L^P-ESTIMATES AND POLYHARMONIC BOUNDARY VALUE PROBLEMS ON THE SIERPINSKI GASKET AND GAUSSIAN FREE FIELDS ON HIGH DIMENSIONAL SIERPINSKI CARPET GRAPHS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2014

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Cornell University 2014

We define a suitable trace space on the set X halving the Sierpinski Gasket, then we prove L^p-estimates for $p > 1$ for the restriction operator on $\text{dom}_{L^p}\Delta(\text{SG})$. We also construct a right inverse to the restriction operator, that is the extension operator, and provide similar L^p-estimates. Then, we consider the polyharmonic boundary value problem which involves finding a biharmonic function with prescribed values and Laplacian values on the bottom line (identified with the interval) and top vertex of the SG. After constructing a suitable orthogonal basis of piecewise biharmonic splines, we express the solution to the BVP in terms of the Haar expansion coefficients of the prescribed data and this basis. After constructing a Sobolev type space on SG, which is analogous to the H^2-Sobolev space in classical analysis, we prove how smoothness of the prescribed data is reflected in the smoothness of the solution to the BVP. In the second part of the thesis, we focus on Gaussian Free Fields on High dimensions Sierpinski Carpet graphs. We assume that a “hard wall” is imposed at height zero so that the field stays positive everywhere. Our first result, in the second part of the thesis, is a large deviation type estimate which identifies the rate of exponential decay for $\mathbb{P}(\Omega^+_{V_N})$, namely the probability that the field stays positive. Then, in our second theorem we prove the leading-order asymptotics for the local sample mean of the free field above the hard wall on any transient Sierpinski carpet graph.

BIOGRAPHICAL SKETCH

Baris Evren Ugurcan was born in Corum, Turkey in 1985. His interest in mathematics started from early ages. After obtaining a B.S. and M.S. degree from Bilkent University (Turkey), he went on to Cornell University to obtain his PhD in Mathematics under the direction of Professor Robert S. Strichartz. His research interests include analysis and stochastic processes on graphs and their natural scaling limits: fractals and operator theory. He has many research papers with several coauthors in this areas.

Dedicated to my family.

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my advisor Professor Robert S. Strichartz. I am grateful to him for introducing me to this field and supporting my research. His patience and full support made possible this thesis. Especially, his putting his full trust to his students made my PhD years a very productive and rewarding experience. As a reflection of his trust, in the future, I will try to achieve and prove even grater things and I am sure that his guidance and support will continue. I am also sure that our collaboration, which was established long before, will continue in the future. It is a great experience learning from him!

I also would like to thank my "big sibling" and collaborator Joe P. Chen. He has always been a model-mathematician for me who never hesitated to share his precious experiences. Our collaboration proved fruitful and I expect it to continue in the future.

I would like to thank Professors Laurent Saloff-Coste and Camil Muscalu for serving on my committee and many fruitful discussions.

My thanks also goes to many mathematicians whose works and as a person has been inspiring for my career. Among them are: Professors Len Gross, Lionel Levine, Palle Jorgensen, Luke Rogers, Sasha Teplyaev and Ben Steinhurst.

It is my pleasure to thank my collaborators Professor Lionel Levine, Professor Yuval Peres, Mathav Murugan, Ben Li and Nick Ryder. I am so happy having the chance to work with these people!

I also would like to thank my family, to whom I dedicated this thesis, for their continuous support since my childhood.

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

In this introduction chapter, we basically give a summary of the relevant literature and results established in each chapter followed by a light introduction section. Our pace in the introduction chapter is lively and intuitive whereas in the following chapters we become more technical and give full detailed proofs of the results introduced before, followed by a thorough background section. We start our journey with a boundary value problem (BVP) on the Sierpinski Gasket (SG).

1.2 L^p-Estimates on SG

The first construction of Laplacian on the Sierpinski Gasket dates back to 1987 [G87] [K89]. The laplacian was constructed as a generator of a stochastic process. However, later on an analytic realization of the Laplacian as a renormalized limit of Graph Laplacians was established by Kigami [Ki1]. Kigami's theory develops many tools and analytical objects which is specific to the fractals such as renormalized graph energies, normal derivatives and renormalized graph laplacians.

Although, there has been much work on analysis on fractals since then, the research on boundary value problems on bounded subsets of fractals has just taken off. We mention some recent work on this area: [OS] [LS] [GQS]. As there is much research on function spaces on fractals [Str03], [Str99], [HK06], [HZ], [LRSU] it is possible to ask questions such as:

- (Extension operators) Given a bounded set Ω on SG and some function f on this subset lying in a certain function space (e.g. $C(\Omega))$). Is it possible to extend this function such that the extension lies in dom$(\mathcal{E})$, where $\mathcal{E}$ denotes the self-similar energy on SG?
- (Restriction Operators) Given a function f in a certain function space on SG (e.g. dom$(\mathcal{E})$) does its restriction on Ω lie in a certain function space on Ω (e.g. $C(\Omega))$)?

  Of course, by the currently established theory, these questions and similar questions not only make sense on SG but also Kigami's PCF (post-critically finite) fractals [Ki1]. Extending this theory to other examples such as the Sierpinski Carpet is certainly a challenge.

1.2.1 Results

In Chapter 2, we report our results obtained in [U]. In our work, building up the work in [LS], we extend the L^p estimates for the extension and restriction operators for all $p > 1$.

As in [LS], we work on the half-Sierpinski Gasket shown in Figure 1.1. For simplicity, using the same notation in Figure 1.1, we put $X = \{x_m\}, a_m := u(x_m), \eta_m := \partial_n u(x_m)$. y_m and Y_m are also defined as shown in Figure 1.1. As we will be interested in extension and restriction operators, we define z_m and Z_m to be the reflections of y_m and Y_m across the symmetry line containing X.

Also, define the restriction map by $Ru := \{(a_m, \eta_m)\}$ where u is a function defined on a set containing X. We say that $u \in \mathrm{dom}_{L^p}\Delta(\mathrm{SG})$ if u is continuous on SG and $\Delta u \in L^p(SG)$. We study the image of the restriction operator R and give estimates on the norm.

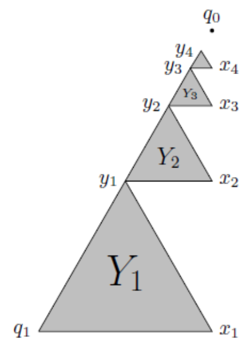

Figure 1.1: Half-Sierpinski Gasket (SG)

Let $p > 1$ and q be its conjugate. We consider the following trace space

$$\mathcal{T}_p := \{(a_m, \eta_m) \mid a_m = A_1 + A_2(3/5)^m + a'_m,\ \|(5^p/3)^{m/p} a'_m\|_p, \|3^{\frac{m(p-1)}{p}} \eta_m\|_p < \infty\}$$

with the norm given by

$$\|(a_m, \eta_m)\|^p_{\mathcal{T}_p} = |A_1|^p + |A_2|^p + \|(5^p/3)^{m/p} a'_m\|^p_p + \|3^{\frac{m(p-1)}{p}} \eta_m\|^p_p.$$

For harmonic functions on SG we have the following result.

Lemma 1.2.1. *If h is a harmonic function, then $h \in \mathcal{T}_p$ with*

$$\|Rh\|^p_{\mathcal{T}_p} = |h(q_0)|^p + \frac{1}{2^p}|h(q_1) + h(q_2) - 2h(q_0)|^p + \frac{1}{2^{p+1}}|h(q_1) - h(q_2)|^p. \tag{1.1}$$

We define the following norm on $\mathrm{dom}_{L^p}\Delta(\mathrm{SG})$:

$$\|u\|_{\mathrm{dom}_{L^p}\Delta(\mathrm{SG})} = \|u\|^p_{L^\infty(SG)} + \|\Delta u\|^p_p.$$

In the following result, we give an estimate on $\|Ru\|_{\mathcal{T}_p}$ in the case u vanishes on the analytical boundary of SG, namely $u|_{V_0} = 0$.

Theorem 1.2.2. *Let $p > 1$, $u \in dom_{L^p}\Delta(SG)$ and $u|_{V_0} = 0$. Then, $Ru \in \mathcal{T}_p$ and*

$$||Ru||_{\mathcal{T}_p} \le C||\Delta u||_{L^p(SG)}. \tag{1.2}$$

Putting together Lemma 1.2.1 and Theorem 1.2.2 we obtain the following estimate on $||Ru||_{\mathcal{T}_p}$ in the general case.

Theorem 1.2.3. *(L^p- Trace Theorem) The restriction operator $R : dom_{L^p}\Delta(SG) \to \mathcal{T}_p$ is bounded and*

$$||Ru||_{\mathcal{T}_p} \le C_1||u||_{L^\infty(SG)} + C_2||\Delta u||_{L^p(SG)}.$$

Of course, the curious question is whether we can find a right inverse to the restriction operator. Answering the question, in the following Theorem we give the existence of a right inverse to the restriction operator R.

Theorem 1.2.4. *(L^p-extension Theorem) Let $p > 1$. There exists a bounded linear extension map $E : \mathcal{T}_p \to dom_{L^p}\Delta(SG)$ with $R \circ E = Id$.*

1.3 Polyharmonic Boundary Value Problems on SG

In this introductory section and in the corresponding chapter (Chapter 3) we denote Sierpinski Gasket by SG or K. L denotes the bottom line of the SG naturally identified with the unit interval $[0,1]$. If v_1 denotes the top vertex of the SG we put $\overline{SG} = SG \smallsetminus L \cup v_1$. Ψ denotes the function $\Psi = 1$ on the relevant domain. ψ denotes the skew-symmetric mother wavelet on $[0,1]$ i.e. 1 on $[0,1/2]$ and -1 on $(-1/2,1]$. If w is the word in $0,1$'s telling us the address of a dyadic interval, ψ_w denotes the scaled down copy of the mother wavelet ψ onto this dyadic interval.

We solve the following boundary value problem on the Sierpinski Gasket

$$\Delta^2 u = 0$$
$$\Delta u|_L = f_2$$
$$\Delta u(v_1) = c'$$
$$u|_L = f_1$$
$$u(v_1) = c.$$

We present a solution to the above BVP in terms of the coefficients of the Haar expansion of f_1 and f_2 on L i.e. on the unit interval. We take our analysis further and also investigate which Sobolev spaces f_1 and f_2 should belong to so that the solution u will belong to the space on the SG which is analogous to the H^2 Sobolev space in the classical case with norm given by

$$(u,u) = \int (u|_L)^2 dx + \mathcal{E}(u,u) + \int_{SG} (\Delta u)^2 d\mu < \infty.$$

1.4 Results

We construct a basis $\{h_0, h_2, h_3, h_\psi^1, h_w^1, h_\psi^2, h_w^2\}$ on SG and express the solution to the BVP as an infinite series in terms of this basis. We provide necessary and sufficient conditions, namely we prove the following main results:

Theorem 1.4.1. *Suppose the Haar series of f_1, f_2 on L is given by*

$$f_1 = c_\Psi(f_1)\Psi + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_1)\psi_w \tag{1.3}$$

$$f_2 = c_\Psi(f_2)\Psi + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_2)\psi_w. \tag{1.4}$$

Then the unique biharmonic function satisfying

$$\Delta^2 u = 0$$

$$\Delta u|_L = f_2$$

$$\Delta u(v_1) = c'$$

$$u|_L = f_1$$

$$u(v_1) = c$$

is given by

$$u(x) = (c' - c_{\Psi(f_2)})h_3(x) + c_\Psi(f_1)\Psi + (a - c_\psi(f_1))h_0(x) + c_\Psi(f_2)h_2 + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_1)h_w^1(x) + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_2)h_w^2(x) \tag{1.5}$$

The $(\cdot,\cdot)$*-norm of* u*, in the case* $c' = 0, c_\Psi(f_2) = 0$*, is given by*

$$(u,u) = |c_\Psi(f_1)|^2 + \sum_{m=0}^{M} \sum_{|w|=m} |c_w(f_1)|^2 + (c - c_\psi(f_1))^2 E_0 + E_1 \sum_{m=0}^{M} \sum_{|w|=m} \left(\frac{10}{3}\right)^m |c_w(f_1)|^2 + E_2 \sum_{m=0}^{M} \sum_{|w|=m} \left(\frac{2}{15}\right)^m |c_w(f_2)|^2 + + L_1 \sum_{m=0}^{M} \sum_{|w|=m} \left(\frac{2}{3}\right)^m |c_w(f_2)|^2 \tag{1.6}$$

where $E_0 = \mathcal{E}(h_0), E_1 = \mathcal{E}(h_\psi^1), E_2 = \mathcal{E}(h_\psi^2), L_1 = (h_\psi^2, h_\psi^2)_4$.

Theorem 1.4.2. *A biharmonic function u on $\overline{SG}$ has*

$$(u,u) = \int (u|_L)^2 dx + \mathcal{E}(u,u) + \int_{SG} (\Delta u)^2 d\mu < \infty$$

if and only if $u|_L = f_1$ is in $H^{s_1}(L)$ for $s_1 = \frac{\log \frac{10}{3}}{\log 4} > 0$ and $\Delta u|_L = f_2$ is in $H^{s_2}(L)$ for $s_2 = \frac{\log \frac{2}{3}}{\log 4} < 0$. In this case, (1.3), (1.5) and (1.6) hold for $M = \infty$ and (1.4) converges in the H^{s_2} norm to f_2.

Theorem 1.4.3. *Let u be a function in $\overline{SG}$ with $(u,u) < \infty$. Then, u has boundary values with $u|_L = f_1$ in $H^{s_1}(L)$ for $s_1 = \frac{\log \frac{10}{3}}{\log 4} > 0$ and $\Delta u|_L = f_2$ in $H^{s_2}(L)$ for $s_2 = \frac{\log \frac{2}{3}}{\log 4} < 0$.*

1.5 Gaussian Free Fields on High Dimensional Sierpinski Carpet Graphs

The discrete GFF on a graph is a centered Gaussian process whose covariance matrix is given by the Green's function of the corresponding Laplacian operator on the graph. As a simple example, if we have just one point, we can assign a random variable to this point in which case we have a single random variable. If we consider 2 points now, we can assign 2 independent Random variables, in which case we have a 2-tuple of independent Gaussian variables. We can go on for sure, but we can do something a little more complicated: we can take a graph G and put a centered Gaussian random variables to each vertex $x \in G$ that we denote by ϕ_x. Rather than having independent Gaussian random variables we can put correlations namely $\mathrm{cov}(\phi_x, \phi_y) = G(x,y)$ where G is the Green's function corresponding to the Laplacian operator on the graph.

Another way to go about is to directly give the Dirichlet energy. Recall that in a general context the Dirichlet energy is defined by $\mathcal{E}(f,f) = (\Delta f, f)$. Let $\mathcal{G} = (V,E)$ be a finite or transient infinite graph, and let

$$E_{\mathcal{G}}(f) = \frac{1}{2}\sum_{\substack{x,y\in V\\ x\sim y}} [f(x) - f(y)]^2 \quad (f : V \to \mathbb{R})$$

be the Dirichlet energy on $\mathcal{G}$. A *Gaussian free field* on $\mathcal{G}$ is a collection of Gaussian random variables $\varphi = \{\varphi_x\}_{x\in V}$ with mean zero and covariance given by the Green's function $G(x,y)$ $(x,y \in V)$ for simple random walk on $\mathcal{G}$. Formally, the law of φ has density proportional to $\exp\left(-\frac{1}{2}E_{\mathcal{G}}(\cdot)\right)$ with respect to the Lebesgue measure on $\mathbb{R}^V$.

1.5.1 Results

In this section, we summarize our results obtained in [CU] (joint with Joe P. Chen).

Now, assume that a "hard wall" is imposed at height zero so that the field stays positive everywhere. We prove the leading-order asymptotics for the local sample mean of the free field above the hard wall on any transient Sierpinski carpet graph. Therefore, we extend the results of Bolthausen, Deuschel, and Zeitouni [BDZ95] for the free field on $\mathbb{Z}^d$, $d \geq 3$, to the fractal setting. In our proofs, we heavily use the theory of transient regular Dirichlet forms together with coarse graining, and conditioning arguments introduced in the previous literature. Thus, our results stands as a fine blend of analytic, in particular potential theoretic, and probabilistic techniques.

In what follows, F is a *transient* generalized Sierpinski carpet, namely a Sierpinski Carpet on which Brownian motion is transient. Recall that the existence

$\mathbb{Z}^d$, $d \geq 3$	Infinite graph	Transient GSC graph
$[-\frac{L}{2}, \frac{L}{2}]^d \cap \mathbb{Z}^d$	Approximating subgraph ("box")	$\mathcal{G}_N = (V_N, \sim)$
L	Side length of box	ℓ_F^N
$\asymp L^d$	Volume of box	$\asymp m_F^N$
$\asymp L^2$	Expected escape time of random walk from box	$\asymp t_F^N$
$\asymp L^{2-d} \in (0,1)$	Resistance across opposite faces of box	$\asymp \rho_F^N = (t_F/m_F)^N \in (0,1)$

Table 1.1: Comparison of relevant parameters on $\mathbb{Z}^d$ and on the Sierpinski carpet graph.

of a Dirichlet form was proved in [BB99] [KusuokaZhou] and [BBKT] as a scaling limit of corresponding Dirichlet forms on Sierpinski Carpet graphs.

Rather than writing out the definitions in full length, we give an impressionist sketch through Table 1.1 and Figure 1.2 for the reader's convenience.

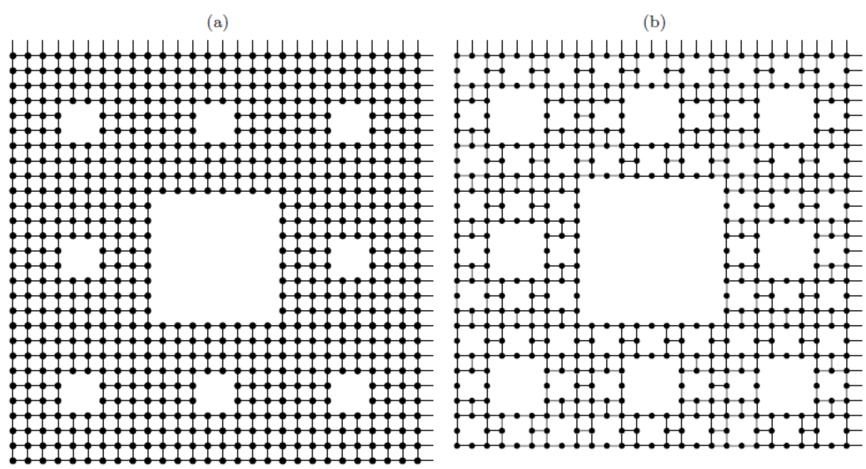

Figure 1.2: The 3rd-level approximation of, respectively, the outer Sierpinski carpet graph $\mathcal{G}_\infty$ and the inner Sierpinski carpet graph $\mathcal{I}_\infty$, here shown for the standard 2-dimensional Sierpinski carpet. According to the conventions in the text, when embedded in $(\mathbb{R}_+)^d$, the least vertex of $\mathcal{G}_\infty$ is situated at the origin, while the least vertex of $\mathcal{I}_\infty$ is situated at $(\frac{1}{2}, \cdots, \frac{1}{2})$. All edges have Euclidean distance 1.

We introduce some notations before we give a summary of our results. As shown in Figure 1.2, $\mathcal{G}_\infty$ denotes the outer Sierpinski Carpet graph and $G_{\mathcal{G}_\infty} : V_\infty \times V_\infty \to \mathbb{R}$ the Green's function for simple random walk thereon. We denote the measure space on the Sierpinski Carpet F by (F, ν), where ν is the constant multiple of the $d_h(F)$-dimensional Hausdorff measure on F. We also consider the unbounded version of the Sierpinski Carpet$F_\infty := \bigcup_{N=0}^\infty \ell_F^N F$ and ν_∞, accordingly, the σ-finite self-similar Borel probability measure on F_∞, assigning mass m_F^N to $\ell_F^N F$.

We also recall the notion of the (0-order) capacity of the compact carpet F with respect to a Dirichlet form $(\mathcal{E}, \mathcal{F}) \in \mathfrak{E}$ on $L^2(F_\infty, \nu_\infty)$, given by

$$\mathrm{Cap}_{\mathcal{E}}(F) := \inf\{\mathcal{E}(f,f) : f \in \mathcal{F} \cap C_c(F_\infty), f \geq 1 \text{ a.e. on } F\},$$

Let $\mathbb{P}$ be the law of the Gaussian free field on $\mathcal{G}_\infty$ with covariance $G_{\mathcal{G}_\infty}$, and let $\Omega^+_{V_N}$ denote the entropic repulsion event $\{\varphi_x \geq 0 \text{ for all } x \in V_N\}$. Our first main result identifies the rate of exponential decay for $\mathbb{P}(\Omega^+_{V_N})$.

Theorem 1.5.1. *There exists positive constants C_1 and C_2 such that*

$$-C_1 \leq \varliminf_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} \log(t_F^N)} \leq \varlimsup_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} \log(t_F^N)} \leq -C_2 \tag{1.7}$$

We elaborate on where the constants come from in Chapter 4, actually we give explicitly what the constants are. The constants C_1 and C_2 depend on $\mathrm{Cap}_{\mathcal{E}}(F)$ and the Green's function $G_{\mathcal{G}_\infty}$. Two other sources which needs to be considered for explicit computation of constants are: comparing the Dirichlet forms on $\mathcal{G}_\infty$ and on $\mathcal{I}_\infty$ and comparing the (maximal or minimal) cluster point of the sequence of renormalized Dirichlet forms on $\mathcal{I}_\infty$ with an element of $\mathfrak{E}$.

Although the upper and lower bounds in Theorem 4.1.1 are different, we are still able to give a precise description of entropic repulsion on $\mathcal{G}_\infty$. In the following Theorem, we prove that conditional on $\Omega^+_{V_N}$, the local sample mean of the free field on V_N is pushed to a height which is proportional to $\sqrt{N}$, and as $N \to \infty$, the rescaled height converges in probability to a constant.

Theorem 1.5.2. *For any $\epsilon > 0$ and $\eta > 0$,*

$$\lim_{N\to\infty} \sup_{\substack{x\in V_N \\ V_{N,\epsilon}(x)\subset V_N}} \mathbb{P}\left(\left| \frac{\bar{\varphi}_{N,\epsilon}(x)}{\sqrt{\log(t_F^N)}} - \sqrt{2\underline{G}} \right| \geq \eta \;\middle|\; \Omega^+_{V_N} \right) = 0, \tag{1.8}$$

where $\bar{\varphi}_{N,\epsilon}(x) := \frac{1}{|V_{N,\epsilon}(x)|} \sum_{z\in V_{N,\epsilon}(x)} \varphi_z$ *and* $V_{N,\epsilon}(x) := \left\{ z \in V_N : \max_{1\leq i\leq d} \left| z_i - [\ell_F^N x_i] \right| \leq \epsilon \cdot \ell_F^N \right\}$.

CHAPTER 2

L^P-ESTIMATES ON SG

2.1 Background and Notation

In Chapters 2 and 3 of this thesis, we mainly work on the Sierpinski Gasket (SG). We can think of it as approximated by a sequence of graphs. In this section, we introduce some known facts and background material taken from [Ki1] and [Str]. We begin this section with the concrete definition of SG.

Definition 2.1.1. *Let* $\{q_0, q_1, q_2\}$ *denote the vertices of an equilateral triangle where* q_0 *is the top vertex,* q_1 *is the lower left and* q_2 *the lower right. Consider three functions* $F_i : \mathbb{R}^2 \to \mathbb{R}^2, i = 0, 1, 2,$ *defined by*

$$F_i x = \frac{1}{2}(x - q_i) + q_i.$$

SG is the unique nonempty compact set which satisfies

$$SG = \cup_{i=0}^{2} F_i(SG).$$

Definition 2.1.2. *We define a word* $w = w_1 w_2 ... w_m$ *of length* $|w| = m$ *on the alphabet* $\{0, 1, 2\}$*, that is each* $w_i \in \{0, 1, 2\}$*. We put* $F_w = F_{w_1} \circ F_{w_2} \circ ... \circ F_{w_m}$ *and call* $F_w(SG)$ *a cell of level m. The standard self-similar measure* μ *is characterized by* $\mu_w := \mu(F_w SG) = \left(\frac{1}{3}\right)^{|w|}$.

Definition 2.1.3. *We define the (analytical) boundary of SG to be* $V_0 = \{q_0, q_1, q_2\}$*. We define* $V_m = \cup_{i=0}^{2} F_i V_{m-1}$ *and* $V_* = \cup_{i=0}^{\infty} V_m$*. Let* Γ_0 *be the complete graph on* V_0*. We construct a graph* Γ_m *with vertices* V_m *by defining the edge relation* $y \underset{m}{\sim} x$ *if there is a cell of level m containing both* x *and* y *(i.e. there exist a word* w *of length* m *such that* $x = F_w q_i$ *and* $y = F_w q_j$ *for some distinct* $i, j \in \{0, 1, 2\}$*)*

We call $V_* \setminus V_0$ *the set of junction points and observe that for each point* $x \in V_* \setminus V_0$ *,* $x = F_w q_i = F_{w'} q_j$ *for distinct words* w, w' *with the same length.*

Definition 2.1.4. *Given a real-valued function* u *defined on* V_**, we define the renormalized graph energy on* Γ_m *by* $\mathcal{E}_m = r^{-m} E_m(u)$*, where* $r = \frac{3}{5}$ *and* $E_m(u) = \sum_{y \underset{m}{\sim} x} (u(x) - u(y))^2$.

For instance, $E_0(u) = (u(q_0) - u(q_1))^2 + (u(q_1) - u(q_2))^2 + (u(q_2) - u(q_0))^2$.

It is easy to show that $\mathcal{E}_m(u)$ *is an increasing sequence so* $\lim_{m \to \infty} \mathcal{E}_m(u)$ *exists. We define*

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u)$$

to be the energy of a function u*. We say* $u \in dom\mathcal{E}$ *if and only if* $\mathcal{E}(u) < \infty$*. Even though the definition of* $\mathcal{E}(u)$ *only involves* V_**, we can regard* u *as function defined on SG because function of finite energy admits a unique continuous extension on SG. We can define the associated bilinear form*

$$\mathcal{E}(u, v) = \frac{1}{4}(\mathcal{E}(u + v) - \mathcal{E}(u - v))$$

for $u, v \in dom\mathcal{E}$*.*

In addition, we define $dom_0\mathcal{E} = \{u \in dom\mathcal{E} : u|_{V_0} \equiv 0\}$*.*

Definition 2.1.5. *Let* ζ *be a positive continuous measure. We can define a Laplacian* Δ_ζ *weakly via bilinear energy form. Let* $u \in dom\mathcal{E}$ *and* f *be continuous on* SG*. Then we say* $u \in dom\Delta_\zeta$ *with* $\Delta_\zeta u = f$ *if*

$$\mathcal{E}(u, v) = -\int_{SG} f v d\zeta \text{ for all } v \in dom_0\mathcal{E}.$$

More generally, if we only assume $f \in L^2(d\zeta)$ *and the above equality holds, then we say* $u \in dom_{L^2}\Delta_\zeta$ *and* $\Delta_\zeta u = f$*.*

We can also define a graph Laplacian Δ_m *on* Γ_m *by*

$$\Delta_m u(x) = \sum_{y \underset{m}{\sim} x} (u(y) - u(x)), \; x \in V_m \setminus V_0.$$

Definition 2.1.6. *The normal derivative of a function u at a boundary point $q \in V_0$ is given by*

$$\partial_n u(q) = \lim_{m\to\infty} \frac{1}{r^m} \sum_{y \underset{m}{\sim} q} (u(q) - u(y)), \text{where } r = \frac{3}{5}, \text{ if the limit exists.}$$

It is known that the normal derivative exists if $u \in dom\Delta_\zeta$. We can also define the normal derivative on junction points. Let $x = F_w q_i$ be a boundary point of the m-cell $F_w(SG)$. We define $\partial_n u(x)$ with respect to the cell $F_w(SG)$ to be $r^{-|w|}\partial_n(u \circ F_w)(q_i)$. Notice that if $x = F_w q_i = F_{w'} q_{i'}$ is a junction point, then the normal derivative defined at x with respect to $F_w(SG)$ and $F_{w'}(SG)$ may not be equal. However, we have

$$\partial_n u(F_w q_i) + \partial_n u(F_{w'} q_{i'}) = 0 \text{ if } u \in dom_{L^2}\Delta_\zeta.$$

We also define the effective resistance between $x, y \in SG$ to be $R(x, y)$, where

$$R(x,y) = \left(\min_u \left(\mathcal{E}(u,u) \mid u(x) = 1, u(y) = 0\right)\right)^{-1}.$$

We can obtain an analogue of Gauss-Green formula in a fractal setting [Ki1, Str].

Theorem 2.1.7. *Suppose $u \in dom_{L^2}\Delta_\zeta$ for some measure ζ.Then $\partial_n u(x)$ exists for all $x \in V_0$ and we have*

$$\mathcal{E}(u,v) = -\int_{SG} (\Delta_\zeta u) v d\zeta + \sum_{V_0} v \partial_n u$$

for all $v \in dom\mathcal{E}$.

We can also get a localized version of this formula.

$$\mathcal{E}_{F_w K}(u,v) = -\int_{F_w K} (\Delta_\zeta u) v d\zeta + \sum_{\partial F_w K} v \partial_n u, \tag{2.1}$$

where K denotes the Sierpinski gasket.

Definition 2.1.8. *Let u_0 be a function initially defined on V_0, with $u_0(q_0) = a, u(q_1) = b, u(q_2) = c$. There exists a unique function u_1, called the harmonic extension of u_0, which extends u_0 to V_1 such that $\mathcal{E}(u_1)$ is minimized. We put $p_1 = F_0(q_1), p_2 = F_0(q_2), p_3 = F_1(q_2)$. The values of u_1 on $V_1 \setminus V_0 = \{p_1, p_2, p_3\}$ is given by*

$$u(p_1) = \frac{1}{5}c + \frac{2}{5}a + \frac{2}{5}b$$
$$u(p_2) = \frac{2}{5}c + \frac{2}{5}a + \frac{1}{5}b$$
$$u(p_3) = \frac{2}{5}c + \frac{1}{5}a + \frac{2}{5}b.$$

Thereby, u_0 can be extended to all cells in V_* by applying the above harmonic extension algorithm recursively. Observe that with this harmonic structure SG has a 3-dimensional space of harmonic functions which is exactly the cardinality of V_0.

The space $\mathcal{S}(\mathcal{H}_0, V_m)$ of piecewise harmonic splines of level m is defined to be the space of continuous functions such that $u \circ F_w$ is harmonic for all $|w| = m$.

We see that $\mathcal{S}(\mathcal{H}_0, V_m)$ is contained in dom$(\mathcal{E})$ and is finite dimensional of dimension $|V_m|$. These functions are obtained by specifying values of u on V_m then extending harmonically, by using the above algorithm, to all higher levels. We have $\mathcal{E}(u) = \mathcal{E}_m(u)$ for these functions.

ψ_x^m denotes the piecewise harmonic spline in $\mathcal{S}(\mathcal{H}_0, V_m)$ satisfying $\psi_x^m(y) = \delta_{xy}$ for $y \in V_m$. In the sequel, we drop the superscript and just write ψ_x when it is clear from the subscript index.

2.2 Definition of the Trace Space

Recall that we have $a_m = u(x_m)$ and $\eta_m = \partial_n u(x_m)$. Let $p > 1$ and q be its conjugate i.e. $\frac{1}{p} + \frac{1}{q} = 1$. We recall the definition of the following trace space

$$\mathcal{T}_p := \{(a_m, \eta_m) \mid a_m = A_1 + A_2(3/5)^m + a'_m,\ \|(5^p/3)^{m/p} a'_m\|_p, \|3^{\frac{m(p-1)}{p}} \eta_m\|_p < \infty\}$$

with the norm given by

$$\|(a_m, \eta_m)\|^p_{\mathcal{T}_p} = |A_1|^p + |A_2|^p + \|(5^p/3)^{m/p} a'_m\|^p_p + \|3^{\frac{m(p-1)}{p}} \eta_m\|^p_p.$$

Remark 2.2.1. *Observe that these norms converge to their counterparts in [LS], namely to $\|5^m a'_m\|_\infty$ and $\|3^m \eta_m\|_{Lip}$ as $p \to \infty$.*

We define the following norm on $\mathrm{dom}_{L^p}\Delta(\mathrm{SG})$:

$$\|u\|_{\mathrm{dom}_{L^p}\Delta(\mathrm{SG})} = \|u\|^p_{L^\infty(SG)} + \|\Delta u\|^p_p.$$

The following sequence of key lemmas are the L^p-versions of the corresponding Lemmas in [LS].

Lemma 2.2.2. *Let a_m be a sequence and $r > 1$. Then, $a_m = A + a'_m$ with $\|r^{m/p} a'_m\|_p < \infty$ iff $\|r^{m/p}(a_{m+1} - a_m)\|_p < \infty$. Furthermore, we have*

$$\|r^{m/p} a'_m\|_p \le C \|r^{m/p}(a_{m+1} - a_m)\|_p$$

Proof. The first statement implies the second statement. Observe that $a_{m+1} = A + a'_{m+1}$ and $a_m = A + a'_m$, which gives

$$\|r^{m/p}(a_{m+1} - a_m)\|_p = \|r^{m/p}(a'_{m+1} - a'_m)\|_p \le C_1 \|r^{(m+1)/p} a'_{m+1}\|_p + \|r^{m/p} a'_m\|_p < \infty$$

In order to get the other direction, we first show that a_m is a Cauchy sequence. For $m > n$ we have

$$a_m - a_n = \sum_{k=n}^{m-1} (a_{k+1} - a_k) r^{k/p} r^{-k/p}$$

applying Holder's inequality yields

$$|a_m - a_n| \le \left(\sum_{k=n}^{m-1} |a_{k+1} - a_k|^p r^k \right)^{1/p} \left(\sum_{k=n}^{m-1} \frac{1}{(r^{q/p})^k} \right)^{1/q} \le C \left(\frac{1}{r^{qn/p}} \right)$$

which implies that a_m is a Cauchy sequence. So, $a_m \to A$ for some A. We can write

$$a_m - A = \sum_{k=m}^{\infty} (a_k - a_{k+1}) = \sum_{k=0}^{\infty} (a_{m+k} - a_{m+k+1}).$$

Multiplying by $r^{m/p}$ yields

$$r^{m/p}(a_m - A) = \sum_{k=0}^{\infty} r^{(m+k)/p} r^{-k/p} (a_{m+k} - a_{m+k+1}).$$

Finally, by Minkowski's inequality we obtain

$$\begin{aligned} \|r^{m/p}(a_m - A)\|_p &\le \sum_{k=0}^{\infty} r^{-k/p} \|(a_{k+m} - a_{k+m+1}) r^{(k+m)/p}\|_p \\ &\le \sum_{k=0}^{\infty} r^{-k/p} \|(a_m - a_{m+1}) r^{m/p}\|_p. \end{aligned}$$

Hence, the result.

□

Lemma 2.2.3. *Let a_m be a sequence. We have*

$$\|(5^p/3)^{m/p}(5a_{m+2} - 8a_{m+1} + 3a_m)\|_p < \infty.$$

if and only if $a_m = A_1 + A_2(3/5)^m + a'_m$ *with* $\|(5^p/3)^{m/p} a'_m\|_p < \infty$. *Also, the following estimate holds*

$$\|(5^p/3)^{m/p} a'_m\|_p \le C\|(5^p/3)^{m/p}(5a_{m+2} - 8a_{m+1} + 3a_m)\|_p$$

Proof. If we do the calculation we obtain

$$\begin{aligned}&\|(5^p/3)^{m/p}(5a_{m+2} - 8a_{m+1} + 3a_m)\|_p = \|(5^p/3)^{m/p}(5a'_{m+2} - 8a'_{m+1} + 3a'_m)\|_p \\ &\le D_1\|(5^p/3)^{(m+2)/p} a_{m+2}\|_p + D_2\|(5^p/3)^{(m+1)/p} a_{m+1}\|_p + D_3\|(5^p/3)^{m/p} a_m\|_p < \infty\end{aligned}$$

which shows that the second part implies the first part.

For the converse implication, we use the substitution $d_m = \frac{5^m}{3^m}(a_{m+1} - a_m)$. We can write

$$\sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m (5a_{m+2} - 8a_{m+1} + 3a_m)^p = 3^p \sum_{m=1}^{\infty} 3^{(p-1)m}(d_{m+1} - d_m)^p < \infty.$$

Observe that d_m satisfies the assumptions of Lemma 3.1.2 with $r = 3^{p-1}$. Hence, $d_m = D + d'_m$ with

$$\sum_{m=1}^{\infty} 3^{(p-1)m}|d'_m|^p \le C \sum_{m=1}^{\infty} 3^{(p-1)m}(d_{m+1} - d_m)^p.$$

Now, we apply the Lemma 3.1.2 for a second time for $e_m = a_m + (5/2)(3/5)^m D$, which yields

$$\sum_{m=1}^{\infty} 3^{(p-1)m}|d'_m|^p = \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m (e_{m+1} - e_m)^p.$$

We obtain $e_m = E + e'_m$ where

$$\sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m |e'_m|^p \le C \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m (e_{m+1} - e_m)^p.$$

Putting everything together yields $a_m = E - (5/2)(3/5)^m D + e'_m$ with

$$\sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m |e'_m|^p \le C \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m (5a_{m+2} - 8a_{m+1} + 3a_m)^p.$$

Hence, the result. □

Lemma 2.2.4. *Let $r < 1$ be a constant and a_m be a sequence. Then $\|r^{\frac{m}{p}} a_m\|_p < \infty$ if and only if $\|r^{\frac{m}{p}}(a_{m+1} - a_m)\|_p < \infty$. Furthermore we have*

$$\|r^{\frac{m}{p}} a_m\|_p \leq C_1|a_1| + C_2\|r^{\frac{m}{p}}(a_{m+1} - a_m)\|_p.$$

Proof. It is easy to see that the first statement implies the second one. For the converse, writing a_m as a telescoping series and multiplying by $r^{\frac{m}{p}}$ yields

$$r^{m/p} a_m = r^{m/p} a_1 + \sum_{k=1}^{m-1} r^{\frac{m}{p}}(a_{k+1} - a_k) = r^{\frac{m}{p}} a_1 + \sum_{k=1}^{m-1} (a_{m-k+1} - a_{m-k}) r^{\frac{m-k}{p}} r^{\frac{k}{p}}. \tag{2.2}$$

We apply the Minkowski's inequality to obtain

$$\begin{aligned}\left\|\sum_{k=1}^{m-1} (a_{m-k+1} - a_{m-k}) r^{\frac{m-k}{p}} r^{\frac{k}{p}}\right\|_p &\leq \sum_{k=1}^{\infty} r^{k/p} \left\|(a_{m-k+1} - a_{m-k}) r^{\frac{m-k}{p}} 1_{k<m}\right\|_p \\ &\leq \sum_{k=1}^{\infty} r^{k/p} \|(a_{m+1} - a_m) r^{m/p}\|_p.\end{aligned}$$

Applying Minkowski's inequality one more time to (2.2) and using the above estimate yields

$$\|r^{m/p} a_m\|_p \leq \|r^{m/p} a_1\|_p + C_2\|(a_{m+1} - a_m) r^{m/p}\|_p.$$

Hence, the result. □

Lemma 2.2.5. *Let η_m be a sequence. Then, $\|3^{\frac{m(p-1)}{p}}(3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m)\|_p < \infty$ if and only if $\eta_m = 5^m A + \eta'_m$ with $\|3^{\frac{m(p-1)}{p}} \eta'_m\|_p < \infty$. Furthermore, we have*

$$\|3^{\frac{m(p-1)}{p}} \eta'_m\|_p \leq C_1|\eta_2 - 5\eta_1| + C_2\|3^{\frac{m(p-1)}{p}}(3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m)\|_p.$$

Proof. The second statement obviously implies the first one. For the converse, let $e_m = 3^m(\eta_{m+1} - 5\eta_m)$. We have

$$\sum_{m=1}^{\infty} 3^{m(p-1)} |3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m|^p = \sum_{m=1}^{\infty} \frac{1}{3^m} |e_{m+1} - e_m|^p.$$

Applying Lemma 2.2.4 to e_m yields

$$\sum_{m=1}^{\infty} \frac{1}{3^m} |e_m|^p \leq \left(C_1 |e_1| + C_2 \| 3^{\frac{m(p-1)}{p}} (3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m) \|_p \right)^p < \infty.$$

We put $5^m d_m = \eta_m$. Then, we can write

$$\sum_{m=1}^{\infty} \frac{1}{3^m} |e_m|^p = \sum_{m=1}^{\infty} 3^{m(p-1)} |\eta_{m+1} - 5\eta_m|^p = 5^p \sum_{m=1}^{\infty} \left(3^{p-1} 5^p\right)^m |d_{m+1} - d_m|^p < \infty$$

Now, we apply 3.1.2 to the sequence d_m to obtain $d_m = D + d'_m$ with

$$\sum_{m=1}^{\infty} \left(3^{p-1} 5^p\right)^m |d'_m|^p \leq C \sum_{m=1}^{\infty} \left(3^{p-1} 5^p\right)^m |d_{m+1} - d_m|^p$$

where $C = C(p)$ is a constant. By definition of d_m we have $\eta_m = 5^m D + 5^m d'_m$. Putting everything together along with the definition $\eta'_m := 5^m d'_m$ yields

$$\sum_{m=1}^{\infty} 3^{m(p-1)} |\eta'_m|^p \leq \left(C_1 |e_1| + C_2 \| 3^{\frac{m(p-1)}{p}} (3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m) \|_p \right)^p.$$

□

Now, having established the key Lemmas, we are ready to prove the first main result of this chapter. The proof is rather technical so we give some motivation. Recall that $a_m = u(x_m)$ and $\eta_m = \partial_n u(x_m)$. The main tool lurking in the background turns out to be the Green's formula. Recall that given any function u on SG for which Δu exists, we can write as

$$u(x) = \int_{SG} G(x, y) \Delta u(y) dy + h(x)$$

where $G(x, y)$ is the Green's function and h is the harmonic function having the same values as u on V_0. By using the results in Chapter 1 of Appendix A, we relate the linear combinations $5a_{m+2} - 8a_{m+1} + 3a_m$ and $3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m$ to integrals which we can bound using the sequence of Lemmas we proved in this section. We also note that the definition of Ψ_m and certain estimates on its size was given in (A.3) and (A.4).

Theorem 2.2.6. *Let $p > 1$, $u \in dom_{L^p}\Delta(SG)$ and $u|_{V_0} = 0$. Then, $Ru \in \mathcal{T}_p$ and*

$$||Ru||_{\mathcal{T}_p} \leq C||\Delta u||_{L^p(SG)}. \tag{2.3}$$

Proof. We follow the ideas in [LS]. We start with $u \in \text{dom}_{L^p}\Delta(\text{SG})$ with $Ru = \{(a_m, \eta_m)\}$. Applying the Green's formula in Theorem A.0.3 to $5a_{m+2} - 8a_{m+1} + 3a_m$ and the equation for $G(x_m, y)$ in Lemma A.5, we obtain

$$5a_{m+2} - 8a_{m+1} + 3a_m = \frac{3^m}{5^m}\int_{SG}\overline{G}_m \Delta u dy. \tag{2.4}$$

where $\overline{G}_m$ is defined to be

$$\frac{1}{50}\left(9\Psi_{m+2}(3,1,1) - 20\Psi_{m+1}(1,0,0) + 25\Psi_m(1,-1,-1)\right).$$

Arguing in a similar manner to [LS], we deduce that $\overline{G}_m$ is supported only on $D_m = Y_m \cup Y_{m+1} \cup Y_{m+2} \cup Z_m \cup Z_{m+1} \cup Z_{m+2}$. So, we have

$$5a_{m+2} - 8a_{m+1} + 3a_m = \frac{3^m}{5^m}\int_{D_m}\overline{G}_m \Delta u dy. \tag{2.5}$$

We observe that $|\psi_{x_m}|^q \leq |\psi_{x_m}|$ and $\int_{SG}|\psi_{x_m}|dy = \frac{2}{3^{m+1}}$. Therefore, for $C = C(a,b,c)$ we obtain

$$\int_{SG}|\Psi_m(a,b,c)|^q \leq \frac{C}{3^m}.$$

Having this estimate at our disposal, we apply Holder's inequality to (2.5) and obtain

$$|5a_{m+2} - 8a_{m+1} + 3a_m|^p \leq C\left(\frac{3}{5}\right)^{mp}||\Delta u||^p_{L^p(D_m)}\frac{1}{3^{pm/q}}.$$

Pluggging in $1 - 1/q = 1/p$ and rearranging yields

$$\left| \left(\frac{5^p}{3} \right)^{m/p} (5a_{m+2} - 8a_{m+1} + 3a_m) \right|^p \le C \|\Delta u\|^p_{L^p(D_m)}. \tag{2.6}$$

Recall that $D_m = Y_m \cup Y_{m+1} \cup Y_{m+2} \cup Z_m \cup Z_{m+1} \cup Z_{m+2}$ which implies

$$\|\Delta u\|^p_{L^p(D_m)} = \sum_{k=m}^{m+2} \|\Delta u\|^p_{L^p(Y_k \cup Z_k)}. \tag{2.7}$$

We also have

$$\|\Delta u\|^p_{L^p(SG)} = \sum_{k=1}^{\infty} \|\Delta u\|^p_{L^p(Y_k \cup Z_k)}. \tag{2.8}$$

Therefore, using (2.6) we can write

$$\left\| \left(\frac{5^p}{3} \right)^{m/p} (5a_{m+2} - 8a_{m+1} + 3a_m) \right\|_p^p \le C \|\Delta u\|^p_{L^p(SG)}.$$

We apply Lemma 2.2.3 to obtain $a_m = A_1 + A_2(3/5)^m + a'_m$ where $A_1 = \lim_{m\to\infty} a_m$ and $A_2 = \lim_{m\to\infty}(5/3)^m(a_m - A_1)$. By the same lemma, we also have the following estimate

$$\left\| \left(\frac{5^p}{3} \right)^{m/p} a'_m \right\|_p \le \left\| \left(\frac{5^p}{3} \right)^{m/p} (5a_{m+2} - 8a_{m+1} + 3a_m) \right\|_p.$$

Combining the above inequalities, we immediately get

$$\left\| \left(\frac{5^p}{3} \right)^{m/p} a'_m \right\|_p \le C \|\Delta u\|_{L^p(SG)}. \tag{2.9}$$

We want to show that $A_1 = 0$ and $|A_2| \le C\|\Delta u\|_{L^2(SG)}$. The Green's formula for a_m reads

$$a_m = u(x_m) = \int_{SG} G(x_m, y)\Delta u(y) dy.$$

By Lemma A.5, we know that

$$G(x_m, y) = \frac{2}{15} \left(\frac{3}{5} \right)^m \sum_{k=1}^{m} \psi_k(1,2,2)(y) + \frac{1}{6} \left(\frac{3}{5} \right)^m \psi_m(1,-1,-1)(y).$$

Then, by using Holder's inequality and $|\psi_k|^q \leq |\psi_k|$ we can write

$$\left(\frac{5}{3}\right)^m a_m \leq \left(\frac{5}{3}\right)^m \int_{SG} |G(x_m, y)||\Delta u(y)|dy$$
$$\leq D_1 \sum_{k=1}^{m} \int_{SG} |\psi_k(1,2,2)||\Delta u|dy + D_2 \int_{SG} |\psi_m(1,-1,1)||\Delta u|dy$$
$$\leq D_1 \sum_{k=1}^{m} \left(|\psi_k|^q\right)^{1/q} ||\Delta u||_p + D_2 \left(|\psi_m|^q\right)^{1/q} ||\Delta u||_p$$
$$\leq ||\Delta u||_p \sum_{k=1}^{m} \frac{1}{3^{k/q}} + D_2 ||\Delta u||_p \frac{1}{3^{m/q}} \leq C_1 ||\Delta u||_p + C_2 ||\Delta u||_p \frac{1}{3^{m/q}}.$$

This estimate finally implies that

$$A_1 = 0 \text{ and } A_2 \leq C||\Delta u||_p \tag{2.10}$$

as asserted.

Now, we want to bound the normal derivative η_m.

We use Lemma A.0.6 to calculate $3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m$ and obtain

$$3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m = \int_{D_m} \Phi_m \Delta u dy - (\phi_{m+2} - 16\phi_{m+1} + 5\phi_m). \tag{2.11}$$

where

$$\phi_m = \int_{Z_m} \psi_m \Delta u dy$$

and

$$\Phi_m = \frac{1}{10}\left(-3\Psi_{m+2}(5,1,-1) + 10\Psi_{m+1}(8,1,-1) - 25\Psi_m(1,-1,1)\right).$$

Observe that $Z_m \subset D_m$, which will be important for the estimates.

Applying Holder's inequality to (2.11) and ϕ_m and taking pth power yields

$$|3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m|^p \leq \frac{1}{3^{\frac{pm}{q}}} ||\Delta u||^p_{L^p(D_m)}.$$

Since $p/q = p - 1$ we obtain

$$3^{m(p-1)}|3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m|^p \leq \|\Delta u\|^p_{L^p(D_m)}.$$

Now, we use again (2.7) and (2.8) to conclude

$$\|3^{\frac{m(p-1)}{p}}(3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m)\|_p \leq C\|\Delta u\|_{L^p(SG)}.$$

Then, the sequence η_m satisfies the assumption of Lemma 2.2.5 which gives us $\eta_m = 5^m A + \eta'_m$ with

$$\|3^{\frac{m(p-1)}{p}}\eta'_m\|_p \leq C_1|\eta_2 - 5\eta_1| + C_2\|3^{\frac{m(p-1)}{p}}(3\eta_{m+2} - 16\eta_{m+1} + 5\eta_m)\|_p.$$

We recall from Lemma A.0.6 that

$$\begin{aligned}\eta_m = \partial u(x_m) = \frac{3}{5}\left(\frac{1}{3^m}\right)\sum_{k=1}^{m} 3^k \int_{SG} \psi_k(0,-1,1)\Delta u dy \\ - \frac{1}{2}\int_{SG}\psi_m(1,-1,1)\Delta u dy - \phi_m.\end{aligned}$$

Applying Holder to the above equation yields

$$\begin{aligned}|\eta_m| &\leq C_1\|\Delta u\|_p \frac{1}{3^m}\sum_{k=1}^{m} 3^k \frac{1}{3^{k/q}} + C_2\|\Delta u\|_p \frac{1}{3^{m/q}} \\ &\leq C_1\|\Delta u\|_p \frac{1}{3^m}\frac{(3^{1/p})^{m+1} - 3^{1/p}}{3^{1/p} - 1} + C_2\|\Delta u\|_p \frac{1}{3^{m/q}} \\ &\leq C_1\|\Delta u\|_p \frac{1}{3^m}(3^{1/p})^{m+1} + C_2\|\Delta u\|_p \frac{1}{3^{m/q}} \\ &\leq C_3\|\Delta u\|\frac{1}{3^{m/q}} + C_2\|\Delta u\|_p \frac{1}{3^{m/q}} \leq C\|\Delta u\|_p \frac{1}{3^{m/q}}\end{aligned}$$

where $C_1, C_2, C_3 = C_3(p)$ are constants. By the above inequality, we obtain $A = 0$ and $\eta_m = \eta'_m$. We also get the bound $|\eta_2 - 5\eta_1| \leq C||\Delta u||_{L^p(SG)}$. Putting everything together yields

$$||3^{\frac{m(p-1)}{p}} \eta'_m||_p \leq C||\Delta u||_{L^p(SG)}. \tag{2.12}$$

Since $a_m = A_2(3/5)^m + a'_m$, by using (2.9), (2.10) and (2.12) we obtain

$$||Ru||_p^p = |A_1|^p + |A_2|^p + \left\| \left(\frac{5^p}{3}\right)^{m/p} a'_m \right\|_p^p + ||3^{\frac{m(p-1)}{p}} \eta'_m||_p^p \leq C||\Delta u||^p_{L^p(SG)}.$$

Hence, the result. □

Lemma 2.2.7. *If h is a harmonic function, then $h \in \mathcal{T}_p$ with*

$$||Rh||^p_{\mathcal{T}_p} = |h(q_0)|^p + \frac{1}{2^p}|h(q_1) + h(q_2) - 2h(q_0)|^p + \frac{1}{2^{p+1}}|h(q_1) - h(q_2)|^p. \tag{2.13}$$

Proof. Writing h as a linear combination of the constant, skew-symmetric and symmetric harmonic function yields $h(x_m) = A_1 + A_2(3/5)^m$ and $\partial_n h(x_m) = A_3/3^m$, where

$$A_1 = h(q_0)$$
$$A_2 = \frac{1}{2}(h(q_1) + h(q_2) - 2h(q_0))$$
$$A_3 = \frac{1}{2}(h(q_1) - h(q_2)).$$

□

Theorem 2.2.8. *(L^p- Trace Theorem) The restriction operator $R : dom_{L^p}\Delta(SG) \to \mathcal{T}_p$ is bounded and*

$$||Ru||_{\mathcal{T}_p} \leq C_1||u||_{L^\infty(SG)} + C_2||\Delta u||_{L^p(SG)}.$$

Proof. Let $u \in \mathrm{dom}_{L^p}\Delta(\mathrm{SG})$ and h be the harmonic function with $h|_{V_0} = u|_{V_0}$. We put $w = u - h$ and observe that $\Delta u = \Delta w$. We have

$$||Ru||_{\mathcal{T}_p} \leq ||Rw||_{\mathcal{T}_p} + ||Rh||_{\mathcal{T}_p}$$

Since $w = 0$ on V_0, by (2.3) we have

$$||Rw||_{\mathcal{T}_p} \leq C_2 ||\Delta u||_{L^p(SG)}.$$

By (2.13) we can write

$$\begin{aligned} ||Rh||^p_{\mathcal{T}_p} &= |u(q_0)|^p + \frac{1}{2^p}|u(q_1) + u(q_2) - 2u(q_0)|^p + \frac{1}{2^{p+1}}|u(q_1) - u(q_2)|^p \\ &\leq ||u||^p_{L^\infty(SG)} + \frac{1}{2^p}(4||u||_{L^\infty(SG)})^p + \frac{1}{2^{p+1}}(2||u||_{L^\infty(SG)})^p \leq C_1 ||u||^p_{L^\infty(SG)} \end{aligned}$$

where $C_1 = C_1(p), C_2 = C_2(p)$ are constants. Hence, the result. □

2.3 Extension Theorem

We quote the following result from [LS]:

Theorem 2.3.1. *([LS]*7.3 & 7.4) Given any sequences a_m and η_m, there exists a piecewise biharmonic function u on SG and sequences C'_m and C_m such that $Ru = \{(a_m, \eta_m)\}$, $\Delta u = C'_m$ on Y_m, $\Delta u = C_m$ on Z_m and the normal matching conditions hold at x_m, y_m and z_m. We have*

$$C'_m = 5^m \left(\frac{3}{8}\right)(5a_{m+1} - 8a_m + 3a_{m-1}) - 3^m \left(\frac{3}{8}\right)(3\eta_{m+1} - 16\eta_m + 5\eta_{m-1}) \qquad (2.14)$$

$$C_m = 5^m \left(\frac{3}{8}\right)(5a_{m+1} - 8a_m + 3a_{m-1}) + 3^m \left(\frac{3}{8}\right)(3\eta_{m+1} - 16\eta_m + 5\eta_{m-1}) \qquad (2.15)$$

The extension operator that maps two sequences $\{(a_m, \eta_m)\}$ to the function u in the above theorem is denoted by E. The following lemma is a standard result that we include without proof.

Lemma 2.3.2. *Let α, β be complex numbers and $p \geq 1$ we have*

$$|\alpha + \beta|^p \leq 2^{p-1}(|\alpha|^p + |\beta|^p).$$

We prove the following theorem.

Theorem 2.3.3. *(L^p-extension Theorem) Let $p > 1$. There exists a bounded linear extension map $E : \mathcal{T}_p \to dom_{L^p}\Delta(SG)$ with $R \circ E = Id$.*

Proof. Let $\{(a_m, \eta_m)\} \in \mathcal{T}_p$ and $u = E\{(a_m, \eta_m)\}$ where E is as defined above. We have $a_m = A_1 + A_2(3/5)^m + a'_m$ with $\|(5^p/3)^{m/p} a'_m\|_p < \infty$ and $\|3^{\frac{m(p-1)}{p}} \eta_m\|_p < \infty$. It follows that $|a'_m| \to 0$ and $|\eta_m| \to 0$. By the same argument for the $\mathcal{T}_\infty$ case in [LS], u is continuous at q_0 thus continuous everywhere by construction. We want to show that $\Delta u \in L^p(SG)$. By using the definitions of C'_m, C_m and the previous Lemma we have

$$
\begin{aligned}
|C'_m|^p &\le \left|5^m\left(\frac{3}{8}\right)(5a'_{m+1} - 8a'_m + 3a'_{m-1}) - 3^m\left(\frac{3}{8}\right)(3\eta_{m+1} - 16\eta_m + 5\eta_{m-1})\right|^p \\
&\le 2^{p-1}\left|5^m\left(\frac{3}{8}\right)(5a'_{m+1} - 8a'_m + 3a'_{m-1})\right|^p + 2^{p-1}\left|3^m\left(\frac{3}{8}\right)(3\eta_{m+1} - 16\eta_m + 5\eta_{m-1})\right|^p \\
&\le K_1 5^{(m+1)p}|a'_{m+1}|^p + K_2 5^{mp}|a'_m|^p + K_3 5^{(m-1)p}|a'_{m-1}|^p \\
&+ H_1 3^{(m+1)p}|\eta_{m+1}|^p + H_2 3^{(m)p}|\eta_m|^p + H_3 3^{(m-1)p}|\eta_{m-1}|^p
\end{aligned}
$$

for some constants $K_i = K_i(p), H_i = H_i(p)$. Then, we can write

$$
\begin{aligned}
\sum_{m=1}^{\infty} \frac{|C'_m|^p}{3^m} &\le 3K \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m |a'_m|^p + 3H \sum_{m=1}^{\infty} 3^{m(p-1)}|\eta_m|^p \\
&\le \frac{D_1}{2} \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m |a'_m|^p + \frac{D_2}{2} \sum_{m=1}^{\infty} 3^{m(p-1)}|\eta_m|^p \\
&= \frac{D_1}{2}\|(5^p/3)^{m/p} a'_m\|_p^p + \frac{D_2}{2}\|3^{\frac{m(p-1)}{p}} \eta_m\|_p^p < \infty
\end{aligned}
$$

A similar estimate also holds for $\sum_{m=1}^{\infty} \frac{|C_m|^p}{3^m}$, hence we obtain

$$
\|\Delta u\|_{L^p(SG)}^p = \sum_{m=1}^{\infty} \frac{|C'_m|^p + |C_m|^p}{3^m} \le D_1 \sum_{m=1}^{\infty} \left(\frac{5^p}{3}\right)^m |a'_m|^p + D_2 \sum_{m=1}^{\infty} 3^{m(p-1)}|\eta_m|^p < \infty
$$

where $D_1 = D_1(p), D_2 = D_2(p)$ are constants. Hence, the result. □

CHAPTER 3

POLYHARMONIC BOUNDARY VALUE PROBLEMS ON SG

Recall that K denotes SG and L the bottom line of SG identified with $[0,1]$. Similarly L_1 denotes $[0,1/2]$ and L_2 denotes $(1/2,1]$. Let $\Psi = 1$ denote the constant function on the relevant domain. The Haar functions give us an orthonormal basis on $L^2[0,1]$. The Haar basis we use consists of the functions Ψ and $\psi_{n,k}$ for $n = 0,1,2,\dots$ and $k = 0,1,\dots,2^{n-1}$ which are defined as follows

$$\psi_{n,k}(t) = \begin{cases} 2^{n/2} & t \in [\frac{k}{2^n}, \frac{k+1/2}{2^n}) \\ -2^{n/2} & t \in (\frac{k+1/2}{2^n}, \frac{k+1}{2^n}] \\ 0 & \text{otherwise} . \end{cases}$$

Observe that $\psi_{n,k}$ are nothing but the dilated and translated versions of the mother wavelet

$$\psi(t) = \begin{cases} 1 & t \in [0,1/2) \\ -1 & t \in (1/2,1] \\ 0 & \text{otherwise} . \end{cases}$$

We have $\psi = \psi_{0,0}$. For, $f \in L^2[0,1]$, we have the Haar coefficients of f as

$$c_\Psi(f) =< f, \Psi >$$

$$c_{n,k}(f) =< f, \psi_{n,k} >$$

where $< \cdot, \cdot >$ denotes the usual inner product. We know that the Haar series

$$c_\Psi(f)\Psi + \sum_{n=0}^{\infty} \sum_{k=0}^{2^{n-1}} c_{n,k}(f)\psi_{n,k}. \tag{3.1}$$

converges to f in the L^2-norm. As in [OS] we use a different notation to have the representation of f comparable to the analysis on SG. As described in section 2.1 w denotes a word in 0 and 1's. Observe that the contractions $F_0(t) = \frac{1}{2}t$ and $F_2(t) = \frac{1}{2}t + \frac{1}{2}$ are the restrictions of the two of the three contraction mappings defining SG. For details we refer the reader to Section 2.1. It follows that $F_w[0,1]$ for a word with $|w| = m$ is a dyadic interval in the form $[k/2^m, (k+1)/2^m]$. Using this notation we have that

$$\psi_w = \begin{cases} 2^{m/2}\psi \circ F_w^{-1} & \text{on } F_w[0,1]. \\ 0 & \text{otherwise .} \end{cases}$$

Similar to the previous case, we put

$$c_\Psi(f) = < f, \Psi >$$
$$c_w(f) = < f, \psi_w > .$$

That is, f has the representation

$$f = c_\Psi(f)\Psi + \sum_{m=0}^{\infty} \sum_{|w|=m} c_w(f)\psi_w. \tag{3.2}$$

Following [OS] we define the following Sobolev-type function spaces and norms on L. For f as in (3.2) we put

$$||f||_{H^s}^2 = |c_\Psi(f)|^2 + \sum_{m=0}^{\infty} \sum_{|w|=m} |2^{sm} c_w(f)|^2 \tag{3.3}$$

and

$$H^s(L) = \{f \ : \ ||f||_{H^s} < \infty\}. \tag{3.4}$$

Observe that putting $s = 0$ yields $H^0(L) = L^2(L)$ in which case $||f||_{H^0} = ||f||_2$ since Haar functions are orthonormal in $L^2(L)$. For $s < 0$, the elements of H^{-s} are distributions on L. H^s and H^{-s} are dual spaces of each other with respect to the usual pairing [J4, J5].

In this Chapter, we denote the 3 boundary vertices of SG by v_1, v_2, v_3. Recall that we put $\overline{SG} = SG \smallsetminus L \cup \{v_1\}$ We want to solve the following BVP on K:

$$\begin{aligned} \Delta^2 u &= 0 \\ \Delta u|_L &= f_2 \\ \Delta u(v_1) &= c' \\ u|_L &= f_1 \\ u(v_1) &= c \end{aligned}$$

Next we write the Haar expansion of both f_1 and f_2.

$$f_1 = c_\Psi(f_1)\Psi + \sum_{m=0}^{\infty} \sum_{|w|=m} c_w(f_1)\psi_w \tag{3.5}$$

$$f_2 = c_\Psi(f_2)\Psi + \sum_{m=0}^{\infty} \sum_{|w|=m} c_w(f_2)\psi_w. \tag{3.6}$$

we want to obtain the following representation for u.

$$\begin{aligned} u(x) = (c' - c_{\Psi(f_2)})h_3(x) + c_\Psi(f_1)\Psi + (a - c_\psi(f_1))h_0(x) + c_\Psi(f_2)h_2 \\ + \sum_{m=0}^{\infty} \sum_{|w|=m} c_w(f_1)h_w^1(x) + \sum_{m=0}^{\infty} \sum_{|w|=m} c_w(f_2)h_w^2(x) \end{aligned}$$

where h_0, h^1_ψ are the harmonic functions as constructed in [OS] and the biharmonic functions h_2, h_3 and h^2_ψ are constructed, in the following section (Section 3.1), so that they satisfy

$$\begin{aligned} h_0(v_1) = 1 \;\; h_0|_L = 0 \;\; \Delta h_0|_L = 0 \\ \Delta h_2 = 1 \;\; h_2(v_1) = 0 \;\; h_2|_L = 0 \\ \Delta h_3(v_1) = 1 \;\; h_3(v_1) = 0 \;\; h_3|_L = 0 \;\; \Delta h_3|_L = 0 \\ h^1_\psi(v_1) = 0 \;\; h^1_\psi|_L = \psi \;\; \Delta h^1_\psi|_L = 0 \\ h^2_\psi(v_1) = 0 \;\; h^2_\psi|_L = 0 \;\; \Delta h^2_\psi|_L = \psi. \end{aligned}$$

Also, we define the miniaturized versions of h^1_ψ and h^2_ψ, as h^1_w and h^2_w.

$$h^1_w = \begin{cases} 2^{m/2} h^1_\psi \circ F_w^{-1} & \text{on } F_w K \\ 0 & \text{otherwise .} \end{cases}$$

$$h^2_w = \begin{cases} \frac{2^{m/2}}{5^m} h^2_\psi \circ F_w^{-1} & \text{on } F_w K \\ 0 & \text{otherwise .} \end{cases}$$

Remark 3.0.4. *The appearance of the factor $\frac{1}{5^m}$ in the definition of h^2_w is because F_w^{-1} scales the Laplacian by 5^m and we want to bring it down to 1.*

We note that h^1_w and h^2_w satisfy

$$\begin{aligned} h^1_w(v_1) = 0 \;\; h^1_w|_L = \psi_w \;\; \Delta h^1_w|_L = 0 \\ h^2_w(v_1) = 0 \;\; h^2_w|_L = 0 \;\; \Delta h^2_w|_L = \psi_w. \end{aligned}$$

3.1 Construction of the Basic Functions

In the following proposition below, we give the rules of how the normal derivatives of a biharmonic function is distorted when miniaturized. For the definition of normal derivatives on SG we refer the reader to Section 2.1. We already know that

$$\Delta^n(\phi \circ F_i) = (\frac{1}{5})^n (\Delta^n \phi) \circ F_i \tag{3.7}$$

By iteration for $|w| = k$ we obtain

$$\Delta^n(\phi \circ F_w) = (\frac{1}{5})^{nk} (\Delta^n \phi) \circ F_w.$$

For $n = 2$, namely in the bilaplacian case, we have

$$\Delta^2(\phi \circ F_w) = (\frac{1}{5})^{2k} (\Delta^2 \phi) \circ F_w.$$

Remark 3.1.1. *Similarly we have* $\Delta(\phi \circ F_w^{-1}) = 5^n (\Delta\phi) \circ F_w^{-1}$.

Lemma 3.1.2. *Let g_i denote the harmonic function with $g_i(v_j) = \delta_{ij}$ and let $|w| = k$. Then we have*

$$\partial_n g_i \circ F_w(v_i) = 2\left(\frac{5}{3}\right)^k$$
$$\partial_n g_i \circ F_w(v_j) = -\left(\frac{5}{3}\right)^k$$

Proposition 3.1.3. *Let u be a biharmonic function and $|w| = k$. Then, we have*

$$\begin{aligned}\partial_n(u \circ F_w^{-1})(F_w v_1) &= \left(\frac{5}{3}\right)^k (2u(v_1) - u(v_2) - u(v_3)) \\ &+ \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{7}{45} + \Delta u(v_2)\frac{4}{45} + \Delta u(v_3)\frac{4}{45}\right).\end{aligned}$$

$$\begin{aligned}\partial_n(u \circ F_w^{-1})(F_w v_2) &= \left(\frac{5}{3}\right)^k (2u(v_2) - u(v_1) - u(v_3)) \\ &+ \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{4}{45} + \Delta u(v_2)\frac{7}{45} + \Delta u(v_3)\frac{4}{45}\right)\end{aligned}$$

$$\partial_n(u\circ F_w^{-1})(F_wv_3) = \left(\frac{5}{3}\right)^k (2u(v_3) - u(v_1) - u(v_2)) + \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{4}{45} + \Delta u(v_2)\frac{4}{45} + \Delta u(v_3)\frac{7}{45}\right).$$

Proof. We use the localized symmetric Gauss-Green formula on $A = F_wK$. We have

$$\int_A (u\circ F_w^{-1})\Delta(v\circ F_w^{-1}) - \int_A (v\circ F_w^{-1})\Delta(u\circ F_w^{-1})$$
$$= \sum_{q\in\{v_1,v_2,v_3\}} ((u\circ F_w^{-1})(F_wq)\partial_n(v\circ F_w^{-1})(F_wq) - (v\circ F_w^{-1})(F_wq)\partial_n(u\circ F_w^{-1})(F_wq)).$$

We first put $v = g_1$, since g_1 is harmonic we obtain

$$\partial_n(u\circ F_w^{-1})(F_wv_1) = \sum_{q\in\{v_1,v_2,v_3\}} u(q)\partial_n(v\circ F_w^{-1})(F_wq) + \int_A (v\circ F_w^{-1})\Delta(u\circ F_w^{-1}).$$

Observe that since u is biharmonic, $\Delta u = h$ with $h = \Delta u(v_1)g_1 + \Delta u(v_2)g_2 + \Delta u(v_3)g_3$. So we obtain, by using Lemma 3.1.2,

$$\partial_n(u\circ F_w^{-1})(F_wv_1) = \left(\frac{5}{3}\right)^k (2u(v_1) - u(v_2) - u(v_3)) + 5^k\left(\Delta u(v_1)\int_{F_wK}(g_1\circ F_w)^2\,d\mu + \Delta u(v_2)\int_{F_wK}(g_2\circ F_w)(g_1\circ F_w)d\mu + \Delta u(v_3)\int_{F_wK}(g_3\circ F_w)(g_1\circ F_w)d\mu\right).$$

Since $\int_K g_ig_jd\mu = \frac{4}{45}$ and $\int_K g_i^2d\mu = \frac{7}{45}$, plugging in yields

$$\partial_n(u\circ F_w^{-1})(F_wv_1) = \left(\frac{5}{3}\right)^k (2u(v_1) - u(v_2) - u(v_3)) + \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{7}{45} + \Delta u(v_2)\frac{4}{45} + \Delta u(v_3)\frac{4}{45}\right).$$

Similar calculation with $v = g_2, g_3$ yields

$$\partial_n(u \circ F_w^{-1})(F_w v_2) = \left(\frac{5}{3}\right)^k (2u(v_2) - u(v_1) - u(v_3))$$
$$+ \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{4}{45} + \Delta u(v_2)\frac{7}{45} + \Delta u(v_3)\frac{4}{45}\right)$$

and

$$\partial_n(u \circ F_w^{-1})(F_w v_3) = \left(\frac{5}{3}\right)^k (2u(v_3) - u(v_1) - u(v_2))$$
$$+ \left(\frac{5}{3}\right)^k \left(\Delta u(v_1)\frac{4}{45} + \Delta u(v_2)\frac{4}{45} + \Delta u(v_3)\frac{7}{45}\right).$$

□

As a Corollary of Lemma 3.1.3, we have:

Corollary 3.1.4. *Assume that $\Delta u = C$ on a cell of level m with boundary p_0, p_1, p_2. Then, the outward normal derivative of u reads*

$$\partial_n u(p_j) = \left(\frac{5}{3}\right)^k (2u(p_j) - u(p_{j+1}) - u(p_{j-1})) + \frac{C}{3^{m+1}}.$$

As shown in Figure 3.1, we start with the prescribed data a, b, c, m, n, q and want to obtain an equation relating these with x, y, z. In order to do so, we use the two basis given in [SU] for the space of biharmonic functions. Again, by the Figure 3.1, we can write

$$u = xf_{01} + af_{02} + mf_{03} + yf_{11} + bf_{12} + nf_{13}$$
$$u = xf_{01}^{(1)} + af_{02}^{(1)} + mf_{03}^{(1)} + zg_{01}^{(1)} + cg_{02}^{(1)} + qg_{03}^{(1)}$$

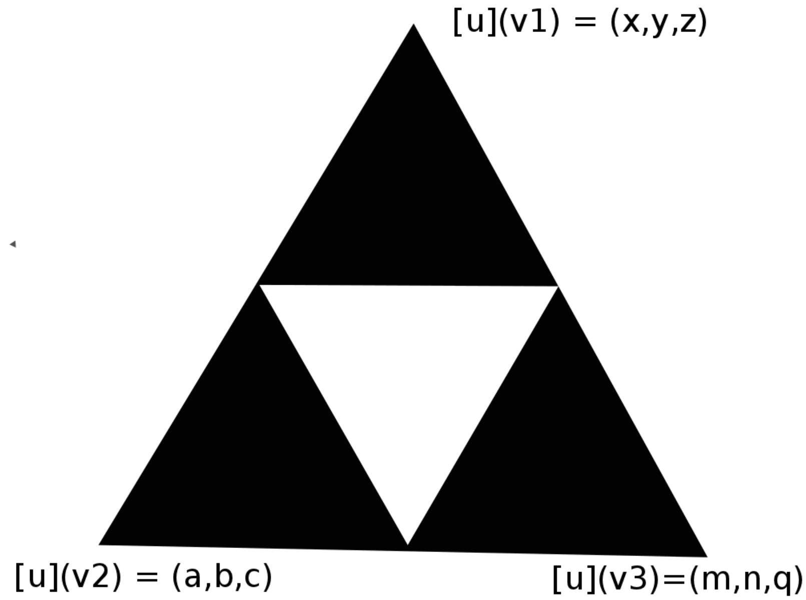

Figure 3.1: We define $[u](x) \coloneqq (u(x), \Delta u(x), \partial_n u(x))$.

We know by [SU] that the two basis are related by the equations

$$f_{0k}^{(1)} = \sum_{l=1}^{3} f_{0k} + b_{kl} f_{1l}$$
$$g_{0k}^{(1)} = \sum_{l=1}^{3} d_{kl} f_{1l}$$

Here, $d_{kl} = J_{kl}$ and $b_{kl} = -\sum_{m=1}^{3} J_{ml} \partial_n f_{0k}(v_m)$ where J is the inverse matrix of the inner products of the easy basis for $\mathcal{H}_0$ and $\partial_n f_{0k}(v_m) = -H_{mk}$ where $\{H_{mk}\}$ denotes the Dirichlet form $\mathcal{E}_0$ on $V_0 \times V_0$. Namely, we have

$$H = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \quad J = \begin{pmatrix} 11 & -4 & -4 \\ -4 & 11 & -4 \\ -4 & -4 & 11 \end{pmatrix}.$$

We obtain $b_{ii} = -30, b_{ij} = 15$ and $d_{ii} = 11, d_{ij} = -4$. The conversion equations between two basis give rise to the equation

$$15\,a - 4\,c + 15\,m - 4\,q - 30\,x + 11\,z = y$$
$$-30\,a + 11\,c + 15\,m - 4\,q + 15\,x - 4\,z = b$$
$$15\,a - 4\,c - 30\,m + 11\,q + 15\,x - 4\,z = n$$

which has the solution

$$x = r_2, \tag{3.8}$$
$$y = -\frac{135}{2}\,a - \frac{11}{4}\,b + \frac{105}{4}\,c - 15\,r_1 + \frac{45}{4}\,r_2 + \frac{225}{4}\,r_3, \tag{3.9}$$
$$z = -\frac{15}{2}\,a - \frac{1}{4}\,b + \frac{11}{4}\,c - r_1 + \frac{15}{4}\,r_2 + \frac{15}{4}\,r_3, \tag{3.10}$$
$$m = r_3, \tag{3.11}$$
$$n = 45\,a + b - 15\,c + 15\,r_1 - 45\,r_3, \tag{3.12}$$
$$q = r_1 \tag{3.13}$$

for three free parameters r_1, r_2, r_3. We will use the equations 3.8 to construct the functions we need.

Lemma 3.1.5. *There exists a biharmonic function h_0 satisfying*

$$h_0(v_1) = 1 \;\; h_0|_L = 0 \;\; \Delta h_0|_L = 0.$$

Proof. Since a harmonic function is fully determined by its values on the boundary, we prescribe certain values on the boundary of the triangle then extend harmonically. Of course, one needs to check the normal derivative matching conditions. As shown in Figure 3.2 we construct a highly symmetric harmonic

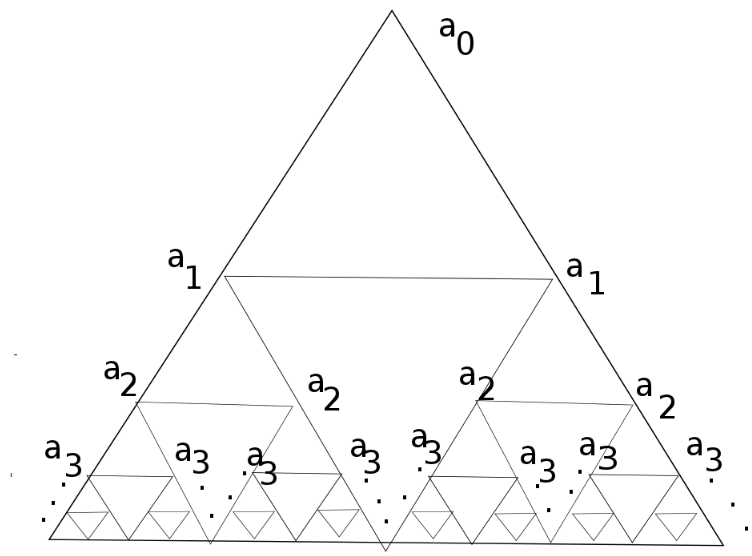

Figure 3.2: a_n denotes the function values on corresponding vertices.

function, this construction is inspired by [OS]. The construction gives rise to the structure shown in Figure 3.3.

The matching conditions for the normal derivative yields

$$2a_{n+1} - a_{n+1} - a_n = \frac{-5}{3}(2a_{n+1} - 2a_{n+1}) \tag{3.14}$$

which we can write as

$$a_{n+1} - a_n = \frac{10}{3}(a_{n+2} - a_{n+1}). \tag{3.15}$$

We apply the transformation $\Delta_n = a_{n+1} - a_n$ and observe that $\sum_{n=0}^{N} \Delta_n = a_{N+1} - a_0$ and $\sum_{n=0}^{N} \Delta_{n+1} = a_{N+2} - a_1$. Observe that we can write (3.15) as

$$\frac{3}{10}\Delta_n = \Delta_{n+1}.$$

Summing both sides from 0 to N yields

$$\frac{3}{10}\sum_{n=0}^{N} \Delta_n = \sum_{n=0}^{N} \Delta_{n+1}.$$

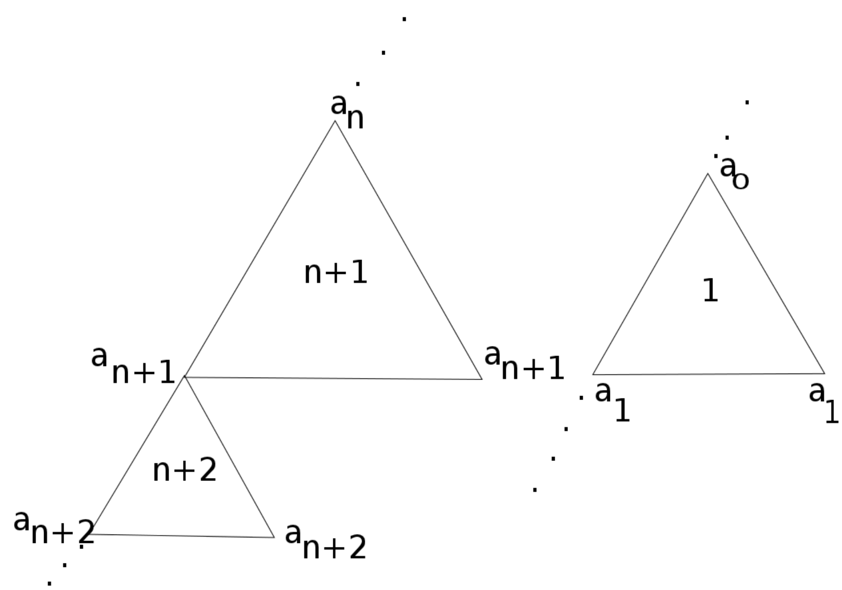

Figure 3.3: a_n denotes the values on corresponding vertices and the numbers in the triangles denote the level numbers.

By our previous observations we obtain the recurrence

$$a_{N+2} = \frac{3}{10}a_{N+1} + a_1 - \frac{3}{10}a_0.$$

Remarks 3.1.6. *1. Observe that for the initial condition* $a_0 = 1, a_1 = \frac{3}{10}$ *one gets* $a_n = \left(\frac{3}{10}\right)^n$. *In this case,* $L := \lim_{n\to\infty} a_n = 0$.

2. We can play with a_0 *and* a_1 *in order to get* $L = a_1 - \frac{3}{10}a_0 \neq 0$.

□

Proposition 3.1.7. *There exists a function* $h_3(x)$ *biharmonic on* $\overline{SG}$ *which satisfies the* *following properties*

$$h_3(v_1) = 0,$$
$$\Delta h_3(v_1) = 1$$
$$h_3|_L = 0$$
$$\Delta h_3|_L = 0.$$

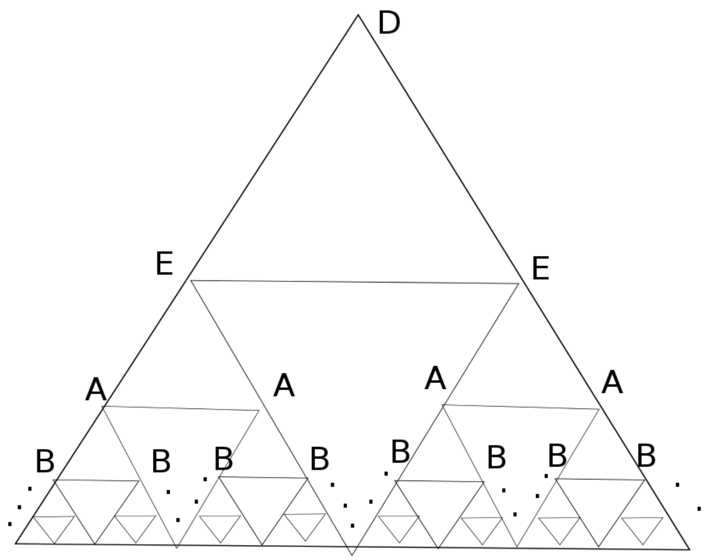

Figure 3.4: Here $D = (x_0, a_0), E = (x_1, a_1), A = (x_2, a_2), B = (x_3, a_3)$ where x_i denote the function and a_i denote the Laplacian values.

Proof. We use the fact from [SU] that a biharmonic function on a triangle is fully determined by its function and Laplacian values on the boundary of the triangle. We construct a symmetric biharmonic function as shown in Figure 3.4 which gives rise to the structure shown in Figure 3.5. In this case, by Proposition 3.1.3, the matching equations for the normal derivatives read, using the same notation as in Figure 3.5,

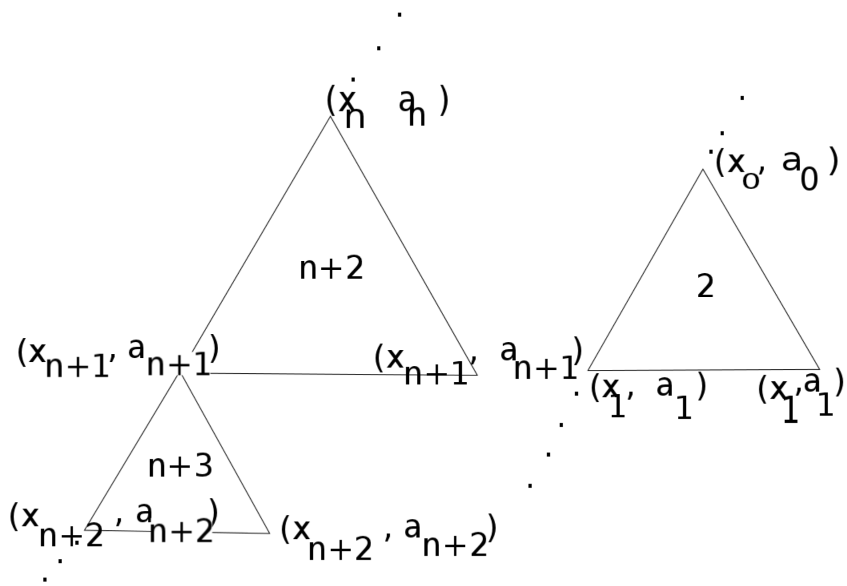

Figure 3.5: In (x_i, a_i), x_i stands for the function values and a_i for the Laplacian values. The numbers inside the triangles denote the level number.

$$
\begin{aligned}
2x_{n+1} - x_{n+1} - x_n + \frac{7}{45}a_{n+1} + \frac{4}{45}a_{n+1} + \frac{4}{45}a_n \\
= \frac{-5}{3}\left[(2x_{n+1} - 2x_{n+2} + \frac{7}{45}a_{n+1} + \frac{4}{45}a_{n+2} + \frac{4}{45}a_{n+2})\right]
\end{aligned}
$$

which we can rearrange as

$$
x_{n+1} - x_n + \frac{11}{45}a_{n+1} + \frac{4}{45} = \frac{10}{3}(x_{n+2} - x_{n+1}) - \frac{5}{3}\left(\frac{7a_{n+1} + 8a_{n+2}}{45}\right). \tag{3.16}
$$

We can apply the transformation $\Delta_n = x_{n+1} - x_n$ and write (3.16) as

$$
\Delta_{n+1} = \frac{3}{10}\Delta_n + \lambda_n \tag{3.17}
$$

where $\lambda_n = \frac{11a_{n+1}+4a_n}{150} + \frac{7a_{n+1}+8a_{n+2}}{90}$.

Now, we sum both sides of (3.17) from 0 to N to get

$$x_{N+2} = \frac{3}{10}x_{N+1} + x_1 - \frac{3}{10}x_0 + \sum_{n=0}^{N} \lambda_n \tag{3.18}$$

Remarks 3.1.8. *1. Observe that when a_0 and a_1 are chosen so that $L < 1$ then we have that $\sum_{n=0}^{\infty} \lambda_n < \infty$ in which case the equation (3.18) has a solution which gives a biharmonic function with non-zero Laplacian.*

In this general construction, in order to get h_3, we simply set $a_0 = 1$ and $a_1 = 3/10$ in which case $a_n = (3/10)^n$. Observe that with these, we have $\sum_{n=0}^{\infty} \lambda_n < \infty$. We simply set $x_0 = 1$ and $x_1 = 3/10$. Of course, we need to subtract off a harmonic function to get the correct boundary values, which does not effect the value of the Laplacian. □

Proposition 3.1.9. *There exists a biharmonic function, h_ψ^2 on $\overline{SG}$ satisfying the following properties*

$$\begin{aligned} h_\psi^2(v_1) &= 0, \\ \Delta h_\psi^2(v_1) &= 0 \\ \partial_n h_\psi^2(v_1) &= 0 \\ h_\psi^2|_L &= 0 \\ \Delta h_\psi^2|_L &= \psi. \end{aligned}$$

Proof. We use the construction in Lemma 3.1.7. Observe that we can construct a biharmonic function with $[u](v_1) = (0,0,0), [u](v_2) = (m,n,q), [u](v_3) = (-m,-n,-q)$. We simply put this function on F_1K as shown in Figure 3.6. Then, apply the construction in a skew-symmetric manner both to F_2K and F_3K by

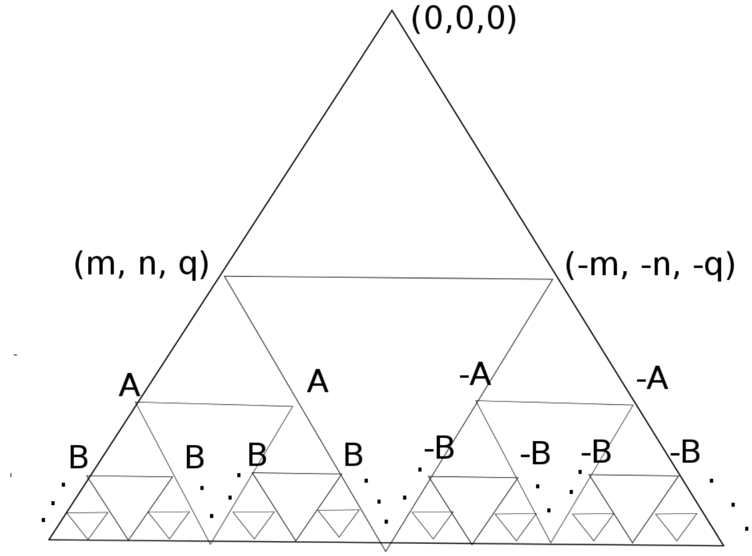

Figure 3.6: Here $A = (x_1, a_1), B = (x_1, a_1)$ where x_i denote the function and a_i denote the Laplacian values. Similarly, $(0,0,0), (m,n,q), (-m,-n,-q)$ are the triples in the form $(u(x), \Delta u(x), \partial_n u(x))$ at the corresponding vertex x

choosing the parameters in a suitable way. We again subtract off a harmonic function to get the correct boundary values. □

Remark 3.1.10. *Observe that when we have a recurrence relation of the following form*

$$x_{n+1} = rx_n + b_n$$

with $|r| < 1$ *and* $|b_n| < C$ *then* $\lim_{n\to\infty} x_n < \infty$. *This follows from the fact that* $x_N = \sum_{i=0}^{N-1} b_i r^{N-1-i}$. *Since,* $|x_N| \leq C \sum_{i=0}^{N-1} |r|^i < \infty$.

Lemma 3.1.11. *There exists a biharmonic function* u *on* K *with the following property*

$$(u(v_1), \Delta u(v_1), \partial_n u(v_1)) = (1/3, 1, 1)$$

$$(u(v_2), \Delta u(v_2), \partial_n u(v_2)) = (0, 1, 0)$$

$$(u(v_3), \Delta u(v_3), \partial_n u(v_3)) = (0, 1, 0).$$

If we put $u_1(x) = (u \circ F_1^{-1})(x), x \in F_1 K$ *then we have*

$$(u_1(F_1(v_1)), \Delta u_1(F_1(v_1)), \partial_n u_1(F_1(v_1))) = (1/3, 5, 5/3)$$
$$(u_1(F_1(v_2)), \Delta u_1(F_1(v_2)), \partial_n u_1(F_1(v_2))) = (0, 5, 0)$$
$$(u_1(F_1(v_3)), \Delta u_1(F_1(v_3)), \partial_n u_1(F_1(v_3))) = (0, 5, 0)$$

Proof. We obtain u by plugging in $a = c = r_3 = r_1 = 0, b = n = 1, r_2 = 1/3$ in the equations (3.8). For the second part, we scale the Laplacian accordingly and change the normal derivatives by using Proposition 3.1.3. □

Proposition 3.1.12. *There exists a function* $h_2(x)$ *biharmonic on* $\overline{SG}$ *which satisfies the following properties*

$$h_2(v_1) = 0$$
$$\Delta h_2 = 1$$
$$h_2|_L = 0.$$

Proof. We first define a function R. We define R to be equal to u_1, as in Lemma 3.1.11, on $F_1 K$. After that keeping the derivative to be 5, we continue in a symmetric way all the way down to the bottom line L, as shown in Figure 3.7.

As shown in Figure 3.7 we miniaturize this function to the top of the SG (we denote the normal derivative by m and Laplacian value by a) and continue all the way to the bottom in a symmetric way. This gives rise to the structure shown in Figure 3.8.

As seen in Figure 3.7, we continue to the bottom the Laplacian in a symmetric way. By using Corollary 3.1.4, the first compatibility equation for normal derivatives reads:

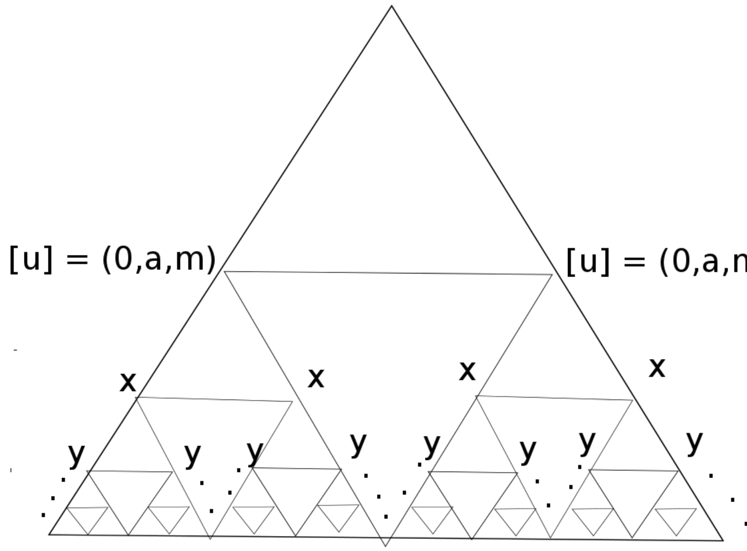

Figure 3.7: Here x and $y = x_1$ are the function values, the Laplacian is equal to a on F_2K and F_3K. Similarly, $(0, a, m)$ is a triple in the form $(u(x), \Delta u(x), \partial_n u(x))$ at the corresponding vertex x

$$x = \frac{a}{2.5^2.3} + \left(\frac{5}{3}\right)^2 \frac{m}{2}.$$

Now, we write the matching condition for the general point x_{n+1} in figure 3.8 to obtain

$$\left(\frac{5}{3}\right)^{n+2} (2x_{n+1} - x_{n+1} - x_n) + \frac{a}{3^{n+3}} + \left(\frac{5}{3}\right)^{n+3} (2x_{n+1} - 2x_{n+2}) + \frac{a}{3^{n+4}} = 0.$$

In order to solve this recursion we put $\Delta_n = x_n - x_{n-1}$. Then the equation becomes

$$\Delta_{n+2} = \frac{3}{10}\Delta_{n+1} + \frac{2a}{3.5^{n+3}}.$$

We observe that $\sum_{n=0}^{N-1} \Delta_{n+1} = x_N$ since we have the initial conditions $x_1 = x, x_0 = 0, \Delta_1 = x$. Putting these together yields the equation

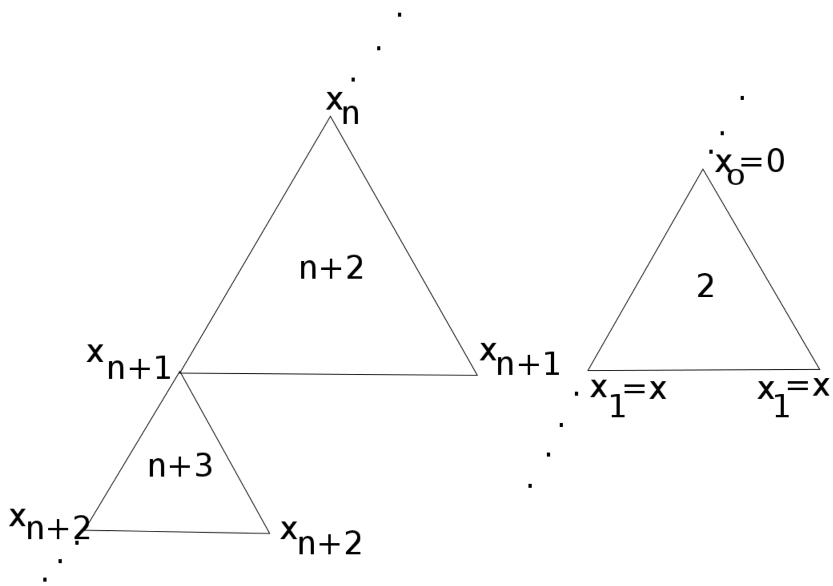

Figure 3.8: The numbers in the triangles denote the level number and x_i are the values of the function on the corresponding vertices.

$$x_{N+1} = \frac{3}{10}x_N + \sum_{n=0}^{N-1} \frac{2a}{3.5^{n+3}} + x.$$

By plugging in $a = 5, m = 0$ we get the recurrence relation

$$x_{N+1} = \frac{3}{10}x_N + \sum_{n=0}^{N-1} \frac{2 \times 5}{3.5^{n+3}} + \frac{2}{15}.$$

with initial conditions $x_0 = 0, x_1 = \frac{2}{15}, \Delta_1 = \frac{2}{15}$. We have $\lim_{N\to\infty} x_N = \frac{5}{21}$.

We consider $K = \frac{R}{5}$ which satisfies $K(v_1) = \frac{1}{15}, \Delta K = 1, K|_L = \frac{1}{21}$. In order to get h_2 we subtract off the harmonic function Y with $Y(v_1) = \frac{1}{15}$ and $Y|_L = \frac{1}{21}$. Hence, the result. □

3.2 Orthogonality of Basic Functions

So far, we have the basic functions: $\mathcal{A} := \{h_0, h_\psi^1, h_w^1, h_\psi^2, h_w^2\}$. We want to show that this family of functions is orthogonal with respect to the inner product

$$(f,g) = \int f|_L g|_L dx + \mathcal{E}(f,g) + \int_{SG} \Delta f \Delta g d\mu.$$

In the sequel we will put

$$(f,g)_2 = \int f|_L g|_L dx$$
$$(f,g)_3 = \mathcal{E}(f,g)$$
$$(f,g)_4 = \int_{SG} \Delta f \Delta g d\mu.$$

That is, $(f,g) = (f,g)_2 + (f,g)_3 + (f,g)_4$. We will show that the family, $\{h_0, h_\psi^1, h_w^1, h_\psi^2, h_w^2\}$, is orthogonal with respect to each $(\cdot,\cdot)_i$.

Lemma 3.2.1. *Let $A \subset SG$. If f is symmetric and g is skew-symmetric on A then $\mathcal{E}_A(f,g) = 0$.*

Proof. It is a standard fact that for each f,g there exits a signed measure (energy measure) $\nu_{f,g}$ bilinear in f and g such that

$$\mathcal{E}_A(f,g) = \int 1_A \nu_{f,g} = \nu_{f,g}(A).$$

Then, if f is symmetric and g is skew-symmetric it follows from bilinearity that $\mathcal{E}_A(f,g) = \nu_{f,g}(A) = 0$. $\square$

Remark 3.2.2. *Observe that the measure $\nu_{f,g}$ does not have any atoms and therefore does not charge points. This follows from the fact that $\nu_{f,f}$ does not have any atoms because*

$$\nu_{f,f}(F_w K) \le r^{|w|} \mathcal{E}(f,f).$$

Together with the polarization identity the above inequality implies

$$|\nu_{f,g}(F_w K)| \le r^{|w|}(\mathcal{E}(f+g) + \mathcal{E}(f-g))$$

which implies that $\nu_{f,g}$ is atomless.

Proposition 3.2.3. *[OS] Let $v \in dom(\mathcal{E})$ then, the Gauss-Green formula holds*

$$\mathcal{E}(h_0, v) = \frac{-7}{3}\int_L v|_L dx + v(v_1)\partial_n h(v_1).$$

That is, if $\int_L v|_L dx = 0$ then we simply have $\mathcal{E}(h_0, v) = v(v_1)\partial_n h(v_1)$.

Lemma 3.2.4.

$$(h_0, h_\psi^1)_2 = 0 \quad (h_0, h_\psi^1)_3 = 0 \quad (h_0, h_\psi^1)_4 = 0$$

$$(h_0, h_\psi^2)_2 = 0 \quad (h_0, h_\psi^2)_3 = 0 \quad (h_0, h_\psi^2)_4 = 0$$

$$(h_0, h_w^1)_2 = 0 \quad (h_0, h_w^1)_3 = 0 \quad (h_0, h_w^1)_4 = 0$$

$$(h_0, h_w^2)_2 = 0 \quad (h_0, h_w^2)_3 = 0 \quad (h_0, h_w^2)_4 = 0$$

Proof. We put $B := \mathcal{A} \setminus h_0$. Recall that h_0 is the harmonic function on $\overline{SG}$ with $h_0(v_1) = 1, h_0|_L = 0$. Given that h_0 is harmonic we readily have $(h_0, f)_4 = 0$ for any $f \in B$. Since $h_0|_L = 0$ we obtain $(h_0, f)_2 = 0$ for all $f \in B$.

By Lemma 3.2.1 we have $(h_0, f)_3 = 0$ for $f \in \{h_\psi^1, h_\psi^2, h_w^1, h_w^2\}$.

□

Lemma 3.2.5.

$$(h_\psi^1, h_\psi^2)_2 = 0 \quad (h_\psi^1, h_\psi^2)_3 = 0 \quad (h_\psi^1, h_\psi^2)_4 = 0$$

$$(h_\psi^1, h_w^1)_2 = 0 \quad (h_\psi^1, h_w^1)_3 = 0 \quad (h_\psi^1, h_w^1)_4 = 0$$

$$(h_\psi^1, h_w^2)_2 = 0 \quad (h_\psi^1, h_w^2)_3 = 0 \quad (h_\psi^1, h_w^2)_4 = 0$$

Proof. This time we put $B = \mathcal{A} \setminus h^1_\psi$.

Since $h^2_\psi|_L = 0$ we have $(h^1_\psi, h^2_\psi)_2 = 0$. By, skew symmetry we easily obtain $(h^1_\psi, h^1_w)_2 = 0$.

Similarly $h^2_w|_L = 0$ implies that $(h^1_\psi, h^2_w)_2 = 0$.

We want to show $(h^1_\psi, h^2_\psi)_3 = \mathcal{E}(h^1_\psi, h^2_\psi) = 0$. We can write

$$\mathcal{E}(h^1_\psi, h^2_\psi) = \sum_{i=1}^{3} \frac{1}{3}\mathcal{E}((h^1_\psi \circ F_i, h^2_\psi \circ F_i).$$

In the above summation we name the three parts as I, II and III. We denote by n_1 and $-n_2$ the normal derivatives $\partial_n h^1_\psi(F_1 v_2)$ and $\partial_n h^1_\psi(F_1 v_3)$ with respect to the cell $F_1 K$. We put $h^2_\psi(F_1 v_2) = x, h^2_\psi(F_1 v_3) = -x$ and recall that $h^2_\psi(v_1) = 0, h^2_\psi|_{L_1} = 0, h^2_\psi|_{L_2} = 0$. Applying local Gauss-Green to the cells $F_1 K, F_2 K, F_3 K$ and noting the cancellations by Lemma 3.2.3 yields

$$I = n_1 x + n_2 x$$

$$II = -n_1 x$$

$$III = -n_2 x.$$

Therefore, $I + II + III = 0$.

By the skew symmetry of h^1_w, h^2_w and symmetry of h^1_ψ on $F_w K$ we have that $(h^1_w, h^1_\psi)_3 = (h^2_w, h^1_\psi)_3 = 0$.

Since h^1_ψ is harmonic on $\overline{SG}$ we have that $(h^1_\psi, h^2_\psi)_4 = (h^1_\psi, h^1_w)_4 = (h^1_\psi, h^2_w)_4 = 0$.

□

Lemma 3.2.6.

$$(h_\psi^2, h_w^1)_2 = 0 \quad (h_\psi^2, h_w^1)_3 = 0 \quad (h_\psi^2, h_w^1)_4 = 0$$
$$(h_\psi^2, h_w^2)_2 = 0 \quad (h_\psi^2, h_w^2)_3 = 0 \quad (h_\psi^2, h_w^2)_4 = 0$$

Proof. Since $h_\psi^2|_L = 0$ we obtain $(h_\psi^2, h_w^1)_2 = 0, (h_\psi^2, h_w^2)_2 = 0$. By skew symmetry of h_w^1 and h_w^2 and symmetry of h_ψ^2 on the cell $F_w K$ we have $(h_\psi^2, h_w^1)_3 = 0, (h_\psi^2, h_w^2)_3 = 0$ and $(h_\psi^2, h_w^2)_4 = 0$. Since, h_w^1 is harmonic we readily obtain $(h_\psi^2, h_w^1)_4 = 0$. □

Lemma 3.2.7.

$$(h_w^1, h_w^2)_2 = 0 \quad (h_w^1, h_w^2)_3 = 0 \quad (h_w^1, h_w^2)_4 = 0$$
$$(h_{w'}^1, h_w^2)_2 = 0 \quad (h_{w'}^1, h_w^2)_3 = 0 \quad (h_{w'}^1, h_w^2)_4 = 0.$$

Proof. The first row is simply the scaled down versions of the previous Lemmas, hence they are all zero. The second row follows from definitions and skew-symmetry. □

3.3 Solution of Polyharmonic BVP and Convergence Theorems

Definition 3.3.1. *Let D_M denote the closed subset of SG which is the union of all the cells $F_w F_0(SG)$ for $|w| < M$, here we take $w_j = 1, 2$. Observe that ∂D_M consists of the points q_0 and $F_w q_0$ for $|w| = M$. See Figure 3.9.*

We put $V_n = D_{n+1} \setminus D_n$. Observe that $\overline{SG} = \cup_{i=0}^{\infty} V_i$.

Lemma 3.3.2. *There exists a unique biharmonic function on D_M when we prescribe values and laplacian values on ∂D_M.*

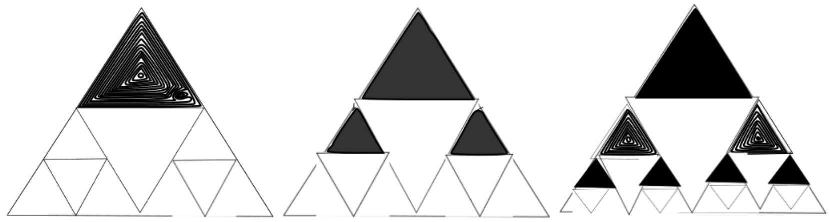

Figure 3.9: The Picture of D_n for the first three levels.

Proof. For the harmonic case, we know from [OS] that the prescribed values on ∂D_M determines a unique harmonic function on D_M. We obtain the biharmonic case by an iterated application of the harmonic case. Suppose there are two biharmonic functions u_1 and u_2 with the same values and Laplacian values on D_M. It follows that $h_1 = \Delta u_1$ and $h_2 = \Delta u_2$ are two harmonic functions on D_M but observe that h_1 and h_2 has the same values on ∂D_M. It follows from the harmonic case that $\Delta u_1 = \Delta u_2$ which implies $\Delta(u_1 - u_2) = 0$. That is to say, $u_1 - u_2$ is harmonic on D_M with $(u_1 - u_2)|_{\partial D_M} = 0$. By another application of the harmonic case, we obtain $u_1 = u_2$. □

Theorem 3.3.3. *Suppose the Haar series of f_1, f_2 on L is given by*

$$f_1 = c_\Psi(f_1)\Psi + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_1)\psi_w \tag{3.19}$$

$$f_2 = c_\Psi(f_2)\Psi + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_2)\psi_w. \tag{3.20}$$

Then the unique biharmonic function satisfying

$$\Delta^2 u = 0$$
$$\Delta u|_L = f_2$$
$$\Delta u(v_1) = c'$$
$$u|_L = f_1$$
$$u(v_1) = c$$

is given by

$$u(x) = (c' - c_{\Psi(f_2)})h_3(x) + c_\Psi(f_1)\Psi + (a - c_\psi(f_1))h_0(x) + c_\Psi(f_2)h_2 + \sum_{m=0}^{M}\sum_{|w|=m} c_w(f_1)h_w^1(x) + \sum_{m=0}^{M}\sum_{|w|=m} c_w(f_2)h_w^2(x) \tag{3.21}$$

The $(\cdot,\cdot)$*-norm of* u*, in the case* $c' = 0, c_\Psi(f_2) = 0$*, is given by*

$$(u,u) = |c_\Psi(f_1)|^2 + \sum_{m=0}^{M}\sum_{|w|=m} |c_w(f_1)|^2 + (c - c_\psi(f_1))^2 E_0 + E_1\sum_{m=0}^{M}\sum_{|w|=m}\left(\frac{10}{3}\right)^m |c_w(f_1)|^2 + E_2\sum_{m=0}^{M}\sum_{|w|=m}\left(\frac{2}{15}\right)^m |c_w(f_2)|^2 + + L_1\sum_{m=0}^{M}\sum_{|w|=m}\left(\frac{2}{3}\right)^m |c_w(f_2)|^2 \tag{3.22}$$

where $E_0 = \mathcal{E}(h_0), E_1 = \mathcal{E}(h_\psi^1), E_2 = \mathcal{E}(h_\psi^2), L_1 = (h_\psi^2, h_\psi^2)_4$.

Corollary 3.3.4. *By using the notation in Theorem 3.3.3 we have*

$$(u,u)_2 \le (u,u)_3 + 2|c_\Psi(f_1)|.$$

Proof. By Theorem 3.3.3 we have

$$(u,u)_2 = |c_\Psi(f_1)|^2 + \sum_{m=0}^{M} \sum_{|w|=m} |c_w(f_1)|^2$$

$$(u,u)_3 = (c - c_\psi(f_1))^2 E_0 + E_1 \sum_{m=0}^{M} \sum_{|w|=m} \left(\frac{10}{3}\right)^m |c_w(f_1)|^2$$
$$+ E_2 \sum_{m=0}^{M} \sum_{|w|=m} \left(\frac{2}{15}\right)^m |c_w(f_2)|^2.$$

Since, $E_0 = \frac{7}{3} > 1$, it follows that

$$(u,u)_2 \leq (u,u)_3 + 2c_\Psi(f_1). \tag{3.23}$$

Hence, the result. □

Remark 3.3.5. *In this remark, we explain how we consider the trace to the boundary, as in $\Delta u|_L = f_2$, when $f \in H^{s_2}$ for $s_2 < 0$. We know that the elements of H^{s_2} are distributions on the unit interval lying in the dual of H^{-s_2}. So, we denote the pairing between two spaces by $< \cdot, \cdot >$. A natural interpretation is the following:*

Consider the Haar series expansion of f_2, which does not really converge to a function so we cut it at M and put

$$S_M f_2 = c_\Psi(f_2)\Psi + \sum_{m=0}^{M} \sum_{|w|=m} c_w(f_2)\psi_w.$$

We will denote convergence in H^{s_2} norm by $\rightrightarrows$. Using this notation, we have that

$$S_M f_2 \rightrightarrows f_2.$$

We define the following function, $u_M : SG \to \mathbb{R}$,

$$u_M = C + Ah_3 + Dh_2 + Bh_0 + \sum_{m=0}^{M} \sum_{|w|=m} c_{1,w}^M h_w^1(x) + \sum_{m=0}^{M} \sum_{|w|=m} c_{2,w}^M h_w^2(x). \tag{3.24}$$

We choose the coefficients $A, B, C, D, c_{1,w}, c_{2,w}$ in such a way that u_M and u have the same values and Laplacian values on ∂D_M (Definition 3.3.1). By Lemma 3.3.2, there is *a unique biharmonic function with prescribed values and Laplacian values on ∂D_M so it follows that, restricted to D_M, u and u_M are the same.*

At this point we make the important observation that actually $c_{1,w}^M = c_{1,w}^{M+1}$ and $c_{2,w}^M = c_{2,w}^{M+1}$ for $|w| \le M$. Namely, as we go to higher levels we do not "overwrite" the existing coefficients. For instance, in the simplest case, if we apply this procedure to the BVP in Theorem 3.3.3, we simply get:

$$\begin{aligned} C &= c_\Psi(f_1) \\ D &= c_\Psi(f_2) \\ A &= c' - c_\Psi(f_2) \\ B &= a - c_\psi(f_1) \\ c_{1,w}^M &= c_w(f_1) \\ c_{2,w}^M &= c_w(f_2). \end{aligned}$$

This happens because h_w^1 and h_w^2 , by definition, do not have support on ∂D_M for $|w| \ge M+1$. That is to say, the coefficients in u_M is fully determined by the basis functions which has support on ∂D_M, namely by Ψ, h_0, h_3, h_w^1 and h_w^2 for $|w| \le M$.

Now we are ready to define. We put

$$\Delta u|_L = f_2 \text{ iff } \Delta u_M|_L = S_M f_2.$$

Observe that with this definition

$$\Delta u_M|_L \rightrightarrows f_2 \tag{3.25}$$

We observe that (3.25) implies that

$$< \Delta u_M|_L, g > \rightrightarrows < f_2, g >$$

as $M \to \infty$*, where* $g \in H^{-s_2}$*. So that,* $\Delta u|_L$ *becomes an element in the dual of* H^{-s_2}*.*

Theorem 3.3.6. *A biharmonic function* u *on* $\overline{SG}$ *has*

$$(u,u) = \int (u|_L)^2 dx + \mathcal{E}(u,u) + \int_{SG} (\Delta u)^2 d\mu < \infty$$

if and only if $u|_L = f_1$ *is in* $H^{s_1}(L)$ *for* $s_1 = \frac{\log \frac{10}{3}}{\log 4} > 0$ *and* $\Delta u|_L = f_2$ *is in* $H^{s_2}(L)$ *for* $s_2 = \frac{\log \frac{2}{3}}{\log 4} < 0$*. In this case, (3.19), (3.21) and (3.22) hold for* $M = \infty$ *and (3.20) converges in the* H^{s_2} *norm to* f_2*.*

Proof. We first show the "if" part. As $M \to \infty$, the contribution to the energy comes from h^1_w and h^2_w. If we put $\mathcal{E}(h^1_\psi, h^1_\psi) = E_1$ we obtain

$$\mathcal{E}(h^1_w, h^1_w) = \left(\frac{10}{3}\right)^m E_1.$$

So, for finite M the total contribution of energy coming from h^1_w's is

$$E_1 \sum_{m=0}^{m=M} \sum_{|w|=m} \left(\frac{10}{3}\right)^m |c_w(f_1)|^2.$$

Therefore, the H^{s_1}-norm is finite as $M \to \infty$ when $s_1 = \frac{\log \frac{10}{3}}{\log 4} > 0$.

Similarly, putting $\mathcal{E}(h^2_\psi, h^2_\psi) = E_2$ we obtain

$$\mathcal{E}(h^2_w, h^2_w) = \left(\frac{2}{15}\right)^m E_2.$$

In this case, the total contribution coming from h^2_w's is

$$E_2 \sum_{m=0}^{m=M} \sum_{|w|=m} \left(\frac{2}{15}\right)^m |c_w(f_2)|^2$$

in which case H^{s_2}-norm stays finite as $M \to \infty$ for $s_2 = \frac{\log \frac{2}{15}}{\log 4} < 0$.

We also investigate the contribution to the total norm coming from h_w^2's as

$$(h_w^2, h_w^2)_4 = \int (\Delta h_w^2)^2 d\mu.$$

We put

$$\int (\Delta h_\psi^2)^2 d\mu = L_1.$$

Therefore, the total contribution ends up being

$$L_1 \sum_{m=0}^{m=M} \sum_{|w|=m} \left(\frac{2}{3}\right)^m |c_w(f_2)|^2.$$

In this case, H^{s_2}-norm stays finite as $M \to \infty$ for $s_2 = \frac{\log \frac{2}{3}}{\log 4} < 0$. Since, $\frac{\log \frac{2}{15}}{\log 4} < \frac{\log \frac{2}{3}}{\log 4} < 0$, we pick the larger one because of the containment for H^s-spaces.

Now, we turn to the "only if" part. Let u be a biharmonic function on $\overline{SG}$, first we suppose that $c' = c_\psi(f_2) = 0$ and then derive the general case as a Corollary. We define u_M as exactly in (3.24). So that our observations, after (3.24), in Remark 3.3.5 hold.

We note that (u_M, u_M) is equal to (3.22). We want to show that $(u - u_M, u - u_M) \to 0$ as $M \to \infty$. We can find constants C_1 C_2 and C_3 such that $\mathcal{E}(h_\psi^1) = C_1 \mathcal{E}(h_\psi^1|_{D_1}), \mathcal{E}(h_\psi^2) = C_2 \mathcal{E}(h_\psi^2|_{D_1}), (h_\psi^2, h_\psi^2)_4 = C_3 (h_\psi^2|_{D_1}, h_\psi^2|_{D_1})_4$. By scaling and orthogonality we can use this to conclude the existance of a constant $C > 1$ such that

$$(u_M, u_M)_3 \leq C(u_M|_{D_M}, u_M|_{D_M})_3$$

$$(u_M, u_M)_4 \leq C(u_M|_{D_M}, u_M|_{D_M})_4.$$

By the above observations we have $u|_{D_M} = u_M|_{D_M}$ so we can write

$$(u_M, u_M)_3 \le C(u_M|_{D_M}, u_M|_{D_M})_3 = C(u|_{D_M}, u|_{D_M})_3 \le C(u,u) \tag{3.26}$$
$$(u_M, u_M)_4 \le C(u_M|_{D_M}, u_M|_{D_M})_4 = C(u|_{D_M}, u|_{D_M})_4 \le C(u,u)$$

Therefore, by (3.22), it follows that $(u_M, u_M)_3$ and $(u_M, u_M)_3$ are bounded and increasing.

Now we want to show that $(u - u_M, u - u_M)_3 \to 0$ as $M \to \infty$. Observe that

$$(u - u_M, u - u_M)_3 = ((u - u_M)|_{D_M}, (u - u_M)|_{D_M})_3 + ((u - u_M)|_{D^c_M}, (u - u_M)|_{D^c_M})_3$$

Since $(u - u_M)|_{D_M} = 0$, we get $((u - u_M)|_{D_M}, (u - u_M)|_{D_M})_3 = 0$. We need to show

$$((u - u_M)|_{D^c_M}, (u - u_M)|_{D^c_M})_3 \to 0$$

as $M \to \infty$. By triangle inequality we have

$$((u - u_M)|_{D^c_M}, (u - u_M)|_{D^c_M})_3 \le (u|_{D^c_M}, u|_{D^c_M})_3 + (u_M|_{D^c_M}, u_M|_{D^c_M})_3. \tag{3.27}$$

As $(u,u)_3 < \infty$ by assumption, we readily obtain $(u|_{D^c_M}, u|_{D^c_M})_3 \to 0$ as $M \to \infty$.

By using the notation in Definition 3.3.1, observe that

$$(u_M|_{D^c_M}, u_M|_{D^c_M})_3 = \sum_{n=M}^{\infty} (u_M|_{V_n}, u_M|_{V_n})_3.$$

We define the following double sequence

$$S^k_N = \sum_{e=1}^{N} (u_k|_{V_e}, u_k|_{V_e})_3$$

which is, by (3.26), uniformly bounded and increasing in both k and N. We have

$$S^k_N - S^k_M = \sum_{e=N}^{M} (u_k|_{V_e}, u_k|_{V_e})_3.$$

Taking $k = N$ and letting $M = \infty$ yields

$$S_N^N - S_\infty^k = \sum_{e=N}^{\infty} (u_k|_{V_e}, u_k|_{V_e})_3 = (u_M|_{D_M^c}, u_M|_{D_M^c})_3.$$

By the fact that the sequence S_N^k is monotone and uniformly bounded both $\lim_{N\to\infty} S_N^N$ and $\lim_{N\to\infty} S_\infty^N$ exists and equal. Therefore we obtain

$$\lim_{M\to\infty} (u_M|_{D_M^c}, u_M|_{D_M^c})_3 = 0.$$

By (3.27), we have $(u - u_M, u - u_M)_3 \to 0$ as $M \to \infty$. A similar argument shows that $(u - u_M, u - u_M)_4 \to 0$ as $M \to \infty$.

We have that (3.21) and (3.22) hold for u_M. Also, in our case (3.23) reads

$$(u_M, u_M)_2 \leq (u_M, u_M)_3 + 2C.$$

That is to say, by (3.26), $(u_M, u_M)_2$ is also uniformly bounded. We obtain $(u - u_M, u - u_M)_2 \to 0$ as $M \to \infty$.

In conclusion, we get $(u - u_m, u - u_m) \to 0$ as $M \to \infty$. It follows that we can take the limit as $M \to \infty$ in (3.21) and (3.22) and this also implies the convergence of $u_M|_L$ to f_1 in H^{s_1} and $\Delta u_M|_L$ to f_2 in H^{s_2}.

In order to get the general case, as above, we solve the case $c' = c_\psi(f_2) = 0$ and then simply replace u by $u + (c' - c_\psi(f_2))h_3 + c_\psi(f_2)h_2$. Observe that adding on a constant does not change the regularity of f_2. □

Theorem 3.3.7. *Let u be a function in $\overline{SG}$ with $(u, u) < \infty$. Then, u has boundary values with $u|_L = f_1$ in $H^{s_1}(L)$ for $s_1 = \frac{\log \frac{10}{3}}{\log 4} > 0$ and $\Delta u|_L = f_2$ in $H^{s_2}(L)$ for $s_2 = \frac{\log \frac{2}{3}}{\log 4} < 0$.*

Proof. We proceed as in the proof of Theorem 3.3.6. We form the functions u_M in a similar way this time being equal to u on ∂D_M. We have $u_M(v_1) = u(v_1)$

and all the other norms $(u_M, u_M)_i$ for $i = 2, 3, 4$ are bounded by (u, u). We take the limit as $M \to \infty$ to obtain a biharmonic function h with $h|_L = f_1 = u|_L$ and $\Delta h|_L = f_2 = \Delta u|_L$. Hence, the result. $\square$

CHAPTER 4

GAUSSIAN FREE FIELDS ON HIGH DIMENSIONAL SIERPINSKI CARPET GRAPHS

This Chapter is mostly drawn from [CU] which is joint work with Joe P. Chen.

4.1 High Dimensional Sierpinski Carpet Graphs

Construction of the fractal

Let $F_0 := [0,1]^d$ be the unit cube in $\mathbb{R}^d$, $d \geq 2$, and fix an $\ell_F \in \mathbb{N}$, $\ell_F \geq 3$. For $N \in \mathbb{Z}$, let $\mathcal{Q}_N$ be the collection of closed cubes of side ℓ_F^{-N} with vertices in $\ell_F^{-N}\mathbb{Z}^d$. For $A \subset \mathbb{R}^d$, let $\mathcal{Q}_N(A) = \{Q \in \mathcal{Q}_N : \mathrm{int}(Q) \cap A \neq \varnothing\}$. Denote by Ψ_Q the orientation-preserving affine map which maps F_0 to $Q \in \mathcal{Q}_N$.

We now introduce a decreasing sequence $(F_N)_N$ of closed subsets of F_0 as follows. Fix $m_F \in \mathbb{N}$, $1 \leq m_F < \ell_F^d$, and let F_1 be the union of m_F distinct elements of $\mathcal{Q}_1(F_0)$. Then by induction we put

$$F_{N+1} = \bigcup_{Q \in \mathcal{Q}_N(F_N)} \Psi_Q(F_1) = \bigcup_{Q \in \mathcal{Q}_1(F_1)} \Psi_Q(F_N)\ ,\ N \geq 1.$$

It is a standard argument to show that $F = \bigcap_{N=0}^{\infty} F_N$ is the unique fixed point of the iterated function system of contractions $\{\Psi_Q\}_{Q \in \mathcal{Q}_1(F_1)}$. Moreover, F has Hausdorff dimension $d_h(F) = \log m_F / \log \ell_F$.

We say that F is a *generalized Sierpinski carpet (GSC)* if and only if F_1 satisfies the following four conditions:

1. (Symmetry) F_1 is preserved under the isometries of the unit cube.

2. (Connectedness) F_1 is connected.

3. (Non-diagonality) Let $m \geq 1$ and $B \subset F_0$ be a cube of side length $2\ell_F^{-m}$, which is the union of 2^d distinct elements of $\mathcal{Q}_m$. Then if $\mathrm{int}(F_1 \cap B)$ is non-empty, it is connected.

4. (Borders included) F_1 contains the segment $\{(x_1, 0, \cdots, 0) \in \mathbb{R}^d : x_1 \in [0,1]\}$.

For alternative ways of stating the non-diagonality condition 3, see [KajinoND]. Throughout the article, we shall refer to ℓ_F and m_F as, respectively, the *length scale factor* and the *mass scale factor* of the carpet F.

The stochastic analysis on the Sierpinski carpet is built upon the measure space (F, ν), where ν is the self-similar Borel probability measure which assigns mass m_F^{-N} to each $\Psi_Q(F)$, $Q \in \mathcal{Q}_N(F_N)$. Note that ν is a constant multiple of the $d_h(F)$-dimensional Hausdorff measure on F. We will also consider the *unbounded carpet* $F_\infty := \bigcup_{N=0}^{\infty} \ell_F^N F$, and let ν_∞ be the σ-finite self-similar Borel probability measure on F_∞, assigning mass m_F^N to $\ell_F^N F$.

We introduce two other important scale factors associated with Sierpinski carpets. Let D_N be the network of diagonal crosswires obtained by connecting each vertex of a cube $Q \in \mathcal{Q}_N$ to the vertex at the center of the cube via a wire of unit resistance. Denote by $\mathcal{R}_{D_N}$ the resistance across two opposite faces of D_N. It was shown in [BB90Resistance, McGillivray] that there exist $\rho_F \in (0, \infty)$ and positive constants $C(d)$ and $C'(d)$ such that

$$C\rho_F^N \leq \mathcal{R}_{D_N} \leq C'\rho_F^N.$$

The constant ρ_F is henceforth referred to as the *resistance scale factor* of the carpet F. As of this writing, there's no known exact formula for ρ_F: the best estimate, obtained via a resistance shorting and a cutting argument, is [BB99]*Proposition 5.1

$$\ell_F^2/m_F \le \rho_F \le 2^{1-d}\ell_F.$$

Next, let $t_F = m_F\rho_F$, which stands for the *time scale factor* of the carpet F. The significance of t_F is due to the fact that the expected time for a d-dimensional Brownian motion to traverse from one face of $\ell_F^N F_N$ to the opposite face scales with t_F^N.

It is often convenient to introduce, respectively, the *Hausdorff*, *walk*, and *spectral* dimensions of F:

$$d_h(F) = \frac{\log m_F}{\log \ell_F}, \quad d_w(F) = \frac{\log t_F}{\log \ell_F}, \quad d_s(F) = 2\frac{\log m_F}{\log t_F}.$$

Under the strict inequality $m_F < \ell_F^d$, one has $1 \le d_s(F) < d_h(F) < d$ and $d_w(F) > 2$. The latter inequality implies that diffusion on F (resp. F_∞) is *sub-Gaussian*, in contrast with Gaussian diffusion which has walk dimension 2.

For each generalized Sierpinski carpet F, we consider two associated graphs. See Figure 1.2.

Let $V_N = \ell_F^N F_N \cap \mathbb{Z}^d$. Introduce the graph $\mathcal{G}_N = (V_N, \sim)$, where throughout the paper, the edge relation "$\sim$" means that two vertices x, x' are connected by an edge if and only if their Euclidean distance $\|x - x'\| = 1$. Put $\mathcal{G}_\infty = \bigcup_{N\in\mathbb{N}} \mathcal{G}_N$, which we call the *outer* Sierpinski carpet graph. Observe that $\mathcal{G}_\infty$ is a subgraph of $(\mathbb{Z}_+)^d$. In this paper we will study the Gaussian free field on $\mathcal{G}_\infty$.

Next, let $I_N = \ell_F^N F_N \cap \left(\mathbb{Z}^d + \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right)\right)$. Introduce the graph $\mathcal{I}_N = (I_N, \sim)$. Put $\mathcal{I}_\infty = \bigcup_{N\in\mathbb{N}} \mathcal{I}_N$, which we call the *inner* Sierpinski carpet graph. It is easy to see that $|I_N| = m_F^N$, and that there exist constants $C_{1.1}$ and $C_{1.2}$, independent of

N, such that $C_{1.1} m_F^N \leq |V_N| \leq C_{1.2} m_F^N$.

For easy reference, we provide in Table 1.1 a side-by-side comparison of the relevant parameters on $\mathbb{Z}^d$ and on the Sierpinski carpet graph ($\mathcal{G}_\infty$ or $\mathcal{I}_\infty$). It is known that simple random walk on the latter is transient if and only if $\rho_F < 1$ [BBSCGraph, McGillivray].

4.1.1 Main results

In what follows, F is a *transient* generalized Sierpinski carpet, with $\rho_F < 1$ (equivalently, $d_s(F) > 2$). This includes any generalized Sierpinski carpet whose cross-section contains a full copy of the 2-plane $[0,1]^2$ (*cf.* [BB99]*§9), as well as other, *but not all*, d-dimensional ($d \geq 3$) carpets, such as the Menger sponge. Our analysis does *not* apply to any generalized Sierpinski carpet in $\mathbb{R}^2$, whereby $\rho_F > 1$ and is hence recurrent.

Let $G_{\mathcal{G}_\infty} : V_\infty \times V_\infty \to \mathbb{R}$ be the Green's function for simple random walk on the outer Sierpinski carpet graph $\mathcal{G}_\infty$ without killing. We denote $\overline{G} := \sup_{x \in V_\infty} G_{\mathcal{G}_\infty}(x,x)$ and $\underline{G} := \inf_{x \in V_\infty} G_{\mathcal{G}_\infty}(x,x)$. Since simple random walk on $\mathcal{G}_\infty$ is transient, both $\underline{G}$ and $\overline{G}$ are positive and finite.

We also need the notion of the (0-order) capacity of the compact carpet F with respect to a Dirichlet form $(\mathcal{E}, \mathcal{F}) \in \mathfrak{E}$ on $L^2(F_\infty, \nu_\infty)$, given by

$$\mathrm{Cap}_{\mathcal{E}}(F) := \inf\{\mathcal{E}(f,f) : f \in \mathcal{F} \cap C_c(F_\infty), f \geq 1 \text{ a.e. on } F\},$$

See Proposition 4.3.2 for a more general definition of the capacity, as well as some important properties.

Let $\mathbb{P}$ be the law of the Gaussian free field on $\mathcal{G}_\infty$ with covariance $G_{\mathcal{G}_\infty}$, and let $\Omega^+_{V_N}$ denote the entropic repulsion event $\{\varphi_x \geq 0 \text{ for all } x \in V_N\}$. Our first main result identifies the rate of exponential decay for $\mathbb{P}(\Omega^+_{V_N})$.

Theorem 4.1.1. *There exists a point $x_0 \in V_\infty$ such that for any Dirichlet form $(\mathcal{E}, \mathcal{F}) \in \mathfrak{E}$, there are positive constants $C_{1.3}(\mathcal{E})$ and $C_{1.4}(\mathcal{E})$ such that*

$$\begin{aligned} -C_{1.3} \cdot \overline{G} \cdot \mathrm{Cap}_{\mathcal{E}}(F) &\leq \varliminf_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} \log(t_F^N)} \\ &\leq \varlimsup_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} \log(t_F^N)} \leq -C_{1.4} \cdot G_{\mathcal{G}_\infty}(x_0, x_0) \cdot \mathrm{Cap}_{\mathcal{E}}(F). \end{aligned} \tag{4.1}$$

The constants $C_{1.3}$ and $C_{1.4}$ are attributed to two sources: one coming from comparing the Dirichlet forms on $\mathcal{G}_\infty$ and on $\mathcal{I}_\infty$ (Lemma 4.2.4), and the other coming from comparing the (maximal or minimal) cluster point of the sequence of renormalized Dirichlet forms on $\mathcal{I}_\infty$ with an element of $\mathfrak{E}$ (Theorem 4.2.3). Due to the lack of precise control of the constants involved in these comparisons, *the authors deem it not possible to determine whether $C_{1.3}$ equals $C_{1.4}$.*

Notwithstanding the small discrepancy between the lower and upper bounds in Theorem 4.1.1, we are still able to give a precise description of entropic repulsion on $\mathcal{G}_\infty$. We shall prove that conditional on $\Omega^+_{V_N}$, the local sample mean of the free field on V_N is pushed to a height which is proportional to $\sqrt{N}$, and as $N \to \infty$, the rescaled height converges in probability to a constant.

Theorem 4.1.2. *For any $\epsilon > 0$ and $\eta > 0$,*

$$\lim_{N\to\infty} \sup_{\substack{x\in V_N \\ V_{N,\epsilon}(x)\subset V_N}} \mathbb{P}\left(\left| \frac{\bar{\varphi}_{N,\epsilon}(x)}{\sqrt{\log(t_F^N)}} - \sqrt{2\underline{G}} \right| \geq \eta \;\middle|\; \Omega^+_{V_N} \right) = 0, \tag{4.2}$$

where $\bar{\varphi}_{N,\epsilon}(x) := \frac{1}{|V_{N,\epsilon}(x)|} \sum_{z\in V_{N,\epsilon}(x)} \varphi_z$ *and* $V_{N,\epsilon}(x) := \left\{ z \in V_N : \max_{1\leq i\leq d} \left| z_i - [\ell_F^N x_i] \right| \leq \epsilon \cdot \ell_F^N \right\}$.

We comment that in the case of $\mathbb{Z}^d$, which has full translational invariance, one can replace $G_{\mathbb{Z}^d}(0,0)$ by $G_{\mathbb{Z}^d}(x,x)$ for any $x \in \mathbb{Z}^d$. On the other hand, in the case of the Sierpinski carpet graph, we have no explicit information about how the on-diagonal Green's function $G_{\mathcal{G}_\infty}(x,x)$ varies with $x \in V_\infty$. Nevertheless, our result says that $\sqrt{2\underline{G}\log t_F}\sqrt{N}$, where $\underline{G} = \inf_{x\in V_\infty} G_{\mathcal{G}_\infty}(x,x)$, sets the leading-order asymptotic height for the free field above the hard wall on V_N. A sketch of the arguments leading to this result will appear at the beginning of §4.5.2.

The rest of the Chapter is organized as follows. In Section 4.2 we recapitulate the construction of local regular Dirichlet forms on Sierpinski carpets via graphical approximations (the Kusuoka-Zhou construction), and prove the convergence of the discrete Green forms, both on $\mathcal{I}_\infty$ and on $\mathcal{G}_\infty$, to a continuum Green form on F_∞. We then proceed to prove Theorem 4.1.1 in Section 4.4, and Theorem 4.1.2 in Section 4.5.

4.2 Dirichlet Forms

In this section we provide the necessary potential theoretic lemmata to prove Theorems 4.1.1 and 4.1.2. Our main results are Theorem 4.2.3 and Lemma 4.2.5; only Lemma 4.2.5 will be used in subsequent sections.

Notations. If (X,m) denotes a measure space, then $\langle f,\mu\rangle_X$ stands for $\int_X f\,d\mu$, pairing a function f on X with a Borel measure μ.

4.2.1 Kusuoka-Zhou construction of Dirichlet forms

Let F be a generalized Sierpinski carpet, and $\mathcal{I}_\infty = (I_\infty, \sim)$ be the inner Sierpinski carpet graph introduced in §4.1. For each $N \in \mathbb{N}$ and each $w \in I_\infty$, let $\Psi_w^{(N)}$ be the closed cube of side ℓ_F^{-N} centered at $\ell_F^{-N} w$. We define the *mean-value operator* $\tilde{P}_N : L^1(F_\infty, \nu_\infty) \to C(I_\infty; \mathbb{R})$ by

$$(\tilde{P}_N f)(w) = \frac{1}{\nu_\infty\left(\Psi_w^{(N)} \cap F_\infty\right)} \int_{\Psi_w^{(N)} \cap F_\infty} f(y) \nu_\infty(dy),$$

Similarly, if μ_∞ is a Radon measure on F_∞ such that $\mu_\infty \ll \nu_\infty$, then define $\tilde{P}_N \mu_\infty = \left(\tilde{P}_N \frac{d\mu_\infty}{d\nu_\infty}\right) \nu_N$, where $\nu_N = \frac{1}{m_F^N} \mathbb{1}_{I_\infty}$ is a self-similar measure on I_∞.

As is customary, we define the discrete Dirichlet form on the graph $\mathcal{I}_\infty$ by

$$E_{\mathcal{I}_\infty}(f_1, f_2) = \frac{1}{2} \sum_{\substack{w, w' \in I_\infty \\ w \sim w'}} (f_1(w) - f_1(w'))(f_2(w) - f_2(w'))$$

for all f_1, f_2 in the natural domain $\mathcal{D}(E_{\mathcal{I}_\infty}) = \{f \in \ell^2(I_\infty) : E_{\mathcal{I}_\infty}(f, f) < \infty\}$. Furthermore, let $\mathcal{E}_N^{\mathcal{I}} = \rho_F^N E_{\mathcal{I}_\infty}$ be the *renormalized* Dirichlet form, where $\rho_F \in (0, \infty)$ is the resistance scale factor identified in §4.1.

Let $\mathcal{F}_0 := \left\{f \in L^2(F_\infty, \nu_\infty) : \sup_N \mathcal{E}_N^{\mathcal{I}}(\tilde{P}_N f, \tilde{P}_N f) < \infty\right\}$. The following convergence result for $(\mathcal{E}_N^{\mathcal{I}})_N$ is originally due to Kusuoka and Zhou [KusuokaZhou]*Proposition 5.2 & Theorem 5.4, and later generalized in [HKKZ]*Lemma 4.1 & Theorem 4.3.

Proposition 4.2.1. 1. *There exists a constant $C_{2.1}$ such that for all $N, M \geq 1$ and all $f \in \mathcal{F}_0$,*

$$\mathcal{E}_N^{\mathcal{I}}(\tilde{P}_N f, \tilde{P}_N f) \leq C_{2.1} \mathcal{E}_{N+M}^{\mathcal{I}}(\tilde{P}_{N+M} f, \tilde{P}_{N+M} f).$$

2. *There exists $(\mathcal{E}, \mathcal{F}_0) \in \mathfrak{E}$ and positive constants $C_{2.2}$ and $C_{2.3}$ such that for all*

$f \in \mathcal{F}_0$,

$$C_{2.2} \sup_N \mathcal{E}_N^{\mathcal{I}}(\tilde{P}_N f, \tilde{P}_N f) \le \mathcal{E}(f,f) \le C_{2.3} \varliminf_{N\to\infty} \mathcal{E}_N^{\mathcal{I}}(\tilde{P}_N f, \tilde{P}_N f). \tag{4.3}$$

Remark 4.2.2. *In their original work [KusuokaZhou], Kusuoka and Zhou identified a family of Dirichlet forms, denoted $\mathcal{D}ch$, which are associated with cluster points of the sequence of suitably rescaled Markov processes on $\mathcal{I}_N$. Then they proved (4.3) for any $\mathcal{E} \in \mathcal{D}ch$, and showed that $(\mathcal{E}, \mathcal{F}_0)$ is a local regular Dirichlet form. Note that $\mathcal{D}ch \subset \mathfrak{E}$ by virtue of [BBKT]*Theorem 3.2.*

4.2.2 Convergence of discrete Green forms

In this subsection we shall consider Dirichlet forms on a class of smooth measures (instead of functions), and derive a convergence result similar to Proposition 4.2.1. From now on let $\mathcal{M}_+(F)$ be the family of all nonnegative finite Borel measures on F, and let

$$\mathcal{M}^{(0)}_{0,\mathrm{ac}}(F) = \left\{ \mu \in \mathcal{M}_+(F) : \mu \ll \nu,\ \frac{d\mu}{d\nu} \in \mathcal{F}_0 \right\}.$$

Let $G_{\mathcal{I}_N} : V_\infty \times V_\infty \to \mathbb{R}$ be the Green's function for simple random walk on $\mathcal{I}_\infty$ killed upon exiting $\mathcal{I}_N$. By the reproducing property of Green's function, $E_{\mathcal{I}_\infty}(G_{\mathcal{I}_N}(w,\cdot), h) = h(w)$ for all $h \in \mathcal{D}(E_{\mathcal{I}_\infty})$ with $\mathrm{supp}(h) \subset I_N$. Therefore, denoting by $U_N^{\mathcal{I}}$ the 0-order potential operator associated with $\mathcal{E}_N^{\mathcal{I}}$, we have

$$\begin{aligned}\mathcal{E}_N^{\mathcal{I}}(U_N^{\mathcal{I}}\mu, h) &= \langle h, \mu \rangle_{\mathcal{I}_N} \\ &= \frac{1}{m_F^N} \sum_{w \in I_N} h(w) \frac{d\mu}{d\nu_N}(w) = \mathcal{E}_N^{\mathcal{I}}\left(\rho_F^{-N} \frac{1}{m_F^N} \sum_{w\in I_N} G_{\mathcal{I}_N}(\cdot, w) \frac{d\mu}{d\nu_N}(w), h \right)\end{aligned}$$

for all $h \in \mathcal{D}(E_{\mathcal{I}_\infty})$ with $\mathrm{supp}(h) \subset I_N$, and all nonnegative measures μ with support in I_N. It follows that

$$\begin{aligned} &\mathcal{E}_N^{\mathcal{I}}(U_N^{\mathcal{I}}\mu, U_N^{\mathcal{I}}\mu) = \\ &= \mathcal{E}_N^{\mathcal{I}}\left(\rho_F^{-N}\tfrac{1}{m_F^N}\textstyle\sum_{w\in I_N} G_{\mathcal{I}_N}(\cdot,w)\tfrac{d\mu}{d\nu_N}(w), \rho_F^{-N}\tfrac{1}{m_F^N}\sum_{w'\in I_N} G_{\mathcal{I}_N}(\cdot,w')\tfrac{d\mu}{d\nu_N}(w')\right) \\ &= \rho_F^{-N}\tfrac{1}{m_F^{2N}}\textstyle\sum_{w,w'\in I_N} G_{\mathcal{I}_N}(w,w')\tfrac{d\mu}{d\nu_N}(w)\tfrac{d\mu}{d\nu_N}(w') \end{aligned} \tag{4.4}$$

for all such measures μ. The expression in (4.4) is what we shall call the **Green form** corresponding to the Dirichlet form $\mathcal{E}_N^{\mathcal{I}}$. It has a kernel given by the (renormalized) Green's function $\rho_F^{-N} G_{\mathcal{I}_N}$, whence the name.

The following result proved in [CU] describes the convergence of the discrete Green forms.

Theorem 4.2.3. *[CU] There exist $(\mathcal{E}, \mathcal{F}) \in \mathfrak{E}$ and constants $C_{2.4}(\mathcal{E}), C_{2.5}(\mathcal{E})$ such that*

$$\begin{aligned} C_{2.4}\mathcal{E}(U\mu, U\mu) \;&\le\; \varliminf_{N\to\infty} \mathcal{E}_N^{\mathcal{I}}\left(U_N^{\mathcal{I}}\tilde{P}_N\mu, U_N^{\mathcal{I}}\tilde{P}_N\mu\right) \\ &\le\; \varlimsup_{N\to\infty} \mathcal{E}_N^{\mathcal{I}}\left(U_N^{\mathcal{I}}\tilde{P}_N\mu, U_N^{\mathcal{I}}\tilde{P}_N\mu\right) \;\le\; C_{2.5}\mathcal{E}(U\mu, U\mu) \end{aligned} \tag{4.5}$$

for all $\mu \in \mathcal{M}_{0,\mathrm{ac}}^{(0)}(F)$, where U is the 0-order potential operator associated with $\mathcal{E}$.

4.2.3 Comparison of discrete Dirichlet & Green forms

Recall that we are considering the free field on the outer Sierpinski carpet graph $\mathcal{G}_\infty$, while the convergence from discrete Dirichlet forms to the continuum one is based on the inner Sierpinski carpet graph $\mathcal{I}_\infty$. To bridge this gap, we shall compare the discrete Dirichlet (and Green) forms on $\mathcal{G}_\infty$ and on $\mathcal{I}_\infty$ in this subsection.

Observe (from Figure 1.2) that for each "center vertex" $w \in I_\infty$, there is a unique set $\mathcal{C}(w)$ of 2^d "corner vertices" in V_∞ which are nearest neighbors of

w, *i.e.*, $\mathcal{C}(w) = \{x \in V_\infty : \|x - w\| = \sqrt{d}/2\}$. Let $\tilde{Q} : C(V_\infty; \mathbb{R}) \to C(I_\infty; \mathbb{R})$ be the projection operator given by

$$(\tilde{Q}f)(w) = \frac{1}{2^d} \sum_{x \in \mathcal{C}(w)} f(x).$$

As is customary, we introduce the discrete Dirichlet form on graph $\mathcal{G}_\infty$ by

$$E_{\mathcal{G}_\infty}(f_1, f_2) = \frac{1}{2} \sum_{\substack{x, x' \in V_\infty \\ x \sim x'}} (f_1(x) - f_1(x'))(f_2(x) - f_2(x'))$$

for all f_1, f_2 in the natural domain $\mathcal{D}(E_{\mathcal{G}_\infty})$. Let $G_{\mathcal{G}_N} : V_\infty \times V_\infty \to \mathbb{R}$ denote the Green's function killed upon exiting $\mathcal{G}_N$.

The following Lemma from [CU] gives the required comparison of the Discrete Dirichlet forms.

Lemma 4.2.4. *[CU] For all $f \in \mathcal{D}(E_{\mathcal{G}_\infty})$,*

$$E_{\mathcal{G}_\infty}(f, f) \geq E_{\mathcal{I}_\infty}\left(\tilde{Q}f, \tilde{Q}f\right). \tag{4.6}$$

It follows that for all nonnegative functions f on V_N,

$$\sum_{x, x' \in V_N} G_{\mathcal{G}_N}(x, x') f(x) f(x') \leq 2^{2d} \sum_{w, w' \in I_N} G_{\mathcal{I}_N}(w, w') (\tilde{Q}f)(w) (\tilde{Q}f)(w'). \tag{4.7}$$

4.2.4 The main lemma

In this subsection, we establish the limsup convergence of discrete Green forms on $\mathcal{G}_\infty$, which will play a crucial role in the main proofs.

Let $G^{\square}_{\mathcal{G}_N} : V_N \times V_N \to \mathbb{R}$ be the restriction of $G_{\mathcal{G}_\infty}$ on $V_N \times V_N$; we have added a superscript $\square$ to distinguish it from the Green's function on $\mathcal{G}_\infty$ killed upon exiting $\mathcal{G}_N$. Also introduce the probability measure $\eta_N := \frac{1}{|V_N|}\mathbb{1}_{V_N}$ on V_N. Define, for any $h \in \ell^1(V_N;\mathbb{R}) \cap \ell^\infty(V_N;\mathbb{R})$,

$$U_N^{\mathcal{G}}(h\eta_N) := \rho_F^{-N}\frac{1}{|V_N|}\sum_{x\in V_N} G_{\mathcal{G}_N}(\cdot,x)h(x), \quad U_N^{\mathcal{G}\square}(h\eta_N) := \rho_F^{-N}\frac{1}{|V_N|}\sum_{x\in V_N} G_{\mathcal{G}_N}^{\square}(\cdot,x)h(x).$$

Writing $\mathcal{E}_N^{\mathcal{G}} = \rho_F^N E_{\mathcal{G}_\infty}$ for the renormalized discrete Dirichlet form on $\mathcal{G}_\infty$, we have

$$\mathcal{E}_N^{\mathcal{G}}\left(U_N^{\mathcal{G}}(h\eta_N), U_N^{\mathcal{G}}(h\eta_N)\right) = \rho_F^{-N}\frac{1}{|V_N|^2}\sum_{x,x'\in V_N} G_{\mathcal{G}_N}(x,x')h(x)h(x')$$

by the reproducing property of $G_{\mathcal{G}_N}$. Meanwhile, let us abuse notations slightly and introduce the quadratic form

$$\mathcal{E}_N^{\mathcal{G}}(U_N^{\mathcal{G}\square}(h\eta_N), U_N^{\mathcal{G}\square}(h\eta_N)) := \rho_F^{-N}\frac{1}{|V_N|^2}\sum_{x,x'\in V_N} G_{\mathcal{G}_N}^{\square}(x,x')h(x)h(x'),$$

as it is suggestive of another Green form.

Lemma 4.2.5 (The main lemma)**.** *[CU] For every $h \in L^1(F,\nu) \cap L^\infty(F,\nu)$, define $h_N : V_N \to \mathbb{R}$ by $h_N(\cdot) = h(\ell_F^{-N}\cdot)$. Then the following hold:*

1. $\overline{\lim}_{N\to\infty} \mathcal{E}_N^{\mathcal{G}}(U_N^{\mathcal{G}\square}(h_N\eta_N), U_N^{\mathcal{G}\square}(h_N\eta_N)) = \overline{\lim}_{N\to\infty} \mathcal{E}_N^{\mathcal{G}}(U_N^{\mathcal{G}}(h_N\eta_N), U_N^{\mathcal{G}}(h_N\eta_N)).$

For some $(\mathcal{E},\mathcal{F}) \in \mathfrak{E}$:

2. *There exists a constant $C_{2.6}(\mathcal{E})$ such that*

$$\overline{\lim}_{N\to\infty} \mathcal{E}_N^{\mathcal{G}}\left(U_N^{\mathcal{G}\square}\left(\left(\frac{d\mu}{d\nu}\right)_N \eta_N\right), U_N^{\mathcal{G}\square}\left(\left(\frac{d\mu}{d\nu}\right)_N \eta_N\right)\right) \le C_{2.6}\mathcal{E}(U\mu, U\mu)$$

   for all $\mu \in \mathcal{M}_{0,\mathrm{ac}}^{(0)}(F)$.

3. *There exists a constant $C_{2.7}(\mathcal{E})$ such that*

$$\overline{\lim}_{N\to\infty} \rho_F^N \left\langle \mathbb{1}_{V_N}, \sum_{x\in V_N} (G_{\mathcal{G}_N}^{\square})^{-1}(\cdot,x)\mathbb{1}_{V_N}(x)\right\rangle_{V_N} \le C_{2.7}\mathrm{Cap}_{\mathcal{E}}(F),$$

   where $\left(G_{\mathcal{G}_N}^{\square}\right)^{-1}$ denotes the matrix inverse of $G_{\mathcal{G}_N}^{\square}$, and $\mathrm{Cap}_{\mathcal{E}}(F)$ denotes the 0-capacity of F with respect to $\mathcal{E}$.

4.3 Different Characterizations of Capacity

In this subsection we introduce the concepts of smooth measures and (0-order) capacity with respect to a (transient) regular Dirichlet form. Much of this can be found in [FOT]*Chapter 2 and [ChenFukushima]*Chapter 2.

Suppose $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form on $L^2(X,m)$. Let $\mathfrak{O}$ denote the family of all open subsets of X, and for each $A\in\mathfrak{O}$, define $\mathcal{L}_A=\{u\in\mathcal{F}:u\geq 1\ m\text{-a.e. on } A\}$. The 1-capacity of the set $A\in\mathfrak{O}$ with respect to $\mathcal{E}$ is given by

$$\mathrm{Cap}_{\mathcal{E},1}(A)=\begin{cases}\inf_{f\in\mathcal{L}_A}\left[\mathcal{E}(f,f)+\|f\|_{L^2}^2\right], & \mathcal{L}_A\neq\varnothing\\ \infty, & \mathcal{L}_A=\varnothing\end{cases}. \tag{4.8}$$

If $A\subset X$ is an arbitrary subset, then put $\mathrm{Cap}_{\mathcal{E},1}(A)=\inf_{B\in\mathfrak{O},A\subset B}\mathrm{Cap}_{\mathcal{E},1}(B)$. A statement is said to hold *quasi-everywhere* (*q.e.*) on A if and only if there exists a set $U\subset A$ with $\mathrm{Cap}_{\mathcal{E},1}(U)=0$ such that the statement holds everywhere on $A\backslash U$. A function $f:X\to\mathbb{R}$ is said to be *quasi-continuous* if for every $\epsilon>0$, there exists an open set Ω with $\mathrm{Cap}_{\mathcal{E},1}(\Omega)<\epsilon$ such that f is continuous on $X\backslash\Omega$. We say that v is a *quasi-continuous modification* of f if v is quasi-continuous and $v=f$ m-a.e, and denote v by $\overline{f}$.

A positive Radon measure μ on X is called a *measure of finite energy integral* (with respect to $\mathcal{E}$) if there exists a constant $C_\mu>0$ such that for all $f\in\mathcal{F}\cap C_c(X)$,

$$\int_X|f|d\mu\leq C_\mu\left[\mathcal{E}(f,f)+\|f\|_{L^2}^2\right]^{1/2}. \tag{4.9}$$

We denote by S_0 the family of all measures of finite energy integral.

If furthermore $(\mathcal{E},\mathcal{F})$ is transient, then one may complete $\mathcal{F}$ in the $\mathcal{E}$-norm, and $(\mathcal{F}_e:=\overline{F}^{\mathcal{E}},\mathcal{E})$ is a Hilbert space called the *extended Dirichlet space*. Then we have the following 0-order counterparts of the above notions: the 0-*capacity* of a set $A \in \mathfrak{O}$, denoted by $\mathrm{Cap}_{\mathcal{E}}(A)$, is given by (4.8) with $\mathcal{F}$ and $\mathcal{E}(f,f) + \|f\|_{L^2}^2$ replaced respectively by $\mathcal{F}_e$ and $\mathcal{E}(f,f)$. The 0-capacity of an arbitrary set A then follows similarly. Likewise, a positive Radon measure μ on X is called a *measure of finite* 0*-order energy integral* if (4.9) holds with the same replacements. Denote by $S_0^{(0)}$ the family of all measures of finite 0-order energy integral.

There is an important connection between $S_0^{(0)}$ and $\mathcal{F}_e$, which is based on the Riesz representation theorem. For every $\mu \in S_0^{(0)}$, there exists a unique $U\mu \in \mathcal{F}_e$ such that $\mathcal{E}(f, U\mu) = \langle \overline{f}, \mu \rangle_X$ for all $f \in \mathcal{F}_e$. We shall refer to $U : S_0^{(0)} \to \mathcal{F}_e$ as the 0-order *potential operator* associated with $\mathcal{E}$. Any $h \in \mathcal{F}_e$ which can be written in the form $h = U\mu$ for some $\mu \in S_0^{(0)}$ is called a 0-order *potential* relative to $\mathcal{E}$.

Let us remark that $S_0^{(0)} \subset S_0 \subset S$, where S is the family of *smooth measures* consisting of all positive Borel measures μ on X such that:

- μ charges no set of zero 1-capacity.
- There exists an increasing sequence $(F_n)_n$ of closed sets such that $\mu(F_n) < \infty$ for all n, and that $\lim_{n\to\infty} \mathrm{Cap}_{\mathcal{E},1}(K \backslash F_n) = 0$ for any compact set K.

In general, elements of S_0 need not be absolutely continuous with respect to m, but each of them can be approximated by a sequence of absolutely continuous measures, *cf.* [FOT]*Lemma 2.2.2. Here we give the 0-order version of this statement.

Proposition 4.3.1. *Let $(\mathcal{E}, \mathcal{F})$ be a transient regular Dirichlet form on $L^2(X,m)$, and let G_β and U denote respectively the β-resolvent and the 0-order potential operator associated with $\mathcal{E}$. Given each $\mu \in S_0^{(0)}$, let $h_\beta := \beta(U\mu - \beta G_\beta(U\mu))$ for each $\beta \in \mathbb{N}$. Then as $\beta \to \infty$, $h_\beta \cdot m$ converges vaguely to μ.*

Proof. This is the Yosida approximation (*cf.* [FOT]*(1.3.18)): $h_\beta \geq 0$ m-a.e., and for all $f \in \mathcal{F}$,

$$(h_\beta, f)_{L^2(m)} = (\beta(U\mu - \beta G_\beta(U\mu)), f)_{L^2(m)} \underset{\beta\to\infty}{\longrightarrow} \mathcal{E}(U\mu, f).$$

Therefore $\lim_{\beta\to\infty}\langle f, h_\beta \cdot m\rangle_X = \langle f, \mu\rangle_X$ for all $f \in \mathcal{F} \cap C_c(X)$. □

Last but not least, let us record several equivalent characterizations of the 0-capacity.

Proposition 4.3.2. *Let $(\mathcal{E}, \mathcal{F})$ be a transient regular Dirichlet form on $L^2(X, m)$. Fix an arbitrary set $B \subset X$ and suppose $\mathcal{L}_B \neq \varnothing$.*

1. *There exists a unique element e_B in $\mathcal{L}_B$ minimizing $\mathcal{E}(\cdot,\cdot)$. In particular, $\mathrm{Cap}_{\mathcal{E}}(B) = \mathcal{E}(e_B, e_B)$.*
2. *e_B is the unique element of $\mathcal{F}_e$ satisfying $\overline{e}_B = 1$ q.e. on B and $\mathcal{E}(e_B, f) \geq 0$ for any $f \in \mathcal{F}_e$ with $\overline{f} \geq 0$ q.e. on B.*
3. *There exists a unique measure $\mu_B \in S_0^{(0)}$ supported in B such that $e_B = U\mu_B$. In particular,*
$$\mathrm{Cap}_{\mathcal{E}}(B) = \mathcal{E}(U\mu_B, U\mu_B) = \langle \overline{U\mu_B}, \mu_B\rangle_X.$$
4. *If B is a compact set, then*
$$\begin{aligned}\mathrm{Cap}_{\mathcal{E}}(B) = \langle \mathbb{1}_B, \mu_B\rangle_X &= \sup\left\{\mathcal{E}(U\mu, U\mu) : \mu \in S_0^{(0)},\ \mathrm{supp}(\mu) \subset B,\ \overline{U\mu} \leq 1 \text{ q.e.}\right\}\\ &= \sup\left\{\frac{\langle \mathbb{1}_B, \mu\rangle_X^2}{\mathcal{E}(U\mu, U\mu)} : \mu \in S_0^{(0)},\ \mathrm{supp}(\mu) \subset B\right\}.\end{aligned}$$

Proof. The first two items are the 0-order version of [FOT]*Theorem 2.1.5, as explained on [FOT]*p. 74. Item (iii) is proved in conjunction with [FOT]*Lemma 2.2.10. The first two equalities in Item (iv) follow directly from (ii) and (iii), while the third equality can be obtained by a variational argument. □

The function e_B and the measure μ_B are known as, respectively, the 0-order *equilibrium potential* and *equilibrium measure* of the set B (with respect to $\mathcal{E}$).

4.4 Proof of Theorem 4.1.1

Notations. In the next two sections, $\Phi : \mathbb{R} \to [0,1]$, defined by

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\xi^2/2} d\xi,$$

stands for the cdf of a standard normal random variable. For any measurable subset S of V_∞, we denote by $\mathcal{F}_S := \sigma\{\varphi_x : x \in S\}$ the sigma-algebra generated by the free field on S, and by $\Omega_S^+ := \{\varphi_x \geq 0 \text{ for all } x \in S\}$ the event that the field is nonnegative everywhere on S. Finally, we fix an element $(\mathcal{E}, \mathcal{F})$ from the family $\mathfrak{E}$ of local, regular, conservative, non-zero Dirichlet forms on $L^2(F_\infty, \nu_\infty)$ which are invariant under the local symmetries of the carpet.

4.4.1 Lower bound

Let $\alpha > 2\overline{G}$, where $\overline{G} := \sup_{x \in V_\infty} G_{\mathcal{G}_\infty}(x,x)$. Denote by $\mathbb{P}_N$ the law of the free field on $\mathcal{G}_\infty$ with mean $\sqrt{\alpha \log t_F}\sqrt{N}$ and covariance $G_{\mathcal{G}_\infty}$.

First we wish to show that $\lim_{N\to\infty} \mathbb{P}_N(\Omega_{V_N}^+) = 1$. Observe that for any $x \in V_\infty$,

$$\mathbb{P}_N(\varphi_x < 0) = \mathbb{P}(\varphi_x < -\sqrt{\alpha N \log t_F}) = \Phi\left(-\sqrt{\frac{\alpha N \log t_F}{G_{\mathcal{G}_\infty}(x,x)}}\right),$$

Using the fact that $G_{\mathcal{G}_\infty}(x,x) \leq \overline{G}$ and $\Phi(a) \leq \frac{1}{2} e^{-a^2/2}$ for $a \leq 0$, we deduce that

$$\mathbb{P}_N(\varphi_x < 0) \leq \frac{1}{2} t_F^{-(N\alpha)/(2\overline{G})}.$$

It follows that

$$\mathbb{P}_N\left((\Omega^+_{V_N})^c\right) = \mathbb{P}_N\left(\bigcup_{x\in V_N}\{\varphi_x < 0\}\right) \le |V_N|\mathbb{P}_N(\varphi_x < 0 : x \in V_N) \le c(t_F^N)^{(1-\frac{\alpha}{2\overline{G}})} \underset{N\to\infty}{\longrightarrow} 0,$$

which is what we want.

Next we adopt the relative entropy argument as used in the proof of [BDZ95]*Lemma 2.3. Let $\Pi_N = \left.\frac{d\mathbb{P}_N}{d\mathbb{P}}\right|_{\mathcal{F}_{V_N}}$. Introduce the relative entropy of $\mathbb{P}_N$ to $\mathbb{P}$ restricted to V_N by

$$\mathrm{Ent}_{V_N}(\mathbb{P}_N|\mathbb{P}) = \int_{\mathbb{R}^{V_\infty}} \Pi_N \log(\Pi_N)d\mathbb{P} = \frac{1}{2}\alpha N \log t_F \left\langle \mathbb{1}_{V_N}, \sum_{x\in V_N}(G^{\square}_{\mathcal{G}_N})^{-1}(\cdot,x)\mathbb{1}_{V_N}(x)\right\rangle_{V_N},$$

where $(G^{\square}_{\mathcal{G}_N})^{-1}$ denotes the matrix inverse of $(G^{\square}_{\mathcal{G}_N}) := G_{\mathcal{G}_\infty}|_{V_N\times V_N}$. Applying the entropy inequality

$$\log\left(\frac{\mathbb{P}(\Omega^+_{V_N})}{\mathbb{P}_N(\Omega^+_{V_N})}\right) \ge -\frac{1}{\mathbb{P}_N(\Omega^+_{V_N})}\left(\mathrm{Ent}_{V_N}(\mathbb{P}_N|\mathbb{P}) + e^{-1}\right),$$

cf. the end of the proof of [BDZ95]*Lemma 2.3, we obtain

$$\begin{aligned}
&\varliminf_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} N \log t_F} \qquad\qquad (4.10)\\
\ge\ & \varliminf_{N\to\infty}\left[-\frac{1}{\mathbb{P}_N(\Omega^+_{V_N})}\left(\frac{\mathrm{Ent}_{V_N}(\mathbb{P}_N|\mathbb{P}) + e^{-1}}{\rho_F^{-N} N\log t_F}\right) + \frac{\log \mathbb{P}_N(\Omega^+_{V_N})}{\rho_F^{-N} N \log t_F}\right]\\
\ge\ & \varliminf_{N\to\infty}\left[-\frac{1}{\mathbb{P}_N(\Omega^+_{V_N})}\frac{\mathrm{Ent}_{V_N}(\mathbb{P}_N|\mathbb{P})}{\rho_F^{-N} N\log t_F}\right] + \varliminf_{N\to\infty}\left[-\frac{1}{\mathbb{P}_N(\Omega^+_{V_N})}\frac{e^{-1}}{\rho_F^{-N} N\log t_F}\right]\\
+\ & \varliminf_{N\to\infty} \frac{\log \mathbb{P}_N(\Omega^+_{V_N})}{\rho_F^{-N} N \log t_F}\\
\ge\ & -\left(\varlimsup_{N\to\infty}\frac{1}{\mathbb{P}_N(\Omega^+_{V_N})}\cdot \varlimsup_{N\to\infty}\frac{\mathrm{Ent}_{V_N}(\mathbb{P}_N|\mathbb{P})}{\rho_F^{-N} N\log t_F}\right) + 0 + 0 \ \ge\ -\frac{1}{2}\alpha C_{2.7}\mathrm{Cap}_{\mathcal{E}}(F)
\end{aligned}$$

by Lemma 4.2.5(iii). By making α arbitrarily close to $2\overline{G}$, we obtain the desired lower bound.

Remark 4.4.1. *If we instead use a constant multiple of the original Dirichlet form* $(\gamma\mathcal{E}, \mathcal{F})$*,* $\gamma > 0$*, inequality (4.10) will hold under the substitutions* $C_{2.7} \to \gamma^{-1}C_{2.7}$ *and* $\mathrm{Cap}_{\mathcal{E}}(F) \to \mathrm{Cap}_{\gamma\mathcal{E}}(F)$*.*

4.4.2 Upper bound

Just as in the $\mathbb{Z}^d$ setting [BDZ95], the proof of the upper bound involves a series of *coarse graining* and *conditioning* arguments on the free field $\{\varphi_x\}_{x\in V_\infty}$, though some modifications are needed to account for the fractal geometry.

Notations. If $\mathcal{G} = (V(\mathcal{G}), \sim)$ is a finite subgraph of a larger graph $\mathcal{G}_0 = (V(\mathcal{G}_0), \sim)$, then we denote the set of *peripheral vertices* of $\mathcal{G}$ by

$$\partial\mathcal{G} := \{x \in V(\mathcal{G}) : x \sim y \text{ for some } y \in V(\mathcal{G}_0)\backslash V(\mathcal{G})\}.$$

The *interior* of the graph $\mathcal{G}$ will thusly be defined by $\mathring{\mathcal{G}} := (V(\mathcal{G})\backslash\partial\mathcal{G}, \sim)$.

Following §4.1, we denote by $\mathcal{Q}_j(F_j)$ ($j \in \mathbb{N}$) the collection of closed cubes of side ℓ_F^{-j} whose vertices are in $\ell_F^{-j}\mathbb{Z}^d$, and which are contained in F_j. Then to each $\bar{Q} \in \mathcal{Q}_j(F_j)$ corresponds a unique vector $\mathbf{p} = (p_1, \cdots, p_d) \in (\mathbb{N}_0)^d$ such that $\bar{Q} = \left[p_1\ell_F^{-j}, (p_1+1)\ell_F^{-j}\right] \times \cdots \times \left[p_d\ell_F^{-j}, (p_d+1)\ell_F^{-j}\right]$. Keeping with this notation, we define two related cubes derived from $\bar{Q}$:

$$\begin{aligned} Q_\llcorner &= \left[p_1\ell_F^{-j}, (p_1+1)\ell_F^{-j}\right) \times \cdots \times \left[p_d\ell_F^{-j}, (p_d+1)\ell_F^{-j}\right), \\ Q &= Q_\llcorner \cup \left(\bar{Q}\backslash \bigcup_{\bar{Q}'\in\mathcal{Q}_j(F_j)} Q'_\llcorner\right). \end{aligned}$$

Let $\mathcal{Q}_j^\circ(F_j)$ be the totality of all Q. Observe that $F_j = \bigcup_{Q\in\mathcal{Q}_j^\circ(F_j)} Q$, and that $Q_1 \cap Q_2 = \varnothing$ for any $Q_1, Q_2 \in \mathcal{Q}_j^\circ(F_j)$ with $Q_1 \neq Q_2$.

Next we introduce, for each $k \le N$, the following collections of kth-level subgraphs of $\mathcal{G}_N$:

$$\begin{aligned} S_k(\mathcal{G}_N) &:= \left\{\ell_F^N\bar{Q} \cap \mathcal{G}_N : \bar{Q} \in \mathcal{Q}_{N-k}(F_{N-k})\right\}, \\ S_k^\circ(\mathcal{G}_N) &:= \left\{\ell_F^N Q \cap \mathcal{G}_N : Q \in \mathcal{Q}_{N-k}^\circ(F_{N-k})\right\}. \end{aligned}$$

By construction, there is a bijection $\iota_k : S_k^\circ(\mathcal{G}_N) \to \mathcal{Q}^\circ_{N-k}(F_{N-k})$ which maps each $\mathfrak{g} \in S_k^\circ(\mathcal{G}_N)$ to a $Q \in \mathcal{Q}^\circ_{N-k}(F_{N-k})$.

Now let us fix a sufficiently large $k \in \mathbb{N}$, and designate a vertex $x_0 \in V_k \backslash \partial\mathcal{G}_k$ as the "representative interior point" of $\mathcal{G}_k$. We don't insist on where x_0 is located within V_k, so long as it stays away from the periphery $\partial\mathcal{G}_k$. (Contrast this setup with previous works on $\mathbb{Z}^d$ [BDZ95, Kurt], where it is natural to designate the center vertex of each block cell as the representative interior point.). Then for any $N > k$, let

$$\begin{aligned} \mathcal{C}_N &= \{x \in V_N : x = x_0 + \ell_F^k \mathbf{p} \text{ for some } \mathbf{p} \in (\mathbb{N}_0)^d\} \quad \text{and} \\ \mathcal{D}_N &= \{x \in V_N : \exists i \in \{1, \cdots, d\} \text{ such that } x_i = p\ell_F^k \text{ for some } p \in \mathbb{N}_0\} \end{aligned}$$

be, respectively, the set of all representative interior points and kth-level boundary points in V_N; see Figure 4.1. Note that $|\mathcal{C}_N| = m_F^{N-k}$.

With the setup complete, we can proceed with the main arguments. *Coarse graining* means that we are sampling the free field ϕ at only one vertex from each subgraph $\mathfrak{g} \in S_k(\mathcal{G}_N)$. On top of that, we will analyze these Gaussian random variables *conditional* upon the sigma-algebra $\mathcal{F}_{\mathcal{D}_N}$ generated by the free field on the "conditioning grid" $\mathcal{D}_N$. The key observation is that under $\mathbb{P}(\cdot|\mathcal{F}_{\mathcal{D}_N})$, $\{\varphi_x : x \in \mathcal{C}_N\}$ are independent Gaussian random variables with mean $\mathbb{E}(\varphi_x|\mathcal{F}_{\mathcal{D}_N}) =: \mu_x$ and identical variance $G_{\mathring{\mathcal{G}}_k}(x_0, x_0)$. It is standard to check for every $x \in \mathcal{C}_N$, μ_x is nonnegative on $\Omega^+_{\mathcal{D}_N}$ via a random walk representation.

Let us now carry out the estimate of $\mathbb{P}(\Omega^+_{V_N})$. First of all,

$$\mathbb{P}(\Omega^+_{V_N}) \le \mathbb{P}\left(\Omega^+_{\mathcal{C}_N} \cap \Omega^+_{\mathcal{D}_N}\right) = \mathbb{E}\left(\prod_{x \in \mathcal{C}_N} \mathbb{P}[\varphi_x \ge 0|\mathcal{F}_{\mathcal{D}_N}] \cdot \mathbb{1}_{\Omega^+_{\mathcal{D}_N}}\right), \tag{4.11}$$

where the equality comes from a basic identity for conditioned random variables and the independence of $\{\varphi_x : x \in \mathcal{C}_N\}$ under $\mathbb{P}(\cdot|\mathcal{F}_{\mathcal{D}_N})$.

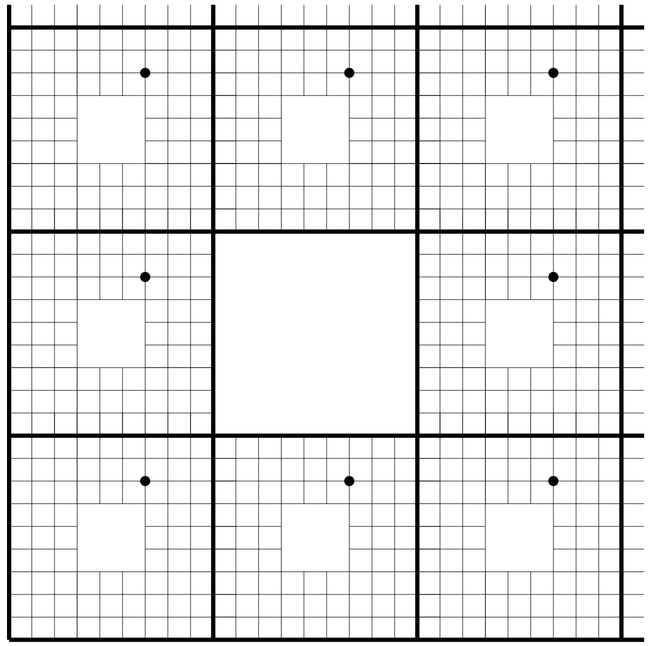

Figure 4.1: The coarse-graining and conditioning scheme on the outer Sierpinski carpet graph $\mathcal{G}_\infty$. Vertices indicated by filled dots are the representative interior points ($\mathcal{C}_N$), while vertices covered by the solid lines (the conditioning grid) are where the free field φ is conditioned upon ($\mathcal{D}_N$).

Now take a $j \in \mathbb{N}$ and consider all $N > j + k$. For each $\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$, let $\mathfrak{B}_\mathcal{C} := \mathfrak{B} \cap \mathcal{C}_N$; observe that $|\mathfrak{B}_\mathcal{C}| = m_F^{N-j-k}$. Let $\kappa > 0$, and $\alpha_{x_0,\kappa} := 2(G_{\mathcal{G}_\infty}(x_0,x_0) - \kappa)\log t_F$. Finally, for $\delta \in (0,1)$, define the event

$$\Gamma_{x_0,\mathfrak{B}} := \left\{ \varphi : \left| \left\{ x \in \mathfrak{B}_\mathcal{C} : \mu_x \le \sqrt{\alpha_{x_0,\kappa} N} \right\} \right| \ge \delta |\mathfrak{B}_\mathcal{C}| \right\}.$$

Set $\Gamma_{x_0} = \bigcup_{\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)} \Gamma_{x_0,\mathfrak{B}}$.

By writing $\Omega^+_{\mathcal{D}_N}$ as the disjoint union of $(\Omega^+_{\mathcal{D}_N} \cap \Gamma_{x_0})$ and $(\Omega^+_{\mathcal{D}_N} \cap \Gamma^c_{x_0})$, we develop (4.11) further as

$$\mathbb{P}(\Omega^+_{V_N}) \le \mathbb{E}\left(\prod_{x \in \mathcal{C}_N} \mathbb{P}[\varphi_x \ge 0 | \mathcal{F}_{\mathcal{D}_N}] \cdot \mathbb{1}_{\Omega^+_{\mathcal{D}_N} \cap \Gamma_{x_0}} \right) + \mathbb{E}\left(\prod_{x \in \mathcal{C}_N} \mathbb{P}[\varphi_x \ge 0 | \mathcal{F}_{\mathcal{D}_N}] \cdot \mathbb{1}_{\Omega^+_{\mathcal{D}_N} \cap \Gamma^c_{x_0}} \right). \tag{4.12}$$

The claim is that the first term on the RHS of (4.12) becomes negligible as

$N \to \infty$. Since this result plays an essential role in Section 4.5, we record it as a separate lemma.

Lemma 4.4.2. *Let $\gamma \in (0,1)$. Then for k large enough, there exists a constant $C_{3.1}(\delta, k)$, independent of N, such that*

$$\mathbb{E}\left(\prod_{x\in\mathcal{C}_N} \mathbb{P}[\varphi_x \geq 0|\mathcal{F}_{\mathcal{D}_N}] \cdot \mathbb{1}_{\Omega^+_{\mathcal{D}_N}\cap\Gamma_{x_0}}\right) \leq \exp\left(-C_{3.1} m_F^N t_F^{-N(1-\gamma)}\right). \qquad (4.13)$$

Proof of Lemma. Since $\uparrow \lim_{k\to\infty} G_{\mathring{\mathcal{G}}_k}(x_0,x_0) = G_{\mathcal{G}_\infty}(x_0,x_0)$, we have $\dfrac{G_{\mathcal{G}_\infty}(x_0,x_0)-\kappa}{G_{\mathring{\mathcal{G}}_k}(x_0,x_0)} \leq 1-\gamma$ for k large enough. So on Γ_{x_0}, there exists at least a $\mathfrak{B}^* \in S^\circ_{N-j}(\mathcal{G}_N)$ such that $\Gamma_{x_0,\mathfrak{B}^*}$ holds, and therefore

$$\begin{aligned}
\prod_{x\in\mathcal{C}_N} \mathbb{P}[\varphi_x \geq 0|\mathcal{F}_{\mathcal{D}_N}] &\leq \mathbb{P}\left[\varphi_{x_0} - \mu_{x_0} \geq -\sqrt{\alpha_{x_0,\kappa} N}\,\middle|\,\mathcal{F}_{\mathcal{D}_N}\right]^{\delta|\mathfrak{B}^*_{\mathcal{C}}|} \\
&= \left[1 - \Phi\left(-\frac{\sqrt{\alpha_{x_0,\kappa}N}}{\sqrt{G_{\mathring{\mathcal{G}}_k}(x_0,x_0)}}\right)\right]^{\delta|\mathfrak{B}^*_{\mathcal{C}}|} \\
&\leq \left[1 - \frac{\sqrt{G_{\mathring{\mathcal{G}}_k}(x_0,x_0)}}{\sqrt{\alpha_{x_0,\kappa}N}} \exp\left(-\frac{\alpha_{x_0,\kappa}N}{2G_{\mathring{\mathcal{G}}_k}(x_0,x_0)}\right)\right]^{\delta|\mathfrak{B}^*_{\mathcal{C}}|} \\
&\leq \left[1 - \frac{1}{\sqrt{2(1-\gamma)N\log t_F}} t_F^{-N(1-\gamma)}\right]^{\delta|\mathfrak{B}^*_{\mathcal{C}}|} \leq \exp\left(-c m_F^N t_F^{-N(1-\gamma)}\right).
\end{aligned}$$

Once again, we used the fact that $\varphi_{x_0} - \mu_{x_0}$ is a centered Gaussian random variable under $\mathbb{P}(\cdot|\mathcal{F}_{\mathcal{D}_N})$, and applied a standard Gaussian estimate. The last inequality derives from the inequality $1 - x \leq e^{-x}$. □

We turn to estimate the second term on the RHS of (4.12). The key is to obtain a lower bound for $\sum_{x\in\mathfrak{B}_{\mathcal{C}}} \mu_x$ ($\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$) on $\Gamma^c_{x_0}$. We have

$$\sum_{x\in\mathfrak{B}_{\mathcal{C}}} \mu_x = \sum_{\substack{x\in\mathfrak{B}_{\mathcal{C}} \\ \mu_x > \sqrt{\alpha_{x_0,\kappa}N}}} \mu_x + \sum_{\substack{x\in\mathfrak{B}_{\mathcal{C}} \\ \sqrt{\alpha_{x_0,\kappa}N} \geq \mu_x \geq 0}} \mu_x.$$

The second summand can be bounded below by 0. As for the first summand, observe that on $\Gamma^c_{x_0}$, there are at least $(1-\delta)|\mathfrak{B}_\mathcal{C}|$ many representative interior points x whose μ_x exceeds $\sqrt{\alpha_{x_0,\kappa}N}$. Therefore

$$\sum_{x\in\mathfrak{B}_\mathcal{C}} \mu_x \geq (1-\delta)|\mathfrak{B}_\mathcal{C}|\sqrt{\alpha_{x_0,\kappa}N} \quad \text{on } \Gamma^c_{x_0}.$$

Introduce arbitrary nonnegative numbers $f_\mathfrak{B} \geq 0$, $\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$. We then have

$$\begin{aligned}
\mathbb{P}(\Gamma^c_{x_0}) &\leq \mathbb{P}\left(\frac{1}{|\mathfrak{B}_\mathcal{C}|}\sum_{x\in\mathfrak{B}_\mathcal{C}} \mu_x \geq (1-\delta)\sqrt{\alpha_{x_0,\kappa}N}\right) \\
&= \mathbb{P}\left(\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\frac{1}{|\mathfrak{B}_\mathcal{C}|}\sum_{x\in\mathfrak{B}_\mathcal{C}} \mu_x \geq (1-\delta)\sqrt{\alpha_{x_0,\kappa}N}\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\right) \\
&\leq \exp\left[-\frac{(1-\delta)^2\alpha_{x_0,\kappa}N\left(\sum\limits_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\right)^2}{2\,\mathrm{Var}\left(\sum\limits_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\frac{1}{|\mathfrak{B}_\mathcal{C}|}\sum\limits_{x\in\mathfrak{B}_\mathcal{C}} \mu_x\right)}\right].
\end{aligned}$$

From the elementary random variable identity $\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}(X|\mathcal{F})) + \mathbb{E}(\mathrm{Var}(X|\mathcal{F}))$ one deduces that $\mathrm{Var}(X) \geq \mathrm{Var}(\mathbb{E}(X|\mathcal{F}))$. Applied to our setting we find

$$\mathrm{Var}\left(\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\sum_{x\in\mathfrak{B}_\mathcal{C}} \mu_x\right) \leq \mathrm{Var}\left(\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\sum_{x\in\mathfrak{B}_\mathcal{C}} \varphi_x\right).$$

Let the function $\Xi_j : F \to \mathbb{R}_+$ be given by $\Xi_j = \sum\limits_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\mathbb{1}_{\iota_j(\mathfrak{B})}$. One verifies that

$$\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B} = \frac{1}{|\mathfrak{B}_\mathcal{C}|}\sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell^N_F}\right) \qquad \text{and} \qquad \sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} f_\mathfrak{B}\sum_{x\in\mathfrak{B}_\mathcal{C}}\varphi_x = \sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell^N_F}\right)\varphi_x.$$

Hence

$$\mathrm{Var}\left(\sum_{\mathfrak{B}\in S^{\circ}_{N-j}(\mathcal{G}_N)} f_{\mathfrak{B}} \frac{1}{|\mathfrak{B}_{\mathcal{C}}|}\sum_{x\in\mathfrak{B}_{\mathcal{C}}}\varphi_x\right) = \frac{1}{|\mathfrak{B}_{\mathcal{C}}|^2}\mathrm{Var}\left(\sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell_F^N}\right)\varphi_x\right)$$

$$= \frac{1}{|\mathfrak{B}_{\mathcal{C}}|^2}\sum_{x,x'\in\mathcal{C}_N} G_{\mathcal{G}_\infty}(x,x')\Xi_j\left(\frac{x}{\ell_F^N}\right)\Xi_j\left(\frac{x'}{\ell_F^N}\right)$$

$$= \frac{1}{|\mathfrak{B}_{\mathcal{C}}|^2}\sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x')\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x}{\ell_F^N}\right)\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x'}{\ell_F^N}\right).$$

Putting things together,

$$\mathbb{P}(\Gamma_{x_0}^c) \le \exp\left[-\frac{(1-\delta)^2\alpha_{x_0,\kappa}N\left(\sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell_F^N}\right)\right)^2}{2\sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x')\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x}{\ell_F^N}\right)\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x'}{\ell_F^N}\right)}\right].$$

It follows that

$$\overline{\lim_{N\to\infty}}\frac{\log\mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N}N\log t_F} \le \overline{\lim_{N\to\infty}}\frac{\log\mathbb{P}(\Gamma^c_{x_0})}{\rho_F^{-N}N\log t_F}$$

$$\le \overline{\lim_{N\to\infty}}\Bigg[-(1-\delta)^2(G_{\mathcal{G}_\infty}(x_0,x_0)-\kappa)$$

$$\frac{\left(\frac{1}{m_F^{N-k}}\sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell_F^N}\right)\right)^2}{\frac{\rho_F^{-N}}{m_F^{2(N-k)}}\sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x')\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x}{\ell_F^N}\right)\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x'}{\ell_F^N}\right)}\Bigg]$$

$$\le -\frac{(1-\delta)^2(G_{\mathcal{G}_\infty}(x_0,x_0)-\kappa)}{C_{1.2}^2 m_F^{2k}}$$

$$\frac{\lim_{N\to\infty}\left(\frac{1}{m_F^{N-k}}\sum_{x\in\mathcal{C}_N}\Xi_j\left(\frac{x}{\ell_F^N}\right)\right)^2}{\overline{\lim_{N\to\infty}}\frac{\rho_F^{-N}}{|V_N|^2}\sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x')\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x}{\ell_F^N}\right)\left(\Xi_j\mathbb{1}_{\ell_F^{-N}\mathcal{C}_N}\right)\left(\frac{x'}{\ell_F^N}\right)}$$

$$\le -\frac{(1-\delta)^2(G_{\mathcal{G}_\infty}(x_0,x_0)-\kappa)}{C_{1.2}^2 C_{2.6}}\left[\frac{\langle\mathbb{1}_F,\Xi_j\nu\rangle_F^2}{\mathcal{E}(U(\Xi_j\nu),U(\Xi_j\nu))}\right] \tag{4.14}$$

for all $\Xi_j \in \mathcal{F}$. In obtaining the convergence for the denominator, we applied Lemma 4.2.5(ii) and identified the limit measure as $m_F^{-k}\Xi_j\nu$.

By varying over the coefficients $f_{\mathfrak{B}}$ in Ξ_j and taking the limit $j \to \infty$, we can obtain any $\Xi\nu \in \mathcal{M}^{(0)}_{0,\text{ac}}(F)$. Then we can recover any $\mu \in S_0^{(0)}$, $\text{supp}(\mu) \subset F$, by an approximating sequence of measures in $\mathcal{M}^{(0)}_{0,\text{ac}}(F)$ *à la* Yosida (Proposition 4.3.1). We supremize the bracketed expression on the RHS of (4.14) over all $\mu \in S_0^{(0)}$ and apply Proposition 4.3.2(iv), then take $\delta, \kappa \to 0$ to get

$$\varlimsup_{N\to\infty} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} N \log t_F} \leq -\frac{1}{C_{1.2}^2 C_{2.6}} \cdot G_{\mathcal{G}_\infty}(x_0, x_0) \cdot \text{Cap}_{\mathcal{E}}(F). \tag{4.15}$$

This essentially proves the upper bound in Theorem 4.1.1, though *a priori* not the sharpest possible bound. In principle, one can choose the interior point $x_0^* \in V_k \backslash \partial\mathcal{G}_k$ with the biggest on-diagonal Green's function value $G_{\mathcal{G}_\infty}(x_0^*, x_0^*)$, and run through the preceding argument to get (4.15) with $G_{\mathcal{G}_\infty}(x_0^*, x_0^*)$ in place of $G_{\mathcal{G}_\infty}(x_0, x_0)$.

4.5 Proof of Theorem 4.1.2

The purpose of this section is to prove that for any $\epsilon > 0$ and any $\eta > 0$,

$$\lim_{N\to\infty} \sup_{\substack{x\in V_N \\ V_{N,\epsilon}(x)\subset V_N}} \mathbb{P}\left(\bar{\varphi}_{N,\epsilon}(x) \leq \left(\sqrt{2\underline{G}\log t_F} - \eta\right)\sqrt{N} \,\middle|\, \Omega^+_{V_N}\right) = 0. \tag{4.16}$$

$$\lim_{N\to\infty} \sup_{\substack{x\in V_N \\ V_{N,\epsilon}(x)\subset V_N}} \mathbb{P}\left(\bar{\varphi}_{N,\epsilon}(x) \geq \left(\sqrt{2\underline{G}\log t_F} + \eta\right)\sqrt{N} \,\middle|\, \Omega^+_{V_N}\right) = 0. \tag{4.17}$$

4.5.1 Lower bound

In this subsection, $\mathcal{L}_S := \frac{1}{|S|}\sum_{x\in S} \delta_{\varphi_x}$ denotes the empirical measure of the free field φ on a measurable subset S of V_∞.

Equation (4.16) is a direct consequence of the following lemma.

Lemma 4.5.1. *For any $\alpha < 2\underline{G}\log t_F$ and $\delta > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\left(\mathcal{L}_{V_N}\left[0, \sqrt{\alpha N}\right] \geq \delta \;\middle|\; \Omega^+_{V_N}\right) = 0. \tag{4.18}$$

Proof. For the sake of clarity, we present the proof in two steps.

Step 1. Fix a representative interior point $x_0 \in V_k \backslash \partial\mathcal{G}_k$ as in Section 4.4.2. Also recall the definition of $\mathcal{C}_N$. Our interim goal is to show that for any $\alpha < 2G_{\mathcal{G}_\infty}(x_0, x_0)\log t_F$ and $\delta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\mathcal{L}_{\mathcal{C}_N}\left[0, \sqrt{\alpha N}\right] \geq \delta \;\middle|\; \Omega^+_{V_N}\right) = 0. \tag{4.19}$$

Following the proof of [BDZ95]*Lemma 4.4, we define, for each $\alpha > 0$, the events

$$\Theta_N(\alpha) = \left\{x \in \mathcal{C}_N : \varphi_x \leq \sqrt{\alpha N}\right\} \quad \text{and} \quad \bar{\Theta}_N(\alpha) = \left\{x \in \mathcal{C}_N : \mu_x \leq \sqrt{\alpha N}\right\}.$$

Then for each $\delta > \delta' > 0$ and $\alpha < \alpha' < 2G_{\mathcal{G}_\infty}(x_0, x_0)\log t_F$,

$$\begin{aligned}
\left\{\mathcal{L}_{\mathcal{C}_N}\left[0, \sqrt{\alpha N}\right] \geq \delta\right\} &= \{|\Theta_N(\alpha)| \geq \delta|\mathcal{C}_N|\} \\
&= \left\{|\Theta_N(\alpha)| \geq \delta|\mathcal{C}_N|, |\bar{\Theta}_N(\alpha')| \geq \delta'|\mathcal{C}_N|\right\} \cup \\
&\quad\; \left\{|\Theta_N(\alpha)| \geq \delta|\mathcal{C}_N|, |\bar{\Theta}_N(\alpha')| < \delta'|\mathcal{C}_N|\right\} \\
&\subset \left\{|\bar{\Theta}_N(\alpha')| \geq \delta'|\mathcal{C}_N|\right\} \cup \left\{|\Theta_N(\alpha) \cap \bar{\Theta}_N(\alpha')^c| \geq (\delta - \delta')|\mathcal{C}_N|\right\} \\
&=: J_0 \cup J_1.
\end{aligned}$$

By Lemma 4.4.2, for each $\gamma \in (0, 1)$ there exists a positive constant $C_{3.1}$ such that

$$\mathbb{P}\left(J_0 \cap \Omega^+_{V_N}\right) \leq \exp\left(-C_{3.1} m_F^N t_F^{-N(1-\gamma)}\right).$$

On the other hand, the lower bound of Theorem 4.1.1 implies that for all sufficiently large N, $\mathbb{P}(\Omega^+_{V_N}) \geq \exp(-c\rho_F^{-N} N \log t_F)$. Therefore

$$\frac{\mathbb{P}\left(J_0 \cap \Omega^+_{V_N}\right)}{\mathbb{P}(\Omega^+_{V_N})} \leq \exp\left(-c\rho_F^{-N}\left(t_F^{N(1-\gamma)} - c'N\right)\right) \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

Thus it remains to show that

$$\frac{\mathbb{P}\left(J_1 \cap \Omega^+_{V_N}\right)}{\mathbb{P}(\Omega^+_{V_N})} \longrightarrow 0 \quad \text{as} \quad N \to \infty.$$

Note first that $\mu_x - \varphi_x \geq (\sqrt{\alpha'} - \sqrt{\alpha})\sqrt{N}$ whenever $x \in \Theta_N(\alpha) \cap \bar{\Theta}_N(\alpha')^c$. So on J_1,

$$\frac{1}{|\mathcal{C}_N|}\sum_{x \in \mathcal{C}_N} |\varphi_x - \mu_x| \geq (\delta - \delta')(\sqrt{\alpha'} - \sqrt{\alpha})\sqrt{N}.$$

Using the fact that under $\mathbb{P}(\cdot|\mathcal{F}_{\mathcal{D}_N})$, $\{\varphi_x - \mu_x : x \in \mathcal{C}_N\}$ are independent centered Gaussian random variables with variance $G_{\mathring{\mathcal{G}}_k}(x_0, x_0)$, we then find

$$\begin{aligned}
\mathbb{P}\left(J_1 \cap \Omega^+_{V_N}\right) &\leq \mathbb{P}\left(J_1 \cap \Omega^+_{\mathcal{D}_N}\right) \\
&\leq \mathbb{E}\left(\mathbb{P}\left(\frac{1}{|\mathcal{C}_N|}\sum_{x \in \mathcal{C}_N}|\varphi_x - \mu_x| \geq (\delta - \delta')(\sqrt{\alpha'} - \sqrt{\alpha})\sqrt{N} \,\middle|\, \mathcal{F}_{\mathcal{D}_N}\right) \cdot \mathbb{1}_{J_1 \cap \Omega^+_{\mathcal{D}_N}}\right) \\
&\leq \exp\left(-\frac{(\delta - \delta')^2(\sqrt{\alpha'} - \sqrt{\alpha})^2 N |\mathcal{C}_N|^2}{2|\mathcal{C}_N| G_{\mathring{\mathcal{G}}_k}(x_0, x_0)}\right) \\
&\leq \exp\left(-CN\rho_F^{-N} t_F^N\right)
\end{aligned}$$

for some positive constant C which depends on anything but N. This shows that $\mathbb{P}(J_1 \cap \Omega^+_{V_N})$ decays faster than $\mathbb{P}(\Omega^+_{V_N})$ as $N \to \infty$, and hence proves (4.19).

Step 2. Observe that the proof in Step 1 continues to hold for any other interior point $x_0 \in V_k \backslash \partial\mathcal{G}_k$ with the obvious replacements. Thus we can deduce that for any $\alpha < 2\left(\min_{x_0 \in V_k \backslash \partial\mathcal{G}_k} G_{\mathcal{G}_\infty}(x_0, x_0)\right) \log t_F$ and $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}\left(\mathcal{L}_{V_N \backslash \mathcal{D}_N}\left[0, \sqrt{\alpha N}\right] \geq \delta \,\middle|\, \Omega^+_{V_N}\right) = 0.$$

This falls short of (4.18) because $\mathcal{D}_N$ has been excluded from the empirical measure. To redress this shortcoming, we need to translate the conditioning grid

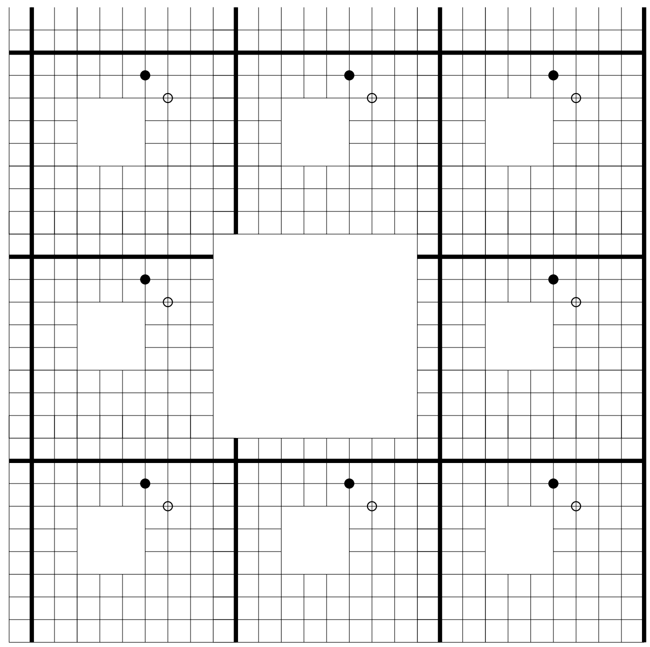

Figure 4.2: The coarse graining and conditioning scheme upon translation. As in Figure 4.1, the filled dots indicate the original representative interior points ($\mathcal{C}_N$). Applying a translation by $z - x_0$ for some $z \in V_k$ (one of the hollow dots), one obtains the new representative interior points ($\tilde{\mathcal{C}}_N^z$, hollow dots) and conditioning grid ($\tilde{\mathcal{D}}_N^z$, solid lines).

relative to the underlying graph V_∞, so that points on $\mathcal{D}_N$ lie within the grid, and then carry out the conditioning scheme. Let us take a moment to describe the translation procedure, as it will be used again in §4.5.2.

As before we fix a representative interior point $x_0 \in V_k \backslash \partial \mathcal{G}_k$. For each $z \in [0, \ell_F^k)^d \cap V_k =: V_k^{\llcorner}$, define

$$\begin{aligned}\tilde{\mathcal{C}}_N^z &= \{x \in V_N : x = z + \ell_F^k \mathbf{p} \text{ for some } \mathbf{p} \in (\mathbb{N}_0)^d\},\\ \tilde{\mathcal{D}}_N^z &= \{x \in V_N : \exists i \in \{1, \cdots, d\} \text{ such that } x_i = p\ell_F^k + (z - x_0)_i \text{ for some } p \in \mathbb{N}_0\}.\end{aligned}$$

In effect, we are translating the set of coarse-graining points and the conditioning grid by a vector $z - x_0$; see Figure 4.2. Since $\tilde{\mathcal{D}}_N^z$ separates points in $\tilde{\mathcal{C}}_N^z$, we can associate to each $x \in \tilde{\mathcal{C}}_N^z$ a unique subgraph $\mathfrak{g}_x = (V(\mathfrak{g}_x), \sim)$ of $\mathcal{G}_N$ such that:

- $\partial \mathfrak{g}_x \subset \tilde{\mathcal{D}}_N^z$.
- $V(\mathfrak{g}_x)$ contains all vertices in V_N which are inscribed by $\partial \mathfrak{g}_x$.
- x is the only element of $\tilde{\mathcal{C}}_N^z$ which lies in $V(\mathfrak{g}_x)$.

The conditioning argument now reads as follows: Under $\mathbb{P}(\cdot|\mathcal{F}_{\tilde{\mathcal{D}}_N^z})$, $\{\varphi_x : x \in \tilde{\mathcal{C}}_N^z\}$ are independent Gaussian random variables, each having mean $\mathbb{E}(\varphi_x|\mathcal{F}_{\tilde{\mathcal{D}}_N^z}) =: \tilde{\mu}_x^z$ and variance $G_{\mathring{\mathfrak{g}}_x}(x,x)$. Keep in mind that the variances are *not all* identical because the subgraphs $(\mathfrak{g}_x)_{x\in\tilde{\mathcal{C}}_N^z}$ no longer retain the symmetries of the original carpet. Nevertheless, we still have the resistance shorting rule

$$G_{\mathring{\mathfrak{g}}_x}(x,x) = R_{\text{eff}}\left(x, V\left((\mathring{\mathfrak{g}}_x)^c\right)\right) \le R_{\text{eff}}(x,\{\infty\}) = G_{\mathcal{G}_\infty}(x,x),$$

where $R_{\text{eff}}(A,B)$ is the effective resistance between two (finite) subsets A, B of V_∞ on the graph $\mathcal{G}_\infty$.

Now define $\tilde{\mathfrak{B}}_{\mathcal{C}}^z = V(\mathfrak{B}) \cap \tilde{\mathcal{C}}_N^z$ for each $\mathfrak{B} \in S_{N-j}^\circ(\mathcal{G}_N)$ and each $z \in V_k^{\llcorner}$. Note that $V(\mathfrak{B})$ equals the disjoint union $\bigcup_{z\in V_k^{\llcorner}} \tilde{\mathfrak{B}}_{\mathcal{C}}^z$, and that the $|\tilde{\mathfrak{B}}_{\mathcal{C}}^z|$ are not the same for all z and $\mathfrak{B}$ due to inclusion/exclusion of kth-level boundary points. Nevertheless we still have $|\tilde{\mathfrak{B}}_{\mathcal{C}}^z| = \mathcal{O}(m_F^{N-k})$.

Let $\kappa > 0$ and $\alpha_\kappa := 2(\underline{G} - \kappa)\log t_F$. Define, for $\delta \in (0,1)$, the event

$$\tilde{\Gamma}_{\mathfrak{B}}^z := \left\{\varphi : \left|\{x \in \tilde{\mathfrak{B}}_{\mathcal{C}}^z : \tilde{\mu}_x^z \le \sqrt{\alpha_\kappa N}\right| \ge \delta|\tilde{\mathfrak{B}}_{\mathcal{C}}^z|\right\}, \tag{4.20}$$

and put $\tilde{\Gamma}^z = \bigcup_{\mathfrak{B}\in S_{N-j}^\circ(\mathcal{G}_N)} \tilde{\Gamma}_{\mathfrak{B}}^z$. We have the following analog of Lemma 4.4.2:

Lemma 4.5.2. *Let $\gamma \in (0,1)$. Then for k large enough, there exists a constant $C_{4.1}(\delta,k)$, independent of N, such that*

$$\mathbb{E}\left(\prod_{x\in\tilde{\mathcal{C}}_N^z} \mathbb{P}[\varphi_x \ge 0|\mathcal{F}_{\tilde{\mathcal{D}}_N^z}] \cdot \mathbb{1}_{\Omega^+_{\tilde{\mathcal{D}}_N^z}\cap\tilde{\Gamma}^z}\right) \le \exp\left(-C_{4.1} m_F^N t_F^{-N(1-\gamma)}\right). \tag{4.21}$$

The proof is essentially identical to that of Lemma 4.4.2, except that we cannot peg the height to be anything higher than $\sqrt{\alpha_\kappa N}$ in the event $\tilde{\Gamma}^z_{\mathfrak{B}}$, due to the unequal variances amongst the conditioned variables.

At last we can describe how to adapt the proof in Step 1 to the translated conditioning grid. The events $\Theta_N(\alpha)$, $\bar{\Theta}_N(\alpha)$, J_0 and J_1 are as before, except that one replaces $\mathcal{C}_N$, $\mathcal{D}_N$, and μ_x with, respectively, $\tilde{\mathcal{C}}^z_N$, $\tilde{\mathcal{D}}^z_N$, and $\tilde{\mu}^z_x$, and puts $\alpha < \alpha' < 2\underline{G}\log t_F$. Then by the aforementioned conditioning argument and Lemma 4.5.2, one shows that $\lim_{N\to\infty}\mathbb{P}(J_0|\Omega^+_{V_N}) = 0$. Similarly, using conditioning and a standard Gaussian estimate, one finds $\lim_{N\to\infty}\mathbb{P}(J_1|\Omega^+_{V_N}) = 0$. Upon varying over all $z \in V_k^{\llcorner}$ one proves Lemma 4.5.1. □

4.5.2 Upper bound

In this subsection we prove the upper bound (4.17). The overall strategy is to show that on $\Omega^+_{V_N}$, the coarse-grained averages of φ_x and of μ_x differ by $\mathcal{O}(1)$ as $N \to \infty$. Since the μ_x are independent under the conditioning, we can use standard Gaussian estimates to bound them below uniformly by a threshold $\sqrt{2\underline{G}\log t_F N}$. It follows that the local sample mean of the actual field φ_x is bounded below by the same threshold plus an $\mathcal{O}(1)$ error. Finally, we invoke the convergence of discrete Green forms (Lemma 4.2.5(ii)) and a capacity argument (like the one used at the end of the proof in §4.4.2) to establish the asymptotic sharpness of the threshold. Our approach is inspired by [Kurt].

In what follows, we fix an $y \in F$ and an $\epsilon > 0$ such that the ϵ cubic neighborhood of y,

$$\mathcal{B}(y,\epsilon) = \left\{y' \in F : \max_{1\le i\le d}|y'_i - y_i| \le \epsilon\right\},$$

is contained in F. We then let

$$V_{N,\epsilon}(y) = \left\{ z \in V_N : \max_{1\le i\le d} \left| z_i - [\ell_F^N y_i] \right| \le \epsilon \cdot \ell_F^N \right\},$$

and denote by $\mathcal{G}_{N,\epsilon}(y) = (V_{N,\epsilon}(y), \sim)$ the corresponding graph. In essence, $\mathcal{G}_{N,\epsilon}(y)$ (relative to $\mathcal{G}_N$) can be viewed as the graphical approximation of $\mathcal{B}(y,\epsilon)$ (relative to F).

The quantity of interest is the average of φ over $V_{N,\epsilon}(y)$, *i.e.*, the local sample mean of the free field,

$$\bar{\varphi}_{N,\epsilon}(y) = \frac{1}{|V_{N,\epsilon}(y)|} \sum_{z\in V_{N,\epsilon}(y)} \varphi_z.$$

For each $\eta > 0$, denote

$$\mathcal{M}_{N,\eta} := \left\{ \bar{\varphi}_{N,\epsilon}(y) \ge (\sqrt{2\underline{G}\log t_F} + \eta)\sqrt{N} \right\}.$$

Our goal is to prove that for any $\eta > 0$,

$$\lim_{N\to\infty} \mathbb{P}\left(\mathcal{M}_{N,\eta} \mid \Omega^+_{V_N} \right) = 0. \tag{4.22}$$

To begin the proof, we fix a sufficiently large $k \in \mathbb{N}$, take a $j \in \mathbb{N}$, and consider all $N > k + j$. Fix a representative interior point $x_0 \in V_k \backslash \partial\mathcal{G}_k$ as usual. Let $\kappa > 0$, and denote $\alpha_\kappa = 2(\underline{G} - \kappa)\log t_F$. Two events are introduced as follows. The first event is $\tilde{\Gamma}^z = \bigcup_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_N)} \tilde{\Gamma}^z_{\mathfrak{B}}$, where $\tilde{\Gamma}^z_{\mathfrak{B}}$ is given in (4.20) and is defined for each $\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$, $z \in V_k^{\llcorner}$ and $\delta \in (0,1)$. The second event, defined for each $s > 0$ and $z \in V_k^{\llcorner}$, is

$$\tilde{D}^z_s := \left\{ \varphi : \text{there exists an } \mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N) \text{ such that } \frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|} \sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}} (\varphi_x - \tilde{\mu}^z_x) < -s \right\}.$$

Observe that $\mathcal{M}_{N,\eta}$ equals the disjoint union $(\mathcal{M}_{N,\eta}\cap J_2)\cup(\mathcal{M}_{N,\eta}\cap J_3)\cup(\mathcal{M}_{N,\eta}\cap J_4)$, where

$$J_2 := \left(\bigcup_{z\in V_k^{\llcorner}} \tilde{\Gamma}^z \right), \quad J_3 := \left(\bigcap_{z\in V_k^{\llcorner}} \left(\tilde{\Gamma}^z\right)^c \right) \cap \left(\bigcup_{z\in V_k^{\llcorner}} \tilde{D}^z_s \right), \quad J_4 := \left(\bigcap_{z\in V_k^{\llcorner}} \left(\tilde{\Gamma}^z\right)^c \right) \cap \left(\bigcap_{z\in V_k^{\llcorner}} \left(\tilde{D}^z_s\right)^c \right).$$

So the task boils down to proving that each of $\mathbb{P}(\mathcal{M}_{N,\eta} \cap J_2 \cap \Omega^+_{V_N})$, $\mathbb{P}(\mathcal{M}_{N,\eta} \cap J_3 \cap \Omega^+_{V_N})$, and $\mathbb{P}(\mathcal{M}_{N,\eta} \cap J_4 \cap \Omega^+_{V_N})$ decays faster than $\mathbb{P}(\Omega^+_{V_N})$ as $N \to \infty$.

For J_2, we combine Lemma 4.5.2 with a union bound to find that for any $\gamma \in (0,1)$,

$$\mathbb{P}(J_2 \cap \Omega^+_{V_N}) \le |V_k^\llcorner| \exp\left(-Cm_F^N t_F^{-N(1-\gamma)}\right),$$

which decays faster than $\mathbb{P}(\Omega^+_{V_N})$.

For J_3, we use the fact that under $\mathbb{P}(\cdot|\mathcal{F}_{\tilde{\mathcal{D}}^z_N})$, $\{\varphi_x - \tilde{\mu}^z_x : x \in \tilde{\mathfrak{B}}^z_{\mathcal{C}}\}$ are independent (though not identically distributed) Gaussian random variables to find

$$\mathrm{Var}\left(\frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}(\varphi_x - \tilde{\mu}^z_x)\,\middle|\,\mathcal{F}_{\tilde{\mathcal{D}}^z_N}\right) = \frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|^2}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}\mathrm{Var}\left(\varphi_x - \tilde{\mu}^z_x\,\middle|\,\mathcal{F}_{\tilde{\mathcal{D}}^z_N}\right) \le \frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|^2}\cdot|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|\overline{G} = \frac{\overline{G}}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}.$$

By applying a union bound followed by a Gaussian estimate, we see that there exists $z' \in V_k^\llcorner$ such that

$$\begin{aligned}
&\mathbb{P}(J_3 \cap \Omega^+_{V_N}) \le |V_k^\llcorner| \cdot \mathbb{P}\left(\left(\tilde{\Gamma}^{z'}\right)^c \cap \tilde{D}^{z'}_s \cap \Omega^+_{V_N}\right)\\
\le\;& |V_k^\llcorner| \cdot \mathbb{E}\left(\mathbb{P}\left(\exists \mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N) : \frac{1}{|\tilde{\mathfrak{B}}^{z'}_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^{z'}_{\mathcal{C}}}\left(\varphi_x - \tilde{\mu}^{z'}_x\right) < -s \,\middle|\, \mathcal{F}_{\tilde{\mathcal{D}}^{z'}_N}\right)\cdot \mathbb{1}_{\Omega^+_{\tilde{\mathcal{D}}^{z'}_N}\cap(\tilde{\Gamma}^{z'})^c}\right)\\
\le\;& |V_k^\llcorner| \cdot \exp\left(\frac{-Cs^2 m_F^N}{2\overline{G}}\right),
\end{aligned}$$

where C depends on anything but N. This decays faster than $\mathbb{P}(\Omega^+_{V_N})$ as $N \to \infty$.

It remains to estimate $\mathbb{P}(\mathcal{M}_{N,\eta} \cap J_4 \cap \Omega^+_{V_N})$. Observe that for every $z \in V_k^\llcorner$, $\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$, and $s > 0$,

$$\frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}(\varphi_x - \tilde{\mu}^z_x) \ge -s \quad \text{on} \quad (\tilde{D}^z_s)^c \cap \Omega^+_{V_N},$$
$$\frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}\tilde{\mu}^z_x \ge (1-\delta)\sqrt{\alpha_\kappa N} \quad \text{on} \quad (\tilde{\Gamma}^z)^c \cap \Omega^+_{V_N}.$$

Therefore

$$\frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}\varphi_x = \frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}\tilde{\mu}^z_x + \frac{1}{|\tilde{\mathfrak{B}}^z_{\mathcal{C}}|}\sum_{x\in\tilde{\mathfrak{B}}^z_{\mathcal{C}}}(\varphi_x - \tilde{\mu}^z_x) \geq (1-\delta)\sqrt{\alpha_\kappa N} - s$$

$$\text{on} \quad (\tilde{\Gamma}^z)^c \cap (\tilde{D}^z_s)^c \cap \Omega^+_{V_N}.$$

We then take the intersection over all $z \in V_k^{\llcorner}$ to conclude that for every $\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N)$,

$$\bar{\varphi}_{\mathfrak{B}} := \frac{1}{|V(\mathfrak{B})|}\sum_{x\in V(\mathfrak{B})}\varphi_x \geq (1-\delta)\sqrt{\alpha_\kappa N} - s \quad \text{on} \quad J_4 \cap \Omega^+_{V_N}. \tag{4.23}$$

From now on put $s = \mathcal{O}(1)$.

Motivated by [Kurt]*§3, we define, for each $\theta \in [0,1)$ and $\kappa' > 0$, the event

$$C_{\theta,\kappa'} := \Big\{\varphi : \text{ there exist } \lfloor m_F^{(1-\theta)j} \rfloor \text{ many } \mathfrak{B}_0 \in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y)) \\ \text{such that } \bar{\varphi}_{\mathfrak{B}_0} \geq \left(\sqrt{\alpha_\kappa} + m_F^{\theta j}\kappa'\right)\sqrt{N}\Big\},$$

where

$$S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y)) = \left\{\mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_N) : \mathfrak{B} \subset \mathcal{G}_{N,\epsilon}(y)\right\}.$$

Denote by S^θ_{N-j} the collection of $\mathfrak{B}_0$ in the event $C_{\theta,\kappa'}$. By (4.23), for every $\eta > 0$ and every $\theta \in [0,1)$, there exists $\kappa' > 0$, independent of j, such that for all sufficiently large N,

$$\begin{aligned}
&\mathbb{P}(\mathcal{M}_{N,\eta}) \\
\leq\ & \mathbb{P}\left(\left\{\bar{\varphi}_{\mathfrak{B}} \geq (1-\delta)\sqrt{\alpha_\kappa N} - \mathcal{O}(1),\ \forall \mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y))\right\} \cap C_{\theta,\kappa'}\right) \\
=\ & \mathbb{P}\Big(\left\{\bar{\varphi}_{\mathfrak{B}} \geq (1-\delta)\sqrt{\alpha_\kappa N} - \mathcal{O}(1),\ \forall \mathfrak{B} \in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y))\right\}, \\
& \left\{\bar{\varphi} \geq (\sqrt{\alpha_\kappa} + m_F^{\theta j}\kappa')\sqrt{N},\ \forall \mathfrak{B}_0 \in S^\theta_{N-j}\right\}\Big) \\
\leq\ & \mathbb{P}\Bigg(\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}}\bar{\varphi}_{\mathfrak{B}} \\
\geq\ & \left[(1-\delta)\sqrt{\alpha_\kappa N} - \mathcal{O}(1)\right]\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} + \lfloor m_F^{(1-\theta)j}\rfloor \kappa' m_F^{\theta j}\sqrt{N}\Bigg),
\end{aligned}$$

where in the last line we inserted arbitrary $f_{\mathfrak{B}} \geq 0, \mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y)) \backslash S^{\theta}_{N-j}$, and fixed $f_{\mathfrak{B}_0} = 1$ for all $\mathfrak{B}_0 \in S^{\theta}_{N-j}$. Since the $\bar{\varphi}_{\mathfrak{B}}$ are centered Gaussian variables, we employ a standard estimate to find

$$\mathbb{P}\left(\mathcal{M}_{N,\eta}\right) \leq \exp\left(- \frac{\left(\left[(1-\delta)\sqrt{\alpha_\kappa N} - \mathcal{O}(1) \right] \sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} + C_{4.2} m_F^j \kappa' \sqrt{N} \right)^2}{2\mathrm{Var}\left(\sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \bar{\varphi}_{\mathfrak{B}} \right)} \right), \tag{4.24}$$

for some constant $C_{4.2}$ independent of N and j.

Now let $\Xi_j : F \to \mathbb{R}_+$ be defined by $\Xi_j = \sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \mathbb{1}_{\iota_{N-j}(\mathfrak{B})}$. (Note that $\mathrm{supp}(\Xi_j) \subset \mathcal{B}(y,\epsilon)$.) Then

$$\sum_{x \in V_{N,\epsilon(y)}} \Xi_j\left(\frac{x}{\ell_F^N}\right) = \sum_{x \in V_{N,\epsilon}(y)} \sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \mathbb{1}_{\iota_{N-j}(\mathfrak{B})}\left(\frac{x}{\ell_F^N}\right) \leq |V_{N-j}| \left(\sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \right),$$

and

$$\sum_{x \in V_{N,\epsilon}(y)} \Xi_j\left(\frac{x}{\ell_F^N}\right) \varphi_x = \sum_{x \in V_{N,\epsilon}(y)} \sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \varphi_x \mathbb{1}_{\iota_{N-j}(\mathfrak{B})}\left(\frac{x}{\ell_F^N}\right) = \sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \sum_{x \in \mathfrak{B}} \varphi_x.$$

Consequently,

$$\begin{aligned} \mathrm{Var}\left(\sum_{\mathfrak{B} \in S^{\circ}_{N-j}(\mathcal{G}_{N,\epsilon}(y))} f_{\mathfrak{B}} \bar{\varphi}_{\mathfrak{B}} \right) &\leq \frac{1}{|V^{\llcorner}_{N-j}|^2} \sum_{x,x' \in V_{N,\epsilon}(y)} G_{\mathcal{G}_\infty}(x,x') \Xi_j\left(\frac{x}{\ell_F^N}\right) \Xi_j\left(\frac{x'}{\ell_F^N}\right) \\ &\leq \frac{C_{4.3}}{|V_{N-j}|^2} \sum_{x,x' \in V_N} G_{\mathcal{G}_\infty}(x,x') \Xi_j\left(\frac{x}{\ell_F^N}\right) \Xi_j\left(\frac{x'}{\ell_F^N}\right) \end{aligned}$$

for some constant $C_{4.3}$ independent of j and N. Plugging these into (4.24) and then applying Lemma 4.2.5(ii), we find

$$
\begin{aligned}
&\overline{\lim_{N\to\infty}} \frac{\log \mathbb{P}(\mathcal{M}_{N,\eta})}{\rho_F^{-N} N \log t_F} \\
\le\ & \overline{\lim_{N\to\infty}} \left[- \frac{\left(\left[(1-\delta)\sqrt{\underline{G}-\kappa} - \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right] \cdot \frac{1}{|V_{N-j}|} \sum_{x\in V_N} \Xi_j\left(\frac{x}{\ell_F^N}\right) + \frac{C_{4.2} m_F^j \kappa'}{\sqrt{2\log t_F}} \right)^2}{C_{4.3} \frac{\rho_F^{-N}}{|V_{N-j}|^2} \sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x') \Xi_j\left(\frac{x}{\ell_F^N}\right) \Xi_j\left(\frac{x'}{\ell_F^N}\right)} \right] \\
\le\ & - \frac{\underline{\lim_{N\to\infty}} \left(\left[(1-\delta)\sqrt{\underline{G}-\kappa} - \mathcal{O}\left(\frac{1}{\sqrt{N}}\right) \right] \cdot \frac{1}{|V_N|} \sum_{x\in V_N} \Xi_j\left(\frac{x}{\ell_F^N}\right) + \frac{C_{4.4}\kappa'}{\sqrt{2\log t_F}} \right)^2}{C_{4.3}\, \overline{\lim_{N\to\infty}} \frac{\rho_F^{-N}}{|V_N|^2} \sum_{x,x'\in V_N} G_{\mathcal{G}_\infty}(x,x') \Xi_j\left(\frac{x}{\ell_F^N}\right) \Xi_j\left(\frac{x'}{\ell_F^N}\right)} \\
\le\ & -\frac{1}{C_{4.5}} \cdot \frac{\left((1-\delta)\sqrt{\underline{G}-\kappa} \langle \mathbb{1}_F, \Xi_j \nu \rangle_F + \frac{C_{4.4}\kappa'}{\sqrt{2\log t_F}} \right)^2}{\mathcal{E}(U(\Xi_j\nu), U(\Xi_j\nu))},
\end{aligned} \tag{4.25}
$$

where $C_{4.5} = C_{4.3}C_{2.6}$ and $C_{4.4}$ are independent of j (and N).

Following the end of §4.4.2, one would expect to optimize the coefficients $f_{\mathfrak{B}}$ and take the $j \to \infty$ limit to retrieve the capacity in the RHS of (4.25). But we have decided prior to (4.24) to fix some of the coefficients $f_{\mathfrak{B}_0}$, in order to leave the κ' term intact, *viz.*

$$
\Xi_j = \left(\sum_{\mathfrak{B}\in S^\circ_{N-j}(\mathcal{G}_{N,\epsilon}(y))\setminus S^\theta_{N-j}} f_{\mathfrak{B}} \mathbb{1}_{\iota_{N-j}(\mathfrak{B})} \right) + \sum_{\mathfrak{B}_0 \in S^\theta_{N-j}} \mathbb{1}_{\iota_{N-j}(\mathfrak{B}_0)} =: \Xi_{j,0} + \mathbb{1}_{\iota_{N-j}(S^\theta_{N-j})},
$$

where the $f_{\mathfrak{B}} \ge 0$ are arbitrary. This is done with intention to create a "rescaling imbalance" between the two terms in the numerator, as the end of the proof reveals. On the other hand, in order to have complete control on Ξ_j, we would like to exclude the fixed coefficients. The next arguments show that this is indeed possible: as $j \to \infty$, Ξ_j can be replaced by $\Xi_{j,0}$ in the RHS of (4.25).

Let's write

$$\begin{aligned}
& \mathcal{E}\left(U(\Xi_j\nu), U(\Xi_j\nu)\right) \\
= \quad & \mathcal{E}\left(U(\Xi_{j,0}\nu), U(\Xi_{j,0}\nu)\right) + 2\cdot\mathcal{E}\left(U(\Xi_{j,0}\nu), U(\mathbb{1}_{\iota_{N-j}(S^\theta_{N-j})}\nu)\right) \\
+ \quad & \mathcal{E}\left(U(\mathbb{1}_{\iota_{N-j}(S^\theta_{N-j})}\nu), U(\mathbb{1}_{\iota_{N-j}(S^\theta_{N-j})}\nu)\right) \\
=: \quad & K_1 + 2K_2 + K_3.
\end{aligned}$$

Suppose, without loss of generality, that K_1 is bounded above by a constant independent of j. The key estimate is on K_3. Let Q_j denote the closure of any one of the $\iota_{N-j}(\mathfrak{B}_0)$, which is a subset of $\mathcal{B}(y,\epsilon)$ inscribed by a hypercube of side ℓ_F^{-j}. Also let $G : F_\infty \times F_\infty \to \mathbb{R}_+$ be the integral kernel associated with U. By [BB99]*Corollary 6.13(a), there exists C_8 such that $G(x,x') \le C_8\|x-x'\|^{d_w-d_h}$ for all $x,x' \in F_\infty$. Therefore

$$\begin{aligned}
K_3 \quad \le \quad & \lfloor m_F^{(1-\theta)j}\rfloor \int_{Q_j\times Q_j} G(x,x')d\nu(x)d\nu(x') \;\le\; Cm_F^{(1-\theta)j}\int_{Q_j\times Q_j}\frac{d\nu(x)d\nu(x')}{\|x-x'\|^{d_h-d_w}} \\
\le \quad & Cm_F^{(1-\theta)j}\int_{Q\times Q}\frac{d\nu(\ell_F^{-j}x)d\nu(\ell_F^{-j}x')}{(\ell_F^{-j}\|x-x'\|)^{d_h-d_w}} \;\le\; Cm_F^{(1-\theta)j}\ell_F^{-j(d_h+d_w)}\int_{Q\times Q}\frac{d\nu(x)d\nu(x')}{\|x-x'\|^{d_h-d_w}} \\
= \quad & \mathcal{O}(m_F^{-\theta j}t_F^{-j}),
\end{aligned}$$

where Q is a cubic region of side $\mathcal{O}(1)$, and the last integral is finite by the same argument as in the proof of Lemma 4.2.5(i). This then allows us to estimate K_2 via Cauchy-Schwarz, namely, $(K_2)^2 \le K_1K_3 \le K_1\mathcal{O}(m_F^{-\theta j}t_F^{-j})$. Hence as $j\to\infty$,

$$\mathcal{E}(U(\Xi_j\nu), U(\Xi_j\nu)) = \mathcal{E}(U(\Xi_{j,0}\nu), U(\Xi_{j,0}\nu)) + \mathcal{O}(m_F^{-\theta j/2}t_F^{-j/2}). \tag{4.26}$$

We can now bound the RHS of (4.25) from above. First use the trivial inequality $\langle \mathbb{1}_F, \Xi_j\nu\rangle_F \ge \langle \mathbb{1}_F, \Xi_{j,0}\nu\rangle_F$ and apply it to the numerator. Then plugging (4.26) into the denominator, and taking the limit $j\to\infty$ on both sides of (4.25), we obtain

$$\varlimsup_{N\to\infty}\frac{\log\mathbb{P}(\mathcal{M}_{N,\eta})}{\rho_F^{-N}N\log t_F} \le -\frac{1}{C_{4.3}}\cdot\frac{\left((1-\delta)\sqrt{\underline{G}-\kappa}\langle\mathbb{1}_F,\Xi_0\nu\rangle_F + \frac{C_{4.4}\kappa'}{\sqrt{2\log t_F}}\right)^2}{\mathcal{E}(U(\Xi_0\nu),U(\Xi_0\nu))}, \tag{4.27}$$

where $\Xi_0 \in \mathcal{F}$ is any nonnegative function supported on $\mathcal{B}(y,\epsilon)$ minus a set of capacity zero. By the Yosida approximation (Proposition 4.3.1), we may then replace $\Xi_0 \nu$ by any $\mu \in S_0^{(0)}$ with the same maximal support set.

Now comes the simple but crucial rescaling argument:

$$\mathcal{E}(U\mu, U\mu) = \gamma \cdot (\gamma\mathcal{E})(U^\gamma \mu, U^\gamma \mu) \quad \text{for each } \gamma > 0 \text{ and for all } \mu \in S_0^{(0)},$$

where $U^\gamma = \gamma^{-1} U$ is the 0-order potential operator associated with $\gamma\mathcal{E}$. Thus the RHS of (4.27) can be rewritten as

$$-\frac{1}{\gamma C_{4.3}} \cdot \frac{\left((1-\delta)\sqrt{\underline{G}-\kappa}\langle \mathbb{1}_F, \mu\rangle_F + \frac{C_{4.4}\kappa'}{\sqrt{2\log t_F}}\right)^2}{(\gamma\mathcal{E})(U^\gamma\mu, U^\gamma\mu)}.$$

Fix a compact set $\mathcal{K}$ within $\mathcal{B}(y,\epsilon)$ minus the aforementioned set of capacity zero, and choose μ to be $\mu_{\mathcal{K}}$, the 0-order equilibrium measure of $\mathcal{K}$ with respect to $\gamma\mathcal{E}$. According to Proposition 4.3.2, $\langle \mathbb{1}_F, \mu_{\mathcal{K}}\rangle_F = (\gamma\mathcal{E})(U^\gamma\mu_{\mathcal{K}}, U^\gamma\mu_{\mathcal{K}}) = \mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K})$. (This achieves the aforementioned "rescaling imbalance.") Then upon taking $\delta, \kappa \to 0$, and combining with the lower bound of Theorem 4.1.1 (see also Remark 4.4.1), we arrive at

$$\begin{aligned}
& \overline{\lim_{N\to\infty}} \frac{\log \mathbb{P}(\mathcal{M}_{N,\eta} | \Omega^+_{V_N})}{\rho_F^{-N} N \log t_F} \\
\leq \; & \overline{\lim_{N\to\infty}} \frac{\log \mathbb{P}(\mathcal{M}_{N,\eta})}{\rho_F^{-N} N \log t_F} - \underline{\lim_{N\to\infty}} \frac{\log \mathbb{P}(\Omega^+_{V_N})}{\rho_F^{-N} N \log t_F} \\
\leq \; & -\frac{1}{\gamma C_{4.3}} \left[\underline{G}\mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K}) + \frac{\sqrt{2\underline{G}}(C_{4.4}\kappa')}{\sqrt{\log t_F}} \right. \\
+ \; & \left. \frac{(C_{4.4}\kappa')^2}{2\log t_F \mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K})} \right] + \gamma^{-1} C_{1.3} \overline{G} \mathrm{Cap}_{\gamma\mathcal{E}}(F).
\end{aligned} \tag{4.28}$$

Solving a quadratic inequality shows that the RHS of (4.28) is negative if

$$\kappa' > C_{4.4}^{-1} \cdot \mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K}) \cdot \sqrt{2\log t_F} \cdot \left(\sqrt{C_{1.3}C_{4.3}\overline{G} \cdot \frac{\mathrm{Cap}_{\gamma\mathcal{E}}(F)}{\mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K})}} - \sqrt{\underline{G}} \right).$$

Observe that $\mathrm{Cap}_{\gamma\mathcal{E}}(\mathcal{K})$ depends linearly on γ, while the rest of the expression on the RHS is manifestly independent of γ. So by tuning γ, we can make $\kappa' > \Delta$ for any $\Delta > 0$, and thus

$$\overline{\lim_{N\to\infty}} \frac{\log \mathbb{P}(\mathcal{M}_{N,\eta} | \Omega^+_{V_N})}{\rho_F^{-N} N \log t_F} < 0$$

for any $\eta > 0$. This proves (4.22) for any $y \in F$ and $\epsilon > 0$, whence (4.17).

APPENDIX A

CHAPTER 1

We collect some of the results about Green's function we used in Chapter 2 here. Most of this material is known to experts and drawn from the Appendix of [LS]. In this chapter, we freely use the notation from Sections 2.1 and 1.2.

Theorem A.0.3. *For a continuous function f, consider the following Dirichlet problem on SG*

$$-\Delta u = f \ \text{ on } \ SG \setminus V_0$$
$$u = 0 \ \text{ on } \ V_0.$$

The Dirichlet problem has a unique solution in dom(Δ) given by

$$u(x) = \int_{SG} G(x,y) f(y) dy$$

for $G(x,y) = \lim_{M\to\infty} \sum_{k=1}^{M} \sum_{s,s' \in V_k \setminus V_{k-1}} g(s,s') \psi_s^k(x) \psi_{s'}^k(y)$ *where*

$$g(s,s') = \begin{cases} \frac{3}{10}\left(\frac{3}{5}\right)^k & \text{for } s = s' \in V_k \setminus V_{k-1}, \\ \frac{1}{10}\left(\frac{3}{5}\right)^k & \text{for } s = s' \in F_w K, |w| = k-1 \ \text{ and } \ s \neq s'. \end{cases}$$

As we also noted in Section 2.1 we drop the superscript "m" in $\psi_{x_m}^m, \psi_{y_m}^m$ and $\psi_{z_m}^m$ and instead write ψ_{x_m}, ψ_{y_m} and ψ_{z_m}. Because, unless otherwise is noted, the superscript index well matched the subscript index. We can get from the definition that

$$\int_{SG} |\psi_{x_m}| dy = \int_{SG} |\psi_{y_m}| dy = \int_{SG} |\psi_{z_m}| dy = \frac{2}{3^{m+1}}. \tag{A.1}$$

since $|\psi_{x_m}|^2 \leq |\psi_{x_m}|$, we can further develop this as

$$\int_{SG} |\psi_{x_m}|^2 dy = \int_{SG} |\psi_{y_m}|^2 dy = \int_{SG} |\psi_{z_m}|^2 dy \leq \frac{2}{3^{m+1}}. \tag{A.2}$$

For simplicity, we define

$$\Psi_m(a,b,c)(x) = a\psi_{x_m}(x) + b\psi_{y_m}(x) + c\psi_{z_m}(x). \tag{A.3}$$

Observe that putting together (A.1) and (A.2) yields

$$\int_{SG} |\Psi_m(a,b,c)(x)| dx \leq \frac{C_1}{3^m} \text{ and } \int_{SG} |\Psi_m(a,b,c)(x)|^2 dx \leq \frac{C_2}{3^m} \tag{A.4}$$

for constants $C_1 = C_1(a,b,c)$ and $C_2 = C_2(a,b,c)$.

Lemma A.0.4. *We obtain*

$$G(x_m, y) = \frac{2}{15}\left(\frac{3}{5}\right)^m \sum_{m=1}^{m} \Psi_k(1,1,1)(y) + \frac{1}{6}\left(\frac{3}{5}\right)^m \Psi_m(1,-1,-1)(y). \tag{A.5}$$

Lemma A.0.5. *We have*

$$G(z_m, y) = \frac{1}{10}\left(\frac{3}{5}\right)^m \sum_{m=1}^{m} \Psi_k(1,2,2)(y) + \frac{1}{10}\left(\frac{3}{5}\right)^m \sum_{k=1}^{m} 3^k \Psi_k(0,-1,1)(y). \tag{A.6}$$

Lemma A.0.6. *Suppose $u = 0$ on V_0 and Δu exists on SG then we have*

$$\begin{aligned}\eta_m = \partial_n u(x_m) = \frac{3}{5}\left(\frac{1}{3}\right)^m \sum_{k=1}^{m} 3^k \int_{SG} \Psi_k(0,-1,1)\Delta u dy \\ + \frac{-1}{2}\int_{SG} \Psi_m(1,-1,1)\Delta u dy - \phi_m\end{aligned} \tag{A.7}$$

where $\phi_m = \int_{Z_m} \psi_{z_m} \Delta u dy$.

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