TOPOLOGICAL MODELS FOR HYPERBOLIC AND SEMI-PARABOLIC COMPLEX HÉNON MAPS

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Consider the parameter space $\mathcal{P}_\lambda \subset \mathbb{C}^2$ of complex Hénon maps

$$H_{c,a}(x, y) = (x^2 + c + ay, ax), \quad a \neq 0$$

which have a fixed point with one eigenvalue a root of unity $\lambda = e^{2\pi i p/q}$; this is a parabola in a^2. Inside the parabola $\mathcal{P}_\lambda$, we look at those Hénon maps that are small perturbations of a quadratic polynomial p with a parabolic fixed point of multiplier λ. We prove that there is an open disk of parameters (inside $\mathcal{P}_\lambda$) for which the semi-parabolic Hénon map is structurally stable on the Julia sets J and J^+. The set J^+ is homeomorphic to an inductive limit of $J_p \times \mathbb{D}$ under an appropriate solenoidal map $\psi : J_p \times \mathbb{D} \to J_p \times \mathbb{D}$, $\psi(\zeta, z) = \left(p(\zeta), \epsilon\zeta - \dfrac{\epsilon^2 z}{p'(\zeta)}\right)$, where J_p is the Julia set of the polynomial p. The set J is homeomorphic to a solenoid with identifications, hence connected.

We also consider the class of Hénon maps that are small perturbations of a hyperbolic (or parabolic) polynomial $p(x) = x^2 + c$. We describe the set J^+ as the quotient of 3-sphere with a dyadic solenoid removed by an equivalence relation. We define a lamination for the Hénon map by lifting the Thurston lamination of the polynomial p from the closed unit disk to the unit 4-ball in $\mathbb{C}^2$, using the inductive limit. "Lifting" the leaves of the lamination of the polynomial gives a lamination for the Hénon map.

BIOGRAPHICAL SKETCH

Remus Radu completed the first year of his undergraduate education at the University of Bucharest, studying mathematics. He then continued his undergraduate studies in Germany, at Jacobs University Bremen. He earned his Bachelor's Degree in Mathematics from Jacobs University, being also a member of the President's List for high academic achievements. Remus earned his Master's Degree in Computer Science from Cornell University.

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During 2010 I have studied in absentia in Europe and I was a Marie Curie Early State Researcher within the E.U. Research Training Network CODY (Conformal Structures and Dynamics). Many thanks to Sebastian van Strien and

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CHAPTER 1

INTRODUCTION

A Hénon map is a polynomial automorphism of $\mathbb{C}^2$ and can be written as

$$H_{c,a}(x,y) = (x^2 + c + ay, ax), \quad \text{for } a \neq 0$$

where a and c are complex parameters. In this parametrization, the Hénon map has constant Jacobian $-a^2$. In order to study the dynamics of polynomial automorphisms of $\mathbb{C}^2$ we need to understand their behavior under forward and backward iterations. The dynamical objects that we need to study are the sets $K^\pm$ (the set of points with bounded forward/backward orbits) and their topological boundaries $J^\pm = \partial K^\pm$. The set $J = J^+ \cap J^-$ is the analogue of the one-dimensional Julia set for polynomials.

We say that $H_{c,a}$ is hyperbolic if it is hyperbolic on its Julia set J. If the Hénon map is hyperbolic and $|a| < 1$ then the interior of K^+ consists of the basins of attraction of finitely many attractive periodic points [BS1]. Each basin of attraction is a Fatou-Bieberbach domain (a proper subset of $\mathbb{C}^2$, biholomorphic to $\mathbb{C}^2$). The common boundary of the basins is the set J^+ [BS1]. The set J^+ is where the most interesting chaotic behavior takes place. For hyperbolic Hénon maps, periodic points are dense in J and the map is structurally stable on J [BS1].

In Chapter 3 we study semi-parabolic Hénon maps. Unlike hyperbolic Hénon maps which exhibit structural stability, semi-parabolic Hénon maps are not expected to be structurally stable. The general assumption is that bifurcations will occur as we perturb from a semi-parabolic Hénon map. Bedford, Smillie, and Ueda show in [BSU] the complications that can arise by describing the phenomenon of "semi-parabolic implosion" in $\mathbb{C}^2$ (discontinuity of J and

J^+ on the parameters). We show that there are classes of semi-parabolic Hénon maps that are structurally stable on the sets J and J^+ inside a parametric region of codimension one in $\mathbb{C}^2$.

A fixed point of $H_{c,a}$ is called semi-parabolic if the derivative of $H_{c,a}$ at the fixed point has two eigenvalues $\lambda = e^{2\pi i p/q}$ and μ, with $|\mu| < 1$. The Hénon map is called semi-parabolic if it has a semi-parabolic fixed point.

The set of parameters $(c, a) \in \mathbb{C}^2$ for which $H_{c,a}$ has a fixed point with one eigenvalue λ is the curve

$$\mathcal{P}_\lambda := \left\{ (c, a) \in \mathbb{C}^2 \mid c = (1 - a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2 \right\}.$$

This is in fact a parabola in a^2. When $a = 0$ we have $c_0 = \lambda/2 - \lambda^2/4$ and the dynamics of the Hénon map $H_{c_0,0}$ reduces to the dynamics of the polynomial $p(x) = x^2 + c_0$, which has a parabolic fixed point of multiplier λ. Let J_p denote the Julia set of the polynomial p.

In Theorem 3.3 we prove a structure theorem for semi-parabolic Hénon maps $H_{c,a}$ that are perturbations of a parabolic polynomial $p(x) = x^2 + c_0$. This generalizes the theorem of Hubbard and Oberste-Vorth [HOV2] (that characterizes Hénon maps that are perturbations of a hyperbolic polynomial) to the semi-parabolic setting. The technique of the proof is quite new and it is inspired by the proof of Douady and Hubbard [DH] that the Julia set of a parabolic polynomial is locally connected. We prove the result as a fixed point theorem on an appropriate function space. The function space is described in Sections 2.4 and 3.11. We show contraction with respect to a metric μ which is the infimum of a pull-back of an Euclidean metric on a tubular neighborhood of the local stable manifold of the semi-parabolic fixed point and a product of two Poincaré metrics. The metric is described in Section 3.8. In order to establish the conjugacy we have used some heavy-duty topology: a theorem of Hamstrom [Ham], which states that if S is a compact surface with nonempty boundary then the components of the group of homeomorphisms which are the identity on the boundary are contractible. This is described in detail in Chapter 4. In order to understand J^+ as a whole, we will use the technique of inductive limits introduced in [HOV2] and briefly outlined in Section 5.1.1. In Chapter 2 we reprove the theorem of Hubbard and Oberste-Vorth [HOV2] as a fixed point theorem.

We show that semi-parabolic Hénon maps with small enough Jacobian are structurally stable (inside the curve $\mathcal{P}_\lambda$) on J and J^+. By structural stability on J^+ we understand that if (c_1, a_1) and (c_2, a_2) belong to $\mathcal{P}_\lambda$ and if $0 < |a_i| < \delta$ then $J^+_{c_1,a_1}$ is homeomorphic to $J^+_{c_2,a_2}$ and $(H_{c_1,a_1}, J^+_{c_1,a_1})$ is conjugate to $(H_{c_2,a_2}, J^+_{c_2,a_2})$. This is consistent with [B]. The same result holds if we replace J^+ with J. Furthermore, the Julia set J of the Hénon map is homeomorphic to a solenoid with identifications $\Sigma := \bigcap_{n\geq 0} \psi^{\circ n}(J_p \times \mathbb{D}_r)$, where $\psi(\zeta, z) = \left(p(\zeta), \epsilon\zeta - \dfrac{\epsilon^2 z}{2\zeta}\right)$, for some small $\epsilon > 0$ that does not depend on a. Thus J is connected.

Suppose $\lambda = 1$ (so $c_0 = 1/4$) and that we are perturbing from the parabolic polynomial $p(x) = x^2 + 1/4$ (the root of the main cardioid of the Mandelbrot set). The Julia set J_p of this polynomial is a Jordan curve and so the set J is homeomorphic to a solenoid (with no identifications). In [B], Bedford asks whether $J_{c,a}$ is homeomorphic to a solenoid for $(c, a) \in \mathcal{P}_1$ and a sufficiently small and whether the semi-parabolic Hénon map is structurally stable on its Julia set $J_{c,a}$ (Questions 1 and 2, [B]). We provide a positive answer to both questions. Moreover, we show in Proposition 5.15 that the set $J^+_{c,a}$ is homeomorphic to a 3-sphere with a dyadic solenoid removed, for all $(c, a) \in \mathcal{P}_1$ and a sufficiently small.

In order to have nontrivial identifications in the description of the Julia set J from Corollary 3.3.2 we need to consider $\lambda = e^{2\pi i p/q}$, different from 1. To do so, we have generalized a theorem of Ueda [U] and Hakim [Ha] regarding the local normal form around the semi-parabolic fixed point from the case $\lambda = 1$ to the case $\lambda = e^{2\pi i p/q}$. This is given in Section 3.3. In Section 3.4 we define *big* attractive petals for semi-parabolic germs of $(\mathbb{C}^2, 0)$. Both sections are of independent interest. In Section 3.6 we show how to control the size of the normalizing neighborhood for a family of semi-parabolic Hénon maps. The precise statement is given in Theorem 3.19.

In one-dimensional dynamics the pinched disk model for polynomial Julia sets (as described by Thurston [Th]) is an important tool in understanding the geometry of connected Julia sets. Thurston models the filled-in Julia set as a quotient of the unit disk $\mathbb{D}$ along the leaves of a lamination defined inside the disk. A recent description of degree d invariant laminations is given in [Th1].

In Chapter 5 we describe a construction of a topological model (*pinched ball model*) for Hénon maps that are small perturbations of hyperbolic polynomials (or parabolic polynomials in the appropriate setting). The underlying model where the nontrivial pinching takes place is the unit 3-sphere with a solenoid removed and is described in Section 5.2. The pinching is done inside the 4-dimensional unit ball, along the leaves of a dynamically defined lamination. Quotienting by the induced equivalence relation gives a global model for the set J^+, which is easier to understand than the inductive limit description from Corollary 3.3.3. The lamination for the Hénon map is described in Section 5.4 as a lift of the Thurston lamination of the polynomial p from the closed unit disk to the unit 4-ball in $\mathbb{C}^2$, using the inductive limit.

CHAPTER 2

HYPERBOLIC HÉNON MAPS

2.1 Preliminaries

In general, a complex Hénon map $H_{p,a} : \mathbb{C}^2 \to \mathbb{C}^2$ is defined by

$$H_{p,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix},$$

where p is a monic polynomial of degree $d \geq 2$. Note that $H_{p,a}$ is a biholomorphism with constant jacobian equal to a, whenever $a \neq 0$.

Define the invariant subsets as in [BS1], [FS] and [HOV1]

$$K^{\pm} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \ : \ \left\| H_{p,a}^{\circ n} \begin{pmatrix} x \\ y \end{pmatrix} \right\| \text{ remains bounded as } n \to \pm\infty \right\},$$

as well as $K = K^- \cap K^+$. Then let $J^{\pm} = \partial K^{\pm}$ and $J = J^- \cap J^+$. The set J is often called the *Julia set* for the Hénon map $H_{p,a}$. Define the escaping sets $U^{\pm} = \mathbb{C}^2 - K^{\pm}$.

In this chapter we describe Hénon maps $H_{p,a}$ that are small perturbations of a hyperbolic polynomial p, for sufficiently small Jacobian a. We will therefore look at hyperbolic maps $H_{p,a}$ with $0 < |a| < 1$. In this situation, it is known that K^- has no interior and so $K^- = J^-$ [BS1], [FM].

Definition 2.1. We say that the Hénon map H is hyperbolic (on J) if there is a continuous splitting $J \ni x \mapsto E^s_x \oplus E^u_x = T_x\mathbb{C}^2$ of the complex tangent space, with $DH_x(E^s_x) = E^s_{H(x)}$, $DH_x(E^u_x) = E^u_{H(x)}$ and

$$\|DH_x^{\circ n}|_{E^s_x}\| \leq C\lambda^n, \quad \|DH_x^{-\circ n}|_{E^u_x}\| \leq C\lambda^n$$

for some constants C and $0 < \lambda < 1$.

For a polynomial p of degree $d \geq 2$, the *filled Julia set* of p is

$$K_p = \{z \in \mathbb{C} \mid |p^{\circ n}(z)| \text{ bounded as } n \to \infty\}.$$

The set $J_p = \partial K_p$ is the *Julia set* of p.

If K_p is connected (or equivalently J_p is connected) then there exists a unique analytic isomorphism

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p$$

such that $\psi_p(z^2) = p(\psi_p(z))$ and $\psi_p(z)/z \to 1$ as $z \to \infty$. Furthermore, if J_p is locally connected then ψ_p extends to the boundary $\mathbb{S}^1$ and defines a continuous, surjective map $\gamma : \mathbb{S}^1 \to J_p$ [M]. The boundary map γ is called the Carathéodory loop.

Assume p is hyperbolic. The filled Julia set K_p is connected and locally connected, and none of the critical points of p belong to the Julia set J_p. Let $D \subset \mathbb{C}$ be a disk of large enough radius such that $J_p \subset D$.

Remark 2.2. If p monic and centered, having connected Julia set J_p, then it is enough to consider D a disk of radius 2 [Bu].

Hubbard and Oberste-Vorth [HOV2] studied the structure of the sets J, J^+ and J^- for Hénon maps that are small perturbations of a hyperbolic polynomial p. The proof relies on telescopes for hyperbolic polynomials and crossed mappings. We will give a new proof of the theorem regarding the sets J and J^+ in the language of a fixed point theorem. We will recover the set J^+ inside the bidisk $\mathbb{D}_r \times \mathbb{D}_r$ as the image of the unique fixed point of a contracting graph-transform operator in some function space $\mathcal{F}$, which we define in Section 2.4. The proof resembles the proof of the Hadamard-Perron theorem [KH]. The starting point is to put a product of two Poincaré metrics on a neighborhood of J^+ inside $\mathbb{D}_r \times \mathbb{D}_r$ with respect to which the derivative of the Hénon map DH expands strongly in horizontal cones and contracts strongly in vertical cones. The approach has the advantage that it can be generalized to complex Hénon maps with a semi-parabolic fixed point (or more generally to the class of Hénon maps discussed in Remark 4.13). The semi-parabolic case will be treated in detail in the next Chapter 3. We will complete the proof of the theorem in Chapter 4, when we establish the conjugacy between the Hénon map and a certain model map.

2.2 Construction of the neighborhood U

We assume for clarity of exposition that p is a quadratic polynomial, $p(x) = x^2 + c$. Define the Hénon map as

$$H_{p,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) + ay \\ ax \end{pmatrix}.$$

It is conjugated to the initial Hénon map by the linear transformation $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ ay \end{pmatrix}$. When $a = 0$, the Hénon map becomes

$$H_{p,0}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) \\ 0 \end{pmatrix}$$

and maps $\mathbb{C}^2$ to the x-axis and $J^+ = J_p \times \mathbb{C}$. When $a \neq 0$, $H_{p,a}$ is a biholomorphism whose inverse is

$$H_{p,a}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a}\begin{pmatrix} y \\ x - p(y/a) \end{pmatrix}.$$

Assume p is hyperbolic with connected Julia set. Since p is quadratic, there is only one attracting cycle Z. Let V_0 be a neighborhood of this cycle such that $p(V_0)$ is relatively compact in V_0; we will consider V_0 as a union of sufficiently small disks centered around the points of the cycle. Set $V_n = p^{-n}(V_0)$, where we only take into account the preimages that belong to the immediate basin of attraction. The only critical point is 0 which is attracted to Z, so there exists a minimal n for which V_n contains the critical value c.

Consider the set

$$U := \mathbb{C} - \overline{V_n} - \{z \in \mathbb{C} - K_p \mid |\psi_p^{-1}(z)| \geq R\}$$

for some $R > 2$.

The set $U' := p^{-1}(U) \subset U$ is relatively compact in U, and $p : U' \to U$ is a covering map. Let μ be the Poincaré metric on U. Measured in the metric μ, the map $p : U' \to U$ is strongly expanding. The construction of the sets U and U' is the same as in [DH] and [H].

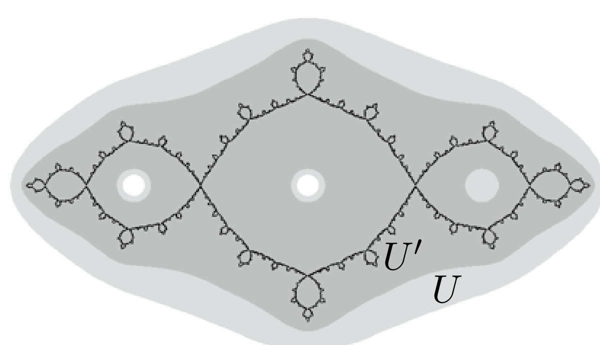

Figure 2.1: A neighborhood U of the Julia set of the polynomial $p(x) = x^2 - 1$. The attracting cycle is $\{0, c\}$ and the set V_n is a union of two small disks centered around the points of this cycle. The set U is the complement of V_n inside an equipotential of the Green's function of p. The set $U' \subset U$, in dark gray, is compactly contained in U.

Let ρ_U denote the density function of the Poincaré metric on U. Then ρ_U is positive and C^∞-smooth on $\overline{U'} = \overline{p^{-1}(U)} \subset\subset U$.

Lemma 2.3. *There exists a positive constant C such that*

$$|\rho_U(z) - \rho_U(z - \delta)| \leq |\delta| C \rho_U(z)$$

for all points $z, z - \delta \in U'$.

Proof. Let

$$C := \frac{\sup_{U'} |\rho'_U(z)|}{\inf_{U'} \rho_U(z)}.$$

The density function is smooth on U' so, by the Mean Value Theorem we have

$$\left| \frac{\rho_U(z) - \rho_U(z - \delta)}{\delta} \right| \leq \sup_{U'} |\rho'_U| \leq \frac{\sup_{U'}(\rho'_U)}{\inf_{U'}(\rho_U)} \rho_U(z) = C\rho_U(z),$$

which concludes the proof of the lemma. □

Lemma 2.4. *Let z_1 and z_2 be any two points in U', and let δ be small enough so that $z_1 - \delta$ and $z_2 - \delta$ are still in U'. Then*

$$d_U(z - \delta, z' - \delta) \leq (1 + |\delta| C) d_U(z, z').$$

Proof. Let γ be a curve connecting z_1 and z_2, for which $l(\gamma) = d_U(z_1, z_2)$. Then, if we translate γ by δ, we get a curve (not necessarily length minimizing) connecting $z_1 - \delta$ to $z_2 - \delta$. For δ small enough, we can assume that the new curve $\gamma - \delta$ is still contained in U'.

It has length

$$l(\gamma - \delta) = \int_{\gamma - \delta} \rho_U(z)|dz| = \int_{\gamma} \rho_U(z - \delta)|dz|.$$

Using the triangle inequality

$$\begin{aligned}\int_\gamma \rho_U(z-\delta)|dz| &\leq \int_\gamma |\rho_U(z-\delta)-\rho_U(z)||dz| + \int_\gamma \rho_U(z)|dz| \\ &\leq \int_\gamma |\delta|C\rho_U(z)|dz| + \int_\gamma \rho_U(z)|dz| = (1+|\delta|C)l(\gamma).\end{aligned}$$

It follows that $d_U(z_1-\delta, z_2-\delta) \leq l(\gamma-\delta) \leq (1+|\delta|C)l(\gamma) = (1+|\delta|C)d_U(z_1, z_2)$. □

One other useful observation is that since U' is compactly contained in U, the Poincaré metric of U is bounded below and above by the Euclidean metric on U'.

Lemma 2.5. *There exist two constants* $m = \inf_{z\in U'} \rho_U(z)$ *and* $M = \sup_{z\in U'} \rho_U(z)$ *such that*

$$m|x-x'| \leq d_U(x,x') \leq M|x-x'|, \text{ for all } x, x' \in U'.$$

Proof. The proof follows directly from the definition of U and U'. □

2.3 Construction of the neighborhood V

Let U' be the neighborhood of J previously constructed, so that $p : U'' = p^{-1}(U') \to U'$ is strictly expanding in the Poincare metric of U.

Set $V := U' \times \mathbb{D}_r$ for $r > 0$, chosen so that

- $\overline{H(V)}$ does not intersect the horizontal boundary of V, so $|ax| < r$, for any $x \in U'$.

- All points in $H(V) - V$ belong to the escaping set U^+.

  One can choose for instance $r > 2$ and $\mathrm{diam}(U') > 2$.

Let the Jacobian a be small enough so that

- $r|a| < |p(x) - c|$ for any x in U'. This is possible because we removed a disc around the critical value c of the polynomial p, hence $\inf_{x \in U'} |p(x) - c| > 0$.
- $r|a| < d(\partial U', \partial U)$. This assures that for any x in U', the disk of radius $r|a|$ around x belongs to U. In other words, the $r|a|$–neighborhood of U' is compactly contained in U.

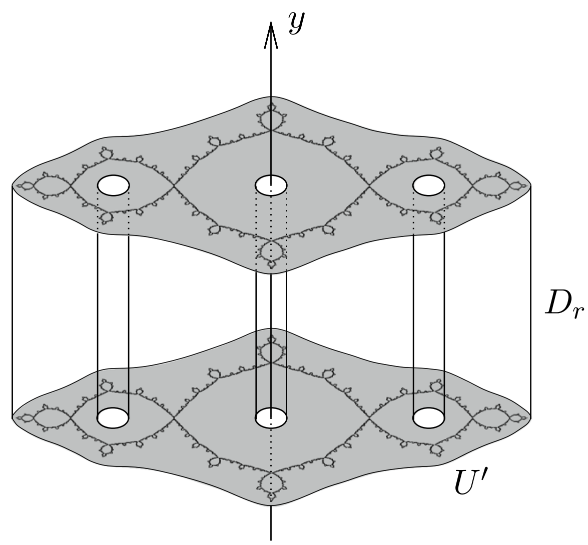

Figure 2.2: A neighborhood $V = U' \times \mathbb{D}_r$ for $J^+ \cap \{|y| < r\}$.

Lemma 2.6. *Let $(x, y) \in V$. Then $H^{-1}(x, y) = (x', y') \in V$, provided that $|y'| < r$.*

Proof. The point

$$H^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a}\begin{pmatrix} y \\ x - p(y/a) \end{pmatrix}$$

belongs to V if $y/a \in U'$ and $\left|\frac{1}{a}(x - p(y/a))\right| < r$.

In this lemma we claim that the first condition is redundant, once the second condition is satisfied. We know $\left|x - p(y/a)\right| < r|a|$. Since $x \in U'$ and we chose a small enough so that the disk of radius $r|a|$ around x is still in U, it follows that $p(y/a) \in U$, hence $y/a \in U'$. Therefore $H^{-1}(x, y)$ belongs to V. □

Proposition 2.7. *Let $(x, y), (x', y')$ be two points in V with $H_{p,a}(x, y) = (x', y')$ and (ξ, η) and (ξ', η') two tangent vectors such that $DH_{(x,y)}(\xi, \eta) = (\xi', \eta')$. Then*

(a) If $|\xi'| < |\eta'|$ then $|\xi| < |\eta|$.

(b) If $|\xi| > |\eta|$ then $|\xi'| > |\eta'|$.

Proof. A direct computation show that $\xi' = 2x\xi + a\eta$ and $\eta' = a\xi$.

(a) If $|\xi'| < |\eta'|$ then $2|x||\xi| - |a||\eta| < |\xi'| < |\eta'| = |a||\xi|$, so $|\xi|(2x - |a|) < |a||\eta|$. The point (x, y) belongs to V, so x is bounded away from 0, in fact we have $|x| > r|a|$ where $r > 2$. So we get $|\xi| < |\eta|$.

(a) If $|\xi| > |\eta|$ then $|\xi'| > 2|x||\xi| - |a||\eta| > (2|x| - |a|)|\xi| > |a||\xi| = |\eta'|$. □

Fix a and p and denote for simplicity the Hénon map $H_{p,a}$ by H. We would like to define two invariant families of cones $C^h_{(x,y)}$ and $C^v_{(x,y)}$ in the tangent bundle of V such that

$$\begin{aligned} C^h_{(x,y)} &= \{(\xi, \eta) \in T_{(x,y)}V \ : \ |(x, \xi)|_U > |(y, \eta)|_{\mathbb{D}_r} \text{ and } |\xi| > |\eta|\} \\ C^v_{(x,y)} &= \{(\xi, \eta) \in T_{(x,y)}V \ : \ |(x, \xi)|_U < |(y, \eta)|_{\mathbb{D}_r} \text{ and } |\xi| < |\eta|\}, \end{aligned}$$

where the lengths are measured with respect to the Poincaré metric on U and $\mathbb{D}_r$, and with respect to the Euclidean metric.

Definition 2.8. An analytic curve $\gamma \subset V$ is vertical-like if for all points (x, y) on γ, the tangent vectors (ξ, η) to γ at (x, y) belong to the vertical cone $C^v_{(x,y)}$.

Lemma 2.9. *Let γ be a vertical-like curve in V. Then $H^{-1}(\gamma) \cap V$ is the union of two vertical-like curves γ_1 and γ_2.*

Proof. Since the curve γ is vertical-like, it is the graph of a holomorphic function $f : \mathbb{D}_r \to U'$, hence $\gamma = \{(f(z), z),\ z \in \mathbb{D}_r\}$. The function f contracts Poincaré length and $|f'(z)| < 1$. Then

$$H^{-1}(\gamma) = \left\{ H^{-1}\begin{pmatrix} f(z) \\ z \end{pmatrix} = \frac{1}{a}\begin{pmatrix} z \\ f(z) - (\frac{z}{a})^2 - c \end{pmatrix},\ z \in \mathbb{D}_r \right\}$$

is an analytic curve whose horizontal foldings do not belong to the strip $\{y \in \mathbb{D}_r\}$. Suppose there is a folding inside $\{y \in \mathbb{D}_r\}$. Then, by Lemma 2.6, the folding point is actually inside V, hence its projection on the first coordinate, $\frac{z}{a} \in U'$ is bounded away from 0 (independent of a). Then $f'(z) - \frac{2z}{a^2} = 0$ cannot have solutions inside $\mathbb{D}_r$, as $\frac{2}{a}(\frac{z}{a})$ gets arbitrarily large when a is small enough, whereas $f'(z)$ remains bounded.

Therefore the degree of the projection of $H^{-1}(\gamma)$ on the second coordinate is constant in $\{y \in \mathbb{D}_r\}$. It is easy to see that the degree is 2, by looking at the number of intersections of $H^{-1}(\gamma)$ with the x-axis. The curve γ is vertical-like in V, hence it intersects H(x-axis) in exactly 2 points. Then $H^{-1}(\gamma)$ intersects the x-axis in 2 points.

Thus $H^{-1}(\gamma) \cap \{(x, y), y \in \mathbb{D}_r\}$ is a union of two analytic curves γ_1 and γ_2. By Lemma 2.6, γ_1 and γ_2 are contained in V, hence $H^{-1}(\gamma) \cap V$ is the union of two analytic curves γ_1 and γ_2.

The map $pr_2 : \gamma_1 \to \mathbb{D}_r,\ pr_2(x, y) = y$ is a degree one cover map, hence by the Implicit Function Theorem, γ_1 is the graph of a holomorphic function $x = \phi(y)$ where $\phi : D_r \to U'$. The map ϕ must also be injective, because $pr_1 : \gamma_1 \to U'$, $pr_1(x, y) = x$ is injective. By the Schwarz-Pick lemma, $\phi : D_r \to U'$ is weekly contracting in the Poincaré metrics of D_r and U', hence strongly contracting if we endow U' with the Poincaré metric of U. By Lemma 2.7 we have $|\phi'(z)| < 1$ for $z \in \mathbb{D}_r$. It follows that γ_1 is vertical-like. □

Corollary 2.9.1. The horizontal cones $C^h_{(x,y)}$ are H-invariant, in the sense that

$$dH(C^h_{(x,y)}) \subseteq C^h_{H(x,y)}, \text{ for any } (x, y) \text{ in } V \cap H^{-1}(V).$$

The vertical cones $C^v_{(x,y)}$ are invariant under H^{-1},

$$dH^{-1}(C^v_{(x,y)}) \subseteq C^v_{H^{-1}(x,y)}, \text{ for any } (x, y) \text{ in } V \cap H(V).$$

Proof. The invariance of the vertical cones follows from Proposition 2.7 and from Lemma 2.9. The invariance of the horizontal cones follows by changing H with H^{-1}. □

2.4 The function space $\mathcal{F}$ and the contraction

Choose $R > 2$ as in the definition of the neighborhood U of J_p and define the sequence of maps (equipotentials) $\gamma_n : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$,

$$\gamma_{n+1}(t) = \psi_p\left(R^{1/2^{n+1}} e^{2\pi i t}\right) = p^{-1}(\gamma_n(2t)). \tag{2.1}$$

Note that $\gamma_{-1}(\mathbb{R}/\mathbb{Z}) \subset \partial U$ and $\gamma_0(\mathbb{R}/\mathbb{Z}) \subset \partial U'$.

Definition 2.10. Let $f_0 : \mathbb{S}^1 \times \mathbb{D}_r \to V$, $f_0(t, z) = (\gamma_0(t), z)$.

For any fixed $t \in \mathbb{S}^1$, $f_0(2t \times \mathbb{D}_r)$ is a vertical disk in V. $H^{-1} \circ f_0(2t \times \mathbb{D}_r)$ is a vertical parabola, whose tip in not in V. Hence $H^{-1} \circ f_0(2t \times \mathbb{D}_r) \cap V$ is a union of two vertical-like disks, and $pr_2 : H^{-1} \circ f_0(2t \times \mathbb{D}_r) \cap V \to \mathbb{D}_r$, $pr_2(x, z) = z$ is a degree two covering map.

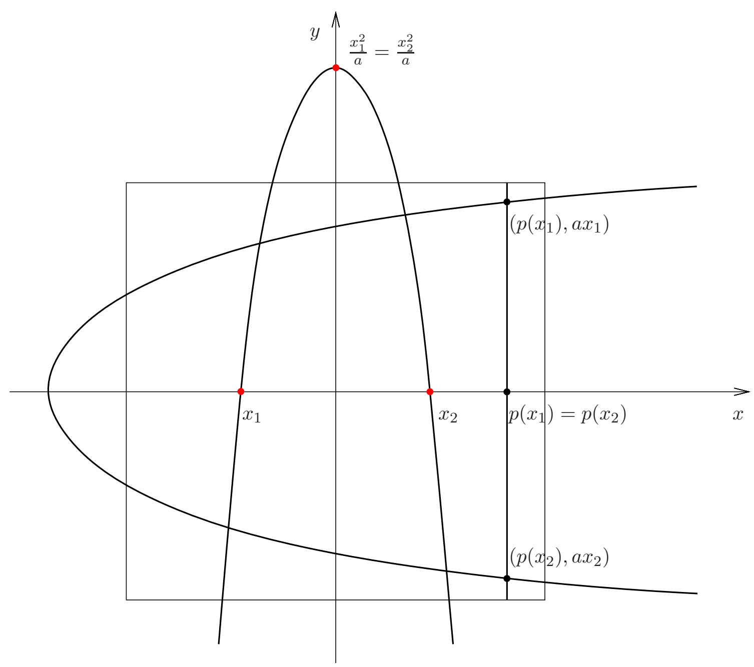

Figure 2.3: The preimage of a fiber of f_0 in the neighborhood V.

Let C_t be the component of $H^{-1} \circ f_0(2t \times \mathbb{D}_r) \cap V$ that crosses the x-axis at $(\gamma_2(t), 0)$. Similarly, let $C_{t+1/2}$ be the component of $H^{-1} \circ f_0(2t \times \mathbb{D}_r) \cap V$ that crosses the x-axis at $(\gamma_2(t + 1/2), 0)$. Notice that $pr_2 : C_t \to \mathbb{D}_r$, $pr_2(x, z) = z$ is a degree one covering map, hence, by the implicit function theorem, C_t is the graph of a holomorphic function $x = \phi_t(z)$. Let us define a new function $\widetilde{f_0} : \mathbb{S}^1 \times \mathbb{D}_r \to V$ as $\widetilde{f_0}(t, z) = (\phi_t(z), z)$. Notice that $\widetilde{f_0}$ is homotopic to f_0 by construction, since $\gamma_2(t)$ and $p^{-1}(\gamma_1(2t))$ are homotopic.

Remark 2.11. Notice that $\widetilde{f_0}$ is homotopic to f_0 by construction, since $\gamma_2(t)$ and $p^{-1}(\gamma_1(2t))$ are homotopic. Moreover, since a is small, $\widetilde{f_0}(\mathbb{S}^1 \times \mathbb{D}_r)$ and $f_0(\mathbb{S}^1 \times \mathbb{D}_r)$ are disjoint.

Consider the space of functions

$$\begin{aligned}\mathcal{F} \;=\; & \Big\{ f : \mathbb{S}^1 \times \mathbb{D}_r \to V \;\Big|\; f \text{ is proper, } f\big|_{t\times\mathbb{D}_r} \text{ is analytic,} \\ & pr_2 \circ f(t,0) = 0,\ f(t,\mathbb{D}_r) \text{ is vertical-like, } pr_2\left(\frac{\partial f}{\partial z}(t,0)\right) > 0, \\ & \text{and } f \text{ is homotopic to } f_0 \Big\}.\end{aligned}$$

Use the Kobayashi metric on V and on $\mathcal{F}$ consider the metric

$$d(f_1, f_2) = \sup_{t\in\mathbb{S}^1} \sup_{z\in\mathbb{D}_r} d\left(pr_1(f_1(t,z)),\ pr_1(f_2(t,z))\right).$$

Notice that the set $\mathcal{F}$ is nonempty: the functions f_0 and $\widetilde{f_0}$ are both in $\mathcal{F}$.

Proposition 2.12. *The space $\mathcal{F}$ is complete in the d-metric defined above.*

Consider the graph transform $F : \mathcal{F} \to \mathcal{F}$, defined as

$$F(f) = \tilde{f},$$

where $\tilde{f}\big|_{t\times\mathbb{D}_r}$ is the conformal map of the component of $H^{-1}(f(2t \times \mathbb{D}_r)) \cap V$ "homotopic to" $\widetilde{f_0}(t \times \mathbb{D}_r)$, normalized so that 0 maps to the intersection of this component with the x-axis, and also $pr_2\left(\frac{\partial \tilde{f}}{\partial z}(t,0)\right) > 0$.

Proposition 2.13. *The map $F : \mathcal{F} \to \mathcal{F}$ is well defined.*

Proof. We have $F(f_0(t,z)) = \widetilde{f_0}(t,z)$. Consider next any map $f \in \mathcal{F}$.

For fixed $t \in \mathbb{S}^1$, $z \to pr_2 \circ f(2t,z)$ is a degree one covering map from $\mathbb{D}_r$ to $\mathbb{D}_r$. Then $pr_2 : (H^{-1} \circ f(2t \times \mathbb{D}_r)) \cap V \to \mathbb{D}_r$ is a degree two covering map over $\mathbb{D}_r$, and the map $pr_1 : (H^{-1} \circ f(2t \times \mathbb{D}_r)) \cap V \to U$ has degree one over its image. So to make $\mathcal{F}(f)$ well defined, we need to indicate a way to select one of the two components of $H^{-1}(f(2t \times \mathbb{D}_r)) \cap V$.

Since f belongs to the space $\mathcal{F}$, f and f_0 must be homotopic, hence $H^{-1} \circ f$ and $H^{-1} \circ f_0$ are also homotopic. Let $G : [0,1] \times \mathbb{S} \times \mathbb{D}_r \to V$ be a homotopy between $G(0,t,z) = f_0(t,z)$, and $G(1,t,z) = f(t,z)$, such that $G((s,t) \times \mathbb{D}_r)$ is vertical-like, and $pr_2(G(s,t,0)) = 0$.

Then $H^{-1} \circ G(s,2t,z)$ is a homotopy between $H^{-1}(f_0(2t,z))$ and $H^{-1}(f(2t,z))$, such that $pr_1(H^{-1} \circ G(s,2t,0)) = 0$. For any s and t, $H^{-1} \circ G(s,2t \times (\mathbb{D}_r - \{0\}))$ consists of two vertical-like disks. When $s = 0$, the set $H^{-1} \circ G(0, 2t \times (\mathbb{D}_r - \{0\}))$ has two components, say L_t^0 and $L_{t+1/2}^0$, whose restrictions to the set V are precisely $\widetilde{f_0}(t \times \mathbb{D}_r)$ and $\widetilde{f_0}((t+1/2) \times \mathbb{D}_r)$. For any s, $H^{-1} \circ G(s, 2t \times (\mathbb{D}_r - \{0\}))$ also has two components. We will label as L_t^s (respectively as $L_{t+1/2}^s$) the component that is homotopically obtained from L_t^0 (respectively from $L_{t+1/2}^0$).

We are now able to use this homotopy to label the two components of $H^{-1}(f(2t \times \mathbb{D}_r)) \cap V$. Denote by $C_t = L_t^1 \cap V$ and $C_{t+1/2} = L_{t+1/2}^1 \cap V$. As before, $pr_2 : C_t \to \mathbb{D}_r$, $pr_2(x,z) = z$ is a degree one covering map, hence, by the implicit function theorem, C_t is the graph of a holomorphic function $x = \phi_t(z)$. Let then $\mathcal{F}(f(t,z)) = (\phi_t(z), z)$.

Notice also that $F(f)$ belongs to $\mathcal{F}$ by construction. In particular, the last condition is satisfied because $F(f)$ is homotopic to $F(f_0)$, and $F(f_0)$ is homotopic to f_0. □

Theorem 2.14. *The map $F : \mathcal{F} \to \mathcal{F}$ is a contraction in the metric defined on $\mathcal{F}$.*

Proof. Consider any two functions f_1 and f_2 in the space $\mathcal{F}$. We will show that there exists a constant $\mu < 1$ such that, for any $t \in \mathbb{S}^1$

$$\sup_{z \in \mathbb{D}_r} d_U\left(pr_1(F(f_1(t,z))), pr_1(F(f_2(t,z)))\right) \leq \mu \sup_{z \in \mathbb{D}_r} d_U\left(pr_1(f_1(2t,z)), pr_1(f_2(2t,z))\right).$$

Recall that $F(f_1(t \times \mathbb{D}_r))$ and $F(f_2(f \times \mathbb{D}_r))$ are two vertical-like disks in $V = U' \times \mathbb{D}_r$, parametrized by the second coordinate, so there exists two conformal maps $\varphi_1, \varphi_2 : \mathbb{D}_r \to U'$ such that $F(f_1(t,z)) = (\varphi_1(z), z)$ and $F(f_2(t,z)) = (\varphi_2(z), z)$.

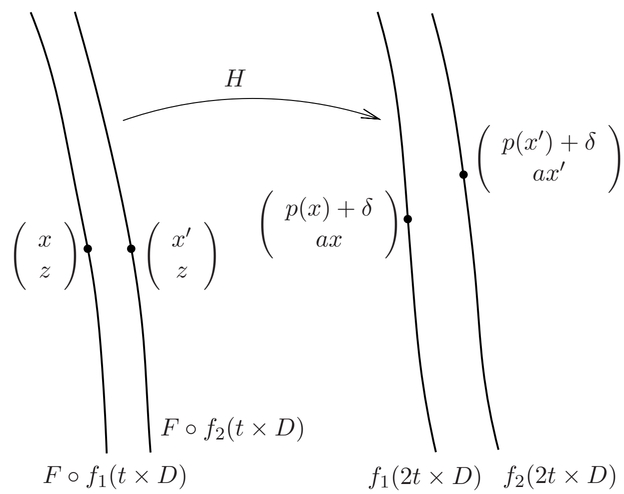

Figure 2.4: Two fibers $F \circ f_1$ and $F \circ f_2$ and their image under H.

Let z_0 be any point in $\mathbb{D}_r$. We have

$$H_{p,a}\begin{pmatrix} \varphi_i(z_0) \\ z_0 \end{pmatrix} = \begin{pmatrix} p(\varphi_i(z_0)) + az_0 \\ a\varphi_i(z_0) \end{pmatrix}, \quad i = 1, 2.$$

If we denote by $x = \varphi_1(z_0)$, $x' = \varphi_2(z_0)$, and $\delta = az_0$, we get

$$H_{p,a}\begin{pmatrix} x \\ z_0 \end{pmatrix} = \begin{pmatrix} p(x) + \delta \\ ax \end{pmatrix}, \text{ and } H_{p,a}\begin{pmatrix} x' \\ z_0 \end{pmatrix} = \begin{pmatrix} p(x') + \delta \\ ax' \end{pmatrix}.$$

The points x, x', $p(x) + \delta$ and $p(x') + \delta$ all belong to U'. Then $p(x)$ and $p(x')$ must belong to a $|\delta|$−neighborhood of U', say $N_{|\delta|}$. Since $|\delta| < r|a|$, $N_{|\delta|}$ is compactly contained in U. The polynomial $p : p^{-1}(N_{|\delta|}) \to N_{|\delta|}$ is strongly expanding in the Poincaré metric of U, i.e.

$$d_U(x, x') \leq K d_U(p(x), p(x')), \quad K < 1.$$

The expansion factor K only depends on the modulus of the annulus $U - N_{|\delta|}$.

By Lemma 2.2, if δ is small,

$$d_U(p(x), p(x')) \leq (1 + |\delta| C) d_U(p(x) + \delta, p(x') + \delta),$$

so we get

$$d_U(x, x') \leq K(1 + |\delta| C) d_U(p(x) + \delta, p(x') + \delta).$$

To link the right hand side with the distance between $f_1(2t \times \mathbb{D}_r)$ and $f_2(2t \times \mathbb{D}_r)$, we need to apply the triangle inequality. Also, notice that both fibers belong to the vertical cone, so the vertical distance between any two points of the fiber is bigger then their horizontal distance.

Let $\psi : \mathbb{D}_r \to f_1(2t \times \mathbb{D}_r)$ be the conformal isomorphism which parametrizes the fiber. Then $\psi(ax) = (p(x) + \delta, ax)$ and $\psi(ax') = (p(x') + \delta, ax')$. The mapping $pr_1 \circ \psi : \mathbb{D}_r \to U'$ is holomorphic and contracting in the Poincaré metrics of $\mathbb{D}_r$ and U. It follows that

$$\begin{aligned} d_U(p(x) + \delta, p(x') + \delta) &\leq \sup_{z \in \mathbb{D}_r} d_U(f_1(2t, z), f_2(2t, z)) + d_U(pr_1 \circ \psi(ax), pr_1 \circ \psi(ax')) \\ &\leq \sup_{z \in \mathbb{D}_r} d_U(f_1(2t, z), f_2(2t, z)) + d_{\mathbb{D}_r}(ax, ax') \end{aligned}$$

We will now use two comparisons of the Poincaré metrics on $\mathbb{D}_r$ and U' with the Euclidean metric. The set $\overline{H(V)}$ does not intersect the vertical boundary of

V, so ax' and ax', belong to a disk W compactly contained in $\mathbb{D}_r$. We can then estimate the Poincaré metrics on $\mathbb{D}_r$ in terms of the Euclidean metric. There are two constants m_r and M_r such that

$$m_r|ax - ax'| \leq d_{\mathbb{D}_r}(ax, ax') \leq M_r|ax - ax'|.$$

Following Lemma 2.3, a similar comparison holds if we consider the Poincaré metric of U on the set U'. In particular since x and x' also belong to U', we know

$$m|x - x'| \leq d_U(x, x') \leq M|x - x'|.$$

So we can the estimate

$$d_{\mathbb{D}_r}(ax, ax') \leq M_r|ax - ax'| = |a|M_r|x - x'| \leq |a|\frac{M_r}{m}d_U(x, x').$$

Combining all previous estimates we get

$$d_U(x, x') \leq K(1 + |\delta|C)\left(\sup_{z\in\mathbb{D}_r} d_U(f_1(2t, z), f_2(2t, z)) + |a|\frac{M_r}{m}d_U(x, x')\right),$$

which yields

$$d_U(x, x') \leq \frac{K(1 + |\delta|C)}{1 - |a|K(1 + |\delta|C)\frac{M_r}{m}}\left(\sup_{z\in\mathbb{D}_r} d_U(f_1(2t, z), f_2(2t, z))\right).$$

Set

$$\mu := \frac{K(1 + |\delta|C)}{1 - |a|K(1 + |\delta|C)\frac{M_r}{m}}.$$

The contraction factor $K < 1$ decreases as a decreases, while C and m are fixed constants, independent of a and r. $M_r > 0$ depends on the radius r which can be chosen independently of a. The factor δ is small, $|\delta| < |a|r$. Choose a_0 small enough such that $(1 + |a_0|rC)(1 + |a_0|\frac{M_r}{m}) < \frac{1}{K}$; this can be done, since $\frac{1}{K}$ is strictly bigger than 1.

It follows that $\mu < 1$ for all a with $0 < |a| < a_0$. Thus for all $t \in \mathbb{S}^1$ we have

$$\sup_{z \in \mathbb{D}_r} d_U(F(f_1(t,z)), F(f_2(t,z))) \leq \mu \sup_{z \in \mathbb{D}_r} d_U(f_1(2t,z), f_2(2t,z)).$$

Taking the supremum after $t \in \mathbb{S}^1$, we get the desired contraction

$$d(F(f_1), F(f_2)) \leq \mu \cdot d(f_1, f_2), \quad \mu < 1$$

The constants δ and μ are local variables. □

Proposition 2.15. *The map $F : \mathcal{F} \to \mathcal{F}$ has a fixed point.*

Proof. F is contracting in the metric defined on $\mathcal{F}$. The existence of a fixed point follows from the Banach fixed point theorem. □

Let f^* be a fixed point of F, so $F(f^*) = f^*$. We have the following lemmas.

Proposition 2.16. *For any fixed $t \in \mathbb{S}^1$, $f^*(t,z) = (\varphi_t(z), z)$, where $\varphi_t : D_r \to U'$ is holomorphic, and either injective or constant.*

Proof. f^* is obtained as an application of the Banach fixed point theorem, hence it is obtained as a limit of the sequence $f_0(t,z) = (\gamma_r(t), z)$, $F^{\circ n}(f_0)(t,z) = (\varphi_t^n(z), z)$, where $\varphi_t^n : \mathbb{D}_r \to U'$ are holomorphic and injective for $n \geq 1$. By Hurwitz's theorem a uniform limit of holomorphic injective mappings is holomorphic and either injective or constant. □

Proposition 2.17. *The function $f^* : \mathbb{S}^1 \times \mathbb{D}_r \to V$ is continuous with respect to $t \in \mathbb{S}^1$.*

Proof. The fixed point $f^*(t,z) = (\varphi_t(z), z)$ is obtained as a uniform limit of the sequence $f_0(t,z) = (\gamma_0(t), z)$, $F^{\circ n}(f_0)(t,z) = (\varphi_t^n(z), z)$, where $\varphi_t^n(z)$ is continuous with respect to the parameter t. Hence $f^*(t,z)$ is continuous with respect to t. □

Proposition 2.18. *The function $f^* : \mathbb{S}^1 \times \mathbb{D}_r \to V$ is holomorphic with respect to the parameter a.*

Proof. It is obvious that $f_0(t,z) = (\gamma_0(t), z)$ does not depend on the parameter a. When a is small, $0 < |a| < \delta$, each function $F^{\circ n}(f_0)(t,z) = (\varphi_t^n(z), z)$ depends holomorphically on the parameter a. The construction of the metric space is uniform in a. Hence the limit $f^*(t,z) = (\varphi_t(z), z)$ is holomorphic with respect to the parameter a. □

We now recover $J^+ \cap V$ as the image of the fixed point f^*. The proofs of the following lemmas are somehow similar to [HOV2].

Lemma 2.19. $J^+ \cap V = \bigcap_{n\geq 0} H^{-n}(V)$.

Proof. Let $q \in \bigcap_{n\geq 0} H^{-n}(V)$. Since all forward iterates of q remain in the bounded set V, q cannot belong to U^+. The set U' does not contain any attractive cycles of the polynomial p, so, for small values of the Jacobian a, $V = U' \times D_r$ does not contain any attractive cycles of H. H is hyperbolic, and the interior of K^+ consists of the basin of attraction of an attractive periodic orbit. Since all forward iterates of q remain in V, q cannot belong to the interior of K^+. Hence $q \in J^+$.

We chose V so that $J \subset V$. $W^s(J) = J^+$, hence if $q \in J^+ \cap V$, then $q \in W^s(y)$, for some $y \in J$. So all forward iterates of q converge to $y \in V$. In particular no forward iterate can exit V, since r was chosen big enough so that all points in $H(V) - V$ belong to the escaping set. Hence $q \in \bigcap_{n\geq 0} H^{-n}(V)$. □

Lemma 2.20. *The set $Im(f^*)$ is forward invariant under the Hénon map H, and*

$$J^+ \cap V \cap \bigcup_{n\geq 0} H^{-n}(Im(f^*)) \subseteq Im(f^*) \subseteq J^+ \cap V.$$

Proof. We have

$$\begin{aligned} H^{-1}(f^*(2t \times \mathbb{D})) \cap V &= f^*(t \times \mathbb{D}) \cup f^*((t+1/2) \times \mathbb{D}) \\ H^{-1}(Im(f^*)) \cap V &= Im(f^*). \end{aligned}$$

It follows that $Im(f^*) \subseteq H^{-1}(Im(f^*))$, hence also $H(Im(f^*)) \subseteq Im(f^*)$. Since H is injective, $H^{-(n+1)}(Im(f^*)) \cap H^{-n}(V) = H^{-n}(Im(f^*))$ for every $n \geq 0$, hence, by induction, we also have

$$H^{-(n+1)}(Im(f^*)) \cap H^{-n}(V) \cap \ldots \cap H^{-1}(V) \cap V = Im(f^*). \tag{2.2}$$

But $\bigcap_{n\geq 0} H^{-n}(V) = J^+ \cap V$, hence $H^{-(n+1)}(Im(f^*)) \cap J^+ \cap V \subset Im(f^*)$. It follows that

$$J^+ \cap V \cap \bigcup_{n\geq 0} H^{-n}(Im(f^*)) \subset Im(f^*).$$

Relation 2.2 also yields

$$Im(f^*) \quad \subset \quad H^{-n}(V) \cap \ldots \cap H^{-1}(V) \cap V \quad \text{for every } n \geq 0.$$

Therefore $Im(f^*) \subset \bigcap_{n\geq 0} H^{-n}(V) = J^+ \cap V$. □

Lemma 2.21. $\bigcup_{n\geq 0} H^{-n}(Im(f^*)) = J^+$.

Proof. We have $Im(f^*) \subseteq J^+$ from the previous lemma, hence

$$\bigcup_{n\geq 0} H^{-n}(Im(f^*)) = W^s(Im(f^*)) \subseteq J^+.$$

For a sufficiently small, the Hénon map is a small perturbation of the hyperbolic polynomial p and, in particular, $J \subset\subset V$, and the number of hyperbolic periodic cycles of period n of H on J is equal to the number of repelling cycles of period n of the polynomial p on J_p.

It follows from the construction of f^* that $H(f^*(t\times\mathbb{D}_r))$ is compactly contained in $f^*(2t \times \mathbb{D}_r)$ for all $t \in \mathbb{S}^1$. For any periodic angle t of the doubling map on $\mathbb{S}^1$, such that $2^n t = t \pmod 1$, $H^n(f^*(t \times \mathbb{D}_r)) \subset\subset f^*(t \times \mathbb{D}_r)$ is a contraction from a holomorphic disk into itself, hence it has a fixed point in $f^*(t \times \mathbb{D}_r)$.

It is easy to see that we have accounted for all periodic points of H. Let q be any point in V such that $H^n(q) = q$. Suppose, for simplicity, that $n = 1$. We can construct a function $g \in \mathcal{F}$ such that $q \in \mathrm{Im}(g)$. One can easily then show that q will still belong to $F(g)$. But the sequence $F^{\circ n}(g)$ converges to f^*, hence $q \in \mathrm{Im}(f^*)$.

The Hénon map H is hyperbolic, so $J = J^*$. But all hyperbolic cycles of the Hénon map belong to $Im(f^*)$. We can conclude, since $Im(f^*)$ is closed in V, that $J \subseteq Im(f^*)$. Moreover, since H is hyperbolic, $W^s(J) = J^+$. This gives $J^+ = W^s(J) \subseteq W^s(Im(f^*))$.

Thus $J^+ = W^s(Im(f^*))$ and the lemma follows. □

Corollary 2.21.1. $Im(f^*) = J^+ \cap V = J^+ \cap \{(x, y) \in \mathbb{C}^2, y \in \mathbb{D}_r\}$.

CHAPTER 3

SEMI-PARABOLIC HÉNON MAPS

3.1 A description of the parameter space

Definition 3.1. A fixed point (x, y) of H is called semi-parabolic if the derivative $DH_{(x,y)}$ has eigenvalues $\lambda = e^{2\pi i p/q}$ and μ, with $|\mu| < 1$.

When p is a hyperbolic polynomial, for small values of the Jacobian a, the Hénon map $H_{p,a}$ is also hyperbolic. This allows us to understand hyperbolic Hénon maps as perturbations of hyperbolic polynomials, at least for small values of the Jacobian. Note that it is not true that if p has a parabolic fixed point (or parabolic cycle), and a is small enough, then $H_{p,a}$ is semi-parabolic.

Proposition 3.2. *The set of parameters $(c, a) \in \mathbb{C}^2$ for which*

$$H_{c,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c + ay \\ ax \end{pmatrix}$$

has a fixed point with one eigenvalue a root of unity λ, is a curve of equation

$$\mathcal{P}_\lambda := \left\{ (c, a) \in \mathbb{C}^2 \mid c = (1 - a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2 \right\}.$$

Proof. Let (x, y) be a fixed point of the Hénon map such that its derivative has an eigenvalue λ. Then $DH_{(x,y)} = \begin{pmatrix} 2x & a \\ a & 0 \end{pmatrix}$, so λ is a root of the characteristic polynomial $\lambda^2 - 2x\lambda - a^2 = 0$. The parameters (c, a) must verify the equations

$$x^2 + c + ay = x, \quad y = ax, \quad \text{and } x = \frac{\lambda}{2} - \frac{a^2}{2\lambda}.$$

The solution set is the curve

$$c = c(a) := (1 - a^2)\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right) - \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)^2. \tag{3.1}$$

Notice that c is of degree 2 as a function of the Jacobian $-a^2$ (and that is why we say that $\mathcal{P}_\lambda$ is a parabola). In fact, the same curve for the Hénon map written in the standard parametrization $H(x, y) = (x^2 + c - ay, x)$ (used in Section 5.1) is

$$c = \left(\frac{\lambda}{2} + \frac{a}{2\lambda}\right)(a + 1) - \left(\frac{\lambda}{2} + \frac{a}{2\lambda}\right)^2.$$

□

Fix $\lambda = e^{2\pi i p/q}$ a root of unity. Let $\mathbf{q}_a$ be the fixed point of H such that the derivative of $DH(\mathbf{q}_a)$ has one eigenvalue λ, and one eigenvalue $\mu = -\frac{a^2}{\lambda}$. When the Jacobian $-a^2$ has absolute value smaller than one, the eigenvalue μ is in absolute value smaller than one, and we call $\mathbf{q}_a$ a *semi-parabolic fixed point*, and $H_{c,a}$ a semi-parabolic Hénon map. The fixed point has an explicit equation

$$\mathbf{q}_a := \left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}, a\left(\frac{\lambda}{2} - \frac{a^2}{2\lambda}\right)\right).$$

We will use this notation throughout this chapter. We will see that for δ small enough and $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$, the semi-parabolic fixed point $\mathbf{q}_a$ has multiplicity $q + 1$.

Discussion when a = 0. The parametric line $a = 0$ intersects the curve $\mathcal{P}_\lambda$ at a point $c_0 = \frac{\lambda}{2} - \frac{\lambda^2}{4}$. Consider the polynomial $p(x) = x^2 + c_0$. It is easy to see that $p(x)$ has a parabolic fixed point $q_0 = \frac{\lambda}{2}$, of multiplier λ, and all other cycles are repelling.

The Hénon map $H_{c_0,0}$ has a very simple form

$$H_{c_0,0}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) \\ 0 \end{pmatrix}.$$

Its dynamics reduces to the dynamics of the polynomial $p(x)$. In particular it has a semi-parabolic fixed point $\mathbf{q}_0 = \left(\frac{\lambda}{2}, 0\right)$, and all its other periodic points are hyperbolic. The relevant sets under forward dynamics can then be easily described:

$$J^+ = J_p \times \mathbb{C}, \quad K^+ = K_p \times \mathbb{C}, \quad U^+ = (\mathbb{C} - K_p) \times \mathbb{C},$$

where K_p and $J_p = \partial K_p$ are the Julia set, respectively the filled-in Julia set of the polynomial p.

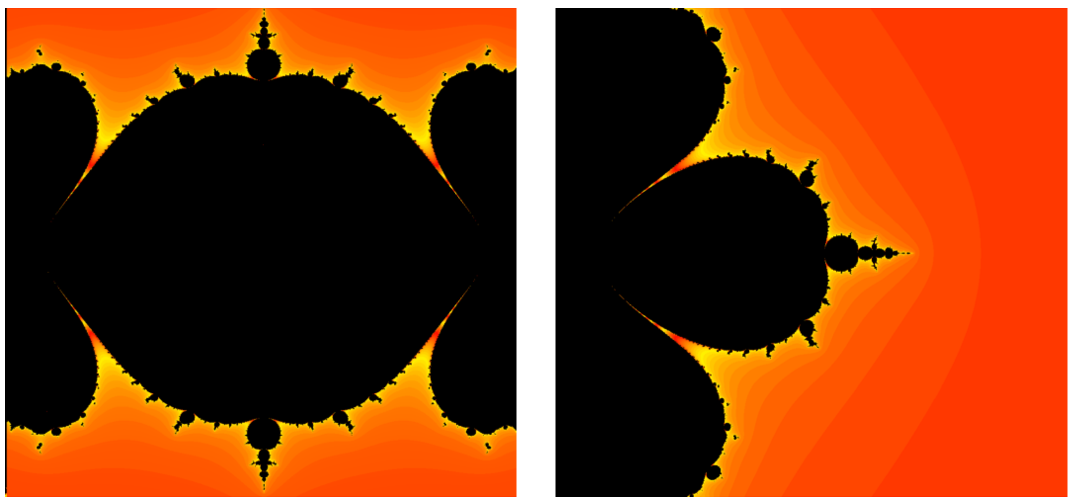

Figure 3.1: A parameter plot inside the parabola $\mathcal{P}_{-1}$. In both pictures the large region in the center contains the disk $|a| < \delta$. The black region represents (a rough approximation of) the set of parameters $(a, c) \in \mathcal{P}_{-1}$ for which J is connected. The pictures were generated using FractalStream. LEFT: The Hénon map is written as $H(x, y) = (x^2 + c + ay, ax)$. RIGHT: The Hénon map is written in the standard form $H(x, y) = (x^2 + c - ay, x)$.

3.2 Main results

We would like to understand the semi-parabolic Hénon maps $H_{c,a}$, where (c, a) lies in a small open disk centered at 0 inside the curve $\mathcal{P}_\lambda$, as a perturbation of the quadratic polynomial $p(x)$ that has a parabolic fixed point of multiplier λ. In [RT1] we prove the following structure theorem, which we present in detail in Chapters 3 and 4. We finish the proof in Section 4.2. An even more general theorem is presented in [T] (Theorem 4.28) and [RT2].

Theorem 3.3 (Main Theorem). *Let* $p(x) = x^2 + c_0$ *be a polynomial with a parabolic fixed point of multiplier* $\lambda = e^{2\pi i p/q}$. *There exists* $\delta > 0$ *such that for all parameters* $(c, a) \in \mathcal{P}_\lambda$ *with* $0 < |a| < \delta$ *there exists a homeomorphism*

$$\Phi : J_p \times \mathbb{D}_r \to J^+ \cap \overline{V}$$

such that the diagram

$$\begin{array}{ccc} J_p \times \mathbb{D}_r & \xrightarrow{\Phi} & J^+ \cap \overline{V} \\ \psi \big\downarrow & & \big\downarrow H_{c,a} \\ J_p \times \mathbb{D}_r & \xrightarrow{\Phi} & J^+ \cap \overline{V} \end{array}$$

commutes. The map ψ *is defined by*

$$\psi(\zeta, z) = \left(p(\zeta), a\zeta - \frac{a^2 z}{2\zeta} \right)$$

and is solenoidal.

The map ψ depends on a, but we will show that all maps ψ are conjugate to each other, for sufficiently small $0 < |a| < \delta$. Thus it does not matter which one we use. This is shown in detail in Chapter 4, more specifically in Lemmas 4.8 and 4.9. The definition of a solenoidal map in this context is the same as in [HOV1].

Corollary 3.3.1. $J^+ \cap \overline{V}$ is a trivial fiber bundle over J_p, the Julia set of the parabolic polynomial $p(x) = x^2 + c_0$, with fibers biholomorphic to $\mathbb{D}_r$. The set J^+ is laminated by Riemann surfaces isomorphic to $\mathbb{C}$.

Corollary 3.3.2. The Julia set J for the Hénon map is (homeomorphic to) a solenoid with identifications

$$J \simeq \bigcap_{n\geq 0} \psi^{\circ n}(J_p \times \mathbb{D}_r),$$

hence connected. The periodic points are dense in J.

Moreover, in Corollary 4.12.1 we prove that $J = J^*$, where J^* is the closure of the saddle periodic points. Passing to the inductive limit we get a global model for J^+. The notion of *inductive limit* is described in detail in Section 5.1.1, where we talk about a global model for J^+.

Corollary 3.3.3. The map Φ extends naturally to a homeomorphism $\breve{\Phi}$ and the following diagram

$$\begin{array}{ccc} \varinjlim(J_p \times \mathbb{D}_r, \psi) & \xrightarrow{\breve{\Phi}} & J^+ \\ \breve{\psi}\big\downarrow & & \big\downarrow H_{c,a} \\ \varinjlim(J_p \times \mathbb{D}_r, \psi) & \xrightarrow{\breve{\Phi}} & J^+ \end{array}$$

commutes.

Theorem 3.3 shows that semi-parabolic Hénon maps with small enough Jacobian (assume $|a| < \delta$) have connected Julia set J and are structurally stable on J and J^+. For $\lambda = 1$, the parabolic polynomial is $p(x) = x^2 + 1/4$ and it has trivial lamination. The Julia set J_p is homeomorphic to a circle. From Corollary 3.3.2 it follows that J is homeomorphic to a solenoid (with no identifications). Therefore, when $\lambda = 1$, Theorem 3.3 and Corollary 3.3.2 answer some questions of Eric Bedford (Questions 1, 2 in [B]).

Remark 3.4. We were able to characterize J without using J^- by creating a tight neighborhood V for J^+ inside the polydisk $\mathbb{D}_r \times \mathbb{D}_r$. We can also make an observation regarding the set J^- as well. Consider first the following theorem.

Theorem 3.5 ([BS8]). *The Hénon map is hyperbolic on J if and only if there is a neighborhood $\mathcal{N}$ of J and Riemann surface laminations $\mathcal{L}^{\pm}$ of $\mathcal{N} \cap J^{\pm}$ such that $\mathcal{L}^{+}$ and $\mathcal{L}^{-}$ intersect transversely at all points of J.*

In our perturbative setting, from Corollary 3.3.1 J^+ is laminar and $J = J^*$, but the Hénon map is semi-parabolic (hence not hyperbolic), so J^- is non-laminar or J^+ and J^- may have points of non-transverse intersection. From Corollary 3.3.2, J is connected and by Theorem 1.5 in [Du] it follows that the set $J^- - K^+$ supports a unique Riemann surface lamination which is uniquely ergodic. In fact, it seems reasonable that J^- is non-laminar precisely at the semi-parabolic fixed point.

The key to proving the Theorem 3.3 is to build a metric on a neighborhood of J^+ which is expanding in the horizontal direction. This is inspired by the original proof of Douady and Hubbard that the (inverse) Böttcher isomorphism extends continuously to the boundary for polynomials with a parabolic fixed point [DH], Section X.

Remark 3.6. Suppose $(c, a) \in \mathcal{P}_{\lambda}$ with $0 < |a| < \delta$ as in the theorem. We can find a whole family of parameters a that go to zero such that the Julia set of the polynomial $p_a(x) = x^2 + c(a)$ is disconnected. However, as we have seen, the Julia set J for the Hénon map $H_{c,a}$ is connected. This is because the dynamics of the semi-parabolic Hénon map $H_{c,a}$ is related to the dynamics of the parabolic polynomial $p(x) = x^2 + c_0$, but not necessarily to the dynamics of $p_a(x)$.

For $(c, a) \in \mathcal{P}_{\lambda}$, Equation 3.1 can be rewritten as

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4} + a^2 w, \quad \text{where } w := \frac{2\lambda - 2\lambda^2 - 1}{4\lambda} + \frac{a^2(2\lambda - 1)}{4\lambda^2}. \tag{3.2}$$

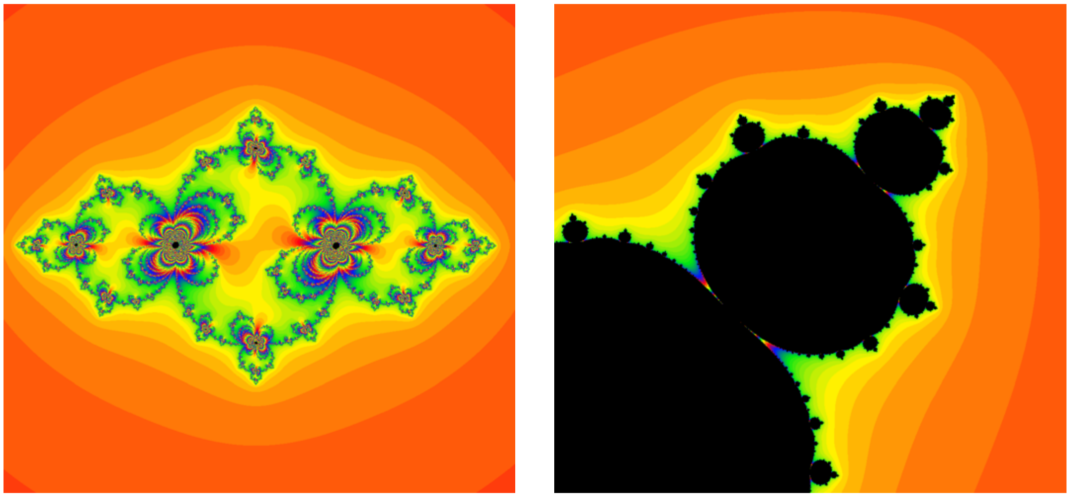

Figure 3.2: LEFT: The disconnected Julia set for $p_a(x) = x^2 + c(a)$, for $a = 0.03348 + 0.033534i$. Then $c(a) = -0.750002 + 0.00336817i$ and $|a| < 0.05$. RIGHT: The unstable manifold of the hyperbolic fixed point for $H(x, y) = (x^2 + c(a) + ay, ax)$ for the same value a.

Thus we can also write the semi-parabolic Hénon map as $H_{p,a}$, but in this case

$$H_{p,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) + a^2 w + ay \\ ax \end{pmatrix}, \tag{3.3}$$

where p is the parabolic polynomial $p(x) = x^2 + c_0$. This would emphasize the dependency on the polynomial p as in the hyperbolic setting.

Remark 3.7. Let $(H_a)_{a \in \mathbb{D}_\delta}$ be a family of complex Hénon maps with a semi-parabolic fixed point with one eigenvalue $\lambda = e^{2\pi i p/q}$. The constant δ is chosen as in Theorem 3.3. Let $a \in \mathbb{D}_\delta$. It follows from [BLS] that H_a admits an invariant measure μ_a which is the unique measure of maximal entropy $\log(2)$. The measure μ_a has two non-zero Lyapunov exponents $\lambda_a^- < 0 < \lambda_a^+$.

Let J_a denote the Julia set of H_a. We have the following dichotomy from [BS5]: $\lambda_a^- = \log(2)$ if and only if J_a is connected. We have shown in Theorem 3.3, that the Julia set J_a is connected for each $a \in \mathbb{D}_\delta$. Thus $\lambda_a^+ = \log(2)$ for this family of semi-parabolic Hénon maps. Moreover $\lambda_a^- = 2\log|a| - \log(2)$, since $\lambda_a^+ + \lambda_a^- = \log|Jac(H_a)|$ and the Jacobian of H_a is $-a^2$ if the Hénon map is written as in Proposition 3.2.

Remark 3.8. Consider a family of semi-parabolic Hénon maps with small enough Jacobian (suppose $|a| < \delta$ with δ as in Theorem 3.3). Then there are no wandering components of $int(K^+)$. We are not using this in the proof of the theorem, but it is an interesting observation. This follows from the construction in Section 3.8 and Section 4.6 from [T] of a metric on the neighborhood $\overline{V}$ with respect to which the Hénon map expands "horizontally".

Definition 3.9. For a point p, the ω-limit set $\omega(p)$ is the set of all limit points of the sequence $(H^n(p))_{n\geq 0}$.

Theorem 3.10. *Let $H_{c,a}$ be a complex Hénon map with a semi-parabolic fixed point and with $0 < |a| < \delta$. Then there are no wandering components of $int(K^+)$.*

Proof. The proof is similar to the one given in the hyperbolic case in [B2] and [BS2]. Let Ω be a wandering component of $int(K^+)$ and let $p \in \Omega$. If the orbit of a point is bounded in forward time, then it must enter $\mathbb{D}_r \times \mathbb{D}_r$ and remain there. Thus for n large, $H^n(p) \in \mathbb{D}_r \times \mathbb{D}_r$. In fact, $H^n(p)$ is in V, because the part that we take out from the bidisk $\mathbb{D}_r \times \mathbb{D}_r$ when building the neighborhood V in Section 3.7 belongs to the basin of attraction of the semi-parabolic fixed point $\mathbf{q}_a$. The component Ω is wandering so its iterates are disjoint from the basin of attraction of the semi-parabolic fixed point, thus $H^n(p) \in V$.

The ω-limit set of p, $\omega(p)$ is a nonempty compact subset of $\overline{V}$. It is invariant under H^{-1}, hence it belongs to $K^- = J^-$. It cannot belong to U^+, so it belongs to K^+. There are two cases to consider.

Case 1. Suppose $\omega(p) \cap int(K^+) \neq 0$. The proof is the same as in [BS2]. Let $q \in \omega(p) \cap int(K^+)$ and let Ω_0 be the component of $int(K^+)$ which contains q. There exists a subsequence of iterates $H^{n_k}(p) \to q$. In particular there exist integers n_1 and n_2 such that $H^{n_1}(p)$ and $H^{n_2}(p)$ belong to Ω_0. Thus $H^{n_1}(\Omega) = H^{n_2}(\Omega)$ and it follows that Ω is pre-periodic (actually periodic because H is invertible).

Case 2. Suppose $\omega(p) \subset J^+$. Then $\omega(p) \in J^+ \cap J^- = J$. This case is almost the same as in [BS2] and is a consequence of the expanding metric μ that is built in Section 3.8 (horizontally) and Section 4.6 from [T] (in horizontal cones). Since $p \in int(K^+)$, the sequence of forward iterates H^n forms a normal family on a neighborhood of p, so $\|DH^n_p\|$ is bounded for any $n \geq 0$.

Let $q \in \omega(p) \in J^+$. Assume further that $q \notin W^s_{loc}(\mathbf{q}_a)$, the local stable manifold of the semi-parabolic fixed point $\mathbf{q}_a$. Let U be a neighborhood of q in V. On V there exists a family of horizontal cones, invariant under DH such that the derivative DH expands with a factor of $1+\epsilon$, for some $\epsilon > 0$ (which depends on how close we are to the stable manifold of the semi-parabolic fixed point). The family of horizontal cones and the expansion is described in detail in Section 4.6 from [T] and [RT2]. We have $DH(C^h_x) \subset C^h_{H(x)}$ and $\|H(x), DH_x v\|_\mu > (1+\epsilon)\|x, v\|_\mu$ for $v \in C^h_x$. There exists a sequence of iterates $H^{n_k}(p) \to q$ so for k large enough $H^{n_k}(p) \in U$. This yields that $\|H^{n_k}(x), DH^{n_k}_x v\|_\mu > (1+\epsilon)^{n_k}\|x, v\|_\mu \to \infty$. Note that the metric μ is an infimum metric and is bounded by the standard Euclidean metric, so there exists a constant $C > 0$ such that $\|DH^{n_k}_p\| \cdot C \geq (1+\epsilon)^{n_k} \to \infty$. This is a contradiction, since $\|DH^n\|$ is bounded. Therefore, $q \in W^s_{loc}(\mathbf{q}_a)$. Since $q \in \omega(p)$, it follows that q is the semi-parabolic fixed point.

Thus all components of the interior of K^+ are non wandering. They fall either in the first or the second case. In the later, the only components are the components of the basin of attraction of the semi-parabolic fixed point. The set J^+ inside $\mathbb{D}_r \times \mathbb{D}_r$ moves holomorphically with a and by letting $a = 0$, we see that the interior of K^+ is generated by the semi-parabolic fixed point $\mathbf{q}_a$. □

3.3 Normal form of semi-parabolic Hénon maps

Hakim in [Ha] and Ueda in [U] have studied normal forms for germs of semi-attractive transformations H of $(\mathbb{C}^n, 0)$ for which $DH_{(0)}$ has one eigenvalue $\lambda = 1$, and the other eigenvalues $\mu_2, \dots \mu_n$ have absolute values $|\mu_j| < 1$, $j = 2, \dots, n$.

The following results are similar to Proposition 2.1, 2.2, and 2.3 from [Ha] and to Section 6 from [U]. We have adapted the propositions in [Ha] to semi-parabolic germs of transformations of $(\mathbb{C}^2, 0)$ with eigenvalues $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. As a consequence we get that 0 is a fixed point with multiplicity $\nu q + 1$ for some constant ν which we call the *(semi) parabolic multiplicity* of the fixed point, like in one-dimensional dynamics.

Proposition 3.11. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ, with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. There exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = \mu y + xh(x, y) \end{cases}, \tag{3.4}$$

where $a_j(\cdot)$ and $h(\cdot, \cdot)$ are germs of holomorphic functions from $(\mathbb{C}, 0)$ to $\mathbb{C}$, respectively from $(\mathbb{C}^2, 0)$ to $\mathbb{C}$, with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

Proof. The proof is the same as in [Ha] and [U] and is based on the straightening of the local stable manifold of the fixed point. □

Proposition 3.12. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ, with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. For every integer m there exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \ldots + a_m x^m + a_{m+1}(y) x^{m+1} + \ldots \\ y_1 = \mu y + x h(x, y) \end{cases} \tag{3.5}$$

where $a_2, \ldots, a_m$ constants.

Proof. The proof is the same as in Proposition 2.2 from [Ha] (proved also in Section 6 of [U]). However, we will refer to this proof when we discuss the domain of convergence of the functions $u(\cdot)$ and $v(\cdot)$ defined below. We know from Proposition 3.11 above that there exist local coordinates (x, y) in which H has the form

$$\begin{cases} x_1 = a_1(y) x + a_2(y) x^2 + \ldots \\ y_1 = \mu y + x h(x, y) \end{cases}$$

The germs $a_i(\cdot)$ and $h(\cdot, \cdot)$ germs of holomorphic functions from $(\mathbb{C}, 0)$ to $\mathbb{C}$, respectively from $(\mathbb{C}^2, 0)$ to $\mathbb{C}$, with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

(1) Reduction to $a_1(y) = \lambda$. Consider as in [Ha] and [U] a coordinate transformation

$$\begin{cases} X = u(y) x \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X/u(Y) \\ y = Y \end{cases}$$

where u is a germ of analytic functions from $(\mathbb{C}, 0)$ to $\mathbb{C}$ with $u(0) = \lambda$. We need to find u such that

$$\begin{aligned} X_1 &= u(y_1)x_1 = u(\mu y + xh(x,y))\left(a_1(y)x + a_2(y)x^2 + \ldots\right) \\ &= u(\mu Y + X/u(Y)h(X/u(Y),Y))\left(a_1(Y)X/u(Y) + a_2(Y)(X/u(Y))^2 + \ldots\right) \\ &= \frac{u(\mu Y)a_1(Y)}{u(Y)}X + O(X^2) = \lambda X + O(X^2). \end{aligned}$$

Thus u satisfies the equation $u(Y) = u(\mu Y)\frac{a_1(Y)}{\lambda}$. We successively substitute μY instead of Y in this equation and obtain the unique solution

$$u(Y) = \prod_{n=0}^{\infty} \frac{a_1(\mu^n Y)}{\lambda}. \tag{3.6}$$

This series converges in a neighborhood of 0 since $\mu < 1$ and $a_1(0) = \lambda$.

(2) Reduction to $a_2(y), \ldots, a_m(y)$ constants. We proceed by induction on m. The base case $m = 1$ was discussed above. Suppose that $m \geq 2$ and that there exist local coordinates (x, y) in which H has the form

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \ldots + a_{m-1}x^{m-1} + a_m(y)x^m + \ldots \\ y_1 = \mu y + xh(x,y) \end{cases},$$

with $a_2, \ldots, a_{m-1}$ constant. We would like to find local coordinates so that $a_m(y)$ is also constant. Consider the transformation

$$\begin{cases} X = x + v(y)x^m \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - v(Y)X^m + \ldots \\ y = Y \end{cases}$$

where v is a germ of analytic functions from $(\mathbb{C}, 0)$ to $\mathbb{C}$ with $v(0) = 0$. Using the coordinates given by this transformation we get

$$\begin{aligned} X_1 &= x_1 + v(y_1)x_1^m \\ &= \lambda x + a_2x^2 + \ldots + a_{m-1}x^{m-1} + (a_m(y) + v(\mu y))\,x^m + O(x^{m+1}) \\ &= X - v(Y)X^m + a_2X^2 + \ldots + a_{m-1}X^{m-1} + (a_m(Y) + v(\mu Y))\,X^m + O(X^{m+1}) \\ &= X + a_2X^2 + \ldots + a_{m-1}X^{m-1} + (a_m(Y) + v(\mu Y) - v(Y))\,X^m + O(X^{m+1}). \end{aligned}$$

We need v such that the coefficient of X^m is constant, i.e. $a_m(Y) + v(\mu Y) - v(Y) = a_m(0)$ is constant. This gives the equation $v(Y) - v(\mu Y) = a_m(Y) - a_m(0)$. We successively substitute μY instead of Y in this equation and obtain

$$v(Y) = \sum_{n=0}^{\infty} (a_m(\mu^n Y) - a_m(0)) . \tag{3.7}$$

The series clearly converges in a neighborhood of 0 since $\mu < 1$. □

Proposition 3.13. *Let H be a semi-parabolic germ of transformation of $(\mathbb{C}^2, 0)$, with eigenvalues λ and μ, with $\lambda = e^{2\pi i p/q}$ and $|\mu| < 1$. There exist local coordinates (x, y) in which H has the form $H(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + Cx^{2\nu q+1} + a_{2\nu q+2}(y)x^{2\nu q+2} + \ldots) \\ y_1 = \mu y + xh(x, y) \end{cases} \tag{3.8}$$

and C a constant. Moreover the multiplicity of the fixed point is $\nu q + 1$.

Proof. Suppose that the map has the form from Equation 3.5, where m is big enough, and fixed

$$\begin{cases} x_1 = \lambda x + a_k x^k + \ldots + a_m x^m + a_{m+1}(y)x^{m+1} + \ldots \\ y_1 = \mu y + xh(x, y) \end{cases}$$

Consider the coordinate transformation

$$\begin{cases} X = x + bx^k \\ Y = y \end{cases} \quad \text{with inverse} \quad \begin{cases} x = X - bX^k + \ldots \\ y = Y \end{cases}$$

In the new coordinate system, we get

$$\begin{aligned} X_1 &= x_1 + bx_1^k = (\lambda x + a_k x^k + \ldots) + b(\lambda x + a_k x^k + \ldots)^k \\ &= \lambda x + a_k x^k + \ldots + b\lambda^k x^k + \ldots \\ &= \lambda x + (a_k + b\lambda^k)x^k + \ldots \\ &= \lambda(X - bX^k + \ldots) + (a_k + b\lambda^k)(X - bX^k + \ldots)^k + \ldots \\ &= \lambda X + (a_k + b(\lambda^k - \lambda))X^k + \ldots \end{aligned}$$

If k is not congruent to 1 modulo q (i.e. $\lambda^k \neq \lambda$), then we can set

$$b = \frac{a_k}{\lambda - \lambda^k}$$

and eliminate the term $a_k x^k$. This proves that by successive coordinate transformations of the form $X = x + bx^k$, $Y = y$ we can eliminate terms with powers that are not congruent to 1 modulo q, so the first term that cannot be eliminated in this way will have a power of the form $\nu q + 1$ for some ν.

Thus the map takes the form

$$\begin{cases} x_1 = \lambda(x + a_{\nu q+1} x^{\nu q+1} + \ldots + a_m x^m + a_{m+1}(y) x^{m+1} + \ldots) \\ y_1 = \mu y + x h(x, y) \end{cases} \tag{3.9}$$

Here we will assume that m was chosen so that $m > 2\nu q + 1$. Thus the coefficients up to order m are still constants. We can further reduce Equation 3.9 to $a_{\nu q+1} = 1$ by considering a transformation of the form $X = Ax$, $Y = y$, where A is a constant such that $A^{\nu q} = a_{\nu q+1}$. Consider therefore the transformation H written as

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + \ldots + a_m x^m + a_{m+1}(y) x^{m+1} + \ldots) \\ y_1 = \mu y + x h(x, y) \end{cases} \tag{3.10}$$

We have previously showed that we can eliminate any term of degree k between 1 and m, which is not congruent to 1 modulo q. One can also eliminate all terms of degree $jq + 1$, where $\nu < j < 2\nu$. Assume a_{jq+1} is the first such coefficient different from 0.

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + a_{jq+1} x^{jq+1} + \ldots) \\ y_1 = \mu y + x h(x, y) \end{cases} \tag{3.11}$$

Consider the transformation

$$\phi \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \quad \text{where} \quad \begin{cases} X = x + b x^{(j-\nu)q+1} \\ Y = y \end{cases} \quad \text{and} \quad b = \frac{a_{jq+1}}{(2\nu - j)q}.$$

Define $G := \phi \circ H \circ \phi^{-1}$ and suppose that $G(X, Y) = (X_1, Y_1)$ with

$$\begin{cases} X_1 = \lambda(X + X^{\nu q+1} + AX^k + \ldots) \\ Y_1 = \mu Y + Xh(X, Y) \end{cases} \tag{3.12}$$

and $k \leq jq$. We show that $A = 0$ by comparing the terms of the power series of $G \circ \phi$ and $\phi \circ H$. We will only need to analyze the x-coordinate. The first coordinate of $\phi \circ H$ is

$$\begin{aligned} X_1 &= x_1 + bx_1^{(j-\nu)q+1} \\ &= \lambda(x + x^{\nu q+1} + a_{jq+1}x^{jq+1} + \ldots) + b\lambda(x + x^{\nu q+1} + a_{jq+1}x^{jq+1} + \ldots)^{(j-\nu)q+1} \\ &= \lambda\left(x + bx^{(j-\nu)q+1} + x^{\nu q+1} + (a_{jq+1} + b((j-\nu)q+1))x^{jq+1} + O_y(x^{jq+2})\right) \end{aligned}$$

The first coordinate of $G \circ \phi$ is

$$\begin{aligned} X_1 &= \lambda(X + X^{\nu q+1} + AX^k + \ldots) \\ &= \lambda\left((x + bx^{(j-\nu)q+1}) + (x + bx^{(j-\nu)q+1})^{\nu q+1} + A(x + bx^{(j-\nu)q+1})^k + \ldots\right) \\ &= \lambda\left(x + bx^{(j-\nu)q+1} + x^{\nu q+1} + b(\nu q+1)x^{jq+1} + Ax^k + O_y(x^{k+1})\right) \end{aligned}$$

We have that $a_{jq+1} + b((j-\nu)q+1) = b(\nu q+1)$ by the choice for b. The two power series are equal so the coefficient of x^k vanishes, so $A = 0$. Thus in the first coordinate of $\phi \circ H \circ \phi^{-1}$ the coefficient of x^{jq+1} is zero and the coordinate transformation did not introduce additional terms of lower powers.

Using similar transformations we can eliminate all terms between $\nu q + 1$ and $2\nu q + 1$ and write $H(x, y) = (x_1, y_1)$ with

$$\begin{cases} x_1 = \lambda(x + x^{\nu q+1} + Cx^{2\nu q+1} + O_y(x^{2\nu q+2})) \\ y_1 = \mu y + xh(x, y) \end{cases}$$

for some constant C.

It is easy to prove that in the last coordinate system, $H^{\circ q}$ takes the form

$$\begin{cases} x_1 = x + qx^{\nu q+1} + \widetilde{C}x^{2\nu q+1} + O_y(x^{2\nu q+2}) \\ y_1 = \mu^q y + x\widetilde{h}(x,y) \end{cases} \tag{3.13}$$

The partial derivative $\frac{\partial y_1}{\partial y}(0,0) = \mu^q < 1$, hence by the Implicit Function Theorem, the equation $\mu^q y + xh(x,y) = y$ has a unique solution $y = \varphi(x)$ in a neighborhood of 0, where φ is a holomorphic function. From the first equation it then follows that $x = 0$ is a fixed point of $H^{\circ q}$ of multiplicity $\nu q + 1$. □

The normalizing form as proven in the previous theorem holds locally in a neighborhood of the semi-parabolic fixed point. The disadvantage of the "local" statement is that it does not allow us to control the size of the neighborhood of the fixed point where we can put on normalizing coordinates. However, in Section 3.6 we will show how to control the size of this neighborhood.

We will consider a class of semi-parabolic Hénon maps which are perturbations of polynomials with a parabolic fixed point, and show how to extend this theorem in order to get uniform bounds (with respect to the parameters) on the size of the normalizing neighborhood.

3.4 Attracting and repelling sectors

Set $m := \nu q$ and let

$$\Delta_R = \left\{ x \in \mathbb{C} \,\Big|\, \left(Re(x^m) + \frac{1}{2R}\right)^2 + \left(|Im(x^m)| - \frac{1}{2R}\right)^2 < \frac{1}{2R^2} \right\}.$$

There are m connected components of Δ_R, which we denote $\Delta_{R,j}$, for $1 \leq j \leq m$.

Define

$$\mathcal{P}_{att} = \left\{(x,y) \in \mathbb{C}^2 \mid x \in \Delta_R, |y| < r\right\}$$

and let

$$\mathcal{P}_{att,j} = \left\{(x,y) \in \mathbb{C}^2 \mid x \in \Delta_{R,j}, |y| < r\right\}$$

be the connected components of $\mathcal{P}_{att}$. These are called *(big) attractive petals* for the Hénon map, similar to the one-dimensional case.

Proposition 3.14. *For R large enough and r small enough*

$$H(\overline{\mathcal{P}_{att,j}}) \subset \mathcal{P}_{att,j+\nu p} \cup \{0\} \times \mathbb{D}_r \quad \text{for } 1 \leq j \leq \nu q.$$

In particular $H(\overline{\mathcal{P}_{att}}) \subset \mathcal{P}_{att} \cup \{0\} \times \mathbb{D}_r$ *and all points of* $\mathcal{P}_{att}$ *are attracted to the origin under iterations by H.*

Proof. The proof is similar to [Ha] but we will need to introduce a formalism similar to the one-dimensional as in [BH] and [DH] to resolve the ambiguity about which branch of $x^{1/m}$ we are talking about.

Assume that R is large enough and r is small enough so that H is well defined and has the expansion from Proposition 3.13. Define the region U_{R_1}

$$U_{R_1} := \{X \in \mathbb{C} \mid R_1 - Re(X) < |Im(X)|\}$$

where $R_1 = R/m$ and set $W_{R_1,r} := U_{R_1} \times \mathbb{D}_r \subset \mathbb{C}^2$.

Consider the Hénon map H written as

$$\begin{cases} x_1 = \lambda(x + x^{m+1} + Cx^{2m+1} + a_{2m+2}(y)x^{2m+2} + \ldots) \\ y_1 = \mu y + xh(x,y) \end{cases}.$$

Suppose $(x,y) \in \mathcal{P}_{att,j}$ and consider the transformation

$$\begin{cases} X = -\dfrac{1}{mx^m} \\ Y = y \end{cases}.$$

It maps each $\mathcal{P}_{att,j}$ to $W_{R_1,r}$ (it maps points $(0,y)$ to (∞,y)). Let $\widehat{H}(X,Y) = (X_1,Y_1)$ be the map in these coordinates

$$\begin{aligned}
X_1 &= -\frac{1}{mx_1^m} = -\frac{1}{m\left(\lambda(x + x^{m+1} + Cx^{2m+1} + a_{2m+2}(y)x^{2m+2} + \dots)\right)^m} \\
&= \frac{X}{(1 + x^m + Cx^{2m} + a_{2m+2}(y)x^{2m+1} + \dots)^m} \\
&= X\left(1 - m(x^m + Cx^{2m} + \dots) + \frac{m(m+1)}{2}x^{2m} + \dots\right) \\
&= X + 1 + \frac{A}{X} + O_Y\left(\frac{1}{|X|^{1+1/m}}\right)
\end{aligned}$$

where $A := \frac{1}{m}\left(\frac{m+1}{2} - C\right)$ is a constant. The notation $O_Y\left(\frac{1}{|X|^\alpha}\right)$ represents a holomorphic function of (X,Y) in $W_{R_1,r}$ which is bounded by $\frac{K}{|X|^\alpha}$ for some constant K. Similarly

$$Y_1 = \mu y + xh(x,y) = \mu Y + O_Y\left(\frac{1}{|X|^{1/m}}\right).$$

Note that $|X| > \frac{R_1}{\sqrt{2}}$ for all $X \in U_{R_1}$. There exists constants K' and K'' such that

$$\begin{aligned}
|X_1 - X - 1| &\leq \frac{K'}{|X|} < \frac{K_1}{R_1} \quad \text{where } K_1 := K'\sqrt{2} \\
|Y_1 - \mu Y| &\leq \frac{K''}{|X|^{1/m}} < \frac{K_2}{R_1^{1/m}} \quad \text{where } K_2 := K''\sqrt{2}^{1/m}.
\end{aligned}$$

Choose R_1 large enough and r small enough so that

$$\begin{cases} \frac{K_1}{R_1} < \frac{1}{2} \\ \frac{K_2}{R_1^{1/m}} < (1 - |\mu|)r\ . \end{cases} \tag{3.14}$$

The first condition gives $|X_1 - X - 1| < \frac{1}{2}$, which implies $Re(X_1) > Re(X) + \frac{1}{2}$ and $|Im(X_1)| > |Im(X)| - \frac{1}{2}$. Thus $R_1 - Re(X_1) < |Im(X_1)|$. The second condition gives

$$|Y_1| \leq |Y_1 - \mu Y| + |\mu||Y| < \frac{K_2}{R_1^{1/m}} + |\mu|r < r.$$

Hence $\widehat{H}(W_{R_1,r}) \subset W_{R_1,r}$.

We need to show that points in $W_{R_1,r}$ are attracted by $(\infty, 0)$ under iterations by $\widehat{H}$. Let $(X,Y) \in W_{R_1,r}$ and set $(X_n, Y_n) = \widehat{H}^{\circ n}(X,Y)$. Assume without loss of generality that $Re(X) > \rho$, where $\rho > 0$ is a constant to be defined later. We can make this assumption since $Re(X_k) > Re(X) + \frac{k}{2}$ for every positive integer k. We take the first integer k_0 such that $Re(X_{k_0}) > \rho$ and let $X := X_{k_0}$ and $Y := Y_{k_0}$.

Clearly

$$Re(X_n) > \rho + \frac{n}{2} \tag{3.15}$$

for every $n \geq 0$. This follows immediately by induction since

$$Re(X_{n+1}) > Re(X_n) + 1/2 > \rho + (n+1)/2.$$

We now show by induction that

$$|Y_n| < 2NrR_1^{1/m}\left(\frac{1}{\rho + \frac{n}{2}}\right)^{1/m}, \quad n \geq 0$$

where N is an integer number such that $NR_1^{1/m} > \rho^{1/m}$. When $n = 0$, $|Y| < r$ and

$$r < 2NrR_1^{1/m}\frac{1}{\rho^{1/m}} \Leftrightarrow \rho^{1/m} < 2NR_1^{1/m}.$$

We now proceed to the induction step. First note that $|X_n| \geq Re(X_n) > \rho + \frac{n}{2}$ and $K'' < K_2$. We get

$$\begin{aligned} |Y_{n+1}| &\leq |Y_{n+1} - \mu Y_n| + |\mu||Y_n| < \frac{K''}{|X_n|^{1/m}} + |\mu||Y_n| \\ &< (K_2 + |\mu| 2NrR_1^{1/m})\left(\frac{1}{\rho + \frac{n}{2}}\right)^{1/m} < (1 + (2N-1)|\mu|)\, rR_1^{1/m}\left(\frac{1}{\rho + \frac{n}{2}}\right)^{1/m} \end{aligned}$$

and we want to show that

$$|Y_{n+1}| < 2NrR_1^{1/m}\left(\frac{1}{\rho + \frac{n+1}{2}}\right)^{1/m}.$$

This inequality is satisfied if

$$\left(\frac{\rho + \frac{1}{2}}{\rho}\right)^{1/m} = \left(1 + \frac{1}{2\rho}\right)^{1/m} < \frac{2}{1 + |\mu|} \leq \frac{2N}{1 + (2N-1)|\mu|}. \tag{3.16}$$

But $|\mu| < 1$, so $2/(1+|\mu|) > 1$. This allows us to choose a number ρ large enough so that Equation 3.16 is satisfied. Then choose an integer N such that $NR_1^{1/m} > \rho^{1/m}$.

It follows that $(X_n, Y_n) \to (\infty, 0)$ as $n \to \infty$. □

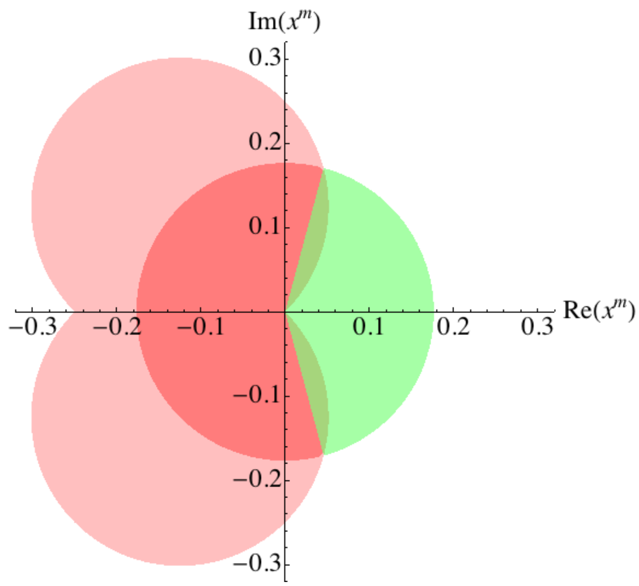

Figure 3.3: The image of $\mathcal{P}_{att}$ under the map $x \mapsto x^m$ at height $y = 0$ is shown in light red. Similarly the attracting sector W^+ is shown in red and the repelling sector W^- in green. The angle opening of the green region is less than π; it is $5\pi/6$.

Let $\epsilon_0 = (\sqrt{3} - 1)/(\sqrt{3} + 1)$. Define *attractive sectors*

$$W^+ := \left\{ x \in \mathbb{C} \mid Re(x^m) \leq \epsilon_0 |Im(x^m)| \text{ and } |x^m| < \frac{1}{\sqrt{2}R} \right\} \times \mathbb{D}_r \subset \mathcal{P}_{att} \qquad (3.17)$$

and *repelling sectors*

$$W^- := \left\{ x \in \mathbb{C} \mid Re(x^m) > \epsilon_0 |Im(x^m)| \text{ and } |x^m| < \frac{1}{\sqrt{2}R} \right\} \times \mathbb{D}_r \qquad (3.18)$$

We will call W^- repelling because as we will see, the Hénon map expands horizontally when the Jacobian is small enough.

There are m components of $W^\pm$ which we denote $W_j^\pm$ for $1 \leq j \leq m$. These are the preimages of the red/green regions in Figure 3.3 under $x \mapsto x^m$. The choice of ϵ_0 means that the angle of the image of W^- under $x \mapsto x^m$ is strictly less than π; in this case it is actually $5\pi/6$.

Furthermore, on W^- we have

$$Re(x^m) > \epsilon_1 |x^m|, \quad \text{where } \epsilon_1 := \frac{\epsilon_0}{1-\epsilon_0^2} > \frac{1}{4}. \tag{3.19}$$

Proposition 3.15. *The basin of attraction of the semi-parabolic fixed point* 0 *is*

$$\bigcup_{n\geq 0} H^{-n}(\mathcal{P}_{att}) = \bigcup_{n\geq 0} H^{-n}(W^+).$$

Proof. The proof follows directly by analyzing the situation at infinity using formula 3.15. □

3.5 Degeneracy of the parametrizing map

This is a self-contained section where we study the degeneracy of the parametrization of the stable manifold of the semi-parabolic fixed point $\mathbf{q}_a$ as $a \to 0$. Consider the Hénon map $H : \mathbb{C}^2 \to \mathbb{C}^2$ written as

$$H\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) + ay \\ ax \end{pmatrix},$$

where $p(x) = x^2 + c$. For semi-parabolic Hénon maps, p should be understood as $p_a(x) = x^2 + c(a)$, but we use p in this section, to simplify notation. When $a \neq 0$, the inverse Hénon map is

$$H^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{a}\begin{pmatrix} y \\ x - p(y/a) \end{pmatrix}.$$

Fix $\lambda = e^{2\pi i p/q}$. Suppose H has a semi-parabolic fixed point at $\mathbf{q}_a$ such that $DH(\mathbf{q}_a)$ has eigenvalues λ and μ, with $|\mu| < 1$. We have $\lambda\mu = -a^2$ so $\mu = -\frac{a^2}{\lambda}$ and $|\mu| = |a|^2$. Set for simplicity

$$q_a := \frac{\lambda}{2} - \frac{a^2}{2\lambda} = \frac{\lambda + \mu}{2}.$$

With this notation, the equation of the fixed point $\mathbf{q}_a$ reduces to

$$\mathbf{q}_a := \begin{pmatrix} q_a \\ a q_a \end{pmatrix} = \frac{\lambda + \mu}{2} \begin{pmatrix} 1 \\ a \end{pmatrix}.$$

Let $v = \begin{pmatrix} -a/\lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \mu/a \\ 1 \end{pmatrix}$ be an eigenvector for the eigenvalue μ. The semi-parabolic fixed point $\mathbf{q}_a$ has a stable manifold $W^s(\mathbf{q}_a) \subset \mathbb{C}^2$,

$$W^s(\mathbf{q}_a) := \{\mathbf{p} \in \mathbb{C}^2 \mid \|H^m(\mathbf{p}) - \mathbf{q}_a\| < C|\mu|^m \text{ for } m \geq 0\},$$

where $C > 0$ is a constant. This is the set of points for which $H^m(\mathbf{p}) \to \mathbf{q}_a$ exponentially as $m \to \infty$. The stable manifold is biholomorphic to $\mathbb{C}$ and has a natural parametrization given by the following proposition.

Proposition 3.16. *The stable manifold $W^s(\mathbf{q}_a)$ has a parametrization $F_a : \mathbb{C} \to W^s(\mathbf{q}_a)$ given by*

$$F_a(z) = \lim_{m\to\infty} H^{-m}(\mathbf{q}_a + \mu^m v z). \tag{3.20}$$

F_a is an injective immersion of $\mathbb{C}$ onto $W^s(\mathbf{q}_a)$ with the property that $F_a(\mu z) = H(F_a(z))$.

Proof. The proof is similar to the proof of Theorem 1 from [H1]. Consider the inverse map H^{-1} instead of H. Then $\mathbf{q}_a$ is a fixed point of H^{-1} and $DH^{-1}(\mathbf{q}_a)$ has eigenvalues $\overline{\lambda}$ and $\mu' = 1/\mu$, where $|\mu'| > 1$. The fixed point $\mathbf{q}_a$ has now an unstable manifold $W^u(\mathbf{q}_a)$ which has a natural parametrization given by F_a as shown in [H1]. □

Proposition 3.17. *The parametrizing function $F_a \to F_0$ as $a \to 0$, where*

$$F_0(z) := \mathbf{q}_0 + \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} \lambda/2 \\ z \end{pmatrix}.$$

Proof. Define a sequence of points

$$\begin{pmatrix} x_i \\ y_i \end{pmatrix} = H^{-i} \begin{pmatrix} x_{i-1} \\ y_{i-1} \end{pmatrix} \quad \text{for } 1 \leq i \leq m,$$

where

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \mathbf{q}_a + \mu^m v z = \begin{pmatrix} q_a + \frac{\mu^{m+1}}{a} z \\ a q_a + \mu^m z \end{pmatrix}.$$

At the first step we have

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = H^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{a} \begin{pmatrix} y_0 \\ x_0 - p(y_0/a) \end{pmatrix},$$

so

$$x_1 = \frac{y_0}{a} = q_a + \frac{\mu^m}{a} z.$$

From $H(\mathbf{q}_a) = \mathbf{q}_a$ we get $p(q_a) + a^2 q_a = q_a$ so $q_a - p(q_a) = a^2 q_a$. Moreover $DH(\mathbf{q}_a)$ has eigenvalues λ and μ so $\lambda + \mu = tr(DH(\mathbf{q}_a))$, which gives $p'(q_a) = \lambda + \mu$. Recall that $\mu = -\frac{a^2}{\lambda}$. We can write

$$p(y_0/a) = p\left(q_a + \frac{\mu^m}{a} z\right) = p(q_a) + p'(q_a)\frac{\mu^m}{a} z + O(\mu^{2m-1})$$

and get

$$\begin{aligned} y_1 = \frac{x_0 - p(y_0/a)}{a} &= \frac{q_a - p(q_a)}{a} + \frac{\mu^{m+1} z - p'(q_a)\mu^m z}{a^2} + O(a\mu^{2m-2}) \\ &= a q_a + \mu^{m-1} z + O(a\mu^{2m-2}). \end{aligned}$$

By induction we can show that for $1 \leq i \leq m$ we have

$$\begin{aligned} x_i &= q_a + \frac{\mu^{m-(i-1)}}{a} z + O(\mu^{2(m-(i-1))}) \\ y_i &= a q_a + \mu^{m-i} z + O(a\mu^{2(m-i)}). \end{aligned}$$

For $i = m$ this reduces to

$$\begin{aligned} x_m &= q_a - a\overline{\lambda}z + O(a^4) \\ y_m &= aq_a + z + O(a). \end{aligned}$$

Thus $x_m \to q_0$ and $y_m \to z$ as $a \to 0$. Therefore $F_0(z) = \mathbf{q}_0 + \begin{pmatrix} 0 \\ z \end{pmatrix}$. Therefore $F_a \to F_0$ and the convergence is uniformly on compacts. $\square$

3.6 Choosing uniform normalizing coordinates

Fix $\lambda = e^{2\pi i p/q}$. We will first look at the normal form for the polynomial

$$p(x) = x^2 + \left(\frac{\lambda}{2} - \frac{\lambda^2}{4}\right)$$

which has a parabolic fixed point $q_0 = \frac{\lambda}{2}$, of multiplier λ.

Lemma 3.18. *There exists a neighborhood V_0 of q_0 and an isomorphism $\phi : V_0 \to \mathbb{D}_\rho$ such that $\widetilde{p}(x) = \phi \circ p \circ \phi^{-1}(x)$ where*

$$\widetilde{p}(x) = \lambda x \left(1 + x^q + Cx^{2q} + O(x^{2q+1})\right).$$

Furthermore, there exists R large enough such that in the region

$$\Delta^- = \left\{ x \in \mathbb{C} \mid Re(x^q) > \epsilon_0 |Im(x^q)| \text{ and } |x^q| < \frac{1}{\sqrt{2}R} \right\}$$

the map $\widetilde{p}$ satisfies $|\widetilde{p}\,'(x)| > 1$. The compact region

$$\Delta^+ = \left\{ x \in \mathbb{C} \mid Re(x^q) \leq \epsilon_0 |Im(x^q)| \text{ and } |x^q| < \frac{1}{\sqrt{2}R} \right\}$$

satisfies $p(\Delta^+) \subset \overset{\circ}{K}_p \cup \{0\}$.

Sketch of Proof. The proof is essentially the same as in [DH]. After a global coordinate change that brings the parabolic fixed point at the origin, we can also write the polynomial as $p(x) = \lambda x + x^2$. Since p is a polynomial of degree 2, the fixed point can only have parabolic multiplicity 1. Hence its multiplicity as a solution of the equation $p^{\circ q}(z) - z = 0$ is $q + 1$. The proof then uses the same coordinate transformations as in Theorem 3.13, in order to eliminate the terms of degree less than $2q + 1$ which are not congruent to 1 mod q. We have

$$\widetilde{p}\,'(x) = \lambda(1 + (q+1)x^q + O(x^{2q})).$$

Since $|\lambda| = 1$ and $Re(x^q) > \epsilon_1 |x|^q$ from Equation 3.19, this gives

$$\begin{aligned} |\widetilde{p}\,'(x)| &= \left|1 + (q+1)x^q + O(x^{2q})\right| \geq \left|1 + (q+1)x^q\right| - m|x|^{2q} \\ &\geq 1 + (q+1)\epsilon_1 |x|^q - m|x|^{2q} > 1 + |x|^q \frac{\epsilon_1}{2}. \end{aligned}$$

for x sufficiently small. The constant m depends only on the polynomial and is chosen so that $\left|\widetilde{p}\,'(x) - \lambda(1 + (q+1)x^q)\right| < m|x|^{2q}$ on $\mathbb{D}_\rho$. It follows that $|\widetilde{p}\,'(x)| > 1$ for $x \in \Delta^-$ if $0 < |x|$ is sufficiently small. □

We have used ρ to measure the size in x rather than x^q. Formally, $\rho^q = \frac{1}{\sqrt{2}R}$ in the definition of the set $\Delta^\pm$. Choose $\rho' > 0$ such that the disk $\mathbb{D}_{2\rho'}(q_0)$ of radius $2\rho'$ centered at q_0 is contained in V_0.

We will study the normal form for the Hénon maps which are small perturbations of the parabolic polynomial $p(x)$ inside the parabola $\mathcal{P}_\lambda$. Since $\mathcal{P}_\lambda$ is parametrized by a, we will simplify our notations and write the Hénon map as H_a, with

$$H_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + c + ax \\ ay \end{pmatrix}.$$

and assume that c is chosen as in Equation 3.2, so that $(c, a) \in \mathcal{P}_\lambda$.

Theorem 3.19. *Let $r > 2$ be a fixed constant. There exists $\delta > 0$ such that for any $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ we can find a coordinate transformation ϕ_a from a tubular neighborhood $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ of the local stable manifold of the semi-parabolic fixed point $\mathbf{q}_a$*

$$\phi_a : B \to \mathbb{D}_\rho \times \mathbb{D}_{r+O(|a|)}$$

in which H_a has the form $H_a(x, y) = (x_1, y_1)$, with

$$\begin{cases} x_1 = \lambda(x + x^{q+1} + Cx^{2q+1} + a_{2q+2}(y)x^{2q+2} + \ldots) \\ y_1 = \mu y + xh(x, y) \end{cases} \tag{3.21}$$

and C is a constant (depending on a) and $xh(x, y) = O(a)$. Moreover the transformations ϕ_a are holomorphic in a, and

$$\lim_{a\to 0} \phi_a = \phi_0(x, y) = (\phi(x), y),$$

where $\phi : \mathbb{D}_{\rho'} \to \mathbb{D}_\rho$ is the change of coordinates for the polynomial $p_0(x) = x^2 + c_0$ with a parabolic fixed point at q_0,

$$\phi \circ p_0 \circ \phi^{-1}(x) = \lambda x(1 + x^q + Cx^{2q} + O(x^{2q+1})).$$

Proof. We will follow the same steps as in Section 3.3.

The degenerate map $H_0(x, y) = (p(x), 0)$ has a semi-parabolic fixed point at $\mathbf{q}_0 = (\frac{\lambda}{2}, 0)$ of multiplicity $q + 1$ and the stable manifold $W^s(\mathbf{q}_0)$ is just a vertical line passing through $\mathbf{q}_0$. The multiplicity of the semi-parabolic fixed point is constant in a neighborhood of $a = 0$ in $\mathcal{P}_\lambda$. When $a \neq 0$, $W^s(\mathbf{q}_a)$ is an analytic submanifold biholomorhic to $\mathbb{C}$. By [H1], $W^s(\mathbf{q}_a)$ depends analytically on a in a neighborhood of $a = 0$ inside $\mathcal{P}_\lambda$, since the fixed point $\mathbf{q}_a$ does not bifurcate.

Definition 3.20. Denote by S_r the horizontal strip $S_r := \{(x, y) \in \mathbb{C}^2 \mid |y| < r\}$ and by $W^s_{r,a}(\mathbf{q}_a)$ the connected component of $W^s(\mathbf{q}_a) \cap S_r$ that contains the fixed point $\mathbf{q}_a$.

Let us choose $\delta > 0$, such that for all $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ the Hénon map $H_{c,a}$ has a fixed point $\mathbf{q}_a$ of multiplicity $q + 1$ and such that the local stable manifold $W^s_{r,a}(\mathbf{q}_a)$ is "vertical-like". Rigorously, we require that the horizontal distance between $W^s_{r,a}(\mathbf{q}_a)$ and the vertical line that contains $\mathbf{q}_a$ is less than $\frac{\rho'}{4}$, and that $W^s_{r,a}(\mathbf{q}_a)$ has no horizontal foldings.

The parametrization function $F_a : \mathbb{C} \to W^s(\mathbf{q}_a)$ is analytic in the parameter a, and by Proposition 3.17, when $a = 0$, it becomes a translation in the horizontal direction

$$F_0(y) = \mathbf{q}_0 + (0, y).$$

F_0 maps the disk $\{y \in \mathbb{C} \mid |y| < r\}$ onto $\{(\frac{\lambda}{2}, y) \in \mathbb{C}^2 \mid |y| < r\} \subset W^s(\mathbf{q}_0)$. Also by Proposition 3.17

$$F_a(y) = F_0(y) + O(a),$$

so F_a will map the disk $\{y \in \mathbb{C} \mid |y| < r\}$ onto a holomorphic disk inside $W^s(\mathbf{q}_a)$ around $\mathbf{q}_a$, of size approximately $r + O(a)$. For a small, fix therefore $2 < r' < r$ such that $W^s_{r',a}(\mathbf{q}_a) \subset F_a(S_r)$. In principle $r' = r + O(|a|)$, but since the vertical size is not a delicate issue, we can think of r' as simply being r.

Proposition 3.21. *Choose $\delta > 0$ as before. For all $(c, a) \in \mathcal{P}_\lambda$ with $|a| < \delta$ there exists a coordinate transformation $\phi^1_a : S_{r'} \to S_r$, such that in the new coordinates, the Hénon map H_a has the form $H_a(x, y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = a_1(y)x + a_2(y)x^2 + \dots \\ y_1 = \mu y + xh(x, y) \end{cases}, \tag{3.22}$$

where $a_j(\cdot)$ and $h(\cdot, \cdot)$ are holomorphic functions from $\{y \in \mathbb{C}, |y| < r'\}$ to $\mathbb{C}$, respectively from $\{(x, y) \in \mathbb{C}^2, |y| < r'\}$ to $\mathbb{C}$, with $a_1(0) = \lambda$ and $h(0, 0) = 0$.

Proof. Suppose $F_a(y) = (f(y), g(y))$, and let $\psi_a : S_r \to \mathbb{C}^2$ be the map

$$\psi_a(x, y) = (x + f(y), g(y)).$$

It is easy to see that ψ_a is an invertible function. The Jacobian matrix is given by

$$D\psi_a|_{(x,y)} = \begin{pmatrix} 1 & f'(y) \\ 0 & g'(y) \end{pmatrix}.$$

$W^s_{r,a}(\mathbf{q}_a)$ is vertical-like, in particular it has no horizontal foldings, hence $g'(y) \neq 0$ for $|y| < r$. This means that ψ_a is invertible in the strip S_r. Define $\phi^1_a(x,y) := \psi_a^{-1}(x,y)$.

The fact that $\phi^1_a(x,y)$ is holomorphic in a follows immediately, since we know that $F_a(y)$ depends holomorphically on a. From Proposition 3.17 we obtain that

$$\phi^1_a(x,y) = \phi^1_0(x,y) + O(a).$$

The transformation ϕ^1_0 is straightforward to compute

$$\phi^1_0(x,y) = (x,y) - \mathbf{q}_0 = \left(x - \frac{\lambda}{2}, y\right).$$

In the new coordinate system the Hénon map H_0 becomes $H_0(x,y) = (x_1, y_1)$, where

$$\begin{cases} x_1 = \lambda x + x^2 \\ y_1 = 0 \end{cases}$$

Therefore when $a \neq 0$ it is easy to control the size of the coefficients in Equation 3.22 in terms of a

$$a_1(y) = \lambda + O(a),\ a_2(y) = 1 + O(a),\ a_i(y) = O(a) \text{ for } i > 2 \text{ and } h(x,y) = O(a). \qquad \square$$

Proposition 3.22. *Choose $\delta > 0$ as before, and let $(c,a) \in \mathcal{P}_\lambda$ with $|a| < \delta$. There exists a coordinate transformation $\phi^2_a : S_r \to S_r$ in which H has the form $H(x,y) = (x_1, y_1)$, with*

$$\begin{cases} x_1 = \lambda x + a_2 x^2 + \ldots + a_{2q+1} x^{2q+1} + a_{2q+2}(y) x^{2q+2} + \ldots \\ y_1 = \mu y + x h(x,y) \end{cases} \tag{3.23}$$

where a_2 is close to 1 and $a_3, \ldots, a_{2q+1}$ are constants close to 0.

Proof. Suppose H_a is already written in the form Equation 3.22. The proof is then the same as in Theorem 3.12 with $m = 2q + 1$. Notice that $a_1(y) = \lambda + O(a)$ for $|y| < r$, so one can therefore perform the same change of coordinates

$$T_1 : (x, y) \to (u(y)x, y), \text{ where } u(y) = \prod_{n \geq 0} a_1(\mu^n y)$$

in order to set $a_1(y) = \lambda$. Since $a_1(y)$ is close to λ on $|y| < r$, it follows that the product converges on $|y| < r$ and $u(y) \neq 0$. Hence $T_1(x, y)$ is invertible.

The coordinate changes that make $a_j(y)$ constant for $2 \leq j \leq 2q + 2$ are of the form

$$T_j : (x, y) \to (x + v(y)x^j, y), \text{ where } v(y) = \sum_{n \geq 0} a_j(\mu^n y) - a_j(0).$$

Clearly the sum is convergent on $|y| < r$ and $v(y) = O(a)$. The transformation T_j is invertible because x is bounded (1/2 would be a reasonable bound for x), so for a small $1 + v(y) j x^{j-1}$ does not cancel out.

The coordinate changes that are done in order to make the first $2q + 1$ coefficients constants are identity on the second coordinate. Denote by $\phi_a^2(x, y)$ their composition. Notice also that in Equation 3.22, $H_0(x, y) = (\lambda x + x^2, 0)$ already has constant coefficients, so ϕ_0^2 is just the identity map. It is easy to check that

$$\phi_a^2(x, y) = (x + O(a), y)$$

and also

$$a_2 = 1 + O(a), a_i = O(a) \text{ for } 2 < i \leq 2q + 1, \; a_i(y) = O(a) \text{ for } i > 2q + 1$$

$$h(x, y) = O(a).$$

□

We are now able to finish the proof of Theorem 3.19. Assume that H_a is written in the form 3.23. We use the same transformations as in Theorem 3.13

in order to eliminate the terms x^i, where $1 < i < q+1$ and $q+1 < i < 2q+1$. Let $\phi_a^3 : \mathbb{D}_{2\rho' - O(|a|)} \times \mathbb{D}_r \to \mathbb{D}_\rho \times \mathbb{D}_r$ denote the coordinate change. When $a = 0$, $H_0(x,y) = (\lambda x + x^2, 0)$, so

$$\phi_0^3(x,y) = (\phi(x), y),$$

where $\phi(x)$ is the coordinate transformation used in Lemma 3.18 to put $p(x) = \lambda x + x^2$ in the normal form $p(x) = \lambda(x + x^{q+1} + Cx^{2q+1} + O(x^{2q+2}))$.

Define $\phi_a(x,y) = \phi_a^3 \circ \phi_a^2 \circ \phi_a^1(x,y)$. Recall that ϕ_0^1 is a horizontal translation by $\frac{\lambda}{2}$ and ϕ_0^2 is the identity map, so when $a = 0$ the composition of the three transformations yields exactly the coordinate transformation used in Lemma 3.18 to put $p(x) = x^2 + \frac{\lambda}{2} - \frac{\lambda^2}{4}$ in the normal form $p(x) = \lambda(x + x^{q+1} + Cx^{2q+1} + O(x^{2q+2}))$.

□ **of Theorem 3.19**

Consider $(c,a) \in \mathcal{P}_\lambda$ with $|a| < \delta$. Then the Hénon map H has a semi-parabolic fixed point $\mathbf{q}_a$ of multiplicity $q+1$. The derivative $DH(\mathbf{q}_a)$ has eigenvalues λ and μ, with $|\mu| < 1$.

Lemma 3.23 (Attractive/Repelling sectors). *Suppose $W^\pm$ are defined as in 3.18 and 3.17. In the region*

$$W^- = \{|x| \leq \rho,\ Re(x^q) > \epsilon_0 |Im(x^q)|\} \times \mathbb{D}_r$$

the derivative DH expands horizontally. The compact region

$$W^+ = \{|x| \leq \rho,\ Re(x^q) \leq \epsilon_0 |Im(x^q)|\} \times \mathbb{D}_r$$

satisfies $H(W^+) \subset int(K^+) \cup \{0\} \times \mathbb{D}_r$.

Proof. Consider the Hénon map written in normal coordinates $\widetilde{H} : \mathbb{D}_\rho \times \mathbb{D}_r \to \mathbb{C}^2$

$$\widetilde{H}_a \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda(x + x^{q+1} + g_a(x,y)) \\ \mu y + x h_a(x,y) \end{pmatrix},$$

where

$$g_a(x,y) = C_a x^{2q+1} + a_{2q+2}(y)x^{2q+2} + \dots$$
$$h_a(x,y) = b_1(y) + \dots + b_k(y)x^k + \dots$$

and $g_a(x,y) = g_0(x) + O(a)$ and $h_a(x,y) = O(a)$.

When $a = 0$, $\widetilde{H}_0(x,y) = (\widetilde{p}(x), 0)$, where

$$\widetilde{p}(x) = \lambda(x + x^{q+1} + g_0(x)) \text{ and } g_0(x) = C_0 x^{2q+1} + a_{2q+1}x^{2q+2} + \dots$$

Let $g_0'(x) = x^{2q}t_0(x)$ and denote by m the supremum of $|t_0(x)|$ on the set Δ^-, where

$$\Delta^- := \{|x| \le \rho,\ \mathrm{Re}(x^q) > \epsilon_0 |Im(x^q)|\}.$$

By eventually reducing $\rho > 0$, we can assume that

$$|1 + (q+1)x^q| - 2m|x^{2q}| > 1 + \frac{\epsilon_1}{2}|x|^q > 1, \text{ for all } x \in \Delta^-, \tag{3.24}$$

where ϵ_1 is given in Equation 3.19. When x is chosen from the repelling sectors Δ^- of the polynomial $\widetilde{p}$ by Lemma 3.18 we have

$$|\widetilde{p}\,'(x)| = |1 + (q+1)x^q + g_0'(x)| > |1 + (q+1)x^q| - m|x^{2q}| > 1$$

hence $\widetilde{p}\,'$ is expanding.

Let $\frac{\partial g_a}{\partial x}(x,y) = x^{2q}t_a(x,y)$. By choosing $|a| < \delta$ small enough, we can assume that

$$\sup_{(x,y)\in W^-} |t_a(x,y)| < 2m.$$

Hence for any (x,y) taken from the repelling sectors $W^- = \Delta^- \times \mathbb{D}_r$ of the Hénon map we have

$$\left|\frac{\partial g_a}{\partial x}(x,y)\right| < 2m|x|^{2q}.$$

When $a = 0$ we also know that $xh_0(x, y) \equiv 0$. Moreover by the construction of the normalizing coordinates we have $xh_a(x, y) = O(a)$. There exists a constant N_a, depending on a, with $0 < N_a < 1$ such that when $|a| < \delta$ the following bounds hold

$$\left|\frac{\partial xh_a}{\partial x}(x, y)\right| < N_a \text{ and } \left|\frac{\partial xh_a}{\partial y}(x, y)\right| < N_a. \tag{3.25}$$

Let (x, y) be a point in the repelling sectors of the Hénon map. Pick $\begin{pmatrix} \zeta \\ 0 \end{pmatrix}$ a horizontal tangent vector at (x, y), and let $D\widetilde{H}_{(x,y)} \begin{pmatrix} \zeta \\ 0 \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \eta_1 \end{pmatrix}$. The derivative is given below

$$D\widetilde{H}_{(x,y)} = \begin{pmatrix} \lambda(1 + (q+1)x^q + \frac{\partial g_a}{\partial x}(x, y)) & \lambda \frac{\partial g_a}{\partial y}(x, y) \\ h_a(x, y) + x\frac{\partial h_a}{\partial x}(x, y) & \mu + x\frac{\partial h_a}{\partial y}(x, y) \end{pmatrix}.$$

Consider now the Euclidean metric on the set $\mathbb{D}_\rho \times \mathbb{D}_r$ and estimate

$$\begin{aligned} |\eta_1| = \left|\frac{\partial xh_a}{\partial x}(x, y)\right| |\zeta| &\leq N_a |\zeta| \\ |\zeta_1| = \left|1 + (q+1)x^q + \frac{\partial g_a}{\partial x}(x, y)\right| |\zeta| &\geq \left(|1 + (q+1)x^q| - \left|\frac{\partial g_a}{\partial x}(x, y)\right|\right) |\zeta| \\ &\geq \left(|1 + (q+1)x^q| - 2m|x|^{2q}\right) |\zeta| \end{aligned}$$

Let $C(x) := \left(|1 + (q+1)x^q| - 2m|x|^{2q}\right)$. We have $C(x) > 1 + \frac{\epsilon_1}{2}|x|^q > 1$ when $x \neq 0$ and $C(x) = 1$ when $x = 0$. In other words, $C(x) = 1$ precisely when the point (x, y) belongs to the stable manifold of the semi-parabolic fixed point $(0, 0)$. When (x, y) is not on the stable manifold, we have

$$|\zeta_1| \geq C(x)|\zeta|, \quad \text{with } C(x) > 1. \tag{3.26}$$

We say that the derivative of the normalized Hénon map $D\widetilde{H}_{(x,y)}$ is expanding in the horizontal direction. □

Remark 3.24. Since $|\eta_1| < N_a|\zeta|$ and $|\zeta_1| \geq C(x)|\zeta| > |\zeta|$ as in Equation 3.26, it also follows that $|\eta_1| < N_a|\zeta_1|$.

Remark 3.25. If the multiplicity of the fixed point is $q + 1$, then there are exactly q connected components of W^- and q components of $int(W^+)$. Thus the parabolic multiplicity ν is equal to 1 for semi-parabolic Hénon maps with small enough Jacobian (assume $|a| < \delta$).

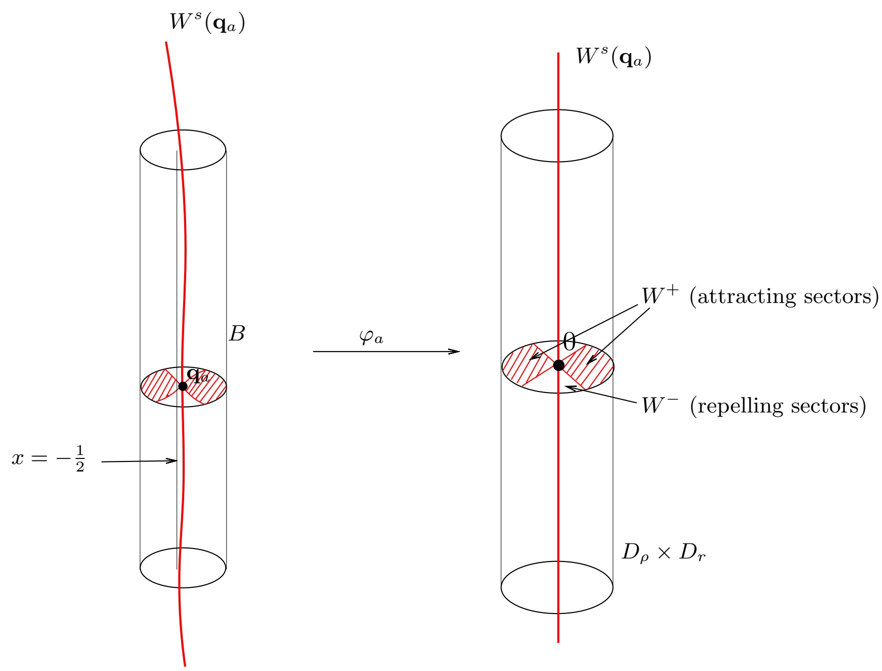

Figure 3.4: The transformation φ_a and the sectors $W^{\pm}$ for $q = 2$.

3.7 Construction of the neighborhood V

We will construct a neighborhood of J^+ for a semi-parabolic Hénon map H_a inside a polydisk $\mathbb{D}_r \times \mathbb{D}_r$. The construction is inspired by the construction of a neighborhood of the Julia set of the parabolic polynomial p on which p is strictly, but not strongly expanding, as in [DH] and [H]. Inside a tubular neighborhood

B of the local stable manifold $W^s(\mathbf{q}_a)$, we want to forget about the dynamics of the polynomial p and construct a neighborhood of $J^+ \cap B$ that is meaningful for the dynamics of the perturbed Hénon map.

Let q_0 be a parabolic fixed point for the polynomial p. Suppose $|a| < \delta$ and consider the normalizing coordinates of the Hénon map H_a on the tubular neighborhood $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ as defined Theorem 3.19 and let W_B^+ be the attractive sectors as defined in Lemma 3.23. Set

$$B' := (H^{-1}(B) - B) \cap \mathbb{D}_r \times \mathbb{D}_r.$$

Define $W_{B'}^+$ to be the preimage of the attractive sectors W_B^+ in B'.

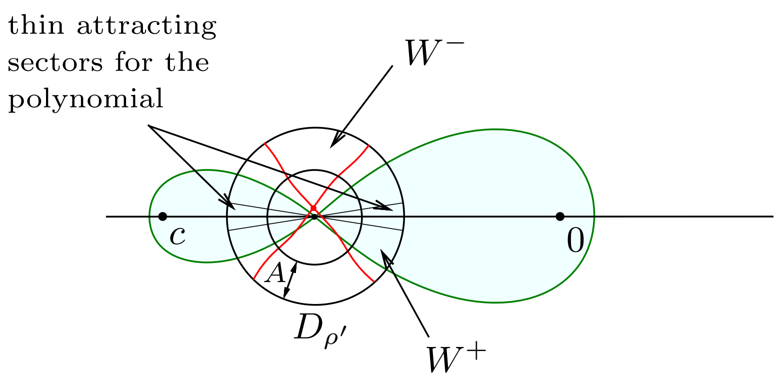

Figure 3.5: Here $q = 2$. This is a cross section around the parabolic fixed point of the polynomial $p(x) = x^2 + c_0$. The red lines are the boundaries of the attractive sectors for the Hénon map. The thin attractive sectors for the polynomial and their preimages are shown in green.

Let n be the first iterate of p, such that $p^{n+1}(0) \in \mathbb{D}_{\rho'}(q_0)$. Construct attractive sectors S_{att} associated with the polynomial p in $\mathbb{D}_{\rho'}(q_0)$, thin enough along the *attractive axes* so that

$$\left(p^{-(n+1)}(S_{att}) \cap A\right) \times \mathbb{D}_r \subset\subset W_B^+. \tag{3.27}$$

Denote by $A = A(q_0; \rho', \rho'/2)$ the annulus between the disk of radius ρ' and the disk of radius $\rho'/2$ centered at q_0.

In the hyperbolic case, we constructed a set U as a complement of a neighborhood of the attracting fixed point. Let us now define the set U to be the complement of $p^{-n}(S_{att})$ inside an equipotential of the Green's function of p, i.e.

$$U := \mathbb{C} - p^{-n}(S_{att}) - \{z \in \mathbb{C} - K_p \mid |\psi_p^{-1}(z)| \geq R\}$$

for some large enough $R > 2$. When writing $p^{-n}(S_{att})$ we only consider the preimages of S_{att} that are contained in the immediate Fatou components of the parabolic fixed point (and contain the parabolic fixed point in the boundary).

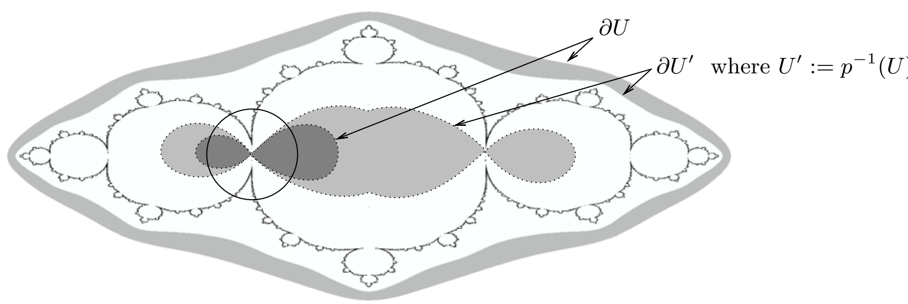

Figure 3.6: The polynomial $p(x) = x^2 - \frac{3}{4}$ has a parabolic fixed point at $-\frac{1}{2}$ and locally connected Julia set J_p. The corresponding neighborhoods U and U' are also shown, but U' is not compactly contained in U, as in the hyperbolic case. Their boundaries touch at the parabolic fixed point.

Then set as before $U' := p^{-1}(U)$. We have $U' \subset U$, and $p : U' \to U$ is a covering map, hence expanding for the Poincaré metric of U, but U' is not relatively compact in U, so there is no constant of uniform expansion.

Define

$$V := (U' \times \mathbb{D}_r - (B \cup B')) \cup (W_B^- \cup W_{B'}^-),$$

where B' is defined above and W_B^- is defined as in Lemma 3.23. We define $W_{B'}^-$ in the same way as $W_{B'}^+$.

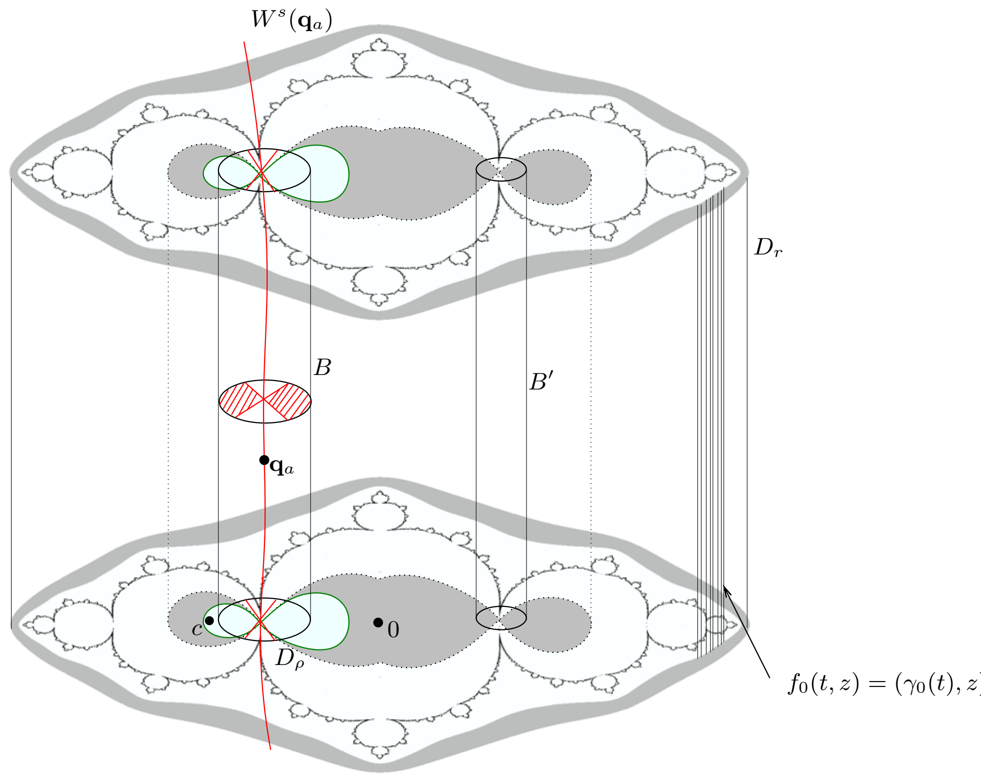

Figure 3.7: A neighborhood $\overline{V}$ of the set J^+ in $\mathbb{D}_r \times \mathbb{D}_r$. The map γ_0 used in the definition of the fiber $f_0(t,z)$ is the same as in Equation 2.1.

The vertical size of the neighborhood V, is $r > 0$, where r can be chosen so that $U' \subset \mathbb{D}_r$ and $\overline{H(V)}$ does not intersect the horizontal boundary of V, so $|ax| < r$, for any $x \in U'$.

The horizontal size of the neighborhood V is given by an equipotential of the parabolic polynomial, contained entirely in the escaping set U^+. One can choose the equipotential so that all points in $H(V) - V$ belong to U^+. One can choose for instance $r > 2$, so that $U' \subset \mathbb{D}_r$ and the outer boundary of U' is outside $\mathbb{D}_2$.

Let a be small enough so that the following two conditions hold

- $r|a| < |p(x) - c_0|$ for any x in U'. This is possible because we removed a disc around the critical value c_0 of the polynomial p, hence $\inf_{x\in U'} |p(x) - c_0| > 0$.

- $r|a| < d(\partial U' - \mathbb{D}_{\rho'}, \partial U - \mathbb{D}_{\rho'})$. This assures that for x in $U' - \mathbb{D}_{\rho'}$, the part of the disk of radius $r|a|$ around x that lies outside $\mathbb{D}_{\rho'}$, belongs to U. In other words, the $r|a|$–neighborhood of $U' - \mathbb{D}_{\rho'}$ is compactly contained in $U - \mathbb{D}_{\rho'}$.

Let $\overline{V}$ denote the set V together with $W^s_{loc}(\mathbf{q}_a)$ and $H^{-1}(W^s_{loc}(\mathbf{q}_a)) \cap B'$.

Lemma 3.26. *The set $\overline{V}$ is a neighborhood of J^+ inside $\mathbb{D}_r \times \mathbb{D}_r$.*

Proof. The outer boundary of the set V is an equipotential of the polynomial cross $\mathbb{D}_r$, which belongs to U^+. From the tubular neighborhood B of the local stable manifold we have removed only the attractive sectors W^+_B, which are contained inside the interior of K^+ union the local stable manifold $W^s_{loc}(\mathbf{q}_a)$. From B' we only removed the attractive sectors $W^+_{B'}$, which are contained inside the interiors of K^+ union a preimage of the local stable manifold $H^{-1}(W^s_{loc}(\mathbf{q}_a)) \cap B'$. Outside of $B \cup B'$, we have removed a vertical tube $p^{-(n+1)}(S_{att}) \times \mathbb{D}_r$ which belongs to the interior of K^+. Therefore

$$J^+ \cap (\mathbb{D}_r \times \mathbb{D}_r) \;=\; (J^+ \cap V) \cup W^s_{loc}(\mathbf{q}_a) \cup \left(H^{-1}(W^s_{loc}(\mathbf{q}_a)) \cap B'\right) \;=\; J^+ \cap \overline{V}.$$

In this sense we say that $\overline{V}$ is a neighborhood of J^+ inside the bidisk $\mathbb{D}_r \times \mathbb{D}_r$. □

Lemma 3.27. $J^+ \cap \overline{V} = \bigcap_{n\geq 0} H^{-n}(\overline{V} \cap \overline{U^+})$.

Proof. Let $q \in \bigcap_{n\geq 0} H^{-n}(\overline{V} \cap \overline{U^+})$. Since all forward iterates of q remain in the bounded set $\overline{V}$, q cannot belong to U^+. Hence $q \in J^+$. Suppose now that $q \in J^+ \cap \overline{V}$. By construction of the neighborhood V, $H(J^+ \cap \overline{V}) \subset J^+ \cap \overline{V}$, so all forward iterates of q remain in $\overline{V}$. Hence $q \in \bigcap_{n\geq 0} H^{-n}(\overline{V})$. □

Based on Proposition 3.26 , we also get that $J = \bigcap_{n\geq 0} H^n(J^+ \cap \overline{V})$.

3.8 Constructing a metric

We will construct a metric μ on the set V with respect to which the derivative of the Hénon map expands horizontal vectors. In the case of hyperbolic Hénon maps, the set V was a product space $V = U' \times \mathbb{D}_r$ and it was enough to endow V with the product of the Poincaré metric of $U = p(U')$ and the Poincaré metric of $\mathbb{D}_r$. The strong expansion of horizontal vectors $((x, y), (\zeta, 0))$ under DH was a consequence of the strong expansion of vectors (x, ζ) under the derivative of the polynomial p.

In the case of semi-parabolic Hénon maps, one can put the same metric on V and show that outside a tubular neighborhood of the stable manifold, horizontal vectors get strongly expanded under DH. Inside the tubular neighborhood however, the dynamics of the parabolic polynomial no longer accurately reflects the dynamics of the Hénon map, so one needs to use a modified metric. We have seen that, after a change of coordinates, the Hénon map written in the normal form expands horizontal tangent vectors, measured with respect to the standard Euclidean metric. So the metric that we want to take on B will be a pull-back of the Euclidean metric from the normalized coordinates.

Recall that $q_0 = \lambda/2$ is the parabolic fixed point of the quadratic polynomial $p(x) = x^2 + c_0$. In order to formalize the definitions, consider $B = \mathbb{D}_{\rho'}(q_0) \times \mathbb{D}_r$ and $B'' = \mathbb{D}_{\rho'/2}(q_0) \times \mathbb{D}_r$.

We have chosen a small enough so that the local stable manifold $W^s_{loc}(\mathbf{q}_a)$ of the semi-parabolic fixed point $\mathbf{q}_a$ is contained in B'' and that Equation 3.27 is satisfied. In addition, the set U' is compactly contained in U outside the disk $\mathbb{D}_{\rho'/2}(q_0)$.

Definition 3.28 (Poincaré metric). Let $(x, y) \in V$ such that $x \in U$, and consider $(\xi, \eta) \in T_{(x,y)}V$ a tangent vector. Then

$$\mu_P\left((x, y), (\xi, \eta)\right) := \max\{|x, \xi|_U, |y, \eta|_{\mathbb{D}_r}\},$$

where $|x, \xi|_U$ and $|y, \eta|_{\mathbb{D}_r}$ are measured with respect to the Poincaré metric on U, respectively on $\mathbb{D}_r$.

Proposition 3.29. *Let $(x, y), (x_1, y_1) \in V$ such that $H(x, y) = (x_1, y_1)$ and (x_1, y_1) does not belong to the tubular neighborhood B'' of the stable manifold. Let $(\zeta, 0)$ be a horizontal tangent vector, and let $(\zeta_1, \eta_1) = DH_{(x,y)}(\zeta, 0)$. Then*

$$\mu_P\left((x_1, y_1), (\xi_1, \eta_1)\right) > k \cdot \mu_P\left((x, y), (\xi, \eta)\right),$$

where the constant $k > 1$ depends only on the distance between ∂U and $\partial U'$ outside of a disk of fixed radius $\rho'/2$ centered at the parabolic fixed point of the polynomial p.

Proof. Using the form of the Hénon map from 3.3, one computes $\zeta_1 = p'(x)\zeta$ and $\eta_1 = a\zeta$. The proof is analogous to the hyperbolic case, see Lemma 2.3, 2.4 and Theorem 2.14. Therefore there exists a constant $k > 1$, which depends only on the distance between ∂U and $\partial U'$ outside of a disc of fixed radius $\mathbb{D}_{\rho'/2}(q_0)$, such that for all $|a| < \delta$ we have

$$\begin{aligned} \mu_P\left((x_1, y_1), (\xi_1, \eta_1)\right) &= |x_1, \zeta_1|_U = |p(x) + ay + a^2 w, p'(x)\zeta| \\ &> k|x, \zeta|_U = k \cdot \mu_P\left((x, y), (\xi, \eta)\right). \end{aligned}$$

□

Let us now define a metric on the normalizing neighborhood B as follows: choose a point $(x, y) \in B$ and a tangent vector $(\zeta, 0) \in T_{(x,y)}B$. By Theorem 3.19, there exists a coordinate transformation $\varphi_a : B \to \mathbb{D}_\rho \times \mathbb{D}_r$ such that $\varphi_a^{-1} \circ \widetilde{H} \circ \varphi_a(x, y) = H(x, y)$. Moreover, by construction, the coordinate transformation φ_a takes horizontal curves to horizontal curves. Hence $\varphi_a(x, y) = (\widetilde{x}, \widetilde{y})$ and $D\varphi_{a,(x,y)}(\zeta, 0) = (\widetilde{\zeta}, 0)$. Set

$$\mu_B\left((x, y), (\zeta, 0)\right) := |\widetilde{\zeta}|$$

where $|\widetilde{\zeta}|$ is the length of $\widetilde{\zeta}$ with respect to the standard Euclidean metric.

More generally, let us give the following definition

Definition 3.30 (Euclidean metric). Let $(x, y) \in B$ and $(\zeta, \eta) \in T_{(x,y)}B$. After a change of coordinates, let $\varphi_a(x, y) = (\widetilde{x}, \widetilde{y})$ and $D\varphi_{a,(x,y)}(\zeta, \eta) = (\widetilde{\zeta}, \widetilde{\eta})$. Define the metric μ_B by the rule

$$\mu_B\left((x, y), (\zeta, \eta)\right) := \max\{|\widetilde{\zeta}|, |\widetilde{\eta}|\}.$$

where $|\widetilde{\zeta}|$ and $|\widetilde{\eta}|$ represent the length of $\widetilde{\zeta}$ and $\widetilde{\eta}$ with respect to the Euclidean metric.

Proposition 3.31. *Let $(x, y), (x_1, y_1) \in B$ such that $H(x, y) = (x_1, y_1)$. Let $(\zeta, 0)$ be a horizontal tangent vector and denote by $(\zeta_1, \eta_1) = DH_{(x,y)}(\zeta, 0)$. Then*

$$\mu_B\left((x_1, y_1), (\zeta_1, \eta_1)\right) > C(x, y) \cdot \mu_B\left((x, y), (\zeta, 0)\right),$$

where $C(x, y) > 1$. The function $C(x, y)$ tends to 1 as (x, y) becomes closer and closer to the local stable manifold.

Proof. Let $\varphi_a(x, y) = (\widetilde{x}, \widetilde{y})$, $\varphi_a(x_1, y_1) = (\widetilde{x}_1, \widetilde{y}_1)$ and $D\varphi_{a,(x,y)}(\zeta_1, \eta_1) = (\widetilde{\zeta}_1, \widetilde{\eta}_1)$. By construction, the coordinate transformation φ_a takes horizontal curves to horizontal curves, so $D\varphi_{a,(x,y)}(\zeta, 0) = (\widetilde{\zeta}, 0)$. By the proof of the Lemma 3.23 it follows that

$$\mu_B\left((x_1, y_1), (\zeta_1, \eta_1)\right) = |\widetilde{\zeta_1}| > C(\widetilde{x}) \cdot |\widetilde{\zeta}| = C(\widetilde{x}) \cdot \mu_B\left((x, y), (\zeta, 0)\right),$$

where $C(\widetilde{x}) > 1$ when $\widetilde{x} \neq 0$ and $C(\widetilde{x}) = 1$ when $\widetilde{x} = 0$, that is when $(\widetilde{x}, \widetilde{y})$ is a point on the local stable manifold. By a slight abuse of notation, we will define $C(x, y) := C(\widetilde{x})$. □

Definition 3.32 (Combined metric). Let $B' = (H^{-1}(B) - B) \cap V$ be one of the preimages of B in V. Choose a number M such that

$$M > \sup_{\substack{(x,y)\in B',\\ (\zeta,0)\in T_{(x,y)}B'}} \frac{2 \cdot \mu_P\left((x, y), (\zeta, 0)\right)}{\mu_B\left(H(x, y), dH_{(x,y)}(\zeta, 0)\right)}.$$

Define a new metric $\mu := \inf\{\mu_P, M\mu_B\}$, where the infimum is taken pointwise between the metrics on V.

Remark 3.33. By choosing M big enough we can assume that on $\mathbb{D}_{\rho'}(q_0) \times \partial\mathbb{D}_r$, the infimum between the two metrics is attained by the Poincare metric μ_P. By choosing a small, we can also assume that there exists $\rho'' > 0$ fixed such that on $\mathbb{D}_{\rho''}(\mathbf{q}_0) \times \partial\mathbb{D}_r$, the infimum is attained by the pull-back metric μ_B. Assume, if necessary, that $\rho'' < \rho'/2$.

Theorem 3.34 (μ-Expansion). *Consider $(x, y), (x_1, y_1) \in V$ with $H(x, y) = (x_1, y_1)$. Let $(\zeta, 0), (\zeta_1, \eta_1)$ be two tangent vectors such that $DH_{(x,y)}(\zeta, 0) = (\zeta_1, \eta_1)$. Then*

$$\mu\left((x_1, y_1), (\zeta_1, \eta_1)\right) > \alpha(x, y) \cdot \mu\left((x, y), (\zeta, 0)\right), \quad \textit{where} \ \ \alpha(x, y) > 1,$$

and $\alpha(x, y)$ is a constant in cases (a), (c) and (d), and $\alpha(x, y) = C(x, y)$, the expansion function of the pull-back metric μ_B, in case (b).

Proof. There are four cases to consider:

(a) $\mu((x,y),(\zeta,0)) = \mu_P((x,y),(\zeta,0))$ and $\mu((x_1,y_1),(\zeta_1,\eta_1)) = \mu_P((x_1,y_1),(\zeta_1,\eta_1))$. Since the Poincaré metric is smaller than the pull-back metric, it means that the points (x,y) and (x_1,y_1) are not very close to $q_0 \times \mathbb{D}_r$. In particular by Remark 3.33 they must lie outside $\mathbb{D}_{\rho''}(q_0) \times \mathbb{D}_r$. By Proposition 3.29,

$$\mu((x_1,y_1),(\zeta_1,\eta_1)) > k \cdot \mu((x,y),(\zeta,0)).$$

(b) $\mu((x,y),(\zeta,0)) = M\mu_B((x,y),(\zeta,0))$ and $\mu((x_1,y_1),(\zeta_1,\eta_1)) = M\mu_B((x_1,y_1),(\zeta_1,\eta_1))$. By Proposition 3.31,

$$\mu((x_1,y_1),(\zeta_1,\eta_1)) > C(x,y) \cdot \mu((x,y),(\zeta,0)).$$

(c) $\mu((x,y),(\zeta,0)) = M\mu_B((x,y),(\zeta,0))$ and $\mu((x_1,y_1),(\zeta_1,\eta_1)) = \mu_P((x_1,y_1),(\zeta_1,\eta_1))$. By Remark 3.33 above, the point (x_1,y_1) cannot be too close to $q_0 \times \mathbb{D}_r$ and it must stay outside the small tube B''. By Proposition 3.29, we have

$$\begin{aligned}\mu_P((x_1,y_1),(\zeta_1,\eta_1)) &> k \cdot \mu_P((x,y),(\zeta,0)) \\ &\geq k \cdot M\mu_B((x,y),(\zeta,0)) = k \cdot \mu((x,y),(\zeta,0)).\end{aligned}$$

(d) $\mu((x,y),(\zeta,0)) = \mu_P((x,y),(\zeta,0))$ and $\mu((x_1,y_1),(\zeta_1,\eta_1)) = M\mu_B((x_1,y_1),(\zeta_1,\eta_1))$.

(i) If $(x,y) \in B'$, then by the choice of the constant M we have

$$\begin{aligned}\mu_P((x,y),(\zeta,0)) &< \frac{2 \cdot \mu_P((x,y),(\zeta,0))}{\mu_B((x_1,y_1),(\zeta_1,\eta_1))} \cdot \frac{1}{2}\mu_B((x_1,y_1),(\zeta_1,\eta_1)) \\ &< \frac{1}{2} \cdot M\mu_B((x_1,y_1),(\zeta_1,\eta_1))\end{aligned}$$

hence $\mu((x_1,y_1),(\zeta_1,\eta_1)) > 2 \cdot \mu((x,y),(\zeta,0))$.

(ii) If $(x,y) \in B$, and the Poincaré metric is smaller than the pull-back metric, then (x,y) must be outside the small tube B'' which encloses the local stable manifold $W^s_{loc}(\mathbf{q}_a)$. If we denote by

$$k' := \inf_{\substack{(x,y)\in V-B'' \\ |a|<\delta}} C(x,y)$$

the infimum of the expansion rate $C(x,y)$ outside B'', then $k' > 1$.

By Proposition 3.31, we know that

$$\mu_B\left((x_1,y_1),(\zeta_1,\eta_1)\right) \;>\; C(x,y)\cdot\mu_B\left((x,y),(\zeta,0)\right).$$

Therefore

$$\begin{aligned} M\mu_B\left((x_1,y_1),(\zeta_1,\eta_1)\right) &> C(x,y)\cdot M\mu_B\left((x,y),(\zeta,0)\right) \\ &\geq k'\cdot\mu_P\left((x,y),(\zeta,0)\right), \end{aligned}$$

hence $\mu\left((x_1,y_1),(\zeta_1,\eta_1)\right) > k'\cdot\mu\left((x,y),(\zeta,0)\right)$. □

3.9 Cone invariance in the Euclidean metric

Definition 3.35. Define the vertical cone at a point (x,y) from the set $\mathbb{D}_\rho\times\mathbb{D}_r$ to be

$$C^v_{(x,y)} = \left\{(\xi,\eta)\in T_{(x,y)}\mathbb{D}_\rho\times\mathbb{D}_r,\ |\xi| < |x|^{2q}|\eta|\right\}.$$

We will show that the vertical cones are invariant under $D\widetilde{H}^{-1}$.

Proposition 3.36. *Consider (x,y) and (x_1,y_1) in the repelling sectors of $\mathbb{D}_\rho\times\mathbb{D}_r$ such that $\widetilde{H}(x,y)=(x_1,y_1)$. Then*

$$D\widetilde{H}^{-1}_{(x_1,y_1)}\left(C^v_{(x_1,y_1)}\right)\subset C^v_{(x,y)}.$$

Proof. Let $(\xi',\eta')\in C^v_{(x_1,y_1)}$ and $(\xi,\eta)=D\widetilde{H}^{-1}_{(x,y)}(\xi',\eta')$. We need to show that $(\xi,\eta)\in C^v_{(x,y)}$. Compute as before

$$\begin{aligned} \xi' &= \lambda\left(1+(q+1)x^q+\frac{\partial g_a}{\partial x}(x,y)\right)\xi+\lambda\frac{\partial g_a}{\partial y}(x,y)\eta \\ \eta' &= \left(\frac{\partial xh_a}{\partial x}(x,y)\right)\xi+\left(\mu+\frac{\partial xh_a}{\partial y}(x,y)\right)\eta \end{aligned}$$

When $a = 0$, the function $g_0(x, y)$ is just a function of the variable x, hence $\frac{\partial g_0}{\partial y}(x, y) \equiv 0$. For $0 < |a| < \delta$ we can assume that there exists a constant $0 < M_a < 1$, such that $\left|\frac{\partial g_a}{\partial y}(x, y)\right| < M_a |x|^{2q+2}$. We can now estimate

$$\begin{aligned} |\xi'| &> \left(|1 + (q+1)x^q| - 2m|x|^{2q}\right)|\xi| - M_a|x|^{2q+2}|\eta| \\ |\eta'| &< N_a|\xi| + (|\mu| + N_a)|\eta| \end{aligned}$$

Since (ξ', η') belongs to the vertical cone at (x_1, y_1), we also know that

$$|\xi'| < |x_1|^{2q}|\eta'| < |x|^{2q}|1 + x^q + g_a(x, y)/x|^{2q}|\eta'| < |x|^{2q} M_1^{2q}|\eta'|,$$

where M_1 is the supremum of $|1 + x^q + g_a(x, y)/x|$ on the repelling sectors W^- of the tubular neighborhood B, that is

$$M_1 := \sup_{(x,y)\in W^-,\ |a|<\delta} \left|1 + x^q + g_a(x, y)/x\right|. \tag{3.28}$$

Clearly $M_1 > 0$. In fact we could take a constant $M_1 > 1$ because $Re(x^q) > \epsilon_1|x|^q$ in the repelling sectors W^-. By combining these inequalities we get

$$\left(|1 + (q+1)x^q| - 2m|x|^{2q}\right)|\xi| - M_a|x|^{2q+2}|\eta| < M_1^{2q}N_a|x|^{2q}|\xi| + M_1^{2q}(|\mu| + N_a)|x|^{2q}|\eta|.$$

After regrouping the terms, we obtain

$$|\xi| < \frac{A_2}{A_1}|x|^{2q}|\eta|$$

where A_1 and A_2 are defined as follows

$$\begin{aligned} A_1 &:= |1 + (q+1)x^q| - (2m + M_1^{2q}N_a)|x|^{2q} \\ A_2 &:= M_1^{2q}(|\mu| + N_a) + M_a|x|^2. \end{aligned}$$

Since x is chosen from the repelling sectors we have $|1 + (q+1)x^q| - 2m|x|^{2q} > 1$. The bounds N_a, M_a and the eigenvalue μ all depend on a, and they tend to 0 as $a \to 0$. For $|a|$ small enough we can assume that $A_1 > \frac{2}{3}$ and $A_2 < \frac{1}{3}$. Hence $(\xi, \eta) \in C^v_{(x,y)}$. □

Remark 3.37. We can show that the following

$$C^{v}_{(x,y)} = \left\{(\xi,\eta) \in T_{(x,y)}\mathbb{D}_{\rho} \times \mathbb{D}_{r},\ |\xi| < 2M_{a}|x|^{2q}|\eta|\right\}$$

verify the proposition above for $|a| < \delta$ small enough. When $a \to 0$ we have $M_a \to 0$ and the tangent vector becomes $(0,\eta)$ and all fibers become vertical.

The vertical cones that we have defined in Definition 3.35 are with respect to the Euclidean metric, in the normalized coordinates around the local stable manifold of the semi-parabolic fixed point.

Definition 3.38. One can pull back the vertical cones from the normalized coordinates $\mathbb{D}_{\rho} \times \mathbb{D}_{r}$ into B, using the change of coordinate function ϕ_a. Let $C^{v,B}_{(x,y)}$ denote this pull-back.

However, one can also define vertical cones in the whole set V with respect to the standard Euclidean metric and show invariance under DH^{-1}.

Definition 3.39. Define the vertical cone at a point (x,y) from the set V to be

$$C^{v,E}_{(x,y)} = \left\{(\xi,\eta) \in T_{(x,y)}V,\ |\xi| < \gamma|\eta|\right\}$$

where γ is a constant $\gamma < \left(\frac{\rho}{2}\right)^{2q}$, chosen so that for a small enough, on a neighborhood of the boundary of B, the vertical cones $C^{v,E}_{(x,y)} \subset C^{v,B}_{(x,y)}$.

We will show that these vertical cones are invariant under DH^{-1}.

Proposition 3.40. *Consider (x,y) and (x',y') in V with $H^{-1}(x,y) = (x',y')$. Then*

$$DH^{-1}_{(x,y)}\left(C^{v,E}_{(x,y)}\right) \subset C^{v,E}_{(x',y')}.$$

Proof. Let $(\xi, \eta) \in C^{v,E}_{(x,y)}$ and denote by $(\xi', \eta') = dH^{-1}_{(x,y)}(\xi, \eta)$. Since

$$DH^{-1} = \begin{bmatrix} 0 & \frac{1}{a} \\ \frac{1}{a} & -\frac{2y}{a^3} \end{bmatrix},$$

we can estimate

$$|\eta'| > \left|\frac{1}{a}\xi - \frac{2y}{a^3}\eta\right| > \left|\frac{2y}{a^2}\right| \frac{|\eta|}{|a|} - \gamma\frac{|\eta|}{|a|} = |\xi'|\left(\frac{2}{|a|}\left|\frac{y}{a}\right| - \gamma\right).$$

The point (x', y') is in V, so its first coordinate $x' = \frac{y}{a}$ must be bounded away from 0, and also bounded above by $\operatorname{diam}(U)$. Therefore there exists $r_0 > 0$ such that $r_0 < \left|\frac{y}{a}\right| < r$. We can choose a small so that $|a| < 2r_0/(\gamma + \frac{1}{\gamma})$.

$$|\eta'| > |\xi'|\left(\frac{2r_0}{|a|} - \gamma\right) > \frac{1}{\gamma}|\xi'|,$$

hence $(\xi', \eta') \in C^{v,E}_{(x',y')}$. □

Finally, one may define vertical cones with respect to the Poincaré metric, analogous to the hyperbolic setting, and ask for invariance under DH^{-1}.

Definition 3.41. Let $(x, y) \in V$ such that (x, y) does not belong to $B'' = \mathbb{D}_{\rho'/2} \times \mathbb{D}_r$. Define

$$C^{v,P}_{(x,y)} = \left\{(\xi, \eta) \in T_{(x,y)}V,\ |x, \xi|_U < |y, \eta|_{\mathbb{D}_r}\right\}.$$

When defining these cones, we need to be outside a small neighborhood of the local stable manifold $W^s_{loc}(\mathbf{q}_a)$ because otherwise the projection on the first coordinate of a point $(x, y) \in V$ may not belong to U, so it would be improper to measure $|x, \xi|_U$.

Proposition 3.42. *Consider (x, y) and (x', y') in V with $H^{-1}(x, y) = (x', y')$. Assume further that $x, x' \in U$. Then*

$$dH^{-1}(C^{v,P}_{(x,y)}) \subseteq C^{v,P}_{H^{-1}(x,y)}.$$

Proof. The proof is the same as in Corollary 2.9.1, if we replace the Hénon map with $H(x, y) = (p(x) + ay + a^2 w, ax)$ and thus treat H as a perturbation of the parabolic polynomial p. □

3.10 Distance between vertical-like curves

In this section we work entirely in the normalized coordinates from Theorem 3.19. The notion of vertical-like curves translates as follows

Definition 3.43. We will call an analytic curve $\gamma \subset \mathbb{D}_\rho \times \mathbb{D}_r$ *vertical-like* if γ is the graph of an analytic function $\phi : \mathbb{D}_r \to \mathbb{D}_\rho$, and for all points (x, y) on γ, the tangent vectors (ξ, η) to γ at (x, y) belong to the vertical cone $C^v_{(x,y)}$, defined in Definition 3.35.

Let us now consider two vertical-like curves in the same repelling sector of $\mathbb{D}_\rho \times \mathbb{D}_r$, that are entirely contained in the escaping set U^+. Denote these vertical curves

$$f_1(z) = (\varphi_1(z), z) \text{ and } f_2(z) = (\varphi_2(z), z).$$

Let $g_1(\mathbb{D}_r)$ be the image under $\widetilde{H}^{-1}$ of $f_1(\mathbb{D}_r)$, contained inside $\mathbb{D}_\rho \times \mathbb{D}_r$. More precisely,

$$\widetilde{H}^{-1}(f_1(\mathbb{D}_r)) \cap (\mathbb{D}_\rho \times \mathbb{D}_r),$$

is a vertical-like fiber, that we can describe as the graph of an analytic function

$$g_1(z) = (\varphi'(z), z), \text{ where } \varphi' : \mathbb{D}_r \to \mathbb{D}_\rho.$$

Similarly, let $g_2(\mathbb{D}_r)$ be $\widetilde{H}^{-1}(f_2(\mathbb{D}_r)) \cap (\mathbb{D}_\rho \times \mathbb{D}_r)$, reparametrized by the second coordinate $g_2(z) = (\varphi''(z), z)$. Notice that $g_1(\mathbb{D}_r)$ and $g_2(\mathbb{D}_r)$ are vertical-like curves (by Proposition 3.36), both contained in some other repelling sector of $\mathbb{D}_\rho \times \mathbb{D}_r$ and in U^+.

Much like in the hyperbolic setting, we would like to show that $\widetilde{H}$ expands the horizontal distance between vertical-like curves. We will measure the horizontal distance with respect to the standard Euclidean metric on $\mathbb{D}_\rho \times \mathbb{D}_r$. We define

$$d(f_1, f_2) = ||\varphi_1 - \varphi_2|| = \sup_{z \in \mathbb{D}_r} |\varphi_1(z) - \varphi_2(z)|$$

Notice that the distance that we define between vertical-like curves is just the distance between the parametrizing functions φ_1 and φ_2 with respect to the sup-norm.

Theorem 3.44. *Let $d(g_1, g_2)$ and $d(f_1, f_2)$ be the horizontal distance between the vertical-like curves g_1, g_2 and respectively f_1, f_2. Then*

$$d(g_1, g_2) < d(f_1, f_2),$$

so the normalized Hénon maps $\widetilde{H}$ expands strictly (but not strongly) the distance between the vertical-like curves g_1 and g_2.

Proof. Let $z \in \mathbb{D}_r$ be arbitrarily chosen and denote by $x' = \varphi'(z)$, and $x'' = \varphi''(z)$. The points (x', z) and (x'', z) lie on the vertical-like curves g_1 and g_2. Let $(x_1, y_1) = \widetilde{H}(x', z)$ and $(x_2, y_2) = \widetilde{H}(x'', z)$ be the corresponding points on f_1 and f_2. Let $(x_3, y_1) = (\varphi_2(y_2), y_2)$ be the point of intersection of the curve f_2 with the horizontal plane $\mathbb{C} \times \{y_1\}$. Suppose without loss of generality that $|x_2| \leq |x_1|$.

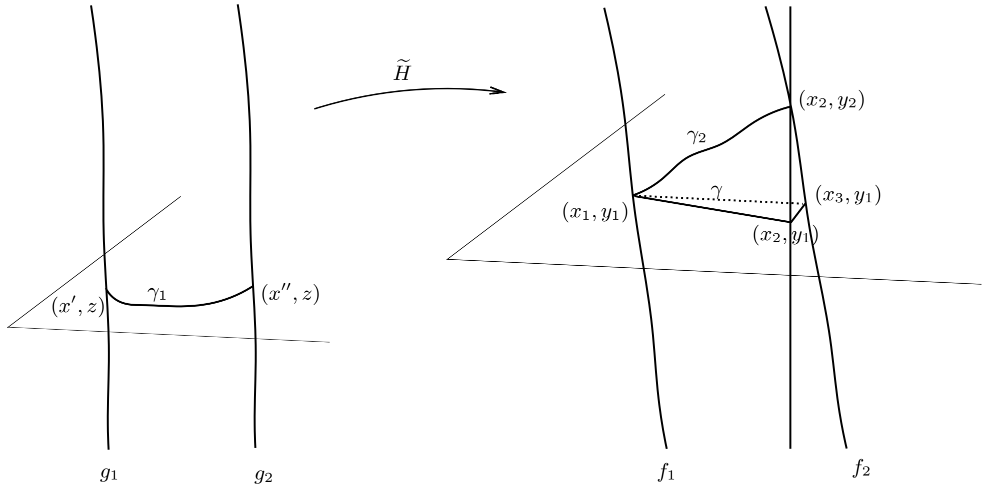

Figure 3.8: Fibers g_1, g_2 and their image fibers f_1, f_2 under $\widetilde{H}$.

Lemma 3.45 (Step 1). *We have*

$$|x' - x''| < \left(1 - \frac{\epsilon}{2M_1} \int_0^1 |tx_1 + (1-t)x_2|^q dt\right) |x_1 - x_2|,$$

where the constants $\epsilon := \epsilon_1/2$ and M_1 are independent of a.

Proof. Choose a straight line in the $\mathbb{C} \times \{y_1\}$ plane,

$$\gamma(t) = (x_\gamma(t), y_1), \quad \text{where } x_\gamma(t) = tx_1 + (1-t)x_2 \text{ and } t \in [0,1],$$

connecting the points (x_1, y_1) and (x_2, y_1). There exists a horizontal curve

$$\gamma_1(t) : [0,1] \to \mathbb{D}_\rho \times \{z\}, \quad \gamma_1(t) = (x_{\gamma_1}(t), z),$$

connecting the points $(x', z) = \gamma_1(0)$ and $(x'', z) = \gamma_2(0)$ and such that if we denote

$$\gamma_2(t) = \widetilde{H}(\gamma_1(t))$$

then the projection of $\gamma_2(t)$ on the plane $\mathbb{C} \times \{y_1\}$ is exactly the straight line $\gamma(t)$. Formally, if we define $pr : \mathbb{D}_\rho \times \mathbb{D}_r \to \mathbb{D}_\rho \times \{y_1\}$, $pr(x, y) = (x, y_1)$, then $pr(\gamma_2(t)) =$

$\gamma(t)$. By Lemma 3.23, we know that $D\widetilde{H}$ expands the horizontal length of vectors in W^{-}, so

$$|\gamma'(t)| > C(x_{\gamma_1}(t))|\gamma_1'(t)|.$$

We will compare the length of the curve γ_1 with the length of γ. Note that $\gamma(t)$ is a just a horizontal line segment, hence $|\gamma'(t)| = |x_1 - x_2|$, for all $t \in [0,1]$ and $l(\gamma) = |x_1 - x_2|$.

$$l(\gamma_1) = \int_0^1 |\gamma_1'(t)|dt < \int_0^1 \frac{1}{C(x_{\gamma_1}(t))}|\gamma'(t)|dt = |x_1 - x_2| \int_0^1 \frac{1}{C(x_{\gamma_1}(t))}dt$$

Recall from 3.24 that $C(x) = |1 + (q+1)x^q| - 2m|x|^{2q} \geq 1 + \epsilon|x|^q$, where $\epsilon := \frac{\epsilon_1}{2}$. The constant $0 < \epsilon < 1$ is independent of a, and $C(x) > 1$ for all $(x, y) \in W^{-}$. We can also estimate

$$\frac{1}{C(x)} \leq \frac{1}{1 + \epsilon|x|^q} \leq 1 - \frac{\epsilon}{2}|x|^q.$$

Recall also that for any $t \in [0,1]$

$$x_\gamma(t) = \lambda x_{\gamma_1}(t)\left(1 + x_{\gamma_1}(t)^q + g_a(x_{\gamma_1}(t), z)/x_{\gamma_1}(t)\right)$$

where $\left|1 + x_{\gamma_1}(t)^q + g_a(x_{\gamma_1(t)}, z)/x_{\gamma_1}(t)\right| < M_1$, as in Equation 3.28. By combining these estimates we obtain

$$\begin{aligned}\int_0^1 \frac{1}{C(x_{\gamma_1}(t))}dt < 1 - \frac{\epsilon}{2}\int_0^1 |x_{\gamma_1}(t)|^q dt \quad &< \quad 1 - \frac{\epsilon}{2M_1}\int_0^1 |x_\gamma(t)|^q dt \\ &= \quad 1 - \frac{\epsilon}{2M_1}\int_0^1 |tx_1 + (1-t)x_2|^q dt.\end{aligned}$$

This proves that

$$|x' - x''| \leq l(\gamma_1) \leq \left(1 - \frac{\epsilon}{2M_1}\int_0^1 |tx_1 + (1-t)x_2|^q dt\right)|x_1 - x_2|.$$

□

Lemma 3.46 (Step 2).

$$|x_2 - x_3| \le N_a|x_1 - x_2|\,(|x_2| + N_a|x_1 - x_2|)^{2q}\,,\ \textit{where}\ \lim_{a\to 0} N_a = 0.$$

Proof. The geometric intuition behind the inequality is that the curve γ_2 connecting (x_1, y_1) and (x_2, y_2) becomes horizontal as $a \to 0$, while the fibers f_1 and f_2 become vertical. The rigorous proof is outlined below.

From Remark 3.24 it follows that $|y_1 - y_2| < N_a|x_1 - x_2|$, where $\lim_{a\to 0} N_a = 0$.

The curve

$$t \to (\varphi_2(ty_1 + (1-t)y_2), ty_1 + (1-t)y_2)\,, \quad t \in [0,1]$$

is vertical-like so in particular the horizontal distance is smaller than the vertical distance and

$$|\varphi(ty_1 + (1-t)y_2) - \varphi(y_2)| < |ty_1 + (1-t)y_2 - y_2| = t|y_1 - y_2|,$$

for $t \in [0,1]$. Using $\varphi(y_2) = x_2$ this gives

$$|\varphi(ty_1 + (1-t)y_2)| < |x_2| + t|y_1 - y_2| < |x_2| + tN_a|x_1 - x_2|.$$

Hence

$$\begin{aligned} |x_2 - x_3| &\le \int_0^1 \left|\frac{\partial\varphi(ty_1 + (1-t)y_2)}{\partial t}\right| dt \le \int_0^1 |y_1 - y_2|\,|\varphi(ty_1 + (1-t)y_2)|^{2q}\,dt \\ &\le |y_1 - y_2|\,(|x_2| + N_a|x_1 - x_2|)^{2q} \le N_a|x_1 - x_2|\,(|x_2| + N_a|x_1 - x_2|)^{2q}\,. \end{aligned}$$

and the lemma follows. □

Lemma 3.47 (Technical estimate). *Let $q \ge 1$ be a natural number and $x_1, x_2 \in \mathbb{C}$ be two complex numbers, with $|x_2| \le |x_1|$. Then*

$$|x_1|^q \le 2(q+1)\int_0^1 |tx_2 + (1-t)x_1|^q dt.$$

Proof. If $x_1 = 0$ then $x_2 = 0$ and we have equality. Otherwise, set $x = x_2/x_1$. Then $|x| \leq 1$ and we have to show that

$$\frac{1}{2(q+1)} \leq \int_0^1 \left| t\frac{x_2}{x_1} + (1-t) \right|^q dt = \int_0^1 |tx + (1-t)|^q dt.$$

For any $t \in [0,1]$ we have $|tx+(1-t)| \geq |t|x|-(1-t)| = |t(1+|x|)-1|$. Let $u = t(1+|x|)-1$. Then $du = (1+|x|)dt$ and

$$\int_0^1 |t(1+|x|)-1|^q \, dt \quad = \quad \frac{1}{|x|+1} \int_{-1}^{|x|} |u|^q \, du = \frac{1}{|x|+1} \frac{|x|^{q+1}+1}{q+1} > \frac{1}{2(q+1)},$$

since $0 \leq |x| \leq 1$. □

We now return to the proof of Theorem 3.44. Let for simplicity

$$I_0 := \int_0^1 |tx_1 + (1-t)x_2|^q dt.$$

In Step 1 we showed that

$$|x' - x''| < \left(1 - \frac{\epsilon}{2M_1} I_0\right) |x_1 - x_2|.$$

We can use the triangle inequality in the $\mathbb{D}_\rho \times \{y_1\}$ disk to connect $|x_1 - x_2|$ to the distance between the curves f_1 and f_2

$$|x_1 - x_2| - |x_2 - x_3| \leq |x_1 - x_3| \leq d(f_1, f_2).$$

In Step 2 we showed that

$$|x_2 - x_3| < N_a |x_1 - x_2| \left(|x_2| + N_a |x_1 - x_2|\right)^{2q}$$

Suppose without loss of generality that $|x_2| \leq |x_1|$ and $|x_1| < 1$. Then from the technical estimate Lemma 3.47 we get

$$\begin{aligned} |x_2 - x_3| &< N_a |x_1 - x_2| (1 + 2N_a)^{2q} |x_1|^{2q} \\ &< N_a |x_1 - x_2| (1 + 2N_a)^{2q} \cdot 2(q+1) I_0. \end{aligned}$$

If a is small enough such that $N_a < \frac{1}{4q}$ then

$$2(q+1)(1+2Na)^{2q} < 2(q+1)\left(1+\frac{1}{2q}\right)^{2q} < 2e(q+1).$$

This is similar to what we previously required for N_a. In conclusion

$$|x_2 - x_3| < 2e(q+1)N_a|x_1 - x_2|I_0.$$

Hence

$$|x' - x''| < \frac{1 - \frac{\epsilon}{2M_1}I_0}{1 - 2e(q+1)N_aI_0}\, d(f_1, f_2)$$

where the quantity

$$C = \frac{1 - \frac{\epsilon}{2M_1}I_0}{1 - 2e(q+1)N_aI_0} < 1,$$

for a small enough. Indeed, the constants ϵ and M_1 are independent of the parameter a whereas $N_a \to 0$ as $a \to 0$, so it can be made small so that

$$N_a < \frac{\epsilon}{4e(q+1)M_1},$$

which is a fixed constant. However, this bound is not optimized. It follows that

$$d(g_1, g_2) \le Cd(f_1, f_2)$$

as claimed, where C depends on the distance between the curves and the y−axis. This dependence is hidden in the previous computations in I_0.

□ **of Theorem 3.44**

3.11 The contraction.

In this section, we construct a function space $\mathcal{F}$ and a graph transform operator $F : \mathcal{F} \to \mathcal{F}$. We endow the space $\mathcal{F}$ with a metric induced by the infimum metric on the set V and show that the operator F is strictly (but not strongly) contracting. We use a generalization of the Banach fixed point theorem, due to Browder, to claim the existence of a unique fixed point f^* of F.

Definition 3.48. An analytic curve $L : \mathbb{D}_r \to V$ is *vertical-like*, if the following conditions are met. Choose $(x, y) \in L$ and (ξ, η) a tangent vector to L at (x, y). If $(x, y) \in B$, then (ξ, η) belongs to the pull-back vertical cone $C^{v,B}_{(x,y)}$ described in Definition 3.38 using Definition 3.35. If (x, y) is outside B'' then (ξ, η) belongs to the standard vertical cones $C^{v,P}_{(x,y)}$ and $C^{v,E}_{(x,y)}$ described in Definitions 3.41 and 3.39.

Consider the space of functions

$$\begin{aligned}\mathcal{F} \;\; = \;\; & \Big\{ f : \mathbb{S}^1 \times \mathbb{D}_r \to V \;\Big|\; f(t,z) = (\varphi_t(z), z), \text{ where } \varphi_t : \mathbb{D}_r \to U \text{ is analytic,} \\ & f \text{ is continuous with respect to t, } f(t \times \mathbb{D}_r) \text{ is vertical-like,} \\ & \text{and } f \text{ is homotopic to } f_0 \Big\}.\end{aligned}$$

Consider the graph transform $F : \mathcal{F} \to \mathcal{F}$, defined as

$$F(f) = \tilde{f},$$

where $\tilde{f}\big|_{t\times\mathbb{D}_r}$ is the conformal map of the component of $H^{-1}\left(f(2t \times \mathbb{D}_r)\right) \cap V$ "homotopic to" $\widetilde{f_0}(t \times \mathbb{D}_r)$, normalized via the implicit function theorem (the projection on the second coordinate). Let

$$\mathcal{F}' = \Big\{ f_n : \mathbb{S}^1 \times \mathbb{D}_r \to V \quad \Big| \quad f_0(t,z) = (\gamma_0(t), z),\; f_n(t,z) = F \circ f_{n-1}(t,z) \text{ for } n \geq 1 \,\Big\}.$$

Use the modified metric μ on V and on $\mathcal{F}$ consider the metric

$$d(f, g) = \sup_{t\in\mathbb{S}} \sup_{z\in\mathbb{D}_r} d_\mu \left(f(t,z),\; g(t,z)\right).$$

where $d_\mu\left(f(t,z),\; g(t,z)\right)$ is the infimum of the length of horizontal rectifiable paths $\gamma : [0,1] \to V$ with $\gamma(0) = f(t,z)$ and $\gamma(1) = g(t,z)$. The length is measured with respect to the metric μ.

Proposition 3.49. *The operator $F : \mathcal{F}' \to \mathcal{F}'$ is a strict contraction.*

$$d(F(f), F(g)) < d(f, g), \text{ for any } f, g \in \mathcal{F}'.$$

The proof is an immediate consequence of the following proposition.

Proposition 3.50. *Let $f, g \in \mathcal{F}'$ and $t \in \mathbb{S}^1$. Then*

$$d\left(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)\right) < d\left(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)\right).$$

Proof. The case where the curves are outside a small neighborhood of the local stable manifold $W^s_{loc}(\mathbf{q}_a)$ can be treated as in the hyperbolic setting, (Theorem 2.14), because by Theorem 3.34 the derivative of the Hénon map expands in the horizontal direction with a fixed expansion factor, independent of a. The delicate case is when the curves enter B'' and come close to $W^s_{loc}(\mathbf{q}_a)$. By Theorem 3.34, case (b), the derivative of the Hénon map still expands in the horizontal direction but the expansion factor goes to 1 as we approach the stable manifold. The proof of this case was already given in Section 3.10. □

Proposition 3.51 (Contracting map). *There exists a monotonically increasing and right continuous function $h : [0, \infty) \to [0, \infty)$ such that $h(s) < s$ for each $s > 0$ and*

$$d\left(F(f), F(g)\right) \leq h\left(d(f, g)\right),$$

for any $f, g \in \mathcal{F}'$.

Proof. Let $h : [0, \infty) \to [0, \infty)$ be

$$h(s) := \sup_{\substack{f, g \in \mathcal{F}', t \in S^1 \\ d(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)) \leq s}} d(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)).$$

It is easy to see that h is increasing and that $h(0) = 0$. Moreover, by definition

$$d\left(F(f), F(g)\right) \leq h\left(d(f, g)\right),$$

for any $f, g \in \mathcal{F}'$.

By Section 3.10 we know that

$$d\left(F \circ f(t \times \mathbb{D}_r), F \circ g(t \times \mathbb{D}_r)\right) < C(f, g, t) d\left(f(2t \times \mathbb{D}_r), g(2t \times \mathbb{D}_r)\right), \tag{3.29}$$

where $C(f, g, t)$ is a contraction factor which depends on the fibers $f(t \times \mathbb{D}_r)$ and $g(t \times \mathbb{D}_r)$ and $0 \leq C(f, g, t) < 1$. It follows that $h(s) \leq s$ for all $s \geq 0$. This right-hand limit $h(s+) := \lim_{\delta \searrow 0} h(s + \delta)$ exists everywhere since the function h is monotonically increasing. We want to show that $h(s+) < s$ for all $s > 0$.

Suppose that $h(s+) = s$ for some $s > 0$. Let $(\delta_n)_{n \geq 1}$ be a strictly decreasing sequence of positive numbers converging to 0. For each n there exists fibers f_n, g_n and a $t_n \in \mathbb{S}^1$ such that

$$d\left(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)\right) > h(s + \delta_n) - \delta_n \tag{3.30}$$

and where $d\left(f_n(2t_n \times \mathbb{D}_r), g_n(2t_n \times \mathbb{D}_r)\right) \leq s + \delta_n$. This follows from the definition of $h(s + \delta_n)$ as a supremum. In view of relation 3.29 we get that

$$\begin{aligned} h(s + \delta_n) - \delta_n &< d\left(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)\right) \\ &< C_n d\left(f_n(2t_n \times \mathbb{D}_r), g_n(2t_n \times \mathbb{D}_r)\right) \leq C_n(s + \delta_n) < s + \delta_n, \end{aligned}$$

where $C_n := C(f_n, g_n, t_n)$ is a number as in Equation 3.29 above, with $0 \leq C_n < 1$ for every $n \geq 1$. Dividing both sides by $s + \delta_n$ and passing to the limit as $n \to \infty$ yields

$$\frac{h(s+)}{s} = 1 \leq \lim_{n \to \infty} C_n \leq 1.$$

Thus $\lim_{n \to \infty} C_n$ exists and is equal to 1. However, this can only happen if for all $n \geq n_0$ the fibers f_n and g_n belong to the normalizing tubular neighborhood of the semi-parabolic fixed point and the distance between the fibers is measured in the Euclidean metric (in fact the pull-back of the Euclidean metric under the normalizing map). Otherwise, the contraction factor $C(f_n, g_n, t_n)$ is bounded by a uniform constant $K < 1$.

The contraction factor C_n is constructed explicitly in Section 3.10. It is of the form

$$C_n = \frac{1 - \alpha I_0(n)}{1 - \beta I_0(n)},$$

where α, β are fixed constants with $0 < \beta < \alpha$ and $I_0(n) = \int_0^1 |tx_{1,n} + (1-t)x_{2,n}|^q dt$. The numbers $x_{1,n}$ and $x_{2,n}$ are the x-coordinates of two points that belong to the fibers $f_n(2t_n \times \mathbb{D}_r)$, respectively $g_n(2t_n \times \mathbb{D}_r)$. If $C_n \to 1$ then $I_0(n) \to 0$. In view of Lemma 3.47 we have $I_0(n) \geq \frac{1}{2(q+1)} \max(|x_{1,n}|^q, |x_{2,n}|^q)$, so $x_{1,n} \to 0$ and $x_{2,n} \to 0$. But then $|x_{1,n} - x_{2,n}| \to 0$. It follows from Lemma 3.45 and the choice of $x_{1,n}$ and $x_{2,n}$ that $d\left(F \circ f_n(t_n \times \mathbb{D}_r), F \circ g_n(t_n \times \mathbb{D}_r)\right) \to 0$.

Passing to the limit in Equation 3.30 yields $0 \geq h(s+) = s$, thus $s = 0$. Contradiction! So $h(s+) < s$ for all $s > 0$. The function $\tilde{h} : s \mapsto h(s+)$ is continuous from the right and verifies all properties of the function h. With a small abuse of notation we will consider this as the function h from the hypothesis. □

Theorem 3.52 (Browder [Br]). *Let (X, d) be a complete metric space and suppose $f : X \to X$ satisfies*

$$d(f(x), f(y)) < h(d(x,y)) \quad \textit{for all} \quad x, y \in X,$$

where $h : [0, \infty) \to [0, \infty)$ is increasing and continuous from the right such that $h(s) < s$ for all $s > 0$. Then f has a unique fixed point x^* and $f^n(x) \to x^*$ for each $x \in X$.

Proof. For a fixed $s > 0$, the sequence $(h^n(s))_{n \geq 0}$ is monotone decreasing (not necessarily strictly) and bounded below, so it has a limit as $n \to \infty$. Since h is continuous from the right, the sequence converges to a fixed point of h. But 0 is the only fixed point of h, so $h^n(s) \to 0$ for each $s > 0$.

Let $x_0 \in X$ be fixed and consider $x_n = f^n(x_0)$, $n = 1, 2, \ldots$. We can show inductively that $d(x_n, x_{n+1}) < h^n(d(x_0, x_1))$ for all $n \geq 0$. Passing to the limit, we get that

$$0 \leq \lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} h^n(d(x_0, x_1)) = 0.$$

Thus $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. We now show that $(x_n)_{n\geq 1}$ is Cauchy. Let $\epsilon > 0$. Since $\epsilon - h(\epsilon) > 0$, we can choose n large enough so that

$$d(x_n, x_{n+1}) < \epsilon - h(\epsilon).$$

Consider the ball of radius ϵ around x_n

$$B(x_n, \epsilon) := \{x \in X \mid d(x_n, x) < \epsilon\}.$$

Suppose $z \in B(x_n, \epsilon)$. Then

$$\begin{aligned} d(x_n, f(z)) &\leq d(x_n, f(x_n)) + d(f(x_n), f(z)) \\ &\leq d(x_n, x_{n+1}) + h(d(x_n, z)) \\ &\leq (\epsilon - h(\epsilon)) + h(\epsilon) = \epsilon. \end{aligned}$$

In the last step, we have used the fact that h is increasing, so $d(x_n, z) < \epsilon$ implies $h(d(x_n, z)) \leq h(\epsilon)$. Therefore $f : B(x_n, \epsilon) \to B(x_n, \epsilon)$. It follows that $d(x_n, x_{n+m}) < \epsilon$ for all $m \geq 0$. Thus our sequence is Cauchy, hence convergent since X is complete. Let $\lim_{n\to\infty} f^n(x) = x^* \in X$. Then $f(x^*) = x^*$ since f is continuous. Uniqueness of x^* follows from the contractive condition. □

This theorem, as well as other fixed-point theorems in the same spirit are presented in [KS]. A variant of this theorem was used in [DH] to prove local connectivity of the Julia set of a parabolic polynomial.

Remark 3.53. We call the function h satisfying the hypothesis of Theorem 3.52 a *Browder function*. For $h(s) = Ks$ with $0 < K < 1$, the theorem reduces to the classical Banach fixed point theorem.

Let $\overline{\mathcal{F}'}$ be the completion of the space $\mathcal{F}'$ in the d-metric defined above.

Proposition 3.54. *The map $F : \overline{\mathcal{F}'} \to \overline{\mathcal{F}'}$ has a unique fixed point f^*.*

Proof. The operator F is contracting in the metric defined on $\overline{\mathcal{F}'}$. The existence and uniqueness of a fixed point follows from the fixed point theorem 3.52. □

The fixed point f^* is a continuous surjection $f^* : \mathbb{S}^1 \times \mathbb{D}_r \to J^+ \cap \overline{V}$. In the parabolic setting, we consider $J^+ \cap \overline{V}$ in place of $J^+ \cap V$ in order to include also the local stable manifold $W^s_{loc}(\mathbf{q}_a)$ which is in the boundary of V. We will analyze the fixed point f^* thoroughly in Section 4. The first general property of f^* that we can list in this chapter is analogous to what we obtained in the hyperbolic case.

Proposition 3.55. *$Im(f^*) = J^+ \cap \overline{V}$.*

Proof. By Lemma 3.27

$$J^+ \cap \overline{V} = \bigcap_{n \geq 0} H^{-n}(\overline{V} \cap \overline{U^+}).$$

By construction, $f_0(t, z) = (\gamma_0(t), z)$, so $f_0(\mathbb{S}^1 \times \mathbb{D}_r)$ is the outer boundary of $\overline{V}$ and is entirely contained in U^+. Moreover, f^* was obtained as a limit of the functions $f_n : \mathbb{S}^1 \times \mathbb{D}_r \to \overline{V}$, where $f_n(\mathbb{S}^1 \times \mathbb{D}_r) = H^{-1}(f_{n-1}(\mathbb{S}^1 \times \mathbb{D}_r)) \cap V$, so $f_n(\mathbb{S}^1 \times \mathbb{D}_r)$ is the outer boundary of the set $\bigcap_{0\le k\le n} H^{-k}(\overline{V}\cap\overline{U^+})$. Hence $Im(f^*) = \bigcap_{n\ge 0} H^{-n}(\overline{V}\cap\overline{U^+})$. □

Proposition 3.56. *The fixed point $f^* : \mathbb{S}^1\times\mathbb{D}_r \to J^+\cap\overline{V}$ has the form*

$$f^*(t,z) = (\varphi_t(z), z),$$

where $\varphi_t(z)$ is continuous with respect to t, holomorphic with respect to z and a.

Proof. The proof is the same as in the hyperbolic setting, see Propositions 2.16, 2.17 and 2.18. □

CHAPTER 4

THE CONJUGACY

Consider $f^*(t,z) = (\varphi_t(z), z)$, where $\varphi_t(z)$ is continuous with respect to $t \in \mathbb{S}^1$ and analytic with respect to $z \in \mathbb{D}_r$. Let $\sigma : \mathbb{S}^1 \times \mathbb{D}_r \to \mathbb{S}^1 \times \mathbb{D}_r$

$$\sigma(t,z) = (2t, a\varphi_t(z))\,. \tag{4.1}$$

For sufficiently small $|a| > 0$ the map σ is well-defined and we will see that is also open, and injective.

Theorem 4.1. *Let $p(x) = x^2 + c$ be a hyperbolic polynomial with connected Julia set. There exists $\delta > 0$ such that if $0 < |a| < \delta$ then the diagram*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap V \\ \sigma \big\downarrow & & \big\downarrow H_{p,a} \\ \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap V \end{array}$$

commutes.

Proof. From the definition of f^*, we have that $H \circ f^*(t \times \mathbb{D}_r)$ is compactly contained in $f^*(2t \times \mathbb{D}_r)$. Thus we can write

$$H \circ f^*(t,z) = \begin{pmatrix} p(\varphi_t(z)) + az \\ a\varphi_t(z) \end{pmatrix} = \begin{pmatrix} \varphi_{2t}(a\varphi_t(z)) \\ a\varphi_t(z) \end{pmatrix} = f^* \circ \sigma(t,z).$$

The last equality follows from $f^* \circ \sigma(t,z) = f^*(2t, a\varphi_t(z)) = (\varphi_{2t}(a\varphi_t(z)), a\varphi_t(z))$. Therefore f^* semiconjugates H on $J^+ \cap V$ with σ on $\mathbb{S}^1 \times \mathbb{D}_r$, as claimed. □

Suppose $(c,a) \in \mathcal{P}_\lambda$, for $\lambda = e^{2\pi i p/q}$. We have the description of Proposition 3.2 and we can write

$$c = \frac{\lambda}{2} - \frac{\lambda^2}{4} + a^2 w,$$

where w is a constant depending on a and λ, as in Equation 3.2. We can write the semi-parabolic Hénon map as

$$H_{c,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) + a^2w + ay \\ ax \end{pmatrix},$$

where $p(x) = x^2 + c_0$ is a polynomial with a parabolic fixed point of multiplier λ.

Theorem 4.2. *Let $p(x) = x^2 + c_0$ be a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi ip/q}$. There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$ the diagram*

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap \overline{V} \\ \sigma\big\downarrow & & \big\downarrow H_{c,a} \\ \mathbb{S}^1 \times \mathbb{D}_r & \xrightarrow{f^*} & J^+ \cap \overline{V} \end{array}$$

commutes.

Proof. From the definition of f^*, we have that $H \circ f^*(t \times \mathbb{D}_r)$ is compactly contained in $f^*(2t \times \mathbb{D}_r)$. Thus we can write

$$H \circ f^*(t, z) = \begin{pmatrix} p(\varphi_t(z)) + a^2w + az \\ a\varphi_t(z) \end{pmatrix} = \begin{pmatrix} \varphi_{2t}(a\varphi_t(z)) \\ a\varphi_t(z) \end{pmatrix} = f^* \circ \sigma(t, z).$$

The last equality follows from $f^* \circ \sigma(t, z) = f^*(2t, a\varphi_t(z)) = (\varphi_{2t}(a\varphi_t(z)), a\varphi_t(z))$. Therefore f^* semiconjugates H on $J^+ \cap \overline{V}$ with σ on $\mathbb{S}^1 \times \mathbb{D}_r$, as claimed. □

4.1 Asymptotic expansion and equivalence classes

Lemma 4.3. *We have the following expansion for $\varphi_t(z)$*

$$\varphi_t(z) = \gamma(t) - \frac{1}{2\gamma(t)}az + O(a^2).$$

Proof. Consider the sequence $f_n(t,z) = F^{\circ n}(f_0)(t,z) = (\varphi_t^n(z), z)$, for all $n \geq 1$, and $f_0(t,z) := (\gamma_0(t), z)$. By construction we have that $H \circ f_{n+1}(t \times \mathbb{D}_r)$ is compactly contained in $f_n(2t \times \mathbb{D}_r)$, hence

$$H \circ f_{n+1}(t,z) = \begin{pmatrix} p(\varphi_t^{n+1}(z)) + a^2 w + az \\ a\varphi_t^{n+1}(z) \end{pmatrix} = \begin{pmatrix} \varphi_{2t}^n(a\varphi_t^{n+1}(z)) \\ a\varphi_t^{n+1}(z) \end{pmatrix}$$

and in particular

$$p(\varphi_t^{n+1}(z)) + a^2 w + az = \varphi_{2t}^n(a\varphi_t^{n+1}(z)). \tag{4.2}$$

Consider the sequence of equipotentials $\gamma_n(t)$ as defined in equation 2.1. Since the Julia set J_p is connected, $p'(\gamma_n(t))$ does not vanish. Moreover, if p is parabolic, $p'(\gamma(t))$ does not vanish either, where γ is the Charatéodory loop of the parabolic polynomial p. We have the following two relations

$$\begin{aligned} \gamma_{n+1}(t) &= p^{-1}(\gamma_n(2t)) \\ \left(p^{-1}\right)'(\gamma_n(2t)) &= \frac{1}{p'(\gamma_{n+1}(t))}. \end{aligned}$$

Note that for $n = 0$, $p(\varphi_t^1(z)) + a^2 w + az = \gamma_0(2t)$ so for a sufficiently small the following expansion holds

$$\begin{aligned} \varphi_t^1(z) &= p^{-1}\left(\gamma_0(2t) - az - a^2 w\right) \\ &= p^{-1}(\gamma_0(2t)) - \left(p^{-1}\right)'(\gamma_0(2t))az + O(a^2) \\ &= \gamma_1(t) - \frac{az}{p'(\gamma_1(t))} + O(a^2). \end{aligned}$$

We show by induction that for $n \geq 1$

$$\varphi_t^n(z) = \gamma_n(t) - \frac{az}{p'(\gamma_n(t))} + O(a^2).$$

Indeed, rearranging equation 4.2 yields

$$\begin{aligned}
\varphi_t^{n+1}(z) &= p^{-1}\left(\varphi_{2t}^n(a\varphi_t^{n+1}(z)) - az - a^2w\right)\\
&= p^{-1}\left(\gamma_n(2t) - \frac{a^2\varphi_t^{n+1}(z)}{p'(\gamma_n(2t))} - az + O(a^2)\right) = p^{-1}\left(\gamma_n(2t) - az + O(a^2)\right)\\
&= p^{-1}(\gamma_n(2t)) - \left(p^{-1}\right)'(\gamma_n(2t))az + O(a^2)\\
&= \gamma_{n+1}(t) - \frac{az}{p'(\gamma_{n+1}(t))} + O(a^2).
\end{aligned}$$

Since the polynomial p is quadratic, $p'(\gamma_n(t)) = 2\gamma_n(t)$. Letting $n \to \infty$ we get the expansion for $\varphi_t(z)$. □

If p is a hyperbolic polynomial and we are in the context of Theorem 4.1 then the same expansion for $\varphi_t(z)$ holds. The proof is the same, except that we do not have the term a^2w.

Proposition 4.4. *Let p be hyperbolic or parabolic. For sufficiently small $|a| > 0$ the map σ is open and injective. Also $\sigma(\mathbb{S}^1 \times \mathbb{D}_r) \subset \mathbb{S}^1 \times \mathbb{D}_{|a|\rho}$, with $\rho < r$.*

Proof. If p be hyperbolic or parabolic then there are no critical points in J_p and there exists $\epsilon > 0$ such that if $\xi_1 \neq \xi_2 \in J_p$ such that $p(\xi_1) = p(\xi_2)$ then $|\xi_1 - \xi_2| > \epsilon$. Thus when p is hyperbolic or parabolic $|\gamma(t) - \gamma(t + 1/2)| > \epsilon$ for $t \in \mathbb{S}^1$. From Lemma 4.3 there exists $M > 0$ such that $|\varphi_t(z) - \gamma(t)| < |a|M$ for all $t \in \mathbb{S}^1$ and $z \in \mathbb{D}_r$. Then for $|a| < \frac{\epsilon}{2M}$ the map σ is injective. It is also open because locally it is a homeomorphism. □

Proposition 4.5. *Consider $f^*(t, z) = (\varphi_t(z), z)$ and suppose that $f^*(t_1, z_1) = f^*(t_2, z_2)$ for some $t_1, t_2 \in \mathbb{S}^1$ and $z_1, z_2 \in \mathbb{D}_r$. Then $\varphi_{t_1}(z) = \varphi_{t_2}(z)$ for all $z \in \mathbb{D}_r$.*

Proof. If $f^*(t_1, z_1) = f^*(t_2, z_2)$ then $(\varphi_{t_1}(z_1), z_1) = (\varphi_{t_2}(z_2), z_2)$, hence $z_1 = z_2$ and $\varphi_{t_1}(z_1) = \varphi_{t_2}(z_1)$.

Denote by $s : \mathbb{D}_r \to \mathbb{D}_r$ the holomorphic function

$$s(z) = \varphi_{t_1}(z) - \varphi_{t_2}(z)$$

and assume that $s(z)$ has an isolated zero at z_1 of order m.

The functions $\varphi_{t_1}(z)$ and respectively $\varphi_{t_2}(z)$ were obtained as the limit of the uniformly convergent sequence of holomorphic functions $\varphi^n_{t_1}(z)$ and respectively $\varphi^n_{t_2}(z)$. By Hurwitz's theorem, there exists $\rho > 0$ such that for sufficiently large $n > n_0$, the function $\varphi^n_{t_1}(z) - \varphi^n_{t_2}(z)$ has exactly m zeros in the disk $|z - z_1| < \rho$. This is a contradiction, since by construction

$$\varphi^n_{t_1}(z) \neq \varphi^n_{t_2}(z) \text{ for any } n \geq 0 \text{ and } z \in \mathbb{D}_r.$$

Hence z_1 cannot be an isolated zero of the function s on $\mathbb{D}_r$. It follows that s vanishes identically on $\mathbb{D}_r$ and so $\varphi_{t_1}(z) = \varphi_{t_2}(z)$ for all $z \in \mathbb{D}_r$. □

The fixed point $f^*(t, z) = (\varphi_t(z), z)$ depends on the parameter a. We will use the notation $f^*_a(t, z) = (\varphi_t(z, a), z)$ whenever we want to stress out the dependence on a. Let $\delta > 0$ be chosen as in Theorem 4.1.

Proposition 4.6. *Fix $z \in \mathbb{D}_r$ and $a' \in \mathbb{D}_\delta$ and assume that $\varphi_{t_1}(z, a') = \varphi_{t_2}(z, a')$ for some $t_1, t_2 \in \mathbb{S}^1$. Then $\varphi_{t_1}(z, a) = \varphi_{t_2}(z, a)$ for any a with $|a| < \delta$.*

Proof. Let $s : \mathbb{D}_\delta \to \mathbb{D}_r$ be the holomorphic function $s(a) = \varphi_{t_1}(z, a) - \varphi_{t_2}(z, a)$. Denote by s_n the holomorphic functions $s_n(a) = \varphi^n_{t_1}(z, a) - \varphi^n_{t_2}(z, a)$.

For any $n \geq 0$, and any a with $|a| < \delta$, we have

$$\varphi^n_{t_1}(z, a) \neq \varphi^n_{t_2}(z, a),$$

by construction. Hence $s_n(a) \neq 0$ for any $n \geq 0$ and any a with $|a| < \delta$.

The sequence s_n converges uniformly to s on $\mathbb{D}_\delta$. By Hurwitz, s has either no zeros on $\mathbb{D}_\delta$ or vanishes identically on $\mathbb{D}_\delta$. Since we know that $s(a_1) = 0$ it follows that s vanishes identically, thus

$$\varphi_{t_1}(z, a) = \varphi_{t_2}(z, a) \text{ for any } a \text{ with } |a| < \delta \qquad \square$$

Proposition 4.7. *Consider $t_1 \neq t_2 \in \mathbb{S}^1$. The following statements are equivalent*

a) $f_a^*(t_1, z) = f_a^*(t_2, z)$ *for some a with $|a| < \delta$ and some $z \in \mathbb{D}_r$.*

b) $f_a^*(t_1, z) = f_a^*(t_2, z)$ *for any $z \in \mathbb{D}_r$ and for any a with $|a| < \delta$.*

c) $\gamma(t_1) = \gamma(t_2)$.

Proof. By Proposition 4.5 we know that $f_a^*(t_1, z) = f_a^*(t_2, z)$ for any $a \in \mathbb{D}_\delta$. In particular when $a = 0$ we must have $f_0^*(t_1, z) = f_0^*(t_2, z)$. This is equivalent to $(\gamma(t_1), z) = (\gamma(t_2), z)$, hence $\gamma(t_1) = \gamma(t_2)$.

By Proposition 4.6, we know that $f_a^*(t_1, z) = f_a^*(t_2, z)$ for any $z \in \mathbb{D}_r$. $\square$

4.2 Conjugating to a model map

This allows us to determine the equivalence classes of f^*. We define an equivalence relation $\sim$ on $\mathbb{S}^1 \times \mathbb{D}_r$ so that $(t_1, z) \sim (t_2, z)$ whenever $\gamma(t_1) = \gamma(t_2)$. By Lemma 4.3 $\varphi_t(z)$ can be written as

$$\varphi_t(z) = \gamma(t) - \frac{az}{2\gamma(t)} + a^2\beta(t, z, a).$$

In view of Proposition 4.7 above, $\beta(t_1, z, a) = \beta(t_2, z, a)$ whenever $\gamma(t_1) = \gamma(t_2)$. Clearly $\sim$ is closed. We would like to identify the quotient space $\mathbb{S}^1 \times \mathbb{D}_r/_\sim$ with $J_p \times \mathbb{D}_r$ and the map σ on $\mathbb{S}^1 \times \mathbb{D}_r$ defined in Equation 4.1 with a similar map σ_p acting on $J_p \times \mathbb{D}_r$.

Consider a map $\sigma_p : J_p \times \mathbb{D}_r \to J_p \times \mathbb{D}_r$ of the form

$$\sigma_p(\xi, z) = \left(p(\xi), a\left(\xi - \frac{az}{2\xi} + a^2\beta\left(\gamma^{-1}(\xi), z, a\right)\right)\right). \tag{4.3}$$

It is well defined, in view of the discussion above.

The map $g : \mathbb{S}^1 \times \mathbb{D}_r/_\sim \to J_p \times \mathbb{D}_r$, $g(t, z) = (\gamma(t), z)$ is a homeomorphism which makes the diagram

$$\begin{array}{ccc} \mathbb{S}^1 \times \mathbb{D}_r/_\sim & \xrightarrow{g} & J_p \times \mathbb{D}_r \\ \sigma\downarrow & & \downarrow\sigma_p \\ \mathbb{S}^1 \times \mathbb{D}_r/_\sim & \xrightarrow{g} & J_p \times \mathbb{D}_r \end{array}$$

commute. The conjugacy follows directly from the fact that $p(\gamma(t)) = \gamma(2t)$.

The map σ_p on $J_p \times \mathbb{D}_r$ has the form

$$\sigma_p(\xi, z) = \left(p(\xi), a\xi - \frac{a^2 z}{2\xi} + O(a^3)\right).$$

and can be further conjugated to a solenoidal map

$$\psi(\xi, z) = \left(p(\xi), a\xi - \frac{a^2 z}{2\xi}\right).$$

For $|a| > 0$ small enough σ_p and ψ are well-defined, open, and injective. Both maps depend on a and we will use the notation ψ_a to mark the dependence of ψ on a, but we will use ψ when there is no confusion. We will show that for $0 < |a| < \delta$ all ψ_a are conjugate to each other. Fix ϵ so that $0 < \epsilon < \delta$. Then ψ_a and ψ_ϵ are conjugate and ψ_ϵ does not depend on a.

Lemma 4.8. *There is a homeomorphism $h : J_p \times \mathbb{D}_r \to J_p \times \mathbb{D}_r$ conjugating σ_p to ψ.*

Proof. We first show that there exists a homeomorphism

$$h : J_p \times \mathbb{D}_r - \sigma_p(J_p \times \mathbb{D}_r) \to J_p \times \mathbb{D}_r - \psi(J_p \times \mathbb{D}_r)$$

which is the identity on the outer boundary $J_p \times \partial\mathbb{D}_r$ and given by the formula

$$h(\xi, z) = \psi \circ \sigma_p^{-1}(\xi, z)$$

on the inner boundary $\sigma_p(J_p \times \partial\mathbb{D}_r)$. Define the space $\mathcal{H}$ of fiber homeomorphisms

$$J_p \times \mathbb{D}_r - \sigma_p(J_p \times \mathbb{D}_r) \to J_p \times \mathbb{D}_r - \psi(J_p \times \mathbb{D}_r)$$

that agree with h on the boundary as a fiber bundle over J_p. Let $\xi \in J_p$ and let $\mathcal{H}_\xi$ be the fiber above ξ in $\mathcal{H}$. The fiber above ξ in the range of the homeomorphism h is a disk of radius r with two disjoint disks of radius $\frac{|a|^2 r}{2}$ removed, that is

$$\mathbb{D}_r - \bigcup_{\zeta \in p^{-1}(\xi)} \mathbb{D}_{\frac{|a|^2 r}{2}}(a\zeta).$$

There are d such disks removed if the polynomial has degree d. Similarly, the fiber above ξ in the domain is the disk $\mathbb{D}_r$ with two disjoint simply connected domains removed. These are topological disks of center $a\zeta + O(a^3)$ and radius at most $\frac{|a|^2 r}{2} + O(|a|^3)$, for all $\zeta \in p^{-1}(\xi)$.

In $\mathcal{H}_\xi$ we consider only those fiber homeomorphisms h' which agree with h on the boundary and which move all points by at most $O(|a|^3)$. Since the term $O(|a|^3)$ is much smaller compared to $\frac{|a|^2 r}{2}$ when a is small, there are no Dehn twists created as ξ moves on J_p. Therefore all such homeomorphisms are homotopic and this defines a preferred class of homeomorphisms. Note that $\mathcal{H}_\xi$ is not empty. Furthermore, $\mathcal{H}_\xi$ is contractible. This argument is similar to Lemma 6.8 in [HOV2] and follows from a theorem of Hamstrom (which states that if S is a compact surface with nonempty boundary – in our case a disk with two disjoint disks removed – then the components of the group of homeomorphisms which are the identity on the boundary are contractible).

$\mathcal{H}$ is a locally trivial fiber bundle over J_p, with contractible fibers. A fiber bundle with contractible fibers over a paracompact base has a continuous section. Hence there exists a map $s : J_p \to \mathcal{H}$, $s(\xi) = h_\xi$, which associates to each ξ a homeomophism h_ξ, so that the choice is continuous with respect to ξ. Set h to be s.

We now extend h on the inner levels by the dynamics, so we can construct a homeomorphism

$$h : J_p \times \mathbb{D}_r - \bigcap_{n\geq 0} \sigma_p^{\circ n}(J_p \times \mathbb{D}_r) \to J_p \times \mathbb{D}_r - \bigcap_{n\geq 0} \psi^{\circ n}(J_p \times \mathbb{D}_r)$$

which conjugates σ_p to ψ. Furthermore, we extend to the Cantor set (in each fiber) by continuity. □

Lemma 4.9. *Let $0 < |a| < \delta$. There is a homeomorphism $h_{a,\epsilon} : J_p \times \mathbb{D}_r \to J_p \times \mathbb{D}_r$ conjugating ψ_a to ψ_ϵ.*

Proof. We can prove this directly, in the same way as Lemma 4.8 above. We would need to consider the space of homeomorphisms $\mathcal{H}$ and construct a preferred class of homeomorphisms. □

Consider the linear change of variables $(\xi, z) \to (\xi, az)$. For $|a| > 0$ this transformation conjugates $\psi_a : J_p \times \mathbb{D}_r \to J_p \times \mathbb{D}_r$ to a map $\psi'_a : J_p \times \mathbb{D}_{r'} \to J_p \times \mathbb{D}_{r'}$, where $r' = \frac{r}{|a|}$ and

$$\psi'_a(\xi, z) = \left(p(\xi), \xi - \frac{a^2 z}{2\xi}\right). \tag{4.4}$$

Similarly ψ_ϵ is conjugate to ψ'_ϵ. Note that all these maps depend on the polynomial p. When p is hyperbolic, Lemma 4.9 is Proposition 6.13 from [HOV2], and the situation is not very different when p is parabolic.

Remark 4.10. The map ψ'_ϵ is the same model map that was used in [HOV2] in understanding Hénon maps that are small perturbations of hyperbolic polynomials. It is the same model map f_p that we will use in the last chapter 5. We also recover the following theorem below.

Theorem 4.11 ([HOV2])**.** *Let p be a hyperbolic polynomial with connected Julia set. There exists $\delta > 0$ such that if $0 < |a| < \delta$ then there exists a homeomorphism $\breve{\Phi}$ that makes the diagram*

$$\begin{array}{ccc} \varinjlim(J_p \times \mathbb{D}_r, \psi_\epsilon) & \xrightarrow{\breve{\Phi}} & J^+ \\ \breve{\psi}_\epsilon \big\downarrow & & \big\downarrow H_{p,a} \\ \varinjlim(J_p \times \mathbb{D}_r, \psi_\epsilon) & \xrightarrow{\breve{\Phi}} & J^+ \end{array}$$

commute.

We now have all the ingredients to complete the proof of the structure theorem for semi-parabolic Hénon maps, Theorem 3.3, and its corollaries.

Proof of Theorem 3.3. The proof follows directly from Theorem 4.2, Lemma 4.8, and Lemma 4.9. □

Note that $J = \bigcap_{n\geq 0} H^{\circ n}(J^+ \cap \overline{V})$. Let $\Sigma := \bigcap_{n\geq 0} \sigma^{\circ n}(\mathbb{S}^1 \times \mathbb{D}_r)$. Then Σ is a (dyadic) solenoid for $0 < |a| < \delta$ and in view of Theorem 4.2, Proposition 4.7, and the above discussion, we can present J as a quotiented solenoid, $J \simeq \Sigma/_\sim$. More directly, we can regard J as

$$J \simeq \bigcap_{n\geq 0} \psi_\epsilon^{\circ n}(J_p \times \mathbb{D}_r).$$

Theorem 4.12. *Let $p(x) = x^2 + c_0$ be a polynomial with a parabolic fixed point of multiplier $\lambda = e^{2\pi i p/q}$. There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_\lambda$ with $0 < |a| < \delta$ there exists a homeomorphism g^* wich is continuous with respect to t and analytic in a and z and which makes the diagram*

$$\begin{array}{ccc} J_p \times \mathbb{D}_r & \xrightarrow{g^*} & J^+ \cap \overline{V} \\ \sigma_p \downarrow & & \downarrow H_{c,a} \\ J_p \times \mathbb{D}_r & \xrightarrow{g^*} & J^+ \cap \overline{V} \end{array}$$

commute.

Proof. The homeomorphism g^* is a composition between f^* and the map g defined above. The model map σ_p is defined in Equation 4.3. □

Corollary 4.12.1. The Julia set J is equal to J^*, the closure of the saddle periodic points.

Proof. The Julia set J is homeomorphic to a solenoid with identifications. Since the periodic points are dense in the solenoid, we get that J is the closure of the periodic points of the Hénon map. Let $x_a \in J$ be a periodic point of period k of the Hénon map H_a, different from the semi-parabolic fixed point $\mathbf{q}_a$. The periodicity of x_a induces a periodicity on the disks that foliate $J_p \times \mathbb{D}_r$, namely there exists a periodic point $\xi \in J_p$, $p^{\circ k}(\xi) = \xi$ of the parabolic polynomial p such that $x_a \in g^*(\xi \times \mathbb{D}_r)$ and $\sigma_p^{\circ k}(\xi \times \mathbb{D}_r)$ is compactly contained inside $\xi \times \mathbb{D}_r$. Note that $\xi \neq q_0$, where q_0 is the parabolic fixed point of p. The conjugacy map $g^*(\xi, z)$ is holomorphic with respect to z, so the stable multipliers of the Hénon map coincide with the stable multipliers of the model map

$$\sigma_p(\xi, z) = \left(p(\xi), a\xi - \frac{a^2 z}{p'(\xi)} + O(a^3)\right).$$

Let $\lambda^{s/u}$ be the eigenvalues of $DH^{\circ k}_{x_a}$. Then $\lambda^s = O(a^{2k})$ and $\lambda^u = (p^{\circ n})'(\xi) + O(a)$. The function g^* is holomorphic with respect to a, so the disks that foliate $J^+ \cap \overline{V}$ move holomorphically with a. The point x_a moves holomorphically with a and we have $x_a \to \xi$ as $a \to 0$.

The polynomial Julia set J_p is the closure of the repelling periodic points [M]. By the Fatou-Shishikura inequality [S], a polynomial of degree $d \geq 2$ has at most $d - 1$ non-repelling cycles. Since p is quadratic and has a parabolic fixed point q_0, all other periodic cycles are repelling. Therefore $|(p^{\circ n})'(\xi)| > 1$. Clearly, when a is small, $|\lambda^u| > 1$ and $|\lambda^s| < 1$, so the periodic point x_a is a saddle point of the Hénon map.

Let δ be as in Theorem 3.3. We show that the periodic point x_a is saddle. It is easy to see that $|\lambda^s| < 1$, so we only need to show that $|\lambda^u| > 1$. Assume that $|\lambda^u| = 1$ for some parameter a_0 with $0 < |a_0| < \delta$. Then we can perturb a_0 so that $|\lambda^u|$ becomes strictly smaller than 1. Otherwise $1/|\lambda^u|$ would have a local maximum at a_0, which is not possible. Thus we can find a parameter a close to a_0 for which x_a is a sink, and as such it must belong to the interior of K^+ and not to J^+. This is a contradiction. Thus all periodic points are saddles, except the semi-parabolic fixed point $\mathbf{q}_a$. It follows that $J = J^*$. □

Remark 4.13 (Higher degrees). This technique and the results from Section 3.2 can be generalized to Hénon maps that are small perturbations of a polynomials p of degree $d \geq 2$ whose critical points are attracted either to attractive or parabolic cycles. The model map becomes

$$\psi(\xi, z) = \left(p(\xi), a\xi - \frac{a^2 z}{p'(\xi)}\right).$$

CHAPTER 5

GLOBAL TOPOLOGICAL MODELS

In one-dimensional dynamics the pinched disk model for polynomial Julia sets (as described by Thurston [Th]) is an important tool in understanding the geometry of connected Julia sets. Thurston models the filled-in Julia set as a quotient of the unit disk $\mathbb{D}$ along the leaves of a lamination defined inside the disk.

So far we have seen how to build topological models for Hénon maps that are small perturbations of a polynomial p by using the Julia set J_p as a building block. In this chapter we wish to build a global model for J^+ by using the lamination of the polynomial p as a building block.

The idea is to define a lamination for the Hénon map by lifting the Thurston lamination of the polynomial p from the closed unit disk to the unit 4-ball in $\mathbb{C}^2$, using the inductive limit. Lifting the leaves of the lamination of the polynomial, gives a lamination for the Hénon map, whose leaves (inside the 4-ball) connect a finite number of stable manifolds in $\mathbb{S}^3 - \Sigma^-$, where Σ^- is a repelling solenoid. The stable manifolds that are identified are the stable manifolds of a periodic orbit on the attractive solenoid Σ^+.

5.1 Preliminary tools

Consider $H_{p,a} : \mathbb{C}^2 \to \mathbb{C}^2$ be a complex Hénon map defined by

$$H_{p,a}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p(x) - ay \\ x \end{pmatrix},$$

where p is a monic quadratic polynomial. We wish to build a global model for Hénon maps $H_{p,a}$ that are small perturbations of a polynomial p. We will consider both cases: p is hyperbolic, as described in Chapter 2, or p is parabolic as described in Chapter 3.

5.1.1 Inductive and projective limits

As we have seen, the inductive limit is an important tool in the characterization of the set J^+.

If $f : X \to X$ is an open, injective map from a space X to itself, then $\varinjlim(X, f) = X \times \mathbb{N}/\sim$, where the equivalence relation is defined by $(x, n) \sim (f(x), n + 1)$.

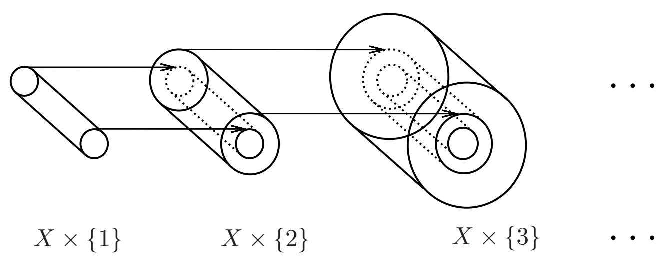

Figure 5.1: The inductive limit of $[0, 1] \times \mathbb{D}$ under $f(t, z) = (t, z/2)$ is $[0, 1] \times \mathbb{C}$.

The inductive limit is an increasing union of sets homeomorphic to X, so locally it looks like X. The limit space $\check{X} = \varinjlim(X, f)$ comes with a natural bijective map $\check{f} : \check{X} \to \check{X}$ given by

$$(x, n) \mapsto (f(x), n) \sim (x, n - 1).$$

Assume the critical points of p are attracted to attractive or parabolic cycles. The filled Julia set K_p is connected and locally connected, and none of the critical points of p belong to the Julia set J_p [DH]. This is enough to say that $J_p \subset D$, where D is a complex disk of radius 2 [Bu].

Consider the map $f_p : J_p \times D \to J_p \times D$, defined by

$$f_p(\xi, z) = \left(p(\xi), \xi + \varepsilon \frac{z}{p'(\xi)}\right). \tag{5.1}$$

For sufficiently small $\varepsilon > 0$ the map f_p is well defined, open, and injective [HOV2, Lemma 1.2]. It is in fact the same map that was used in Theorem 3.3 and Equation 4.4 (up to an appropriate conjugacy). Furthermore, let

$$\check{\mathbb{C}}_p := \varinjlim (J_p \times D, f_p)$$

and consider the induced map $\check{f}_p : \check{\mathbb{C}}_p \to \check{\mathbb{C}}_p$ acting on it.

Projective limits. For a map $f : X \to X$ the space $\widehat{X} = \varprojlim (X, f)$ is

$$\widehat{X} = \{(\ldots, z_{-2}, z_{-1}, z_0) \mid z_0 \in X \text{ and } f(z_{-i-1}) = z_{-i}, \text{ for all } i \in \mathbb{N}\}.$$

The map f induces a bijective map $\widehat{f} : \widehat{X} \to \widehat{X}$,

$$\widehat{f}(\ldots, z_{-2}, z_{-1}, z_0) = (\ldots, f(z_{-2}), f(z_{-1}), f(z_0)) = (\ldots, z_{-1}, z_0, f(z_0)).$$

5.1.2 Pinched disk model

A complete description of quadratic invariant laminations and the pinched disk model is given in [Th]. We only give a brief outline here. Let $p(z) = z^2 + c$. Let K_p and $J_p = \partial K_p$ be the filled Julia set, respectively the Julia set of p.

If K_p is connected then there exists a unique analytic isomorphism

$$\psi_p : \mathbb{C} - \overline{\mathbb{D}} \to \mathbb{C} - K_p$$

such that $\psi_p(z^2) = p(\psi_p(z))$ and $\psi_p(z)/z \to 1$ as $z \to \infty$.

The set $\{\psi_p(re^{2\pi i\theta}) : r > 1\}$ is called the *external ray* of angle θ. A ray lands at a point $z \in J_p$ if the limit $\lim_{r \searrow 1} \psi_p(re^{2\pi i\theta})$ exists and is equal to z. If J_p locally connected then all rays land and ψ_p extends to the boundary $\mathbb{S}^1$ and defines a continuous, surjective map $\gamma : \mathbb{S}^1 \to J_p$ [M].

Define the associated invariant lamination $\mathcal{L}$ as follows:

- if two rays of angles θ, θ' land at a common point, then there is a leaf $\overline{\theta\theta'}$ in the lamination $\mathcal{L}$
- if more than two rays land together, then take a leaf for each pair of adjacent angles to form a polygonal gap in $\mathcal{L}$.

The quotient space $\tilde{\mathcal{L}}$ is obtained by collapsing each leaf and polygonal gap to a point. There exists a homeomorphism $\pi : \widetilde{\mathcal{L}} \to K_p$ and the space $\widetilde{\mathcal{L}}$ is the *pinched disk* model of K_p.

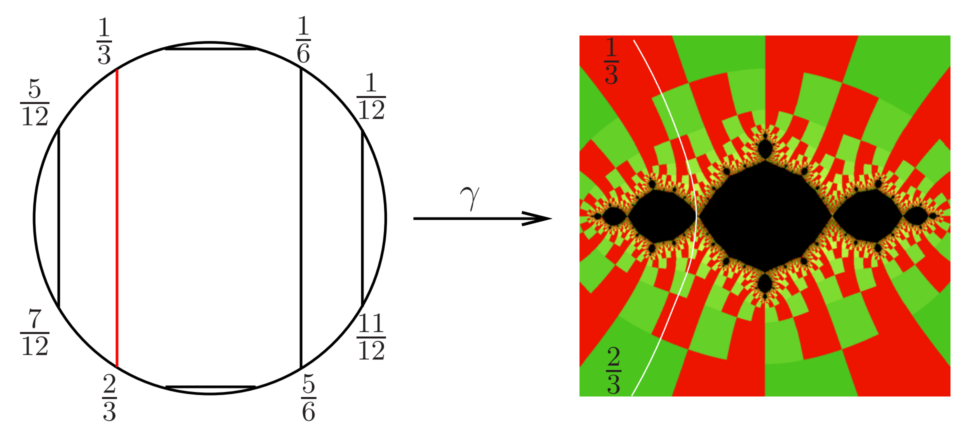

Figure 5.2: A set of leaves of the lamination for $z \mapsto z^2 - 1$. The minor leaf is shown in red.

5.2 The model space $\mathbb{S}^3 - \Sigma^-$

The unit sphere $\mathbb{S}^3$ can be written as a union of two solid tori $\mathbb{T}_0$ and $\mathbb{T}_1$, glued along their boundaries. After rescaling, we can assume that $\mathbb{T}_0$ is in the standard position $\mathbb{T}_0 = \mathbb{S}^1 \times D$.

The map $f_0 : \mathbb{T}_0 \to \mathbb{T}_0$ is a solenoidal map and it extends to a homeomorphism $\sigma : \mathbb{S}^3 \to \mathbb{S}^3$ [HOV1], defined by

$$\sigma(\xi, z) = \left(\xi^2, \xi + \epsilon \frac{z}{\xi}\right) \quad \text{on } \mathbb{T}_0, \tag{5.2}$$

which has two invariant solenoids

$$\Sigma^+ = \bigcap_{n \geq 0} \sigma^n(\mathbb{T}_0) \text{ and } \Sigma^- = \bigcap_{n \geq 0} \sigma^{-n}(\mathbb{T}_1) \tag{5.3}$$

attracting, respectively repelling.

The limit space $\varinjlim(\mathbb{S}^1 \times D, f_0)$ is homeomorphic to $\mathbb{S}^3 - \Sigma^-$ [HOV1]. The map $\breve{f}_0$ acting on it is identified as the restriction of the solenoidal map σ:

$$\begin{array}{ccc} \varinjlim(\mathbb{S}^1 \times D, f_0) & \xrightarrow{\simeq} & \mathbb{S}^3 - \Sigma^- \\ \breve{f}_0 \downarrow & & \downarrow \sigma \\ \varinjlim(\mathbb{S}^1 \times D, f_0) & \xrightarrow{\simeq} & \mathbb{S}^3 - \Sigma^- \end{array}$$

We write $\mathbb{R}^4$ in polar coordinates as $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^3$ and consider the cones over the two solenoids $cone(\Sigma^\pm) = \{(r, \theta) \mid r \geq 1 \text{ and } \theta \in \Sigma^\pm\}$. The model space is

$$X = \mathbb{R}^4 - cone(\Sigma^-).$$

Bonnot [Bo] gave a complete characterization on $\mathbb{C}^2$ of those Hénon maps that are small perturbations of $p(x) = x^2 + c$, where c is taken from the interior of the Mandelbrot set. The lamination of the polynomial p is trivial in this case (no pinching), so the lamination of the Hénon map has no pinching.

Theorem 5.1 (Bonnot [Bo]). *Let $p(x) = x^2 + c$, with c from the interior of the main cardioid of the Mandelbrot set. Then there is $\delta > 0$ such that if $0 < |a| < \delta$ there exist a homeomorphism Φ for which the diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & \mathbb{C}^2 \\ {\scriptstyle (r,\theta)\mapsto(r^2,\sigma(\theta))}\downarrow & & \downarrow{\scriptstyle H_{p,a}} \\ X & \xrightarrow{\Phi} & \mathbb{C}^2 \end{array}$$

In this model, $J^+ = \mathbb{S}^3 - \Sigma^-$ while K^+ is the closed unit 4-ball with Σ^- removed. Moreover, $K^- = J^- = cone(\Sigma^+)$.

If c is taken from the interior of another hyperbolic component of the interior of the Mandelbrot set, then the Thurston lamination of $p(x) = x^2 + c$ is nontrivial. When we perturb from such a polynomial, then the set J^+ is a complicated fractal object and it is nowhere a topological manifold [FS].

5.3 The model for J^+

Let p be a hyperbolic quadratic polynomial with connected Julia set. The map f_p has the form

$$f_p(\xi, z) = \left(p(\xi), \xi + \varepsilon\frac{z}{\xi}\right). \tag{5.4}$$

For simplicity, $f_0 : \mathbb{S}^1 \times D \to \mathbb{S}^1 \times D$ will denote f_p when $p(x) = x^2$, that is

$$f_0(\xi, z) = \left(\xi^2, \xi + \varepsilon\frac{z}{\xi}\right). \tag{5.5}$$

For $\epsilon > 0$ small enough f_0 and f_p are well-defined, open, and injective.

Standard assumption: Let γ be the Carathéodory loop of p. We assume that $\arg(\gamma(\xi)) - \arg(\xi) \neq \pm\pi/2$, for all $\xi \in \mathbb{S}^1$. We believe this to be true for p hyperbolic or parabolic, but in any case, we will work under this assumption. Note that it can be checked by hand that this condition on γ is satisfied for $p(x) = x^2 - 1$.

Theorem 5.2. *Let p be a hyperbolic quadratic polynomial with connected Julia set. There exists $\delta > 0$ such that for all $0 < |a| < \delta$ there exists a continuous surjective map $\Phi : \mathbb{S}^3 - \Sigma^- \to J^+$ which makes the diagram*

$$\begin{array}{ccc} \mathbb{S}^3 - \Sigma^- & \xrightarrow{\Phi} & J^+ \\ \sigma\big\downarrow & & \big\downarrow H_{p,a} \\ \mathbb{S}^3 - \Sigma^- & \xrightarrow{\Phi} & J^+ \end{array}$$

commutative.

Proof. The proof follows from the following theorem.

Theorem 5.3. *There is a continuous surjective map $\varphi : \mathbb{S}^1 \times D \to J_p \times D$ such that the diagram*

$$\begin{array}{ccc} \mathbb{S}^1 \times D & \xrightarrow{\varphi} & J_p \times D \\ f_0\big\downarrow & & \big\downarrow f_p \\ \mathbb{S}^1 \times D & \xrightarrow{\varphi} & J_p \times D \end{array}$$

commutes.

Passing to the inductive limit

$$\begin{array}{ccccccc} \mathbb{S}^3 - \Sigma^- & \xrightarrow{\simeq} & \varinjlim(\mathbb{S}^1 \times D, f_0) & \xrightarrow{\breve{\varphi}} & \varinjlim(J_p \times D, f_p) & \xrightarrow{\simeq} & J^+ \\ \sigma\big\downarrow & & \breve{f}_0\big\downarrow & & \big\downarrow \breve{f}_p & & \big\downarrow H \\ \mathbb{S}^3 - \Sigma^- & \xrightarrow{\simeq} & \varinjlim(\mathbb{S}^1 \times D, f_0) & \xrightarrow{\breve{\varphi}} & \varinjlim(J_p \times D, f_p) & \xrightarrow{\simeq} & J^+ \end{array}$$

this gives $J^+ = \mathbb{S}^3 - \Sigma^- / \sim_{\breve{\varphi}}$ and so the Hénon map H is conjugated on J^+ to a solenoidal map σ on $\mathbb{S}^3 - \Sigma^- / \sim_{\breve{\varphi}}$. □

Corollary 5.3.1. J^+ is homeomorphic to $(\mathbb{S}^3 - \Sigma^-)/_{\sim_\Phi}$, where $\sim_\Phi$ is the equivalence relation induced by the map Φ.

The simplest case where nontrivial identifications occur on $\mathbb{S}^3 - \Sigma^-$ is for small perturbations of a polynomial p, where p is selected from the second hyperbolic component of the Mandelbrot set. Throughout this section, for simplicity, whenever we draw pictures, we assume $p(x) = x^2 + c$, where $|c+1| < 1/4$.

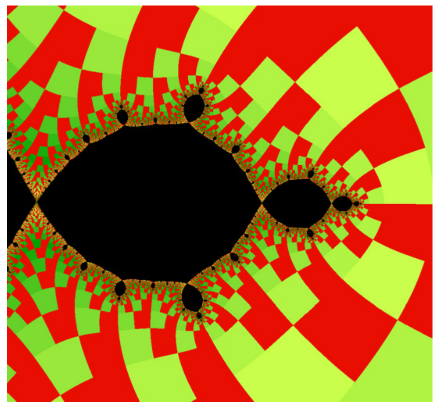

Figure 5.3: The picture of an unstable manifold $W^u(q)$ of a saddle fixed point q for a Hénon map that is a small perturbation of $z \mapsto z^2 - 1$.

Remark 5.4. External rays for the Hénon map were defined in [BS7] to understand the set J. The identifications of rays in the picture above are very similar to Figure 5.2 and were also described combinatorially in [O].

Lemma 5.5 (Main Lemma). *There exists a continuous surjective map*

$$\varphi : \mathbb{S}^1 \times D - f_0(\mathbb{S}^1 \times D) \to J_p \times D - f_p(J_p \times D)$$

defined on the boundary as follows:

(i) $\varphi(\xi, z) = (\gamma(\xi), z)$ *on the outer boundary* $\mathbb{S}^1 \times \partial D$;

(ii) $\varphi(\xi, z) = f_p \circ \varphi \circ f_0^{-1}(\xi, z)$ *on the inner boundary* $f_0(\mathbb{S}^1 \times \partial D)$.

Proof. Notice that $\gamma(-\xi) = -\gamma(\xi)$ and $\gamma(\xi^2) = \gamma(\xi)^2 + c$. The map f_0 maps the disks $D_{\sqrt{\xi}}$ and $D_{-\sqrt{\xi}}$ of radius 2 to two smaller disks of radius 2ϵ centered at $\sqrt{\xi}$, respectively at $-\sqrt{\xi}$, inside the disk D_ξ.

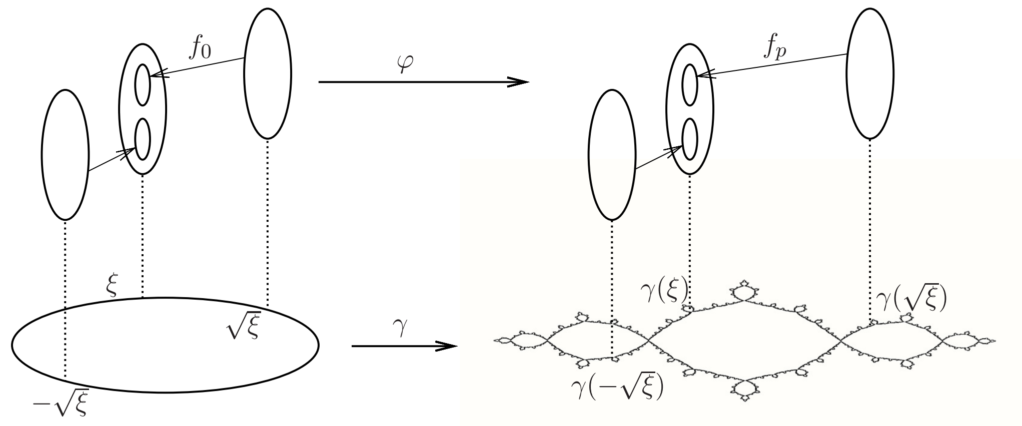

Figure 5.4: A fiber above ξ in $\mathbb{S}^1 \times D - f_0(\mathbb{S}^1 \times D)$. Its image under φ is a fiber above $\gamma(\xi)$ in $J_p \times D - f_p(J_p \times D)$.

Similarly, the map f_p maps the disks $D_{\gamma(\sqrt{\xi})}$ and $D_{\gamma(-\sqrt{\xi})}$ of radius 2 to two smaller disks of radius $\dfrac{2\epsilon}{|\gamma(\xi)|}$ centered at $\gamma(\sqrt{\xi})$, respectively at $-\gamma(\sqrt{\xi})$, inside the disk $D_{\gamma(\xi)}$.

Consider the straight line AB in $D_{\gamma(\xi)}$ passing through the centers of the two smaller disks inside. It intersects the boundary of $D_{\gamma(\xi)}$ in $A = 2\frac{\gamma(\sqrt{\xi})}{|\gamma(\sqrt{\xi})|}$ and $B = -A$. The line AB intersects the boundary of $f_p(D_{\gamma(\sqrt{\xi})})$ in P' and Q' where

$$\begin{aligned} P' &= \gamma(\sqrt{\xi}) + \frac{2\epsilon}{|\gamma(\sqrt{\xi})|^2}\gamma(\sqrt{\xi}) \\ Q' &= \gamma(\sqrt{\xi}) - \frac{2\epsilon}{|\gamma(\sqrt{\xi})|^2}\gamma(\sqrt{\xi}). \end{aligned}$$

The line AB intersects the boundary of $f_p(D_{\gamma(-\sqrt{\xi})})$ in P'' and Q'' where

$$\begin{aligned} P'' &= -\gamma(\sqrt{\xi}) - \frac{2\epsilon}{|\gamma(\sqrt{\xi})|^2}\gamma(\sqrt{\xi}) \\ Q'' &= -\gamma(\sqrt{\xi}) + \frac{2\epsilon}{|\gamma(\sqrt{\xi})|^2}\gamma(\sqrt{\xi}). \end{aligned}$$

The function φ is already defined on the outer and inner boundary. On the inner boundary, one can find the preimages P_1, Q_1, P_2, Q_2 in D_ξ of P', Q', P'', Q''.

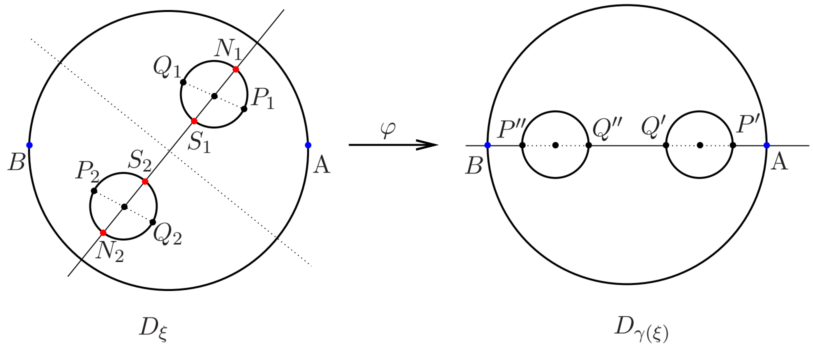

Figure 5.5: A fiber D_ξ above ξ in $\mathbb{S}^1 \times D - f_0(\mathbb{S}^1 \times D)$ and its corresponding fiber $D_{\gamma(\xi)}$ above $\gamma(\xi)$ in $J_p \times D - f_p(J_p \times D)$.

In order to find P_1, we set $\varphi(P_1) = P'$ and solve

$$\gamma(\sqrt{\xi}) + \epsilon\frac{z}{\gamma(\sqrt{\xi})} = \gamma(\sqrt{\xi}) + \frac{2\epsilon}{|\gamma(\sqrt{\xi})|^2}\gamma(\sqrt{\xi}).$$

This gives $z = 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2}$ and

$$P_1 = \sqrt{\xi} + \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2}.$$

Similarly we obtain

$$\begin{aligned} Q_1 &= \sqrt{\xi} - \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} \\ P_2 &= -\sqrt{\xi} - \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} \\ Q_2 &= -\sqrt{\xi} + \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2}. \end{aligned}$$

Consider the straight line in D_ξ passing through the centers of the two smaller disks inside. It intersects the boundary of $f_0(D_{\sqrt{\xi}})$ in $N_1 = (1 + 2\epsilon)\sqrt{\xi}$ and $S_1 = (1 - 2\epsilon)\sqrt{\xi}$. It intersects the boundary of $f_0(D_{-\sqrt{\xi}})$ in $N_2 = (-1 - 2\epsilon)\sqrt{\xi}$ and $S_2 = (-1 + 2\epsilon)\sqrt{\xi}$.

Remark 5.6. Symmetries of the picture inside D_ξ.

- The two smaller disks $f_0(D_{\sqrt{\xi}})$ and $f_0(D_{-\sqrt{\xi}})$ are symmetric with respect to the origin inside D_ξ. The points P_1 and P_2 are symmetric with respect to the origin, as well as Q_1 and Q_2.

- The points P_1 and Q_1 on the boundary $\partial f_0(D_{\sqrt{\xi}})$ are symmetric with respect to the center of the smaller disk $f_0(D_{\sqrt{\xi}})$, which is $\sqrt{\xi}$.

Lemma 5.7. *For any ξ on $\mathbb{S}^1$, $P_1 \neq S_1$ and $Q_1 \neq N_1$. Similarly $P_2 \neq S_2$ and $Q_2 \neq N_2$.*

Proof of Lemma. If $P_1 = S_1$ then

$$\sqrt{\xi} + \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} = (1 - 2\epsilon)\sqrt{\xi} \;\Rightarrow\; \frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} = -(\sqrt{\xi})^2.$$

If $Q_1 = N_1$ then

$$\sqrt{\xi} - \frac{\epsilon}{\sqrt{\xi}} \cdot 2\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} = (1 + 2\epsilon)\sqrt{\xi} \;\Rightarrow\; \frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} = -(\sqrt{\xi})^2.$$

However this equality gives

$$\frac{\gamma(\sqrt{\xi})^2}{|\gamma(\sqrt{\xi})|^2} = -(\sqrt{\xi})^2 \;\Leftrightarrow\; \left(\frac{\frac{\gamma(\sqrt{\xi})}{|\gamma(\sqrt{\xi})|}}{\sqrt{\xi}}\right)^2 = (\pm i)^2,$$

which is equivalent to $\arg(\gamma(\sqrt{\xi})) - \arg(\sqrt{\xi}) = \pm\pi/2$, which does not hold under our assumption on γ. □

Remark 5.8. The points P_1 and Q_1 are symmetric with respect to the center of the circle $f_0(D_{\sqrt{\xi}})$. It is possible that $P_1 = N_1$ and $Q_1 = S_1$, however it is not possible that $P_1 = S_1$ or $Q_1 = N_1$.

Remark 5.9. We can consider points A and B on the boundary of D_ξ as well, since the map $\varphi : D_\xi \to D_{\gamma(\xi)}$ is the identity on the (outer) boundary of D_ξ. For any ξ on $\mathbb{S}^1$, $A \neq \pm 2i\sqrt{\xi}$ and $B \neq \pm 2i\sqrt{\xi}$. This means that, as ξ moves on $\mathbb{S}^1$, point A moves on ∂D_ξ in the semicircle that encloses the small disk $f_0(D_{\sqrt{\xi}})$, while point B moves on ∂D_ξ in the semicircle that encloses the small disk $f_0(D_{-\sqrt{\xi}})$, as shown on the figure.

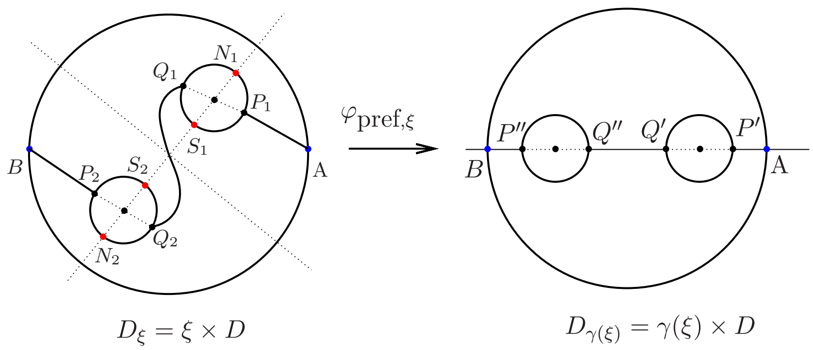

Figure 5.6: A fiber D_ξ above ξ in $\mathbb{S}^1 \times D - f_0(\mathbb{S}^1 \times D)$ and its corresponding fiber $D_{\gamma(\xi)}$ above $\gamma(\xi)$ in $J_p \times D - f_p(J_p \times D)$. The associated paths need to be shown in blue and red.

We construct two curves C_1 and C_2 joining A and B in the disk D_ξ. We form a path C_1 (shown in blue) in $\mathbb{D}_\xi$ by joining A to P_1 by a simple curve, P_1 to Q_1 by an arc (moving counterclockwise on the inner circle), Q_1 to Q_2 by a simple curve, Q_2 to P_2 by an arc (moving counterclockwise on the inner circle), and finally P_2 to B by a simple curve. In the same way we form a path C_2 in $\mathbb{D}_\xi$ by moving clockwise on the inner circles when joining P_1 and Q_1 and Q_2 to P_2.

We construct two curves C' and C' joining A and B in the disk $D_{\gamma(\xi)}$. We form a path C' (shown in blue) in $\mathbb{D}_{\gamma(\xi)}$ by joining A to P' by a line segment, P' to Q' by a semicircle (moving counterclockwise on the inner circle), Q' to Q'' by a line segment, Q'' to P'' by a semicircle (moving counterclockwise on the inner circle), and finally P'' to B by a line segment. Similarly, we form a path C'' in $\mathbb{D}_{\gamma(\xi)}$ by moving clockwise on the inner circles when joining P' to Q' and Q'' to P''.

Extend φ continuously so that it sends the path C_1 to C' and the path C_2 to C''. The function φ is now defined on the boundary of the blue disk, and can be extended radially to the interior. Call the extension function $\varphi_{\text{pref},\xi}$. Obviously the curves can be chosen continuously with respect to ξ.

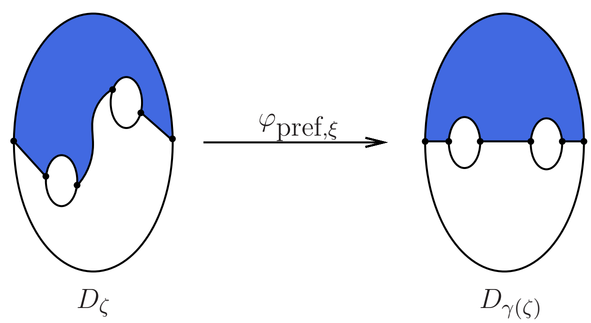

Figure 5.7: Fiber homeomorphism $\varphi_{\text{pref},\xi}$. It sends the center of mass of the blue region in D_ξ to the center of mass of the blue region in $D_{\gamma(\xi)}$ and extends radially.

One needs to check that $\varphi_{\text{pref},\xi}$ does not change homotopy type when ξ does a full turn on $\mathbb{S}^1$. Denote the last for simplicity by $\varphi_{\text{pref},\xi+1}$.

Lemma 5.10. *The functions $\varphi_{\text{pref},\xi}$ and $\varphi_{\text{pref},\xi+1}$ are homotopic, as functions from*

$$D_\xi - \left(f_0(D_{\sqrt{\xi}}) \cup f_0(D_{-\sqrt{\xi}})\right) \to D_{\gamma(\xi)} - \left(f_0(D_{\gamma(\sqrt{\xi})}) \cup f_0(D_{\gamma(-\sqrt{\xi})})\right).$$

First notice that after ξ does a full turn on $\mathbb{S}^1$, the inner disks in $\mathbb{D}_{\gamma(\xi)}$ interchange, as well as the ones in $\mathbb{D}_\xi$. Notice also that $\varphi_{\text{pref},\xi}$ and $\varphi_{\text{pref},\xi+1}$ coincide on $\partial(D_\xi) \cup \partial(f_0(D_{\sqrt{\xi}})) \cup \partial(f_0(D_{-\sqrt{\xi}}))$.

The point P_1 has moved continuously with respect to ξ on the boundary of the inner disk $f_0(\mathbb{D}_{\sqrt{\xi}})$, but it always avoids one point $P_1 \neq S_1$. Hence, after a full turn on $\mathbb{S}^1$, the curve AP_1 has produced no Dehn twist around the inner disks $f_0(\mathbb{D}_{\sqrt{\xi}})$. There is no Dehn twist around the two inner disk, because the curve AP_1 is only contained in the semi-disk around $\sqrt{\xi}$, so it never wraps around both inner disks $D_{\sqrt{\xi}}$ and $D_{-\sqrt{\xi}}$.

Denote by $\mathcal{H}$ the space of fiber homeomorphisms defined on

$$\mathbb{S}^1 \times D - f_0(\mathbb{S}^1 \times D) \to J_p \times D - f_p(J_p \times D)$$

that agree with φ on the boundary, such that restricted to each fiber $D_\xi \to D_{\gamma(\xi)}$ are homotopic to a preferred one (which we call φ_{pref} as below). The space $\mathcal{H}$ is a locally trivial fiber bundle over $\mathbb{S}^1$, with contractible fibers [Ham]:

$$\mathcal{H}_\xi = \Big\{\varphi_\xi : D_\xi - f_0(D_{\sqrt{\xi}} \cup D_{-\sqrt{\xi}}) \to D_{\gamma(\xi)} - f_p(D_{\gamma(\sqrt{\xi})} \cup D_{\gamma(\sqrt{\xi})}),$$
$$\varphi_\xi = \varphi \text{ on the boundary, and } \varphi_\xi \text{ homotopic to } \varphi_{\text{pref}}\Big\}.$$

A fiber bundle with contractible fibers over a paracompact base has a continuous section. Hence there exists $s : \mathbb{S}^1 \to \mathcal{H}$, $s(\xi) = \varphi_\xi$, which associates to each ξ a homeomophism φ_ξ, so that the choice is continuous with respect to ξ. Set φ to be s. □ **of Lemma 5.5**

We now extend φ on the inner levels by the dynamics and to the Cantor set by continuity. The proofs of the following propositions follow iteratively from the construction above.

Proposition 5.11. *The map φ extends to a continuous surjection*

$$\varphi : \mathbb{S}^1 \times D - \widehat{\mathbb{S}^1} \to J_p \times D - \widehat{J_p}.$$

Proposition 5.12. *The map φ extends to a continuous surjection $\varphi : \mathbb{S}^1 \times D \to J_p \times D$.*

Proof of Theorem 5.3. The proof of Theorem 5.3 follows from the propositions above and the main lemma 5.5. □

5.4 Laminations for the Hénon map

Denote by $L_p \subset \mathbb{D}$ the collection of the leaves of the lamination of the polynomial p. For simplicity, we represent the leaves as straight lines (instead of hyperbolic geodesics) inside the unit disk. Set

$$X_p := L_p \cup \mathbb{S}^1.$$

We would like to define a lamination similar to L_p for Hénon maps which are small perturbations of the polynomial p. We will "lift" the lamination of the polynomial p and create an appropriate lamination for the Hénon map $H_{p,a}$. The lamination for the Hénon map will be supported on $\mathbb{S}^3 - \Sigma^- := \varinjlim(\mathbb{S}^1 \times D, f_p)$. The rigorous definition of the lift will require taking an inductive limit of $L_p \times D$, under an appropriate extension of the solenoidal map f_p.

There is a natural extension of the solenoidal map f_p from $\mathbb{S}^1 \times D$ to the space $X_p \times D$, which is compatible with the dynamics of the polynomial p on L_p.

In general, one component of L_p is a convex n-gon with vertices $\xi_1, \xi_2, \ldots, \xi_n$ on $\mathbb{S}^1$. Let $g_p : X_p \to X_p$ be the function that maps convex combinations of the vertices $\xi_1, \xi_2, \ldots, \xi_n$ of a component of L_p to the same convex combination of the images of these vertices under $z \to z^2$, i.e.

$$g_p(t_1\xi_1 + \ldots + t_n\xi_n) = t_1\xi_1^2 + \ldots + t_n\xi_n^2,$$

where $t_1 + \ldots + t_n = 1$ and $0 \leq t_i \leq 1$. If $\xi \in \mathbb{S}^1$ then $g_p(\xi) = \xi^2$.

Suppose $p(z) = z^2 + c$ and $|c + 1| < 1/4$. In this case the lamination consists only of straight lines and there are no polygonal gaps. If ξ is a point on the leaf connecting ξ_1 and ξ_2, then $\xi = t\xi_1 + (1 - t)\xi_2$ for some $t \in [0, 1]$ and

$$g_p(\xi) = t\xi_1^2 + (1 - t)\xi_2^2.$$

Define $f_0 : X_p \times D \to X_p \times D$ as

$$f_0\,(\xi, z) = \left(g_p(\xi), \xi + \epsilon z \frac{|\xi|}{\xi}\right). \tag{5.6}$$

Note that this extension is well defined since p is hyperbolic and so X_p does not contain the critical point zero.

We can also naturally extend the conjugacy function φ from $\mathbb{S}^1 \times D$ to $X_p \times D$. Let φ be the identity on the outer boundary $X_p \times \partial\mathbb{D}$. Extend φ to the inner boundary of $X_p \times D - f_0(X_p \times \mathbb{D})$ by the dynamics. Let $\xi = t_1\xi_1 + \ldots + t_n\xi_n$ be a point in X_p for some $0 \leq t_i \leq 1$ with $t_1 + \ldots + t_n = 1$. Then

$$\varphi\left(g_p(\xi), \xi + \epsilon z \frac{|\xi|}{\xi}\right) := \left(\gamma(\xi_1^2), \gamma(\xi_1) + \frac{\epsilon z}{\gamma(\xi_1)}\right). \tag{5.7}$$

This is well defined since if $\xi_1, \ldots, \xi_n$ define a polygonal gap in the quadratic lamination, then $\gamma(\xi_1) = \gamma(\xi_i)$ for all $1 < i \leq n$.

Define φ on the interior of $X_p \times D - f_0(X_p \times \mathbb{D})$ by a homotopy relative to the boundary $\partial\left(X_p \times D - f_0(X_p \times \mathbb{D})\right)$, between φ on the components of $(X_p \cap \mathbb{S}^1) \times D$.

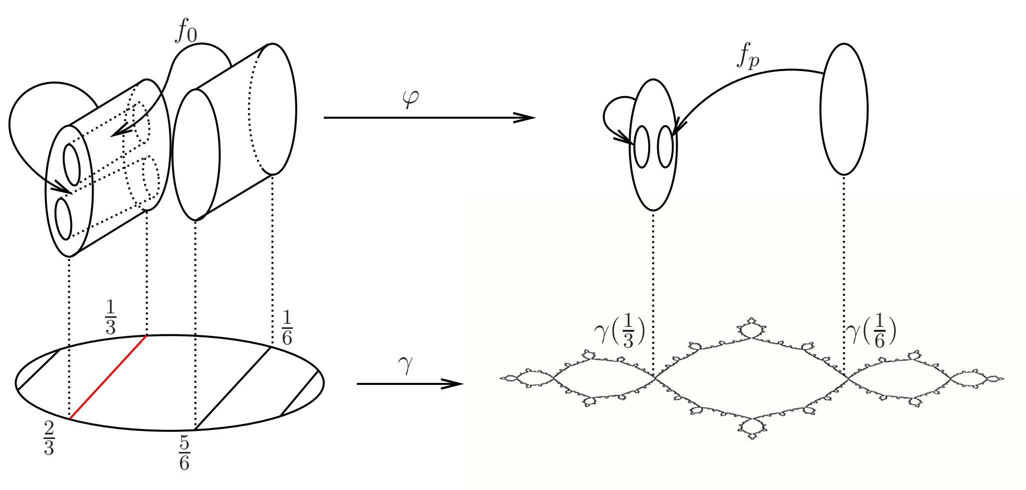

Figure 5.8: A fiber above a leaf of the lamination for the polynomial p is a tube with two interior tubes removed. It maps under φ to a disk with two interior disks removed.

Figure 5.9: The fiber above the leaf joining 1/3 and 2/3. The red tube is the image under f_0 of the tube above the leaf [1/6, 5/6]. The green tube is the image under f_0 of the tube above the leaf [1/3, 2/3].

Theorem 5.13. *There is a continuous surjective map $\varphi : X_p \times D \to J_p \times D$ such that the diagram*

$$\begin{array}{ccc} X_p \times D & \xrightarrow{\varphi} & J_p \times D \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_p} \\ X_p \times D & \xrightarrow{\varphi} & J_p \times D \end{array}$$

commutes.

Proof. The proof is similar to the proof of Theorem 5.3 using X_p as the base space, instead of $\mathbb{S}^1$. □

As before, we can pass to the inductive limit and get a global model for J^+, but we also need to understand the equivalence classes of φ. The pinching of the 4-ball is done along the equivalence classes of the function φ that we construct in Theorems 5.2 and 5.13.

Definition 5.14. The set of all identifications for the Hénon map is given by

$$\mathcal{L}_p := \varinjlim(L_p \times D, f_0).$$

This is the lamination for the Hénon map.

The lamination of the polynomial $p(x) = x^2 + c$, with c from the interior of the main cardioid of the Mandelbrot set is empty. Thus $\mathcal{L}_p$ is empty as well and we have the 4-ball model described by Bonnot [Bo].

Let $p_{1/4}(x) = x^2 + 1/4$. This is a parabolic polynomial with a fixed point of multiplier $\lambda = 1$. The Julia set $J_{1/4}$ is homeomorphic to a circle, so the Thurston lamination is empty as well. We consider small enough perturbations of semi-parabolic Hénon maps as in the context of Theorem 3.3. We have the following characterization: the set J^+ of semi-parabolic Hénon maps that are small perturbations of $p_{1/4}$ is homeomorphic to a 3-sphere with a solenoid removed.

Proposition 5.15. *There exists $\delta > 0$ such that for all parameters $(c, a) \in \mathcal{P}_1$ with $0 < |a| < \delta$, the set $J^+_{c,a}$ is homeomorphic to $\mathbb{S}^3 - \Sigma^-$.*

Proof. It follows from Theorem 3.3 and 5.2. □

Suppose now that the Hénon map $H_{p,a}$ is a small perturbation of the polynomial $p(x) = x^2 + c$ with $|c + 1| < 1/4$. The lamination of this polynomial is generated by the leaf connecting 1/3 and 2/3 and its preimages. The lamination $\mathcal{L}_p$ is also easy to describe in this case.

Proposition 5.16. *Suppose $p(x) = x^2 + c$ with $|c + 1| < 1/4$. The limit space $\mathcal{L}_p$ is closed, connected, and homeomorphic to $\mathbb{C} \times [0, 1]$. The set $\mathbb{C} \times \{0\}$ is the stable manifold $W^s(\overline{10}.\overline{10})$, and $\mathbb{C} \times \{1\}$ is the stable manifold $W^s(\overline{01}.\overline{01})$, where $\overline{10}.\overline{10}$ and $\overline{01}.\overline{01}$ is a periodic two orbit on the attractive solenoid Σ^+.*

Proof. The fact that the space $\mathcal{L}_p$ is homeomorphic to $\mathbb{C} \times [0, 1]$ follows from the definition of the inductive limit of $L_p \times D$ under the solenoidal map f_0. Pictures 5.8, 5.9 and 5.10 illustrate this. □

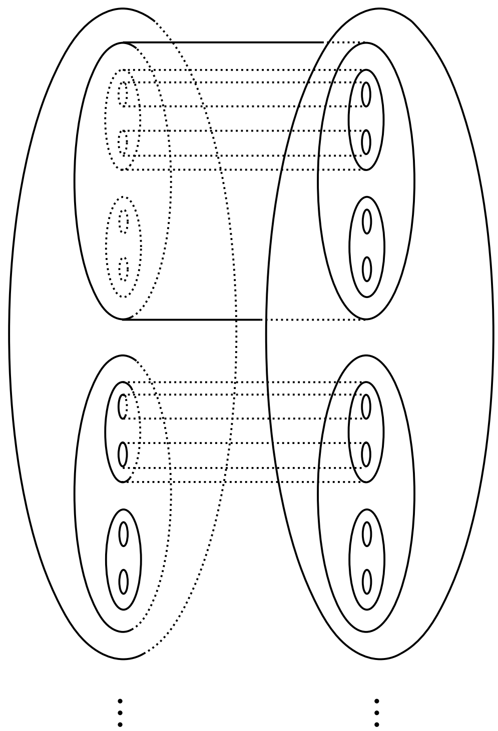

Figure 5.10: The nontrivial identifications in $\mathbb{S}^3 - \Sigma^-$ occur only in the stable manifolds of $\overline{01}.\overline{01}$ (corresponding to 1/3) and $\overline{10}.\overline{10}$ (corresponding to 2/3). This set is homeomorphic to $\mathbb{C} \times [0, 1]$.

More generally, if c belongs to a hyperbolic component (or its root) attached to the main cardioid of the Mandelbrot set then the limit space $\mathcal{L}_p$ is homeomorphic to $\mathbb{C} \times \Delta$, where Δ is the "ideal polygon" in the Thurston lamination of the polynomial p. For example, for the rabbit, Δ would be a triangle.

The construction that we did so far models the dynamics of the Hénon map on the set J^+. The lamination of the Hénon map is a lift of the lamination of the polynomial supported on the model space $\mathbb{S}^3 - \Sigma^-$. The space $\mathcal{L}_p$ described in Lemma 5.16 sits inside the unit four ball. The set J^+ is $\mathbb{S}^3 - \Sigma^-$ quotiented along the leaves of $\mathcal{L}_p$.

We would like to extend the model to represent the dynamics of the Hénon map on the interior of the set K^+. However, note that the map f_0 does not extend in an obvious way to $D \times D$ because it is not defined when $\xi = 0$. We will give a descriptive account for this model.

Let G be the critical gap in the lamination and G' be its image under $z \to z^2$. Then $G \times D$ is a topological four dimensional ball. Its boundary is $\partial(G \times D) = \mathbb{T}_0 \cup \mathbb{T}_1$, a union of two tori where

$$\mathbb{T}_0 = \partial G \times D \quad \text{and} \quad \mathbb{T}_1 = G \times \partial D.$$

The set $G' \times D$ is a topological four dimensional ball, with boundary $\partial(G \times D) = \mathbb{T}'_0 \cup \mathbb{T}'_1$, where

$$\mathbb{T}'_0 = \partial G' \times D \quad \text{and} \quad \mathbb{T}'_1 = G' \times \partial D.$$

The function f_0 is already defined on $\mathbb{T}_0 \to \mathbb{T}'_0$. It maps the solid torus $\mathbb{T}_0$ in a solid torus wrapped twice inside $\mathbb{T}'_0$. Note that the map f_0 as defined in Equation 5.6 is solenoidal of the appropriate type and has a $1/\xi$ in the second component. Thus it extends to the whole 3-sphere $\partial(G \times D) \to \partial(G' \times D)$. To see this, we can use the following lemma:

Lemma 5.17 ([HOV1])**.** *If $\mathbb{T}_0$ and $\mathbb{T}_1$ are two solid tori and $\psi : \partial\mathbb{T}_0 \to \partial\mathbb{T}_1$ is a homeomorphism which sends curves on $\partial\mathbb{T}_0$ which bound discs in $\mathbb{T}_0$ into curves which bound disks in $\mathbb{T}_1$, then ψ extends to a homeomorphism $\mathbb{T}_0 \to \mathbb{T}_1$.*

The map f_0 is defined on $\partial\mathbb{T}_1 \to \partial\mathbb{T}'_1$ since $\partial T_0 = \partial T_1$ and $\partial T'_0 = \partial T'_1$. Thus it extends to $\mathbb{T}_1 \to \mathbb{T}'_1$ as well. We have $f_0 : \partial(G \times D) \to \partial(G' \times D)$ and we can extend it radially to the interior. However, this would not be a correct representation of the dynamics of the Hénon map. We regard $G \times D$ and $G' \times D$ as analogs of the immediate basins of attraction of the cycle of period two of the Hénon map, restricted to a bi-disk $\mathbb{D}_r \times \mathbb{D}_r$. Construct $f_0 : G \times D \to G' \times D$ as follows:

(a) it is given by the lamination on $\partial G \times D \to \partial G' \times D$;

(b) it has the appropriate winding number and so it extends to

$$f_0 : G \times \partial D \to G' \times \partial D;$$

(c) adjust φ on the horizontal boundary to model the dynamics;

(d) extend radially to the interior of $G \times D$.

Choose a center P of $G' \times D$ and use polar coordinates on $G' \times D$. There exists a positive continuous function η on $\partial(G' \times D)$ such that

(a) $\eta = 1$ on $\varphi(\partial G \times D) \subset \partial G' \times D$;

(b) $\eta < 1$ on $\partial(G' \times D) - \varphi(\partial G \times D)$.

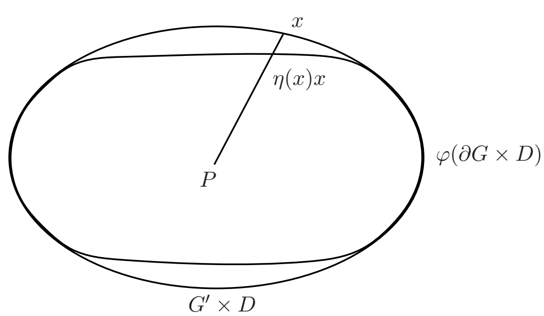

Figure 5.11: The function η acting on $\partial(G' \times D)$.

Construct now $f_0 : G' \times D \to G \times D$. Using the same notations as before, f_0 is already defined on $\mathbb{T}'_0 = \partial G' \times D$ and it maps the solid torus $\mathbb{T}'_0$ to a smaller solid torus wrapped around once inside the solid torus $\mathbb{T}_0 = \partial G \times D$. More precisely, $f_0(\mathbb{T}'_0) = \partial G \times D_\epsilon$. Thus f_0 extends trivially on $\mathbb{T}'_1$ and $f_0(\mathbb{T}'_1) = G \times \partial D_\epsilon$. We then extend radially to the interior of $G' \times D$.

5.5 Further comments

Consider the class of Hénon maps that are small perturbations of $p(x) = x^2 + c$ and $|c + 1| < 1/4$. These have the simplest nontrivial lamination. We are in the context of Lemma 5.16 and Corollary 5.3.1. Suppose that we keep the same model as in Corollary 5.3.1, but instead we identify $x \sim_\Phi \sigma^{2k}(y)$, for some integer $k \neq 0$. These are pinched ball models, but it is not known whether there are Hénon maps that realize them. If such Hénon maps exist, then they cannot come from perturbations of polynomials. Otherwise, from the construction, the leaves of the lamination of the polynomial will be crossing, which is not possible.

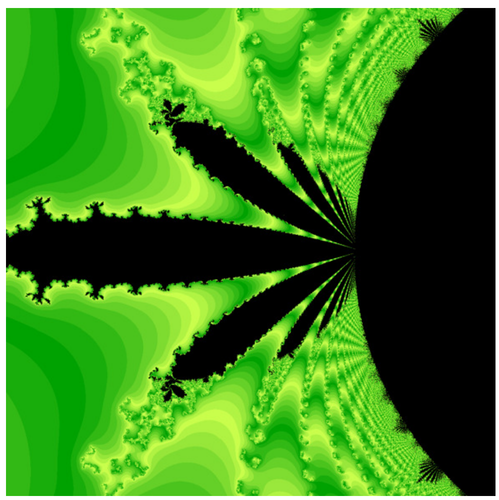

Figure 5.12: A parameter plane of $H(x, y) = (x^2 + c + ay, ax)$, for $a = -3/7$ fixed. In parameter space this corresponds to a transverse on the curve $\mathcal{P}_{-1}$, at $a = -3/7$. The black region represents a rough approximation of the region where the Julia set $J_{c,a}$ is connected. The root of these regions (with "finger-like" appearance) corresponds to a semi-parabolic Hénon map with a fixed point with an eigenvalue -1. The regions to the left (conjecturally) correspond to "fingers", i.e. pinched ball models with $x \sim_\Phi \sigma^{2k}(y)$. The picture was generated with FractalStream.

A similar phenomenon as in the picture above also occurs in the parameter space of cubic polynomials [RT3].

BIBLIOGRAPHY

[B] E. Bedford, *Complex Hénon maps with semi-parabolic fixed points*, J. Difference Equ. Appl. 16, No. 5-6, 425-426 (2010).

[B2] E. Bedford, *Dynamics of rational surface automorphisms*, in *Holomorphic dynamical systems*, Vol. 1998 of *Lecture Notes in Math.*, 57-104. Springer, Berlin, 2010.

[BH] X. Buff, J. H. Hubbard, *Dynamics in One Complex Variable*, to appear.

[Bo] S. Bonnot, *Topological model for a class of complex Hénon mappings*, Comment. Math. Helv. 81 (2006), 827-857.

[Br] F. E. Browder, *On the convergence of successive approximations for nonlinear functional equations*, Indag. Math. 30 (1968), 27-35.

[Bu] X. Buff, *On the Bieberbach conjecture and holomorphic dynamics*, Proc. Amer. Math. Soc., 131 (2003), 755–759.

[BLS] E. Bedford, M. Lyubich, J. Smillie, *Polynomial diffeomorphisms of C^2. IV: the measure of maximal entropy and laminar currents*, Invent. Math. 112, 77125 (1993).

[BS1] E. Bedford, J. Smillie, *Polynomial difeomorphisms of $\mathbb{C}^2$: currents, equilibrium measure and hyperbolicity*, Invent. Math. 103 (1991), no. 1, 69-99.

[BS2] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$ II: Stable manifolds and recurrence*, J. Am. Math. Soc. 4(4) (1991), 657-679.

[BS5] E. Bedford, J. Smillie, *Polynomial diffeomorphism of $\mathbb{C}^2$. V: Critical points and Lyapunov exponents*, J. Geom. Anal. 8(3) (1998), 349-383.

[BS7] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$. VII: Hyperbolicity and external rays*, Ann. Sci. Ecole Norm. Sup. 32, 455-497 (1999).

[BS8] E. Bedford, J. Smillie, *Polynomial diffeomorphisms of $\mathbb{C}^2$. VIII: Quasi-expansion*, Amer. J. Math. 124 (2002), 221-271.

[BSU] E. Bedford, J. Smillie, T. Ueda, *Parabolic Bifurcations in Complex Dimension* 2, arXiv:1208.2577.

[DH] A. Douady, J. H. Hubbard, *Étude dynamique des polynômes complexes. Partie II.* Publications Mathématiques d'Orsay [Mathematical Publications of Orsay], vol. 85, Université de Paris-Sud, Département de Mathématiques, Orsay, 1985.

[Du] R. Dujardin, *Some remarks on the connectivity of Julia sets for 2-dimensional diffeomorphisms*, Complex dynamics, 6384, Contemp. Math., 396, AMS, 2006.

[FM] S. Friedland, J. Milnor, *Dynamical properties of plane polynomial automorphisms*, Ergodic Theory Dynam. Systems 9 (1989).

[FS] J.-E. Fornæss and N. Sibony, *Complex Hénon mappings in $\mathbb{C}^2$ and Fatou-Bieberbach domains*, Duke Math. J. 65 (1992), no. 2, 345-380.

[Ha] M. Hakim, *Attracting domains for semi-attractive transformations of $\mathbb{C}^p$*, Publ. Mat. 38 (1994), no. 2, 479-499.

[Ham] M. Hamstrom, *Homotopy groups of the space of homeomorphisms, III*, J. Math. 10 (1966), 563-573.

[H] J. H. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2: Four Theorems by William Thurston*, to appear.

[H1] J. H. Hubbard, *Parametrizing unstable and very unstable manifolds*, Mosc. Math. J. 5 (2005), no. 1, 105-124.

[HOV1] J. H. Hubbard, R.W. Oberste-Vorth, *Hénon mappings in the complex domain I: The global topology of dynamical space*, Pub. Math. IHES 79 (1994), 5-46.

[HOV2] J.H. Hubbard, R.W. Oberste-Vorth, *Hénon mappings in the complex domain II: Projective and inductive limits of polynomials*, in *Real and Complex Dynamical Systems*, Branner and Hjorth, eds., Kluwer Academic Publishers (1995), 89-132.

[KH] A. Katok (ed.), B. Hasselblatt (ed.), *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press (1995).

[KS] W. A. Kirk (ed.), B. Sims (ed.), *Handbook of Metric Fixed Point Theory*, Kluwer Academic Publishers, Dordrecht, 2001.

[M] J. Milnor, *Dynamics in one complex variable*, 3rd edn., Princeton Univ. Press, 2006.

[O] R. A. Oliva, *On the combinatorics of external rays in the dynamics of the complex Henon map*, Ph.D. Thesis, Cornell University, 1998.

[RT1] R. Radu, R. Tanase, *Semi-parabolic Hénon maps*, preprint, 2012.

[RT2] R. Radu, R. Tanase, *Semi-parabolic tools for hyperbolic Hénon maps*, preprint, 2012.

[RT3] R. Radu, R. Tanase, *Fingers for cubic polynomials*, in preparation.

[S] M. Shishikura, *On the quasiconformal surgery of rational functions*, Ann. Sci. Ecole Norm. Sup., 20 (1987), 1-29.

[T] R. Tanase, *Hénon maps, discrete groups, and continuity of Julia sets*, Ph.D. Thesis, Cornell University, 2013.

[Th] W. Thurston, *On the geometry and dynamics of iterated rational maps*, in *Complex Dynamics: Families and Friends*, A.K. Peters, Wellesley/MA (2009), 3-137. Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher.

[Th1] W. Thurston, *Degree d Invariant Laminations*, Manuscript 2011.

[U] T. Ueda, *Local structure of analytic transformations of two complex variables I*, J. Math. Kyoto Univ., 26(2) (1956), 233-261.

[U2] T. Ueda, *Local structure of analytic transformations of two complex variables II*, J. Math. Kyoto Univ. 31 (1991), no. 3, 695-711.