for $|z_1|^K \geq |(y, z_1, z_2)|$,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \widetilde{m_2}(\xi, \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\phi_1}(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2 \\
=&O(|z_1|^{-(M+3)K}) + C\frac{1}{z_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} m_1(\xi + \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\widetilde{\phi_1}}(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2 \\
\lesssim&O(|(y, z_1, z_2)|^{-M-3}) \\
&+ \frac{C}{z_1} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} m_1(\xi + \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi) \widehat{\widetilde{\phi_1}}(\eta_1) \widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2
\end{aligned}$$

for some smooth compactly supported function $\widehat{\widetilde{\phi_1}}$ and some constant C.

The term $O(|(y, z_1, z_2)|^{-M-3})$ is clearly acceptable for our estimate.

For the remaining term, by Lemma 3.1.1, we can see that

$$\mathcal{F}^{-1}\left(m_1(\xi+\eta_1,\eta_2)\right)(y,z_1,z_2)=\delta_{\{y=z_1\}}\frac{1}{y^2}\chi_{[0,y]}(z_2).$$

Therefore, the term we need to analyze is, up to a constant multiple,

$$\begin{aligned}
&\frac{1}{z_1}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} m_1(\xi+\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\widetilde{\phi_1}}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2\\
=&\frac{1}{z_1}\left(\left[\delta_{\{y=z_1\}}\frac{1}{y^2}\chi_{[0,y]}(z_2)\right]*[\phi(y)\widetilde{\phi_1}(\eta_1)\phi_2(z_2)]\right)\\
:=&\langle z_1\rangle^{-1}\left(\left[\delta_{\{y=z_1\}}\frac{1}{y^2}\chi_{[0,y]}(z_2)\right]*\Phi\right)
\end{aligned}$$

By a change of coordinate, the absolute value of the above expression can be written as

$$\begin{aligned}
&\left|\langle x_1-x_2\rangle^{-1}\left(\left[\delta_{\{x_2=0\}}\frac{1}{x_1^2}\chi_{[0,x_1]}(x_3)\right]*\Phi\right)\right|\\
=&\langle x_1-x_2\rangle^{-1}\left|\left(\left[\delta_{\{x_2=0\}}\mathcal{F}^{-1}(m_1)(x_1,x_3)\right]*\Phi\right)\right|\\
\le&\langle x_1-x_2\rangle^{-1}\left|([\delta_{\{x_2=0\}}k_1(x_1,x_3)]*\Phi)\right|+\langle x_1-x_2\rangle^{-1}\left|([\delta_{\{x_2=0\}}h_1(x_1,x_3)]*\Phi)\right|
\end{aligned}$$

By lemma 2, the first term is bounded by $\langle x_1-x_2\rangle^{-1}\langle\vec{x}\rangle^{-M}$, which is clearly acceptable. Now it remains to bound the second term. Since Φ is a Schwartz function, we have

$$|\Phi(\vec{x})|\lesssim\langle x_2\rangle^{-M}\langle(x_1,x_3)\rangle^{-M}.$$

Hence

$$\begin{aligned}
&\langle x_1-x_2\rangle^{-1}\left|([\delta_{\{x_2=0\}}h_1(x_1,x_3)]*\Phi)\right|\\
\le&\langle x_1-x_2\rangle^{-1}\left(\left[\delta_{\{x_2=0\}}\langle(x_1,x_3)\rangle^{-2}\right]*\left[\langle x_2\rangle^{-M}\langle(x_1,x_3)\rangle^{-M}\right]\right)\\
\eqsim&\langle x_1-x_2\rangle^{-1}\langle x_2\rangle^{-M}\langle(x_1,x_3)\rangle^{-2}\\
=&\langle z_1\rangle^{-1}\langle y-z_1\rangle^{-M}\langle(y,z_2)\rangle^{-2}
\end{aligned}$$

This ends the estimate for the first range of (y, z_1, z_2).

The second range where $|z_1|^K < |(y, z_1, z_2)|$ and $|(y, z_1, z_2)| > 1$ is easier and follows the same idea:

$$\begin{aligned}&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \widetilde{m_2}(\xi, \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2\right| \\ :=&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} m_2(\xi, \eta_1, \eta_2) e^{-2\pi i(y\xi + z_1\eta_1 + z_2\eta_2)} \widehat{\phi}(\xi)\widehat{\widetilde{\phi}_1}(\eta_1)\widehat{\phi_2}(\eta_2) d\xi d\eta_1 d\eta_2\right|\end{aligned}$$

By an analogue of Lemma 3.1.1 with exactly the same proof, we have

Lemma 3.1.3.

$$\mathcal{F}^{-1}(m_2)(y, z_1, z_2) = p.v.\frac{1}{y^3}\chi_{[0,y]}(z_1)\chi_{[0,y]}(z_2).$$

We can similarly let

$$k_2(y, z_1, z_2) = b(y)\mathcal{F}^{-1}(m_2)(y, z_1, z_2)$$

and

$$h_2(y, z_1, z_2) = \mathcal{F}^{-1}(m_2)(y, z_1, z_2) - k_2(y, z_1, z_2)$$

and similarly we have

- k_2 is supported in a ball around the origin in $\mathbb{R}^3$.
- $|h_2(y, z)| \lesssim \langle (y, z_1, z_2) \rangle^{-3}$.

By the same proof of Lemma 3.1.2, we have

Lemma 3.1.4. *Let Φ be smooth on $\mathbb{R}^3$ and*

$$|D\Phi(\vec{x})| \lesssim \langle \vec{x} \rangle^{-M},$$

then

$$\left|(k_2 * \Phi)(\vec{x})\right| \lesssim \langle \vec{x} \rangle^{-M}$$

Therefore,

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} m_2(\xi,\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\widetilde{\phi_1}}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2\right| \\
=&\left| \left[(k_2+h_2)(y,z_1,z_2)\right] * \left[(\phi(y)\widetilde{\phi_1}(z_1)\phi_2(z_2))\right]\right| \\
\leq&\left| \left[k_2(y,z_1,z_2)\right] * \left[(\phi(y)\widetilde{\phi_1}(z_1)\phi_2(z_2))\right]\right| + \left| \left[h_2(y,z_1,z_2)\right] * \left[(\phi(y)\widetilde{\phi_1}(z_1)\phi_2(z_2))\right]\right|
\end{aligned}$$

By Lemma 3.1.4, the first term is bounded by $\langle (y,z_1,z_2)\rangle^{-M}$ and hence is acceptable. It suffices to bound the second term.

But since we have $|h_2(y,z)| \lesssim \langle (y,z_1,z_2)\rangle^{-3}$ and $\phi(y)\widetilde{\phi_1}(z_1)\phi_2(z_2)$ is Schwartz function, we see the second term is bounded by $\langle (y,z_1,z_2)\rangle^{-3}$.

Hence, we can draw the conclusion that, for the second case (i.e. $|z_1| \leq 1$),

$$\begin{aligned}
&\left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} m_2(\xi,\eta_1,\eta_2)e^{-2\pi i(y\xi+z_1\eta_1+z_2\eta_2)}\widehat{\phi}(\xi)\widehat{\widetilde{\phi_1}}(\eta_1)\widehat{\phi_2}(\eta_2)d\xi d\eta_1 d\eta_2\right| \\
\lesssim&\langle (y,z_1,z_2)\rangle^{-3} \\
\lesssim&\langle z_1\rangle^{-K\gamma}\langle (y,z_1,z_2)\rangle^{-3+\gamma}
\end{aligned}$$

When K is chosen to be large enough, we get the desired estimate. This ends the proof of the proposition. □

Observe that for any $\delta > 0$, both $\langle z_1\rangle^{-1}\langle y - z_1\rangle^{-M}\langle (y,z_2)\rangle^{-2}$ and $\langle z_1\rangle^{-M}\langle (y,z_1,z_2)\rangle^{-3+\gamma}$ is in $L^{\frac{2}{3}+\delta}(\mathbb{R}^3)$ when γ is chosen to be small enough. Furthermore, by the exact same calculation above, we have the following similar bound for the decay of localized fourier coefficients of m_d, for all $d \geq 2$:

Proposition 3.1.2. *For a whitney box cut-off where ϕ_1 is a lacunary function, for any* *large $M > 0$ and small $\gamma > 0$,*

$$\left|\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} m_d(\xi,\eta_1,\cdots,\eta_d)e^{-2\pi i(y\xi+z_1\eta_1+\cdots+z_d\eta_d)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\cdots\widehat{\phi_d}(\eta_d)d\xi d\eta_1\cdots d\eta_d\right|$$
$$\lesssim\langle z_1\rangle^{-1}\langle y-z_1\rangle^{-M}\langle(y,z_2,\cdots,z_d)\rangle^{-d}+\langle z_1\rangle^{-M}\langle(y,z_1,z_2,\cdots,z_d)\rangle^{-d-1+\gamma}$$

Hence, as a function of $(y, z_1, \ldots, z_d)$,

$$\left|\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} m_d(\xi,\eta_1,\cdots,\eta_d)e^{-2\pi i(y\xi+z_1\eta_1+\cdots+z_d\eta_d)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta_1)\cdots\widehat{\phi_d}(\eta_d)d\xi d\eta_1\cdots d\eta_d\right|$$

lies in

$$L^{\frac{d}{d+1}+\delta}(\mathbb{R}^{d+1})$$

for any $\delta > 0$.

3.2 Boundedness of the Calderón Commutators

Now Theorem 1.1.4 will be immediate given Theorem 2.3.2 and Proposition 3.1.2,

To see why, suppose now we are considering C_d and we do the same Littlewood-Paley decomposition on each variable as in (2.1). So we have restricted m_d, the symbol of C_d, to families of whitney boxes along cones. If the family of whitney boxes have to property that its cut off function has $\left\{\phi_I^0\right\}_I$ being the only lacunary family, then m_d is smooth and homogeneous in the cone and we can just apply Coifman-Meyer Theorem (1.1.1) to conclude the boundedness of that part.

If there are some other family $\left\{\phi_I^j\right\}_I$ for some $1 \leq j \leq d$ being lacunary, we can without loss of generality assume $j = 1$. By Proposition 3.1.2, in a Whitney box in $\mathbb{R}^{d+1}$ where the corresponding family family $\left\{\phi_I^j\right\}_I$ is lacunary, the localized fourier coefficent of m_d satisfies

$$\begin{aligned}\left|C^k_{n_0,\ldots,n_d}\right| \lesssim & \langle n_1\rangle^{-1}\langle n_0-n_1\rangle^{-M}\langle (n_0,n_2,\cdots,n_d)\rangle^{-d}+\langle n_1\rangle^{-M}\langle (n_0,n_1,n_2,\cdots,n_d)\rangle^{-d-1+\gamma}\\ \lesssim & \langle n_1\rangle^{-(2-\epsilon')}\langle n_0-n_1\rangle^{-\epsilon}\prod_{j=2}^{d}\left\langle n_j\right\rangle^{-(1+\epsilon)},\end{aligned}$$

where γ, ϵ and ϵ' can be arbitrarily close to 0. Therefore, in this case, the boundedness we want will be implied by Theorem 2.3.1 (Actually a more concrete version of it, Theorem 2.3.2). Since the above expression lies in $L^{\frac{d}{d+1}+\epsilon}(\mathbb{R}^{d+1})$ for any $\epsilon > 0$. we can actually prove the boundedness in Theorem 1.1.4 for the range $\frac{d}{d+1} < r$ by the method in Muscalu [16]. Therefore, it now suffices to prove the weak boundedness

$$C_d : L^{p_0}\times\cdots\times L^{p_d}\to L^{r,\infty}$$

for $p_0,\ldots,p_d$ close to 0 and r close to $\frac{1}{d+1}$ and then apply standard interpolation arguments such as in [17] with the above known bound for $\frac{d}{d+1} < r$.

3.3 Generalizations

3.3.1 Calderón Commutators with Higher Order of Difference

The dth Calderón Commutator can also be written as

$$C_d(a_0,a_1,\cdots,a_d)(x) = \text{p.v.}\int_{\mathbb{R}} a_0(x+t)\prod_{j=1}^{d}\frac{\Delta_t}{t}A_j(x)\frac{dt}{t},$$

where

$$A_j' = a_j$$

and

$$\frac{\Delta_t}{t} g(x) := \frac{g(x+t) - g(x)}{t}.$$

Therefore, a natural generalization of the Calderón Commutators is

$$\widetilde{C}_d(a_0, a_1, \cdots, a_d)(x) = \text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}} a_0(x+t+s) \prod_{j=1}^{d} \frac{\Delta_s}{s} \circ \frac{\Delta_t}{t} A_j(x) \frac{dt}{t} \frac{ds}{s},$$

where

$$A_j'' = a_j.$$

By a straightforward calculation, we have

$$\widetilde{C}_d(a_0, a_1, \cdots, a_d)(x) = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \widetilde{m}_d(\xi_0, \cdots, \xi_d) \widehat{a_0}(\xi_0) \cdots \widehat{a_d}(\xi_d) e^{2\pi i x \cdot (\xi_0 + \cdots + \xi_d)} d\xi_0 \cdots d\xi_d,$$

where

$$\begin{aligned}\widetilde{m}_d(\xi_0, \cdots, \xi_d) =& \left(\int_0^1 \cdots \int_0^1 \text{sgn}(\xi_0 + \alpha_1 \xi_1 + \cdots + \alpha_d \xi_d) d\alpha_1 \cdots d\alpha_d \right)^2 \\ =& m_d(\xi_0, \cdots, \xi_d)^2.\end{aligned}$$

That is, the symbol for $\widetilde{C}_d$ is the square of m_d, the symbol for C_d. By Proposition 3.1.2, the localized fourier coefficients of m_d satisfies (2.9). It is a robust fact that localized fourier coefficients of m_d satisfies (2.9) implies the localized fourier coefficients of m_d^2 also satisfies (2.9). For if we define

$$H(n_0, \ldots, n_d) = \langle n_k \rangle^{-(2-\epsilon_k')} \langle n_{k'} - n_k \rangle^{-(1+\epsilon)} \prod_{j \neq k, k'} \left\langle n_j \right\rangle^{-(1+\epsilon)},$$

then we have

$$(H * H)(n_0, \ldots, n_d) \lesssim H(n_0, \ldots, n_d).$$

Therefore, by Theorem 2.3.3, we have the boundedness of $\widetilde{C}_d$ with the full range of exponents for free:

Theorem 3.3.1.
$$\left\|\widetilde{C}_d(f, a_1, \ldots, a_d)\right\|_r \lesssim \|f\|_{p_0} \prod_{k=1}^{d} \|a_k\|_{p_k}$$
for all $1 < p_0, p_1, \ldots, p_d < \infty$, $\frac{1}{d+1} < r < \infty$ *and* $\frac{1}{p_0} + \cdots + \frac{1}{p_d} = \frac{1}{r}$.

3.3.2 A generalization of the Calderón Commutators in higher dimensions

Define
$$C^n(f)(x) = \text{p.v.} \int_{\mathbb{R}^n} f(y) \frac{A(x) - A(y)}{\left[|x-y|^2 + (\phi(x) - \phi(y))^2\right]^{\frac{n+1}{2}}} dy,$$
where A, ϕ are Lipschitz functions on $\mathbb{R}^n$.

This is an analogue of the Cauchy integral on Lipschitz curve, which was first studied by Coifman, McIntosh and Meyer [5].

Then $C^n(f)(x)$ can be similarly written as
$$\sum_{k=0}^{\infty} c_k \text{ p.v.} \int_{\mathbb{R}^n} f(y) \frac{A(x) - A(y)}{|x-y|} \left(\frac{\phi(x) - \phi(y)}{|x-y|}\right)^{2k} \frac{dy}{|x-y|^n}$$
for some bounded sequence of numbers $\{ c_k \}_k$.

This suggests us to study, for any odd number d, the following analogue of the dth commutator in $\mathbb{R}^n$:
$$C_d^n(f, A_1, \cdots A_d) = \text{p.v.} \int_{\mathbb{R}^n} f(y) \prod_{j=1}^{d} \frac{A_j(x) - A_j(y)}{|x-y|} \frac{dy}{|x-y|^n}$$
and to investigate its boundedness in the following sense:

For odd number d, does C_d^n satisfy any boundedness of type
$$\left\|C_d^n(f, A_1, \cdots, A_d)\right\|_r \lesssim \|f\|_p \|\nabla A_1\|_{p_1} \cdots \|\nabla A_d\|_{p_d}$$

for some range of $1 < p, p_1, \cdots, p_k \leq \infty$, $0 < r < \infty$ and $\frac{1}{p} + \frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$?

A careful calculation shows that $C_d^n(f, A_1, \cdots, A_d)(x)$ is equal to

$$\int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} M_d^n(\xi, \eta_1, \cdots, \eta_d) \widehat{f}(\xi) \prod_{j=1}^{d} \widehat{A^j}(\eta_j) e^{2\pi i x \cdot (\xi + \sum_{j=1}^d \eta_d)} d\xi d\eta_1 \cdots d\eta_d,$$

where

$$M_d^n(\xi, \eta_1, \cdots, \eta_d) = \int_0^1 \cdots \int_0^1 \text{p.v.} \int_{\mathbb{R}^n} \frac{\prod_{j=1}^d t \cdot \eta_j}{|t|^{n+d}} e^{2\pi i t \cdot (\xi + \sum_{j=1}^d \alpha_j \eta_j)} dt d\alpha_1 \cdots d\alpha_d.$$

By using the same technique and carefully estimating decay of localized fourier coefficients, we have an affirmative answer for some range of quasi-Banach estimate for the first commutator C_1^n.

First, we need a lemma about the geometric calculation involved in studying our symbol:

Lemma 3.3.1. *For $\xi, \eta \in \mathbb{R}^n$ with $\eta \neq 0$,*

$$\int_0^1 \frac{\xi + \alpha\eta}{|\xi + \alpha\eta|} d\alpha = \left(\frac{|\xi + \eta|}{|\eta|} - \frac{|\xi|}{|\eta|} \right) \frac{\eta}{|\eta|} + u,$$

where $u \in \mathbb{R}^n$ is a vector such that $u \cdot \eta = 0$.

Proof. Without Loss of generality, we can assume $|\eta| = 1$. We can also assume ξ is not parallel to η and then draw the conclusion for the whole range of ξ by continuity.

Decompose ξ as

$$\xi = -\beta\eta + sw,$$

where $\beta, s \in \mathbb{R}$ and w is a unit vector perpendicular to η.

Let $\theta(\alpha)$ be the angle between the vectors $\xi + \alpha\eta$ and w. All vectors we are considering in this proof lie in the same plane. Therefore, we can assign an orientation for $\theta(\alpha)$ as α varies from 0 to 1. Then we see that, with one of the two orientations of $\theta(\alpha)$,

$$\tan\theta(\alpha) = \frac{\alpha-\beta}{s}$$

and hence

$$\sin\theta(\alpha) = \frac{\alpha-\beta}{\sqrt{(\alpha-\beta)^2+s^2}}.$$

Geometrically we can see that

$$\frac{\xi+\alpha\eta}{|\xi+\alpha\eta|} = \sin\theta(\alpha)\eta + \cos\theta(\alpha)w$$

and therefore

$$\begin{aligned}\int_0^1 \frac{\xi+\alpha\eta}{|\xi+\alpha\eta|}d\alpha &= \left(\int_0^1 \sin\theta(\alpha)d\alpha\right)\eta + \left(\int_0^1 \cos\theta(\alpha)d\alpha\right)w \\ &= \left(\int_0^1 \frac{\alpha-\beta}{\sqrt{(\alpha-\beta)^2+s^2}}d\alpha\right)\eta + \left(\int_0^1 \cos\theta(\alpha)d\alpha\right)w \\ &= \left(\sqrt{(1-\beta)^2+s^2} - \sqrt{\beta^2+s^2}\right)\eta + \left(\int_0^1 \cos\theta(\alpha)d\alpha\right)w \\ &= (|\xi+\eta|-|\xi|)\,\eta + \left(\int_0^1 \cos\theta(\alpha)d\alpha\right)w\end{aligned}$$

This gives the conclusion for $|\eta| = 1$, with $u = \left(\int_0^1 \cos\theta(\alpha)d\alpha\right)w$. Since the expression

$$\int_0^1 \frac{\xi+\alpha\eta}{|\xi+\alpha\eta|}d\alpha$$

is homogenous of degree 0 in $\mathbb{R}^n\times\mathbb{R}^n$, the desired conclusion follows. □

Now we are ready to prove the boundness of C_1^n:

Theorem 3.3.2.

$$\left\|C_1^n(f,A)\right\|_r \lesssim \|f\|_p \,\|\nabla A\|_{p_1}$$

for any $1 < p, p_1 < \infty$, $\frac{n}{n+1} < r < \infty$ *and* $\frac{1}{p}+\frac{1}{p_1}=\frac{1}{r}$.

Proof. Again, by a direct calculation, we have

$$C_1^n(f,A)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_0^1 \frac{\xi+\alpha\eta}{|\xi+\alpha\eta|}d\alpha\cdot\eta\,\widehat{f}(\xi)\widehat{A}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta. \tag{3.1}$$

By Lemma 3.3.1,

$$\begin{aligned} C_1^n(f,A)(x) =& \sum_{k=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(|\xi+\eta|-|\xi|)\eta_k}{|\eta|^2}\,\widehat{f}(\xi)\eta_k\widehat{A}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \\ =& \frac{1}{2\pi i}\sum_{k=1}^n \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(|\xi+\eta|-|\xi|)\eta_k}{|\eta|^2}\,\widehat{f}(\xi)\widehat{\partial_k A}(\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta \end{aligned}$$

Since the symbol

$$m_{1,k}^d(\xi,\eta) := \frac{(|\xi+\eta|-|\xi|)\eta_k}{|\eta|^2}$$

is homogenous of degree 0, to estimate its localized fourier coefficients on whitney boxes, it suffices to consider only a finite family of whitney boxes, as in the case of C_d before.

Consider a whitney box B in $\mathbb{R}^n\times\mathbb{R}^n$. For such a B, we want to estimate the localized fourier coefficient of $m_{1,k}^d$ in B, which we denote by

$$c_{1,k,m_0,m_1}^n = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} m_{1,k}^n(\xi,\eta)e^{-2\pi i(m_0\cdot\xi+m_1\cdot\eta)}\widehat{\phi}(\xi)\widehat{\phi_1}(\eta)d\xi d\eta,$$

where $\widehat{\phi}(\xi)\widehat{\phi_1}(\eta)$ is a smooth cut-off function for B.

- Case 1: η may be zero in B. i.e. $B\cap\{(\xi,\eta)\in\mathbb{R}^n\times\mathbb{R}^n : \eta=0\}\neq\emptyset$. In this case, by replacing B with a smaller whitney box if necessary, we can assume $|\xi|$ is always bigger than $|\eta|$ in B. Therefore,

  $$\frac{(|\xi+\eta|-|\xi|)\eta_k}{|\eta|^2} = \int_0^1 \frac{\xi_k+\alpha\eta_k}{|\xi+\alpha\eta|}d\alpha.$$

  Since $|\xi|>|\eta|$ in B, $|\xi+\alpha\eta|$ is never zero in the integral. Therefore, the symbol $\frac{(|\xi+\eta|-|\xi|)\eta_k}{|\eta|^2}$ is a smooth function in B and its localized fourier coefficients decays arbitrarily polynomially fast. That is,

  $$\left|c_{1,k,m_0,m_1}^n\right| \lesssim \langle(m_0,m_1)\rangle^{-M}$$

for any $M > 0$.

- Case 2: η is never 0 in B. In this case, $\dfrac{\eta_k}{|\eta|^2}$ is a smooth function in B. Therefore, we only have to estimate the decay of the localized fourier coefficients of

$$|\xi + \eta| - |\xi|$$

in B. Up to a multiple constant, the function

$$x \mapsto |x|$$

on $\mathbb{R}^n$ has fourier transform

$$\xi \mapsto |\xi|^{-n-1}.$$

Therefore, we have

$$\left|c^n_{1,k,m_0,m_1}\right| \lesssim \langle m_0\rangle^{-n-1}(\langle m_1\rangle^{-M} - \langle m_0 - m_1\rangle^{-M})$$

for any $M > 0$.

Concluding the two cases, we see that for any whitney box B, we have

$$\sum_{m_0,m_1 \in \mathbb{Z}^n} \left(\log\langle m_0\rangle \log\langle m_1\rangle \left|c^n_{1,k,m_0,m_1}\right|\right)^s < \infty$$

for all $s > \frac{n}{n+1}$. By the straightforward n-dimensional generalization of Theorem 2.3.3, if we let $C^n_{1,k}$ be the bilinear operator with symbol $m^n_{1,k}$, then we see

$$\begin{aligned}
\left\|C^n_1(f, A)\right\|_r &= \left\|\sum_{k=1}^n C^n_{1,k}(f, \partial_k A)\right\|_r \\
&\leq \sum_{k=1}^n \left\|C^n_{1,k}(f, \partial_k A)\right\|_r \\
&\lesssim \sum_{k=1}^n \|f\|_p \|\partial_k A\|_{p_1} \\
&\lesssim \|f\|_p \|\nabla A\|_{p_1}
\end{aligned}$$

□

The anaylsis of C_1^n is a special case of a wider class of singular integral studied by Christ and Journé [3], whose result has been recently improved to a wider range of exponents by Seeger, Smart and Street [26]. Recall that, up to a constant multiple,

$$C_1^n(f, A)(x) = \sum_{k=1}^{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left[\int_0^1 \frac{(\xi + \alpha\xi_1)_k}{|\xi + \alpha\xi_1|} d\alpha \right] \widehat{f}(\xi) \widehat{\partial_k A}(\xi_1) e^{2\pi i x \cdot (\xi + \xi_1)} d\xi d\xi_1$$

Since $\frac{\xi_k}{|\xi|}$ is a Marcinkiewicz-Mikhlin-Hormander symbol (i.e. a symbol satisfying (1.1)), the boundedness of C_1^n can be boiled down to the boundedness of multilinear operators with symbol in the form

$$\int_0^1 m(\xi + \alpha\xi_1) d\alpha$$

where m is a Marcinkiewicz-Mikhlin-Hormander symbol on $\mathbb{R}^n$. Similarly, the boundedness of C_d^n can be obtained by the boundedness of multilinear operators with symbol of type

$$\int_0^1 \cdots \int_0^1 m(\xi + \alpha_1\xi_1 + \cdots + \alpha_d\xi_d) d\alpha_1 \cdots d\alpha_d, \tag{3.2}$$

where again m is a Marcinkiewicz-Mikhlin-Hormander symbol on $\mathbb{R}^n$.

In [26], it is proved that if T is a $(d+1)$-linear operator on $\mathbb{R}^n$ with symbol of type (3.2), then it satisfy the Banach estimate

$$\|T(f_0, f_1, \cdots, f_d)\|_r \leq C_{m,\delta} d^2 \log(d+2) \prod_{j=0}^{d} \left\|f_j\right\|_{p_j}$$

for $1 < p_i \leq \infty$, $1 \leq r < \infty$, $\frac{1}{p_0} + \cdots \frac{1}{p_d} = \frac{1}{r}$ and $C_{m,\beta}$ is a constant depending on only m and δ, where $\min\{p_0, p_1, \cdots, p_d, (1 - \frac{1}{r})^{-1}\} \geq 1 + \delta$.

Therefore, it is natural to ask the following question:

Can we prove boundedness of multilinear operators on $\mathcal{S}(\mathbb{R}^n)$ with symbol of type (3.2), using time-frequency analysis, especially for quasi-Banach bounds that are not available in [26]?

This question will need more studies to answer. One of the main reasons why it is harder than Theorem 3.3.2 is that, the orthogonal part u in Lemma 3.3.1 behaves much worse than the parallel part which is calculated explicitly in Theorem 3.3.2, when $n \geq 2$. u is canceled out in the inner product in (3.1) and this does not happened in the calculation when we consider one singular integral operator of Christ and Journé type.

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