BU-194-M

ON THE INADMISSIBILITY OF s^2 AS AN ESTIMATOR OF σ^2*

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Abstract

If $\nu s^2/\sigma^2$ is distributed as chi-square with ν degrees of freedom and if the risk function of an estimator of σ^2 is taken to be either the mean absolute error or mean squared error of estimate then there exists an estimator of σ^2 which has uniformly smaller risk than s^2. When risk is mean absolute error, a better estimator of σ^2 is $\nu s^2/$(median value of $\chi^2_{\nu+2}$) and when risk is measured by mean squared error a better estimator is $\nu s^2/(\nu+2)$. These results are obtained very simply by minimizing the risk function of the estimator $\nu s^2/k$.

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Introduction and summary

If $\nu s^2/\sigma^2$ is distributed as chi-square with ν degrees of freedom and if the risk function of an estimator of σ^2 is taken to be either the mean absolute error or mean squared error of estimate then there exists an estimator of σ^2 which has uniformly smaller risk than s^2. When risk is mean absolute error, a better estimator of σ^2 is $\nu s^2/(\text{median value of } \chi^2_{\nu+2})$ and when risk is measured by mean squared error a better estimator is $\nu s^2/(\nu+2)$. These results are obtained very simply by minimizing the risk function of the estimator $\nu s^2/k$.

Inadmissibility with respect to mean absolute error

For any $k > 0$ the risk function $E|\sigma^2-\nu s^2/k|$ is zero at the point $\sigma^2=0$, and for $\sigma^2 > 0$

$$\inf_k E|\sigma^2-\nu s^2/k| = \sigma^2 \inf_k E|1-\chi^2_\nu/k|$$

when k is neither a function of s^2 or σ^2. Letting $F_{\chi^2_\nu}(y)$ denote the chi-square distribution function

$$F_{\chi^2_\nu}(y) = \int_0^y \frac{z^{\frac{\nu-2}{2}}}{2^{\frac{\nu}{2}}\,\Gamma\left(\frac{\nu}{2}\right)}\, e^{-z/2}\, dz$$

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and noting that

$$(1) \qquad y\,dF_{\chi^2_\nu}(y) = \nu\,dF_{\chi^2_{\nu+2}}(y)$$

we get

$$E|1-\chi^2_\nu/k| = \frac{\nu}{k}\left[1-2F_{\chi^2_{\nu+2}}(k)\right] - \left[1-2F_\nu(k)\right]$$

The derivative with respect to k is then

$$\frac{d}{dk} E|1-\chi^2_\nu/k| = -\frac{2\nu}{k} f_{\chi^2_{\nu+2}}(k) - \frac{\nu}{k^2}\left[1-2F_{\chi^2_{\nu+2}}(k)\right] + 2f_\nu(k) \ .$$

According to (1),

$$\frac{\nu}{k} f_{\chi^2_{\nu+2}}(k) = f_\nu(k)$$

so the minimizing value of k satisfies the equation

$$F_{\chi^2_{\nu+2}}(k) = \frac{1}{2} \ .$$

Tables of the chi-square distribution show the median value of $\chi^2_{\nu+2}$ falling between $\nu+1$ and $\nu+2$; hence we infer that $\nu s^2/\nu$ is inadmissible for all ν.

Inadmissibility with respect to mean squared error

The mean squared error

$$E\left[1-X_\nu^2/k\right]^2 = E\left[(1-\nu/k) + (\nu-X_\nu^2)/k\right]^2$$

$$= (1-\nu/k)^2 + 2\nu/k^2$$

clearly attains its minimum value at $k=\nu+2$, and

$$\inf_k E\left[\sigma^2-\nu s^2/k\right]^2 = \frac{2\sigma^4}{\nu+2}$$

in contrast to the risk $2\sigma^4/\nu$ of the estimator s^2.

In this case we may note more generally that if $X_1,\cdots,X_n$ are independent and identically distributed then

$$E\left[\frac{\sum_1^n (X_i-\bar{X})^2}{n+1} - \sigma^2\right]^2 = \frac{2\sigma^4}{n+1} + \frac{(n-1)^2}{(n+1)^2}\frac{\kappa_4}{n}$$

which may exceed the risk

$$E\left[\frac{\sum_1^n (X_i-\bar{X})^2}{n-1} - \sigma^2\right]^2 = \frac{2\sigma^4}{n-1} + \frac{\kappa_4}{n}$$

if the fourth cumulant is negative.