CHAPTER IV.

SLIDER-CRANK CHAINS.

34. **Slider-crank Chain.**—A very important chain is obtained from the quadric crank-chain by substituting a sliding pair for one of the turning pairs. It is obvious that the links will undergo the same relative change of position in Fig. 60 (*b*) as in Fig. 60 (*a*), although the lever *c* has been replaced by a block sliding in a circularly curved slot of the same radius as the original lever. The chain as thus transformed may be called a cylindric slider-crank chain, although this name is generally applied to the particular case in which O_{cd} is at an infinite distance and the block slides in a straight slot. It is plain that the mechanism of Fig. 60 (*c*) may be obtained from that of Fig. 60 (*b*) by continually increasing the radius of the pair *cd* until it becomes infinite. The pair *cd* may have prismatic surfaces of any form so long as the sliding motion is properly constrained; thus, for example, *c* may be a hollow block sliding on a prismatic rod *d*, Fig. 60 (*c*). The slider-crank chain in its cylindric form has of course plane motion, and is of special importance, since its different inversions form amongst others the mechanisms of various types of reciprocating steam-engines.

The six virtual centres of the slider-crank chain are easily found, exactly as in the case of the quadric crank-chain, but O_{cd} is always inaccessible. Fig. 61 shows the centrodes of the links *b* (representing the connecting-rod of a direct-acting engine) and *d* (representing the frame or bedplate). The centrode of *b* with respect to *d* (i.e., if *d* is considered as the fixed link) is shown by the full line; the dotted curve represents the centrode described by O_{bd} if b

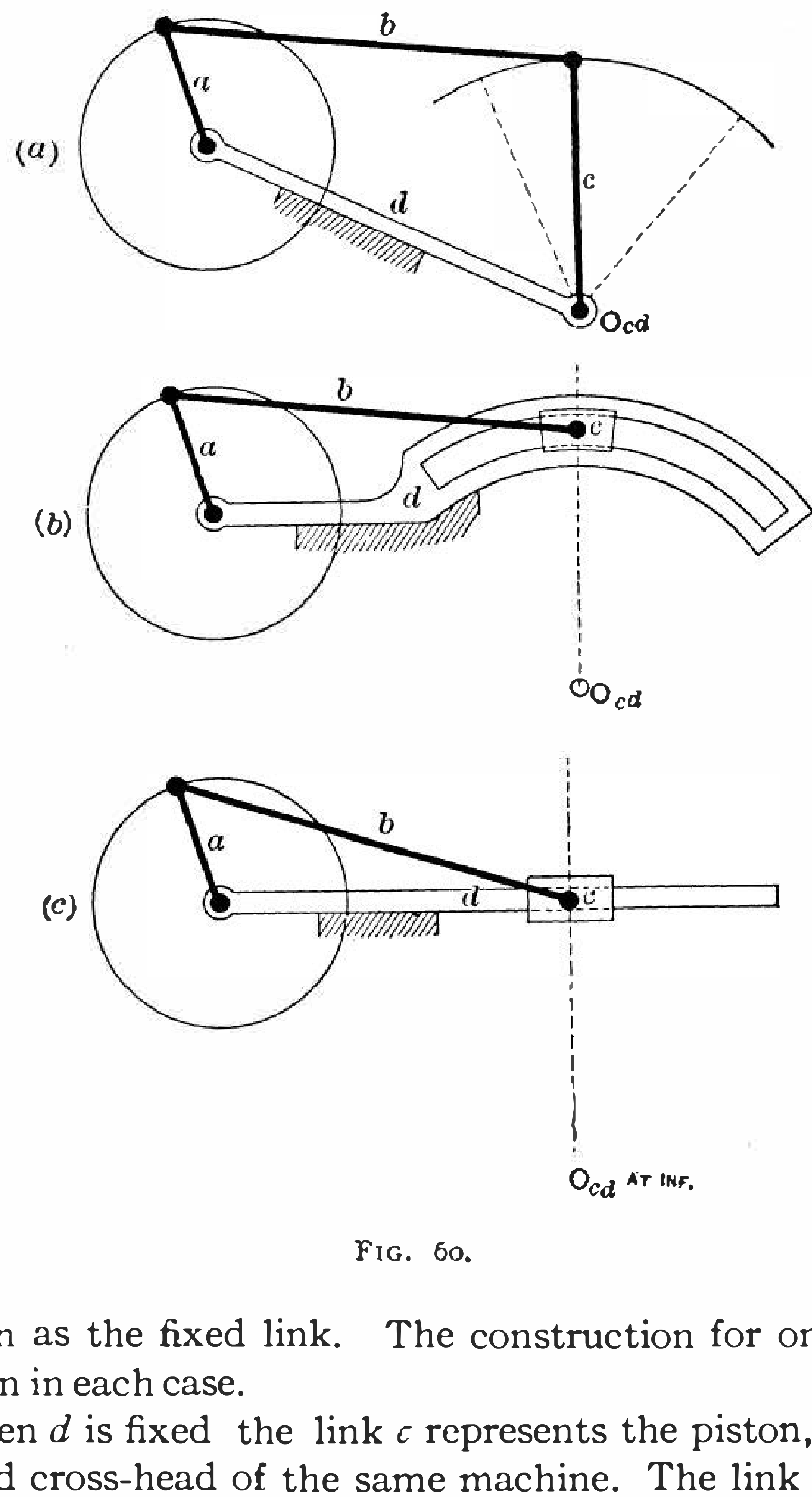


FIG. 60.

is taken as the fixed link. The construction for one point is shown in each case.

When d is fixed the link c represents the piston, piston-rod, and cross-head of the same machine. The link a represents the crank, and b the connecting-rod. A point on the link b between A and B describes an oval curve with reference to d, the shape depending on the position of the point selected, and on the ratio of the lengths of crank and con-

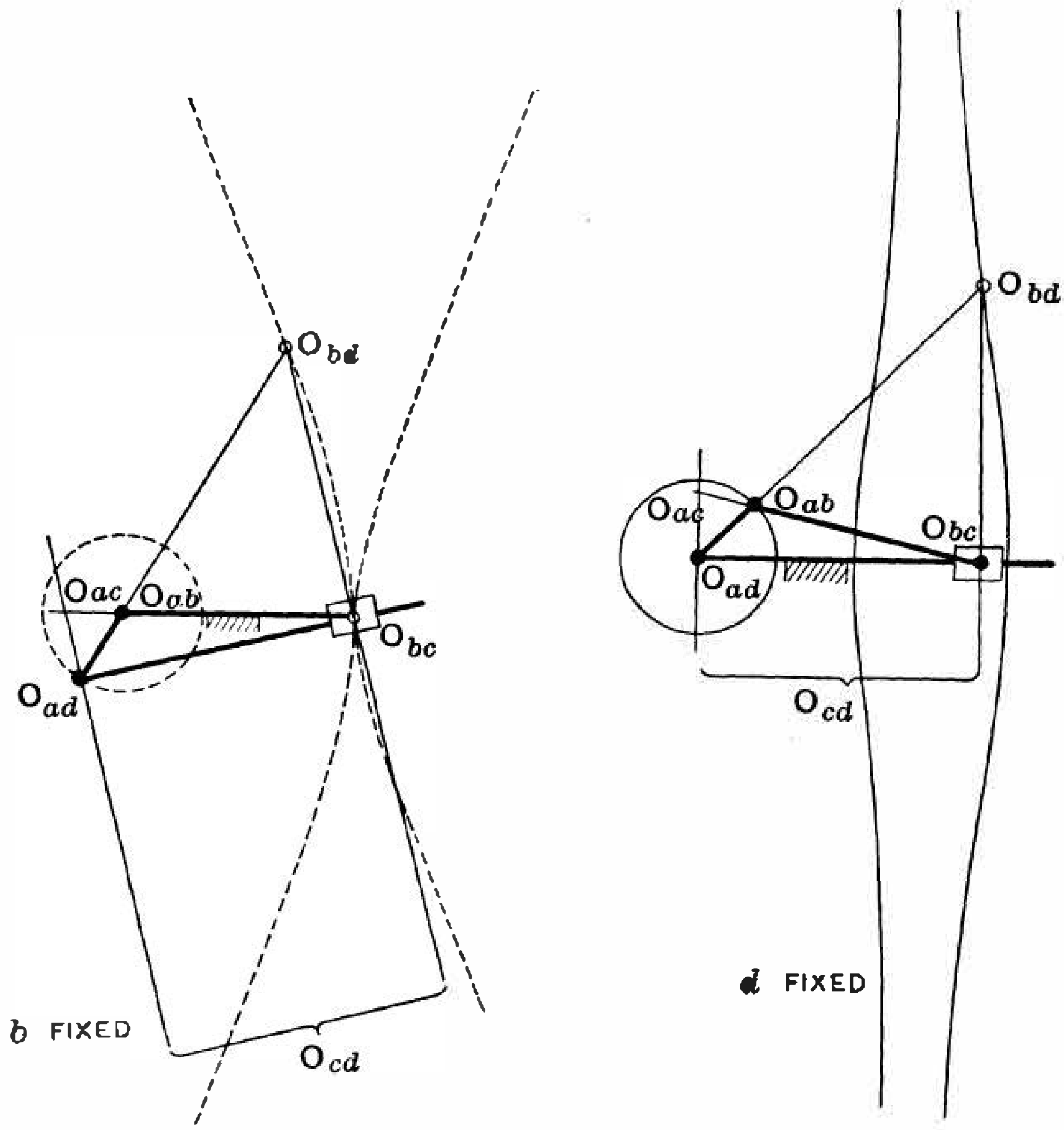


FIG. 61.

necting-rod. This fact is utilized in the design of certain valve-gears.

35. Displacement, Velocity, and Acceleration of Cross-head in Direct-acting Engine. (First Inversion of Slider-crank Chain.)—One of the most important problems in connection with the slider-crank chain is the determination of the velocity and acceleration of the link c, Fig. 60, supposing d to be fixed, and a to rotate with uniform angular velocity. This is approximately the case in a direct-acting steam-engine, where c would represent the cross-head and b the connecting-rod.

It is in general most convenient to deal with these problems graphically, but we shall first give an analytical investigation.

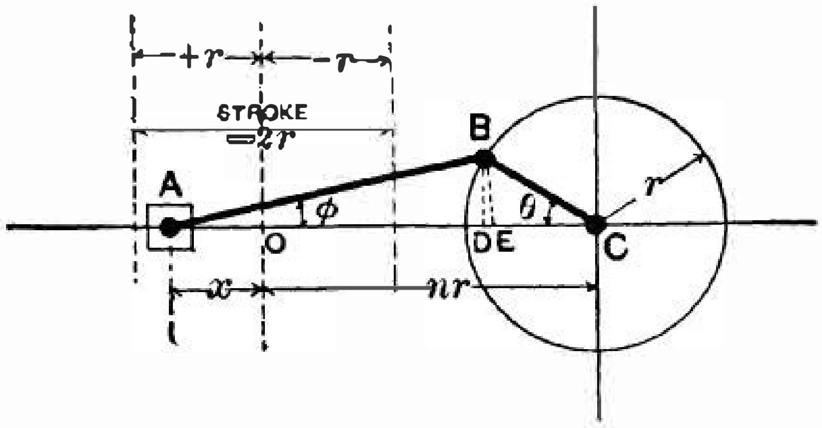


FIG. 62.

In Fig. 62, suppose the line of stroke AO to pass through C, the centre of the crank-shaft. Let BC (the throw of the crank) $=r$, and let $\frac{AB}{BC}=n$, so that the length of connecting-rod $=nr=AB$. When the crank makes any angle θ with the centre line AC, let x be the distance of the cross-head A from O, the middle of its stroke. Draw BD perpendicular to AC, and mark off $AE=AB$. If φ is the angle of obliquity of the connecting-rod,

$$\sin\varphi=\frac{\sin\theta}{n},\quad \text{and}$$

$$\cos\varphi=\frac{1}{n}\sqrt{n^2-\sin^2\theta}.$$

Now
$$\begin{aligned} x&=AC-OC=AC-AB\\ &=CD+DA-AB\\ &=r\cos\theta+nr\cos\varphi-nr\\ &=r(\cos\theta-n+\sqrt{n^2-\sin^2\theta}). \quad . \quad . \quad . \quad . \quad (1)\end{aligned}$$

This gives x in terms of the crank angle θ. It is to be noticed that when $\theta=\frac{\pi}{2}$ the cross-head is not at the middle of its stroke, but at a distance

$$\begin{aligned} x_0&=r(\sqrt{n^2-1}-n)\\ &=-\frac{r}{\sqrt{n^2-1}+n},\end{aligned}$$

the negative sign indicating that A is now to the right of O, Fig. 62.

In the case of a cross-head having simple harmonic motion we should have simply

$$x = r\cos\theta.$$

The term $r(\sqrt{n^2 - \sin^2\theta} - n)$ in equation (1) thus gives what is called the "error due to obliquity" of the connecting-rod. Its values for $\theta = \frac{\pi}{2}$ are shown below for some usual values of n.

$n =$	4	5	6
$\sqrt{n^2 - \sin^2\theta} - n =$	-0.13	-0.11	-0.09

The error due to obliquity is thus seen to diminish rapidly as n increases.*

Next, to determine the velocity of the piston at any instant we differentiate x with regard to time and obtain

$$\frac{dx}{dt} = r\left[-\sin\theta\frac{d\theta}{dt} + \frac{1}{2}(n^2 - \sin^2\theta)^{-\frac{1}{2}}\frac{d}{dt}(n^2 - \sin^2\theta)\right]$$

$$= -r\frac{d\theta}{dt}\left[\sin\theta + \frac{2\sin\theta\cos\theta}{2\sqrt{n^2 - \sin^2\theta}}\right].$$

This is not very convenient for use in practice, but for ordinary values of n we may write without large error n instead of $\sqrt{n^2 - \sin^2\theta}$. For example, if $\theta = \frac{\pi}{2}$, and $\sin\theta$ has its greatest value,

	$\sqrt{n^2 - \sin^2\theta} =$	3.87	4.89	5.91
when	$n =$	4	5	6

Further, we may write V_c, the linear velocity of the crank-

* For a discussion of the problem of the connecting-rod see Hill, Min. Proc. Inst. C. E., Vol. CXXIV, p. 390. Also consult Unwin, Min. Proc. Inst. C. E., CXXV, p. 363, and a paper by G. A. Burls, Min. Proc. Inst. C. E., Vol. CXXXI, p. 338.

pin, instead of $r\frac{d\theta}{dt}$, and, omitting the negative sign, which simply shows that x diminishes at first while θ increases, we have very approximately for the velocity of the piston or cross-head

$$V_p = V_c\left(\sin\theta + \frac{\sin 2\theta}{2n}\right). \quad . \quad . \quad . \quad . \quad (2)$$

As an example, suppose an engine 12 inches stroke running at 250 revolutions per minute, the length of connecting-rod being 3 feet. The crank-pin velocity will be $\frac{250 \times 3.14}{60} =$ 13.08 feet per second. When $\theta = 45^\circ$, the value of n being 6, we have, from equation (2),

$$\begin{aligned} V_p &= 13.08(0.70711 + 0.08333) \\ &= 13.08 \times 0.79044 \\ &= 10.340 \text{ feet per second.} \end{aligned}$$

If the velocity were calculated from the accurate expression previously obtained, we should get

$$\begin{aligned} V_p &= 13.08\left(0.70711 + \frac{1}{2\sqrt{36 - 0.4998}}\right) \\ &= 13.08 \times 0.79103 \\ &= 10.348 \text{ feet per second.} \end{aligned}$$

The approximation, therefore, has led to an error of only 0.008 foot per second in this case.

Proceeding to determine the acceleration of the piston or cross-head for any crank angle, we find very approximately from equation (2), remembering that V_c is constant,

$$\frac{d}{dt}(V_p) = V_c\left(\cos\theta\frac{d\theta}{dt} + \frac{1}{2n}\left[2\cos 2\theta\frac{d\theta}{dt}\right]\right)$$

Now $\frac{d\theta}{dt} = \frac{V_c}{r}$; thus

$$\text{acceleration of piston or cross-head} = \frac{V_c^2}{r}\left(\cos\theta + \frac{\cos 2\theta}{n}\right). \quad (3)$$

The following table gives the value of $\cos\theta + \cos\frac{2\theta}{n}$ for different values of θ and n.

θ.	Value of n.					
	4	4.5	5	5.5	6	∞
0° or 360°	1.250	1.222	1.200	1.182	1.167	1.000
30° or 330°	0.991	0.977	0.966	0.957	0.949	0.866
60° or 300°	0.375	0.389	0.400	0.409	0.417	0.500
90° or 270°	−0.250	−0.222	−0.200	−0.182	−0.167	0.000
120° or 240°	−0.625	−0.611	−0.600	−0.591	−0.583	−0.500
150° or 210°	−0.741	−0.755	−0.766	−0.775	−0.783	−0.866
180°	−0.750	−0.778	−0.800	−0.818	−0.833	−1.000
	Values of $\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$					

36. Graphic Methods for Cross-head Velocity and Acceleration.—We proceed to consider graphic means of determining velocity and acceleration for the cross-head or piston of a direct-acting engine. It is of course possible to draw first a curve of displacement on a time base, and then use the methods described in Chapter II, but simpler means can be employed in this case. In Fig. 63 let AB, BC repre-

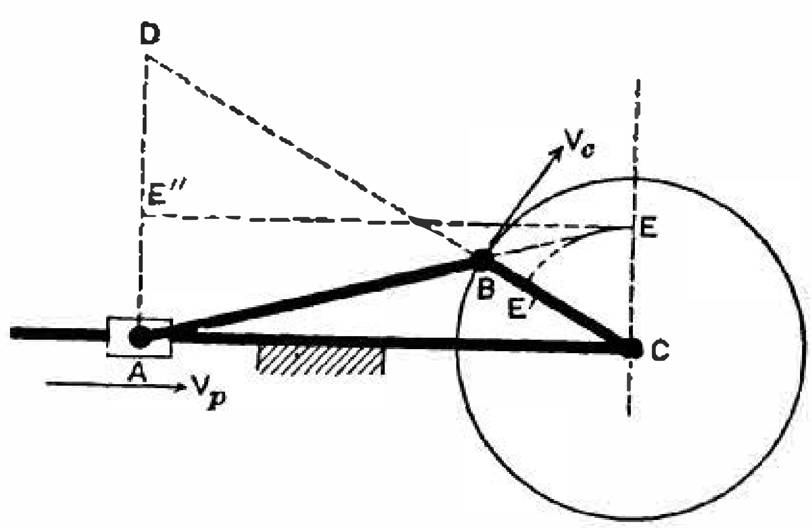


FIG. 63.

sent the connecting-rod and crank in any given position. The point A is moving along the straight line AC, while B is moving for the instant in a direction perpendicular to BC. Hence D, the virtual centre of AB with regard to the fixed link, is easily found at the intersection of the virtual radii of the points A and B. Through C draw a line perpendicular to AC, and therefore parallel to AD, and produce AB to meet it in E. Then the triangles ADB, ECB are similar.

Now the angular velocity of AB about D is measured either by the ratio $\frac{V_p}{AD}$ or by $\frac{V_c}{BD}$, so that

$$\frac{V_p}{V_c} = \frac{AD}{BD} = \frac{CE}{CB}.$$

In many positions of the mechanism D is inaccessible, but E can always be found, and the relation just obtained tells us that CE *represents the velocity of the piston* at the instant for which the diagram is drawn, to the same scale as that to which CB represents the velocity of the crank-pin.

It is generally most convenient to make a polar diagram of piston velocity by marking off a series of points such as E' (where $CE' = CE$) for a number of different crank positions, or, if required, a velocity diagram on a distance base may be constructed by marking off the distance CE along AD, so that a series of points such as E'' are obtained, and a curve drawn whose ordinate at any point is proportional to the velocity of the piston when in that position. Such diagrams have been drawn in Fig. 64, together with a linear velocity diagram on a time base, so as to show the difference between a simple harmonic motion and that which the piston actually possesses. The example taken is that for which the velocity and acceleration have been calculated in the last section. In order to determine the scale to which the ordinates of the curves represent the velocity, it is only necessary to remember that if the length BC were 1 inch, the velocity scale would be 1 inch = 13.08 feet per second, since the crank-pin velocity is 13.08 feet per second. In the figure the construction lines are shown for one position of the mechanism only; in drawing such diagrams care should be taken only to draw those portions of the construction lines which are absolutely necessary, so as to avoid useless complication. Of course accuracy in drawing is indispensable if the numerical results obtained are to be reliable. A line whose length is proportional to the *piston acceleration*

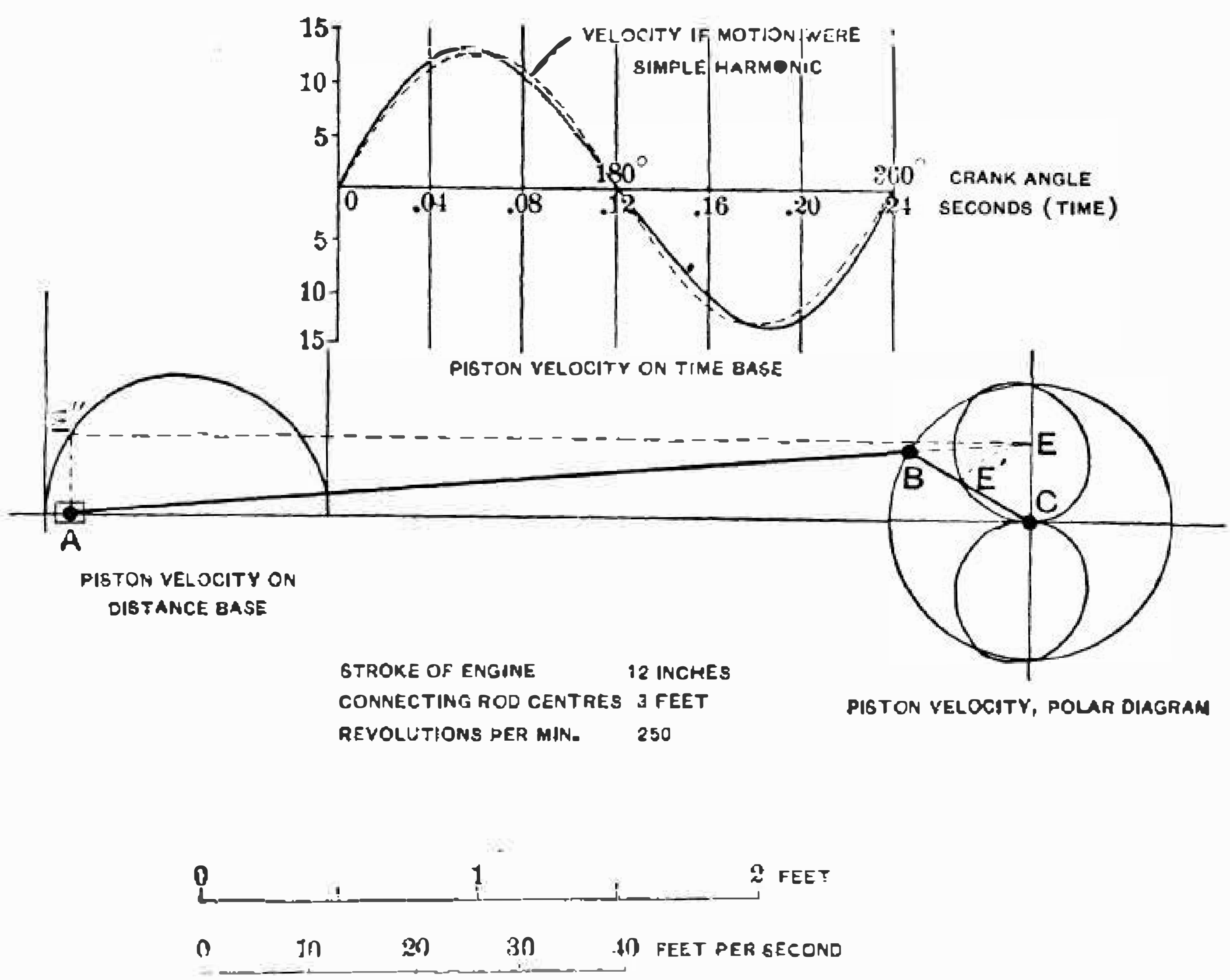


FIG. 64.

may be obtained as follows (see Fig. 65): Take any given position of the cross-head A, and produce AB to cut CE in E. Then, as before,

$$\frac{CE}{CB} = \frac{V_p}{V_c}.$$

Notice that this is true whether the path of the point A passes through C or not, when produced.

The desired acceleration is the rate of change of V_p, which is of course proportional to the rate at which the distance CE is increasing or diminishing at the instant considered. In fact the piston *acceleration* may be considered as being proportional to the *velocity* of the point E along CE at any instant while the engine is in motion, supposing BE always to be in a straight line with AB.

Let this velocity along CE be u_0. The real velocity of the point E, regarded as a point on the connecting-rod, is in a direction perpendicular to DE, its virtual radius. Calling this velocity u_1, we see that u_1 may be resolved into two

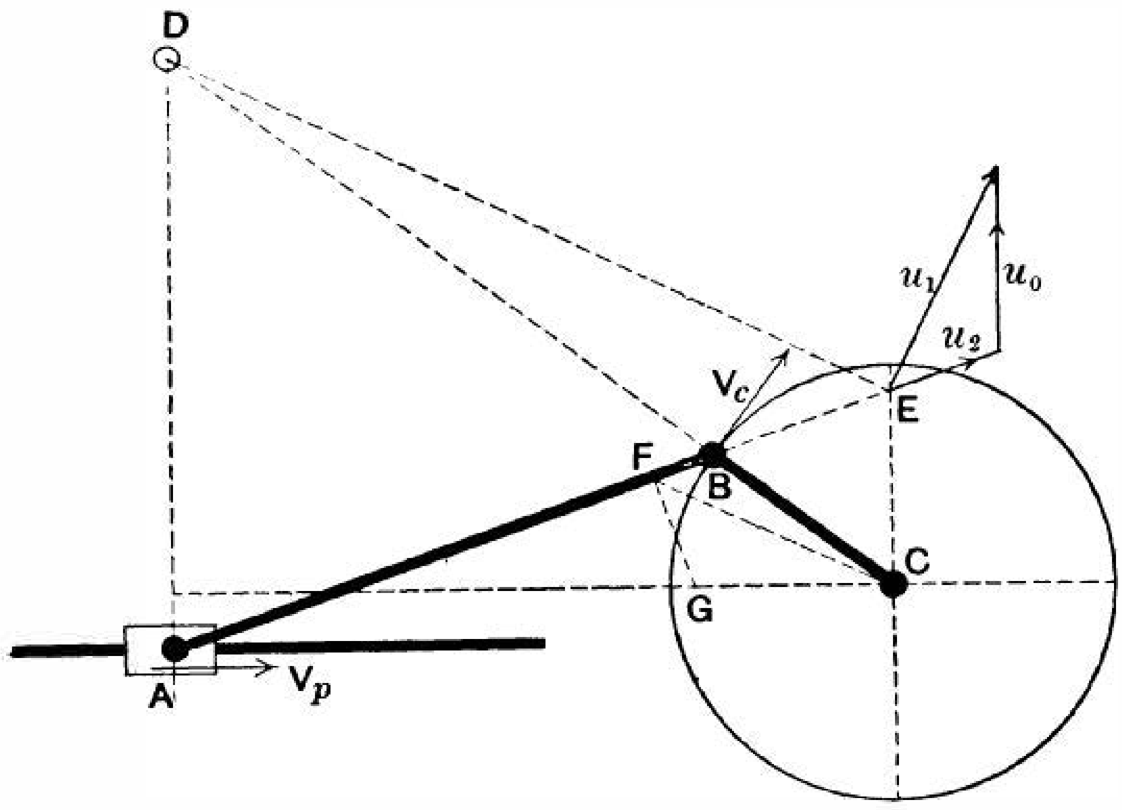


FIG. 65.

components, namely, u_0 in a direction along CE, and u_2 in a direction along BE.

From C draw CF parallel to DE, and draw FG perpendicular to AB. Then the sides FC, CG, GF, of the triangle FCG are respectively perpendicular to the directions of u_1, u_0, u_2. Thus FCG is a triangle of velocities and $\frac{u^0}{u_1}=\frac{CG}{FC}$, or $u_0=u_1\frac{CG}{FC}$. But $\frac{u_1}{V_c}=\frac{DE}{DB}=\frac{FC}{CB}$, therefore $u_1=V_c\frac{FC}{CB}$, and

$$u_0=V_c\frac{CG}{CB}=\text{rate of change of length } CE.$$

Now it has been shown that

$$\text{piston velocity}=V_p=CE.\frac{V_c}{CB},$$

and V_c and CB are constantt hence it follows that the rate of change of the piston velocity must be equal to

$$\text{(rate of change of } CE) \times \frac{V_c}{CB}, \text{ that is,}$$

$$\text{piston acceleration} = u_0 \frac{V_c}{CB} = \frac{V_c^2}{CB^2} . CG.$$

Thus to obtain the numerical value of the piston acceleration we must multiply the length of CG (measured to scale in feet) by $\left(\frac{V_c}{r}\right)^2$, where V_c is the crank-pin velocity in feet per second and r is the crank throw, or radius of the crank-pin circle, in feet.

Hence it follows that

$$\frac{CG}{CB} = \frac{\text{acceleration of piston}}{V_c^2/r},$$

or, in other words, CG represents the piston acceleration to the same scale as that on which CB represents V_c^2/r, the radial acceleration of the crank-pin.

When drawing such a diagram as Fig. 65 it happens that for many positions of the crank the point D becomes inaccessible. Accordingly some other construction must be found to obtain the position of the point F, so that CG may be determined for any crank angle.

Consider the triangles BEC and BAD.

Evidently

$$\frac{BE}{BA} = \frac{BC}{BD}.$$

But $\frac{BC}{BD} = \frac{BF}{BE}$, because the triangles BDE, BCF are similar.

Therefore

$$\frac{BE}{BA} = \frac{BF}{BE},$$

or

$$BA . BF = BE^2.$$

Hence any construction which will make BE a mean proportional between BA and BF will determine the point F.

A number of such constructions have been given; of these perhaps the most convenient in practice is that of Kisch.*

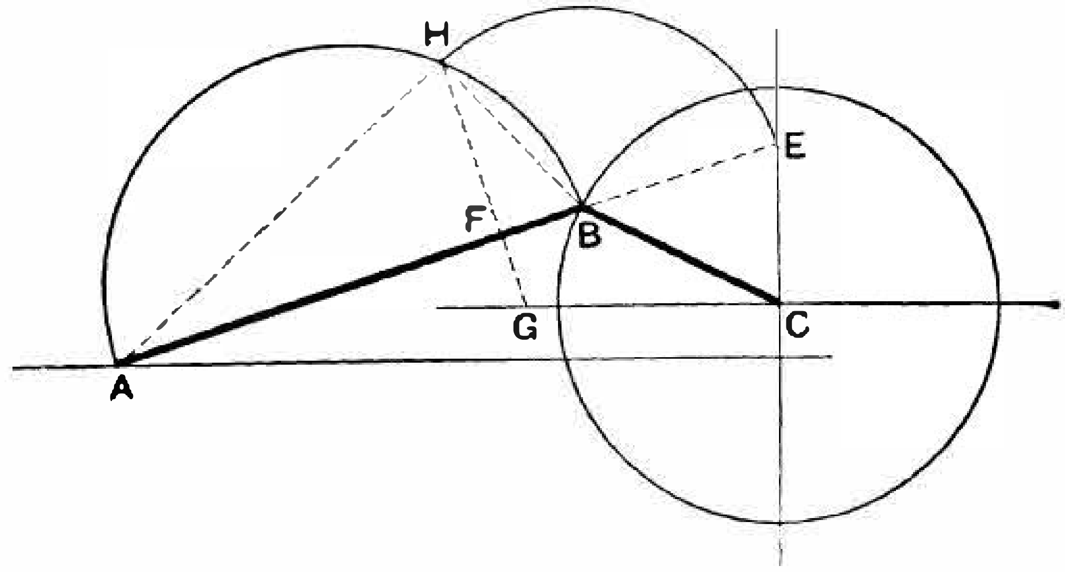


FIG. 66.

On AB describe a semicircle AHB. With centre B and radius BE cut the semicircle in H. Draw HFG perpendicular to AB, cutting AB in F and CG in G. Join BH, HA. Then

$$\frac{BF}{BH}=\frac{BH}{BA}.$$

But $BE=BH$. Hence $BA.BF=BE^2$, and CG represents the acceleration.

The method of determining the acceleration scale of such a diagram may be shown by a numerical example. Fig. 67 has been drawn for the engine for which the velocity of the piston has been previously calculated, taking a crank angle of 45°. The crank-pin velocity being 13.08 feet per second, and the connecting-rod being 6 cranks in length, we have for the acceleration of the piston at that particular crank angle

$$\text{acceleration}=\frac{13.08^2}{0.5}\left(\cos 45^\circ+\frac{\cos 90^\circ}{6}\right)$$

$$=\frac{13.08\times 13.08\times 0.70711}{0.5}$$

$$=242.1 \text{ feet per second per second.}$$

* See Zeitschrift des Vereines Deutscher Ingenieure, Dec. 13, 1890. Given also by Klein, Journal of Franklin Inst., Vol. CXXXII, Sept. 1891.

In Fig. 67 the actual length of the line CG, if the figure were drawn the full size of the engine, would be 0.351 foot. The radius of the crank-pin circle CB is 0.5 foot and repre-

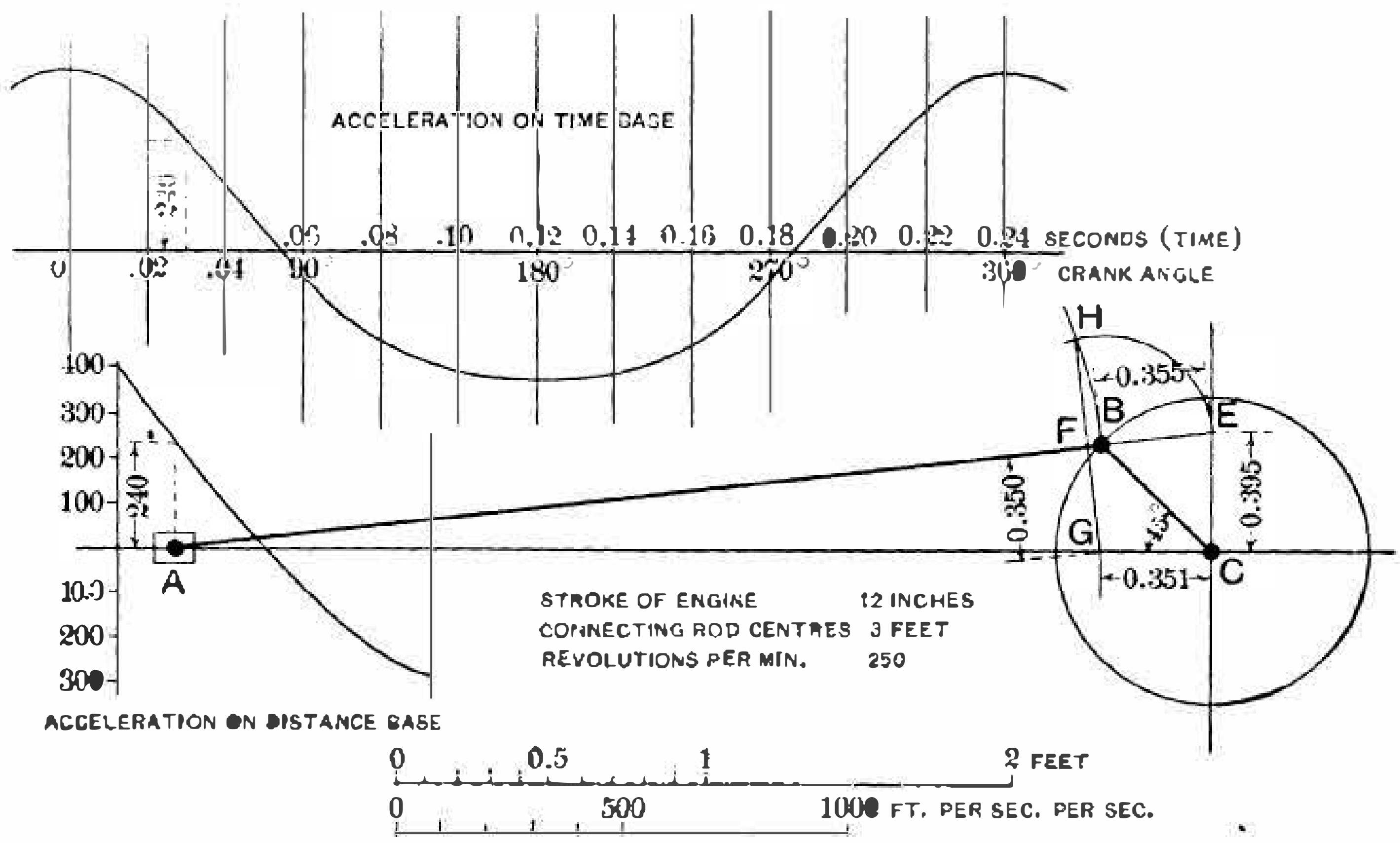


FIG. 67.

sents a velocity of 13.08 feet per second. Hence the velocity scale is 1 foot = 26.16 feet per second, or $\frac{V_c}{CB} = \frac{13.08}{0.5} = 26.16$.

It has been shown that the rate at which the distance CE is changing is

$$u_0 = V_c \cdot \frac{CG}{CB}$$

$$= 0.351 \times 26.16 = 9.18 \text{ feet per second.}$$

This, however, is not the numerical value of the acceleration required, for it represents the rate of change of a length CE, each foot of which stands for a velocity of 26.16 feet per second. Therefore, expressing u_0 in feet per second per second, we have

$u_0 = 0.351 \times 26.16 \times 26.16 = 240$ feet per second per second, a result agreeing (within the limits of accuracy for a small scale drawing) with that just calculated.

It is thus seen that to determine the scale to which CG

represents the piston acceleration, we find, first, the piston velocity represented by unit length of CE (in this case 26.16 feet per second); then it follows that a change of length of CE at the rate of one unit per second represents a change of piston velocity at the rate of 26.16 units per second, or a piston acceleration of 26.16 units. But each unit of length of CG has been shown to represent a change of length of CE at the rate of 26.16 units per second, so that, finally, unit length of CG represents a piston acceleration of 26.16×26.16 units.

This relation may be expressed by saying that if the engine were drawn out full size and the linear velocity scale were 1 foot $= n$ feet per second, then the acceleration scale would be 1 foot $= n^2$ feet per second per second. In this case, as in the case of all graphic methods of determining velocities and accelerations, the manner of finding the velocity and acceleration scales must be thoroughly understood; if this is not done, the diagram becomes almost useless, since no numerical values can be obtained from it.

A number of other constructions for the piston acceleration in the direct-acting engine have been devised.*

37. Angular Velocity and Acceleration of Connecting-rod.—To study the movement of the connecting-rod, adopting the same notation as in § 35, we have, as before,

$$\sin \varphi = \frac{\sin \theta}{n},$$

$$\cos \varphi = \frac{\sqrt{n^2 - \sin^2 \theta}}{n}.$$

The angular velocity of the connecting-rod is the rate of change of φ with regard to time, and we obtain at once

$$\frac{d\varphi}{dt} = \frac{\cos \theta}{\sqrt{n^2 - \sin^2 \theta}} \frac{d\theta}{dt}.$$

* See a paper by Prof. Elliott, Engineering, Vol. LIX, pp. 587 and 711, and Zeitschrift des V. Deutscher Ingenieure, Oct. 13, 1894.

Since $\frac{d\theta}{dt}$ is the angular velocity of the crank, we have

$$\text{angular velocity of connecting-rod} = \frac{V_c}{r} \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}}.$$

Differentiating again to find the angular acceleration, we obtain

$$\frac{d^2\varphi}{dt^2} = \frac{V_c}{r} \cdot \frac{d}{d\theta} \frac{\cos\theta}{\sqrt{n^2-\sin^2\theta}} \cdot \frac{d\theta}{dt}$$

$$= \frac{V_c^2}{r^2}\left\{\cos\theta \frac{d}{d\theta}(n^2-\sin^2\theta)^{-\frac{1}{2}} - \sin\theta(n^2-\sin^2\theta)^{-\frac{1}{2}}\right\}$$

$$= -\frac{V_c^2}{r^2} \cdot \frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}} \cdot \frac{n^2-1}{n^2-\sin^2\theta}. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For ordinary values of n it is sufficiently accurate to write approximately

$$\text{angular acceleration} = -\frac{V_c^2}{r^2} \cdot \frac{\sin\theta}{\sqrt{n^2-\sin^2\theta}}. \quad . \quad (2a)$$

Taking the same example as before, at a crank angle of 45° we have

$$\sin\theta = 0.70711, \quad n = 6, \quad V_c = 13.08, \quad r = 0.5.$$

Thus $\sqrt{n^2-\sin^2\theta} = 5.96$ and $\cos\theta = 0.70711$. Therefore

$$\text{angular velocity} = \frac{13.08 \times 0.70711}{0.5 \times 5.96} = 3.11 \text{ radians per second,}$$

and, from equation (2),

$$\text{angular acceleration} = -\frac{13.08^2}{0.25} \cdot \frac{0.70711}{5.96} \cdot 0.987$$

$$= -80.2 \text{ radians per second per second.}$$

Using equation (2a), we should obtain -81.2 as a result.

The simple construction of Figs. 65, 66, and 67 gives us the angular velocity of the connecting-rod. For

$$\text{angular velocity of crank} = \frac{V_c}{BC},$$

and

$$\text{angular velocity of connecting-rod} = \frac{V_c}{BD}.$$

But $\frac{BC}{BD}=\frac{BE}{BA}$; hence

$$\text{angular velocity of connecting-rod} = \frac{V_c}{r} \cdot \frac{BE}{BA}.$$

If BE is taken at the length it would have were the engine drawn out full size, $BA = nr$, and

$$\text{angular velocity} = BE \cdot \frac{V_c}{nr^2}.$$

(Note that the real lengths of BE and r must be used, measured in feet, V_c being in feet per second.)

In Fig. 67, for example, BE scales 0.355 foot, hence the angular velocity will be

$$0.355 \times \frac{13.08}{6 \times 0.25} = 3.10 \text{ radians per second},$$

a result agreeing with the calculated value.

As regards the angular acceleration we have seen that

$$FC : CG : GF :: u_1 : u_0 : u_2.$$

The velocity u_2 is the rate at which the length BE is changing, and is therefore proportional to the rate of change of the angular velocity of the connecting-rod. Hence it may be shown (just as in the case of the velocity u_0) that

$$\text{angular acceleration} = FG \cdot \frac{V_c^2}{nr^3}.$$

In our example (Fig. 67) FG is 0.350 feet; hence

$$\text{angular acceleration} = 0.350 \times \frac{13.08 \times 13.08}{6 \times 0.125}$$

$$= 79.9 \text{ radians per second per second,}$$

a result agreeing closely with that previously obtained.

Notice that when the crank angle is 90° FG becomes $\frac{nr}{\sqrt{n^2-1}}$ and therefore, if $\theta=\frac{\pi}{2}$,

$$\text{angular acceleration} = \frac{V_c^2 \cdot nr}{nr^3\sqrt{n^2-1}} = \frac{V_c^2}{r^2\sqrt{n^2-1}}.$$

38. Angular Velocity of Cylinder in Oscillating Engine. Second Inversion of Slider-crank Chain.—The second inversion of the slider-crank chain is that in which the link b

(represented by the connecting-rod in a direct-acting steam-engine) is the fixed link. This mechanism is known as the swinging-block slider-crank and is employed as an oscillating steam-engine, of which the link d becomes the piston and rod, while b is the framework. The link c is the cylinder and a is the crank, the cylinder swinging to and fro on trunnions as the crank-shaft revolves. We proceed to compare the angular velocity of the cylinder with that of the crank, the latter being supposed to rotate uniformly.

Let Fig. 68 represent this mechanism. The distance AB is the length of the fixed link, measured from the centre of the cylinder-trunnions to the centre of the crank-shaft, while BC is the half-stroke of the piston. Let $\frac{AB}{BC}=n$. Let the angle the crank has turned through from its lowest position be θ, φ being the angle at which the centre line of the cylinder is inclined to AB. Then

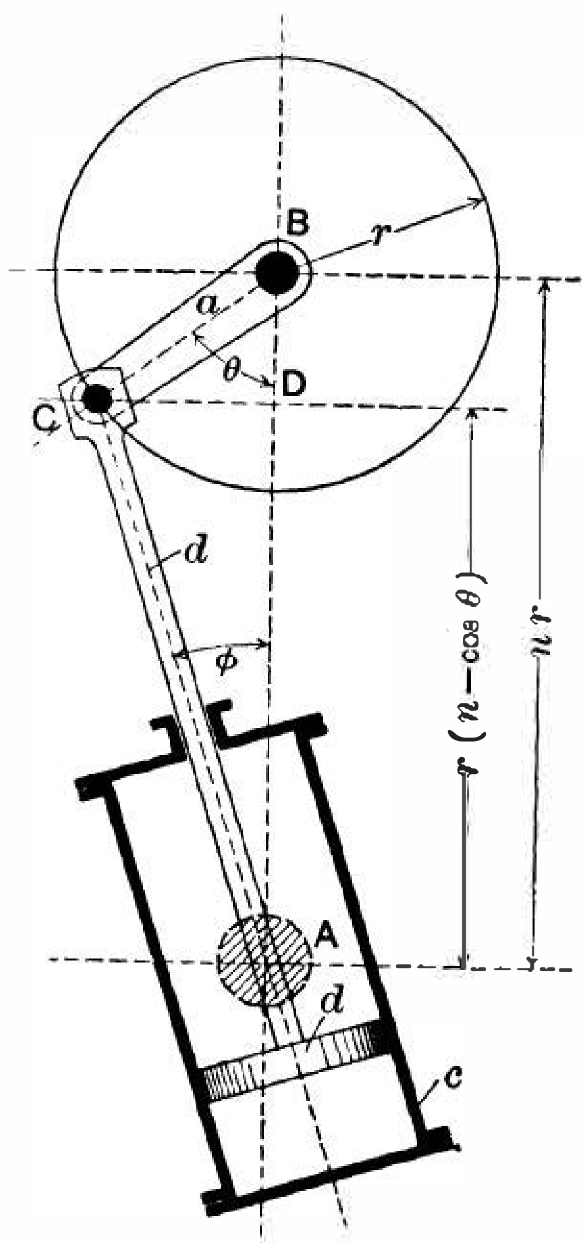


FIG. 68.

$$\tan\varphi=\frac{CD}{DA}=\frac{\sin\theta}{n-\cos\theta}.$$

The angular velocity of the cylinder is $\frac{d\varphi}{dt}$, so that on differentiating

$$\frac{d\varphi}{dt}=\cos^2\varphi\,.\,\frac{d}{d\theta}\frac{\sin\theta}{n-\cos\theta}\cdot\frac{d\theta}{dt}$$

$$=\cos^2\varphi\,\frac{n\cos\theta-1}{(n-\cos\theta)^2}\cdot\frac{d\theta}{dt}.$$

But $\cos^2\varphi=\frac{AD^2}{AC^2}=\frac{AD^2}{AD^2+DC^2}=\frac{(n-\cos\theta)^2}{(n-\cos\theta)^2+\sin^2\theta}.$

Thus $\frac{d\varphi}{dt}=\frac{n\cos\theta-1}{n^2-2n\cos\theta+1}\cdot\frac{d\theta}{dt}.$

From this we find by again differentiating

$$\frac{d^2\varphi}{dt^2} = -\frac{n \sin\theta(n^2-1)}{(n^2-2n\cos\theta+1)^2}\cdot\left(\frac{d\theta}{dt}\right)^2,$$

which is the value of the angular acceleration of the cylinder for any crank angle θ.

Notice that since the angular velocity of the crank is uniform, the cylinder executes its forward and backward swings in unequal times. By assigning suitable proportions this particular inversion of the slider-crank chain may be utilized as a quick-return motion (see Fig. 72), by

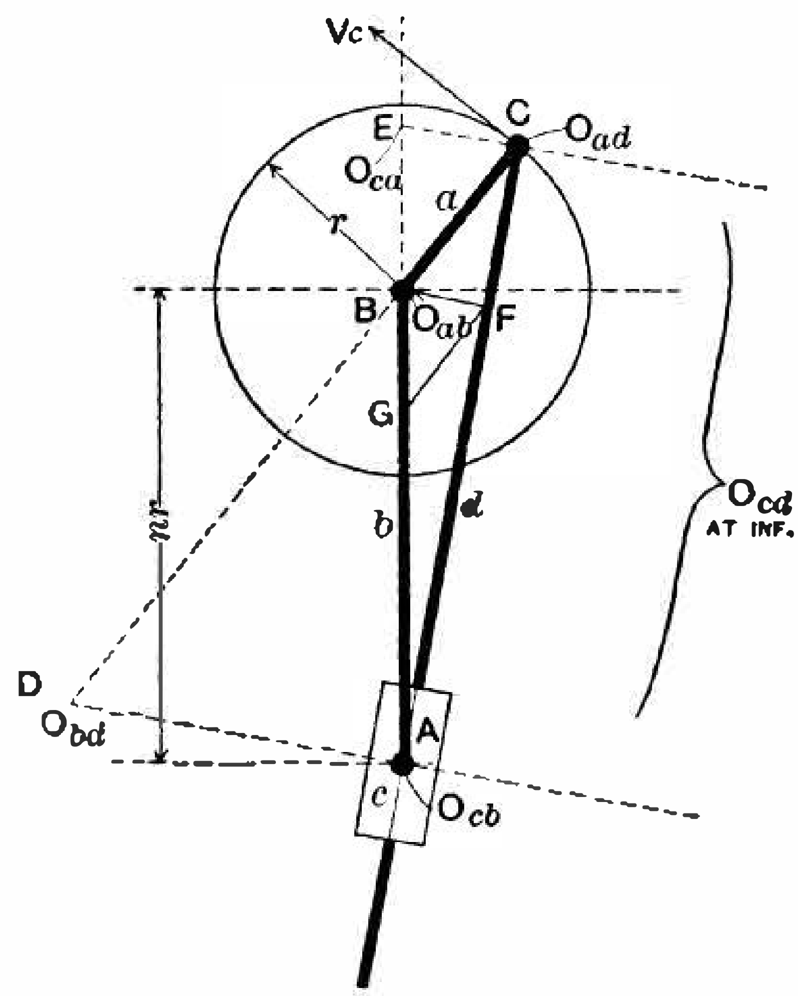


FIG. 69.

causing the swinging link c to actuate the tool-box, say, of a shaping-machine, which can thus be made to perform its return or non-cutting stroke at a quicker rate, and in less time, than its forward or cutting stroke.

The velocity ratio of cylinder and frame may readily be obtained graphically. The positions of the six virtual centres of the mechanism are shown in Fig. 69. Let ω_{ab} represent the angular velocity of the crank with respect to the frame b; we wish to find ω_{cb}. E (the virtual centre of c with regard to a) is a point common to the two bodies a and c for the instant considered. Its linear velocity may be expressed either as $\omega_{cb} \times AE$ or as $\omega_{ab} \times EB$. Hence

$$\frac{\omega_{cb}}{\omega_{ab}} = \frac{EB}{AE}.$$

Draw BF parallel to CE, and FG parallel to CB. Then

$$\frac{\omega_{cb}}{\omega_{ab}} = \frac{EB}{AE} = \frac{CF}{AC} = \frac{BG}{BA};$$

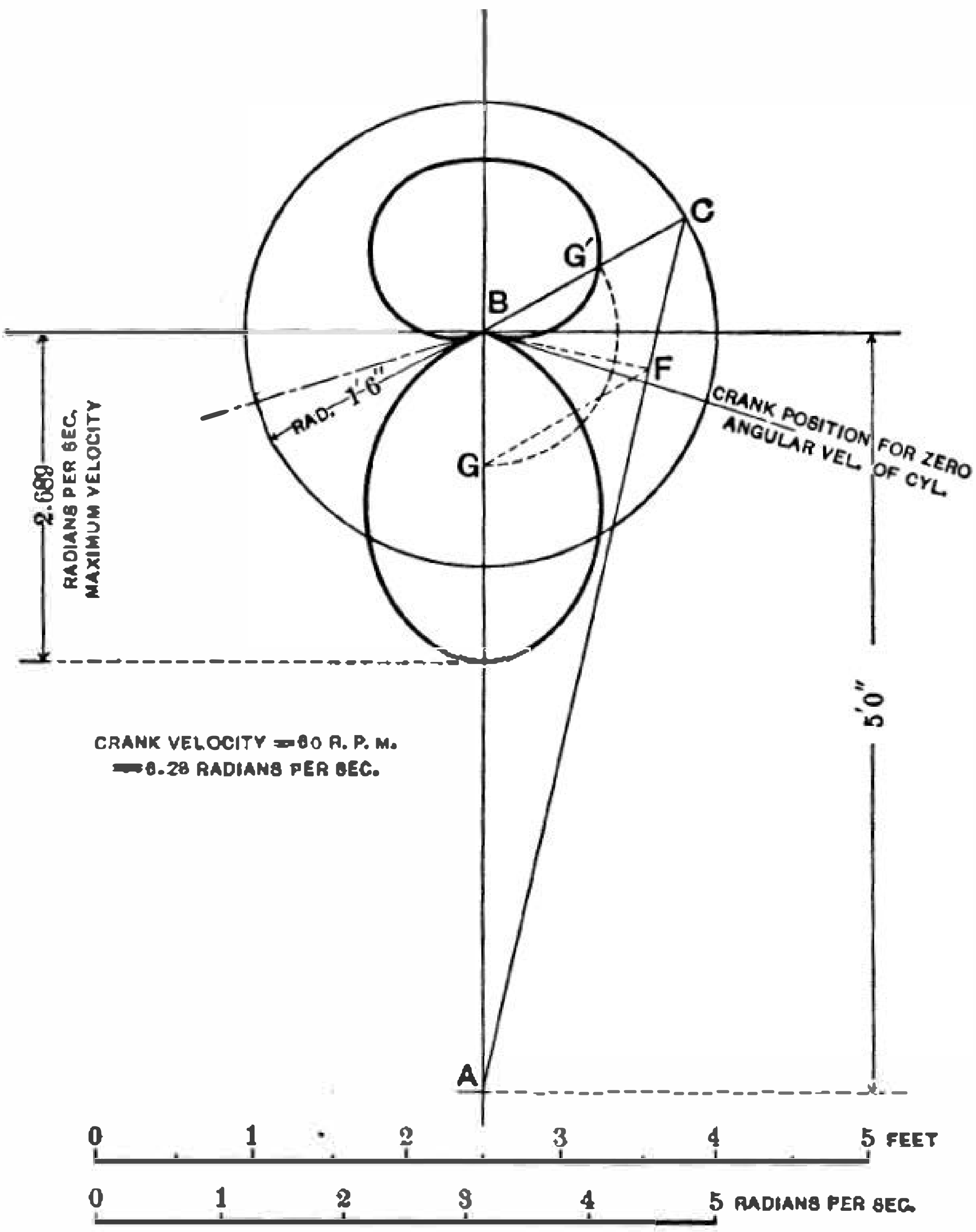


FIG. 70.

and since BA is constant, the length BG is proportional to the angular velocity of the cylinder. A polar diagram may be drawn by marking a distance BG' along BC equal to BG and repeating the construction. This has been done in Fig. 70 for an oscillating engine in which the stroke is 3 feet and the distance AB is 5 feet. At 60 revolutions per minute the maximum angular velocity of the cylinder is represented on the diagram by a distance measuring 2.14 feet to scale, hence its numerical value (the angular velocity of the crank being 2π radians per second) will be

$$6.28\times\frac{2.14}{5}=2.689 \text{ radians per second.}$$

The angular acceleration may easily be obtained by construction from the velocity diagram, as is shown in Fig. 71. The value of the velocity and acceleration should be calculated for one or two positions of the crank, as an exercise, and compared with the diagrams.

Notice that the value of the angular-velocity ratio when $\theta=0°$ or $180°$ is

$$\frac{\omega_{cb}}{\omega_{ab}}=\frac{V_c}{r(n\pm 1)}\cdot\frac{r}{V_c}=\frac{1}{n\pm 1}.$$

Notice also that the angular velocity and angular acceleration will be the same for the piston and rod as for the cylinder. It is easy to show that $\frac{\omega_{db}}{\omega_{ab}}=\frac{CB}{CD}=\frac{BG}{AB}$ (Fig. 69).

Fig. 71 shows the angular velocity and acceleration (in the same example) as plotted on a polar diagram (the acceleration curve being found from the velocity curve as in § 22), and also shows the corresponding linear diagrams on a time base, the scales being marked. The linear diagram of acceleration could be obtained from the velocity-time curve by the method of § 19, but the acceleration scale would not then be the same as that shown on the figure.

A number of problems dealing with velocity ratios and accelerations in the oscillating engine have been worked out by Professor Elliott in a communication to *Engineering*, Vol. LXIII, page 665, to which the reader is referred.

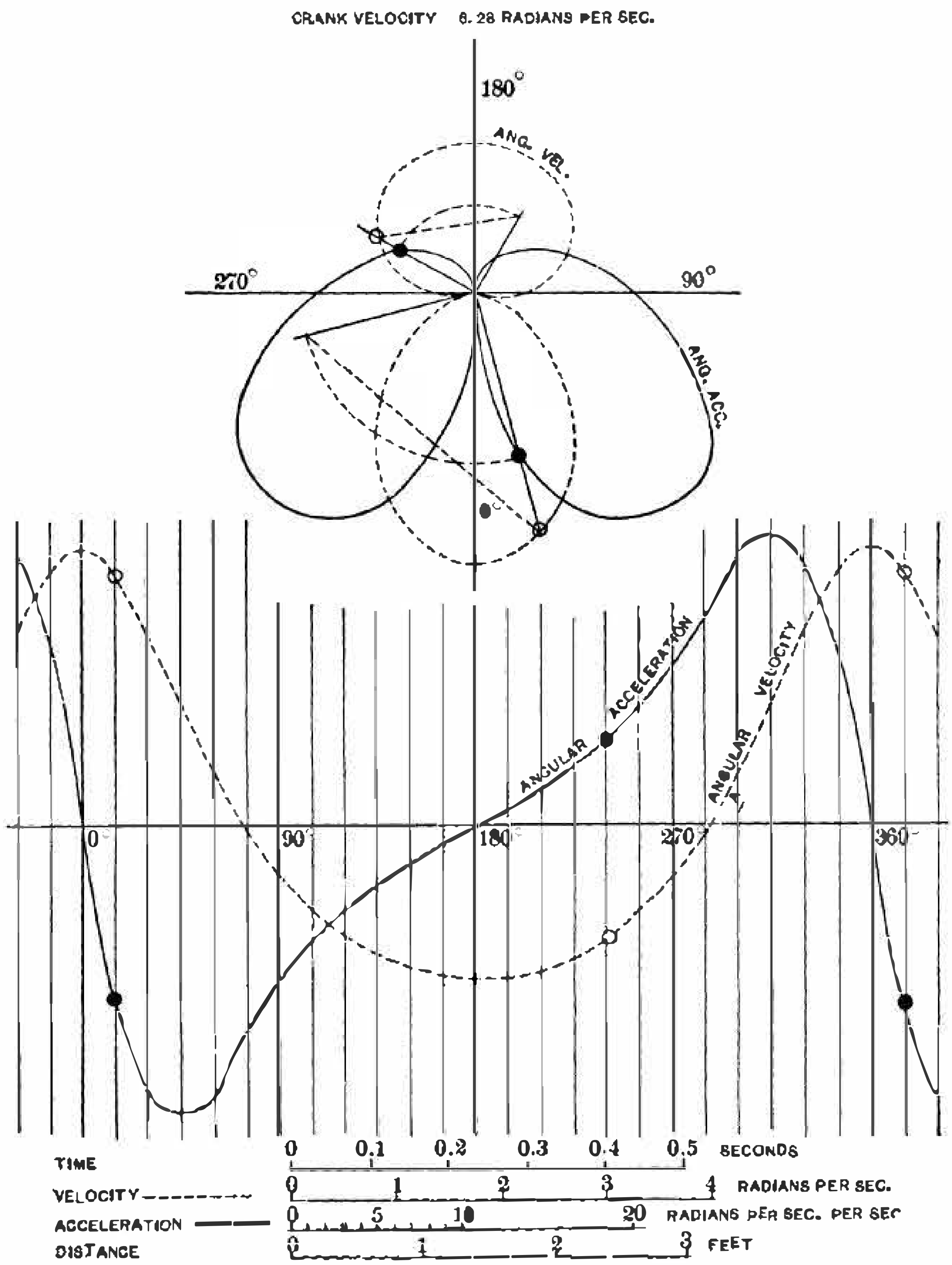


FIG. 71.

A general method will be given later (Chapter V) by which the linear or angular velocity or acceleration may be found graphically for any point on a link of a mechanism of the kind discussed in this chapter.

A good example of the kinematic identity of mechanisms which at first sight appear to be very different is afforded in Fig. 72. The links which correspond in the two cases have the same letters attached. The sketch (*a*) represents the oscillating engine, while (*b*) gives a diagrammatic view of the corresponding quick-return motion. Both are derived

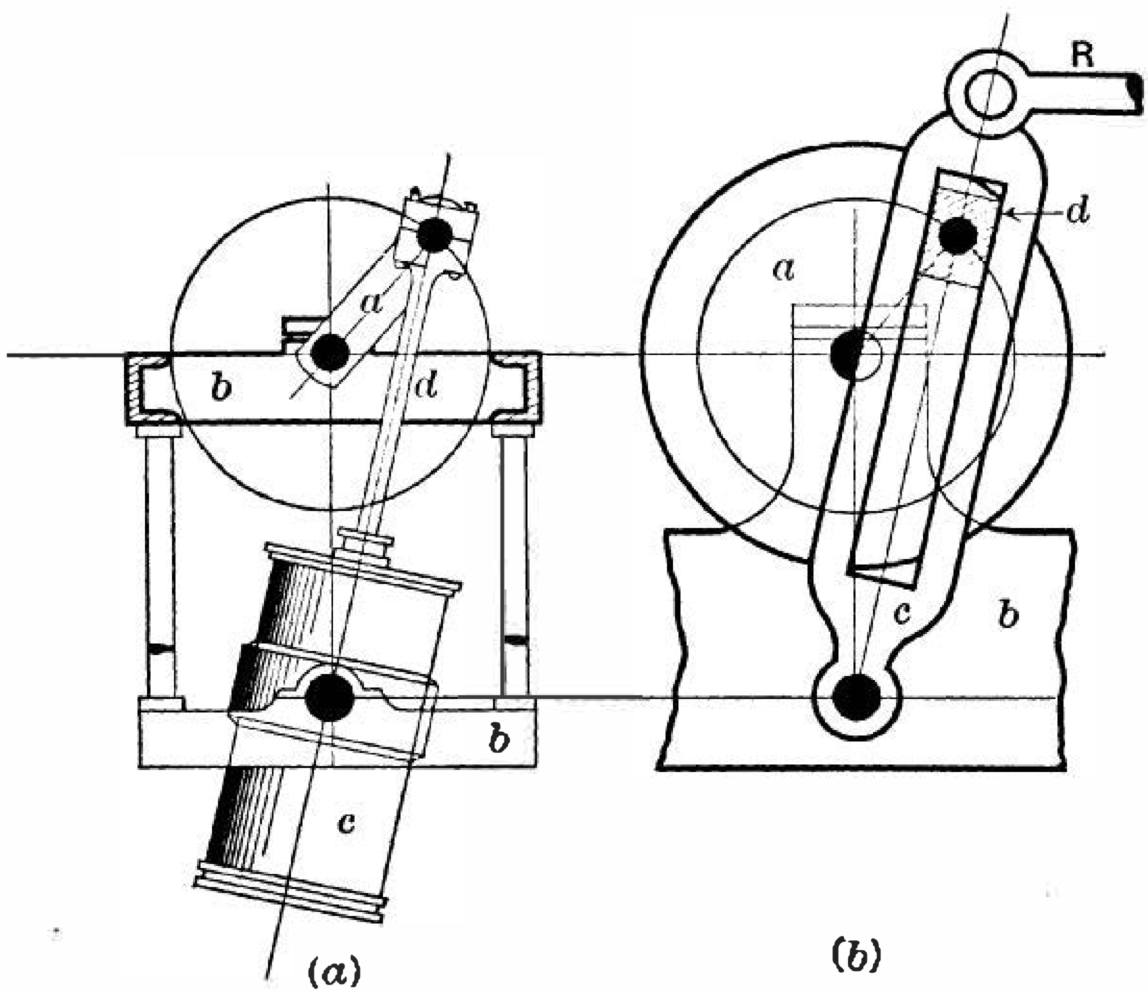


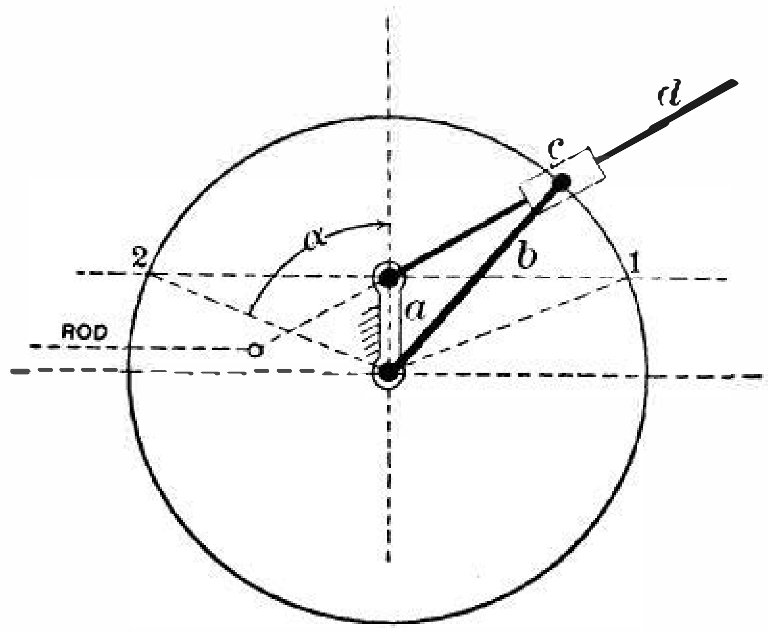
FIG. 72.

from the same inversion of the slider-crank chain. The swinging link *c* has the same relative motions in regard to the links *b* and *d* (with which it pairs) in the quick-return motion as in the engine. The framing of the engine corresponds to the fixed framework of the machine-tool. The rod *R*, which of course does not appear in the engine, communicates the variable motion of the swinging link *c* to the tool-carriage. The crank-shaft of the engine is represented by the disc *a*, to which rotary motion is imparted by the driving mechanism of the tool (not shown).

39. Whitworth Quick-return Motion. Third Inversion of Slider-crank Chain.—Passing to the next inversion, we now have *a* as the fixed link, and the resulting mechanism is one which was applied by Whitworth* as a quick-return motion for the same purposes as have already been mentioned. It has been called by Reuleaux the *turning-block slider-crank chain.*

The velocity ratio of the links b and d may be obtained

FIG. 73*a*.



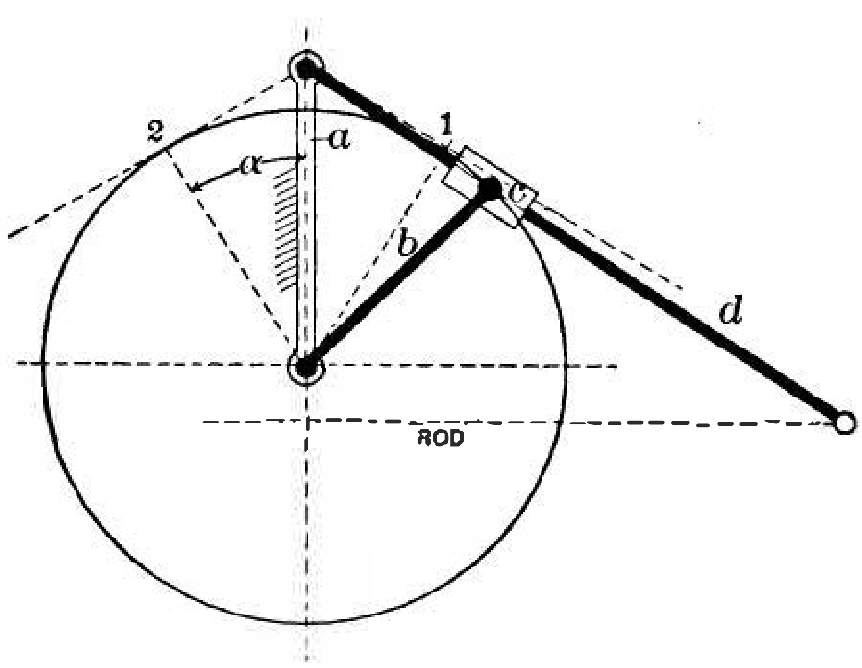


FIG. 73*b*.

exactly as in the case of the oscillating engine. In fact an alteration in the relative lengths of the links a and b, Fig. 70, converts the mechanism there shown into one not differing in any essential particular from the quick-return motion of Fig. 73*b*.

* See Cotterill, Applied Mechanics, p. 108, § 50.

Figs. 73*a* and 73*b* show this quick-return motion with different proportions of the lengths of the links *a* and *b*. It will be seen that while *d* executes complete rotations in the first case, it only swings in the second case.

The relative angular velocities of *b* and *d* are easily found by the construction shown in Fig. 69. Evidently if *a* is the fixed link

$$\frac{\omega_{da}}{\omega_{ba}} = \frac{BD}{CD} = \frac{BA}{EA} = \frac{GA}{BA}.$$

Thus if AB represents the angular velocity of the link *b*, AG represents that of *d* to the same scale.

Supposing the link *b* to have a uniform angular velocity, the positions 1 and 2 are those in which the tool-box is at one or the other end of its travel. Accordingly it is easily seen that the times of the cutting and return strokes will be in the ratio $\frac{180-\alpha}{\alpha}$. Hence in designing such a motion we have only to proportion *a* and *b* so that $\frac{a}{b} = \cos\alpha$ in the Whitworth motion, or $\frac{b}{a} = \cos\alpha$ in the other form, where α has such a value as to make $\frac{180^\circ-\alpha}{\alpha}$ the desired ratio. The times of the cutting and return strokes are often in the ratio 2 : 1 or 3 : 1.

The centrodes for the links *c* and *a* are found by similar constructions to those already shown in Fig. 61 for the links *b* and *d*, and are drawn in Fig. 74. The reader should construct them for himself for the case shown (in which the length of the link *a* is less than that of *b*), and also for the case in which the link *a* is longer than *b*, when the centrodes take quite different forms.

40. Pendulum Pump. Fourth Inversion of Slider-crank Chain.—The last of the four possible inversions of the chain, the *swinging slider-crank*, in which *c* is the fixed link, hash only a very limited application in practice, but has been employed as a small steam donkey-pump. It is shown diagrammatically in Figs. 75*a* and 75*b*, and is shown also in outline in the second sketch in Fig. 74.

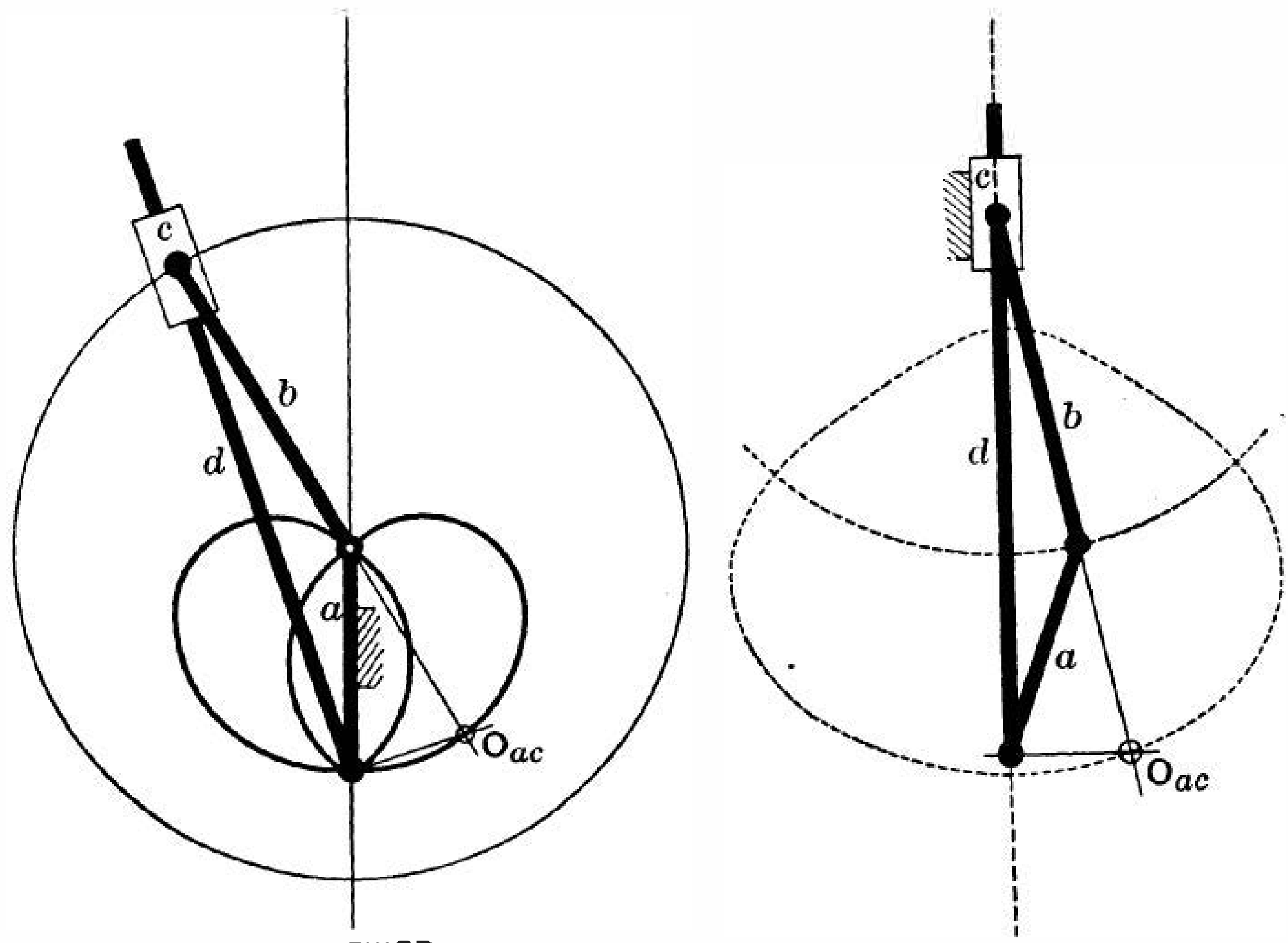


Fig. 74.

It is known as the "pendulum pump," from the motion of the link b. The link c now represents the cylinders (steam and water) and their connecting framework, while d is the piston, rod, and plunger. The crank a takes the form of a small fly-wheel, which rotates about O_{ab}, while that point swings along the dotted arc. The relative angular velocity of b and d is easily found graphically, the virtual centres being known. Let V = linear velocity of d; then, since the link a is turning for the instant about E, we have

$$\text{angular velocity of } a = \frac{\text{linear velocity of point } B}{BE} = \frac{V}{CE},$$

and

$$\text{linear velocity of point } B = V \cdot \frac{BE}{CE}.$$

Therefore

$$\text{angular velocity of } b = V.\frac{BE}{BA \times CE}$$

$$= V.\frac{BA}{BA \times AD} = \frac{V}{AD}.$$

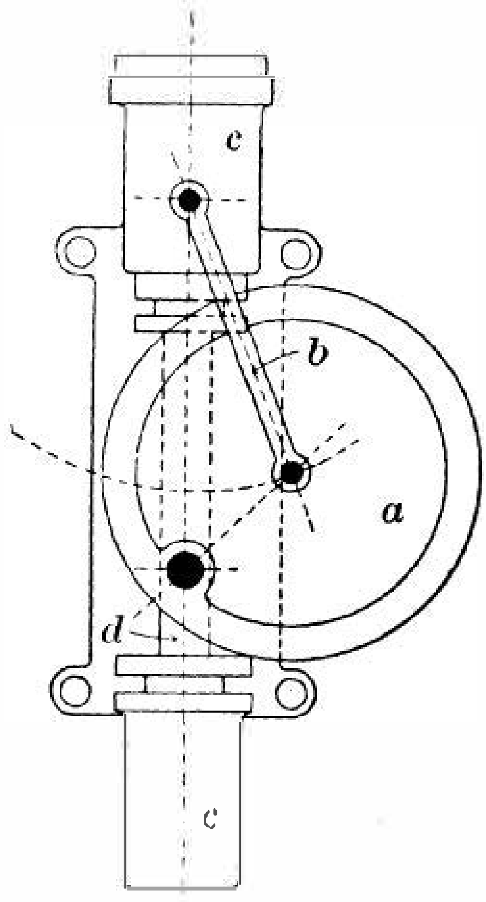


FIG. 75a.

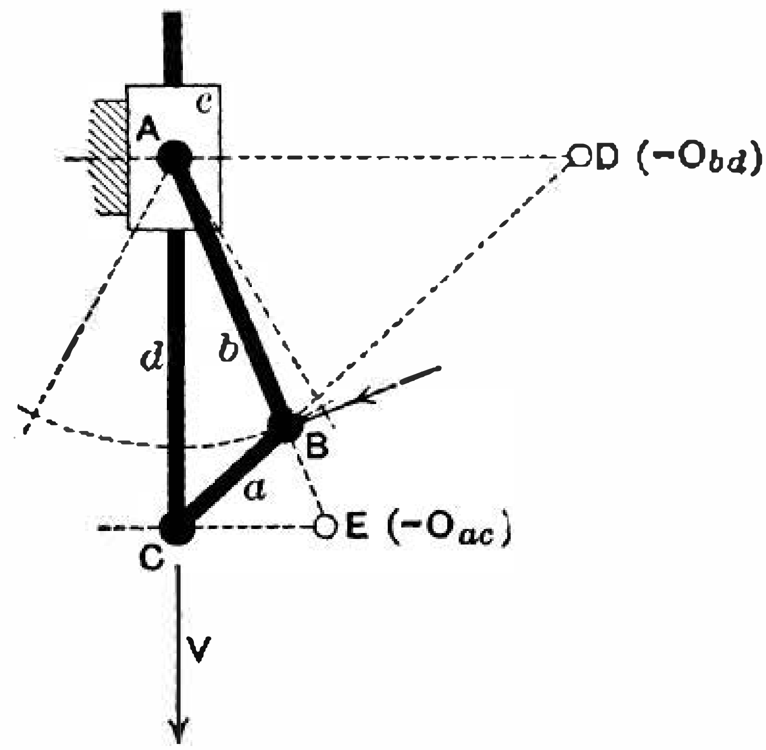


FIG. 75b.

The various inversions of the slider-crank chain may be summarized thus:

Link Fixed.	Name of Chain.	Resulting Mechanism.
d	Turning slider-crank	Direct-acting engine
b	Swinging-block slider-crank	Oscillating engine. Quick-return motion
a	Turning-block slider-crank	Whitworth quick-return motion
c	Swinging slider-crank	Pendulum pump

41. Crossed Slider-crank Chains.—The slider-crank chains hitherto discussed have been arranged so that the straight line in which O_{bc} moves relatively to d passes through O_{ad}. If this is not the case, we obtain a further series of mechanisms known as crossed slider-crank chains, shown in Fig. 76.

The crossed turning slider-crank has been used in certain single-acting high-speed steam-engines, with a view of lessening the effect of the obliquity of a short connecting-rod during the working stroke; the obliquity during the return stroke is of course correspondingly increased. In such a case the determination of the acceleration or velocity of the

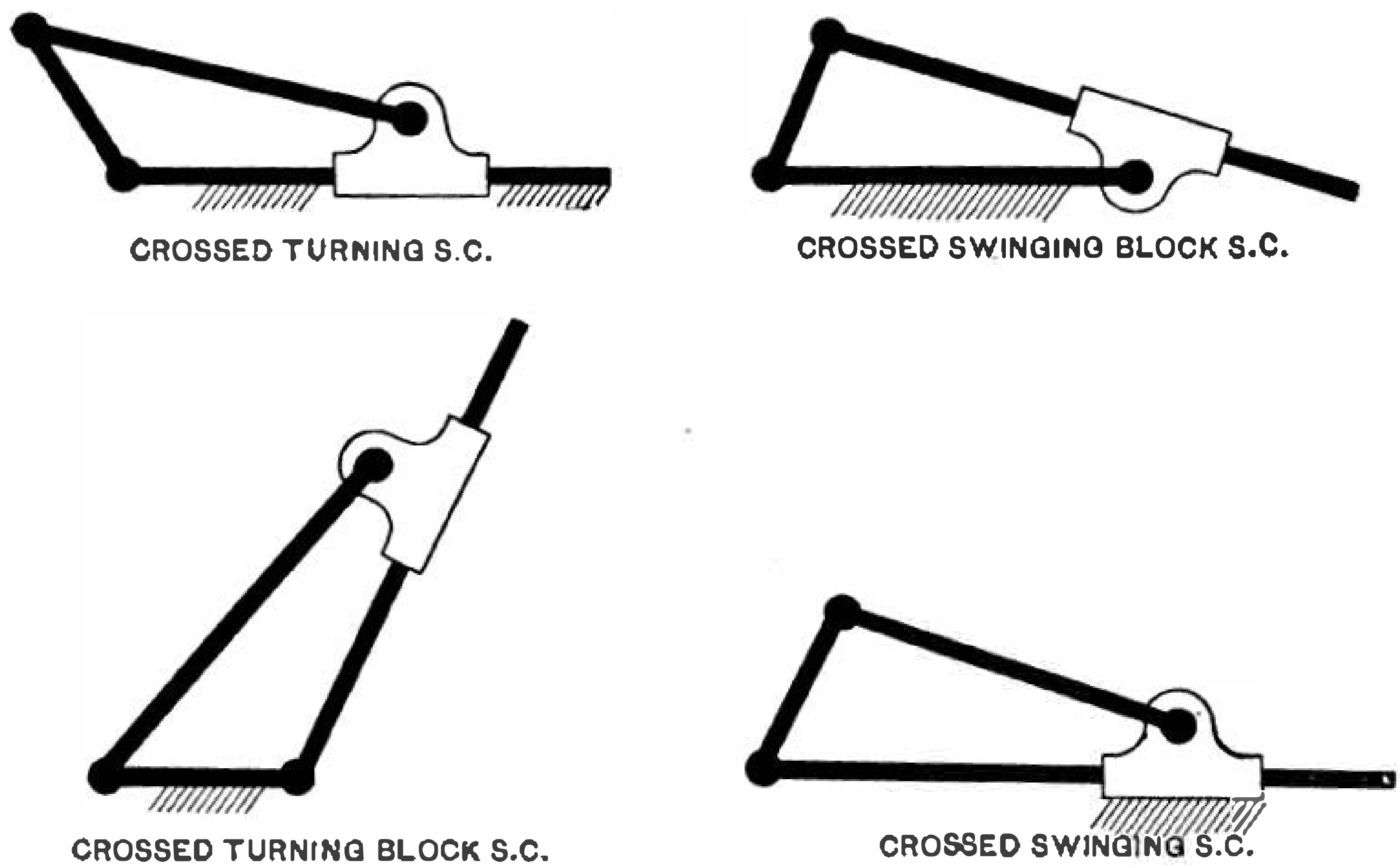


FIG. 76.

piston does not present any difficulty, as it can be carried out by the constructions already given.

42. Double Slider-crank Chain.—We consider next the simple chain formed by two turning and two sliding pairs. It has been already shown that from the quadric crank-chain the slider-crank chain may be derived by substituting a sliding pair for one of the turning pairs, such sliding pair being equivalent to a turning pair of infinitely great radiuse This substitution may be repeated, and Fig. 77 shows the result in the case where the directions of motion of the two sliding pairs are at right angles, and where one link carries an element of each of the two sliding pairs. Such a chain is called a *double slider-crank chain.* The link b has now become compressed into a block sliding in a slot formed in c.

Since the relative motion of b and c is the same as if the pair bc were a turning pair of infinite radius, the velocity ratios and accelerations in this chain will all be found exactly as in the case of a slider-crank chain in which the link b is of

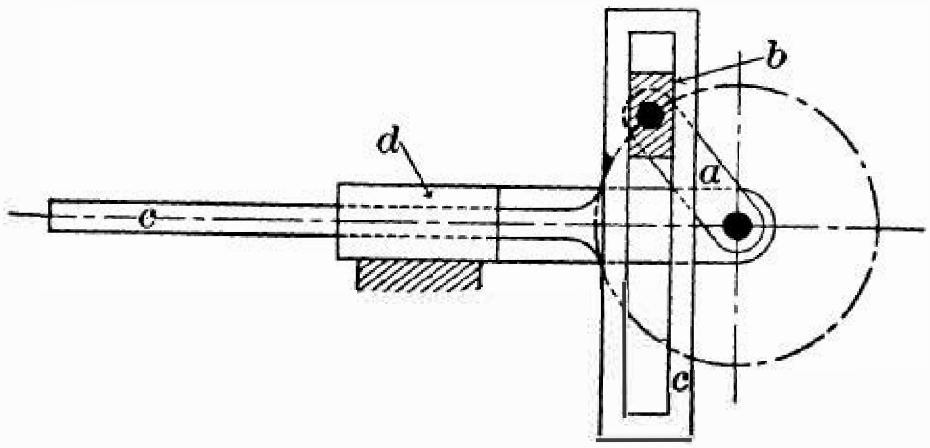


FIG. 77.

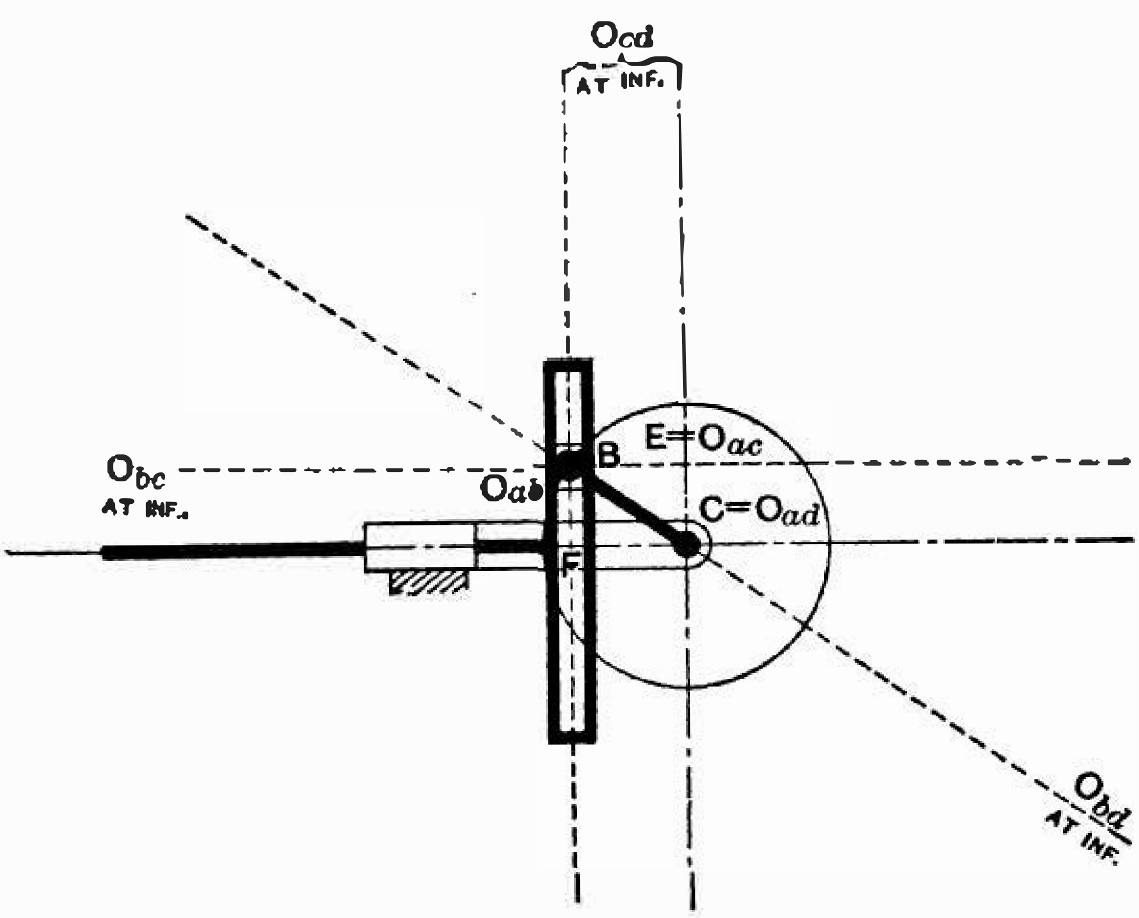


FIG. 78.

infinite length, i.e. when $n=\infty$. Accordingly we may write at once (supposing the link d to be fixed), with the same notation as before,

$$x=r\cos\theta,$$

and

$$\text{linear velocity of } c=V_p=V_c\sin\theta,$$

while

$$\text{acceleration of } c = \frac{V_c^{\,2}}{r} \cos \theta.$$

Notice that if the crank rotates uniformly the motion of *c* with regard to *d* is simple harmonic, and that the link *b* has no angular velocity with regard to *d*.

Fig. 78 shows the six virtual centres of the chain. It is plain that the centrodes of *b* and *d* are now altogether inaccessible.

The distance *CE* is seen to be proportional to the linear velocity of the link *c*, while *CF* is proportional to its linear acceleration as given above; the scales are readily determined.

This inversion of the double slider-crank chain is frequently employed in the construction of steam-pumps. The link *c* represents the steam-piston and pump-plunger, *d* the cylinder, framing, and pump-barrel, and *a* the crank-shaft. The total height of the pump may be made small, on account of the absence of a connecting-rod, thus making the arrangement a very convenient and compact one for certain purposes.

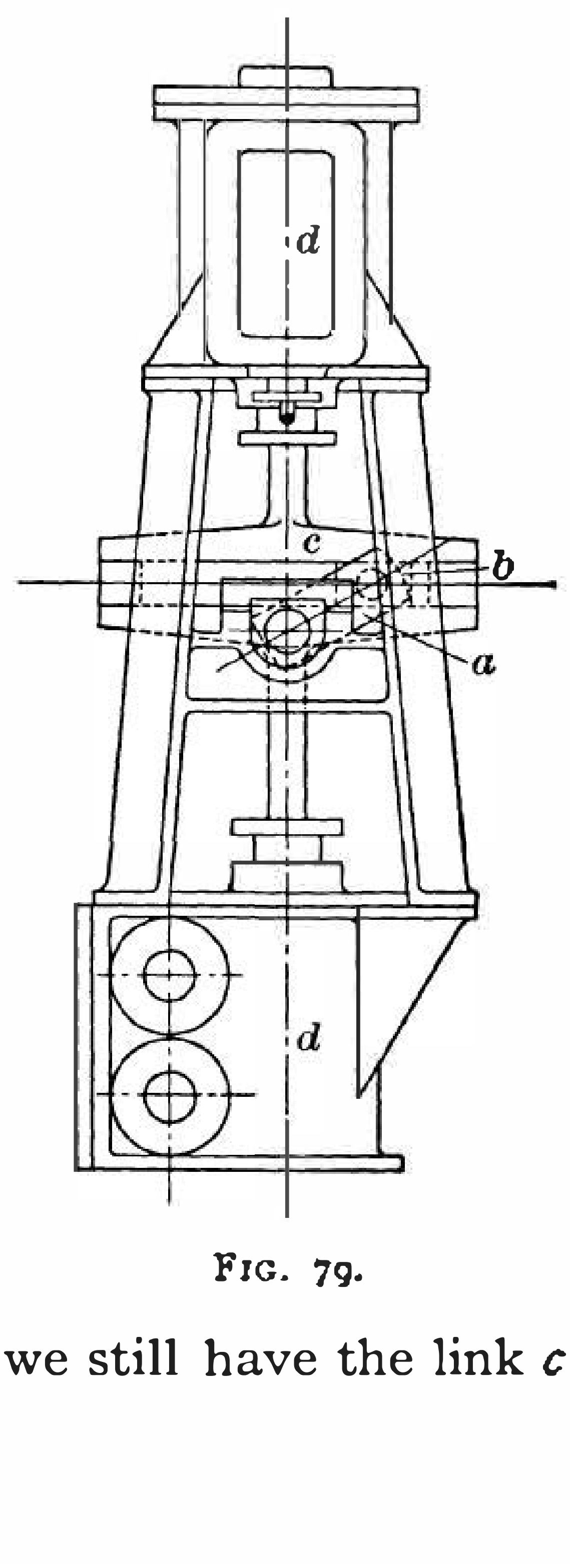


FIG. 79.

The linear and polar diagrams of piston displacement, velocity, and acceleration, supposing that the angular velocity of the crank-shaft is uniform, are precisely those already given for simple harmonic motion.

If the link *b* be supposed fixed instead of *d*, the resulting mechanism is the same as before, for the reason that the relative motion of *b* and *a* is exactly the same as that of *d* and *a*. Thus on fixing *b* we still have the link *c* executing simple harmonic vibrations as the crank rotates uniformly, but the direction of motion is now along the line FB instead of FC (Fig. 78).

43. **Elliptic Trammels.**—If c, the link containing one element of each of the two sliding pairs, is the fixed link, we obtain a mechanism used for the purpose of drawing ellipses, and shown in Fig. 80. The bar a, carrying an element of each of the two turning pairs, now carries a movable tracing-point; the blocks b and d slide in a pair of grooves intersecting at right angles and formed in the link c.

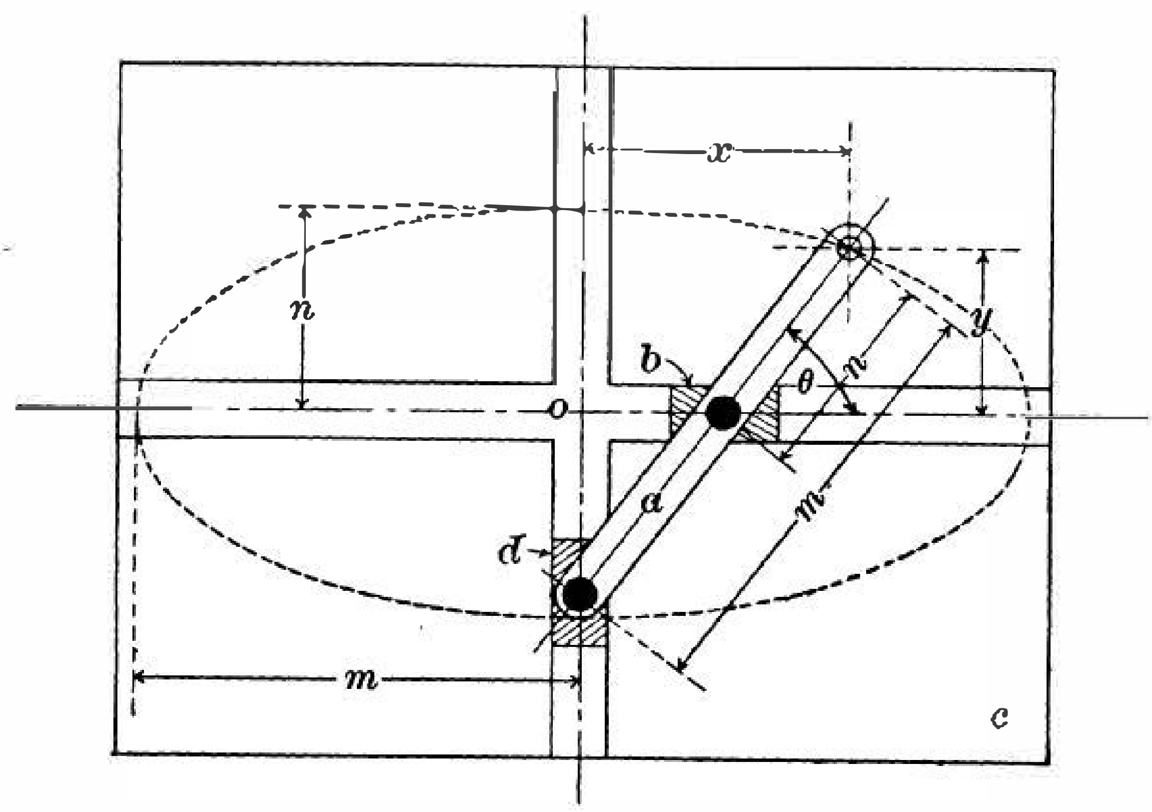


FIG. 80.

The path of the tracing-point is easily seen to be an ellipse, for, with the notation of Fig. 80, we have

$$\sin\theta = y/n,$$
$$\cos\theta = x/m.$$

Hence $$\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1.$$

This equation is seen to represent an ellipse having O as its centre and m and n as its major and minor semi-axes.

From the position of the point O_{ac} (Fig. 81) it is evident that the centrodes of a relatively to c, and c relatively to a, form a pair of circles of which the length of the link a is respectively the radius and the diameter. Hence it follows that the relative motion of a and c may be represented by the rolling together of circular curves of the same sizes as the centrodes in question—a point to which attention is again drawn (see § 57).

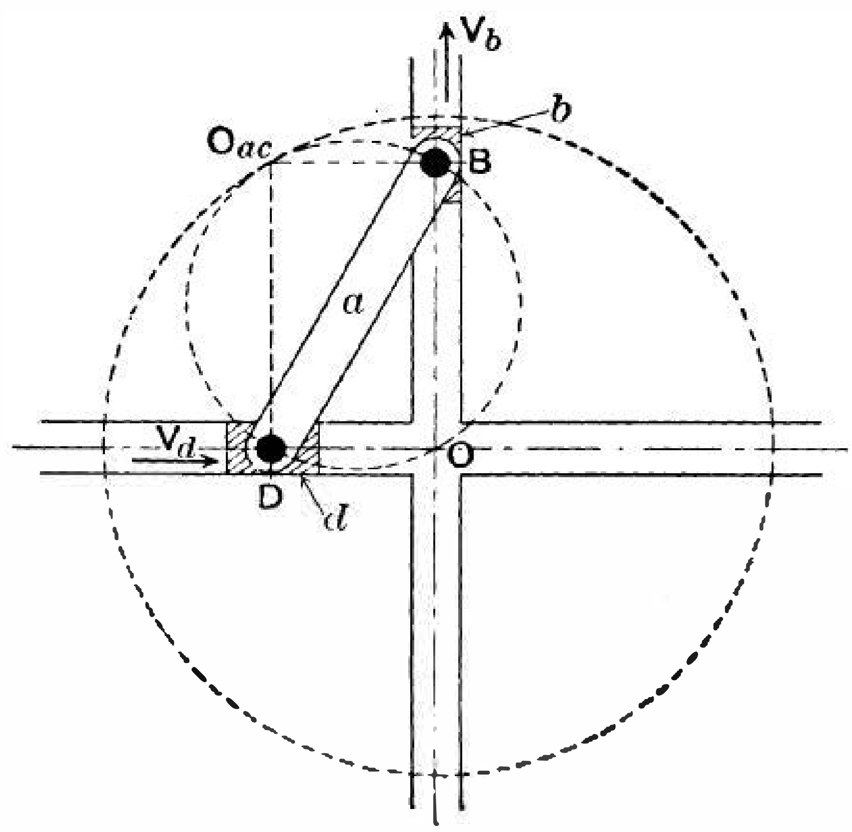


FIG. 81.

Notice that if V_d, V_b be the linear velocities of d and b respectively with regard to c, then, since $\omega_{ac} = \frac{V_d}{OB} = \frac{V_b}{OD}$, we have

$$\frac{V_d}{V_b} = \frac{OB}{OD}.$$

44. Oldham's Coupling.—The fourth inversion of the double slider-crank chain, when a is the fixed link, gives rise to a mechanism which has been used as a coupling for connecting shafts whose axes are parallel, and as an elliptic chuck, by means of which objects of elliptical cross-sections may be turned in an ordinary lathe.

Let Fig. 82 represent the chain when a is fixed. Evidently the locus of O is a circle of diameter BD.

Let O_1, O_2 be two positions of O; then, since O_1DO_2 and

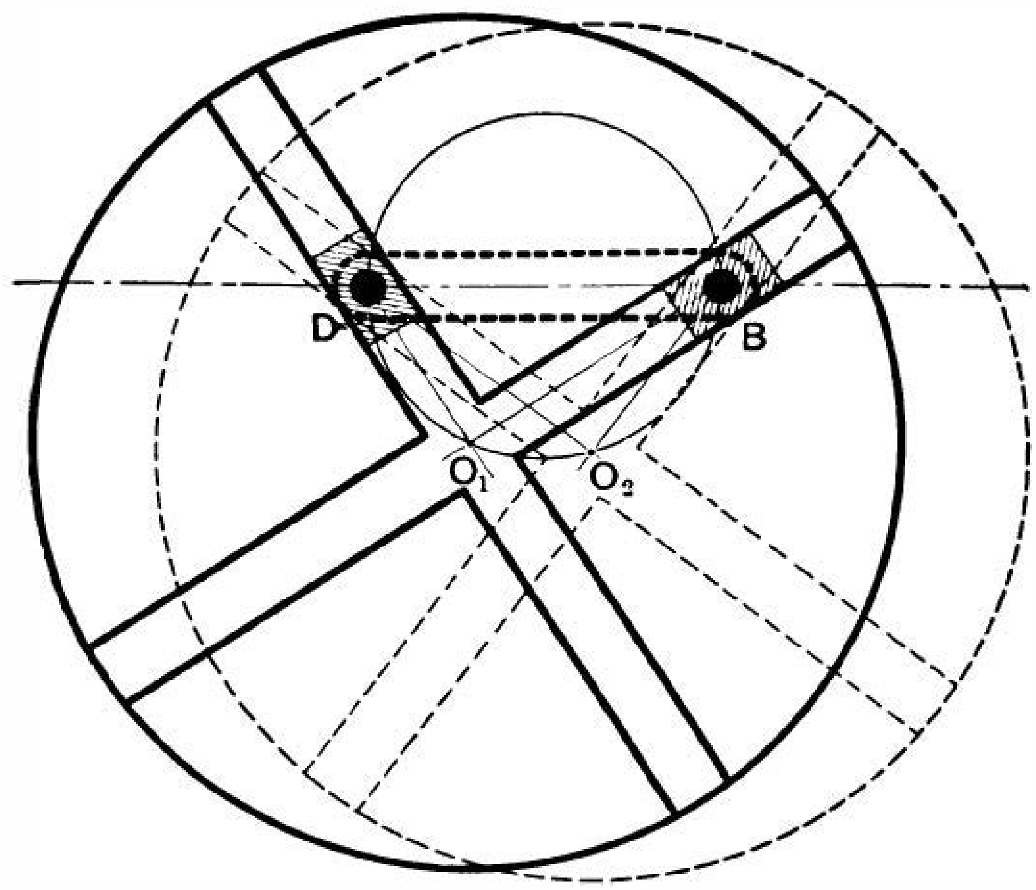


FIG. 82.

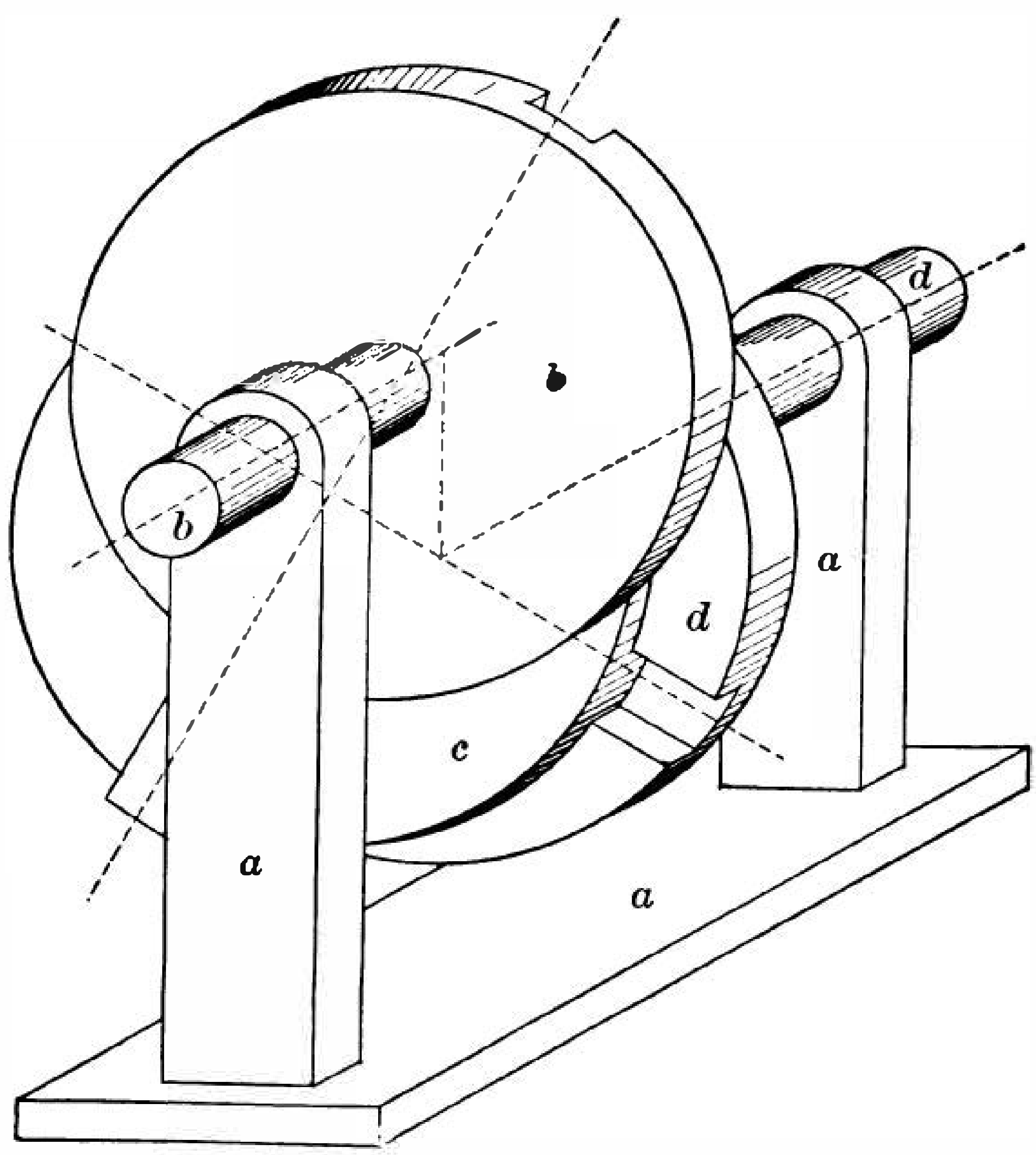


FIG. 83.

O_1BO_2 are angles in the same segment of a circle, they are equal; hence if b turns through any angle O_1BO_2, d turns through an equal angle O_1DO_2. By attaching a shaft to each of the links b and d we are thus enabled to communicate rotation from one to the other with uniform angular velocity ratio. Fig. 83 shows the form actually taken by

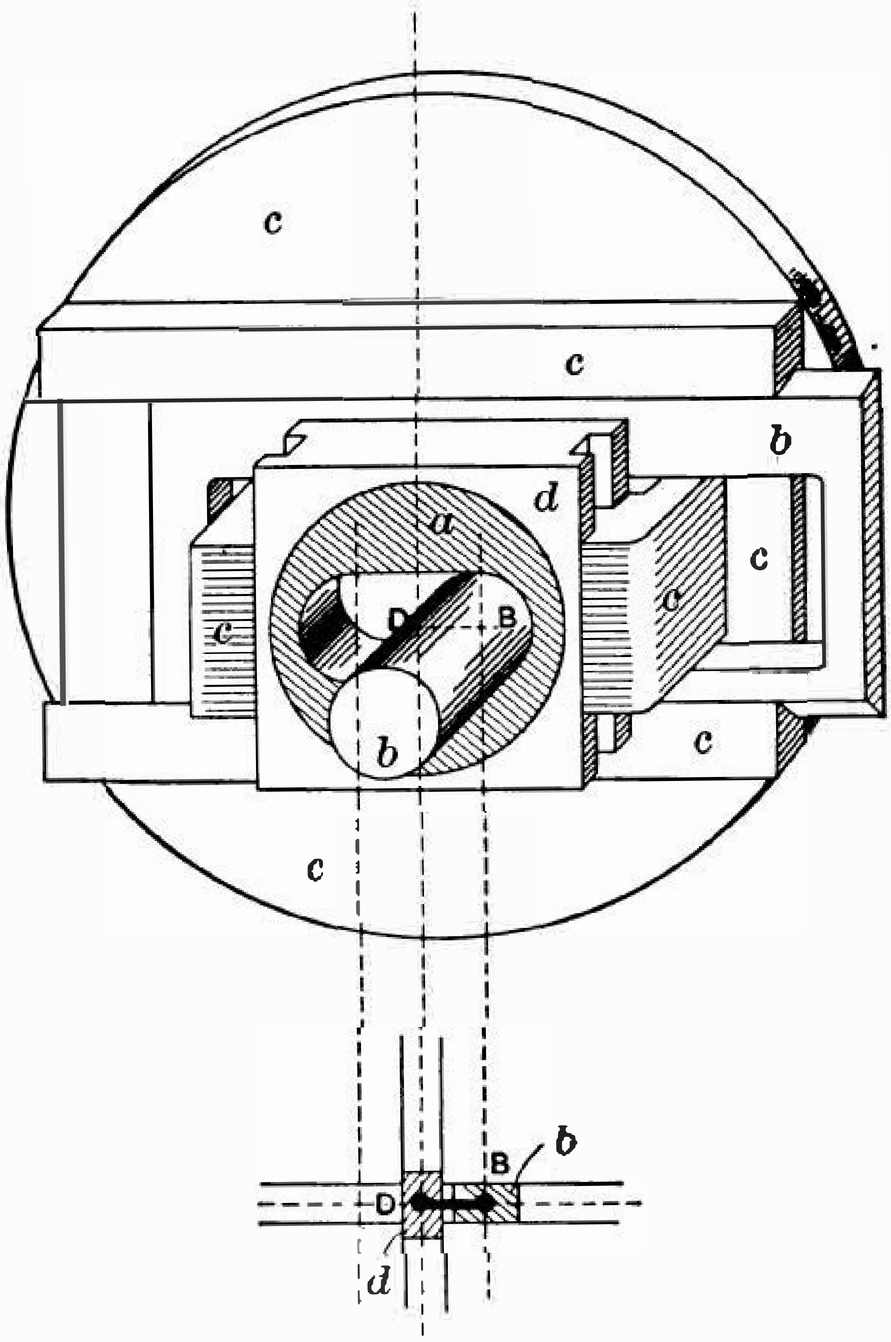


FIG. 84a.

this mechanism when used as *Oldham's coupling.* The link c becomes a disc having projecting feathers or keys on its faces, these keys being at right angles to one another and fitting into corresponding grooves on the enlarged ends of the two shafts b and d. The link a becomes a frame carrying the bearings of the two shafts.

Precisely the same kinematic chain is used in the case of the elliptic chuck, which was probably invented by Leonardo da Vinci.

Figs. 84*a* and 84*b* represent this contrivance, seen from the back, the face of the plate *c* to which the work is attached being turned away from view.

The plate *c* has behind it two straight pairs of guides at right angles to one another; the block *b* slides be-

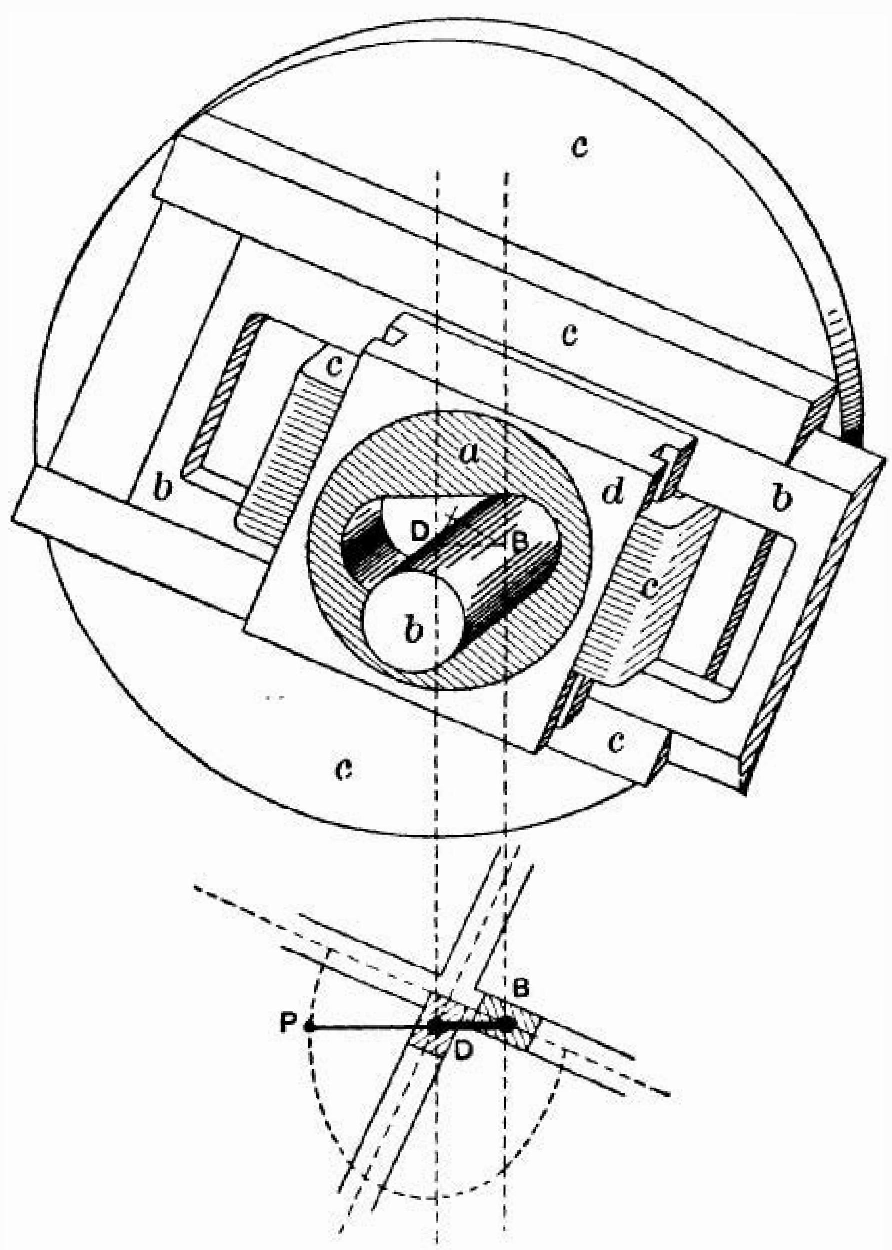


FIG. 84*b*.

tween one of these pairs of guides, while the block *d* moves between the other pair, which pass through slots cut for the purpose in *b*. The block *b* is secured rigidly by being screwed on to the nose of the lathe mandrel, with which it rotates. The mandrel passes through an oval hole in the eccentric *a*, which is clamped firmly (by screws which are not shown) to the lathe headstock, in such a way that the distance BD between the axis of a and that of b can be varied as required. That distance is the effective length of the fixed link in the mechanism, and upon it depends the eccentricity of the ellipse to be described.

It will be seen that this construction corresponds exactly to the arrangement of Fig. 82. Accordingly it is evident that a point at rest with regard to the link a (as the point of a cutting-tool would be) will describe an ellipse on a piece of work attached to, and rotating with, the link c, just as a tracing-point attached to a (Fig. 80) was shown to describe an ellipse with respect to c in that case. It will be seen that the distance from the tracing-point P to D (Fig. 84b) is the semi-minor axis, while the length BD is the difference between the semi-axes.

It is obvious that a number of fresh mechanisms may be derived by changing the angle between the directions of motion of the two sliding pairs; in this case the chain would be known as a skew double slider-crank chain. Fig. 85 shows an example of such a chain, but space does not permit of the discussion of such mechanisms.

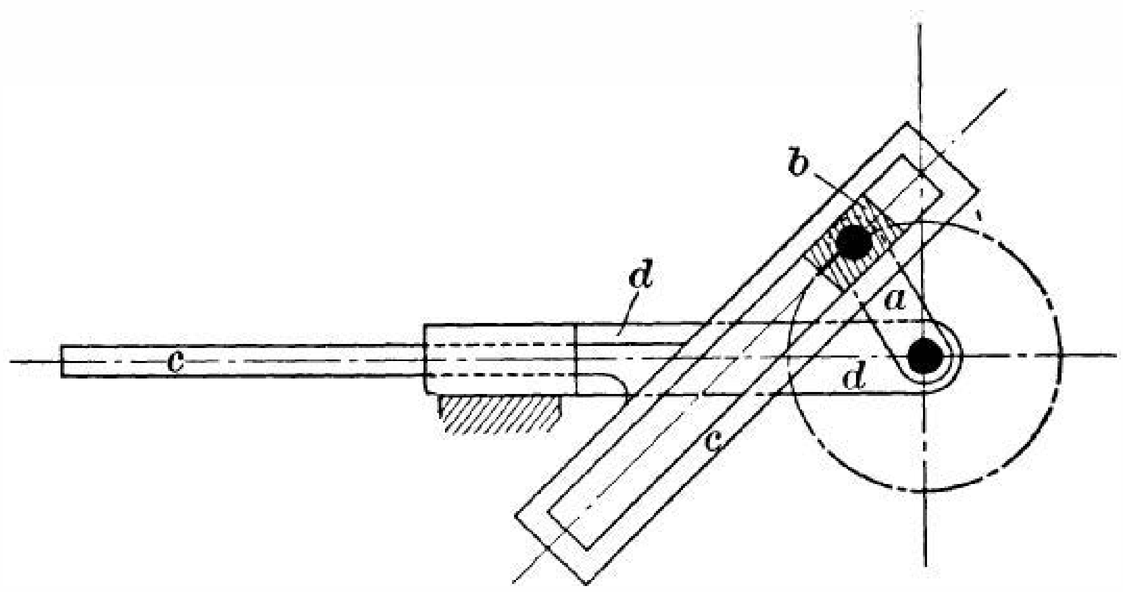


FIG. 85.

45. Crossed-slide Chains.—We proceed to consider the chain derived from the slider-crank chain by introducing a second sliding pair in such a way that each link contains one element of a sliding and one of a turning pair. As distinguished from the chain just discussed (the double slider-crank chain), in which one link contains elements of each of two turning pairs, and another contains elements of each of two sliding pairs, we may call this the *crossed-slide chain*. It is essentially a crossed chain, just as the crossed slider-crank was, because the straight line in which the centre of one turning pair moves does not pass through the centre of the second turning pair. One of its forms is shown diagrammatically in Fig. 86, and is occasionally employed for

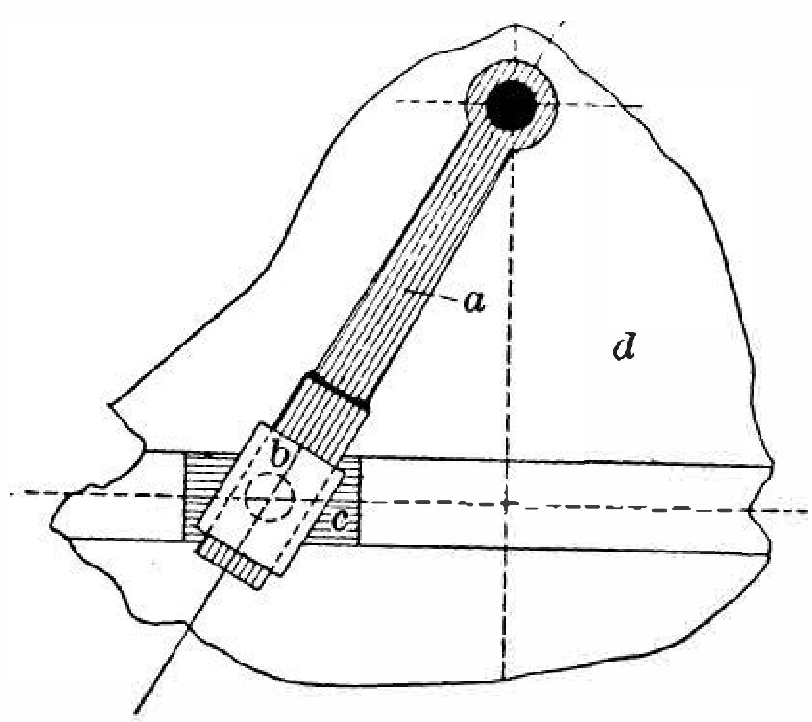


FIG. 86.

working the rudder of large ships, under the name of Rapson's slide. For this purpose it has the great advantage that the leverage increases as the helm is put over. In the figure the fixed link d represents the framework of the ship, a the tiller and rudder-head, and b is a block sliding on a and turning on c. The steering-gear moves the block c between guides secured to d, and thus actuates the rudder.

The same mechanism is employed for working the valves of duplex steam-pumps, in which each of the two steam-pistons works the valve of the neighboring cylinder, and it occurs again in a slightly different form in the arrangement of the compensating cylinders used in the Worthington high-duty pump for storing up the excess of energy exerted by the steam during the first portion of the stroke of the piston, and restoring that energy during the later part (Fig. 87). Here d is the pump framework, a a are the compensating cylinders, rocking on trunnions attached to d; b b are the plungers which enter the cylinders against pressure during the first half of the stroke, and return during the later half; c is the pump-piston, rod, and plunger.

The virtual centres of the chain are shown in Fig. 88, and

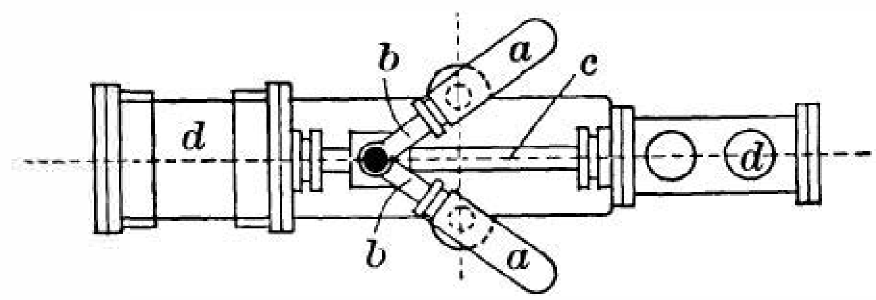


FIG. 87.

the pair of centrodes corresponding to the relative motion of b and d are shown in Fig. 89, construction lines being given for one point on each centrode.

Certain velocity ratios in this chain are of importance; for example, the ratio of the angular velocity of the tiller a to the linear velocity (V_c) of the block c relatively to its guides.

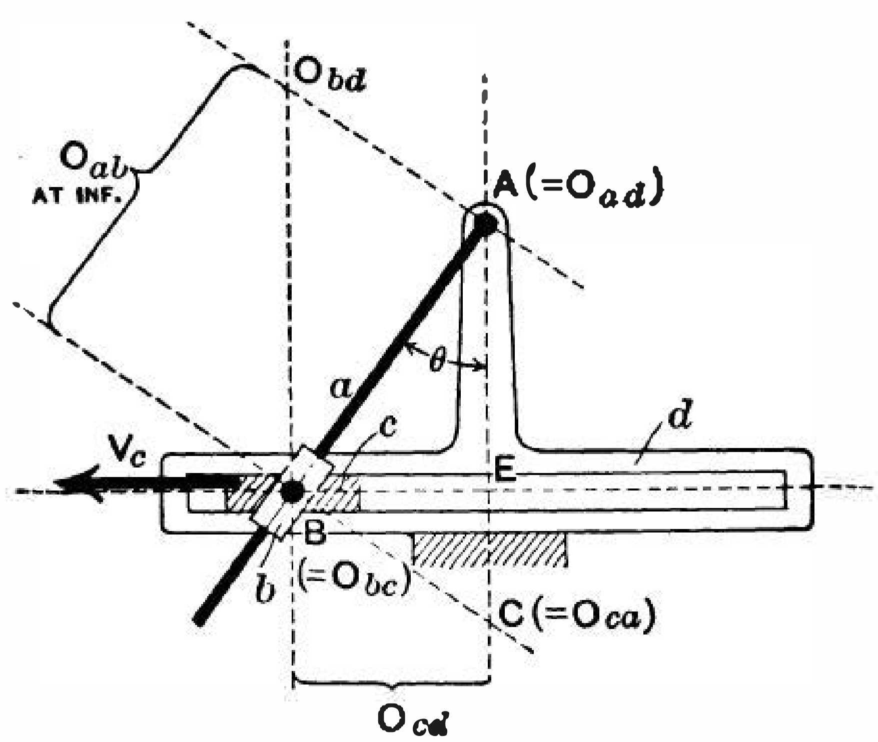


FIG. 88.

In Fig. 88 let the angle $BAC = \theta$; then, since O_{ac} is a point common for the instant to a and c, as a point in c it is moving in a direction perpendicular to AC with velocity V_c. Its angular velocity (and therefore the angular velocity of a in which it is a point) about A is therefore $\frac{V_c}{AC}$, which is easily seen to be equal to

$$\frac{V_c}{AE}\cos^2\theta.$$

Hence if the block c has a uniform linear velocity, the angular velocity of the tiller varies as the square of the cosine of the

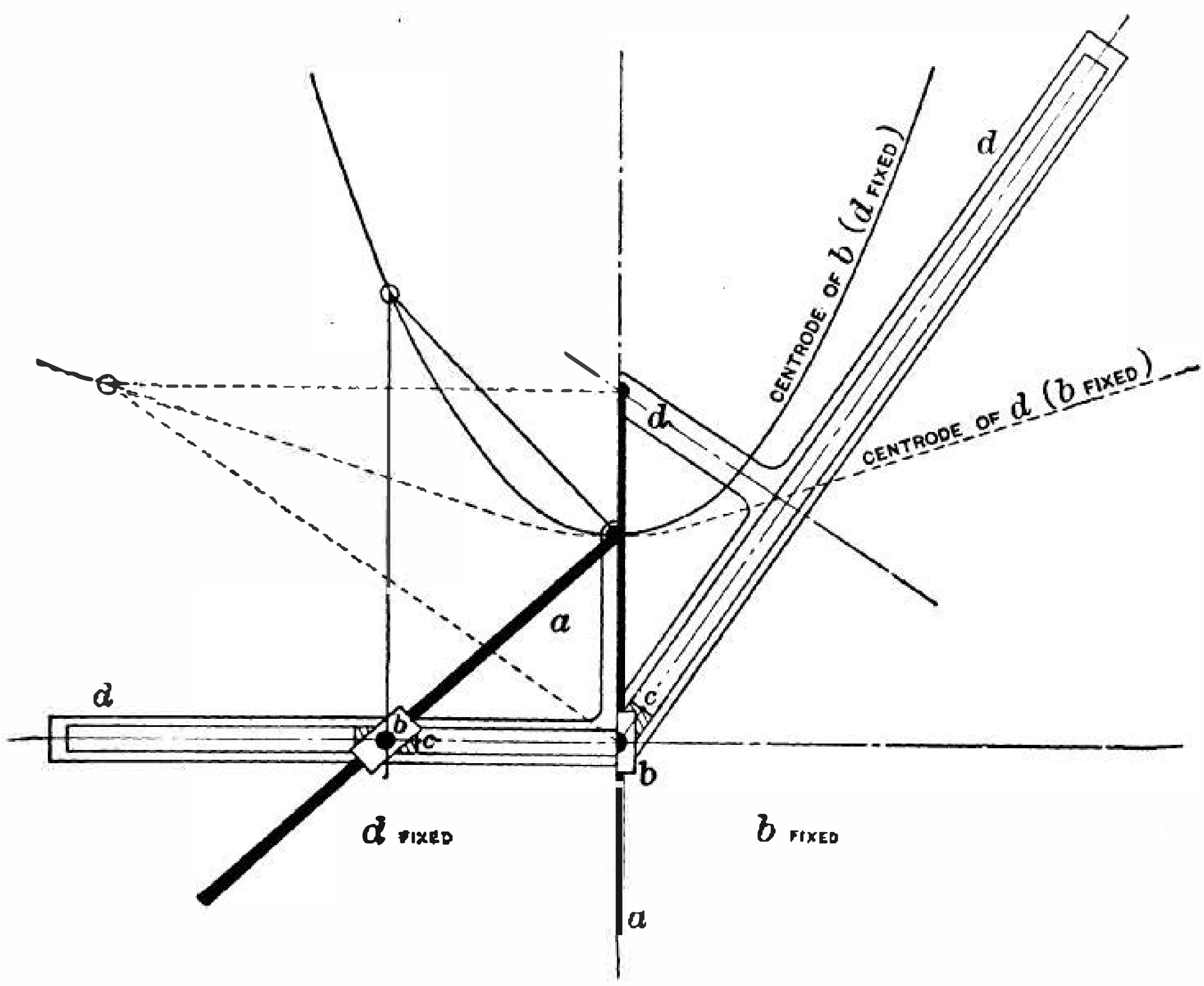


FIG. 89.

angle of helm. It is this property which gives the apparatus its value as a steering-gear; for it may readily be shown that if a constant force be applied to c, the turning moment on the rudder-head increases as the helm is put over; in fact the turning moment varies as $\frac{1}{\cos^2\theta}$.

It is easy to draw a curve of angular velocity for the link a. In Fig. 90 make $AE' = AE$ and draw BC perpendicular to AB. Draw EF parallel to CE', then $\dfrac{AF}{AE} = \dfrac{AE'}{AC}$;

therefore $\dfrac{AF}{AE^2} = \dfrac{1}{AC}$, and

$$\text{angular velocity of tiller} = AF \cdot \frac{V_c}{AE^2}.$$

Thus a series of points such as F will give us a polar diagram of the angular velocity of the tiller.

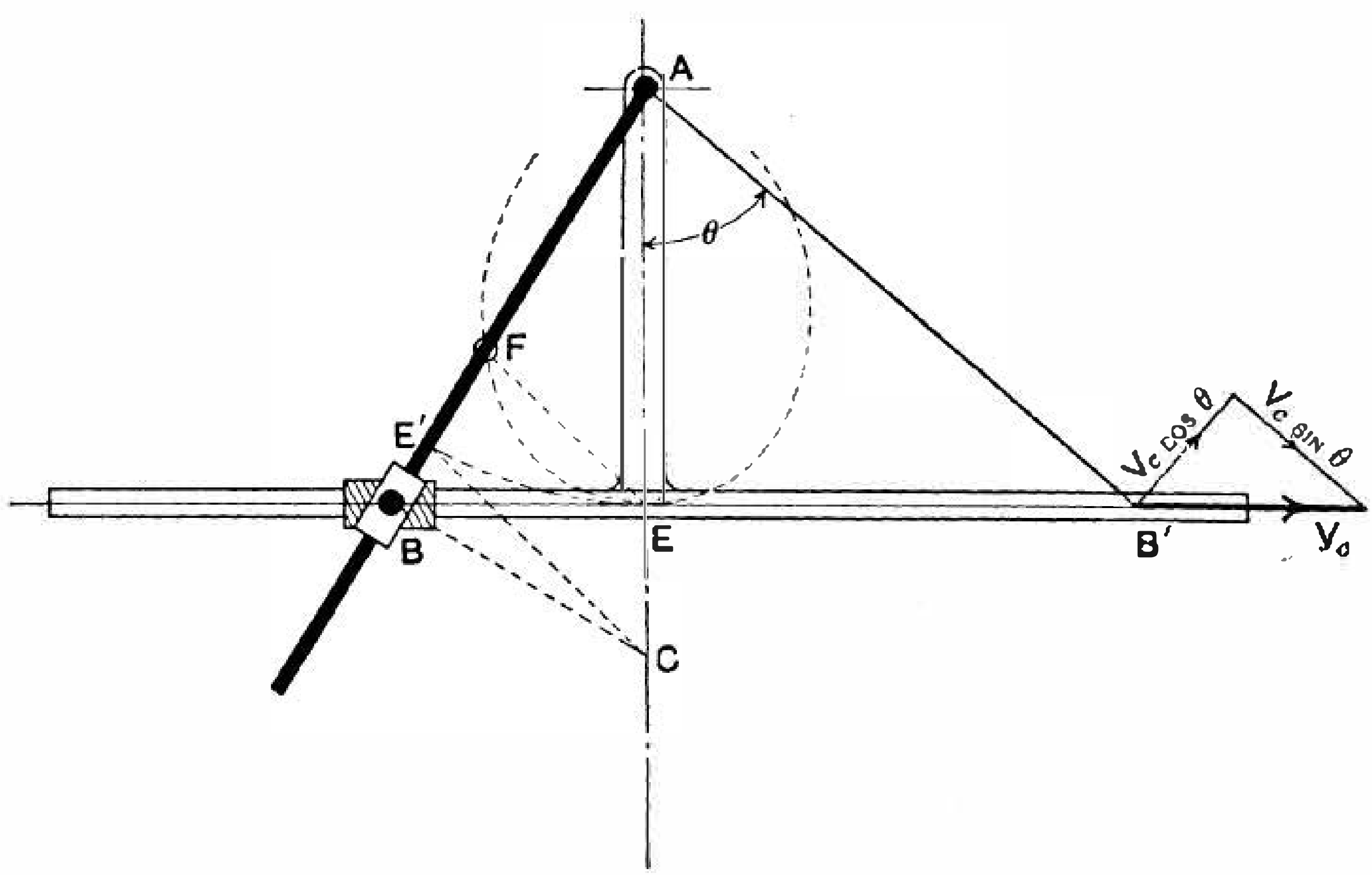


FIG. 90.

Notice that in any position we may look upon V_c, the linear velocity of the point B' along EB', as being the resultant of two velocities, $V_c \sin\theta$ along AB', and $V_c \cos\theta$ at right angles to AB'. The former gives the speed with which the block b is sliding along a; the latter shows that the angular velocity of a is

$$\frac{V_c \cos\theta}{AB'} = \frac{V_c \cos^2\theta}{AE},$$

the same result as that obtained previously.

Rapson's slide is only a particular case of the crossed-slide chain. It may be noticed, however, that we obtain the same mechanism whichever link is the fixed one, since each link has on it an element of a turning pair and also an element of a sliding pair.

46. Straight-line Motions Derived form Slider-crank Chain.—A number of straight-line motions have been devised which are really slider-crank chains. In such mechanisms the line described by the tracing-point is often only approximately straight, and when it is exactly so, its straightness depends upon the accuracy with which the flat surfaces of the sliding pair have been formed.

To this class belongs Scott Russell's straight-line motion, represented in Fig. 91. The link b in an ordinary slider-

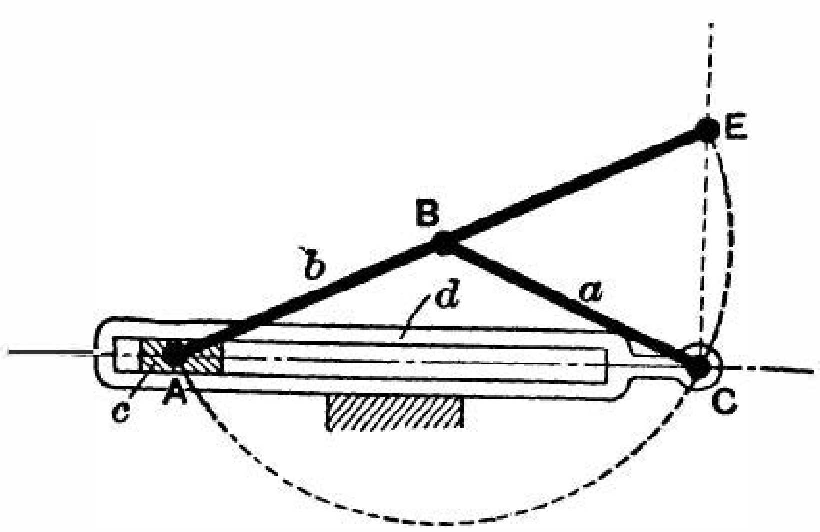


FIG. 91.

crank chain is extended to E, and $AB = BE = BC$. It is then evident that the angle ACE is the angle in a semicircle, and that the point E describes a straight line CE so long as A describes a straight line AC.

With other proportions of the lengths AB, BC, BE, approximate straight-line motions may be obtained. In Fig. 92, for example, suppose A and E to lie on the straight lines AC, EC, respectively; it has been seen that a point B will describe an ellipse (shown by the dotted curve), of which C is the centre, and AB and BE the lengths of the semi-axes. A circle may be drawn so as to cut this ellipse in four points, as at P, Q, R, S, and if we connect B and F, the centre of the circle, by a rigid link, the path of the point E will cut the straight line CD in four places, supposing A traverses the straight line AC. By a suitable choice of the point F, the circular path of B may be made to differ

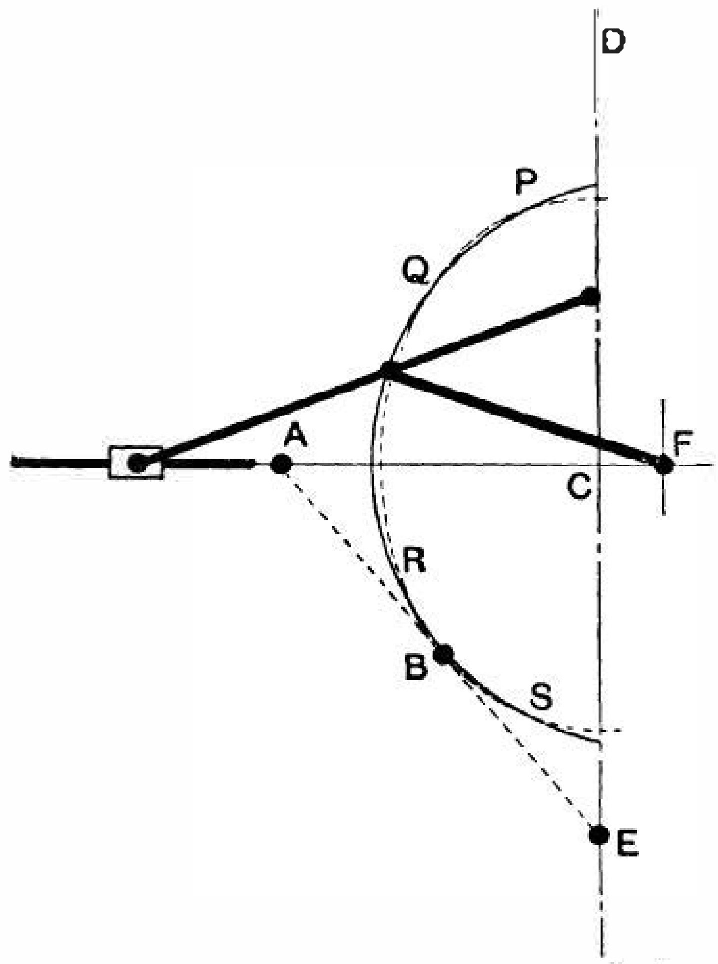


FIG. 92.

very little from the ellipse during a considerable range of movement, and the actual path of the tracing-point E will nearly coincide with the straight line CE.

In the second inversion of the slider-crank chain, in which b is the fixed link, a point on the link d may be chosen such that its path is approximately straight.

Thus in Fig. 93a suppose that a straight line AB, of fixed length, passes through a fixed point O, while a point C on it is compelled to traverse a straight line DE. The curves described by A and B are known as conchoids, and are shown by the dotted lines. It is possible in a swinging-block slider-crank chain to find a point P on the link d in such a position that while the circular path of O_{ad} coincides nearly with the part BB' of the conchoid, and the centre of the pair bc cor-

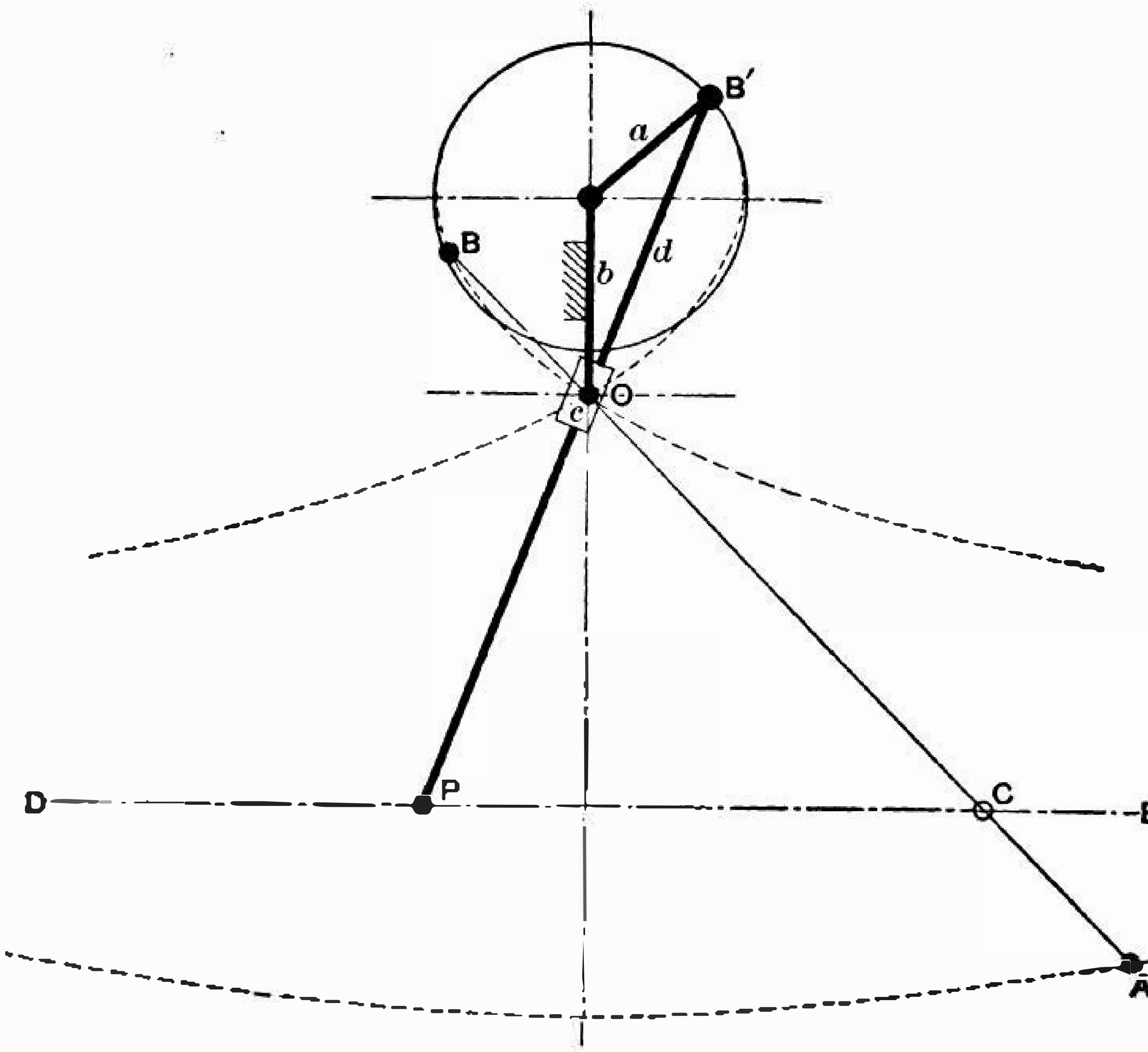


FIG. 93*a*.

responds with the point O, the path of the tracing-point P will nearly coincide with DE. Fig. 93*b* represents a model of such a mechanism, in which the point P is guided in an approximately straight line.*

47. Chain Containing Sliding Pairs only.—It is possible to construct a closed kinematic chain containing only sliding pairs. Such a chain consists of three links, a, b, and c. Of these b and c are blocks sliding on a common link a. A projection on b slides in a slot cut in c, thus completing the chain. If c is the fixed link, the chain will evidently be

* For discussion of a number of straight-line motions see Kennedy, Mechanics of Machinery, p. 417.

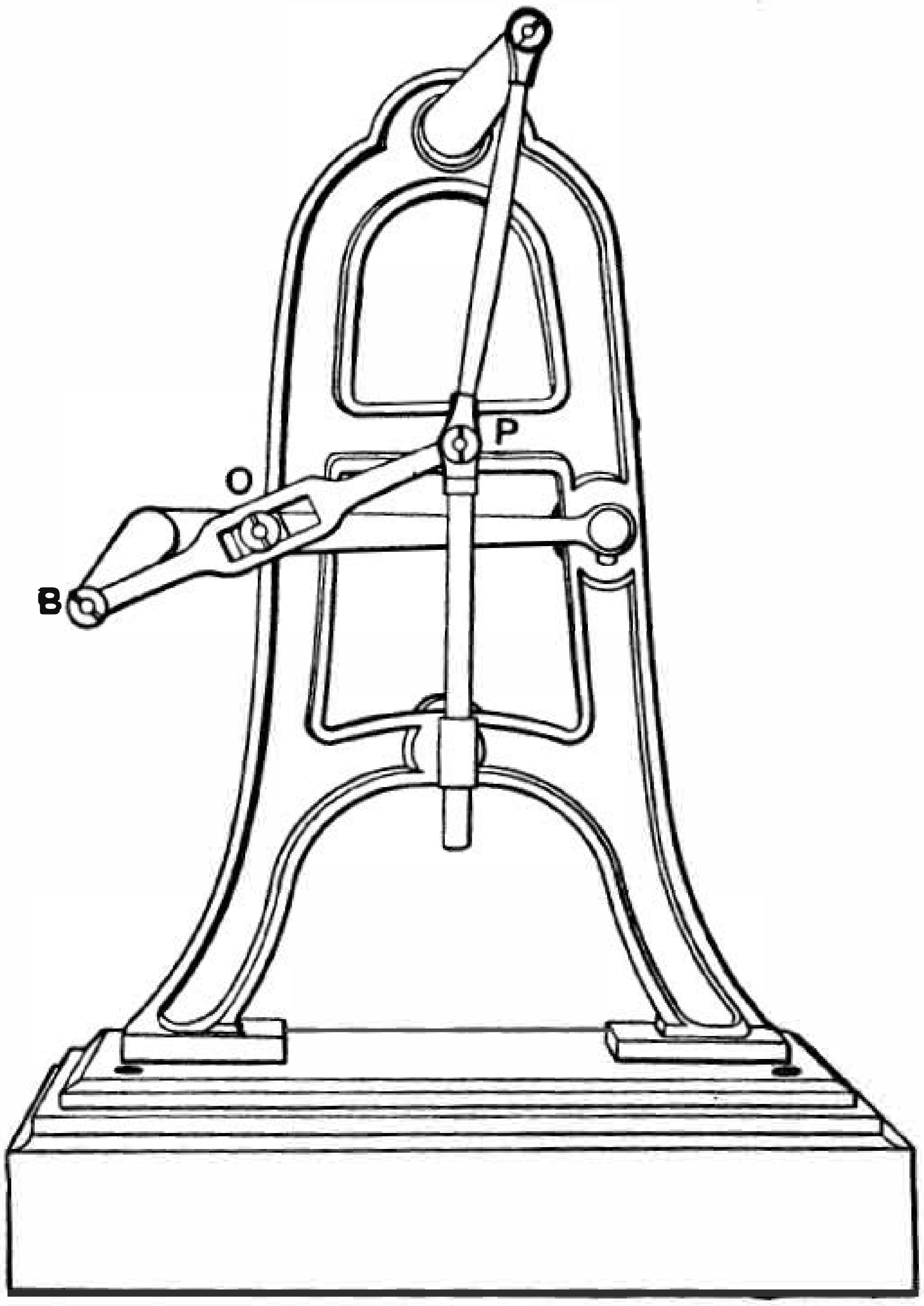


FIG. 93*b*.

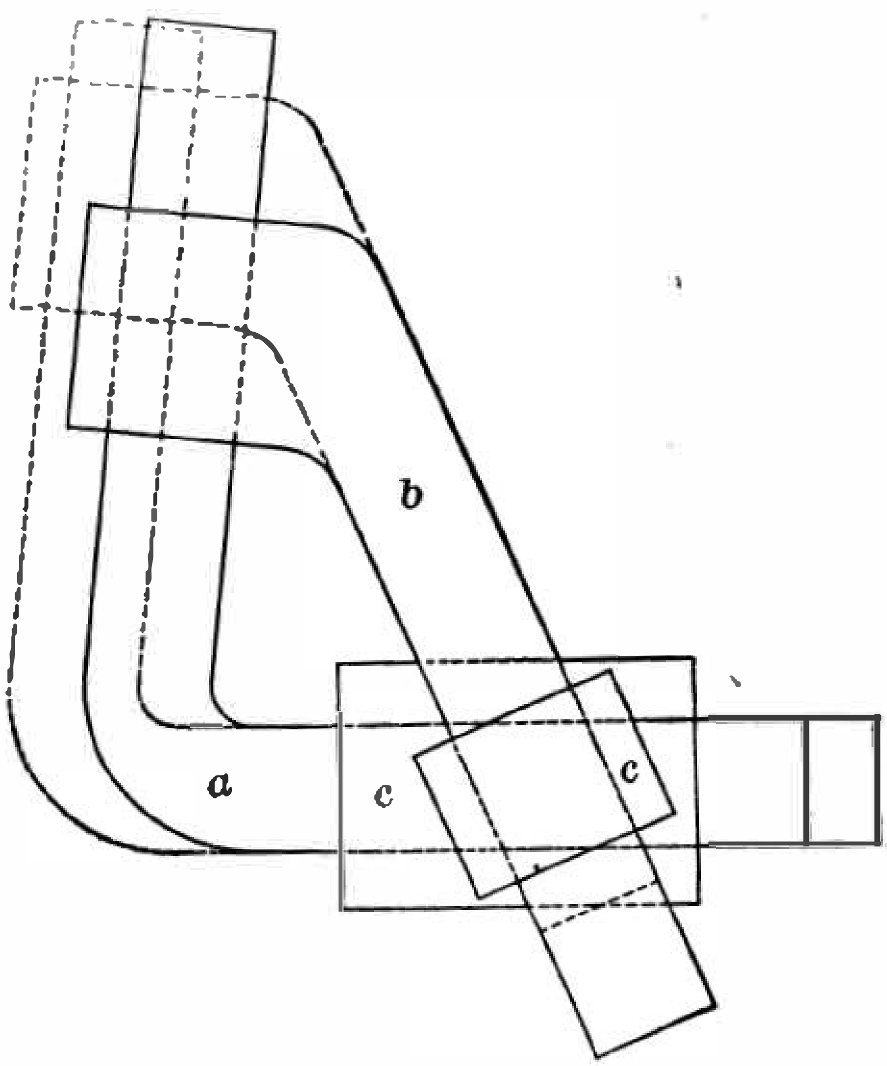


FIG. 94.

capable of being moved into such a position as that shown by the dotted lines in Fig. 94.

Such a chain exists, for instance, in most arrangements for adjusting bearings by means of wedges or cotters, as in the double-adjustment plummer-block (sketched diagrammatically in Fig. 95), in which the brass *c* has to be capable of slight movement in the direction of the arrow, to allow for wear, and is pushed forward by drawing down the wedge *b*. The pedestal itself and its cap form the link *a*, and the upward movement of the block or wedge *b* is prevented by some form of force- or chain-closure (see Chapter VI).

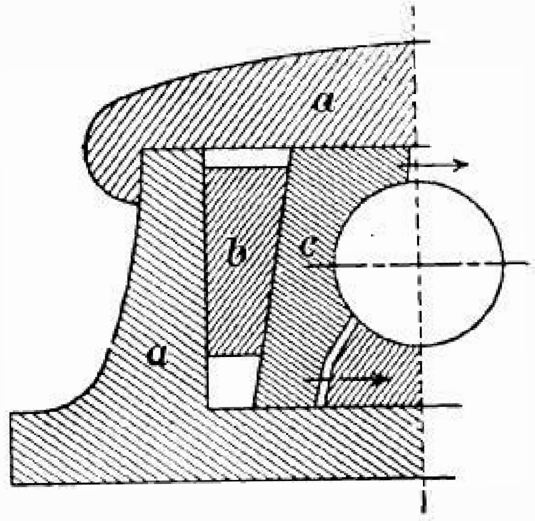


FIG. 95.

A chain containing four links and having four sliding pairs can also be devised, but, like the chain containing five turning pairs, it is not constrainedly closed.