SOME NOVEL COVARIANCE MODELS*

BU-604-M

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Summary

Following a general description of covariance models, consideration is given to models wherein the "slopes" for the covariates differ according to the levels of some of the factors.

1. The General Linear Model

Model: $y = Xb + e \;, \quad e \sim N(0, \sigma^2 I)$

Normal equations: $X'Xb^0 = X'y \;\Rightarrow\; b^0 = (X'X)^- X'y$

Reduction s/s: $R(b) = y'X(X'X)^- X'y$

Error s's: $SSE = y'y - y'X(X'X)^- X'y$

Hypothesis: $H : Xb = 0$

$$F(H) = \frac{R(b)}{r(X)} \bigg/ \frac{SSE}{N - r(X)} \sim F_{r(X), N-r(X)}$$

Design model: X : incidence matrix, 0's and 1's, not of full column rank

Regression: X : observed values, regressor variables, usually of full column rank

Covariance: Combination of design and regression

Design : Xa, X incidence, not full column rank

Regression: Zb, Z regressors, full column rank

columns LIN columns of X

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2. Covariance

Model: $\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{a} + \underset{\sim}{Z}\underset{\sim}{b} + \underset{\sim}{e}\ , \quad e \sim N(0,\sigma^2 I)$

Normal equations:

$$\begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} \\ \underset{\sim}{Z}'\underset{\sim}{X} & \underset{\sim}{Z}'\underset{\sim}{Z} \end{bmatrix} \begin{bmatrix} \underset{\sim}{a}^0 \\ \underset{\sim}{b}^0 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{y} \\ \underset{\sim}{Z}'\underset{\sim}{y} \end{bmatrix}$$

Solutions: $\underset{\sim}{a}^0 = (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{y} - (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{b}^0$

Write $\underset{\sim}{a}^* \equiv (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{y}$, from fitting $E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{a}$

$\equiv \underset{\sim}{a}^*(\underset{\sim}{y})$.

Then $\underset{\sim}{a}^0 = \underset{\sim}{a}^* - (\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{b}^0$

$= \underset{\sim}{a}^*(\underset{\sim}{y}) - \underset{\sim}{a}^*(\underset{\sim}{Z}\underset{\sim}{b}^0)$

$= \underset{\sim}{a}^*(\underset{\sim}{y} - \underset{\sim}{Z}\underset{\sim}{b}^0)$

and $\underset{\sim}{b}^0 = \hat{\underset{\sim}{b}} = (\underset{\sim}{R}'\underset{\sim}{R})^{-1}\underset{\sim}{R}'\underset{\sim}{y}$

with $\underset{\sim}{R} = [\underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}']\underset{\sim}{Z}$

$= \underset{\sim}{P}\underset{\sim}{Z}$, with the familiar $\underset{\sim}{P} = \underset{\sim}{P}' = \underset{\sim}{P}^2 = \underset{\sim}{I} - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'$

$= \underset{\sim}{P}\{\underset{\sim}{z}_j\}$

$= \{\underset{\sim}{z}_j - \underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-}\underset{\sim}{X}'\underset{\sim}{z}_j\}$

$= \{\underset{\sim}{z}_j - \hat{\underset{\sim}{z}}_j\}$ equivalent to fitting $E(\underset{\sim}{z}_j) = \underset{\sim}{X}\underset{\sim}{a}$

$= \{\underset{\sim}{r}_j\}$.

ANALYSES OF VARIANCE

(1) Factors and interactions adjusted for covariates

Mean	1	$N\bar{y}^2$
Covariates (adj for mean)	$r(\mathbf{Z})$	$\mathbf{y}'\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'\mathbf{y}$ [deviations i.e., regression]
Factors (and int.) adj. for covariates	$r(\mathbf{X}) - 1$	$\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y} - N\bar{y}^2 + \mathbf{y}'\mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{y} - \mathbf{y}'\mathbf{z}(\mathbf{z}'\mathbf{z})^{-1}\mathbf{z}'\mathbf{y}$
Residual	$N - r(\mathbf{X}) - R(\mathbf{Z})$	$SSE \equiv \mathbf{y}'\mathbf{y} - \mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}\mathbf{y} - \mathbf{y}'\mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{y}$
TOTAL	N	$\mathbf{y}'\mathbf{y} = \Sigma\, y_i^2$

(2) Covariates adjusted for factors and interactions

Factors (and int.)	$r(\mathbf{X})$	$\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y}$
Mean	1	$N\bar{y}^2$
Factors (adj. for μ)	$r(\mathbf{X}) - 1$	$\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{y} - N\bar{y}^2$
Covariates (adj. for factors)	$r(\mathbf{Z})$	$\mathbf{y}'\mathbf{R}(\mathbf{R}'\mathbf{R})^{-1}\mathbf{R}'\mathbf{y}$
Residual	$N - r(X) - r(Z)$	SSE = as above
TOTAL	N	$\mathbf{y}'\mathbf{y} = \Sigma y_i^2$

HYPOTHESIS TESTING

Use $$\hat{\sigma}^2 = SSE/[N - r(\underset{\sim}{X}) - r(\underset{\sim}{Z})]$$

<u>$H : \underset{\sim}{b} = \underset{\sim}{0}$</u>

$$F = \frac{\underset{\sim}{y}'\underset{\sim}{R}(\underset{\sim}{R}'\underset{\sim}{R})^{-1}\underset{\sim}{R}'\underset{\sim}{y}}{r(\underset{\sim}{Z})\hat{\sigma}^2}$$

<u>$H : \underset{\sim}{K}'\underset{\sim}{a} = \underset{\sim}{m}$</u>

$$F = \frac{Q}{r(\underset{\sim}{K})\hat{\sigma}^2}$$

$$Q = (\underset{\sim}{K}'\underset{\sim}{a}^0 - \underset{\sim}{m})'[\underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{K} + \underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{X}'\underset{\sim}{Z}(\underset{\sim}{R}'\underset{\sim}{R})^{-1}\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{K}]^{-1}(\underset{\sim}{K}'\underset{\sim}{a}^0 - \underset{\sim}{m})$$

<u>$H : \underset{\sim}{K}'[\underset{\sim}{a} + (\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{b}] = \underset{\sim}{0}$</u>

$$F = \frac{\underset{\sim}{a}^{*}{}'\underset{\sim}{K}[\underset{\sim}{K}'(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{K}]^{-1}\underset{\sim}{K}'\underset{\sim}{a}^{*}}{r(\underset{\sim}{K})\hat{\sigma}^2}$$

3. Special Case: Completely Randomized Design

Consider adding covariates to well known design models. The key elements are $\underset{\sim}{b}$, $\underset{\sim}{Z}$, $\underset{\sim}{R}$, $\underset{\sim}{R}'\underset{\sim}{R}$, $\underset{\sim}{R}'\underset{\sim}{y}$.

Example: Completely randomized design

No covariates:

$$y_{ij} = \mu + \alpha_i + e_{ij} \quad \text{with} \quad i = 1,2,3, \quad n_i = 3,2,2$$

$$\begin{bmatrix} 74 \\ 68 \\ 77 \\ \hline 76 \\ 80 \\ \hline 85 \\ 93 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} + \underset{\sim}{e}$$

With covariates:

$$y_{ij} = \mu + \alpha_i + bz_{ij} + e_{ij}$$

or

$$y_{ij} = \mu + \alpha_i + b_i z_{ij} + e_{ij}$$

For bz_{ij}

$$\underset{\sim\sim}{Zb} = \begin{bmatrix} 3 \\ 4 \\ 2 \\ \hline 2 \\ 4 \\ \hline 4 \\ 6 \end{bmatrix} b$$

For $b_i z_{ij}$, $i = 1,2,3$

$$\underset{\sim\sim}{Zb} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & 0 & 0 \\ \hline 0 & 2 & 0 \\ 0 & 4 & 0 \\ \hline 0 & 0 & 4 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For $b_i z_{ij}$, $b_1 \equiv b_3$

$$\underset{\sim\sim}{Zb} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \\ 2 & 0 \\ \hline 0 & 2 \\ 0 & 4 \\ \hline 4 & 0 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The model

$$y_{ij} = \mu + \alpha_i + bz_{ij} + e$$

gives

$$\underset{\sim}{R}'\underset{\sim}{R} = \sum_{i=1} \left(\sum_{j=1}^{n_i} z_{ij}^2 - n_i \bar{z}_{i.} \right)^2$$

$$\underset{\sim}{R}'\underset{\sim}{y} = \sum_{i=1} \left(\sum_{j} y_{ij} z_{ij} - n_i \bar{y}_{i.} \bar{z}_{i.} \right)$$

and leads to

$$\hat{b} = SSE_{yz}/SSE_{zz}$$

and

$$R(b|\mu,\alpha) = (SSE_{yz})^2/SSE_{zz}$$

in the usual manner. Similarly

$$y_{ij} = \mu + \alpha_i + b_i z_{ij} + e$$

with

$$\underset{\sim}{R}'\underset{\sim}{R} = \text{diag} \left\{ \left(\sum_{j=1}^{n_i} z_{ij}^2 - n_i \bar{z}_{i.}^2 \right) \right\} \qquad i = 1,\cdots,a$$

$$\underset{\sim}{R}'\underset{\sim}{y} = \left\{ \sum_{j=1}^{n_i} y_{ij} z_{ij} - n_i \bar{y}_{i.} \bar{z}_{i.} \right\} \qquad i = 1,\cdots,a$$

gives

$$\hat{b}_i = (SSE_{yz})_i / (SSE_{zz})_i$$

and

$$R(\underset{\sim}{b}|\mu,\alpha) = \sum_i (SSE_{yz})_i^2 / (SSE_{zz})_i$$

of the familiar intra-class regression model.

4. Special Case: Rows-By-Columns (Randomized Complete Blocks)

4.1. The usual case. For $i = 1,\cdots,a$, $j = 1,\cdots,b$, $k = 1,\cdots,n$

$$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + bz_{ijk} + e_{ijk}$$

$$\text{Other possibilities: } \begin{cases} b_i z_{ijk} \quad (\text{or } b_j z_{ijk}) \\ b_{ij} z_{ijk} \\ (b_i + b_j^*) z_{ijk} \\ b_{i+j} z_{ijk} \end{cases}$$

and combinations, for several covariates:

$$b_i z_{ijk} + b_j^* u_{ijk} + (\dot{b}_i + \ddot{b}_j) w_{ijk} \quad .$$

Computing for all these is governed by the general algorithm

— whether equal numbers of observations per subclass, or unequal

— whether interaction or not.

Certain special cases are interesting.

4.2. <u>One covariate: A slope for each row.</u> $[n_{ij} = 1]$

Model: $y_{ij} = \mu + \alpha_i + \beta_j + b_i z_{ij} + e_{ij}$

$$\mathbf{R}'\mathbf{R}: \quad \mathbf{S}_1 = \begin{cases} \text{diagonal:} & \frac{a-1}{a} \sum_j (z_{ij} - \bar{z}_{i.})^2 \\ \text{off-diagonal:} & \frac{-1}{a} \sum_j (z_{ij} - \bar{z}_{i.})(z_{i'j} - \bar{z}_{i'.}) \end{cases}$$

$$\mathbf{R}'\mathbf{y}: \quad \mathbf{u}_1 = \left\{ \sum_j (z_{ij} - \bar{z}_{i.})(y_{ij} - \bar{y}_{i.}) \right\} \qquad i = 1, \cdots, a$$

4.3. <u>One covariate: A slope for each column</u> $[n_{ij} = 1]$

Model: $y_{ij} = \mu + \alpha_i + \beta_j + b_j z_{ij} + e_{ij}$

$$\mathbf{R}'\mathbf{R}: \quad \mathbf{S}_2 = \begin{cases} \text{diagonal:} & \frac{b-1}{b} \sum_i (z_{ij} - \bar{z}_{.j})^2 \\ \text{off-diagonal:} & \frac{-1}{b} \sum_i (z_{ij} - \bar{z}_{.j})/(z_{ij'} - \bar{z}_{.j'}) \end{cases}$$

$$\mathbf{R}'\mathbf{y}: \quad \mathbf{u}_2 = \left\{ \sum_i (z_{ij} - \bar{z}_{.j})(y_{ij} - \bar{y}_{.j}) \right\}$$

4.4. <u>Two covariates: one with a slope for each row, one with a slope for each column</u> $[n_{ij}=1]$

Model: $y_{ij} = \mu + \alpha_i + \beta_j + b_i z_{ij} + b_j^* w_{ij} + e_{ij}$

$$\mathbf{R}'\mathbf{R} = \begin{bmatrix} \mathbf{S}_1(z) & \mathbf{S}_{12} \\ \mathbf{S}_{12}' & \mathbf{S}_2(w) \end{bmatrix}$$

with $$\mathbf{S}_{12} = \left\{ (z_{ij} - \bar{z}_{i.})(w_{ij} - \bar{w}_{.j}) \right\} \qquad i = 1, \cdots, a, \quad j = 1, \cdots, b$$

$$\mathbf{R}'\mathbf{y} = \begin{bmatrix} \mathbf{u}_1(z) \\ \mathbf{u}_2(w) \end{bmatrix} .$$

4.5. One covariate: a slope for each cell, composed of a row slope plus a column slope

$[n_{ij} = 1]$

Model: $y_{ij} = \mu + \alpha_i + \beta_j + (b_i^* + b_j)z_{ij} + e_{ij}$

Caution: Need to avoid singularity w.r.t. $(b_i^* + b_j)$'s.

Example 3 rows and 3 columns

$$\underset{\sim}{Zb} = \begin{bmatrix} z_{11} & 0 & 0 & z_{11} & 0 & 0 \\ z_{12} & 0 & 0 & 0 & z_{12} & 0 \\ z_{13} & 0 & 0 & 0 & 0 & z_{13} \\ 0 & z_{21} & 0 & z_{21} & 0 & 0 \\ 0 & z_{22} & 0 & 0 & z_{22} & 0 \\ 0 & z_{23} & 0 & 0 & 0 & z_{23} \\ 0 & 0 & z_{31} & z_{31} & 0 & 0 \\ 0 & 0 & z_{32} & 0 & z_{32} & 0 \\ 0 & 0 & z_{33} & 0 & 0 & z_{33} \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Define $b_1^* \equiv 0$

Estimation

$$\underset{\sim}{R}'\underset{\sim}{R} = \begin{bmatrix} \underset{\sim}{S}_1^*(z) & \underset{\sim}{S}_{12}^*(z,z) \\ \underset{\sim}{S}_{12}^{*\prime}(z,z) & \underset{\sim}{S}_2^*(z) \end{bmatrix} \quad \text{and} \quad \underset{\sim}{R}'\underset{\sim}{y} = \begin{bmatrix} \underset{\sim}{u}_1^*(z) \\ \underset{\sim}{u}_2(z) \end{bmatrix}$$

$\underset{\sim}{S}_1^*$ is $\underset{\sim}{S}_1$ omitting first row and column.

$\underset{\sim}{S}_{12}^*$ is $\underset{\sim}{S}_{12}$ omitting first row.

$\underset{\sim}{u}_1^*$ is $\underset{\sim}{u}_1$ omitting first element.

A typical element of $\underset{\sim}{S}_{12}(z,z)$ is $(z_{ij} - \bar{z}_{i.})(z_{ij} - \bar{z}_{.j})$.

4.5a. One covariate: a slope for each cell, composed of a row slope plus a column slope

[$n_{ij} > 1$, with interaction]

Model: $$y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + (b_i^* + b_j)z_{ijk} + e_{ijk}$$

Definition: $$b_1^* \equiv 0$$

Estimation:

$$\underset{\sim}{R}'\underset{\sim}{R} = \begin{bmatrix} \text{diag}\{s_{i\cdot}\} & \{s_{ij}\} \\ \{s_{ij}\}' & \text{diag}\{s_{\cdot j}\} \end{bmatrix} \quad \begin{matrix} i = 2,\cdots,a \\ j = 1,2,\cdots,b \end{matrix}$$

with

$$s_{ij} = \sum_{k=1}^{n} (z_{ijk} - \bar{z}_{ij\cdot})^2 \ ,$$

$$s_{i\cdot} = \sum_{j=1}^{b} s_{ij} \quad \text{and} \quad s_{\cdot j} = \sum_{i=1}^{a} s_{ij}$$

$$\underset{\sim}{R}'\underset{\sim}{y} = \begin{bmatrix} \{p_{i\cdot}\} & i = 2,\cdots,a \\ \{p_{\cdot j}\} & j = 1,2,\cdots,b \end{bmatrix}$$

with

$$p_{ij} = \sum_{k=1}^{n} (z_{ijk} - \bar{z}_{ij\cdot})(y_{ijk} - \bar{y}_{ij\cdot}) \ ,$$

$$p_{i\cdot} = \sum_{j=1}^{b} p_{ij} \quad \text{and} \quad p_{\cdot j} = \sum_{i=1}^{a} p_{ij} \ .$$

Note: Although $\underset{\sim}{R}'\underset{\sim}{R}$ and $\underset{\sim}{R}'\underset{\sim}{y}$ have elements defined only for $i = 2,\cdots,a$, the summations for elements $s_{\cdot j}$ and $p_{\cdot j}$ are over $i = 1,2,\cdots,a$.