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Abstract

In this paper we have exhibited a set of three mutually orthogonal latin squares of order 15 associated with a Kirkman-Steiner triple system of the corresponding order.

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Summary

In this paper we have exhibited a set of three mutually orthogonal latin squares of order 15 associated with a Kirkman-Steiner triple system of the corresponding order.

1. Introduction and Background

In the following, by an O(n,t) set we mean a set of t mutually orthogonal latin squares of order n. In 1782 Euler [5], the founder of the theory of orthogonal latin squares, conjectured that there is no O(n,2) set, $n \equiv 2 \pmod{4}$. Clausen (see Gunther [8] and Ahrens [1]), Tarry [20], Fisher and Yates [7] and Saxena [19] have all demonstrated the truth of Euler's conjecture for order 6.

MacNeish [14] proved the following theorem:

Theorem. *Let* $n = p^{t_1} p^{t_2} \cdots p^{t_s}$ *be the prime power decomposition of* n. *Then, there exists an* $O(n,\gamma)$ *set where* $\gamma = \min(p_1^{t_1}, p_2^{t_2}, \cdots, p_s^{t_s}) - 1$.

Note that if $n \equiv 2 \pmod{4}$, then $\gamma = 1$, meaning that MacNeish's theorem is incapable of producing an O(n,2) set. After some topological arguments MacNeish conjectured that there is no $O(n,\alpha)$ set, $\alpha > \gamma$. Levi [13, p. 14]

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pointed out the inadmissibility of the MacNeish's topological arguments. However Levi could not disprove MacNeish's claim.

Parker [15] disproved the MacNeish conjecture by proving that for some n,s one can construct an $O(n,\gamma)$ set, $\lambda > \gamma$. Thus, Parker obtained counter-examples to MacNeish's conjecture, the first two cases being $O(21,4)$ and $O(57,7)$. However, no counter-example to Euler's conjecture can be generated using the theorems proved by Parker [15]. But, obviously, the results obtained by Parker cast a serious doubt on the validity of Euler's conjecture.

After this breakthrough by Parker, Bose and Shrikhande [2], Parker [16] and Bose, Shrikhande and Parker [3] disproved Euler's conjecture for all orders excluding 6, and that was the end of Euler's conjecture. Recently Hedayat and Seiden [6,9] among other results, have found an easy method of constructing pairs of orthogonal latin squares of orders $n \doteq 2 \pmod{4}$, n different from 6.

How about MacNeish's conjecture? Previous results could not, by any means, totally demolish the MacNeish's conjecture. The next smallest order viz 12 was undecided. Johnson et.al. [12] and Bose et.al. [4] independently were able to construct an $O(12,5)$ set, thus disproving MacNeish's conjecture for order 12. There are still many other orders which MacNeish's conjecture has not settled. Order 15 is the smallest one. Our object in this paper is to exhibit an $O(15,3)$ set and thus show the falsity of MacNeish's conjecture for this order.

Before closing this section we should mention that the theory of orthogonal latin squares owes its importance to the fact that many well-known combinatorial systems are actually equivalent to $O(n,t)$ sets, viz., error correcting codes, finite projective planes, finite Euclidean planes, finite nets, orthogonal arrays, Hadamard matrices, finite graphs, BIB, PBIB, balanced 1-restrictional lattice designs, difference sets and many others (for more details see Hedayat [10].

2. Definition

A latin square of order n on a set Σ containing n distinct elements is an n×n matrix each of whose rows and columns is a permutation of the set Σ. Two latin squares of order n are said to be orthogonal if, when they are superposed, each symbol of the first square occurs just once with each symbol of the second square. A set of t mutually orthogonal latin squares of order n is a set of t latin squares of order n any two of which are orthogonal.

3. A counter example

In the following we shall exhibit a set of three mutually orthogonal latin squares of order 15. This set disproves the MacNeish conjecture for order 15 because $\gamma = 2$ in MacNeish's theorem.

Hedayat and Raktoe [11] have shown that every resolvable balanced incomplete block design with parameters $b = v(v-1)/6$, $v \equiv 3 \pmod{6}$, $r = (v-1)/2$, $k = 3$ and $\lambda = 1$ (also called Kirkman-Steiner triple system of order v) implies a special O(v,2) set. Here we want to show that the case v = 15 leads to a new result, namely the existence of an O(15,3) set, by showing that one of the latin squares in the derived O(15,2) set (see table 1) corresponding to the Kirkman-steiner triple system given on page 102 of [18] admits a special orthogonal mate (see table 2) which can be embedded in an O(15,3) set (see table 3). Whether or not this O(15,3) set can be embedded in a larger set is under investigation. It is hoped to discuss this matter in a later paper.

Table 1

A	G	L	J	M	K	B	N	O	D	F	C	E	H	I
G	B	J	L	K	M	A	O	N	C	E	D	F	I	H
L	J	C	G	I	H	D	F	E	B	N	A	O	K	M
J	L	G	D	H	I	C	E	F	A	O	B	N	M	K
M	K	I	H	E	G	F	D	C	N	B	O	A	J	L
K	M	H	I	G	F	E	C	D	O	A	N	B	L	J
B	A	D	C	F	E	G	I	H	L	M	J	K	O	N
N	O	F	E	D	C	I	H	G	M	L	K	J	A	B
O	N	E	F	C	D	H	G	I	K	J	M	L	B	A
D	C	B	A	N	O	L	M	K	J	I	G	H	E	F
F	E	N	O	B	A	M	L	J	I	K	H	G	C	D
C	D	A	B	O	N	J	K	M	G	H	L	I	F	E
E	F	O	N	A	B	K	J	L	H	G	I	M	D	C
H	I	K	M	J	L	O	A	B	E	C	F	D	N	G
I	H	M	K	L	J	N	B	A	F	D	E	C	G	O

O	C	E	H	M	K	J	I	G	A	D	L	F	B	N
J	O	D	I	G	L	C	M	A	K	N	B	E	H	F
L	K	O	M	I	N	F	G	B	D	J	E	A	C	H
A	B	F	O	D	C	M	K	J	H	E	I	G	N	L
F	N	B	K	O	A	H	D	I	E	G	C	M	L	J
D	E	G	J	H	O	A	N	C	B	K	F	L	M	I
C	J	M	F	A	H	O	E	L	G	B	N	I	D	K
B	F	N	D	K	G	L	O	E	C	A	H	J	I	M
N	H	I	C	B	J	E	L	O	M	F	K	D	A	G
H	D	K	A	L	I	N	J	F	O	M	G	C	E	B
K	G	C	L	N	D	I	H	M	F	O	A	B	J	E
E	I	L	B	J	M	G	A	D	N	H	O	K	F	C
M	L	H	N	F	E	B	C	K	J	I	D	O	G	A
I	A	J	G	E	F	K	B	H	L	C	M	N	O	D
G	M	A	E	C	B	D	F	N	I	L	J	H	K	O

Table 2

A	G	L	J	M	K	B	N	O	D	F	C	E	H	I
G	B	J	L	K	M	A	O	N	C	E	D	F	I	H
L	J	C	G	I	H	D	F	E	B	N	A	O	K	M
J	L	G	D	H	I	C	E	F	A	O	B	N	M	K
M	K	I	H	E	G	F	D	C	N	B	O	A	J	L
K	M	H	I	G	F	E	C	D	O	A	N	B	L	J
B	A	D	C	F	E	G	I	H	L	M	J	K	O	N
N	O	F	E	D	C	I	H	G	M	L	K	J	A	B
O	N	E	F	C	D	H	G	I	K	J	M	L	B	A
D	C	B	A	N	O	L	M	K	J	I	G	H	E	F
F	E	N	O	B	A	M	L	J	I	K	H	G	C	D
C	D	A	B	O	N	J	K	M	G	H	L	I	F	E
E	F	O	N	A	B	K	J	L	H	G	I	M	D	C
H	I	K	M	J	L	O	A	B	E	C	F	D	N	G
I	H	M	K	L	J	N	B	A	F	D	E	C	G	O

E	A	K	G	N	D	M	O	C	F	H	B	J	L	I
N	E	H	M	L	G	J	I	K	D	O	A	F	B	C
G	D	E	I	M	O	C	L	A	H	N	F	K	J	B
K	C	J	E	H	A	N	D	M	B	F	I	L	O	G
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
H	K	F	N	O	E	L	G	J	A	D	C	B	I	M
C	I	M	A	K	B	E	F	G	L	J	O	N	H	D
F	J	B	H	D	K	O	E	L	C	A	M	I	G	N
M	G	N	C	J	I	K	B	E	O	L	D	H	F	A
O	H	D	L	I	N	B	M	F	E	G	K	A	C	J
D	F	A	O	G	H	I	N	B	K	E	J	C	M	L
L	N	O	J	B	M	F	C	H	G	I	E	D	A	K
I	O	G	B	C	L	A	J	D	N	M	H	E	K	F
B	L	I	F	A	J	D	K	O	M	C	N	G	E	H
J	M	L	K	F	C	H	A	N	I	B	G	O	D	E

Table 3

E	A	K	G	N	D	M	O	C	F	H	B	J	L	I
N	E	H	M	L	G	J	I	K	D	O	A	F	B	C
G	D	E	I	M	O	C	L	A	H	N	F	K	J	B
K	C	J	E	H	A	N	D	M	B	F	I	L	O	G
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
H	K	F	N	O	E	L	G	J	A	D	C	B	I	M
C	I	M	A	K	B	E	F	G	L	J	O	N	H	D
F	J	B	H	D	K	O	E	L	C	A	M	I	G	N
M	G	N	C	J	I	K	B	E	O	L	D	H	F	A
O	H	D	L	I	N	B	M	F	E	G	K	A	C	J
D	F	A	O	G	H	I	N	B	K	E	J	C	M	L
L	N	O	J	B	M	F	C	H	G	I	E	D	A	K
I	O	G	B	C	L	A	J	D	N	M	H	E	K	F
B	L	I	F	A	J	D	K	O	M	C	N	G	E	H
J	M	L	K	F	C	H	A	N	I	B	G	O	D	E

D	I	E	A	O	K	C	H	M	G	B	L	N	J	F
L	H	N	G	F	E	B	M	O	J	K	A	C	D	I
O	N	J	I	K	G	L	C	B	H	A	F	D	M	E
N	A	F	B	D	O	H	G	J	K	M	C	I	E	L
J	F	K	L	M	B	N	I	H	A	C	D	E	G	O
C	G	I	K	L	F	M	J	D	E	O	B	H	N	A
E	L	O	C	J	M	I	A	G	B	H	N	F	K	D
H	K	A	O	C	L	J	E	N	F	D	M	G	I	B
F	C	D	H	E	J	A	B	K	M	G	I	L	O	N
B	J	H	E	A	I	O	D	L	N	F	K	M	C	G
M	E	G	F	B	A	D	N	C	I	L	O	J	H	K
A	M	C	J	I	N	K	O	F	D	E	G	B	L	H
K	D	M	N	G	H	F	L	A	C	I	E	O	B	J
G	O	B	D	H	C	E	F	I	L	N	J	K	A	M
I	B	L	M	N	D	G	K	E	O	J	H	A	F	C

I	C	D	H	F	N	K	A	J	L	O	G	M	B	E
A	O	G	J	C	K	D	L	I	H	E	B	F	N	M
B	J	E	D	H	C	O	N	G	K	I	M	L	F	A
O	N	H	F	B	A	L	E	C	M	K	J	G	I	D
J	F	K	L	M	B	N	I	H	A	C	D	E	G	O
E	K	J	C	N	L	H	M	O	F	D	I	B	A	G
C	B	L	M	G	I	A	O	F	E	N	H	D	J	K
D	L	C	N	A	M	F	G	K	B	H	O	I	E	J
N	I	M	G	J	O	E	K	B	D	A	F	H	C	L
M	D	I	O	K	H	J	F	E	C	G	A	N	L	B
G	H	O	K	L	F	C	B	D	N	J	E	A	M	I
L	E	B	I	O	D	G	H	A	J	M	N	C	K	F
F	G	A	E	D	J	I	C	M	O	B	L	K	H	N
H	M	N	A	E	G	B	J	L	I	F	K	O	D	C
K	A	F	B	I	E	M	D	N	G	L	C	J	O	H

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