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God, Indivisibles, and Logic in the Later Middle Ages: Adam Wodeham's Response to Henry of Harclay

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As its modern edition appears in the Synthese Historical Library, Adam Wodeham's *Tractatus de indivisibilibus* does not appear to belong to any one discipline. With regard to its intended audience, the notice of the book appearing on the back cover states that "This book is an important contribution to the history of philosophy." But it continues, "It will be of interest to all medievalists, particularly to those concerned with medieval science, philosophy, and logic. Theologians and historians of mathematics will also find it useful."[1] In its medieval context as well, *Tractatus de indivisibilibus* had ambiguous disciplinary status. It begins with the question, "Whether charity or [any] other incorruptible form is composed of indivisible forms."[2] Such a reference to charity signals a connection to the *Sentences* of Peter Lombard, Book I, dist. 17. In introducing his answer to this question, however, Wodeham states, "Because this difficulty is the same for all composite divisible things, whether intensive or extensive, which are of one and the same species or homogeneous, therefore I will briefly inquire indifferently concerning the former and the latter."[3] The solutions Wodeham then proposes to the questions he asks rely nearly always on logic.

In his *Physics* Aristotle had argued forcefully for the isomorphism between continua of all sorts, whether physical or mathematical, for instance the extension of bodies, geometrical planes and lines, and motion or time. Medieval Aristotelians extended the isomorphism of continua to qualitative distances or the latitudes of forms. If any of these continua were composed of indivisibles, so the argument went, so were they all. The first principal argument in Wodeham's first question makes use of this isomor-

1. Adam de Wodeham, *Tractatus de indivisibilibus. A Critical Edition with Introduction, Translation, and Textual Notes*, ed. Rega Wood (Dordrecht: Kluwer Academic Publishers, 1988).
2. Wodeham, *Tractatus de indivisibilibus*, p. 33.
3. Wodeham, *Tractatus de indivisibilibus*, pp. 34–35. Here and elsewhere, I occasionally modify Rega Wood's translation in the interests of exactness.

phism by arguing that it appears that charity is composed of indivisible forms, because

> time, in which this kind of forms is intended by continuous motion—if it is a successive measure as is usually assumed—is composed of indivisibles, since distance and motion are composed of such. Therefore, a form successively acquired in some time is composed of indivisibles in respect to time, that is of intensive indivisibles.[4]

For us in the twentieth century, there is no such isomorphism between the structure of bodies and of mathematical entities, because we assume that bodies are composed of atoms—or of indivisible subatomic particles—that have size. Whereas a geometrical line may be infinitely divisible into ever smaller lines, the division of bodies ends when one arrives at indivisible particles. Moreover, what were for Aristotelians qualitative forms inhering in bodies are for us phenomena of perception, so the question whether they are continuous in the outside world or not becomes moot. In the twentieth century, then, the continuum belongs primarily to mathematics. More than that, the contemporary conceptualization of the continuum arithmeticizes it, so that we think of the continuum of real numbers, not of lines or points.

By contrast, for medieval Aristotelians continua were most fundamentally physical. Indeed, Aristotle's definition of continuity—the condition in which successive parts of a thing have their limits conjoined—hardly applies to geometrical entities. If two lines AB and BC are joined together end to end, the point B may be said to connect them, but B is one point, not two points unified with each other, as Aristotle's definition of continuity would seem to require.[5] In the fourteenth century, some Aristotelians of nominalistic stripe believed that mathematical entities existed only in the imaginations of mathematicians and not in outside reality. Those who believed that mathematical entities existed in reality did not suppose them to exist in some Platonic otherworld. Rather they took them to be quantitative aspects of physical bodies.

Medieval faculties of arts recognized distinct disciplines, each having its proper principles and conclusions. As Artistotle stated at the beginning of the *Physics*, practitioners of a given discipline, whether it be geometry or natural philosophy, are not expected to dispute with those who deny the principles of their discipline.[6] Having different principles, natural philoso-

4. Wodeham, *Tractatus de indivisibilibus*, pp. 32–33. Translation modified.

5. Averroes had pointed out this difference between mathematical and physical entities. See Edith Sylla, "Infinite Indivisibles and Continuity in Fourteenth Century Theories of Alteration," in *Infinity Continuity in Antiquity and the Middle Ages*, ed. Norman Kretzmann (Ithaca: Cornell University Press, 1982), pp. 233–34.

6. See Edith Sylla, "The A Posteriori Foundations of Natural Science; Some Medieval Commentaries on Aristotle's *Physics*, Book I, chapters 1 and 2," *Synthese* 40 (1979): 147–87.

phers and geometricians may disagree. Faculties of theology, however, sometimes made claims that upset this peaceful coexistence among arts disciplines. In the period preceding Wodeham's *Tractatus de indivisibilibus*, questions about the relations of indivisibles and continua were raised in theology that had reverberations throughout the disciplines. In considering the possible eternity of the world, Henry of Harclay, referring to God's clear and distinct knowledge, concluded that points in a line must be immediate to one other. This contradicted well-established positions both in Euclidean geometry and in Aristotelian natural philosophy and led to efforts at refutation (as well as to further steps down the same anti-Aristotelian, anti-Euclidean road, for instance, to Walter Chatton's argument that continua are composed of finitely many immediate indivisibles).[7] If one wanted to reject Harclay's conclusion, there seemed to some to be little choice but to deny altogether that points or other indivisibles exist. Others tried to refute Harclay on the assumption that indivisibles like points do exist.

From the point of view of modern mathematics, Harclay's conclusion, that continua are composed of infinitely many indivisibles immediate to each other, is false. Mathematicians have a tendency to believe that if something is false in modern mathematics, it is, simply speaking, false. In the fourteenth century, Harclay's conclusion aroused suspicion and suffered rejection because it contradicted both Euclid and Aristotle. Yet Harclay's arguments, drawing upon God's knowledge or sight, were difficult to refute. Mathematicians did not have sufficient prestige to refute Harclay simply on the grounds that his conclusion contradicted Euclidean geometry. Nor was Aristotle's authority in physics alone sufficient.

Thomas Bradwardine, in his *Tractatus de continuo*, attempted to refute not only Harclay, but also alternative theories of the composition of continua from infinitely many mediate indivisibles (Robert Grosseteste's view) or of finitely many immediate indivisibles (Walter Chatton's view).[8] In his refutations, Bradwardine drew upon resources from every discipline of the medieval university. Although John Murdoch, the modern editor of *De continuo*, has characterized Bradwardine's methodology as primarily mathematical, in my opinion it was most fundamentally physical. Bradwardine assumed, as did most of his contemporaries, that all continua are isomor-

7. For Chatton, see John Murdoch and Edward Synan, "Two Questions on the Continuum: Walter Chatton (?), O.F.M. and Adam Wodeham, O.F.M.," *Franciscan Studies* 26 (1966): 212– 88.

8. See John Murdoch, *Geometry and the Continuum in the Fourteenth Century: A Philosophical Analysis of Thomas Bradwardine's Tractatus de continuo* (unpub. Ph.D. diss., University of Wisconsin, 1957). The definitions, suppositions, and conclusions are published in John Murdoch, "Thomas Bradwardine: Mathematics and Continuity in the Fourteenth Century," in *Mathematics and its Applications to Science and Natural Philosophy in the Middle Ages*, ed. Edward Grant and John E. Murdoch (Cambridge: Cambridge University Press, 1987), pp. 119–30. In what follows, I rely on John Murdoch's unpublished edition and translation of the *Tractatus de continuo* (versions subsequent to his Ph.D. dissertation).

phic—if one continuum is composed of indivisibles in any way, finite or infinite, immediate or mediate, so are they all. In his proofs, Bradwardine assumed the truth of Euclidean geometry, but he argued that he did not thereby commit a *petitio principii*, because, so he claimed, Euclidean geometry is consistent with the possibility that continua are composed of infinitely many mediate indivisibles as well as with the assumption that continua are divisible *in infinitum*.[9]

Here, however, Bradwardine made a logical *faux pas* unless his reasoning is interpreted more narrowly than might at first appear. He said that Euclidean geometry is consistent with infinitely many mediate indivisibles, but assumed that a continuum is not composed of finite and immediate divisibles. He went on to argue, however, that if a continuum is composed of infinitely many mediate indivisibles, it follows that it is composed of finitely many immediate indivisibles. He did this by a physical argument concerning two liquids that come together to form a continuum:

> Conclusion 120. If so [i.e., if continuous natural substance is composed of infinite indivisible substances], then atoms of any sort of continuum are immediately joined. Let two liquids run into each other (*Concurrant duo liquida ad continuationem*). Then two indivisibles of prime matter, which previously were the termini of those liquid bodies, are not corrupted nor is another generated between them. Therefore they remain immediate, and therefore also the points of quantity contained in them [remain immediate].[10]

By Conclusion 30 ("If one continuum were to have immediate atoms, whether finite or infinite, any continuum would have such"), Conclusion 120 is immediately generalized for all continua.[11] Now if in "all continua" Bradwardine includes geometric continua, then he has contradicted himself. He has said that A (Euclidean geometry) is consistent with B (the assumption that a continuum is composed of infinitely many mediate indivisibles), but inconsistent with C (that a continuum is composed of immediate indivisibles). But then he has argued that B implies C, arguing first for physical continua, and then generalizing his conclusion via Conclusion 30 and Supposition 3 to all continua. If B implies C and if A is inconsistent with C, then A is inconsistent with B. If Bradwardine is to be defended from this apparent mistake, then Conclusion 120 must be restricted to physical continua. When, however, Bradwardine came to his crowning 140th conclu-

9. See Edith Sylla, "Thomas Bradwardine's *De continuo* and the Structure of Fourteenth-Century Learning," in *Texts and Contexts in Ancient and Medieval Science. Studies on the Occasion of John E. Murdoch's Seventieth Birthday*, ed. Edith Sylla and Michael McVaugh, (Leiden: E. J. Brill, 1997), pp. 158–59.

10. My translation from Murdoch's text.

11. "Therefore by [Conclusion] 30 the same is true for all continua." Conclusion 30 is in turn based on Supposition 3: "Where there is no cause of diversity or dissimilarity, the judgment is assumed to be similar."

sion, arguing especially against Robert Grosseteste, that "If a continuum is composed of infinitely many mediate indivisibles, it is composed of immediate [indivisibles]," from which it followed as a corollary that: "No continuum is composed of mediate indivisibles," he claimed that Conclusion 140 followed from Conclusions 30 and 120. The bottom line is that either Bradwardine contradicted himself or his arguments against Grosseteste hold only for physical and not geometrical continua. The latter would seem to be the preferable alternative, all things considered. So much for Bradwardine's physical and/or geometrical arguments against Harclay, Grosseteste, and Chatton.

In the rest of this essay, then, I will examine not Bradwardine's, but Adam Wodeham's efforts, in *Tractatus de indivisibilibus*, to refute Harclay's arguments. If Bradwardine's arguments were primarily physical, Wodeham's tools of refutation were logical. That this was so is understandable, given that it was reference to God's knowledge of the points of a line that impelled Harclay's argument. In this case we are talking about something really existing in the world and not only in human conception. But the discussion is not necessarily limited to the normal course of physical nature, but applies rather to God's knowledge of the physical world. This is a case study of what I would call "Aristotelian mathematics," in which mathematical entities, if they exist outside the mind at all, are assumed to inhere in physical bodies. In the twist put on this by Harclay, these entities, in particular points and lines, must be "objective" in the sense that they are there for God to see or know. As Reuben Hersh argues in *What is Mathematics Really?*, mathematical objects are not timeless Platonic entities, but rather entities conceived by humans for their own special needs and ends.[12] In fourteenth-century discussions of the relations of indivisibles, infinites, and continua, we see efforts to conceptualize mathematical systems that are unlike the one that eventually triumphed with Georg Cantor, but ones that do not therefore deserve our immediate scorn as false or wrongheaded. In these fourteenth-century efforts, conceptions of God's knowledge played a decisive role in setting up the problems to be solved; an even more decisive role in solving these problems was played, however, by logic, that is by the theory of the supposition of terms.

HENRY OF HARCLAY AND INFINITE IMMEDIATE INDIVISIBLES

What, then, was the argument made by Henry of Harclay that aroused Adam Wodeham's concern in his *Tractatus de indivisibilibus*? Briefly put, "atomism" or "indivisibilism" had raised its head when Henry of Harclay, in

12. Reuben Hirsh, *What is Mathematics Really?* (New York: Oxford University Press, 1997).

his question whether the world may be eternal *a parte post*, attempted to argue that—contra Philoponus and a long line of thinkers following him—purported paradoxes of infinites do not prevent the possibility of an eternal world, in the past or in the future.[13] Philoponus had argued, for instance, that if the world is eternal there will be infinitely many past revolutions of the sun and infinitely many past revolutions of the moon, but twelve times as many revolutions of the moon as of the sun—a contradiction, he said, in its implication that the solar and lunar revolutions are both equal (infinite) and unequal (one twelve times the other). To the contrary, said Harclay, one infinite can be a multiple of another. Robert Grosseteste, in his marginalia to the *Physics* and elsewhere, had proposed that God knows times absolutely by the numbers of instants they contain, even if there is no other standard of measurement. To God, two years contains twice as many instants as one year, even though both one year and two years contain infinitely many instants.[14] Just so, said Harclay, if the infinite points in lines of different lengths can have ratios to each other, so too the infinites that arise in eternal times can have ratios to each other.

But is this "God's eye view" of the matter possible? If there are in fact infinitely many indivisibles in any continuum, won't Aristotle's arguments against the composition of continua from indivisibles still apply? What indeed will be the relationship between instants in a time period, or between points in a line, or between indivisibles in any continuum? In reply to such worries, Harclay concluded that on this view of the world and God's knowledge of it, indivisibles will be immediate. He argued:

> It is certain that God now knows every point that can be designated in a continuum. Take, then, the first inchoative point of a line. God perceives that point and any point in this line different from it (*quodlibet aliud punctum ab isto in hac linea*). It follows, then, that either up to that more immediate point which God sees there intervenes some line (*usque ad illum punctum inmediatiorem quem Deus videt intercipit aliqua*[15] *linea*) or one does not. If not, then God perceives this point to be immediate to another one. If such a line does intercede, then, since points can be assigned in the line [which falls between the first inchoative point and the other point], these mean points have not been perceived by God. Proof of this consequence: by hypothesis, a line falls between the first point and any other (*quodlibet*) point that God per-

13. See John Murdoch, "Henry of Harclay and the Infinite," in A. Maierù and A. Paravicini-Bagliani, *Studi sul XIV secolo in memoria di Anneliese Maier* (Rome: Èdizioni di Storia e Letteratura, 1982), pp. 219–61.

14. See Sylla, "Thomas Bradwardine's *De continuo* and the Structure of Fourteenth-Century Learning," pp. 164–66.

15. For texts of Harclay, I am relying upon unpublished texts and translations by John Murdoch, except where he has published short excerpts of the text or translation. Although I do not have a microfilm to check, I suggest emending "alia" to read "aliqua," changing the English translation from "another" to "some."

> ceives. Thus, you maintain, God has not perceived the mean point just now discovered [but it was assumed that He perceived every point of the line, so there is a contradiction].[16]

Of course, this conclusion contradicts accepted truth about the relations of indivisibles and continua, as Harclay realized. Although he would have a difficult time defending his conclusion against Aristotle, it nevertheless seemed to him to follow:

> These arguments . . . persuade me of the truth of the matter more than the arguments for the opposing point of view, howsoever little they may agree with Aristotle's way of doing things and in spite of the fact that the [contrary] arguments of the Stagirite and others are extremely difficult [to refute], undoubtedly because our intellect is dull when it comes to understanding the infinite or the infinite division of a continuum.[17]

ADAM WODEHAM'S LOGICAL RESPONSE

In his *Tractatus de indivisibilibus* Adam Wodeham tried at several reprises to refute Harclay's argument. While he himself believed that there are no indivisibles in reality, Wodeham wanted to refute Harclay on his own terms, or at least by replacing "points" with "possible cuts," retaining as much of the rest of the argument as still followed. Rather than reading Harclay directly, Wodeham apparently learned of his argument through the medium of William of Alnwick's previous attempt to refute it. As William of Alnwick analyzed Harclay's argument, it involved the fallacy of arguing from merely confused to determinate supposition. It is true, Alnwick said, that between the end point of the line and any other point, there is a mediate line, where the term "mediate line" has merely confused supposition—for each point chosen there is a different mediate line. But it is not true that there is some line between the end point and any other point, if "some line" is taken with determine supposition—there is no one line between the end point and any other point.[18] Alternatively, Alnwick said, Harclay has committed the fallacies of a figure of speech and of affirming the consequent.

16. Translation based on the translation of John Murdoch from his edition. See Murdoch, "Henry of Harclay and the Infinite." The Latin of this passage is in n. 24.

17. Cf. Latin texts in Murdoch, "Henry of Harclay," p. 230 n.27.

18. "To the first—where, in arguing, it is asked whether or not there is a mean line between the first point of a line and every other point known by God. If not [it is replied], then God perceives this [first] point to be immediate to the other [point]. If there is, then since points can be assigned in the [mean] line, these mean points will not be perceived by God, which is false. To this argument, I have replied

But Alnwick's phrasing of Harclay's argument was not exactly as it was found in Harclay himself. As Alnwick had it, Harclay's argument went as follows:

> God actually sees or knows the first inchoative point of a line and any other point that may be designated in the same line. Therefore, God either sees that between this inchoative point of the line and any other point in the same line a line can intercede, or He does not. If not, then God sees point immediate to point, which is that proposed. If so, then, since in the mean line points could be assigned, these mean points would not be seen by God, which is false. The consequence is evident: For, by hypothesis, a line falls between the first point and any other point of the same line seen by God, and consequently there exists some mean point between this [first] point and any other point seen by God; therefore, that mean point is not seen by God.[19]

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at length in the question, 'Whether God knows an infinity of things,' while excluding the unsatisfactory replies of others who are ignorant of logic. However, I reply in brief that this is true: 'Between the first point of the line and every other point of the same line known by God (*omnem alium punctum eiusdem lineae cognitum a Deo*) there is a mean line.' For any singular [of this universal] is true, and, moreover, its contradictory is false. And this is so because the term 'mean line' in the predicate mediately following the universal sign has merely confused supposition. On the other hand, this is false: 'There is [some one] mean line between the first point and every other point (*omnem alium punctum*) of the same line seen by God,' since there is no [one] mean line between the first point and every other point seen by God. For no such mean line can be given (*non enim contingit dare aliquam talem lineam*), for it would fall between the first point and itself; nor would that line be seen by God. And therefore, when it is inferred: 'If there is [such a mean line], then, as points can be assigned in the line, etc.,' the term 'line' there has particular supposition. And hence an inference is made affirmatively from a superior to an inferior and thus the fallacy [of affirming] the consequent is committed. Similarly, an inference is made from a term having merely confused supposition to the same term having determinate or particular supposition, and *quale quid* is changed into *hoc aliquid*, and a fallacy of a figure of speech occurs. To the proof [of the consequence in Harclay's argument]: When he takes as that assumed 'a line falls between this first point and any other point of the same line seen by God (*quodlibet aliud punctum eiusdem lineae visum a Deo*),' it should be said that this is not what is posited or conceded. But this has been conceded if 'mean line' is placed in the predicate, namely [if we assert]: 'between the first point and any other (*quodlibet aliud*) point seen by God there is a mean line.' The other [proposition above] does not follow from this, as is clear for the reason stipulated. And thus it is clear that this doctor, howsoever subtle, has cozened by [committing] fallacies of the consequent and of a figure of speech." I have made some minor changes in John Murdoch's translation. For the Latin text see John Murdoch, "Naissance et Développement de l'Atomisme au Bas Moyen Âge Latin," *Cahiers d'études médiévales II: La science de la nature: théories et pratiques* (Montreal: Bellarmin; Paris: Vrin, 1974), p. 26 n.41; or Adam de Wodeham, *Tractatus de indivisibilibus*, ed. Rega Wood, p. 292 n.4.

19. Unpublished translation by Murdoch (typescript p. 34). Latin in Adam de Wodeham, *Tractatus de indivisibilibus*, p. 289 n.2.

Where Harclay rather awkwardly writes "up to that more immediate point that God sees there intervenes some line or not" *(usque ad illum punctum immediatiorem quem Deus videt intercipit ali[qu]a linea aut non),* Alnwick cites Harclay as referring not to the more immediate point that God sees but to any other point in the same line *(Aut igitur Deus videt quod inter hoc punctum inchoativum lineae et quodlibet aliud punctum in eadem linea potest linea intercipi, aut non)*. Moreover, at the start of the passage, Alnwick phrases Harclay as referring not only to any other point you please, but to any other point that may be designated *(quodlibet aliud punctum possible signari).*

From the awkwardness of Harclay's Latin, one could well suspect that the text has been miscopied in some way, yet it appears that he is not making exactly the mistake that Alnwick accuses him of—he is not asking whether there is a mean line between the first point and any other point of the line that might be designated, but rather whether there is a mean line between the first point and the nearer (nearest) of the remaining points *(illum punctum immediatiorem),* or between the first point and, collectively, all the other points of the line, where it is assumed, most importantly, that God sees which, of all the other points, is closest to the end point.

In an earlier version of a similar argument, Alnwick had phrased it in yet another way:

> Further, it is true that God actually perceives or knows all points which exist in a continuous line. For nothing is hidden from Him; "All things lie naked and unconcealed before His eyes." Hence, I ask whether between the first point of a line and any other point of the same line known by God there is, or is not, a mean point.[20] If there is, then there is some mean point between the first point of the line and any other point of the same line perceived by God, and consequently that mean point is neither the first point nor some other point perceived by God in the line, and thus it follows that that point is not perceived by God, which is false. If it be said that there is no mean point between the first point of the line and any other point perceived by God, then the first point and the other points known by God are immediate, and consequently a continuum is composed of indivisibles and is not divisible into always divisible [parts].[21]

This variant of the argument, using "any other point of the same line known by God," is much closer to Harclay's intent than the version that Alnwick directly ascribes to Harclay. The points Harclay had in mind are not poten-

20. Note that here the text says 'point' rather than 'line,' which may be a scribal error.

21. Unpublished translation by Murdoch (typescript pp. 8–9). I have not risked emending the translation to be consistent with my other translations, because I do not have the Latin text handy. Yet a third version of Harclay's argument appears in a shorter question on the continuum by Wodeham. See Murdoch and Synan, "Two Questions on the Continuum," p. 274.

tial points, not points that might be designated or chosen, but points all actually seen, perceived, or known by God. As God sees all these points clearly and distinctly, God can see which point is closest to the end point. Thus, in Harclay's version, the line in question (supposing it to exist) would be a determinate line—the line up to the more immediate point—and so the word 'line' would not have merely confused supposition, as Alnwick claimed. Translating medieval logical terms into more modern mathematical conceptions, where Alnwick says that Harclay's argument is fallacious because it converts the term 'mediate line' illegitimately from merely confused supposition to determinate supposition, we might say that Harclay supposes, without a proper warrant, that it is possible to designate, from all the rest of the points of the line other than the end point, that point which is nearest the end point. Reference to God provides Harclay with something like an axiom of choice. To Alnwick's claim that there is no one line between the first point and all the other points, Harclay might reply that humans might not be able to designate such a line (or the point nearest the end point), but God, since He sees everything naked and unconcealed, surely can designate anything that exists.

Although Wodeham, in his *Tractatus de indivisibilibus*, addressed Harclay's arguments as they were reported by Alnwick, nevertheless he noticed that Alnwick's criticism did not meet the entire force of Harclay's argument. Wodeham repeated Alnwick's claim that Harclay's argument commits the fallacies of a figure of speech and of affirming the consequent, but then he continued:

> But the argument can be made in the plural, and then it would be more tedious (*taediosius*). For I ask whether between the point *a* and all the other points of this line, there are some points or none. If there are none, and . . . *a* and all the other points of this line are some points, therefore between some points of this line there is nothing intermediate, and consequently some points of this line are immediate. This argument is tedious in some respects, and for the sake of [this] argument some deny points; others, who posit that the continuum is composed of points, concede it. And if I held [that there were] points or such indivisibles, I would willingly study carefully for a good solution. But since I do not posit indivisibles, I care less (*ideo minus curo*).[22]

22. Translation (slightly modified, with the omission of some clauses that seem duplicative) of Rega Wood, in Adam de Wodeham, *Tractatus de indivisibilibus*, pp. 104–5. As Wood notes (p. 293 n.5), Walter Chatton makes a similar point: "Argumentum est ad hoc commune in villa [Oxoniae]: inter primum punctum huius lineae et quodlibet aliud punctum eius aut est aliquod medium aut non. Si sic . . . . Respondetur [MS: respondeo] quod haec particularis est falsa: Aliquod punctum est medium inter primum et quodlibet aliud etc.; quia quaelibet singularis eius est falsa. Sed haec responsio non sufficit quia accipio totam multitudinem punctorum praeter primum, quia [aliquis] potest demonstrare quodlibet totius multitudinis simul, et signare [*MS adds as an alternate reading:* hoc punctum vel]

Here the argument is tedious because Wodeham seems to worry how he infers "some points" from "*a* and all the other points of this line" (in my translation I have replaced by ellipses some clauses that seem to go nowhere). Wodeham seems to be defensive about his assertion that all the other points of the line are some points, as well he might be, since his fellow nominalists commonly asserted in such a case that there is no such thing as "all the other points"—just as one cannot point to the multitude of all possible integers because any set of integers never contains all possible integers, so one cannot point to all the points of the line other than the first. In medieval logical terminology, one may say that "all the other points of this line" has merely confused supposition, making it impossible to descend to individual points. But the problem goes further. Medieval authors do not have the concept of a mathematical set. *A fortiori*, they do not have the concepts of fuzzy or open-ended sets. While it might be possible to consider a line missing its end point, "all the points of the line except the end point" raises greater problems. When a term posits for a collection of things with merely confused supposition, the medieval logical rule for merely confused supposition says that it is impossible to descend to true propositions concerning individual things. But what about the collection of things from which one is not allowed to choose individual ones? If one denies oneself the concept of set, how is it possible to point out an unended multitude of things?

In his shorter *Questio de Divisione et Compositione Continui*—which Rega Wood believes came before the *Tractatus de indivisibilibus*—Wodeham makes a similar argument, but he gives an example that implies that if the argument is taken in the plural, the predicate will not have merely confused supposition, but will rather be taken as singular. So, for example, he says, if the proposition is "All men are the world [*animal*[23]]," then it follows, "Therefore some world is all men." This is clear, he says, because it follows,

inter hoc punctum et primum etiam aliquod medium. Complete [MS: completum] igitur tota multitudine per talem demonstrationem, ipse inveniet aliquod punctum inter quod et primum non est medium." Thus the manuscripts of both Chatton and Harclay seem a bit garbled, but apparently both men have the idea that God sees all the other points of the line together and is able to see which of these points is closest to the end point. I wonder whether the manuscript might read "conspectum" at the beginning of the last sentence, with the sense: Take the whole multitude of points except the first (a person can point out any whole multitude at once). Take one point from the multitude and see whether there is a point between it and the first point. Take another point and see whether there is a point between it and the first point. Continue through the whole multitude. When one has completed looking over the whole multitude (conspectum), he will have found some point between which and the first there is no mean.

23. I cannot understand this example unless 'animal' means in some sense a singular thing, as in the recent song with the phrase, "We are the world."

"All men are the world, therefore these men"—pointing to all men— "are the world," and, consequently, "the world is these men."[24]

But if this logical form is applied to the case at hand, Wodeham argues, the response is false. It follows, if the argument is made in the plural, that between all the other points of the line (taken collectively) and *a* there is a mean point, therefore there is a mean point between *a* and all the other points of the line, but the antecedent of this inference is false. That is, it is not true that between all the other points of the line, taken collectively, and *a* there is a mean point.[25] If the argument is made in the plural and divisively, then even if, when all the points other than *a* are taken together, there is no medium between *a* and other points in the line, nevertheless the points will not be immediate to *a*. Just as it does not follow all these three are a triple, therefore any one of them is a triple, so it does not follow that all these points have no medium between themselves taken as a group and *a*, therefore they are immediate to *a*.[26]

In the shorter question, Wodeham then goes on:

> And Felthorp says that the multitude of all these points is immediate to *a*, but all these points are not immediate to *a*, because then it would follow that any one of them would be immediate to *a*. But this reply is

24. See Murdoch and Synan, "Two Questions on the Continuum," pp. 281–82: "Verumtamen istud argumentum potest reduci in aliam formam arguendo in plurali sic: quero utrum inter primum punctum huius linee quod sit *a*, et omnia alia puncta huius linee cadit aliquod punctum medium vel non? Si sic, igitur aliquod punctum cadit inter *a* et omnia alia puncta huius linee; consequentia patet quia terminus distribuens in plurali non facit terminum communem mediate sequentem stare confuse tantum, sed determinate quia bene sequitur: *omnes homines sunt animal, igitur aliquod animal est omnes homines*, quod patet. sequitur enim: *omnes homines sunt animal, igitur isti homines*, omnibus hominibus demonstratis, *sunt animal* et, per consequens, *animal est isti homines*. consimiliter in proposito. Si non, igitur *a*, et alia puncta sunt inmediata. ad quod respondentes dicunt aliqui quod non sunt aliqua puncta, alii vero, propter argumentum, concedunt quod continuum componitur ex indivisibilibus, asserentes hoc argumentum demonstrative hoc probare, sicud hercele."

25. Murdoch and Synan, "Two Questions on the Continuum," p. 282, "dico, igitur, pro argumento quod ly *omnia* potest teneri collective vel divisive; si collective, quando dicitur inter omnia puncta huius continui et *a* est punctus medius, neganda est."

26. Murdoch and Synan, "Two Questions on the Continuum," p. 282: "si divisive, dico quod hoc verum est de qualibet multitudine punctorum incepta ex alio extremo huius linee et terminata secundum aliud extremem citra *a* punctum [i.e. that there is a point between them and *a*]. si tamen loquamur de punctis inter quorum quodlibet et *a* est aliquis punctus istorum medius, dicendum est quod inter omnia puncta et *a* non est medium. et si arguitur ultra: igitur omnia illa et *a* sunt immediata, concedunt quidam consequenciam, quia sicud sequitur: *omnia illa animalia sunt alba, igitur quodlibet istorum est album*, ita in proposito. Sed, si predicatum sit terminus inportans multitudinem, non valeret consequencia quia ista consequencia non valet: *omnia tria sunt ternarius, igitur quilibet istorum est ternarius*, et ideo antecedens est falsum."

> false according to me. This may be proved expositorily as follows: these points are not immediate to *a*; but these points are this multitude; therefore, this multitude is not immediate to point *a*.[27]

Thus, again in the shorter question, Wodeham rejects a solution based on a distinction between a multitude or set and its members.

Let me return then to the discussion in the *Tractatus de indivisibilibus*. Although he says he cares less than those who posit indivisibles, Wodeham nevertheless wants to be sure that Harclay's argument is dealt with. Consequently, he reformulates Harclay's argument not in terms of points, which he rejects, but in terms of possible cuts *(incisiones possibiles)*, which he does accept. Then he asks, if a line is everywhere divisible and if one makes a first cut, is there a next possible cut to the first? Is there always a line between any two possible cuts, or can cuts be immediate to each other? He replies:

> when asked either about points or divisions or possible cuts, we should say: Either between a given possible and all other possibles in the same direction there is some possible medium, or there is not. It is said that the 'all' can be taken collectively or dividedly. If dividedly, we should say that yes, there would be [a possible medium] regarding all finites and also many infinites, since [there would be a medium] for all those that do not begin exclusively with *a*, but somewhere else determinate. . . . But if 'all' is taken collectively, so that it distributes the term for all taken together apart from *a*, then it is more difficult. But some say that none are 'all,' nor are some points [all] apart from *a*. But this reply is not pleasing to me.[28]

Up to this point, the argument for cuts has followed, step by step, that for points. Wodeham again raises but rejects a solution denying that there is any such thing as "all points other than *a*." He then proposes yet another solution. Accepting that nothing is between the first point or cut and all the other points or possible cuts taken collectively, Wodeham nevertheless denies that the first point and all the other points are immediate:

> I say alternatively and briefly that it does not follow that between these [and *a*] there is no intermediate, pointing to all the rest [*residuis*] apart from *a*, therefore these [and *a*] are immediates, pointing to *a* and the rest. Second I concede the further conclusion, namely that between some points of this line, if they are posited, there is nothing intermediate; nor does it follow on this account, as I have already said, that they are immediates.[29]

27. Murdoch and Synan, "Two Questions on the Continuum," p. 282.

28. Wodeham, *Tractatus de indivisibilibus*, pp. 104–7. I have modified the translation somewhat. The Latin of the last phrase is "mihi non placet."

29. Wodeham, *Tractatus de indivisibilibus*, pp. 106–7. I have modified the translation slightly.

Wodeham thus concludes in both the longer and the shorter work that even though all the points other than the end point, taken collectively, have no intermediate between themselves and the end point, nevertheless, since each individual point has something[30] between it and the end point, all of them together are not immediate to the end point.[31] Wodeham's arguments for these conclusions are confusing, in part because they are supposed to hold both for points and for possible cuts. To make it seem reasonable that things that have nothing intermediate between them may nevertheless not be immediate, Wodeham gives the example of lines that overlap, seeing that things that are immediate must be distinct. Another example might be a line of three points with spaces in between them. Then the points would have no intermediates, but they would not be immediate (in Aristotle's terms they would be successive, like houses along a street).

How successfully then, in the end, has Wodeham attempted to disprove Harclay's argument for the immediacy of indivisibles—always assuming, conterfactually, that indivisibiles exist, or, alternatively, replacing talk of points with talk of possible cuts? If Harclay's argument is made in the singular, as Alnwick reports it, then Wodeham will answer, just like Alnwick, that Harclay's argument commits the fallacy of arguing from the term "line" with merely confused supposition to the term "line" with determinate supposition. But if the argument is taken more nearly as Harclay made it, saying, in effect, that the end point of a line is immediate to all the rest of the points of the line taken collectively and therefore immediate to that point of the other points that is nearest to the end point, Wodeham's answer in effect denies that one can go through the multitude of points one by one until one finds that point between which and the end point there is no intermediate. Although Wodeham is not satisfied with the argument of those who say that you cannot refer to all the rest of the points other than the end point, he resists Harclay's (and Chatton's) next step, which is to assume that God can examine the points one by one until He finds that one between which and the end there is no intermediate. To the contrary, any point that one might pick out of the rest is not immediate to the end point. Whereas we might be inclined to say that the set of all points other than the end point is immediate to the end point—or at least to ask about the relation of the end point to all the other points—Wodeham makes use of a logical distinction. When a predicate imports a multitude, he claims, then inference from a universal affirmative to the individuals referred to is illegitimate. If three things are a triple, it does not follow that any one of them is a triple. Likewise, it may be true that the points of the line other than the end point may have no point

30. Wodeham says "indivisible," which makes no sense for those who deny indivisibles.

31. Wodeham, *Tractatus de indivisibilibus*, p. 106, "Nec tamen videtur quod sint immediate ipsi *a*, cum per casum nulla demonstrentur nisi talia inter quorum quodlibet et *a* aliquod indivisible mediat."

intermediate between themselves and the end point and yet false that these points are immediate to the end point. These points collectively are nothing but these points singly, and none of these points is immediate to the end point.

How the argument works out concerning all the possible cuts of a line becomes clearer in Wodeham's answer to his fourth question, "whether the infinite divisibility of a continuum in its extension . . . can be reduced to actuality." To answer this question, Wodeham again considers the relations of a universal affirmative proposition to the singulars contained under it. Normally, if a universal affirmative is true, then each singular falling under it is true and vice versa. If all crows are black, then each individual crow is black and vice versa. But sometimes a different relation between a universal proposition and the singulars falling under it occurs: even if all the apostles are twelve, it does not follow that any individual apostle is twelve. And even if each proper part of Socrates is less than Socrates, it does not follow that all the parts of Socrates taken together are less than Socrates. The differences in these cases are explained by the well-known distinction between compounded and divided senses.

In reply to Question IV, then, Wodeham uses the example, "Every ass is seen by a man." Supposing that each ass is seen by a different man, then it is true that "For every ass, a man sees him," but it is false that "a man sees every ass."[32] To the question, then, Wodeham argues analogously:

> I maintain that if we put 'distinct' before the verb 'can', then into all its parts, finite in number, totally distinct from one another, a continuum can at once be divided and have been divided, speaking according to the Philosopher's understanding. This is one universal whose every particular is true. For this valid proposition is true: 'All completely distinct, finite, lengthwise parts of a continuum can be divided and have been divided.'[33]

This is to be understood on the model of the asses each seen by a different man—it is true of the singulars falling under the universal, although not for all of them taken together: if you take any distinct parts, such as halves, they can have been divided from each other; if you take one-thousandths, they can have been divided, and so forth. But from this it does not follow that all the distinct parts can have been divided all at once, because some distinct parts are not distinct from other distinct parts, as the halves are not distinct from the thousandths.

32. Wodeham, *Tractatus de indivisibilibus,* pp. 224–25.

33. Wodeham, *Tractatus de indivisibilibus,* pp. 224–25. Wodeham says "speaking according to the Philosopher's understanding," because he has argued that when a continuum is divided, it ceases to exist. He concludes, "it does not seem that they would be immediate to *a* itself, since in this case nothing is pointed to except those things between any one of which and *a* some indivisible mediates."

With this rationale, Wodeham concluded further [Conclusion 2]:

> notwithstanding what has been said above, speaking according to the Philosopher's understanding, a continuum can at once be divided and also at once have been divided into parts totally distinct from each other, infinite in multitude, even taking 'infinite' categorematically. That is, infinite parts of a continuum, totally distinct from one another, can be divided and have been divided from each other; indeed, infinitely times infinite.[34]

Granted that all the various parts into which a continuum might be divided are actually or categorematically infinite, though not all distinct from each other, then any set of distinct parts can have been divided from each other, and any other set of distinct parts, and so forth—running through the infinitely many parts contained within the whole. Speaking according to the Philosopher's understanding, all these parts may be referred to together as parts of a single continuum—and, as such, they may be categorematically infinite—but in fact they are never divided at one and the same time. Thus, Wodeham can go on to conclude, speaking not according to Aristotle's understanding, that no continuum can be divided at the same time into all its (categorematically infinitely many) parts (Conclusion 3).[35] Nor, on Wodeham's view, can all the totally distinct parts of a continuum be divided at once or have been divided from each other (Conclusion 4).[36] What differentiates Conclusion 2 from Conclusions 3 and 4 is that Conclusion 2 refers to the continuum as if it continues to exist once it has been divided—which Aristotle assumes but Wodeham himself denies—whereas Conclusions 3 and 4 refer only to the distinct parts. In a doubt following just after these conclusions, Wodeham considers Scotus's distinctions between the force of '*quodcumque*,' '*omnis*,' and '*quilibet*.' According to Scotus, one can concede that a continuum may be divided at '*quodcumque signum*,' because '*quodcumque*' is not only distributive, but also partitive.[37] Then for a proposition containing '*quodcumque*' to be true the predicate need not be attributed to every singular contained under the subject at once.

Working in the plural and accepting actual infinites, but denying that a continuum continues to exist after it is cut, Wodeham accepted much of the force of Grosseteste's and Harclay's points of view, including the view that one infinite can be larger than another, though both are infinite. By insisting upon talking about the parts of a continuum that can be divided

34. Wodeham, *Tractatus de indivisibilibus*, pp. 224–25. I have modified the translation very slightly.

35. Wodeham, *Tractatus de indivisibilibus,* p. 226, "nullum continuum simul dividi possit seu [habere] partes ab invicem divisas."

36. Wodeham, *Tractatus de indivisibilibus*, p. 226, "nec praeponendo signum huic verbo 'potest' est verum dicere quod in omnes partes suas totaliter distinctas abinvicem potest continuum dividi vel divisum esse, id est non omnes tales partes continui possunt simul ab invicem dividi, nec esse divisae."

37. Wodeham, *Tractatus de indivisibilibus*, p. 226.

from each other, rather than talking about the infinite divisibility of a single continuum, Wodeham created a framework in which he could use distinctions between compounded and divided senses to distinguish what he was willing to accept from what he wanted to reject. If the problem was phrased in terms of what would be for us infinite sets of points, then Wodeham wriggled out of Harclay's conclusion by denying that there is a transparency between all the points and individual points. If the problem was phrased in terms of possible cuts, then Wodeham denied that one could refer to all the distinct parts actually divided from each other at the same time: in other words, all the distinct parts were not compossible with each other.

In his essay "Adam Wodeham's Anti-Aristotelian Anti-Atomism," Norman Kretzmann presents Wodeham's position in Question IV of the *Tractatus de indivisibilibus* as frivolous and perhaps deserving of the sort of criticism often heaped upon the scholastics in general.[38] Although he recognizes that Wodeham's basic point in denying that a continuum can be divided at all is that in any change there must be a persistence of the subject—a central Aristotelian position—Kretzmann seems to think that Wodeham's solution to the problem of the infinitely divisible continuum is to deny that any division at all can occur, or what Kretzmann calls "macro-indivisibilism."[39] When Wodeham restricts a proper description of division to a division of pre-existing parts from each other (rather than speaking about cutting up the original whole), Kretzmann concludes that "Wodeham is thereby committed unwittingly and unwillingly to the actual existence of infinities."[40] This, Kretzmann calls a scandal.

But was Wodeham really so opposed to the actual existence of infinities? I think not. Like Grosseteste and Harclay, Wodeham accepted that in a larger continuum there are more proportional parts than in a smaller one, even though there are infinitely many parts in each—in a part of the larger continuum equal to the smaller one there will be just as many parts as in the smaller one, and then there will be other parts besides.[41] Like Ockham, Wodeham agreed that actually infinitely many proportional parts exist within a continuum, although not infinitely many parts of the same size.

But what was accomplished by Wodeham's insistence on talking of the division of parts from each other rather than talking of the cutting of the original continuum was to require that propositions about cutting a continuum be taken divisively—at any given time all one can speak of is the division of one set of distinct parts from each other or another set from each

38. Norman Kretzmann, "Adam Wodeham's Anti-Aristotelian Anti-Atomism," *History of Philosophy Quarterly* 1 (1984): 389. This paper, submitted well before Norman Kretzmann's recent death, was written with the expectation that he would be able to consider my suggestions and, if he saw fit, dispute them or offer alternatives. I very much regret that his death has deprived the scholarly community of his continuing insight into the enigmas of fourteenth-century thought.

39. Kretzmann, "Adam Wodeham's Anti-Aristotelian Anti-Atomism," p. 389.

40. Kretzmann, "Adam Wodeham's Anti-Aristotelian Anti-Atomism," p. 391.

41. Wodeham, *Tractatus de indivisibilibus,* pp. 236–37.

other—but not all together in the compounded sense, because some of the parts at issue overlap with, and are not distinct from, other parts. If Wodeham's conclusion to Question IV, then, is used to clarify his conclusions to Question I, it is easier to see how talk about possible cuts differs from talk about points. If points exist in reality and if all things that exist are clearly seen by God, then Harclay's argument, with its conclusion that points must be immediate to each other, might have some force. If, on the other hand, one is talking only about possible cuts, it does not so clearly follow that God must see clearly all at once all the possible distinct parts collectively—instead they appear divisively, as alternatives. Suppose that over an infinite future time, in each unit of time, smaller and smaller parts are divided from each other. Then, given parts of any size, at some future time those parts will be divided and will have been divided from each other, but all the divisions will never have occurred simultanously. If, following Aristotle, one speaks improperly of all these parts being parts of the original continuum, then there will seem to be an actual or categorematic infinite multitude of parts in the continuum. But if one admits that it is improper to speak as Aristotle does, because there is no one thing that continues to exist while it is divided into more or fewer parts, then there is never one thing of which it is true to say that it is actually infinite, as in modern mathematics the set of all the possible parts of the continuum may be said to be infinite.

Wodeham's most direct evasion of Harclay's theological argument was to deny that indivisibles exist in reality. If parts actually exist within a whole continuum, nevertheless their actual division from each other is something that can occur only successively. While for Harclay all the infinitely many points coexisting within a line are clearly seen by God and therefore seen to be next to each other, for Wodeham even God need not see all the distinct parts within the continuum at once because they exist only successively or in a divided and partitive sense, and not all at once.

In a footnote to his article on Wodeham, Norman Kretzmann refers to John Murdoch's presentation of "evidence that medieval indivisibilism may have arisen in connection with the consideration of angelic motion (in Scotus) or of the eternity of the world (in Harclay)."[42] He comments, "The motivations provided in those considerations strike me as insufficient to explain the adherence of philosophers such as Henry of Harclay, Walter Chatton, Gerard of Odo, and Nicholas Bonet to indivisibilism in the face of all its embarrassing consequences." Kretzmann then asks:

> Is it thinkable that having recovered the *Posterior Analytics* and Euclid's *Elements*, and taking Euclid to have supplied the paradigm of an Aristotelian science, men were led to think that geometry could not be a science unless its primary ingredients—e.g. points, lines, and planes—were real? Such a geometric realism strikes me as a possible

42. Kretzmann, "Adam Wodeham's Anti-Aristotelian Anti-Atomism," p. 398 n.46.

> further motivation for fourteenth-century indivisibilism, but I can offer it only as a likely story, having no evidence that that line of thought actually led any indivisibilist to his position.[43]

As far as I can see, the situation was, ironically, almost the direct contrary of what Kretzmann here suggests. If we examine the manuscript sources, it is clear that so-called atomism did arise in the contexts of the motion of angels and the eternity of the world. On the other hand, far from a desire to preserve Euclidean geometry leading Harclay and others to embrace mathematical realism and hence the existence of indivisibles like lines, points, and planes, their mathematical realism—supposing, for instance, that points really exist—led them to anti-Euclidean conclusions. It was then in part a desire to protect Euclid and Aristotle from Harclay's consequences that led such men as Adam Wodeham and John Buridan to deny the real existence of indivisibles like points. Then mathematical entities exist only in the imaginations of mathematicians and not in the outside world. It is not necessary to concede that God sees them all clearly and distinctly.

There then seem to have been two viable positions: to follow Aristotle and Euclid, but to suppose that mathematical entities exist only in the minds of mathematicians; or, alternatively, to follow Grosseteste and Harclay in accepting God's knowledge of infinitely many indivisibles within continua, but then to accept the non-Aristotelian conclusion that infinites can have ratios to each other or even that one indivisible could be immediate to another. While the latter position contradicts the views of modern mathematics, it was no scandal—it may even have developed patterns of thought useful in the development of the calculus or the analysis of infinites. In attempting to disarm Harclay's arguments, Wodeham used logic rather than physics, theology, or mathematics. Modern mathematics also uses logic to lay its foundations. As compared to the dominant current approach to the mathematics of the continuum, Harclay's position seems closer to intuitionism, in the sense that he is working on the assumption that mathematical entities can be clearly conceived—if not by humans, then certainly by God. If, instead of immediately judging that Harclay was wrong, we suspend disbelief; then there is much to learn about the nature of fourteenth-century philosophy, theology, logic, and mathematics, through a patient examination of scholastic debates on indivisibles and the continuum.

43. Kretzmann, "Adam Wodeham's Anti-Aristotelian Anti-Atomism."