CONTINUOUS TIME SKIP-FREE MARKOV PROCESS AND STUDY OF BRANCHING PROCESS WITH IMMIGRATION

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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August 2020

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We first develop the potential and fluctuation theories of continuous-time skip-free Markov processes, extending the recent work from Choi and Patie [20] for skip-free Markov chains. On the one hand, this enables us to revisit in a simple manner the fluctuation theory of continuous-time skip-free random walk on $\mathbb{Z}$. This was originally developed by Spitzer [34] by means of the Wiener-Hopf factorization and, up to now, was the only class of Markov processes with jumps for which such a characterization was attainable. As the second application, we solve the two-sided exit time problems for continuous-time branching processes with immigration (CBI process), which was left open in the literature of this classical family of Markov processes. Next we aim to extend the results to continuous state space branching process with immigration. We identify an intertwining relationship between the discrete and continuous branching process with immigration. By applying the intertwining relation to the results in discrete CBI, we can derive the first hitting and first passage time of continuous CBI process. Lastly, we briefly introduce the main idea of scaled limit approach, which is an alternative way to study the continuous CBI process.

BIOGRAPHICAL SKETCH

Jian Wang was born on July 28, 1992. He grew up in China and went to University of California, Berkeley in 2011 for undergraduate study. He received a Bachelor of Science in Applied Mathematics, Bachelor of Science in statistics and Bachelor of Arts in Economics in December 2014.

In Fall 2015, he came to Cornell to pursue a Ph.D degree in the School of Operations Research and Information Engineering, advised by Professor Pierre Patie, with a concentration in Applied Probability and Statistics.

To my family, Daisy, all my professors and fellow students. To everyone I met in my life and became part of my life.

ACKNOWLEDGEMENTS

First of all, I want to extend my special appreciation and thanks to my advisor, Pierre Patie, for his unparallel knowledge, enthusiasm and invaluable guidance I received throughout the research work over the past five years at Cornell. He inspired me in the exploration dedication of my research direction and always very patient to me with consistent encouragement. My acadenic achievements would not be possible without his tremendous mentorship and advice. In addition, he is one of the most charming gentleman I've ever met and I can feel his love to his family and everyone around him.

I would also like to tahnk my committee member, Robert A. Jarrow and Gennady Samorodnitsky, for their great support, insightful discussions and advice. It's my honor to take everal course by Professor Jarrow in financial engineering, which has opened up new horizons of opportunities in front of me and inspired me to further pursue the career of quantitative finance after graduation. I would also appreciate the help from Professor Samorodnitsky, who was also the chair of department of Operations Research and Information Engineering. Besides the tremendous academic help and advice, he also provided me a great amount of support from the department.

Next I would appreciate all other professors , staff and my college students in the department. We inspired each other on academic ideas, and to be honest, we spend more time beyond the research, to share the personal experience and encourage each other. Special thanks to all the staff member especially in the first half of 2020, when the Covid-19 pandemic interrupt everyone's life. During the crisis, they provide flexible and great support to help me arrange all the things smoothly.

Last but not least, I am so grateful to my parents Aimin Wang and Yun Wang.

I came to U.S. alone nine years ago, which was not an easy decision. However, they always support me and trust me for all of my choices. With their love, I am confident to overcome any difficulties. I would also express my love to my girlfriend, Qiujun Li. We met in Berkeley seven years ago, and most of the time, we endure the long-distance love, in California, DC, New York. Every moment we spend together is my unforgotten memory. I cannot finish my Ph.D and enjoy my life at Cornell without their support and love.

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CHAPTER 1

INTRODUCTION

First hitting times and first passage times problems are ubiquitous in probability theory and, more broadly, in many fields of sciences, such as epidemiology, physics, neurology, insurance and financial mathematics, see the monograph for a nice account. In spite of the wide applicability of first-passage phenomena, the law and statistical properties of these random variables have been characterized only for very few and specific classes of stochastic processes. Besides some isolated examples, this includes the birth-death chains (or one-dimensional diffusion their continuous state space analogue) and Lévy processes for which tailor-made approaches have been developed, see [**?**, 31, 20]. In this vein, in the recent paper [20], Choi and Patie develop an original and comprehensive approach based on a combination of techniques such as the theory of Martin boundary and potential theory to provide an expression for the q-potential of discrete-time skip-free Markov chains. The first and natural step is to extend the idea to the continuous-time skip-free Markov process.

Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov process on the countable state space $E = [[\mathfrak{l}, \mathfrak{r}]] \subseteq \mathbb{Z}$, We assume further that X is irreducible, i.e. for all

$$\text{for all } x, y \in \dot{E} = [\mathfrak{l}, \mathfrak{r}], \mathbb{P}_x(X_t = y) > 0 \text{ for some } t \geq 0,$$

and *downward skip-free*, i.e.

$$\text{for all } x \in E, p(x, x-1) > 0 \text{ and } p(x, x-y) = 0 \;\text{ for }\; y \geq 2.$$

We denote by $\mathcal{M}$ be the set of such downward skip-free Markov chains (or transition operators) on E. It's a special case of the general Markov process and still has a wide range of application. An important problem arising in the analysis of such a process is to characterize the law of its first hitting time and the first passage time, that is the stopping time

$$T_a = \inf\{t > 0; X_t = a\} \tag{1.1}$$

and

$$T_{[b} = \inf\{t > 0; X_t \leq b\}. \tag{1.2}$$

These quantities for Markov processes are central in many problems of applied mathematics ranging from biology, neurology, epidemiology, physics to economics, mathematical finance and insurance mathematics. For instance, the time T_0 corresponds to the time of distinction of a group that only have positive jumps, which is a classical assumption for the branching process model without immigration. Yet their distribution are attainable only for a few families. This includes, for instance, birth-death process (both upwards and downwards skip-free) and random walks, for which interesting techniques, relying on the specific properties on these chains, have been designed. Our approach characterize the first hitting time of the continuous-time Markov process. The purpose of the paper is to thoroughly generalize and extend the ideas in discrete time Markov chain to the continuous-time Markov process, that is to characterize the distribution of the first exit time from an interval and the expression for different important quantities.

Among many applications, we give a comprehensive study on the application of continuous-time branching process with immigration, based on the background studied by Yanev [38], Ney and Bran [39]. By applying the main theory in the general Markov process on branching process and some computational techniques we can have the explicit expression of the first passage time and first exit time. In addition we gave a example of Neveu process as an illustration of the specific branching process. With the result in continuous-time discrete state space Markov process, we further extend the results to the continuous state space from two different approach. The continuous-state space branching processes with immigration (continuous CBI for short) are a class of time-homogeneous Markov process with values in $\mathbb{R}^+$. The first approach is to identify an intertwining relationship between the discrete and continuous CBI. By applying the intertwining relation to the results in discrete CBI, we can define the associated fundamental q-excessive function and therefore derive the first hitting and first passage time of continuous CBI process. The second approach use the idea of scaled-limit. The time change binding the two processes together is called the Lamperti transform, following the foundational work of Lamperti. We can derive the quantities in continuous state space as the limit of associated quantities in discrete state space.

CHAPTER 2

PURELY EXCESSIVE FUNCTIONS AND HITTING TIMES OF CONTINUOUS-TIME BRANCHING PROCESSES

2.1 Introduction and the main result

Let $Z = (Z_t)_{t\geq 0}$ be a continuous-time Galton-Watson process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$ where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by Z satisfying the usual conditions. For $x \geq 0$, we denote by $\mathbb{P}_x$ the law of Z when it is started at x and write simply $\mathbb{P} = \mathbb{P}_1$. Accordingly we shall write $\mathbb{E}_x$ and $\mathbb{E}$ for the associated expectation operators. An important problem arising in the analysis of such a process is to characterize the law of their first passage times, that is the one of the stopping time, for $y \geq 0$,

$$T_y = \inf\{t > 0; Z_t \leq y\}. \tag{2.1}$$

Although first passage times problems for Markov chains are central in many problems of applied mathematics ranging from biology, neurology, epidemiology, physics to economics, mathematical finance and insurance mathematics, see e.g. [4], [9], [10] and [12], their distribution are attainable only for a few families. This includes, for instance, birth-death chains and random walks, for which interesting techniques, relying on the specific properties on these chains, have been designed.

It remains a difficult and fascinating challenge to find a comprehensive approach that enables to characterize the distributions of first passage times of general Markov chains. In this direction, we mention that, recently in [20], the authors develop, by means of the Martin boundary theory, the potential theory for the class of discrete-time skip-free Markov chains yielding to general and closed-form formula for their fluctuation identities. It is the purpose of this work to illustrate the usefulness of this methodology by deriving an explicit expression along with fine properties of the excessive function that characterizes the Laplace transform of the downward first passage time T_y, as defined in (2.1) above, for continuous-time branching processes. It is worth pointing out that we do not exploit here the full potential of the ideas developed in [20] as it also allows to characterize the distribution of the first exit time from an interval. We let the lengthy details for such an extension to a forthcoming work. However, the interested reader may already consult [14] where the aforementioned methodology is applied to obtain the fluctuation theory of continuous state space branching processes with immigration. In this framework, we also mention the interesting paper by Duhalde et al. [25] where the Laplace transform of the downward first passage times of these Markov processes is provided.

Coming back to the setting of a continuous-time and discrete state-space branching process, we recall that it starts with a specified number of particles. If a given particle is alive at a certain time, its additional life length is a random variable which is exponentially distributed with parameter say $a > 0$. Upon death it leaves k offsprings with probability $\lambda_k, k = 0, 1, 2, ...$As usual, particles act independently of each other and of the history of the process. As by-product, one gets, compared to the discrete Galton-Watson process, that a continuous time branching process is downwards skip-free, that is it decreases by at most 1 at each jump. The detailed construction of the continuous-time branching process can be found in Athreya and Ney's book [39, Chap. III].

Next, let $P = (P_t)_{t\geq 0}$ denotes the semigroup of Z, i.e. for a function f on $\mathbb{N}$,

$$P_t f(x) = \mathbb{E}_x[f(Z_t)]. \tag{2.2}$$

It is a well established fact that P defines a Feller semigroup, i.e. P is a Markov semigroup with $P_t C_0(\mathbb{N}) \subseteq C_0(\mathbb{N})$ where $C_0(\mathbb{N})$ stands for the space of continuous functions on $\mathbb{N}$ vanishing at infinity. Next, let G_q be the q-resolvent associated to the semigroup P, that is, for f a bounded function on $\mathbb{N}$,

$$G_q f(x) = \int_0^\infty e^{-qt} P_t f(x) dt = \sum_{y\in\mathbb{N}} f(y) g_q(x,y), \tag{2.3}$$

where $g_q(x,y)$ denotes its associated kernel.

We proceed by writing $\phi_t(s)$ the generating function of the variable Z_t which is the population size at time t, when started with a single individual, i.e. for any $t \geq 0$ and $|s| < 1$,

$$P_t p_s(1) = \mathbb{E}[s^{Z_t}] = \phi_t(s), \tag{2.4}$$

where $p_s(x) = s^x, x \in \mathbb{N}$. Observe that the semigroup property (i.e. the Markov property of Z) yields the following useful semigroup property of the so-called branching mechanism ϕ_t, for any $t, r \geq 0$,

$$\phi_{t+r}(s) = P_{t+r} p_s(1) = P_t \circ P_r p_s(1) = P_t p_{\phi_r(s)}(1) = \phi_t \circ \phi_r(s). \tag{2.5}$$

Next, we define $\mathfrak{f}(s) = \mathbb{E}(s^N)$ as the generating function of the offspring variable N. Note that since we exclude the trivial case when the branching process has a.s. increasing sample paths, we have $\mathfrak{f}(0) = \mathbb{P}(N = 0) > 0$. It is well known, see the aforementioned book [39, III.3(5)], that the mapping $t \mapsto \phi_t(s)$ is solution to the Riccati equation

$$\partial_t \phi_t(s) = F_a(\phi_t(s)) \text{ and } \phi_0(s) = s, \tag{2.6}$$

where

$$F_a(s) = a(\mathfrak{f}(s) - s). \tag{2.7}$$

From the differential equation (4.2), we easily deduce that, for all $t \geq 0$,

$$\int_s^{\phi_t(s)} \frac{dr}{F_a(r)} = t. \tag{2.8}$$

Next, with the fundamental insight that the branching process starting with one particle determines the branching process in general, we get that the collection of mechanisms $(\phi_t)_{t\geq 0}$ characterises the semigroup P. More specifically the branching property entails that, for any $x = 0, 1, \ldots$,

$$P_t p_s(x) = (\phi_t(s))^x = \phi_t^x(s), \tag{2.9}$$

where the second identity serves as a notation and we set $\phi_t^1(s) = \phi_t(s)$.

We proceed by recalling that, for $q > 0$, a function $h_q : \mathbb{N} \to \mathbb{R}^+$ is called q-excessive (resp. q-purely excessive) if for all $t \geq 0$, $e^{-qt} P_t h_q(x) \leq h_q(x)$ with $\lim_{t\downarrow 0} e^{-qt} P_t h_q(x) = h_q(x)$ (resp. and $\lim_{t\uparrow\infty} e^{-qt} P_t h_q(x) = 0$). Moreover, a q-excessive function f is said minimal if $f = f_1 + f_2$ implies $f = c_i f_i$ for $i = 1, 2$, where f_1, f_2 are q-excessive and c_1 and c_2 are constants.

We are now ready to state our main result which includes some interesting facts regarding the stopping time T_y. We point out that the expression (2.13) below of the extinction probability is a classical result, see again [39] and the references therein. However, to the best of our knowledge, the other claims seem to be new in this literature.

Theorem 2.1.1. *For any $q > 0$ and $x \in \mathbb{N}$, let $\phi_t^x = \phi_t^x(0) = \mathbb{P}_x(Z_t = 0)$ denote the probability of extinction, and set*

$$H_q(x) = x \int_0^\infty e^{-qt} F_a(\phi_t) \phi_t^{x-1} dt = q \int_0^\infty e^{-qr} \phi_r^x dr. \tag{2.10}$$

Then, the mapping $x \mapsto H_q(x)$ is a minimal q-purely excessive function which is decreasing on $\mathbb{N}$. Moreover, for any state $y \leq x$,

$$\mathbb{E}_x[e^{-qT_y}] = \frac{H_q(x)}{H_q(y)}. \tag{2.11}$$

Consequently,

$$\mathbb{P}_x(T_y < \infty) = \phi_\infty^{x-y}, \tag{2.12}$$

where $\phi_\infty = \lim_{t\to\infty} \phi_t$ is the smallest root in $(0, 1]$ of the equation $f(s) = s$. Finally, for any $x \in \mathbb{N}$ and $t > 0$,

$$\mathbb{P}_x(T_0 \leq t) = \phi_t^x. \tag{2.13}$$

2.2 Proof of the main result

First, by invoking an argument based on the theory of the Martin boundary for Markov chains, it is shown in [20], that the function H_q appearing in (2.11), the expression of the Laplace transform of the hitting T_y, is characterized by

$$H_q(x) = \lim_{y\to 0} \frac{g_q(x, y)}{g_q(0, y)} = \frac{g_q(x, 0)}{g_q(0, 0)}, \tag{2.14}$$

where we choose 0 as reference point, meaning, in particular, that H_q is normalized such that $H_q(0) = 1$. Note first that in [20], the methodology is in fact detailed for discrete-time Markov chain but, alike for the Martin boundary theory, it extends readily to the continuous-time skip-free Markov chains. Note also that compared to [20] where the original Markov chain is assumed to be upward skip-free, we consider here the problem of characterizing the fundamental excessive function of its dual process, $\widehat{H}_q$ in the notation of the aforementioned paper [20], as we use the facts that here the branching process Z is skip-free downward with state space $\mathbb{N}$. In order to compute this limit we start by observing, with the notation introduced in (3.9), that

$$\lim_{y\to 0}\frac{g_q(x,y)}{g_q(0,y)} = \lim_{s\to 0}\frac{G_q p_s(x)}{G_q p_s(0)} = \frac{G_q p_0(x)}{G_q p_0(0)}. \tag{2.15}$$

On the other hand, we have

$$\begin{aligned} G_q p_s(x) &= \sum_{y=0}^{\infty} s^y g_q(x,y) = \int_0^\infty e^{-qt} P_t p_s(x)dt \\ &= \int_0^\infty e^{-qt}\phi_t^x(s)dt. \end{aligned} \tag{2.16}$$

Hence, combining (2.14) with (3.43) and as for all $t \geq 0$ and $|s| > 1$, $|\phi_t^x(s)| \leq 1$, we obtain, by a dominated convergence argument, that

$$H_q(x) = q\int_0^\infty e^{-qt}\phi_t^x dt.$$

Since 0 is an absorbing boundary point, note that for all $t \geq 0$,

$$x \mapsto \phi_t^x = \mathbb{P}_x(Z_t = 0) = \mathbb{P}_x(T_0 \leq t) \in [0,1],$$

ensuring that for all $q > 0$, the function H_q is well defined and positive on $\mathbb{N}$. As Z is skip-free downward, an application of the strong Markov property yields that for any $0 < x < y$,

$$\mathbb{P}_x(T_0 \leq t) = \int_0^t \mathbb{P}_x(T_y \leq t-s)\mathbb{P}_y(T_0 \in ds) \leq \mathbb{P}_y(T_0 \leq t),$$

implying that $x \mapsto \phi_t^x$ is non-decreasing on $\mathbb{N}$. Next, the integration by parts formula gives that

$$q\int_0^\infty e^{-qt}\phi_t^x dt = \int_0^\infty e^{-qt}\partial_t\phi_t^x dt - [e^{-qt}\phi_t^x]_0^\infty = x\int_0^\infty e^{-qt}F_a(\phi_t)\phi_t^{x-1}dt, \tag{2.17}$$

where, for the last idendity, we used the Ricatti equation (4.2), $\lim_{t\to\infty}\phi_t \leq 1$ and $\lim_{t\to 0}\phi_t = \phi_0(0) = 0$ according to (4.2) again. This completes the proof of (2.11). The properties of H_q can be derived from [20, Theorem 3.1] which is based on a classical argument of the theory of Martin boundary. For sake of completeness, we provide an alternative path here. An application of Fubini's (Tonelli) theorem yields that

$$\begin{aligned} P_t H_q(x) &= q\int_0^\infty e^{-qr} P_t p_{\phi_r}(x) dr \\ &= q\int_0^\infty e^{-qr} \left(\phi_t(\phi_r)\right)^x dr \\ &= q\int_0^\infty e^{-qr} \phi^x_{t+r} dr \\ &= e^{qt} q\int_t^\infty e^{-qu} \phi^x_u du, \end{aligned} \tag{2.18}$$

where we used successively (2.9), (2.5) and performed an obvious change of variables. Thus, as $\phi^x_u \geq 0$ for all $u \geq 0$, we deduce from the last identity that $e^{-qt} P_t H_q(x) \leq H_q(x)$ for all $x \in \mathbb{N}$ and $t > 0$. Finally, we clearly have

$$\lim_{t\downarrow 0} e^{-qt} P_t H_q(x) = \lim_{t\downarrow 0} q\int_t^\infty e^{-qr} \phi^x_r dr = H_q(x),$$

and, by monotone convergence

$$\lim_{t\to\infty} e^{-qt} P_t H_q(x) = \lim_{t\to\infty} q\int_t^\infty e^{-qr} \phi^x_r dr = 0.$$

As H_q is a positive function, this shows that it is a q-purely excessive function. We complete the proof of the first claim by means of the definition of H_q in (2.14) and [20, Theorem 2.3] which, since 0 belongs to the state space, states that H_q is minimal. Next, the expression (1) combined with a dominated convergence argument entail that

$$\lim_{q\downarrow 0} H_q(x) = \lim_{q\downarrow 0} q\int_0^\infty e^{-qt} \phi^x_t dt = \lim_{q\downarrow 0} \int_0^\infty e^{-r} \phi^x_{\frac{r}{q}} dr = \phi^x_\infty. \tag{2.19}$$

The fact that ϕ_∞ is the smallest solution in $(0, 1]$ to the functional equation $f(s) = s$ is classical and a simple proof can be found in [39]. Since $\mathbb{P}_x(T_y < \infty) = \lim_{q\downarrow 0} \mathbb{E}_x[e^{-qT_y}] = \lim_{q\downarrow 0} \frac{H_q(x)}{H_q(y)}$, we deduce, from the identity (2.19) above, the expression (2.12).

We proceed with the following lemma which reveals that indeed the q-purely excessive function H_q defines a supermartingale.

Lemma 2.2.1. *Denoting by* $\mathbf{L}$ *the infinitesimal generator of* P*, we have on* $\mathbb{N}$ *that, for any* $q > 0$

$$\mathbf{L}H_q(x) = qH_q(x)\mathbf{1}_{\{x>0\}}. \tag{2.20}$$

Consequently, the process $(e^{-qt}H_q(Z_t))_{t\geq 0}$ *is a supermartingale but not a martingale.*

Proof. We shall in fact provide two proofs of the identity (2.20). One relies on the relationship between the infinitesimal generator and the resolvent as inverse operators while the other one is based on the backward Kolmogorov's equation. First, since $x \mapsto P_t\mathbf{1}(x) = \mathbf{1}(x) \in C_b(\mathbb{N})$, where $C_b(\mathbb{N})$ stands for the space of continuous bounded functions on $\mathbb{N}$. Thus, P is also a C_b-Feller semigroup. Since plainly $p_0(x) \in C_b(\mathbb{N})$, we have that for all $q > 0$, $G_q p_0$ belongs to the domain of the generator $\mathbf{L}$ of the C_b-Feller semigroup. On the other hand, comparing the expression (3.44) with the definition of H_q, we deduce that

$$H_q(x) = qG_q p_0(x),$$

that is H_q belongs to the domain of $\mathbf{L}$ as well. Hence, recalling that $p_0(x) = \mathbf{1}_{\{x=0\}}$ and G_q is the inverse on $C_b(\mathbb{N})$ of the linear operator $qI - \mathbf{L}$, with I the identity operator, see e.g. [13], we get

$$(\mathbf{L} - q)H_q(x) = q(\mathbf{L} - q)G_q p_0(x) = -qp_0(x),$$

which completes the first proof of the identity (2.20). We now provide an alternative proof of this identity which is based on the backward Kolmogorov's equation. More precisely, observing, from (3.44), that

$$H_q(x) = q\int_0^\infty e^{-qt}P_t p_0(x)dt, \tag{2.21}$$

with $H_q(0) = 0$, we get by resorting successively to a Fubini's argument, to the identity $\mathbf{L}P_t p_0(x) = \partial_t P_t p_0(x)$, see [13] again, that, for $x > 0$,

$$\begin{aligned}\mathbf{L}H_q(x) &= q\int_0^\infty e^{-qt}\mathbf{L}P_t p_0(x)dt = q\int_0^\infty e^{-qt}\partial_t P_t p_0(x)dt \\ &= q\int_0^\infty e^{-qt}\partial_t\phi_t^x dt = q^2\int_0^\infty e^{-qt}\phi_t^x dt \\ &= qH_q(x),\end{aligned}$$

where the penultimate line follows after an integration by parts as in (2.19). However, for $x = 0$,

$$\mathbf{L}H_q(x) = q\int_0^\infty e^{-qt}\mathbf{L}P_t p_0(0)dt = q\int_0^\infty e^{-qt}\partial_t \mathbf{1}dt = 0,$$

which completes the second proof of (2.20). The function H_q being q-excessive, the last claim follows readily from the previous one.

□

2.3 Examples

In this part, we detail the computation of the excessive function H_q for two important instances of continuous-time branching processes.

2.3.1 Reproduction with bifurcations

2.3.2 Reproduction with bifurcations

For the first one, we set $\lambda_1 = \lambda_k = 0, \forall k \geq 3,\ \lambda_0 + \lambda_2 \leq 1$, to get a binary splitting reproduction law whose generating function takes the form

$$\mathfrak{f}(s) = \lambda_0 + \lambda_2 s^2,$$

with possible explosion if $\lambda_0 + \lambda_2 < 1$. The corresponding branching process Z is the linear birth-death process with killing studied in [7], which is both upward and downward skip-free. In this case we have an explicit generating function.

In the non-critical case, putting

$$\alpha = \sqrt{1 - 4\lambda_0\lambda_2} > 0, \qquad \beta = \frac{1-\alpha}{2\lambda_2}, \qquad \overline{\beta} = \frac{1+\alpha}{2\lambda_2},$$

one finds, see for example [15, Sec. 5] for details, that

$$\phi_t(s) = \frac{(\overline{\beta} - s)\beta + (s-\beta)\overline{\beta}e^{-\alpha t}}{\overline{\beta} - s + (s-\beta)e^{-\alpha t}},$$

and in particular $\phi_t(0) = \overline{\beta}\beta\frac{1-e^{-\alpha t}}{\overline{\beta}-\beta e^{-\alpha t}} = \frac{\lambda_0}{\lambda_2}\frac{1-e^{-\alpha t}}{\overline{\beta}-\beta e^{-\alpha t}}$. Integrating with the help of Mathematica and using (1) yields, writing $q_\alpha = \frac{q}{\alpha}$,

$$\begin{aligned} H_q(x) &= q_\alpha \beta^x \frac{x!\Gamma(1+q_\alpha)}{\Gamma(x+q_\alpha+1)}\, {}_2F_1\left(x, q_\alpha; x+q_\alpha+1; \frac{1-\alpha}{1+\alpha}\right) \\ &= q_\alpha \beta^x \frac{x!}{(x+q_\alpha)(x-1+q_\alpha)...(1+q_\alpha)}\, {}_2F_1\left(x, q_\alpha; x+q_\alpha+1; \frac{1-\alpha}{1+\alpha}\right), \end{aligned}$$

where ${}_2F_1(a, b; c; z)$ is the Gauss-Hypergeometric function. The critical case $\lambda_0 = \lambda_2 = \frac{1}{2}, \alpha = 0$, requires a separate derivation [15, (24)], and the generating function is

$$\phi_t(s) = 1 - \frac{1-s}{1 + \frac{t}{2}(1-s)},$$

with $\phi_t(0) = \frac{t}{2+t}$. Integrating yields

$$H_q(x) = x!U(x, 0, 2q),$$

where U is the confluent hypergeometric function.

2.3.3 Continuous Neveu branching process

As another instance of continous-time branching process, we consider the Neveu process and refer, for instance, to the recent paper [28] for a thorough and interesting study of this process. Its branching mechanism is given, for $a > 0$ and $0 < \lambda \leq 1$, by

$$\mathfrak{f}(z) = (1-\lambda) + \lambda\left((1-z)\log(1-z) + z\right).$$

Expanding $a(\mathfrak{f}(z) - z)$ in power series one finds that the rate λ_k at which one individual gives birth to $k \neq 1$ individuals are

$$\lambda_0 = a(1-\lambda), \quad \lambda_k = \frac{a\lambda}{k(k-1)}, \; k \geq 2.$$

Each particle lives a random time with exponential distribution with frequency parameter $\sum_{k\neq 1} \lambda_k = a$. At the time of its death, it gives birth to the random number $N \neq 1$ of particles of the same type with law

$$\mathbb{P}(N = k) = \frac{\lambda_k}{a}, \; k \neq 1.$$

As above, we write $\phi_t(s) = P_t p_s(1) = \mathbb{E}_1[s^{Z_t}], \; t \geq 0$, where we recall that Z_t is the population size at time t started with a single individual obeys

$$\partial_t \phi_t(s) = \mathfrak{f}(\phi_t(s)), \; \phi_0(s) = s,$$

which can be solved explicitly to give

$$\phi_t(s) = 1 - e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda at})}(1-s)^{e^{-\lambda at}}.$$

To provide an explicit expression of H_q, from (1), we get that

$$H_q(x) = \frac{\int_0^\infty e^{-qt}\phi_t^x(0)dt}{\int_0^\infty e^{-qt}dt} = q\int_0^\infty e^{-qt}(1-e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda at})})^x dt.$$

Writing $\overline{\lambda} = \frac{1-\lambda}{\lambda} \in (0,1)$, we get

$$\begin{aligned}\int_0^\infty e^{-qt}(1-e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda at})})^x dt &= \int_0^\infty e^{-qt}\sum_{k=0}^{x}\binom{x}{k}(-1)^k e^{-\overline{\lambda}k(1-e^{-\lambda at})}dt \\ &= \sum_{k=0}^{x}\binom{x}{k}(-1)^k e^{-\overline{\lambda}k}\int_0^\infty e^{-qt+\overline{\lambda}ke^{-\lambda at}}dt. \qquad (2.22)\end{aligned}$$

Observe that, with $\overline{\lambda}_k = \overline{\lambda}k, k \geq 1$, putting $r = e^{-\lambda at}$, we have

$$\begin{aligned}q\int_0^\infty e^{-qt+\overline{\lambda}_k e^{-\lambda at}}dt &= q\int_0^1 r^{\frac{q}{\lambda a}}e^{\overline{\lambda}_k r}\frac{1}{\lambda ar}dr \\ &= \frac{q}{\lambda a}\int_0^1 r^{\frac{q}{\lambda a}-1}e^{\overline{\lambda}_k r}dr \\ &= \frac{q}{\lambda a}\gamma_1\left(\frac{q}{\lambda a},\overline{\lambda}_k\right), \qquad (2.23)\end{aligned}$$

where γ_1 stands for the modified lower incomplete gamma function. Note that for $k = 0$, the integral is equal to 1. Combining (2.22) and (2.23), one finally gets

$$H_q(x) = 1 + \frac{q}{\lambda a}\sum_{k=1}^{x}\binom{x}{k}(-1)^k e^{-\overline{\lambda}k}\gamma_1\left(\frac{q}{\lambda a},\overline{\lambda}_k\right).$$

CHAPTER 3

CONTINUOUS TIME SKIP-FREE MARKOV PROCESS

3.1 Introduction

First passage times problems are ubiquitous in probability theory and, more broadly, in many fields of sciences, such as epidemiology, physics, neurology, insurance and financial mathematics, see the monograph for a nice account. In spite of the wide applicability of first-passage phenomena, the law and statistical properties of these random variables have been characterized only for very few and specific classes of stochastic processes. Besides some isolated examples, this includes the birth-death chains (or one-dimensional diffusion their continuous state space analogue) and Lévy processes for which tailor-made approaches have been developed, see [**?**, 31, 20]. Unfortunately, all attempts to carry over techniques from these cases have had a limited success, providing some evidence towards the need to find a novel way to tackle this issue. In this vein, in the recent paper [20], Choi and Patie develop an original and comprehensive approach based on a combination of techniques such as the theory of Martin boundary and potential theory to provide an expression for the q-potential of discrete-time skip-free Markov chains. Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov process on the countable state space $E = [[\mathfrak{l}, \mathfrak{r}]] \subseteq \mathbb{Z}$, where we use the notation [[(resp.]]) to denote that $\mathfrak{l}$ (resp. $\mathfrak{r}$) may or may not be in E, defined on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P} = (\mathbb{P}_x)_{x\in\mathbb{E}})$. We denote its generator matrix by $P = (P_t)_{t\geq 0}$ where $P_t = (p_t(x, y))_{x,y\in E}$. We assume further that X is irreducible, i.e. for all

$$\text{for all } x, y \in \dot{E} = [\mathfrak{l}, \mathfrak{r}], \mathbb{P}_x(X_t = y) > 0 \text{ for some } t \geq 0,$$

and *downward skip-free*, i.e.

$$\text{for all } x \in E, p(x, x-1) > 0 \text{ and } p(x, x-y) = 0 \text{ for } y \geq 2.$$

We denote by $\mathcal{M}$ be the set of such downward skip-free Markov chains (or transition operators) on E. An important problem arising in the analysis of such a process is to characterize the law of its first hitting time and the first passage time, that is the stopping time

$$T_a = \inf\{t > 0; X_t = a\} \tag{3.1}$$

and

$$T_{[b} = \inf\{t > 0; X_t \leq b\}. \tag{3.2}$$

These quantities for Markov processes are central in many problems of applied mathematics ranging from biology, neurology, epidemiology, physics to economics, mathematical finance and insurance mathematics. For instance, the time T_0 corresponds to the time of distinction of a group that only have positive jumps, which is a classical assumption for the branching process model without immigration. Yet their distribution are attainable only for a few families. This includes, for instance, birth-death process (both upwards and downwards skip-free) and random walks, for which interesting techniques, relying on the specific properties on these chains, have been designed.

It remains a difficult and fascinating challenge to find a comprehensive approach that enables to characterize the distributions of first passage times of general Markov chains. In this direction, we mention that, recently in [20], the authors develop, by means of the Martin boundary theory, the potential theory for the class of discrete-time skip- free Markov chains yielding to general and closed-form formula for their fluctuation identities. In the authors' previous work [17] it characterize the first hitting time of the continuous-time Markov process. The purpose of the paper is to thoroughly generalize and extend the ideas in discrete time Markov chain to the continuous-time Markov process, that is to characterize the distribution of the first exit time from an interval and the expression for different important quantities. Also the paper gives a comprehensive study on the application of continuous-time branching process with immigration, based on the background studied by Yanev [38], Ney and Bran [39]. By applying the main theory in the general Markov process on branching process and some computational techniques we can have the explicit expression of the first passage time and first exit time. In addition we gave a example of Neveu process as an illustration of the specific branching process. With the result in continuous-time discrete state space Markov process, we should be able to extend the results to the continuous state space by introduction of the scaling limit definition. We let the details for such an extension to a forthcoming work.

The following parts of the paper is organized as following. In section 2 we introduce some notations and the classical results about Martin Boundary condition. In section 3 we stated the results for general downward skip-free Markov process based on Choi and Patie's work [20], the proof is included in the Appendix. Next we use the random walk on $\mathbb{Z}$ as an example to illustrate the application of result above. In section 5 we introduce the notations and preliminary results for continuous-time branching process and state our main theorem, followed by the detailed proof parts by parts. In last section we introduce the Neveu process as a specific case of the main theorem.

3.2 Preliminaries

In this section, we review some classical concepts on continuous-time Markov processes that will play a central role throughout the paper. This include some facts of the potential theory and the Martin boundary theory of Markov chains that are discussed in details in [26, 29, 37, 20]. We shall state the main theorem and results used in the later sections.

3.2.1 Basic facts on continuous-time Markov chains

Let $X = (X_t)_{t\geq 0}$ be a continuous-time Markov process on the countable state space $E = [[\mathfrak{l}, \mathfrak{r}]] \subseteq \mathbb{Z}$, and defined on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P} = (\mathbb{P}_x)_{x\in\mathbb{E}})$. We use the notation [[(resp.]]) to indicate that $\mathfrak{l}$ (resp. $\mathfrak{r}$) may or may not be in E. We denote its generator matrix by $P = (P_t)_{t\geq 0}$ where $P_t = (p_t(x, y))_{x,y\in E}$. We assume further that X is irreducible, i.e.

$$\text{for all } x, y \in E,\ p_t(x, y) > 0 \text{ for some } t \geq 0,$$

and *downward skip-free*, i.e.

$$\text{for all } t \geq 0 \text{ and } x \in E,\ p_t(x, x-1) > 0 \text{ and } p_t(x, x-y) = 0 \ \text{ for } \ y \geq 2.$$

We denote by $\mathcal{M}$ be the set of such downward skip-free Markov chains (or transition operators) on E. Note that if a Markov chain is skip-free both downward and upward, it is called a birth-death process. We use the convention that $\mathfrak{l} \in E$ if the boundary point $\mathfrak{l}$ is not absorbing. Otherwise, if $\mathfrak{l}$ is absorbing or $\mathfrak{l} = \infty$, we say that $X \in \mathcal{M}_\infty$. We denote by ζ the lifetime of X. An important problem arising in the analysis of such a process is to characterize the law of its first hitting time and the first passage time, that is the stopping time

$$T_a = \inf\{t > 0; X_t = a\} \tag{3.3}$$

and

$$T_{[b} = \inf\{t > 0; X_t \leq b\}. \tag{3.4}$$

Since X is irreducible, there exists π a positive excessive measure for P, that is, $\pi P_t \leq \pi$ for all $t \geq 0$, see [29, Section 5.2 and 6.8], and that will serve as a reference measure. Let $g_q, q > 0$, be the q-potential kernel of X (or its Green function) with respect to the reference measure π. That is, for any $q > 0$ and $x, y \in E$,

$$g_q(x,y)\pi(y) \;=\; \int_0^\infty e^{-qt}\mathbb{P}_x(X_t = y)dt = \int_0^\infty e^{-qt}p_t(x,y)dt, \;\; x, y \in E. \tag{3.5}$$

The π-dual matrix $\widehat{P}_t = (\widehat{p}_t(y,x))_{y,x\in E}$ is defined as

$$\widehat{p}_t(y,x)\pi(y) \;=\; p_t(x,y)\pi(x). \tag{3.6}$$

Next, denote by T_A the first hitting time of the set $A \subset E$, that is

$$T_A = \inf\{t \geq 0; X_t \in A\},$$

with the usual convention that $\inf\{\emptyset\} = \infty$. If $A = \{a\}$, we write $T_a = T_A$. Similarly, if $A = [a, \mathfrak{r})$, we use $T_{[a} = T_A$. Let $P = (P_t)_{t\geq 0}$ denotes the semigroup of X, it means that for a function f on $\mathbb{N}$,

$$P_t f(x) = \mathbb{E}_x[f(X_t)]. \tag{3.7}$$

It is a well established fact that P defines a Feller semigroup, i.e. P is a Markov semigroup with $P_t C_b(E) \subseteq C_b(E)$ where $C_b(E)$ stands for the space of bounded functions on E. Next, let $G_q, q > 0$, stands for the q-resolvent associated to the semigroup P, that is, for f a bounded function on E and μ a measure on E

$$G_q f(x) \;=\; \int_0^\infty e^{-qt}P_t f(x)dt = \sum_{y\in E} f(y)g_q(x,y)\pi(y) \tag{3.8}$$

$$\mu P(y) \;=\; \sum_{x\in E}\mu(x)p(x,y) \text{ and } \mu P f = \sum_{y\in E} f(y)\mu P(y). \tag{3.9}$$

Note that the duality formula (3.6) reads in terms of q-potential kernel, for any $y, x \in E$, as

$$g_q(x, y) \quad = \quad \hat{g}_q(y, x). \tag{3.10}$$

3.2.2 Martin boundary theory continuous-time Markov chains

In this subsection, we review some essential results of Martin boundary theory of continuous-time Markov chains based on [26, 29, 37]. We start by recalling a few definitions. We write, for $q > 0$,

$$\mathcal{E}_q = \{h : E \to \mathbb{R}^+;\ e^{-qt}P_t h \le h, t \ge 0,\ \lim_{t\to 0} e^{-qt}P_t h = h\}$$

to be the set of q-excessive functions and

$$\mathcal{H}_q = \{h \in \mathcal{E}_q; e^{-qt}P_t h = h, t \ge 0\} \text{ and } \mathcal{P}_q = \{h \in \mathcal{E}_q; \lim_{t\to\infty} e^{-qt}P_t h = 0\}$$

for the set of q-harmonic and q-potential functions on E, respectively. We say that h is harmonic (resp. excessive) if h is 0-harmonic (resp. 0-excessive). We simply write $\mathcal{E}$ (resp. $\mathcal{H}$, $\mathcal{P}$) to denote the set of excessive (resp. harmonic, potential) functions. We will also use the notation $\widehat{\mathcal{E}_q}$ (resp. $\widehat{\mathcal{H}_q}, \widehat{\mathcal{P}_q}$) to denote the set of q-excessive (resp. q-harmonic, q-potential) functions associated with the dual semigroup $\widehat{P}$, which was defined in (3.6). Note that the irreducibility property implies that $h \in \mathcal{E}_q$ is positive on E, since if there exists $y \in E$ such that $h(y) > 0$, then by irreducibility there exists t such that for $x \in E$ $h(x) \ge e^{-qt}p_t(x, y)h(y) > 0$. We point out that if $\hat{h} \in \widehat{\mathcal{E}_q}$, then $\hat{h}\pi$ is a q-excessive measure for P in the sense that $e^{-qt}\ \hat{h}\pi P_t f \le \hat{h}\pi f$. Another commonly used terminology for excessive (resp. harmonic, potential) function is superharmonic (resp. invariant, purely-excessive) function.

We further recall the definition of minimal q-harmonic function. A non-zero function f on E is minimal q-excessive if $f = f_1 + f_2$ implies $f = c_i f_i$ for $i = 1, 2$, where $f_1, f_2 \in \mathcal{E}_q$, c_1 and c_2 are constants and $q > 0$. We say that f is minimal excessive if f is minimal 0-excessive. We write $\mathcal{E}_q^{min}$ (resp. $\mathcal{H}_q^{min}$) to be the set of minimal q-excessive (resp. minimal q-harmonic) functions on E. Next, we state the classical Riesz representation theorem, which gives an unique decomposition of excessive function, the proof can be found in [**?**].

Theorem 3.2.1 (Riesz representation theorem)**.** *For $q \geq 0$, every q-excessive function can be written uniquely as the sum of a q-potential and a q-harmonic function. That is, if $f \in \mathcal{E}_q$, then*

$$f(x) = \sum_{y \in E} g_q(x, y) k_q(y) \pi(y) + h_q(x)$$

where $k_q(x) = f(x) - \lim_{t \to 0} e^{-qt} P_t f(x)$ and $h_q(x) = \lim_{t \to \infty} e^{-qt} P_t f(x) \in \mathcal{H}_q$.

Suppose $(X, \mathbb{P})$ is a transient Markov chain with transition matrix P and reference measure π on a denumerable state space E. For a measure μ on E, define $E_\mu = \{y \in E;\ \mu G(y) > 0\}$, where $\mu G(y) = \sum_{x \in E} \mu(x) G(x, y)$. We say that the measure μ is a *standard measure* if $E_\mu = E$. For $x, y \in E$ and $q \geq 0$, the q-Martin kernel associated to a standard measure μ is defined to be

$$_\mu K_q(x, y) = \tfrac{G_q(x,y)}{\mu G_q(y)}, \tag{3.11}$$

where the denominator is positive since μ is standard. Since X is transient, $_\mu K_q(x, y)$ is finite for any $x, y \in E$. Next, define a metric $_\mu d_q$ on the space E by

$$_\mu d_q(y, z) = \sum_{x \in E} w_x \frac{|_\mu K_q(x, y) - {_\mu K_q}(x, z)| + |\mathbb{1}_{\{x=y\}} - \mathbb{1}_{\{x=z\}}|}{C_x^\mu + 1},$$

where the weights $(w_x)_{x \in E}$, with $w_x > 0$, are chosen such that $\sum_{x \in E} w_x < \infty$, and C_x^μ is a function that depends only on x and satisfies $_\mu K_q(x, y) \leq C_x^\mu$. We can obtain the completion of E with respect to the metric ${}_{\mu}d_q$, namely $\overline{E}$, and the boundary of E in $\overline{E}$ is denoted as $\partial E = \overline{E} - E$. $\overline{E}$ is the Martin compactification of E and ∂E is the Martin boundary. The set

$$\partial_{\mathbb{P}} E = \{y \in \partial E;\ {}_{\mu}K_q(x, y) \text{ is minimal } q\text{-harmonic in } x\}$$

is known as the *minimal Martin boundary*. When there is no ambiguity in the probability measure, we write $\partial_m E = \partial_{\mathbb{P}} E$. The inclusion of the indicator terms $\mathbb{1}_{\{x=y\}}$ and $\mathbb{1}_{\{x=z\}}$ in the metric ${}_{\mu}d_q$ ensures that E is an open set in the Martin compactification $\overline{E}$, and the Martin boundary ∂E is closed. Next, fix a reference point $\mathfrak{o} > \mathfrak{l} + 1$, and we write the q-Martin kernel ${}_{\delta_{\mathfrak{o}}}K_q(x, y)$ as

$$ {}_{\delta_{\mathfrak{o}}}K_q(x, y) \ = \tfrac{\mu G_q(y)}{G_q(\mathfrak{o},y)} {}_{\mu}K_q(x, y) = \tfrac{{}_{\mu}K_q(x,y)}{{}_{\mu}K_q(\mathfrak{o},y)}.$$

Similarly, we have

$$ {}_{\mu}K_q(x, y) \ = \tfrac{G_q(\mathfrak{o},y)}{\mu G_q(y)} {}_{\delta_{\mathfrak{o}}}K_q(x, y) = \tfrac{{}_{\delta_{\mathfrak{o}}}K_q(x,y)}{{}_{\mu}K_q(y)}.$$

A sequence $(y_n)_{n\geq 0}$ is a Cauchy sequence in the metric space $(E, {}_{\mu}d_q)$ if and only if $({}_{\mu}K_q(x, y_n))_{n\geq 0}$ is a Cauchy sequence of real numbers for every x if and only if $({}_{\delta_{\mathfrak{o}}}K_q(x, y_n))_{n\geq 0}$ is a Cauchy sequence of real numbers for every x if and only if $(y_n)_{n\geq 0}$ is a Cauchy sequence in the metric space $(E, {}_{\delta_{\mathfrak{o}}}d_q)$. Thus, up to homeomorphism, $\overline{E}$ is independent of the choice of the reference point $\mathfrak{o}$. It can also be shown that $\overline{E}$ is independent of the choice of the weights $(w_x)_{x\in E}$ (see e.g. Proposition 10.13 in [29]). From now on, we fix the reference point $\mathfrak{o}$ and write

$$K_q(x, y) \ = {}_{\delta_{\mathfrak{o}}}K_q(x, y) = \tfrac{G_q(x,y)}{G_q(\mathfrak{o},y)}.$$

Let $\Omega_\infty = \{\omega;$ there is $x_\infty \in \partial E$ such that $x_t \to x_\infty$ in the Martin topology$\}$. Ω_∞ can be interpreted as the set of *non-terminating* trajectories of X that converges to ∂E. If (X, P) is transient and P is a stochastic matrix then there is a random variable

X_∞ taking values in ∂E such that for each $x \in E$, $\mathbb{P}_x(\lim_{t\to\infty} X_t = X_\infty) = 1$. In terms of trajectory space, we have $\mathbb{P}_x(\Omega_\infty) = 1$. If P is strictly substochastic at some states, we should extend the trajectories to $E \cup \{\nabla\}$, where ∇ is the graveyard state. Define

$$\Omega_\nabla = \{\omega;\ \text{there is } r > 0 \text{ such that } x_t \in E, \forall t \le r \text{ and } x_t = \nabla, \forall t > r\}.$$

Ω_∇ is the set of trajectories that eventually reach ∇. Denote ζ to be the lifetime of X, that is, $\zeta(\omega) = \mathfrak{R}$ for $\omega \in \Omega_\nabla$, where $\mathfrak{R}$ is the last time that X is in the state space E (as defined in Ω_∇), and $\zeta(\omega) = \infty$ otherwise. Define $\Omega_\zeta = \Omega_\nabla \cup \Omega_\infty$. If $(X, \mathbb{P})$ is transient then there is a random variable X_ζ taking values in $\overline{E}$ such that for each $x \in E$, $\mathbb{P}_x\left(\lim_{t\to\zeta} X_t = X_\zeta\right) = 1$. In terms of trajectory space, this means that $\mathbb{P}_x\left(\Omega_\zeta\right) = 1$.

Next, suppose now that $h \in \mathcal{E}_q$. The h-process of X is defined to be the Markov chain on $E^h = \{y \in E;\ h(y) > 0\} = E$, by irreducibility, with the canonical measure $\mathbb{P}^h_x$ such that $\mathbb{P}^h_x(X_t = y) = \frac{p_t(x,y)h(y)}{h(x)}$. Recalling that $\mathfrak{o}$ is the fixed reference point, the q-potential and q-Martin kernels associated with the h-process take respectively the form

$$G^h_q(x,y) = \frac{G_q(x,y)h(y)}{h(x)}, \tag{3.12}$$

$$K^h_q(x,y) = \frac{K_q(x,y)h(\mathfrak{o})}{h(x)}. \tag{3.13}$$

From (3.13) and the definition of the metric ${}_\mu d_q$, we observe that the Martin compactification $\overline{E}$ is homeomorphic to the Martin compactification of the h-process.

Next, we state the following which are the main claims of Theorem 6 and 7 in [26].

Theorem 3.2.2 (Uniqueness of the representation)**.** *Let $q \geq 0$. If $h_q \in \mathcal{E}_q$ such that $h(\mathfrak{d}) = 1$ then h_q has a unique representation of the form*

$$h_q(x) = \int_{E \cup \partial_m E} K_q(x, y)\, \mu_{h_q}(dy) = K_q \mu_{h_q}(x)$$

where $\mu_{h_q}(\cdot) = \mathbb{P}^{h_q}(X_\zeta \in \cdot)$ defines a probability measure. Conversely, for any finite measure μ, the mapping $x \mapsto \int_{E \cup \partial_m E} K_q(x, y)\, d\mu(y)$ defines an q-excessive function, which is q-harmonic if and only if $\mu(E) = 0$. Finally, for all $y \in E \cup \partial_m E$, let $h_q^y(\cdot) = K_q(\cdot, y)$. Then $y \in E \cup \partial_m E$ if and only if $\mathbb{P}^{h_q^y}(X_\zeta = y) = 1$. Moreover we have $\mathbb{P}^{h_q^y}(\zeta = \infty) = 1$ if and only if $y \in \partial_m E$.

The previous claim means that X is forced to terminate at the point $y \in E \cup \partial_m E$ $\mathbb{P}_x^{h_q^y}$-almost surely.

Theorem 3.2.3. *We have $\mathcal{E}_q^{min} = \{h_q;\ h_q(x) = CK_q(x, y), C > 0$ and $y \in E \cup \partial_m E\}$.*

Finally, we recall the following useful result whose proof follows readily from [21, Theorem 11.9].

Lemma 3.2.4. *Suppose that $h_q \in \mathcal{E}_q$, and T is a stopping time with respect to $(\mathcal{F}_t)_{t \geq 0}$. For $x \in E^{h_q}$,*

$$\mathbb{P}_x^{h_q}(T < \infty) = \frac{1}{h_q(x)} \mathbb{E}_x \left[e^{-qT} h_q(X_T) \mathbf{1}_{\{T < \infty\}} \right].$$

3.3 Potential Theory and Fluctuation Identities of continuous-time Skip-free Markov Process

In this section, we provide an explicit representation of the q-potential kernel g_q, of a continuous-time downward skip-free Markov chain $X \in \mathcal{M}$. We start by expressing the q-potential kernel in terms of the so-called fundamental q-excessive (for short FqE) functions of the following three processes: $(X, \mathbb{P})$, $(X, \widehat{\mathbb{P}})$ and $(X, \mathbb{P}^{y]})$, where $(X, \mathbb{P}^{y]})$ is the Markov process $(X, \mathbb{P})$ killed upon entering the half-line $y], \mathfrak{r}]$, which is plainly an downward skip-free Markov chain on the state space $E^{y]} = [[\mathfrak{l}, y)$, with transition kernel denoted by $P^{y]}$. We are now ready to state the two main result of this Section. The detailed proof follows very closed with Choi, Patie [20], which is included in the Appendix.

Theorem 3.3.1. *1. Writing, for any $x \in E$ and $q > 0$,*

$$H_q(x) = K_q \delta_{\mathfrak{l}}(x),$$

(resp. $\widehat{H}_q(x) = \widehat{K}_q \delta_{\mathfrak{r}}(x)$) with $\delta_{\mathfrak{l}}$ is the Dirac mass at $\mathfrak{l}$, we have, for all $q > 0$, that

$$H_q \in \mathcal{E}_q^{min}$$

(resp. $\widehat{H}_q \in \widehat{\mathcal{E}}_q^{min}$) and it is the unique minimal decreasing (resp. increasing) q-excessive for P (resp. $\widehat{P}$) such that $H_q(\mathfrak{o}) = 1$ (resp. $\widehat{H}_q(\mathfrak{o}) = 1$).

Moreover, if $X \in \mathcal{M}_\infty$ (resp. $X \in \mathcal{M} \setminus \mathcal{M}_\infty$) then H_q is the unique decreasing function in $\mathcal{H}_q$ (resp. $H_q \in \mathcal{P}_q$ with $H_q(\mathfrak{l}) < \infty$). We also have the following additional characterization of H_q

$$H_q(x) = \lim_{b \to \mathfrak{l}} \frac{g_q(x, b)}{g_q(\mathfrak{o}, b)} = \lim_{b \to \mathfrak{l}} \frac{\mathbb{E}_x[e^{-qT_b}]}{\mathbb{E}_{\mathfrak{o}}[e^{-qT_b}]}. \tag{3.14}$$

2. *For any $y > \mathfrak{l}$, $0 < \kappa_q^{y]} = \lim_{x \to \mathfrak{l}} \frac{K_q \delta_{\mathfrak{l}}(x)}{K_q^{y]} \delta_{\mathfrak{l}}(x)} < \infty$. Then the function $\mathbf{H}_q^{y]}$ defined, for any $x \in E^{y]}$, by*

$$\mathbf{H}_q^{y]}(x) = \kappa_q^{y]} K_q^{y]} \delta_{\mathfrak{l}}(x)$$

has (with respect to $P^{y]}$) the same property than H_q but with the normalization

$$\lim_{x \to \mathfrak{l}} \frac{\mathbf{H}_q^{y]}(x)}{H_q(x)} = 1$$

(recall that by convention $\mathbf{H}_q^{y]}(x) = 0$ for any $x > y$).

3. *Finally, let* $X \in \mathcal{M}_\infty$ *and set* $C_q = g_q(\mathfrak{d}, \mathfrak{d}) > 0$, *with a fixed reference point* $\mathfrak{d}$. *Then, we have, for all* $x, y \in E$,

$$g_q(x, y) = C_q\, \widehat{H}_q(y)\left(H_q(x) - \mathbf{H}_q^{y]}(x)\right). \tag{3.15}$$

Theorem 3.3.2. 1. *For any* $x, y \in E$, $x \geq y$,

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{H_q(x)}{H_q(y)}.$$

2. *Let* $a \in E$ *and choose a reference point* $\mathfrak{d}^{a]} \in E^{a]}$. *Then, for all* $q > 0$, *the function* $H_q^{a]}(x) = K_q^{a]}\delta_1(x)$ *defined on* $E^{a]}$ *is positive on* $E^{a]}$, *minimal, increasing* q-*harmonic for* $P^{a]}$ *with* $H_q^{a]}(\mathfrak{d}^{a]}) = 1$. *Moreover, for any* $b \leq x \leq a$,

$$\mathbb{E}_x^{a]}[e^{-qT_b}] = \mathbb{E}_x[e^{-qT_b}\mathbb{1}_{\{T_b < T_{a]}\}}] = \frac{H_q^{a]}(x)}{H_q^{a]}(b)}.$$

3. *For any* $x, y \in E$ *and* $q > 0$, *we have*

$$\mathbb{E}_x[e^{-qT_y}] = \frac{H_q(x) - \mathbf{H}_q^{y]}(x)}{H_q(y)}.$$

4. *For all* $q > 0$ *and any* $y \leq x$,

$$\widehat{\mathbb{E}}_y[e^{-qT_x}] = \frac{\widehat{H}_q(y)}{\widehat{H}_q(x)}.$$

5. *Suppose that* $X \in \mathcal{M}_\infty$. *Then, for any* $a \in E$, $x \in E^{a]}$ *and* $q \geq 0$, *we have*

$$\mathbb{E}_x[e^{-qT_{a]}}] = 1 - qC_q \sum_{y \in E^{a]}} \widehat{H}_q(y)\left(\mathbf{H}_q^{a]}(x) - \mathbf{H}_q^{y]}(x)\right)\pi(y),$$

and, for any $b > a$, $x \in E^{(b,a)^c}$,

$$\mathbb{E}_x[e^{-qT_{(a,b)^c}}] = 1 - qC_q \sum_{y \in E^{(b,a)^c}} \widehat{H}_q(y)\left(\frac{\mathbf{H}_q^{y]}(a)}{\mathbf{H}_q^{[b}(a)}\mathbf{H}_q^{a]}(x) - \mathbf{H}_q^{y]}(x)\right)\pi(y).$$

We now turn to the case when $X \in \mathcal{M} \setminus \mathcal{M}_\infty$. We derive an important relationship between the fundamental excessive function associated to the minimal process, that is $X^{[]}$, and the one of X.

Corollary 3.3.3. *Let $X \in \mathcal{M} \setminus \mathcal{M}_\infty$. Then, for any $x \in E$, we have*

$$H_q(x) = \lim_{b \to \mathfrak{l}} \frac{g_q(x,b)}{g_q(b,b)} = \lim_{b \to \mathfrak{l}} \frac{g_q^{[]}(x,b)}{g_q^{[]}(b,b)} = H_q^{[]}(x).$$

3.4 Downward skip-free compound Poisson processes

In this section, we apply the methodology described above to the continuous-time random walks on $\mathbb{Z}$ that are irreducible and downward skip-free. We recall that, for this class of Markov processes, which corresponds to the class of spectrally negative Lévy processes on lattice, this problem has been well studied and has found an impressive range of applications, such as insurance mathematics, epidemiology and queuing theory ..., see [31, 23]. Spitzer [34] solved this problem by means of the celebrated Wiener-Hopf factorization and alternative interesting proofs, based either on excursion theory or martingales devices have been proposed, see again [31, 23] and the references therein. However, all of these approaches rely on the stationarity and independent increments properties of the random walks, making difficult their extension to a wider context. Let now $X = (X_t)_{t \geq 0}$ be a skip-free random walk such that $\mathbb{P}_x(X_0 = x) = 1, x \in \mathbb{Z}$, that is, for any $t > 0$,

$$X_t = X_0 + \sum_{i=1}^{N_t} S_i$$

where $S = (S_i)_{i \geq 1}$ is a sequence of i.i.d. random variables with common distribution F which is supported on $\{y \in \mathbb{Z}; y \geq -1\}$, and $N = (N_t)_{t \geq 0}$ is a Poisson process of parameter $a > 0$ and independent of S. Let

$$\mathcal{G}_F(s) = \sum_{y=-1}^{\infty} s^y \mathbb{P}(S_1 = y),\ 0 < s \leq 1,$$

where, here and for the remaining part of the paper, for a real valued function f on $\mathbb{N}$, we write its generating function by $\mathcal{G}_f(s) = \sum_{y=0}^{\infty} s^y f(y)$. We also write

$$G_a(s) = a(\mathcal{G}_F(s) - 1).$$

Moreover, we assume that $0 < \mathbb{P}(S_1 = -1) < 1$ which entails that X is irreducible. By writing $p_s(x) = s^x, 0 < s \leq 1$, we have, by independency of S and N, that

$$P_t p_s(x) \quad = \quad \mathbb{E}_x[s^{X_t}] = s^x \mathbb{E}_0\left[s^{\sum_{i=1}^{N_t} S_i}\right] = p_s(x) e^{at(\mathcal{G}_F(s)-1)} = p_s(x) e^{tG_a(s)}. \tag{3.16}$$

One easily notes that $\lim_{s\downarrow 0} G_a(s) = \infty, \lim_{s\uparrow 1} G_a(s) = 0$ and, by differentiating, $\frac{d^2}{ds^2}G_a(s) > 0$ on $(0,1)$. Hence, the mapping $s \to G_a(s)$ is continuous, convex and decreasing on $(0, h(0))$, where $h(0)$ is the smallest root of the equation $G_a(s) = 0$. Note that 1 is always a root and $h(0) < 1$ when $\lim_{s\uparrow 1} \frac{d}{ds}G_a(s) > 0$. Therefore, G_a has a well-defined (decreasing) inverse $h : (0,\infty) \to (0, h(0))$. We also recall that, in this case, the dual pocess, with respect to the reference measure $\pi \equiv 1$, is $(X, \widehat{\mathbb{P}}) \stackrel{d}{=} (-X, \mathbb{P})$.

Theorem 3.4.1. *We take the reference point to be* $\mathfrak{d} = 0$ *and let* $q > 0$.

1. $H_q \in \mathcal{H}_q$ *and* $\widehat{H}_q \in \widehat{\mathcal{H}}_q$*, where for* $x \in \mathbb{Z}$,

$$H_q(x) = h^x(q), \quad \widehat{H}_q(x) = h^{-x}(q). \tag{3.17}$$

2. *For any* $y \geq x$*, we have* $\mathbf{H}_q^{y]}(x) = h(q)^y \mathbf{H}_q^{y]}(x-y)$ *where for* $s \in \mathbb{R}$ *such that* $s < h(q)$,

$$\sum_{x>0} s^x \mathbf{H}_q^{0]}(-x) = \frac{1}{C_q(G_a(s) - q)} \tag{3.18}$$

where

$$C_q = \frac{-h'(q)}{h(q)}. \tag{3.19}$$

3. *For any* $x < 0$,

$$\mathbb{E}_x\left[e^{-qT_{0]}}\right] = 1 + qC_q \sum_{y=x+1}^{-1} \mathbf{H}_q^{0]}(y) + \frac{qh'(q)}{1-h(q)}\mathbf{H}_q^{0]}(x).$$

Proof. First, from (3.16) and the definition of h, we get, after noticing that, for any $q > 0$, $x \mapsto H_q(x) = h^x(q) = p_{h(q)}(x)$ is positive on $\mathbb{Z}$,

$$e^{-qt}P_tH_q(x) = e^{-qt}p_{h(q)}(x)e^{tG_a(h(q))} = p_{h(q)}(x) = H_q(x), \tag{3.20}$$

that is $H_q \in \mathcal{H}_q$. Since h has valued in $(0,1)$, $x \mapsto H_q(x)$ is decreasing on $\mathbb{Z}$ for all $q > 0$. As $(X, \widehat{\mathbb{P}}) \stackrel{d}{=} (-X, \widehat{\mathbb{P}})$, the first claims follow readily from Theorem 3.3.1. Next, by means of Tonelli Theorem, we have, for $s \in \mathbb{R}$ such that, $0 < q - G_a(s) < \infty$, which holds at least for $h(q) < s \leq 1$,

$$\sum_{x\in\mathbb{Z}} s^x g_q(0,x) = \int_0^\infty e^{-qt}\mathbb{E}_0\left[s^{X_t}\right]dt = \int_0^\infty e^{-(q-G_a(s))t}dt = \frac{1}{q-G_a(s)}. \tag{3.21}$$

On the other hand, the translation invariant property of X yields that $g_q(-x,0) = g_q(0,x)$ and thus using Theorem 3.3.1(3), one gets

$$\begin{aligned}\sum_{x\in\mathbb{Z}} s^x g_q(-x,0) &= \sum_{x>0} s^x g_q(-x,0) + \sum_{x\geq 0} s^{-x} g_q(x,0)\\ &= \sum_{x>0} s^x C_q\widehat{H}_q(0)\left(H_q(-x) - \mathbf{H}_q^{0]}(-x)\right) + \sum_{x\geq 0} s^{-x}C_q\widehat{H}_q(0)\left(H_q(x) - \mathbf{H}_q^{0]}(x)\right)\\ &= \sum_{x>0} s^x C_q\left(H_q(-x) - \mathbf{H}_q^{0]}(-x)\right) + \sum_{x\geq 0} s^{-x}C_qH_q(x)\end{aligned}$$

where we used in the last identity that $\mathbf{H}_q^{0]}(x) = 0$ for $0 < x$. Rearranging the terms, using (3.21) and the identity $\sum_{x\geq 0} s^{-x}H_q(x) = \frac{s}{s-h(q)}$, yield, for any $h(q) < s \leq 1$,

$$\sum_{x>0} s^x\left(H_q(-x) - \mathbf{H}_q^{0]}(-x)\right) = \frac{1}{C_q(q-G_a(s))} - \frac{s}{s-h(q)}. \tag{3.22}$$

Since, from Theorem 4.3.3(3), the left-hand side can be treated as $\sum_{x>0} s^x\mathbb{E}_{-x}\left[e^{-qT_0}\right] \leq \sum_{x>0} s^x < \infty$ for $|s| < 1$, it is analytical on the unit disc and by the principle of analytical continuation, one gets that

$$C_q = \lim_{s \to h(q)} \frac{1 - h(q)/s}{q - G_a(s)} = \frac{-h'(q)}{h(q)} \tag{3.23}$$

which shows (3.19). Next, from (3.22) and using $\sum_{x \geq 0} s^{-x} H_q(x) = \frac{s}{s-h(q)}$, we get

$$\sum_{x>0} s^x \mathbf{H}_q^{0]}(-x) = \frac{1}{C_q(G_a(s) - q)}. \tag{3.24}$$

Then, note that by the translation invariance property of X, $g_q(x, y) = g_q(x - y, 0)$ for any $x, y \in E$, which after some easy algebra yields $\mathbf{H}_q^{y]}(x) = h^y(q)\mathbf{H}_q^{0]}(x - y)$ for any $x \leq y$. Finally, using the first claim of Theorem 4.3.3(5), we have

$$\begin{aligned}
\mathbb{E}_x[e^{-qT_{0]}}] &= 1 - qC_q \sum_{y \in E^{0]}} \widehat{H}_q(y) \left(\mathbf{H}_q^{0]}(x) - \mathbf{H}_q^{y]}(x) \right) \\
&= 1 - qC_q \sum_{y \in E^{0]}} h^{-y}(q) \left(\mathbf{H}_q^{0]}(x) - h^y(q)\mathbf{H}_q^{0]}(x - y) \right) \\
&= 1 + qC_q \sum_{y=x+1}^{-1} \mathbf{H}_q^{0]}(y) + \frac{qh'(q)}{1 - h(q)} \mathbf{H}_q^{0]}(x)
\end{aligned}$$

which completes the proof of the theorem.

□

3.5 Branching Processes with Immigration

In this section, we apply the methodology developed above to establish the fluctuation theory of branching processes with immigration, which are downward skip-free continuous-time Markov chains. We emphasize that although this class of processes have been intensively studied, only the law of the downward first passage time, that is the one with continuous crossing, has been characterized through its Laplace transform, see [17] and in the continuous state space setting [25]. We start by introducing some notations and preliminary results that will be useful and mention that a classical reference on branching processes is the monograph of Ney and Bran [39]. In section 5.2 we state our main result as the application of Theorem 3.3.1 and Theorem 4.3.3. In the last section 5.3 we present the detailed proof for the main theorem.

3.5.1 Notation and preliminary results

Let us start with $Z = (Z_t)_{t\geq 0}$ a continuous-time Galton-Watson process starting from $x \in \mathbb{N}_0 = \{0, 1, 2, ...\}$ which corresponds to the initial number of particles. The lifetime of each particle is independent to each other and follows a Poisson distribution of parameter $a > 0$. At the end of its life, each particle reproduces independently $k \geq 0$ offspring according to the distribution $\lambda = (\lambda_k)_{k\geq 0}$. Let us write, for $0 \leq s \leq 1$, its probability generating function

$$\mathcal{G}_\lambda(s) = \sum_{k=0}^{\infty} s^k \lambda_k.$$

We also denote by

$$\phi_t^{(x)}(s) = \mathbb{E}_x \left[s^{Z_t} \right]$$

the generating function of the variable Z_t when Z starts from x and we write simply $\phi_t(s) = \phi_t^{(1)}(s)$. Since we exclude the trivial case when the branching process has a.s. increasing sample paths, we have $f(0) > 0$. It is well known, see the aforementioned book [39, III.3(5)], that the mapping $t \mapsto \phi_t(s)$ is the solution to the Riccati equation

$$\frac{d}{dt}\phi_t(s) = F_a(\phi_t(s)), \quad \phi_0(s) = s, \tag{3.25}$$

where

$$F_a(s) = a(\mathcal{G}_\lambda(s) - s). \tag{3.26}$$

Moreover, it is well-known, see e.g. [17], that the branching properties yields that, for any $t, r \geq 0$,

$$\phi_{t+r}(s) = \phi_t \circ \phi_r(s) = \phi_t(\phi_r(s)) \tag{3.27}$$

and, for any $x \in \mathbb{N}_0$,

$$\phi_t^{(x)}(s) = \phi_t^x(s). \tag{3.28}$$

Next, let now $X = (X_t)_{t\geq 0}$ be a continuous-time branching process with immigration (for short CBI). The (random) times at which immigration occurs is independent of the continuous-time Galton-Watson process and follows a Poisson distribution of parameter $b > 0$ (the case $b = 0$ corresponds to the simple branching process). The distribution of the immigrant size is $\nu = (\nu_k)_{k\geq 1}$ we write $\mathcal{G}_\nu(s) = \sum_{k=1}^\infty s^k \nu_k$, for its generating function. When $Y_0 = x \geq 0$, the dynamics of X can be described as follows

$$X_t = Z_t + \sum_{j=1}^{\infty} Z_{t-\tau_j}^{(j)} \mathbb{1}_{\{\tau_j \leq t\}}, \quad t \geq 0,$$

where $Z = (Z_t)_{t\geq 0}$ is a continuous-time Galton-Watson process starting from x, and the τ_j's are the jump times of a Poisson process $N = (N_t)_{t\geq 0}$ with parameter b, the $Z^{(j)}$'s are independent copies of Z, independent of N and $Z_0^{(j)}$ is distributed according to ν. Next, we denote by $\Phi_t(s; x)$ the generating function of $X_t, t \geq 0$, and that is with the notation of (3.7), and, writing $p_s(x) = s^x$ for $x \in \mathbb{N}$, one has, with $0 \leq s \leq 1$,

$$\Phi_t(s; x) = P_t p_s(x) = \mathbb{E}_x[s^{X_t}], \tag{3.29}$$

and we simply write $\psi_t(s) = \Phi_t(s; 0)$. It is well-known, see e.g. [38], that

$$\Phi_t(s; 0) = \psi_t(s) = e^{-b\int_0^t (1-\mathcal{G}_\nu(\phi_u(s)))du}. \tag{3.30}$$

By the branching property we easily get that

$$\Phi_t(s; x) = \phi_t^x(s)\psi_t(s) = \phi_t^x(s) e^{-b\int_0^t (1-\mathcal{G}_\nu(\phi_u(s)))du}. \tag{3.31}$$

Next, recall that for $0 \leq s \leq 1, F_a(s) = a(\mathcal{G}_\lambda(s) - s)$ and thus $F_a(0) = a\mathcal{G}_\lambda(0) > 0$ and $F_a(1) = a(\mathcal{G}_\lambda(1) - 1) = 0$. Furthermore, it is easily seen that the mapping $s \mapsto F_a(s)$ is convex and therefore, $F_a(s)$ has at most one root in the interval $(0, 1)$. Denote this unique root by s_0 if it exists, otherwise let $s_0 = 1$. Then, the mapping $s \mapsto F_a(s)$ is convex and non-increasing on $(0, s_0)$.

We proceed by providing some general results on CBI processes which we will be useful in the sequel. We point out some of these results, which are the discrete analogue of the work of Ogura [32, 33], differ from the classical presentation on CBI processes and, for the sake of completeness, we provide the detailed proofs in the discrete-state space framework.

Proposition 3.5.1. *Let Z be a continuous-time branching process.*

(1) For $0 \leq s \leq 1$,

$$\lim_{t\to\infty} \phi_t(s) = s_0$$

where we recall that s_0 *is the smallest root of* $s \mapsto F_a(s)$ *on* $(0, 1]$.

(2) There exist a non-negative stationary measure π *on* $\mathbb{N}$*, i.e.* $\pi P_t = \pi$*, whose generating function* $\mathcal{G}_\pi$ *is given, for any* $0 < s < s_0$*, by*

$$\mathcal{G}_\pi(s) = \int_0^s \frac{dr}{F_a(r)}. \tag{3.32}$$

Moreover, $\mathcal{G}_\pi$ *satisfies the functional equation, for any* $t \geq 0$ *and* $0 < s < s_0$,

$$\mathcal{G}_\pi(\phi_t(s)) = \mathcal{G}_\pi(s) + t. \tag{3.33}$$

(3) ϕ_t *has the following representation*

$$\phi_t(s) = \mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s) + t) \tag{3.34}$$

where $\mathcal{G}_\pi^{-1}$ *is the inverse function of* $\mathcal{G}_\pi$*, showing that it has the semigroup property.*

Remark 3.5.2. $\mathcal{G}_\pi$ maps $(0, s_0)$ onto $(0, \infty)$ monotonously and there exists a well-defined inverse function $\mathcal{G}_\pi^{-1}$ mapping $(0, \infty)$ onto $(0, s_0)$ monotonously.

Proof. From (3.25), we have that the mapping $t \mapsto \frac{d}{dt}\phi_t(s)$ is positive (resp. negative) if $0 < \phi_t(s) < s_0$ (resp. $s_0 < \phi_t(s) < 1$) implying that $t \mapsto \phi_t(s)$ is non-decreasing (resp. non-increasing) if $0 < s < s_0$ (resp. $s_0 < s < 1$). The limit exist as $\phi_t(s)$ is monotone and bounded. In addition, $\lim_{t\to\infty} \phi_t'(s) = 0$. If $\lim_{t\to\infty} \phi_t(s) = s_0' \neq s_0$, by (3.25), $0 = \lim_{t\to\infty} \phi_t'(s) = F_a(s_0') \neq 0$, which is a contradiction. Next in order to prove Proposition 4.4.1(2), note (3.25) implies $\int_s^{\phi(s)} \frac{dr}{F_a(r)} = t$, with $\mathcal{G}_\pi(s) = \int_0^s \frac{dr}{F_a(r)}$, for $s < s_0$, we have

$$\mathcal{G}_\pi(\phi_t(s)) = \int_0^{\phi_t(s)} \frac{dr}{F_a(r)} = \int_0^s \frac{dr}{F_a(r)} + \int_s^{\phi_t(s)} \frac{dr}{F_a(r)} = \mathcal{G}_\pi(s) + t.$$

On the other hand, by means of Tonelli Theorem, one gets

$$\begin{aligned}
\mathcal{G}_\pi(\phi_t(s)) &= \sum_{x=0}^{\infty} \phi_t^x(s)\pi(x) = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} s^y P_t(x,y)\pi(x) \\
&= \sum_{x=0}^{\infty}\sum_{y=1}^{\infty} s^y P_t(x,y)\pi(x) + \sum_{x=0}^{\infty} P_t(x,0)\pi(x) \\
&= \sum_{y=0} s^y \left(\sum_{x=0}^{\infty} P_t(x,y)\pi(x)\right) + \sum_{x=0}^{\infty} \pi(x)\phi_t^x(0) \\
&= \sum_{y=0} s^y \left(\sum_{x=0}^{\infty} P_t(x,y)\pi(x)\right) + \mathcal{G}_\pi\left(\phi_t(0)\right) \\
&= \sum_{y=0} s^y \left(\sum_{x=0}^{\infty} P_t(x,y)\pi(x)\right) + t.
\end{aligned}$$

π is a stationary measure by definition. Finally, as $\mathcal{G}_\pi$ has a well defined inverse as explained in Remark 5.1, the expression (4.61) is obvious from (4.60). □

Proposition 3.5.3. *let X be a CBI process and let $\alpha = b(1 - \mathcal{G}_\nu(s_0)) \geq 0$ (note that $\alpha = 0$ if X is a simple branching process, i.e. $b = 0$, or $s_0 = 1$).*

(1) *For* $0 \le s \le 1$*,*

$$\lim_{t\to\infty} \phi_t(s) = s_0.$$

(2) *A nonnegative measure* π_α *on* $\mathbb{N}_0$ *is an* α*-stationary measure, i.e.* $\pi_\alpha P_t = e^{-\alpha t}\pi_\alpha$ *for all* $t \ge 0$*, if and only if its generating function* $\mathcal{G}_{\pi_\alpha}(s) < \infty$ *for all* $0 < s < s_0$ *and satisfies the functional equation*

$$\mathcal{G}_{\pi_\alpha}(s) = e^{\alpha t}\psi_t(s)\mathcal{G}_{\pi_\alpha}(\phi_t(s)). \tag{3.35}$$

(3) *There exists a unique, up to a multiplicative constant,* α*-stationary measure whose generating function* $\mathcal{G}_{\pi_\alpha}$ *has the following expression, for* $0 < s < s_0$*,*

$$\mathcal{G}_{\pi_\alpha}(s) = e^{\int_0^s \frac{b(1-\mathcal{G}_\nu(r))-\alpha}{F_a(r)}dr}. \tag{3.36}$$

Hence, π_α *is a reference measure for X.*

(4) *The generating function of X takes the form, for any* $0 < s < s_0$*,*

$$\Phi_t(s;x) = \phi_t^x(s)\psi_t(s) \tag{3.37}$$

where $\phi_t(s) = \mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s)+t))$ *and* $\psi_t(s) = \frac{e^{-\alpha t}\mathcal{G}_{\pi_\alpha}(s)}{\mathcal{G}_{\pi_\alpha}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s)+t)))}$*.*

Remark 3.5.4. Without lose of generality, we can assume that $s_0 = 1$. Otherwise there is an equivalent transformation as stated in Proposition 1.9 in [18] to transfer s_0 to 1. Therefore here we can set $\alpha = b(1 - \mathcal{G}_\nu(1)) = 0$.

Proof. For the stationary measure π_α, (4.64) follows from the definition of α-stationary measure by taking the generating function on both sides. If π_α is an α-stationary measure, $\mathcal{G}_{\pi_\alpha}(r) < \infty$ for some $0 < r < 1$. By Proposition 4.4.1(1), $\phi_t(r) \to s_0$ as $t \to \infty$, therefore $\mathcal{G}_{\pi_\alpha}(s) < \infty$ for all $0 < s < s_0$ by (4.64), which concludes the first part of the proposition. Using the same techniques in the proof of Proposition 4.4.1(2), the converse is obvious. Next (4.65) follows from (4.64), (3.25) and the expression of $\psi_t(s)$. By differentiating both sides of (4.64) at $t = 0$, we get

$$\mathcal{G}'_{\pi_\alpha}(s)F_a(s) - b\mathcal{G}_{\pi_\alpha}(s)\left(1 - \mathcal{G}_\nu(s)\right) = -\alpha\mathcal{G}_{\pi_\alpha}(s).$$

The differential equation above has a unique solution in the form of (4.64) up to a constant multiple. Finally, the explicit expression of $\Phi_t(s; x)$ follows readily from (4.61) and (4.64). □

3.5.2 First exit-time identities for branching processes with immigration

We now provide a characterization of the law of the two-sided exit times for branching processes with immigration. Although this an intensively studied and central class of Markov processes, only the Laplace transform of the downward first passage time, that is the easiest case without overshoot, can be found in the literature, see [17, 36] and [25] for the continuous state space analogue. This example reveals the comprehensive aspect of the methodology presented in this paper. We present first the representation or characterization of the key objects (fundamental excessive functions and constants) that determined the green functions, and, later we In addition we compute the expression for first hitting time $\mathbb{E}[e^{-qT_y}]$ and first passage time $\mathbb{E}[e^{-qT_{[b}}]$.

Theorem 3.5.5. *Let $q > 0$ and $x \in \mathbb{N}_0$.*

(1) Write, for all $t \geq 0$, $\Phi_t(x) = \Phi_t(0; x) = \phi_t^x \psi_t$ where we have set $\phi_t = \phi_t(0)$ and

$\psi_t = \psi_t(0)$. *Note that* $\mathbb{P}_x(X_t = 0) = \Phi_t(x) > 0$ *and let*

$$H_q(x) = C_q^{-1} \int_0^\infty e^{-qt}\Phi_t(x)dt$$

where $C_q = \int_0^\infty e^{-qt}\Phi_t(0)dt$. *Then,* H_q *is the unique decreasing q-purely excessive function such that* $H_q(0) = 1$ *and it is minimal.*

(2) *Let* $\widehat{H}_q$ *be defined on* $\mathbb{N}_0$ *by*

$$\widehat{H}_q(y) = \frac{\gamma_q \star \pi_\alpha(y)}{\pi_\alpha(y)}$$

where $\star$ *stands for the additive convolution and* γ_q *is the discrete measure on* $\mathbb{N}_0$ *defined by*

$$\gamma_q(y) = \sum_{k=0}^{\infty} \pi^{\star k}(y)\frac{q^k}{k!}$$

with $\pi^{\star k}$ *the* k^{th}*-convolution of the stationary measure* π *and the series is convergent for all* $y, q \in \mathbb{N}_0$. *Then,* $\widehat{H}_q$ *is the unique increasing q-invariant function for* $\widehat{P}$.

(3) *Given* $x, y \in \mathbb{N}_0$, *let* $\mathbf{H}_{q,x}(y) = \widehat{H}_q(y)\mathbf{H}_q^{y]}(x)$, *the generating function of* $\pi\mathbf{H}_{q,x}$ *takes the form*

$$\mathcal{G}_{\pi_\alpha \mathbf{H}_{q,x}}(s) = H_q(x)\, e^{\int_0^s \frac{b(1-\mathcal{G}_\nu(r))+q-\alpha}{F_\alpha(r)}dr} - C_q^{-1}H_q(s;x)$$

where $H_q(s;x) = \int_0^\infty e^{-qr}\Phi_r(s;x)dr$. *Note that* $H_q(0;x) = H_q(x)$.

Remark 3.5.6. From Theorem 4.3.1(3), given a specific CBI process we can invert the generating function $\mathcal{G}_{\pi_\alpha \widehat{\mathbf{H}}_{q,x}}$ and get the explicit expression for $\widehat{\mathbf{H}}_q^{y]}(x)$, as illustrated in section 6.

Theorem 3.5.7. *For any* $a, x \in \mathbb{N}_0$ *such that* $x < a$ *and* $q > 0$, *we have*

$$\mathbb{E}_x\left[e^{-qT_{a]}}\right] \;=\; 1 - qC_q\left(\mathbf{H}_q^{a]}(x)\sum_{y=0}^{a-1}\gamma_q \star \pi_\alpha(y) - \sum_{y=x+1}^{a-1}\gamma_q \star \pi_\alpha(y)\mathbf{H}_q^{y]}(x)\right).$$

Remark 3.5.8. Since the process is downward skip-free, we have $T_{a]} = T_a$, the finite summation of $\widehat{H}_q(y)\pi(y)$ and $\widehat{H}_q(y)\mathbf{H}_q^{y]}(x)\pi(y)$ can be computed from Theorem 4.3.1(2) and Theorem 4.3.1(3). In particular, for $x = b-1$,

$$\mathbb{E}_x[e^{-qT_{[b}}] = 1 - qC_q\left(\mathbf{H}_q^{a]}(x)\sum_{y=0}^{b-1}\gamma_q \star \pi_\alpha(y)\right).$$

Corollary 3.5.9. *Let $x, y \in \mathbb{N}_0$, $x < y, 0 < s < s_0$ and $q > 0$. Let $\mathfrak{g}_s$ be the discrete random variable on $\mathbb{N}_0$ defined by*

$$\mathbb{P}(\mathfrak{g}_s = y) = \frac{s^y g_q(y,y)\pi_\alpha(y)}{\sum_{y=0}^{\infty} s^y g_q(y,y)\pi_\alpha(y)}. \tag{3.38}$$

Assuming that $\mathfrak{g}_s$ is independent of Y, then we have

$$\mathbb{E}_x[e^{-qT_{\mathfrak{g}_s}}] = \frac{1}{C_{\mathfrak{g}}(q;s)}\int_0^\infty e^{-qr}\Phi_r(s;x)dr \tag{3.39}$$

where

$$C_{\mathfrak{g}}(q;s) = \sum_{y=0}^{\infty} s^y g_q(y,y)\pi_\alpha(y) = C_q\int_0^\infty e^{-qr}\psi_r\, \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_r)dr. \tag{3.40}$$

3.5.3 Proofs of the results of Section 4.2

Proof of Theorem 4.3.1(1)

First, recalling that $\Phi_t(s;x) = \mathbb{E}_x[s^{X_t}]$ and $\Phi_t(x) = \Phi_t(0;x)$, we get, for any $t > 0$ and $x \in \mathbb{N}_0$,

$$\Phi_t(x) = \mathbb{P}_x(X_t = 0) \geq \mathbb{P}_x(\tau_1 > t)\,\mathbb{P}_x(Z_t = 0) = e^{-bt}\mathbb{P}_x(Z_t = 0) > 0. \tag{3.41}$$

Moreover the strong Markov property combines with the downward skip-free property entails that, for any $x > y > 0$, $\Phi_t(x) = \mathbb{P}_x\left(X_t = 0, T_y < t\right) = \mathbb{P}_x\left(T_y < t\right)\Phi_t(y) > \Phi_t(y)$, that is

$$x \mapsto \Phi_t(x) \text{ is decreasing.} \tag{3.42}$$

Next, since X is an irreducible continuous-time downward skip-free Markov process on $\mathbb{N}_0$, one can use Theorem 3.3.1(1) with $\mathrm{I} = 0 \in \mathbb{N}_0$, to express the function H_q (and choosing 0 as the reference point, that is $H_q(0) = 1$), for any $x \in \mathbb{N}_0$ and $q > 0$, as follows

$$H_q(x) = \lim_{y \to 0} \frac{g_q(x, y)}{g_q(0, y)} = \lim_{s \to 0} \frac{G_q p_s(x)}{G_q p_s(0)} = \frac{G_q p_0(x)}{G_q p_0(0)} \tag{3.43}$$

where, using the notation of (3.9), we have

$$G_q p_s(x) = \int_0^\infty e^{-qt} P_t p_s(x) dt = \int_0^\infty e^{-qt} \Phi_t(s; x) dt \tag{3.44}$$

which is clearly valid for any $|s| < 1$ as, for all $t \geq 0$ and $x \in \mathbb{N}_0$, $|\Phi_t(s; x)| \leq 1$ on the unit disc. Hence, combining (3.43) with (3.44), we obtain that

$$H_q(x) = C_q^{-1} \int_0^\infty e^{-qt} \Phi_t(x) dt$$

and $C_q^{-1} = \int_0^\infty e^{-qt} \Phi_t(0) dt$. We proceed by showing that $H_q \in \mathcal{P}_q$. First, from (3.41), we deduce that $H_q > 0$ on $\mathbb{N}_0$. Next, the expression of $\psi_t(s)$ in (3.30) and the semigroup property (3.27) yield, for all $t, r \geq 0$,

$$\begin{aligned}
\psi_r \psi_t(\phi_r) &= \psi_r(0) \psi_t(\phi_r(0)) = \psi_r(0) e^{-b \int_0^t 1 - \mathcal{G}_\nu(\phi_u(\phi_r(0))) du} \\
&= \psi_r(0) e^{-b \int_0^t 1 - \mathcal{G}_\nu(\phi_{u+r}(0)) du} = e^{-b \int_0^r 1 - \mathcal{G}_\nu(\phi_u(0)) du} e^{-b \int_r^{t+r} 1 - \mathcal{G}_\nu(\phi_u(0)) du} \\
&= e^{-b \int_0^{t+r} 1 - \mathcal{G}_\nu(\phi_u(0)) du} = \psi_{t+r}(0) = \psi_{t+r}.
\end{aligned} \tag{3.45}$$

This combines with an application of Tonelli Theorem and the definition of $\Phi_t(s; x)$ in (3.31) give

$$\begin{aligned}
C_q P_t H_q(x) &= \int_0^\infty e^{-qr} \psi_r P_t p_{\phi_r}(x) dr = \int_0^\infty e^{-qr} \phi_t^x(\phi_r) \psi_r \psi_t(\phi_r) dr \\
&= \int_0^\infty e^{-qr} \phi_{t+r}^x \psi_{t+r} dr = e^{qt} \int_t^\infty e^{-qr} \Phi_r(x) dr \leq e^{qt} C_q H_q(x)
\end{aligned}$$

where we used successively (3.27) and (3.45), performed an obvious change of variable and used the fact that for all $x \in \mathbb{N}_0$, the mapping $r \mapsto \Phi_r(x)$ is non-negative on $\mathbb{R}^+$. We deduce from the last identity that $P_t H_q(x) \leq e^{qt} H_q(x)$ because $C_q > 0$ for $q > 0$, as $\Phi_t(0) > 0$ for all t. Finally, we clearly have $\lim_{t\downarrow 0} e^{-qt} P_t H_q(x) = \lim_{t\downarrow 0} C_q^{-1} \int_t^\infty e^{-qr}\Phi_r(x)dr = H_q(x)$ and by monotone convergence $\lim_{t\to\infty} e^{-qt} P_t H_q(x) = \lim_{t\to\infty} \int_t^\infty e^{-qr}\Phi_r(x)dr = 0$. Being a positive function, this shows that $H_q \in \mathcal{P}_q$ and, since from (3.42), it is decreasing, we complete the proof of Theorem 4.3.1(1) after invoking Theorem 3.3.1(1).

Proof of Theorem 4.3.1(2)

To get the expression of $\widehat{H}_q$, we shall need the following lemmas.

Lemma 3.5.10. *For any $q > 0$, the mapping $\mathcal{G}_{\pi_\alpha \widehat{H}_q}$ defined on $(0, s_0)$ by*

$$\mathcal{G}_{\pi_\alpha \widehat{H}_q}(s) = e^{q\mathcal{G}_\pi(s)} \mathcal{G}_{\pi_\alpha}(s) = e^{\int_0^s \frac{b(1-\mathcal{G}_\nu(r))+q-\alpha}{Fa(r)} dr} \tag{3.46}$$

is the generating function of $\pi \widehat{H}_q$, where $\widehat{H}_q$ is the unique increasing function in $\widehat{\mathcal{H}}_q$ and it is minimal.

Remark 3.5.11. Note that our approach enables to deduce that the function $\widehat{H}_q$ defines through the generating $\mathcal{G}_{\pi \widehat{H}_q}$ is positive and decreasing, the last property does not seem easy to derive from its representation given in Lemma 3.5.12 below.

Proof. As above, using (the dual formulation of) Theorem 3.3.1(1), we have, for all $q > 0$,

$$\widehat{H}_q(y) = \lim_{x\to\infty} \frac{\widehat{g}_q(y, x)}{\widehat{g}_q(0, x)} = \lim_{x\to\infty} \frac{g_q(x, y)}{g_q(x, 0)} \tag{3.47}$$

where we choose again 0 as the reference point, meaning in particular that $\widehat{H}_q$ is normalized such that $\widehat{H}_q(0) = 1$. However, note that for any $|s| < 1$,

$$\lim_{x\to\infty}\sum_{y\geq 0} s^y \frac{g_q(x,y)}{g_q(x,0)} = \lim_{x\to\infty}\sum_{0\leq y\leq x} s^y \frac{g_q(x,y)}{g_q(x,0)} + \lim_{x\to\infty}\sum_{y>x} s^y \frac{g_q(x,y)}{g_q(x,0)} = \lim_{x\to\infty}\sum_{0\leq y\leq x} s^y \frac{g_q(x,y)}{g_q(x,0)} \tag{3.48}$$

since $\sum_{y\geq 0} s^y \frac{g_q(x,y)}{g_q(x,0)}$ is absolutely convergent for $|s| < 1$ and any $x \in \mathbb{N}_0$. Next, by means of Theorem 3.3.1(3) and recalling that $\mathbf{H}_q^{y]}(x) = 0$ for $y < x$, combined with the identity $g_q(x,y) = C_q H_q(x)\widehat{H}_q(y)$, we get

$$\begin{aligned}\lim_{x\to\infty}\sum_{y\geq 0} s^y \frac{g_q(x,y)}{g_q(x,0)} &= \lim_{x\to\infty}\sum_{0\leq y\leq x} s^y \frac{g_q(x,y)}{g_q(x,0)} = \lim_{x\to\infty}\sum_{0\leq y\leq x} s^y \frac{C_q\widehat{H}_q(y)H_q(x)}{C_qH_q(x)} \\ &= \lim_{x\to\infty}\sum_{0\leq y\leq x} s^y\widehat{H}_q(y) = \sum_{y\geq 0} s^y\widehat{H}_q(y).\end{aligned}$$

Thus, we have, writting $\bar{\pi}_\alpha(y) = \frac{\pi_\alpha(y)}{\pi_\alpha(0)}$,

$$\begin{aligned}\mathcal{G}_{\bar{\pi}_\alpha\widehat{H}_q}(s) &= \lim_{x\to\infty}\sum_{y\geq 0} s^y \frac{g_q(x,y)}{g_q(x,0)}\frac{\pi_\alpha(y)}{\pi_\alpha(0)} = \lim_{x\to\infty}\frac{G_q p_s(x)}{g_q(x,0)\pi_\alpha(0)} \\ &= \lim_{x\to\infty}\frac{\int_0^\infty e^{-qt}\Phi_t(s;x)dt}{\int_0^\infty e^{-qt}\Phi_t(x)dt}.\end{aligned}$$

Performing a change of variable $r = \phi_t(s) = \mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s) + t)$, with the expression (4.65) in Proposition 4.4.4(2), one gets $t = \mathcal{G}_\pi(r) - \mathcal{G}_\pi(s)$ and

$$\begin{aligned}\mathcal{G}_{\bar{\pi}_\alpha\widehat{H}_q}(s) &= \lim_{x\to\infty}\frac{\int_0^\infty e^{-qt}\phi_t^x(s)\psi_t(s)dt}{\int_0^\infty e^{-qt}\phi_t^x(0)\psi_t(0)dt} = \lim_{x\to\infty}\frac{\int_0^\infty e^{-qt}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s)+t))^x\frac{e^{-\alpha t}\mathcal{G}_{\pi_\alpha}(s)}{\mathcal{G}_{\pi_\alpha}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s)+t)))}dt}{\int_0^\infty e^{-qt}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(0)+t))^x\frac{e^{-\alpha t}\mathcal{G}_{\pi_\alpha}(0)}{\mathcal{G}_{\pi_\alpha}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(0)+t)))}dt} \\ &= \lim_{x\to\infty}\frac{\int_s^{s_0} e^{-q(\mathcal{G}_\pi(r)-\mathcal{G}_\pi(s))}r^x\frac{\mathcal{G}_{\pi_\alpha}(s)}{\mathcal{G}_{\pi_\alpha}(r)}\mathcal{G}'_\pi(r)dr}{\int_0^{s_0} e^{-q(\mathcal{G}_\pi(r))}r^x\frac{\mathcal{G}_{\pi_\alpha}(0)}{\mathcal{G}_{\pi_\alpha}(r)}\mathcal{G}'_\pi(r)dr} \\ &= e^{q\mathcal{G}_\pi(s)}\frac{\mathcal{G}_{\pi_\alpha}(s)}{\mathcal{G}_{\pi_\alpha}(0)}\lim_{x\to\infty}\frac{\int_s^{s_0} e^{-q\mathcal{G}_\pi(r)}r^x\frac{\mathcal{G}'_\pi(r)}{\mathcal{G}_{\pi_\alpha}(r)}dr}{\int_0^{s_0} e^{-q\mathcal{G}_\pi(r)}r^x\frac{\mathcal{G}'_\pi(r)}{\mathcal{G}_{\pi_\alpha}(r)}dr}.\end{aligned}$$

Next, observing that the maximum of $r \mapsto \ln r$ on $[0, s_0]$ is at the point s_0, an application of the Laplace's approximation yields that

$$\lim_{x\to\infty}\frac{\int_s^{s_0} e^{-q\mathcal{G}_\pi(r)}r^x\frac{\mathcal{G}'_\pi(r)}{\mathcal{G}_{\pi_\alpha}(r)}dr}{\int_0^{s_0} e^{-q\mathcal{G}_\pi(r)}r^x\frac{\mathcal{G}'_\pi(r)}{\mathcal{G}_{\pi_\alpha}(r)}dr} = 1.$$

Finally, since $\mathcal{G}_{\pi_\alpha}(0) = \pi_\alpha(0)$, from Proposition 4.4.1(2) and Proposition 4.4.4(3), we have, for any $0 < s < s_0$,

$$\mathcal{G}_{\pi_\alpha\widehat{H}_q}(s) = e^{q\mathcal{G}_\pi(s)}\mathcal{G}_{\pi_\alpha}(s) = e^{q\int_0^s\frac{1}{F_\alpha(r)}dr}e^{\int_0^s\frac{b(1-\mathcal{G}_\nu(r))-\alpha}{F_\alpha(r)}dr} \tag{3.49}$$

which completes the proof of the lemma. □

We proceed by providing an explicit representation of the function $\widehat{H}_q$ by inverting the generating function given in Lemma 3.5.10.

Lemma 3.5.12. *We have on* $\mathbb{N}_0$

$$\widehat{H}_q(y) = \frac{\gamma_q \star \pi_\alpha(y)}{\pi_\alpha(y)}.$$

Proof. Since $\mathcal{G}_\pi(s) = \int_0^s \frac{dr}{F_a(r)} < \infty$ for any $0 < s < s_0$, we can apply Tonelli theorem to get, for any $q > 0$,

$$e^{q\mathcal{G}_\pi(s)} = \sum_{k=0}^{\infty} \mathcal{G}_\pi^k(s)\frac{q^k}{k!} = \sum_{k=0}^{\infty}\sum_{y=0}^{\infty} s^y \pi^{\star k}(y)\frac{q^k}{k!} = \sum_{y=0}^{\infty} s^y \gamma_q(y)$$

where we recall that $\gamma_q(y) = \sum_{k=0}^{\infty} \pi^{\star k}(y)\frac{q^k}{k!}$ and the series is absolutely convergent for $y \in \mathbb{N}_0$ and $q \in \mathbb{R}$. We complete the proof by observing that $\mathcal{G}_{\pi_\alpha \widehat{H}_q}(s) = e^{q\mathcal{G}_\pi(s)}\mathcal{G}_{\pi_\alpha}(s)$ is the generating function of the convolution of γ_q and π_α. □

The proof of Theorem 4.3.1(2) will completed after the following claim.

Lemma 3.5.13. $\widehat{H}_q \in \mathcal{H}_q$.

Proof. First, using twice Lemma 3.5.10, observe that, for any $0 < s < s_0$, $t > 0$ and $q > 0$,

$$\psi_t(s)\mathcal{G}_{\pi_\alpha \widehat{H}_q}(\phi_t(s)) = \psi_t(s)e^{q\mathcal{G}_\pi(\phi_t(s))}\,\mathcal{G}_{\pi_\alpha}(\phi_t(s)) = e^{q\mathcal{G}_\pi(s)}e^{qt}\mathcal{G}_{\pi_\alpha}(s) = \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s)e^{qt} \quad (3.50)$$

where we used, for the second identity, the relations (4.60) and (4.64). Next, define the duality product $\langle f, g\rangle_{\pi_\alpha} = \sum_{y\geq 0} f(y)g(y)\pi_\alpha(y)$ and recall that $p_s(y) =$

$s^y, y \geq 0$. Then, for any $0 < s < s_0$, $t > 0$ and $q > 0$, we get, using successively the duality relation and (3.37),

$$\begin{aligned}
\langle \widehat{P}_t \widehat{H}_q, p_s \rangle_{\pi_\alpha} &= \langle \widehat{H}_q, P_t p_s \rangle_{\pi_\alpha} = \sum_{y \geq 0} \widehat{H}_q(y) P_t p_s(y) \pi_\alpha(y) = \sum_{y \geq 0} \widehat{H}_q(y) \phi_t^y(s) \psi_t(s) \pi_\alpha(y) \\
&= \psi_t(s) \sum_{y \geq 0} \widehat{H}_q(y) \phi_t^y(s) \pi_\alpha(y) = \psi_t(s) \mathcal{G}_{\pi_\alpha \widehat{H}_q}(\phi_t(s)) = e^{qt} \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s) \\
&= e^{qt} \langle \widehat{H}_q, p_s \rangle_{\pi_\alpha}.
\end{aligned}$$

$$\begin{aligned}
\langle \widehat{P}_t p_r, p_s \rangle_{\pi_\alpha} &= \langle p_r, P_t p_s \rangle_{\pi_\alpha} = \sum_{y \geq 0} r^y P_t p_s(y) \pi_\alpha(y) = \sum_{y \geq 0} r^y \phi_t^y(s) \psi_t(s) \pi_\alpha(y) \\
&= \psi_t(s) \sum_{y \geq 0} (r\phi_t(s))^y \pi_\alpha(y) = \psi_t(s) \mathcal{G}_{\pi_\alpha}(r\phi_t(s)) = e^{qt} \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s) \\
&= e^{qt} \langle \widehat{H}_q, p_s \rangle_{\pi_\alpha}.
\end{aligned}$$

where we used for the identity before the last one the relation (3.50). By injectivity of the generating function, we obtain, for all $t \geq 0, q > 0$ and $y \in \mathbb{N}_0$,

$$\widehat{P}_t \widehat{H}_q(y) = e^{qt} \widehat{H}_q(y). \tag{3.51}$$

Since plainly $\lim_{t \downarrow 0} e^{-qt} P_t \widehat{H}_q(y) = \widehat{H}_q(y)$ and $\widehat{H}_q$ is positive, we have proved that $\widehat{H}_q \in \mathcal{H}_q$. □

Proof of Theorem 4.3.1(3)

First, using (3.44), one gets that, for any $0 < s < s_0$,

$$H_q(s; x) = \sum_{y=0}^{\infty} s^y g_q(x, y) \pi_\alpha(y) = G_q p_s(x) = \int_0^\infty e^{-qt} \Phi_t(s; x) dt.$$

On the other hand, using Theorem 3.3.1(3) and the notation of Theorem 4.3.1(3), for any given $x \in \mathbb{N}_0, q > 0$ and $0 < s < s_0$, one has

$$\begin{aligned} H_q(s;x) &= \sum_{y=0}^{\infty} s^y g_q(x,y)\pi_\alpha(y) = \sum_{y=0}^{\infty} s^y \left[C_q \widehat{H}_q(y)(H_q(x) - \mathbf{H}_q^{y]}(x)) \right] \pi_\alpha(y) \\ &= C_q H_q(x) \sum_{y=0}^{\infty} s^y \widehat{H}_q(y)\pi_\alpha(y) - C_q \sum_{y=0}^{\infty} s^y \widehat{H}_q(y)\mathbf{H}_q^{y]}(x)\pi_\alpha(y) \\ &= C_q H_q(x)\mathcal{G}_{\pi_\alpha \widehat{H}_q}(s) - C_q \mathcal{G}_{\pi_\alpha \mathbf{H_{q,x}}}(s). \end{aligned}$$

We complete the proof after some easy algebra and using the expression (3.46) of $\mathcal{G}_{\pi_\alpha \widehat{H}_q}$.

Proof of Theorem 3.5.7

The direct application of Theorem 4.3.3(5) and simple algebra provide the result.

Proof of Theorem 3.5.9

An application of Tonelli Theorem and Theorem 3.3.1(3) yield that, for $q > 0, 0 < s < s_0$,

$$\begin{aligned} C_g(q;s) &= \sum_{y=0}^{\infty} s^y g_q(y,y)\pi_\alpha(y) = \sum_{y=0}^{\infty} s^y C_q H_q(y)\widehat{H}_q(y)\pi_\alpha(y) \\ &= C_q \sum_{y=0}^{\infty} s^y \int_0^\infty e^{-qr}\Phi_r(y)dr\widehat{H}_q(y)\pi_\alpha(y) = C_q \sum_{y=0}^{\infty} s^y \int_0^\infty e^{-qr}\phi_r^y\psi_r dr\widehat{H}_q(y)\pi_\alpha(y) \\ &= C_q \int_0^\infty e^{-qr}\psi_r \sum_{y=0}^{\infty} s^y \phi_r^y \widehat{H}_q(y)\pi_\alpha(y)dr \\ &= C_q \int_0^\infty e^{-qr}\psi_r \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_r)dr \end{aligned}$$

Since $r \mapsto \psi_r \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_r)$ is plainly bounded on $\mathbb{R}^+$, we deduce that $C_g(q;s) < \infty$. Next, fix s and let $h_s(r) = \psi_r \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_r)$ for $r \geq 0$. Note that $\psi_0 = 1$ and $\mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_0) =$

$\mathcal{G}_{\pi_\alpha \widehat{H}_q}(0) = \pi_\alpha(0)\widehat{H}_q(0) < \infty$ because $\widehat{H}_q(0) = 1$ and π_α is a finite measure, we get $h_s(0)$ is finite. In addition, $\lim_{r\to\infty} \phi_r = s_0$ from Proposition 4.4.1(1). We get $\lim_{r\to\infty} \psi_r = \lim_{r\to\infty} e^{-b\int_0^r (1-\mathcal{G}_\nu(\phi_u(0)))du} < \infty$, and $\lim_{r\to\infty} \mathcal{G}_{\pi_\alpha \widehat{H}_q}(s\phi_r) = \mathcal{G}_{\pi_\alpha \widehat{H}_q}(ss_0) < \infty$, we get $\lim_{r\to\infty} h_s(r) < \infty$. Because $h_s(r)$ is continuous and finite at 0 and infinity, $h_s(r)$ is bounded, and $C_{\mathfrak{g}}(q;s)$ is finite as the Laplace transform of a bounded function. Therefore $\mathfrak{g}_s$ is indeed a random variable. Applying Theorem 3.3.1(3) and Theorem 4.3.1(3) we get

$$\begin{aligned}
\mathbb{E}_x\left[e^{-qT_{\mathfrak{g}_s}}\right] &= \sum_{y=0}^{\infty} \mathbb{P}(\mathfrak{g}_s = y)\mathbb{E}_x\left[e^{-qT_y}\right] = \sum_{y=0}^{\infty} \mathbb{P}(\mathfrak{g}_s = y)\, \frac{g_q(x,y)}{g_q(y,y)} \\
&= \sum_{y=0}^{\infty} \mathbb{P}(\mathfrak{g}_s = y)\, \frac{C_q\widehat{H}_q(y)(H_q(x) - \mathbf{H}_q^{y]}(x))}{g_q(y,y)} \\
&= \frac{1}{C_{\mathfrak{g}}(q;s)} \sum_{y=0}^{\infty} s^y \pi_\alpha(y) C_q \widehat{H}_q(y)(H_q(x) - \mathbf{H}_q^{y]}(x)) \\
&= \frac{1}{C_{\mathfrak{g}}(q;s)} \left(C_q H_q(x)\mathcal{G}_{\pi_\alpha \hat{H}_q}(s) - C_q \mathcal{G}_{\pi_\alpha \mathbf{H}_{q;x}}(s)\right) = \frac{1}{C_{\mathfrak{g}}(q;s)} H_q(s;x) \\
&= \frac{1}{C_{\mathfrak{g}}(q;s)} \int_0^\infty e^{-qr}\Phi_r(s;x)dr
\end{aligned}$$

So we completes the proof.

3.6 Example of Neveu Branching Process in continuous-time

We consider the Neveu branching process and refer to Huillet [28] for a thorough study. We use this example to illustrate the computation of our main result. A Neveu process is a continuous-time branching process on $\mathbb{N}_0$ characterized by its branching mechanism, for $0 < \lambda < 1$ and $0 \le s \le 1$,

$$f(s) = (1-\lambda) + \lambda\left((1-s)\log(1-s) + s\right) \tag{3.52}$$

which gives, according to (3.26), for any $a > 0$,

$$F_a(s) = a(f(s) - s) = a(1 - s)\left((1 - \lambda) + \lambda \log(1 - s)\right). \tag{3.53}$$

One observes that $0 < s_0 = 1 - e^{-\overline{\lambda}^{-1}} < 1$, where $\overline{\lambda} = \frac{\lambda}{1-\lambda} > 0$. By expanding $F_a(s)$ one get that the rates λ_k at which one individual gives birth to $k \neq 1$ individual are

$$\lambda_0 = \mu(1 - \lambda), \quad \lambda_k = \frac{a\lambda}{k(k-1)}, \; k \geq 2. \tag{3.54}$$

As before, let $\phi_t(s) = P_t p_s(1) = \mathbb{E}_1[s^{Z_t}],\; t \geq 0$. By the Riccati equation (3.25), $\phi_t(s)$ can be solved explicitly as

$$\phi_t(s) = 1 - e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda a t})}(1 - s)^{e^{-\lambda a t}}. \tag{3.55}$$

$$\phi_t = \phi_t(0) = 1 - e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda a t})}. \tag{3.56}$$

$$\lim_{r\to\infty} \frac{\phi_r}{\phi_{t+r}} = 1. \tag{3.57}$$

We have the following results for the Neveu process.

Proposition 3.6.1. *Denote* $\bar{\lambda} = \frac{\lambda}{1-\lambda} > 0$, $c_k = \bar{\lambda}k > 0$ *and* $\beta_q = \frac{q}{\lambda a} > 0$.

(1) For $x \in \mathbb{N}_0$, the expression of H_q can be written as

$$H_q(x) \;=\; \sum_{k=0}^{x} \binom{x}{k} (-1)^k I_{\beta_q}(-c_k)$$

where $I_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k \Gamma(\beta+1)}{\Gamma(\beta+k+1)}$ *is the Bessel function.*

(2) For $y \in \mathbb{N}_0$, we have

$$\pi(y)\widehat{H}_q(y) \;=\; \sum_{k=0}^{\infty} \binom{-\beta_q}{k} (-\bar{\lambda})^k \upsilon^{*k}(y),$$

and the stationary measure can be expressed as

$$\pi(y) \quad = \quad \frac{\bar{\lambda}}{a\lambda} \sum_{k=1}^{\infty} \frac{\upsilon^{*k}(y)}{k},$$

where $\upsilon^{*1}(y) = \upsilon(y) = 1/y, \quad for\ y \in \mathbb{N}$ *and* υ^{*k} *is the* k^{th} *convolution of* υ*.*

(3) For $x, y \in \mathbb{N}_0, x \leq y$*, we have*

$$\pi(y)\widehat{\mathbf{H}}_q^{y]}(x) = \left(H_q(x) \sum_{k=0}^{\infty} \binom{-\beta_q}{k} (-\bar{\lambda})^k \upsilon^{*k}(y) - \Gamma(\beta_q + 1) \sum_{k=0}^{x} \binom{x}{k} (-1)^k \sum_{j=0}^{\infty} \frac{k^j (\omega^{*j} \star u_k)(y)}{\Gamma(\beta_q + n + 1)}\right)$$

where ω *is the measure on* $\mathbb{N}_0$ *defined by* $\omega(0) = -\frac{1-\lambda}{\lambda}, \omega(y) = \frac{1}{y}$ *for* $y \in \mathbb{N}$*.* u_k *is the measure on* $\mathbb{N}$ *defined by* $u_k(y) = (-1)^y \binom{k}{y}$ *for* $y \in \mathbb{N}$*.*

(4) For any $x = b - 1$*,* $b > 1$ *and any* $q \geq 0$*, the first passage time is*

$$\mathbb{E}_x[e^{-qT_{[b}}] \quad = \quad 1 - \sum_{y=0}^{b-1} \sum_{n=0}^{\infty} \binom{-\beta_q}{k} (-\bar{\lambda})^k \upsilon^{*k}(y)$$

(5) For any $x \geq 0, q > 0$ *and* $T_{\mathfrak{g}_s}$ *the random stopping time defined in Theorem 3.5.9, we have*

$$\mathbb{E}_x[e^{-qT_{\mathfrak{g}_s}}] \quad = \quad q \frac{1 + \sum_{k=1}^{x} \binom{x}{k} (-1)^k I_{\beta_q}(-d_k(s))}{\int_0^{\infty} e^{-qr} \left(1 + \bar{\lambda} \log(1 - s\phi_r)\right)^{-\beta_q} dr}$$

where $d_k(s) = \frac{k}{\bar{\lambda}} + k \log(1 - s)$*.*

Proof. Proposition 3.6.1(1) was proved in Avram et al. [17]. To show the claim of Proposition 3.6.1(2), we first apply Theorem 4.3.1(2) to get, for $0 \leq s < 1 - e^{-\bar{\lambda}^{-1}}, \bar{\lambda} = \frac{\lambda}{1-\lambda} > 0$,

$$\begin{aligned}
\mathcal{G}_{\pi\widehat{H}_q}(s) &= \exp\left(q \int_0^s \frac{1}{F_a(r)} dr\right) = \exp\left(\frac{q}{a\lambda} \int_0^s \frac{1}{(1-r)(\bar{\lambda}^{-1} + \log(1-r))} dr\right) \\
&= \exp\left(\frac{-q}{a\lambda}\left(\log\left(\bar{\lambda}^{-1} + \log(1-s)\right) + \log\bar{\lambda}\right)\right) \\
&= \left[1 + \bar{\lambda}\log(1-s)\right]^{-\frac{q}{a\lambda}} = \left[1 + \bar{\lambda}\log(1-s)\right]^{-\beta_q} \\
&= \sum_{k=0}^{\infty} \binom{-\beta_q}{k} \bar{\lambda}^k (\log(1-s))^k = \sum_{k=0}^{\infty} \binom{-\beta_q}{k} (-\bar{\lambda})^k \left(s + \frac{s^2}{2} + \frac{s^3}{3} + \ldots\right)^k
\end{aligned}$$

where $\beta_q = \frac{q}{a\lambda} > 0$. Next, define the measure $\upsilon(y) = \frac{1}{y}$ for $y \in \mathbb{N}$. By term-by-term inversion we get

$$\pi(y)\widehat{H}_q(y) = \sum_{k=0}^{\infty} \binom{-\beta_q}{k} (-\bar{\lambda})^k \upsilon^{*k}(y).$$

Then, for $0 \le s < 1 - e^{-\bar{\lambda}^{-1}}$, by (3.53) and (4.59) we can compute

$$\begin{aligned}
\mathcal{G}_\pi(s) &= \sum_{y=0}^{\infty} s^y \pi(y) = \int_0^s \frac{1}{F_a(r)} dr = \int_0^s \frac{1}{a((1-\lambda)(1-r) + \lambda(1-r)\log(1-r))} dr \\
&= -\frac{1}{a\lambda} \log\Big(1 + \frac{\lambda}{1-\lambda} \log(1-s)\Big) = -\frac{1}{a\lambda} \log\Big(1 + \bar{\lambda} \log(1-s)\Big) \\
&= -\frac{1}{a\lambda} \log\Big(1 - \bar{\lambda}(s + \frac{s^2}{2} + \frac{s^3}{3} + \ldots)\Big).
\end{aligned}$$

Another term-by-term inversion yields for $y \in \mathbb{N}$,

$$\pi(y) = \frac{1}{a\lambda} \bar{\lambda}\Big(\upsilon + \frac{\upsilon_2}{2} + \frac{\upsilon_3}{3} + \ldots\ldots\Big) = \frac{1}{a((1-\lambda)} \sum_{k=1}^{\infty} \frac{\upsilon^{*k}(y)}{k}$$

To prove Proposition 3.6.1(3), we apply Theorem 4.3.1(3) to get for any $x \in \mathbb{N}$,

$$\begin{aligned}
\mathcal{G}_{\pi \mathbf{H_{q,x}}}(s) &= H_q(x)\mathcal{G}_{\pi \hat{H}_q}(s) - \frac{1}{C_q} \int_0^\infty e^{-qr} \phi_r^x(s) dr \\
&= H_q(x)\mathcal{G}_{\pi \hat{H}_q}(s) - q \int_0^\infty e^{-qr} \phi_r^x(s) dr.
\end{aligned}$$

We proceed by computing $H_q(s; x) = q \int_0^\infty e^{-qr} \phi_r^x(s) dr$. First, note that

$$\int_0^\infty e^{-qr} \phi_r^x(s) dr = \int_0^\infty e^{-qr} (1 - e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda\mu r})} (1-s)^{e^{-\lambda\mu r}}) dr.$$

Writing $\overline{\lambda} = \frac{\lambda}{1-\lambda}$, which is a constant between 0 and 1, we get

$$\begin{aligned}
\int_0^\infty e^{-qr} (1 - e^{-\frac{1-\lambda}{\lambda}(1-e^{-\lambda a r})} (1-s)^{e^{-\lambda a r}}) dr &= \int_0^\infty e^{-qr} \sum_{k=0}^{x} \binom{x}{k} (-1)^k e^{-\frac{1}{\bar{\lambda}} k (1-e^{-\lambda a r})} (1-s)^{k e^{-\lambda a r}} dr \\
&= \sum_{k=0}^{x} \binom{x}{k} (-1)^k e^{-\frac{1}{\bar{\lambda}} k} \int_0^\infty e^{-qr + \frac{1}{\bar{\lambda}} k e^{-\lambda a r}} (1-s)^{k e^{-\lambda a r}} dr.
\end{aligned}$$

Observe that, with $d_k(s) = \frac{k}{\lambda} + k\log(1-s), k \geq 1$, letting $u = e^{-\lambda\mu r}$, we have

$$\begin{aligned} q\int_0^\infty e^{-qr+\frac{1}{\lambda}ke^{-\lambda ar}}(1-s)^{ke^{-\lambda ar}}dr &= q\int_0^1 u^{\frac{q}{\lambda a}}e^{d_k(s)u}\frac{1}{\lambda au}du = \frac{q}{\lambda a}\int_0^1 u^{\frac{q}{\lambda a}-1}e^{d_k u}du \\ &= \frac{q}{\lambda a}(-d_k(s))^{-\frac{q}{\lambda a}}\gamma(\frac{q}{\lambda a}, -d_k(s)) \\ &= \frac{q}{\lambda\mu}(-d_k(s))^{-\frac{q}{\lambda a}}(-d_k(s))^{\frac{q}{\lambda a}}\Gamma(\frac{q}{\lambda a})e^{d_k(s)}\sum_{j=0}^\infty \frac{(-d_k(s))^j}{\Gamma(\frac{q}{\lambda a}+j+1)} \\ &= e^{d_k(s)}I_{\beta_q}(-d_k(s)). \end{aligned}$$

Therefore we get the expression

$$H_q(s;x) = \sum_{k=0}^x \binom{x}{k}(-1)^k(1-s)^k I_{\beta_q}(-d_k(s)).$$

By means of a term-by-term inversion, we obtain

$$\pi(y)\widehat{\mathbf{H}}_q^{y]}(x) = \left(H_q(x)\sum_{k=0}^\infty \binom{-\beta_q}{k}(-\bar{\lambda})^k \upsilon^{*k}(y) - \Gamma(\beta_q+1)\sum_{k=0}^x \binom{x}{k}(-1)^k \sum_{j=0}^\infty \frac{k^j(\omega^{*j}\star u_k)(y)}{\Gamma(\beta_q+n+1)}\right).$$

Next, we turn to the proof of Proposition 3.6.1(4). By Theorem 3.5.7 and following remark, for any $b > 1$, $x = b-1$ and any $q \geq 0$,

$$\begin{aligned} \mathbb{E}_x\left[e^{-qT_{[b}}\right] &= 1 - qC_q\mathbf{H}_q^{a]}(x)\sum_{y=0}^{b-1}\widehat{H}_q(y)\pi(y) \\ &= 1 - q\frac{1}{q}\sum_{y=0}^{b-1}\sum_{k=0}^\infty \binom{-\beta_q}{k}(-\bar{\lambda})^k\upsilon^{*k}(y) \\ &= 1 - \sum_{y=0}^{b-1}\sum_{k=0}^\infty \binom{-\beta_q}{k}(-\bar{\lambda})^k\upsilon_k(y). \end{aligned}$$

Lastly to show the result of Proposition 3.6.1(5), we use Theorem 3.5.9 to get

$$\begin{aligned} \mathbb{E}_x[e^{-qT_{\mathfrak{g}_s}}] &= \frac{1}{C_{\mathfrak{g}_s}}\int_0^\infty e^{-qr}\Phi_r(s;x)dr \\ &= q\frac{1 + \left(\frac{q}{\lambda a}\right)\sum_{k=1}^x \binom{x}{k}(-1)^k\Gamma(\frac{q}{\lambda a})\sum_{j=0}^\infty \frac{(-d_k)^j}{\Gamma(\frac{q}{\lambda a}+j+1)}}{\int_0^\infty e^{-qr}\left(1+\bar{\lambda}\log(1-s\phi_r)\right)^{-\beta_q}dr}, \end{aligned}$$

which concludes the whole proof. ◻

3.7 Appendix: Proof of Theorem 3.3.1 and Theorem 4.3.3

3.7.1 Proof of Theorem 3.3.1

We proceed with the proof of these statements which is split into several intermediate results. We start with the following result which relates the Martin kernel to the hitting time distribution.

Lemma 3.7.1. *For any $x, y \in E$,*

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{K_q\delta_y(x)}{K_q\delta_y(y)}.$$

Proof. By Theorem 3.2.1, for any $y \in E$, $K_q\delta_y \in \mathcal{P}_q$ which leads, by Theorem 3.2.2, to $\mathbb{P}^{K_q\delta_y}(X_\zeta = y, \zeta < \infty) = 1$, that is, for any $x, y \in E$,

$$\mathbb{P}_x^{K_q\delta_y}(T_y < +\infty) = 1.$$

Since, on the other hand, by Lemma 3.2.4, we have, for any $x, y \in E$,

$$\mathbb{P}_x^{K_q\delta_y}(T_y < +\infty) = \mathbb{E}_x\left[e^{-qT_y}\right]\frac{K_q\delta_y(y)}{K_q\delta_y(x)}$$

we complete the proof. □

Proof of Theorem 3.3.1(1)

Suppose that $x \wedge \mathfrak{o} \geq y \geq b$, where $x \wedge \mathfrak{o} = \min\{x, \mathfrak{o}\}$. By means of Lemma 3.7.1, the downward skip-free property and the strong Markov property, we obtain that

$$K_q\delta_b(x) = \frac{\mathbb{E}_x\left[e^{-qT_b}\right]}{\mathbb{E}_\mathfrak{o}\left[e^{-qT_b}\right]} = \frac{\mathbb{E}_x\left[e^{-qT_y}\right]\mathbb{E}_y\left[e^{-qT_b}\right]}{\mathbb{E}_\mathfrak{o}\left[e^{-qT_y}\right]\mathbb{E}_y\left[e^{-qT_b}\right]} = K_q\delta_y(x). \tag{3.58}$$

Thus, for any $y \leq x \wedge \mathfrak{o}$, $K_q\delta_y(x) = K_q(x, x \wedge \mathfrak{o})$ and one can trivially define the function H_q as the extended Martin kernel at $\mathfrak{l}$, that is, for $x \in E$,

$$H_q(x) = \lim_{y \to \mathfrak{l}} K_q(x, y) = \int_{E \cup \partial_m E} K_q(x, y)\delta_{\mathfrak{l}}(dy). \tag{3.59}$$

Hence if $X \in \mathcal{M}_\infty$ (resp. $X \in \mathcal{M} \setminus \mathcal{M}_\infty$) then $\mathfrak{l} \in \partial_{\mathbb{P}} E$ (resp. $\mathfrak{l} \in E$) and thus by theorems 3.2.2 and 3.2.3 (resp. and Theorem 3.2.1), we get that $H_q \in \mathcal{H}_q \cap \mathcal{E}_q^{min}$ (resp. $H_q \in \mathcal{P}_q \cap \mathcal{E}_q^{min}$). Next note, from the first identity in (3.58), that $H_q(\mathfrak{o}) = 1$, and, for $x \geq y$, $H_q(x) = K_q\delta_y(x)$. Hence, by Lemma 3.7.1 we have, for any $x \geq y$,

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{H_q(x)}{H_q(y)}, \tag{3.60}$$

which entails, by the irreducibility of X, that H_q is positive everywhere since for any $x \in E$, the ratio $H_q(x) = \frac{\mathbb{E}_x\left[e^{-qT_b}\right]}{\mathbb{E}_{\mathfrak{o}}\left[e^{-qT_b}\right]} > 0$. To see that the mapping $x \mapsto H_q(x)$ is decreasing, one observes from again the strong Markov property and the downward skip-free property that (recall that $x \geq y \geq b$)

$$H_q(x) = \frac{\mathbb{E}_x\left[e^{-qT_y}\right]\mathbb{E}_y\left[e^{-qT_b}\right]}{\mathbb{E}_{\mathfrak{o}}\left[e^{-qT_b}\right]} = \mathbb{E}_x\left[e^{-qT_y}\right]H_q(y) < H_q(y).$$

To prove the uniqueness, we proceed by contradiction and thus assume that there exists a positive function $h_q \in \mathcal{E}_q^{min}$ (resp. in $\mathcal{H}_q$ when $X \in \mathcal{M}_\infty$) which differs from H_q and which is also an decreasing function on E. Then, according to Theorem 3.2.3, there exists $y_0 \in E$ (resp. $y_0 = \mathfrak{r}$ or $y_0 = \mathfrak{l}$) such that for all $x \in E$, $h_q(x) = \frac{K_q(x,y_0)}{K_q(o,y_0)}$. Thus, on the one hand, from Lemma 3.7.1 we deduce that for any $x \geq y_0$,

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{K_q\delta_{y_0}(x)}{K_q\delta_{y_0}(y_0)} = \frac{h_q(x)}{h_q(y_0)}.$$

This combines with (3.60) yield that $h_q(x) = H_q(x)$ for any $x \geq y_0$, which proves the claim when $H_q \in \mathcal{H}_q$ and $y_0 = \mathfrak{l}$. In the other cases, choose $x < y_0$ such that $h_q(x) \neq H_q(x)$. Then, observe from Theorem 3.2.2 that $\mathbb{P}_x^{h_q}(T_{y_0} < +\infty) = 1$. As $\mathbb{P}_x^{h_q}(T_{y_0} < +\infty) = \frac{h_q(y_0)}{h_q(x)}\mathbb{E}_x\left[e^{-qT_{y_0}}\right]$ and h_q is decreasing and $x < y_0$, we get that

$\mathbb{E}_x\left[e^{-qT_{y_0}}\right] > 1$ which is impossible. This completes the uniqueness property of H_q. Note that we use similar arguments for deriving the stated properties of $\widehat{H}_q$ after recalling that the dual chain $(X, \widehat{\mathbb{P}})$ is irreducible and upward skip-free. To complete the proof of Theorem 3.3.1(1), we observe from (3.61), that

$$H_q(x) = \lim_{y\to\mathfrak{l}} K_q(x,y) = \lim_{y\to\mathfrak{l}} \frac{g_q(x,y)}{g_q(\mathfrak{o},y)} = \lim_{y\to\mathfrak{l}} \frac{\mathbb{E}_x\left[e^{-qT_y}\right]}{\mathbb{E}_{\mathfrak{o}}\left[e^{-qT_y}\right]}. \tag{3.61}$$

Proof of Theorem 3.3.1(2)

We start with the following claim which is a straightforward reformulation of Theorem 3.3.1(1) for the killed process.

Lemma 3.7.2. *Let $a \in E$ and choose a reference point $\mathfrak{o}^{a]} \in E^{a]} = [[\mathfrak{l}, a[$. Then, for all $q > 0$, the function $H_q^{a]}(x) = K_q^{a]}\delta_{\mathfrak{l}}(x)$ defined on $E^{a]}$ is positive on $E^{a]}$, minimal, decreasing q-harmonic for $P^{a]}$ with $H_q^{a]}(\mathfrak{o}^{a]}) = 1$. Moreover, for any $a \geq x \geq b$,*

$$\mathbb{E}_x^{a]}\left[e^{-qT_b}\right] = \mathbb{E}_x\left[e^{-qT_b}\mathbb{1}_{\{T_b<T_{a]}\}}\right] = \frac{H_q^{a]}(x)}{H_q^{a]}(b)}.$$

Proof. Under $\mathbb{P}^{a]}$, X is an irreducible and downward skip-free Markov chain on $E^{a]}$, the results follows from Theorem 3.3.1(1) and the identity (3.60). □

The following lemma provides an expression for the Laplace transform of the *upward* hitting times $(T_b, \mathbb{P}_x^{a]})$, where $b < x < a$, in terms of the FqE functions of $(X, \mathbb{P})$ and $(X, \mathbb{P}^{a]})$.

Lemma 3.7.3. *For any $b < x < a$ and $q > 0$, we have*

$$\mathbb{E}_x\left[e^{-qT_a}\mathbb{1}_{\{T_a<T_b\}}\right] = \frac{H_q(x)}{H_q(a)} - \frac{H_q(b)}{H_q(a)}\frac{H_q^{a]}(x)}{H_q^{a]}(b)}.$$

Proof. Since by definition $H_q = K_q \delta_{\mathfrak{l}}$, we have, from Theorem 3.2.2 that,

$$\mathbb{P}^{H_q}(X_\zeta = \mathfrak{l}) = 1. \tag{3.62}$$

This combines with the downward skip-free property, see Lemma 3.2.4, yields that under $\mathbb{P}^{H_q}$ the sample paths of X that goes above b before hitting a must hit b before reaching a. That is

$$\mathbb{P}^{H_q}_x(T_a < T_b) = \mathbb{P}^{H_q}_x(T_{a]} < T_b) = 1 - \mathbb{P}^{H_q}_x(T_b < T_{a]}) \tag{3.63}$$

where we used again (3.62) for the second identity. Hence, an application of Lemma 3.2.4 gives

$$\mathbb{P}^{H_q}_x(T_a < T_b) = \frac{H_q(a)}{H_q(x)}\mathbb{E}_x\left[e^{-qT_a}\mathbb{1}_{\{T_a<T_b\}}\right] = 1 - \frac{H_q(b)}{H_q(x)}\mathbb{E}_x\left[e^{-qT_b}\mathbb{1}_{\{T_b<T_{a]}\}}\right] = 1 - \frac{H_q(b)}{H_q(x)}\frac{H_q^{a]}(x)}{H_q^{a]}(b)}$$

where the last equality follows from Lemma 3.7.2. Rearranging the terms provides the desired result. □

The proof of Theorem 3.3.1(2) follows readily after the following claim.

Lemma 3.7.4. *For $x \in E^{a]}$, define*

$$\kappa_q^{a]}(x) = \frac{H_q(x)}{H_q^{a]}(x)}.$$

Then the mapping $x \mapsto \kappa_q^{a]}(x)$ is non-decreasing on $E^{a]}$ with $0 < \kappa_q^{a]}(x) < \infty$. Furthermore, $0 < \kappa_q^{a]} = \lim_{x\to\mathfrak{l}} \kappa_q^{a]}(x) < \infty$.

Proof. It is clear that, for all $x \in E^{a]}$, $0 < \kappa_q^{a]}(x) < \infty$, since both H_q and $H_q^{a]}$ are positive and finite on $E^{a]}$. Next, for any $x \in E^{a]}$ and $t \geq 0$,

$$e^{-qt}P_t^{a]}H_q(x) \leq e^{-qt}P_tH_q(x) \leq H_q(x)$$

where the first inequality follows from the fact that $P_t^{a]}$ is the restriction of P_t to $E^{a]}$, and we use that $H_q \in \mathcal{E}_q$ in the second inequality. Therefore, H_q (restricted on $E^{a]}$) is q-excessive for $P_t^{a]}$. Thus, one may define the Doob H_q-transform of $e^{-qt}P_t^{a]}$ by

$$ {}^{H_q}P_t^{a]}(x,y) = \frac{H_q(y)}{H_q(x)} e^{-qt} P_t^{a]}(x,y) $$

where $x, y \in \{x \in E^{a]} : H_q(x) > 0\} = E^{a]}$ by Theorem 3.7.1. Using Lemma 3.2.4 and Lemma 3.7.2, we have, for any $x \geq b$,

$$ {}^{H_q}\mathbb{P}_x^{a]}(T_b < \infty) = \frac{H_q(b)}{H_q(x)} \mathbb{E}_x^{a]}[e^{-qT_b}] = \frac{H_q^{a]}(x)}{H_q(x)} \frac{H_q(b)}{H_q^{a]}(b)} = \frac{\kappa_q^{a]}(b)}{\kappa_q^{a]}(x)}. \tag{3.64} $$

Since $(X, {}^{H_q}\mathbb{P}^{a]})$ is plainly an irreducible transient downward skip-free Markov chain, using Lemma 3.7.1 and Theorem 3.3.1(1) for $q = 0$, one easily deduces, with the obvious notation, that $\frac{1}{C\kappa_q^{a]}(x)} = {}^{H_q}K^{a]}\delta_{\mathfrak{r}}(x)$, for some $C > 0$. Thus the mapping $x \mapsto \kappa_q^{a]}(x)$ is non-decreasing. Henceforth $\kappa_q^{a]} = \lim_{x \to \mathfrak{l}} \kappa_q^{a]}(x) \leq \kappa_q^{a]}(a-1) < \infty$. Observing that both $H_q(\mathfrak{l})$ and $H_q^{a]}(\mathfrak{l})$ are finite when $X \in \mathcal{M} \setminus \mathcal{M}_\infty$, we readily get that $\kappa_q^{a]} > 0$ which completes the proof in this case. It remains to show that

$$ \lim_{b \to \mathfrak{l}} \frac{\kappa_q^{a]}(b)}{\kappa_q^{a]}(x)} = \lim_{b \to \mathfrak{l}} {}^{H_q}\mathbb{P}_x^{a]}(T_b < \infty) > 0 \tag{3.65} $$

when $X \in \mathcal{M}_\infty$. To this end, we assume the contrary, that is, $\lim_{b \to \mathfrak{l}} {}^{H_q}\mathbb{P}_x^{a]}(T_b < \infty) = 0$. Since ${}^{H_q}\mathbb{P}_x^{a]}(T_b < \infty) = \mathbb{P}_x^{H_q}(T_b < T_{a]})$, the assumption becomes $\lim_{b \to \mathfrak{l}} \mathbb{P}_x^{H_q}(T_b < T_{a]}) = 0$ and (3.63) leads to

$$ \lim_{b \to \mathfrak{l}} \mathbb{P}_x^{H_q}(T_b < T_a) = 0. \tag{3.66} $$

Next, let τ be the first time of jump, which follows an exponential distribution with parameter $\mathfrak{e} > 0$. Conditioning on the state of first jump and using the downward skip-free property, we obtain, writing $T_a^{(1)} = \inf\{t > \tau;\ X_t = a\}$ for the return time to a,

$$ \begin{aligned} \mathbb{P}_a^{H_q}(T_a^{(1)} < T_b) &= \mathbb{P}_a^{H_q}(X_\tau = a-1)\mathbb{P}_{a-1}^{H_q}(T_a^{(1)} < T_b) + \mathbb{P}_a^{H_q}(X_\tau = a) \\ &\quad + \sum_{y > a} \mathbb{P}_a^{H_q}(X_\tau = y)\mathbb{P}_y^{H_q}(T_a^{(1)} < T_b). \end{aligned} \tag{3.67} $$

Taking $b \to \mathfrak{l}$, the left-hand side converges to $\mathbb{P}_b^{H_q}(T_a^{(1)} < \infty)$ due to the monotone convergence theorem, the downward skip-free property and the fact that $X \in \mathcal{M}_\infty$. On the right-hand side of (3.67), the first term converges to $\mathbb{P}_a^{H_q}(X_\tau = a-1)$ as a result of (3.66), while the third term converges to $\sum_{y>a} \mathbb{P}_b^{H_q}(X_\tau = y)\mathbb{P}_y^{H_q}(T_a^{(1)} < \infty)$ by invoking the dominated convergence theorem. Therefore, we arrive at

$$\begin{aligned}\mathbb{P}_a^{H_q}(T_a^{(1)} < \infty) &= \mathbb{P}_a^{H_q}(X_\tau = a-1) + \mathbb{P}_a^{H_q}(X_\tau = a) + \textstyle\sum_{y>a} \mathbb{P}_a^{H_q}(X_\tau = y)\mathbb{P}_y^{H_q}(T_a^{(1)} < \infty) \\ &= \mathbb{P}_a^{H_q}(X_\tau = a-1) + \textstyle\sum_{y>a} \mathbb{P}_a^{H_q}(X_\tau = y) = \mathbb{P}_a^{H_q}(\zeta > 1) = 1,\end{aligned}$$

where the second equality comes from the identity $\mathbb{P}_y^{H_q}(T_a^{(1)} < \infty) = 1$ which holds since $\mathbb{P}_y^{H_q}(X_\zeta = \mathfrak{l}) = 1$ and $y > a$, while the last equality is due to Theorem 3.2.2 with the fact that $H_q \in \mathcal{H}_q$ since $X \in \mathcal{M}_\infty$. This is not possible since X is transient. Therefore, we conclude that $\kappa_q^{a]} > 0$. □

3.7.2 Proof of Theorem 3.3.1(3)

We start with the following extension of Lemma 3.7.3.

Lemma 3.7.5. *For any $x, y \in E$ and $q > 0$, we have*

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{H_q(x) - \mathbf{H}_q^{y]}(x)}{H_q(y)}. \tag{3.68}$$

Proof. The case when $x > y$ is proved in (3.60). Next assume that $x \le y$. Thanks to the downward skip-free property of X, for any $a \in E$ the mapping $b \mapsto \mathbb{1}_{\{T_a < T_b\}}$ is increasing for b large enough. Then, the monotone convergence theorem and the fact that $X \in \mathcal{M}_\infty$ give that $\lim_{b\to\mathfrak{l}} \mathbb{E}_x[e^{-qT_a}\mathbb{1}_{\{T_a<T_b\}}] = \mathbb{E}_x[e^{-qT_a}]$. The sought result follows immediately from Lemma 3.7.3 and Lemma 3.7.4. □

We are now ready to prove the expression (3.15). First using (3.60), Lemma 3.7.1 and the definition of the Martin kernel in (3.11), we obtain, for any $x \geq \mathfrak{o}$,

$$\mathbb{E}_x\left[e^{-qT_\mathfrak{o}}\right] = H_q(x) = \frac{G_q(x,\mathfrak{o})}{G_q(\mathfrak{o},\mathfrak{o})} = \frac{G_q(x,\mathfrak{o})}{C_q}. \tag{3.69}$$

Next, for sake of clarity we state the analogue of the identity (3.60) for the dual chain $(X, \widehat{\mathbb{P}})$.

Lemma 3.7.6. *For all $q > 0$ and any $x \geq y$,*

$$\widehat{\mathbb{E}}_y\left[e^{-qT_x}\right] = \frac{\widehat{H}_q(y)}{\widehat{H}_q(x)}. \tag{3.70}$$

A specific application of the previous result yields, for any $x \geq \mathfrak{o}$, that

$$\widehat{\mathbb{E}}_\mathfrak{o}\left[e^{-qT_x}\right] = \frac{1}{\widehat{H}_q(x)} = \frac{\widehat{g}_q(\mathfrak{o},x)}{\widehat{g}_q(x,x)} = \frac{g_q(x,\mathfrak{o})}{g_q(x,x)}, \tag{3.71}$$

where we use the identity $\widehat{H}_q(\mathfrak{o}) = 1$ in the first equality, the dual version of (3.69) for the second one and the integrated version of the dual identity (3.6) for the last one. Combining (3.69) and (3.71), we get, for any $x \geq \mathfrak{o}$,

$$g_q(x,x) = C_q H_q(x)\widehat{H}_q(x). \tag{3.72}$$

For any $x \leq \mathfrak{o}$, we reverse the role of x and $\mathfrak{o}$ to obtain, respectively,

$$\mathbb{E}_\mathfrak{o}\left[e^{-qT_x}\right] = \frac{1}{H_q(x)} = \frac{g_q(\mathfrak{o},x)}{g_q(x,x)}, \tag{3.73}$$

$$\widehat{\mathbb{E}}_x\left[e^{-qT_\mathfrak{o}}\right] = \widehat{H}_q(x) = \frac{\widehat{G}_q(x,\mathfrak{o})}{\widehat{g}_q(\mathfrak{o},\mathfrak{o})} = \frac{g_q(\mathfrak{o},x)}{g_q(\mathfrak{o},\mathfrak{o})}. \tag{3.74}$$

Combining (3.73) and (3.74), we again arrive at (3.72), which shows that (3.72) holds for all $x \in E$. Note that (3.72) holds regardless of the boundary condition at $\mathfrak{r}$. In particular, when $X \in \mathcal{M}_\infty$, (3), (3.69) (replacing y by $\mathfrak{o}$), (3.72) and Lemma 3.7.1 give 3.3.1(3.15), which complete the proof of Theorem 3.3.1.

3.7.3 Proof of Theorem 4.3.3(5)

We need the following classical result that enables to connect the q-potential of g_q and $g_q^{a]}$ (resp. $\widehat{g}_q$ and $\widehat{g}_q^{a]}$).

Lemma 3.7.7. *We have, for any $q > 0$,*

$$g_q(x,y) = g_q^{a]}(x,y) + \mathbb{E}_x[e^{-qT_a}]g_q(a,y), \quad x,y \in E^{a]}, \tag{3.75}$$

$$\widehat{g}_q(x,y) = \widehat{g}_q^{a]}(x,y) + \widehat{\mathbb{E}}_x[e^{-qT_b}]\widehat{g}_q(b,y), \quad x,y \in E^{a]}. \tag{3.76}$$

Proof. Since X is downward skip-free, $T_{a]} = T_a$ and for any $t > 0$, $a \in E$, $x, y \in E^{a]}$, we have

$$\begin{aligned}\mathbb{P}_x(X_t = y) &= \mathbb{P}_x^{a]}(X_t = y) + \int_0^t \mathbb{P}_x(X_t = y | T_a = r)\mathbb{P}_x(T_a = r)dr \\ &= \mathbb{P}_x^{a]}(X_t = y) + \int_0^t \mathbb{P}_a(X_{t-r} = y)\mathbb{P}_x(T_a = r)dr\end{aligned}$$

where the second equality follows from strong Markov property. Next, multiplying by e^{-qt}, dividing by $\pi(y)$, which is positive by irreducibility, and integral over t, we obtain

$$\begin{aligned}g_q(x,y) &= g_q^{a]}(x,y) + \int_0^\infty \int_0^t e^{-q(t-r)} \frac{\mathbb{P}_a(X_{t-r}=y)}{\pi(y)} e^{-qr}\mathbb{P}_x(T_a = k)drdt \\ &= g_a^{b]}(x,y) + \int_0^\infty \int_r^\infty e^{-q(t-r)} \frac{\mathbb{P}_a(X_{t-r}=y)}{\pi(y)} e^{-qr}\mathbb{P}_x(T_a = k)dtdr \\ &= g_q^{a]}(x,y) + \mathbb{E}_x[e^{-qT_a}]g_q(a,y)\end{aligned}$$

which proves (3.75). (3.76) is the dual statement of (3.75). □

We proceed with the proof of Theorem 4.3.3(5). First, let $B \subset E$ and denote g_q^B (resp. $\widehat{g}_q^B$) to be the q-potential of the X (resp. its dual $\widehat{X}$) killed upon entering into the set B. We recall the Hunt's switching identity for Markov chains, which can be found in [29, page 140], and says that, for any $x, y \in E \backslash B$,

$$g_q^B(x,y) = \widehat{g}_q^B(y,x). \tag{3.77}$$

For sake of simplicity, we simply write $g_q^B = g_q^{a]}$ (resp. $g_q^A = g_q^{a]}$) if $B = a], \mathfrak{r})$ (resp. if $A = (\mathfrak{l}, a]$). With this notation in mind, we express the q-potential kernels of $(X, \mathbb{P}^{a]})$ and $(X, \mathbb{P}^{a]})$ in terms of FqE functions of the three processes $(X, \mathbb{P})$, $(X, \widehat{\mathbb{P}})$ and $(X, \mathbb{P}^{y]})$.

Lemma 3.7.8. *Suppose that $X \in \mathcal{M}_\infty$.*

$$g_q^{a]}(x, y) = C_q \widehat{H}_q(y) \left(\frac{\mathbf{H}_q^{y]}(a)}{H_q(a)} H_q(x) - \mathbf{H}_q^{y]}(x) \right), \quad x, y \in E^{a]}, \tag{3.78}$$

$$g_q^{a]}(x, y) = C_q \widehat{H}_q(y) \left(\mathbf{H}_q^{a]}(x) - \mathbf{H}_q^{y]}(x) \right), \quad x, y \in E^{a]}, \tag{3.79}$$

$$g_q^{(b,a)^c}(x, y) = C_q \widehat{H}_q(y) \left(\frac{\mathbf{H}_q^{y]}(a)}{\mathbf{H}_q^{a]}(a)} \mathbf{H}_q^{a]}(x) - \mathbf{H}_q^{y]}(x) \right), \quad x, y \in E^{(a,b)^c}. \tag{3.80}$$

Proof of Lemma 3.7.8. To show (3.78), we use (3.75) and (3.15) to get

$$\begin{aligned} g_q^{a]}(x, y) &= g_q(x, y) - \mathbb{E}_x \left[e^{-qT_a} \right] g_q(a, y) \\ &= C_q (H_q(x) - \mathbf{H}_q^{y]}(x)) \widehat{H}_q(y) - \frac{H_q(x)}{H_q(a)} C_q (H_q(a) - \mathbf{H}_q^{y]}(a)) \widehat{H}_q(y) \\ &= C_q \widehat{H}_q(y) \left(\frac{H_q(x) \mathbf{H}_q^{y]}(a)}{H_q(a)} - \mathbf{H}_q^{y]}(x) \right). \end{aligned}$$

Next, using the Hunt's switching identity (3.77) and (3.76), we have

$$\begin{aligned} g_q^{a]}(x, y) &= \widehat{g}_q^{a]}(y, x) = \widehat{g}_q(y, x) - \widehat{\mathbb{E}}_y [e^{-qT_a}] \widehat{g}_q(a, x) = g_q(x, y) - \frac{\widehat{H}_q(y)}{\widehat{H}_q(a)} g_q(x, a) \\ &= C_q (H_q(x) - \mathbf{H}_q^{y]}(x)) \widehat{H}_q(y) - \frac{\widehat{H}_q(y)}{\widehat{H}_q(b)} C_q (H_q(x) - \mathbf{H}_q^{a]}(x)) \widehat{H}_q(b) \\ &= C_q \widehat{H}_q(y) \left(\mathbf{H}_q^{a]}(x) - \mathbf{H}_q^{y]}(x) \right), \end{aligned}$$

which proves (3.79). Finally, to get (3.80), we use the identity $g_q^{a]}(x, y) = g_q^{(a,b)^c}(x, y) + \mathbb{E}_x^{a]}[e^{-qT_a}] G_q^{a]}(a, y)$ and (3.79) combined with Lemma 3.7.2. □

With Lemma 3.7.8 in mind, the proof of Theorem 4.3.3(5) follows readily by applying the second claim stated in the following classical results which we prove for sake of completeness.

Lemma 3.7.9. *For any $a > x$ and $t > 0$, we have*

$$\mathbb{P}_x(T_{a]} > t) = P_t^{a]}\mathbb{1}(x) = \frac{\pi\widehat{P}_t^{a]}\delta_x}{\pi(x)} \tag{3.81}$$

and thus, for any $q > 0$,

$$\mathbb{E}_x\left[e^{-qT_{a]}}\right] = 1 - qG_q^{a]}\mathbb{1}(x) = 1 - q\frac{\pi\widehat{G}_q^{a]}\delta_x}{\pi(x)}, \tag{3.82}$$

where $G_q^{a]}f(x) = \sum_{y\in E^{a]}} f(y)g_q^{a]}(x,y)\pi(y)$.

Proof. First, an application of Fubini's theorem yields, that for any $a > x$ and $q > 0$,

$$\begin{aligned} G_q^{a]}\mathbb{1}(x) &= \sum_{y\in E^{a]}} g_q^{a]}(x,y)\pi(y) = \sum_{y\in E^{a]}} \int_0^t e^{-qt}\mathbb{P}_x(X_t = y, t < T_{a]}) \\ &= \int_0^t \sum_{y\in E^{a]}} e^{-qt}\mathbb{P}_x(X_t = y, t < T_{a]}) = \int_0^t e^{-qt}\mathbb{P}_x(T_{a]} > t)dt = \frac{1}{q}\left(1 - \mathbb{E}_x\left[e^{-qT_{a]}}\right]\right). \end{aligned} \tag{3.83}$$

Moreover, from the Hunt's switching identity (3.77), we observe that

$$G_q^{a]}\mathbb{1}(x) \quad = \textstyle\sum_{y\in E^{a]}} g_q^{a]}(x,y)\pi(y) = \sum_{y\in E^{a]}} \pi(y)\widehat{g}_q^{a]}(y,x) = \frac{\pi\widehat{G}_q^{a]}\delta_x}{\pi(x)}$$

which completes the proof of the first claim. Finally, the second statement follows readily from the previous one and the identity (3.83). □

The proof of Corollary 3.3.3 follows readily from the characterization of the fundamental q-excessive functions in terms of the q-potential kernel and the skip-free property which entails that, for any $x \geq b > \mathfrak{l}$

$$\mathbb{E}_x^{[\mathfrak{l}]}\left[e^{-qT_b}\right] = \mathbb{E}_x\left[e^{-qT_b}\mathbb{1}_{\{T_b < T_{\mathfrak{l}]}\}}\right] = \mathbb{E}_x\left[e^{-qT_b}\right].$$

CHAPTER 4

INTERTWINING RELATION

4.1 Introduction

The continuous-state space branching processes with immigration (continuous CBI for short) are a class of time-homogeneous Markov process with values in $\mathbb{R}^{+}$. They can be viewed as the scaled limit of Galton-Watson process. The time change binding the two processes together is called the Lamperti transform, following the foundational work of Lamperti. In the work [40], we studied the continuous-time skip-free Markov process and characterize the distribution of the first exit time from an interval and the expression for different important quantities. In particular, the paper gave a comprehensive study on the application of continuous-time discrete-state space branching process with immigration (discrete CBI for short). In this paper, we identify an intertwining relationship between the discrete and continuous CBI. By applying the intertwining relation to the results in discrete CBI, we can derive the first hitting and first passage time of continuous CBI process. In addition, we can also extend the spectral theorem from discrete CBI process to the continuous CBI process using the intertwining ideas.

The following parts of the paper is organized as following. In section 2, we build the intertwining relationship between continuous CBI and discrete CBI processes. For selected continuous CBI process satisfying certain conditions, we state and prove the intertwining theorem to connect their semigroups. In section 3, we recall the results of first hitting time and first passage time in discrete CBI process, and extend them to continuous CBI process by applying the intertwining relationship. Lastly, in section 4, we use apply the intertwining theorem to extend the spectral theorem on discrete CBI to continuous CBI.

4.2 Branching versus Galton-Watson processes

Let $P = (P_t)_{t\geq 0}$ be the semigroup of a continuous-state space branching process with immigration (CBI, denoted by X_t). Writing $e_\lambda(x) = e^{-\lambda x}$, one has, for any $t, x, \lambda \geq 0$,

$$P_t e_\lambda(x) = e^{-\Phi_t(\lambda)x - \Psi_t(\lambda)}, \tag{4.1}$$

where the CBI exponents Ψ_t and Φ_t solve, for any fixed $\lambda \geq 0$, the so-called generalized Riccati equations,

$$\frac{\partial}{\partial t}\Phi_t(\lambda) = -\psi(\Phi_t(\lambda)), \quad \Psi_0(\lambda) = \lambda, \tag{4.2}$$

$$\Psi_t(\lambda) = \int_0^t \phi(\Phi_s(\lambda))ds, \tag{4.3}$$

with the branching and immigration mechanisms given respectively by

$$\psi(u) = \sigma^2 u^2 + \gamma u - q - \int_0^\infty (1 - e^{-uy} - u(1 \wedge y))\,\Pi(dy), \tag{4.4}$$

$$\phi(u) = bu + \int_0^\infty (1 - e^{-uy})\,\mu(dy), \tag{4.5}$$

where $\sigma, b, q \geq 0$, $\gamma \in \mathbb{R}$ and Π and μ are nonnegative Borel measures on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge y^2)\Pi(dy) + \int_0^\infty (1 \wedge y)\mu(dy) < \infty.$$

Then for all positive $f \in C_0(\mathbb{R}^+)$, i.e. the space of positive continuous function on $\mathbb{R}^+$ vanishing at infinity, define the linear operator Λ that maps f to the function on $\mathbb{N}$,

$$\Lambda f(n) = \mathbb{E}\left[f(G_{n+1})\right] = \int_0^\infty f(x)e^{-x}x^n \frac{dx}{\Gamma(n+1)}, \tag{4.6}$$

where G_{n+1} follows $Gamma(n+1,1)$ distribution.

Recall the results for discrete state space CBI process $Y = (Y_t)_{t\geq 0}$ starting at $Y_0 = x$, with homogeneous exponential split time of parameter a, branching probability generating function f on $\mathbb{N}_0$, homogeneous Poisson immigration time of parameter b, and immigrant size with pgf g on $\mathbb{N}$. For $0 < s \leq 1, t \geq 0$ and $x \in \mathbb{N}$, one has

$$\mathbb{E}_x\left[s^{Y_t}\right] = e^{-v_t(s)}\varphi_t^x(s) \tag{4.7}$$

where the CBI components φ_t and v_t solve the Riccati equations for all $0 < s \leq 1$,

$$\frac{\partial}{\partial_t}\varphi_t(s) = F_a(\varphi_t(s)), \quad \varphi_0(s) = s \tag{4.8}$$

$$v_t(s) = b\int_0^t (1 - g(\varphi_r(s)))dr \tag{4.9}$$

with the branching mechanism given by

$$F_a(s) = a(f(s) - s)$$

First, writing $\overline{\Pi}(y) = \int_y^\infty \Pi(dy), y > 0$, and assuming that $d_1 = \int_0^\infty (1 \wedge y)\,\Pi(dy) < \infty$, we introduce the following sets

$$\mathcal{B} = \{\psi \text{ of the form (4.4); } \Pi(dy) = e^{-y}\sum_{n=0}^{\infty}\frac{m_n}{n!}y^n dy, m_n \geq 0, \overline{\Pi}(0) = \sum_{n=0}^{\infty} m_n < \infty,$$
$$\text{and } \gamma + d_1 + \overline{\Pi}(0) \leq \sigma^2 \leq \frac{1}{2}(\gamma + d_1 + 1))\}$$

$$\mathcal{I} = \{\phi \text{ of the form (4.5); } b = 0, \mu(dy) = e^{-y}\sum_{n=0}^{\infty}\frac{\ell_n}{n!}y^n dy, \ell_n \geq 0 \textit{ and } \sum_{n=0}^{\infty}\ell_n = 1\}$$

Theorem 4.2.1. *Let $P = (P_t)_{t\geq 0}$ be the semigroup of a continuous state space CBI with branching and immigration mechanisms $\psi \in \mathcal{B}$, $\phi \in \mathcal{I}$ respectively. Let $\mathbb{Q} = (\mathbb{Q}_t)_{t\geq 0}$ be the family of linear operators defined as*

$$\mathbb{Q}_t\Lambda = \Lambda P_t \tag{4.10}$$

Then $\mathbb{Q}$ is the semigroup of a discrete-state space CBI, i.e. for any $n \in \mathbb{N}$ and $0 < s \leq 1$, write $p_s(x) = s^x$,

$$\mathbb{Q}_t p_s(n) = e^{-v_t(s)} \varphi_t^n(s),$$

where $v_t(s) = \Psi_t(\frac{1-s}{s})$ and $\varphi_t(s) = (1 + \Phi_t(\frac{1-s}{s}))^{-1}$.

The proof of Theorem 4.2.1 can be done in several steps.

Lemma 4.2.2. *The linear operator Λ is injective and bounded on the set of non-negative functions in $C_0(\mathbb{N})$.*

Proof. Let $f, g \in C_0(\mathbb{N})$ and non-negative such that $\Lambda f = \Lambda g$, that is, for all $n \in \mathbb{N}_0$,

$$\int_0^\infty f(x) e^{-x} x^n dx \quad = \quad \int_0^\infty g(x) e^{-x} x^n dx.$$

Observing that for all $0 \leq a < 1$, $0 \leq \int_0^\infty f(x) e^{-(1-a)x} dx < \infty$ and $0 \leq \int_0^\infty g(x) e^{-(1-a)x} dx < \infty$, we deduce by uniqueness of the moment problem that $f = g$ almost everywhere and thus Λ is injective on the set of non-negative functions in $C_0(\mathbb{N})$, and therefore Λ^{-1} is well-defined. □

Lemma 4.2.3. *For $n \in \mathbb{N}, 0 < s \leq 1$ and $t \geq 0$, we have*

$$\mathbb{Q}_t p_s(n) = e^{-v_t(s)} \varphi_t^n(s),$$

where $v_t(s) = \Psi_t(\frac{1-s}{s})$, and $\varphi_t(s) = \left(1 + \Phi_t(\frac{1-s}{s})\right)^{-1}$.

Proof. First, we observe that for any $\lambda \geq 0$ and $n \in \mathbb{N}_0$, by the property of Gamma distribution, we have

$$\Lambda e_\lambda(n) = \mathbb{E}\left[e^{-\lambda G_{n+1}}\right] = (1 + \lambda)^{-(n+1)} = e_{\ln(1+\lambda)}(n + 1). \tag{4.11}$$

Thus, we get, from (4.1) and by linearity of Λ, that

$$\Lambda P_t e_\lambda(n) = \Lambda e^{-\Psi_t(\lambda)-n\Phi_t(\lambda)} = e^{-\Psi_t(\lambda)}\Lambda e_{\Phi_t(\lambda)}(n) = e^{-\Psi_t(\lambda)} e_{\ln(1+\Phi_t(\lambda))}(n+1) \quad (4.12)$$

Apply $\mathbb{Q}_t$ on both sides of equation (4.11), use the definition of Λ and (4.12), we get

$$\mathbb{Q}_t e_{\ln(1+\lambda)}(n+1) = \mathbb{Q}_t \Lambda e_\lambda(n) = \Lambda P_t e_\lambda(n) = e^{-\Psi_t(\lambda)} e_{\ln(1+\Phi_t(\lambda))}(n+1) \quad (4.13)$$

With $s = \frac{1}{1+\lambda}$, one can rewrite the equation above as

$$\mathbb{Q}_t p_s(n+1) = \mathbb{Q}_t e_{\ln(\lambda+1)}(n) = e^{-v_t(s)}\varphi_t^{n+1}(s) \quad (4.14)$$

for $0 < s \leq 1$, where $v_t(s) = \Psi_t(\frac{1-s}{s})$ and $\varphi_t(s) = \left(1 + \Phi_t\left(\frac{1-s}{s}\right)\right)^{-1}$.
In other words, we've showed

$$\mathbb{Q}_t p_s(n) = \mathbb{Q}_t e_{\ln(\lambda+1)}(n) = e^{-v_t(s)}\varphi_t^{n}(s), \quad (4.15)$$

for all $n \in \mathbb{N}$, which conclude the proof of lemma. □

Lemma 4.2.4. *Let $\psi \in \mathcal{B}$, then the mapping $s \mapsto f(s) = \psi\left(\frac{1-s}{s}\right)s^2 + s$ for $0 < s \leq 1$, is the generating function of a probability measure on $\mathbb{N}$.*

Proof. First let us write $\widetilde{\psi}(u) = \psi(u-1) + u$ and observe that

$$f(s) = \psi\left(\frac{1-s}{s}\right)s^2 + s = \tilde{\psi}\left(\frac{1}{s}\right)s^2. \quad (4.16)$$

Denote $d_1 = \int_0^\infty (1 \wedge y)\,\Pi(dy)$, and $d_2 = \int_0^\infty ((1 \wedge y) + 1)\,\Pi(dy)$, observe $d_2 = d_1 + \int_0^\infty \Pi(dy) = d_1 + \overline{\Pi}(0)$. Both d_1 and d_2 are finite by the definition of $\mathcal{B}$. Then, one has

$$\begin{aligned}
\widetilde{\psi}(u) &= \psi(u-1) + u & (4.17)\\
&= \sigma^2(u-1)^2 + \gamma(u-1) - \int_0^\infty \left(1 - e^{-(u-1)y} - (u-1)(1 \wedge y)\right)\Pi(dy) + u & (4.18)\\
&= \sigma^2 u^2 + (\gamma - 2\sigma^2 + 1 + d_1)u + (\sigma^2 - \gamma - d_2) + \int_0^\infty e^{-uy}e^y\Pi(dy) & (4.19)\\
&= \sigma^2 u^2 + \gamma' u + q' + \int_0^\infty e^{-uy}e^y\Pi(dy), & (4.20)
\end{aligned}$$

where we have set $\gamma' = \gamma - 2\sigma^2 + 1 + d_1$ and $q' = \sigma^2 - \gamma - d_2$. Let now write $s = \frac{1}{u}$ for $u \geq 1$ and then

$$
\begin{aligned}
f(s) &= \widetilde{\psi}\left(\frac{1}{s}\right)s^2 = \widetilde{\psi}(u)u^{-2} = \left(\sigma^2 u^2 + \gamma' u + q' + \int_0^\infty e^{-uy}e^y\Pi(y)dy\right)u^{-2} \\
&= \sigma^2 + \gamma' u^{-1} + q' u^{-2} + u^{-2}\int_0^\infty e^{-uy}\sum_{n=0}^\infty y^n \frac{m_n}{\Gamma(n+1)}dy \\
&= \sigma^2 + \gamma' u^{-1} + q' u^{-2} + u^{-2}\sum_{n=0}^\infty m_n \frac{1}{\Gamma(n+1)}\int_0^\infty e^{-uy}y^n dy \\
&= \sigma^2 + \gamma' u^{-1} + q' u^{-2} + u^{-2}\sum_{n=0}^\infty m_n u^{-n+2} = \sigma^2 + \gamma' s + q' s^2 + \sum_{n=0}^\infty m_n s^n
\end{aligned}
$$

where we use the identity $\frac{1}{\Gamma(a)}\int_0^\infty e^{-uy}y^{a-1}dy = u^{-a}$ for $a, u > 0$ in the last line, and appeals to Tonelli theorem to interchange the integral and the sum. As σ^2, γ', q' and m_n are all non-negative, and the sum of the coefficients are equal to 1 because

$$
\begin{aligned}
\sigma^2 + \gamma' + q' + \sum_{n=0}^\infty m_n &= \sigma^2 + (\gamma - 2\sigma^2 + 1 + d_1) + (\sigma^2 - \gamma - d_2) + \sum_{n=0}^\infty m_n \\
&= 1 - \int_0^\infty \Pi(dy) + \sum_{n=0}^\infty m_n = 1 - \int_0^\infty e^{-y}\sum_{n=0}^\infty y^n \frac{m_n}{\Gamma(n+1)}dy + \sum_{n=0}^\infty m_n \\
&= 1 - \sum_{n=0}^\infty m_n + \sum_{n=0}^\infty m_n = 1,
\end{aligned}
$$

we conclude that f is indeed the probability generating function of a discrete random variable on $\mathbb{N}$. □

We are now ready to complete the proof of Theorem 4.3.2. From Lemma 4.2.3 we have

$$
\mathbb{Q}_t p_s(n) = e^{-v_t(s)}\varphi_t^n(s) \tag{4.21}
$$

for $n \in \mathbb{N}$, where $v_t(s) = \Psi_t(\frac{1-s}{s})$ and $\varphi_t(s) = \left(1 + \Phi_t\left(\frac{1-s}{s}\right)\right)^{-1}$. Next, we show that the right-hand side of (4.21) is indeed the generating function of some discrete-state space CBI. First note for all $0 < s \leq 1$,

$$\varphi_0(s) = \left(1 + \Phi_0(\frac{1-s}{s})\right)^{-1} = \left(1 + \frac{1-s}{s}\right)^{-1} = s$$

and

$$\partial_t \varphi_t(s) = -\partial_t \Phi_t(\lambda)(1 + \Phi_t(\lambda))^{-2} = \psi(\Phi_t(\lambda))\varphi_t^2(s) = \psi\left(\frac{1-\varphi_t(s)}{\varphi_t(s)}\right)\varphi_t^2(s).$$

Thus, writing

$$F_a(s) = \psi\left(\frac{1-s}{s}\right)s^2 = a(f(s) - s) \text{ with } f(s) = \frac{1}{a}\psi\left(\frac{1-s}{s}\right)s^2 + s, \tag{4.22}$$

for any $a > 0$, because $\psi(0) = 0$, we get, for a fixed s,

$$\partial_t \varphi_t(s) = F_a(\varphi_t(s)), \quad \varphi_0(s) = s.$$

By the result of Lemma 4.2.4, f is a pgf of a discrete variable on $\mathbb{N}$, thus $\varphi_t(s)$ satisfies Reccati equation (4.8).

Next note for $\phi \in \mathcal{I}$, by Tonelli's theorem,

$$\begin{aligned}
\phi(u) &= bu + \int_0^\infty (1 - e^{-uy})\,\mu(dy) = \int_0^\infty (1 - e^{-uy})\,e^{-y} \sum_{n=0}^\infty \frac{\ell_n y^n}{n!} dy \\
&= \sum_{n=0}^\infty \int_0^\infty e^{-y} \frac{\ell_n y^n}{n!} dy - \sum_{n=0}^\infty \int_0^\infty e^{-(u+1)y} \frac{\ell_n y^n}{n!} dy \\
&= \sum_{n=0}^\infty \ell_n - \sum_{n=0}^\infty \frac{\ell_n}{(u+1)^{n+1}} = 1 - \sum_{n=0}^\infty \frac{l_n}{(u+1)^{n+1}}
\end{aligned}$$

Then for all $0 < s \leq 1$,

$$\begin{aligned}
v_t(s) &= \Psi_t(\lambda) = \int_0^t \phi\,(\Phi_r(\lambda))\,dr = \int_0^t \phi\left(\frac{1-\varphi_r(s)}{\varphi_r(s)}\right)dr \\
&= \int_0^t (1 - g(\varphi_r(s)))\,dr
\end{aligned}$$

where

$$g(s) = 1 - \phi\left(\frac{1-s}{s}\right) = 1 - \left(1 - \sum_{n=0}^\infty \frac{\ell_n}{(\frac{1-s}{s} + 1)^{n+1}}\right) = \sum_{n=0}^\infty \ell_n s^{n+1} \tag{4.23}$$

is a probability generating function for a discrete random variable on $\mathbb{N}$. Combining the above results and invoking the injectivity of the Laplace transform, we deduce that $\mathbb{Q}$ is the semigroup of a discrete state space CBI, which completes the proof.

Example: Consider the continuous CBI with branching and immigration mechanism defined by $\psi(u) \in \mathcal{B}$ and $\phi(u) \in \mathcal{I}$, with

$$\begin{aligned}
\sigma^2 &= 1 - p, \\
\gamma &= 2(1 - p) - 1 - d_1, \\
\Pi(dy) &= \frac{p}{\Gamma(k-2)} e^{-y} y^{k-3} dy \\
\mu(dy) &= e^{-y} dy
\end{aligned}$$

for some $k \in \mathbb{N}, k \geq 3$ and $0 < p < 1$, where $d_1 = \int_0^\infty (1 \wedge y)\Pi(dy) = \frac{p}{(k-3)!}(\Gamma(k-2,1) + \gamma(k-1,1))$, with Γ and γ as the upper and lower incomplete Gamma functions respectively.

Both Π and μ are finite nonnegative Borel on $(0, \infty)$. We can find the intertwining CBI process on discrete state space.

For Π defined above, from equation (4.20) we have

$$\begin{aligned}
\widetilde{\psi}(u) &= (1-p)u^2 + (\gamma - 2\sigma^2 + 1 + d_1)u + (\sigma^2 - \gamma - d_2) + \int_0^\infty e^{-uy} e^{y} \Pi(dy) \\
&= (1-p)u^2 + ((1-p) - 2(1-p) + 1 + d_1 - d_2) + \int_0^\infty e^{-uy} e^{y} \frac{p}{\Gamma(k-2)} e^{-y} y^{k-3} dy \\
&= (1-p)u^2 + \left(\int_0^\infty \frac{p}{\Gamma(k-2)} e^{-y} y^{k-3} dy - p \right) + pu^{2-k} \\
&= (1-p)u^2 + pu^{2-k}
\end{aligned}$$

and then from equation (4.16)

$$f(s) = \tilde{\psi}\left(\frac{1}{s}\right) s^2 = (1-p) + ps^k,$$

which is a probability generating function on the discrete state space $\{0, k\}$. We can interpret the branching mechanism in the way that, after some exponential time, a particle will either split into k offspring, with probability p, or die, with probability $(1-p)$.

Next we compute its immigration mechanism

$$\phi(u) = \int_0^\infty \left(1 - e^{-uy}\right) e^{-y} dy = 1 - \frac{1}{u+1},$$

hence

$$g(s) = 1 - \phi(\frac{1-s}{s}) = s$$

which is the pgf for constant immigrant size 1.

Remark 4.2.5. For $p = 0$, the underlying branching mechanism for the discrete CBI can be regarded as generalized Yule process, i.e., each particle will live with some time $\tau \sim exp(a)$ and then split to k offspring. We can compute

$$\varphi_t(s) = se^{-at}\left(1 - (1 - e^{-a(k-1)t}s\right)^{-\frac{1}{k-1}}$$

4.3 First Hitting Time and First Passage Time of Continuous CBI

Next we study the expression of first hitting time and first passage time of continuous CBI, along with the expression of associated fundamental q-excessive functions, using the intertwining relation between discrete and continuous CBIs.

Let $P = (P_t)_{t\geq 0}$ be the semigroup of a continuous state-space branching process with immigration (continuous CBI, denoted by $X = (X_t)_{t\geq 0}$). By Theorem 4.2.1, $\mathbb{Q}$, defined as $\mathbb{Q}_t\Lambda = \Lambda P_t$, is the semigroup of some discrete-state space CBI (discrete CBI, denoted by $Y = (Y_t)_{t\geq 0}$).

First for a real valued function F on $\mathbb{N}$, for $0 \leq s \leq 1$, we write its generating function as

$$\mathcal{G}_F(s) = \sum_{y=0}^{\infty} s^y F(y).$$

Recall the results of discrete CBI that for $q > 0$ and a fixed reference point, there exists a function $\mathbb{H}_q(x)$ (resp. $\widehat{\mathbb{H}}_q(x)$, which is unique minimal decreasing (resp. increasing) q-excessive function for $\mathbb{Q}$ (resp. $\widehat{\mathbb{Q}}$, the semigroup of the dual process). And for any state y there exists a function $\mathbb{H}_q^{[y}(x)$ which has (with respect to $\mathbb{Q}^{[y}$, semigroup of process killed upon entering $[y,\infty)$) the same properties than $\mathbb{H}_q$ but with normalization. The expression of $\mathbb{H}_q(x)$, $\widehat{\mathbb{H}}_q(x)$ and $\widehat{\mathbb{H}}_q^{[y}(x)$ is shown in the following theorem represented in Constantinescu, Loeffen, Patie, Wang [40].

Theorem 4.3.1. *Let Y_t be the discrete state space continuous time CBI process, from (4.7), for $q > 0, t \geq 0$ and $x \in \mathbb{N}_0$, write*

$$\Phi_t^{(d)}(s;x) = \mathbb{E}_x\left[s^{Y_t}\right] = e^{-v_t(s)}\varphi_t^x(s)$$

(1) Write, $\Phi_t^{(d)}(x) = \Phi_t^{(d)}(0;x) = e^{-v_t}\varphi_t$, where we have set $v_t = v_t(0)$ and $\varphi_t = \varphi_t(0)$. Note that $\mathbb{P}_x(Y_t = 0) = \Phi_t^{(d)}(x) > 0$ and let

$$\mathbb{H}_q(x) = C_q^{-1}\int_0^{\infty} e^{-qr}\Phi_r^{(d)}(x)dr$$

where $C_q^{-1} = \int_0^{\infty} e^{-qr}\Phi_r^{(d)}(0)dr$. Then the mapping $x \mapsto \mathbb{H}_q(x)$ is a minimal q-purely excessive function for $\mathbb{Q}$.

(2) Let $\widehat{\mathbb{H}}_q(y)$ *be defined on* $\mathbb{N}_0$ *by*

$$\widehat{\mathbb{H}}_q(y) = \frac{\gamma_q \star \pi_\alpha(y)}{\pi(y)}$$

where the reference measure $\pi(y)$ *is positive,* $\gamma_q \star \pi_\alpha$ *is the convolution of the two measures* γ_q *and* π_α*, i.e.* $\gamma_q \star \pi_\alpha(y) = \sum_{m\leq y} \gamma_q(y-m)\pi_\alpha(m)$ *and* γ_q *is the discrete measure on* $\mathbb{N}_0$ *defined by*

$$\gamma_q(y) = \sum_{k=0}^{\infty} \frac{q^k}{k!} \pi^{\star k}(y)$$

with $\pi^{\star k}$ *the* k^{th}*-convolution of the stationary measure* π*. Then,* $\widehat{\mathbb{H}}_q(x)$ *is the unique increasing and q-invariant function for* $\widehat{\mathbb{Q}}$*. Moreover, we have*

$$\mathcal{G}_{\pi_\alpha \widehat{\mathbb{H}}_q}(s) = e^{q\mathcal{G}_\pi(s)} \mathcal{G}_{\pi_\alpha}(s) = e^{\int_0^s \frac{q+(1-g(r))-\alpha}{F_\alpha(r)} dr}$$

up to some constant.

(3) Given $x, y \in \mathbb{N}_0$*, let* $\mathbf{H}_{q,x}(y) = \widehat{\mathbb{H}}_q(y)\mathbb{H}_q^{[y}(x)$*, the generating function of* $\pi\mathbf{H}_{q,x}$ *takes the form*

$$\mathcal{G}_{\pi\mathbf{H}_{q,x}}(s) = \mathbb{H}_q(x)\, \mathcal{G}_{\pi\widehat{\mathbb{H}}_q}(s) - C_q^{-1}\mathbb{H}_q(x; s)$$

where $\mathbb{H}_q(x; s) = \sum_{y=0}^{\infty} s^y g_q(x, y)\pi(y) = \int_0^\infty e^{-qr}\Phi_r^{(d)}(s; x)dr$*. Note that* $\mathbb{H}_q(x; 0) = \mathbb{H}_q(x)$*.*

Furthermore the first hitting time $T_y = \inf_{t>0}\{Y_t = y, Y_0 = x\}$ can be expressed as

$$\mathbb{E}_x\left[e^{-qT_y}\right] = \frac{\mathbb{H}_q(y) - \mathbb{H}_q^{[y}(x)}{\mathbb{H}_q(x)}$$

We first recall the intertwining result in the proof of Lemma 4.2.3 to link the generating function of discrete CBI and Laplace transform of continuous CBI. For $q > 0$ and for positive function $f \in C_0(\mathbb{R}^+)$, let $U_q f = \int_0^\infty e^{-qt}\mathbb{Q}_t f dt$ and

$V_q f = \int_0^\infty e^{-qt} P_t f dt$ be the q resolvent of $\mathbb{Q}_t$ and P_t respectively. Because the linear operator Λ is bounded and closed, by the definition of $\mathbb{Q}$, we have

$$U_q \Lambda f = \int_0^\infty e^{-qt} \mathbb{Q}_t \Lambda f dt = \int_0^\infty e^{-qt} \Lambda P_t f dt = \Lambda V_q f \tag{4.24}$$

Next we have the following theorem for the continuous CBI from the intertwining relations.

Theorem 4.3.2. *Let $P = (P_t)_{t\geq 0}$ be the semigroup of a continuous CBI process $X = (X_t)_{t\geq 0}$, and $\mathbb{Q} = (\mathbb{Q}_t)_{t\geq 0}$ be the semigroup of its intertwining discrete CBI process $Y_t = (Y_t)_{t\geq 0}$, defined in section 4.2. For any $q > 0$, and a fix state b, let $\mathbb{H}_q$, $\widehat{\mathbb{H}}_q$ and $\mathbb{H}_q^{\dagger b}$ be the corresponding q-excessive functions associated with Y.*

(1) For $t \geq 0$, $\lim_{\lambda\to\infty} \Phi_t(\lambda)$ and $\lim_{\lambda\to\infty} \Psi_t(\lambda)$ exist, denoted by Φ_t and Ψ_t respectively. Let

$$H_q(x) = C_q^{-1} \int_0^\infty e^{-qr} e^{-\Phi_r x - \Psi_r} dr,$$

where $C_q = \int_0^\infty e^{-qr} e^{-\Psi_r} dr > 0$. Then the mapping $x \mapsto H_q(x)$ is a decreasing and purely q-excessive function for P.

(2) Let $\widehat{H}_q(y) = \Lambda^* \widehat{\mathbb{H}}_q(y) = \sum_{n=0}^\infty \widehat{\mathbb{H}}_q(n) \frac{y^n}{n!} e^{-y}$ for $y \geq 0$, where Λ^* is the adjoint operator of Λ with respect to π_α, defined as

$$\langle \Lambda f, g\rangle_{\pi_\alpha} = \langle f, \Lambda^* g\rangle$$

for positive functions $g \in C_0(\mathbb{Z}^+)$ and $f \in C_0(\mathbb{R}^+)$. Then the mapping $y \mapsto \widehat{H}_q(y)$ is an increasing and q-invariant for $\widehat{P}$, the semigroup of the dual process. Moreover, for $\lambda \geq 0$, we can express the Laplace transform of $\widehat{H}_q(y)$ as

$$\mathcal{L}_{\widehat{H}_q}(\lambda) = \int_0^\infty e^{-\lambda y} \widehat{H}_q(y) dy = \frac{1}{1+\lambda} \exp\left(\int_\lambda^\infty \frac{\phi(u) + q - \alpha}{\psi(u)} du\right)$$

(3) Fix $b \in \mathbb{R}^+$ and $y \in \mathbb{R}^+$, define

$$\begin{aligned}\mathbf{H}_q^{[b}(y) &= \frac{1}{\widehat{H}_q(b)}\Lambda^*\Lambda^{-1}\widehat{\mathbb{H}}_q(x)\mathbb{H}_q^{[x}(y) \\ &= \frac{1}{\widehat{H}_q(b)}\Lambda^*\Lambda^{-1}\widehat{\mathbb{H}}_q(x)\mathbb{H}_q^{[x}(y)\end{aligned}$$

$\mathbf{H}_q^{[b}$ is a decreasing and q-excessive for $P^{[b}$, the semigroup for the continuous CBI killed upon entering $[b, \infty)$

Here the operator Λ^{-1} applies to y and Λ^* applies to x, i.e.

$$\Lambda^*\Lambda^{-1}\widehat{\mathbb{H}}_q(x)\mathbb{H}_q^{[x}(y) = \sum_{n\in\mathbb{N}}\left(\Lambda^{-1}\widehat{\mathbb{H}}_q(n)\mathbb{H}_q^{[n}(y)\right)\frac{e^{-x}x^n}{n!} \tag{4.25}$$

$$= \sum_{n\in\mathbb{N}}\widehat{\mathbb{H}}_q(n)\left(\Lambda^{-1}\mathbb{H}_q^{[n}(y)\right)\frac{e^{-x}x^n}{n!} \tag{4.26}$$

Proof. (1) First, note that for any fix $t > 0$ and $x \geq 0$, the mapping $\lambda \mapsto P_t e_\lambda(x)$ is decreasing and bounded on $\mathbb{R}^+$, therefore one can write

$$P_t e_\infty(x) = \lim_{\lambda\to\infty} P_t e_\lambda(x).$$

Next, Lemma 4.2.2 entails that Λ^{-1} exists on the range on Λ and thus using (4.13), we get

$$\lim_{\lambda\to\infty} P_t e_\lambda(x) = \lim_{\lambda\to\infty}\Lambda^{-1}\mathbb{Q}_t e_{\ln(1+\lambda)}(x) = \lim_{\lambda\to\infty}\Lambda^{-1}\mathbb{Q}_t p_{\frac{1}{1+\lambda}}(x) = \Lambda^{-1}\mathbb{Q}_t p_0(x)$$

where we used for the last identity that Λ^{-1} is closed as the inverse of a bounded and closed operator. Let X'_t be the continuous branching process without immigration, Athreya, Ney [39] showed the following results in their book chapter 6.1 , that, for all $x \geq 0$ and $t \geq 0$, we have

$$0 < \mathbb{P}_x\left(X'_t = 0\right) = e^{-x\lim_{\lambda\to\infty}\Phi_t(\lambda)},$$

therefore the limit exist for all $t \geq 0$, and we denote by $\lim_{\lambda\to\infty}\Phi_t(\lambda) = \Phi_t$.

From equation (4.3), by Tonelli Theorem, we have

$$\lim_{\lambda\to\infty}\Psi_t(\lambda) = \lim_{\lambda\to\infty}\int_0^t \phi(\Phi_s(\lambda))ds = \int_0^t \phi(\Phi_s)ds$$

Let $\lim_{\lambda\to\infty}\Psi_t(\lambda) = \Psi_t$. Putting pieces together, we obtain

$$P_t e_\infty(x) = e^{-x\Phi_t-\Psi_t}$$

Define $H_q(x) = \Lambda^{-1}\mathbb{H}_q(x)$, using Theorem 4.3.1 and equation (4.24), we have for all the values of $x \in \mathbb{R}^+$

$$\begin{aligned} H_q(x) &= \Lambda^{-1}\mathbb{H}_q(x) = C_q^{-1}\Lambda^{-1}\int_0^\infty e^{-qr}\mathbb{Q}_t p_0(x)dr \\ &= C_q^{-1}\int_0^\infty e^{-qr}\Lambda^{-1}\mathbb{Q}_t p_0(x)dr = C_q^{-1}\int_0^\infty e^{-qr}P_t e_\infty(x)dr \\ &= C_q^{-1}\int_0^\infty e^{-qr-\Phi_t x-\Psi_t}dr, \end{aligned}$$

where we used for the result that Λ^{-1} is closed as the inverse of a bounded and closed operator.
Note here we define the constant $C_q = \int_0^\infty e^{-qr-\Psi_t}dr$, so H_q is normalized at 0, i.e. $H_q(0) = 1$.

The mapping $x \mapsto H_q(x)$ is decreasing as $\Phi_r \in (0,1)$ for all $r \geq 0$. Next we show $H_q(x)$ is q-excessive, i.e. $e^{-qt}P_t H_q(x) \leq H_q(x)$.

First from the Riccati equation (4.2) we have for all $\lambda > 0$, $t, s \geq 0$,

$$\Phi_{t+s}(\lambda) = \Phi_t(\Phi_s(\lambda)),$$

and as $\lambda \mapsto \infty$, we have

$$\Phi_{t+s} = \Phi_t(\Phi_s). \tag{4.27}$$

Next from Reccati equation (4.3) and equation (4.27), we have

$$\begin{aligned}
\Psi_t(\Phi_r) + \Psi_r &= \int_0^t \phi(\Phi_s(\Phi_r))ds + \int_0^r \phi(\Phi_s)ds \\
&= \int_0^t \phi(\Phi_{s+r})ds + \int_0^r \phi(\Phi_s)ds = \int_r^{t+r} \phi(\Phi_s)ds + \int_0^r \phi(\Phi_s)ds \\
&= \int_0^{t+r} \phi(\Phi_s)ds = \Psi_{t+r}.
\end{aligned}$$

Using the equation above and change of variables, we get for $q > 0$, $t \geq 0$ and $x \geq 0$,

$$\begin{aligned}
C_q e^{-qt} P_t H_q(x) &= e^{-qt} P_t \int_0^\infty e^{-qr} e^{-\Phi_t x - \Psi_r} dr \\
&= e^{-qt} \int_0^\infty e^{-qr} e^{-\Psi_r} P_t e^{-\Phi_r x} dr \\
&= \int_0^\infty e^{-q(t+r)} e^{-\Psi_r} P_t e_{\Phi_r}(x) dr \\
&= \int_0^\infty e^{-q(t+r)} e^{-\Psi_r} e^{-\Phi_t(\Phi_r)x} e^{-\Psi_t(\Phi_r)} dr \\
&= \int_0^\infty e^{-q(t+r)} e^{-\Phi_t(\Phi_r)x} e^{-(\Psi_t(\Phi_r)+\Psi_r)} dr \\
&= \int_0^\infty e^{-q(t+r)} e^{-x\Phi_{t+r}} e^{-\Psi_{t+r}} dr \\
&= \int_t^\infty e^{-qr} e^{-\Phi_t x - \Psi_r} dr \\
&\leq \int_0^\infty e^{-qr} e^{-\Phi_t x - \Psi_r} dr = C_q H_q(x).
\end{aligned}$$

We deduce from the last identity that $P_t H_q(x) \leq e^{qt} H_q(x)$ because $C_q > 0$ for $q > 0$, as $\Phi_r^{(d)}(0) > 0$ for given r. Finally, we clearly have

$$\lim_{t \downarrow 0} e^{-qt} P_t H_q(x) = \lim_{t \downarrow 0} q \int_t^\infty e^{-qr} e^{-\Phi_t x - \Psi_r} dr = H_q(x)$$

and by monotone convergence

$$\lim_{t \to \infty} e^{-qt} P_t H_q(x) = \lim_{t \to \infty} q \int_t^\infty e^{-qr} e^{-\Phi_t x - \Psi_r} dr = 0.$$

Being a positive function, this shows that H_q is a purely q-excessive function for P and completes the proof of Theorem 4.3.2(1)

(2) From Theorem 4.2.1, we have the intertwining relation between the semigroups of discrete and continuous CBI, i.e. $\mathbb{Q}_t\Lambda = \Lambda P_t$.

Let $\widehat{\mathbb{Q}}_t$, $\widehat{P}$ be the semigroups of the corresponding dual processes of discrete and continuous CBI respectively, and let Λ^* be the adjoint operator of Λ with respect to discrete measure π_α introduced in Theorem 4.3.1(2) , defined as following,

$$\langle \Lambda f, g\rangle_\pi = \langle f, \Lambda^* g\rangle,$$

for positive functions $g \in C_0(\mathbb{Z}^+)$ and $f \in C_0(\mathbb{R}^+)$, where

$$\langle g_1, g_2\rangle_{\pi_\alpha} = \sum_{n=0}^{\infty} g_1(n)g_2(n)\pi_\alpha(n) \tag{4.28}$$

and

$$\langle f_1, f_2\rangle = \int_0^\infty f_1(x)f_2(x)dx \tag{4.29}$$

Apply Tonelli theorem, We have

$$\begin{aligned}\langle f, \Lambda^* g\rangle &= \langle \Lambda f, g\rangle_{\pi_\alpha} = \sum_{n=0}^{\infty} \Lambda f(n)g(n)\pi_\alpha(n) = \sum_{n=0}^{\infty}\left(\int_0^\infty e^{-x}x^n\frac{dx}{\Gamma(n+1)}\right)g(n)\pi(n)\\ &= \int_0^\infty f(x)e^{-x}\sum_{n=0}^{\infty} x^n g(n)\pi(n)\frac{dx}{n!}\end{aligned}$$

Since this is true for any $f \in C_0(\mathbb{R}^+)$, we deduce that for any $x \geq 0$,

$$\Lambda^* g(x) = e^{-x}\sum_{n=0}^{\infty} x^n \frac{g(n)\pi(n)}{n!} \tag{4.30}$$

Define $\widehat{H}_q(y) = \Lambda^*\widehat{\mathbb{H}}_q(y)$, from Theorem 4.3.1(2) and Tonelli theorem, for $y \geq 0$, we have

$$\widehat{H}_q(y) = \Lambda^*\widehat{\mathbb{H}}_q(y) = e^{-y}\sum_{n=0}^{\infty}\widehat{\mathbb{H}}_q(n)\pi(n)\frac{y^n}{n!} \tag{4.31}$$

We first show that $\widehat{P}_t\Lambda^* = \Lambda^*\widehat{\mathbb{Q}}_t$. For any positive function $f, g \in C_0(\mathbb{N})$, by the definition of dual operator, we have

$$\langle f, \widehat{P}_t\Lambda^* f\rangle = \langle P_t f, \Lambda^* g\rangle = \langle \Lambda P_t f, g\rangle_{\pi_\alpha} = \langle \mathbb{Q}_t\Lambda f, g\rangle_{\pi_\alpha} = \langle \widehat{\mathbb{Q}}_t f, \Lambda g\rangle_\pi \tag{4.32}$$

$$= \langle f, \Lambda^*\widehat{\mathbb{Q}}_t g\rangle, \tag{4.33}$$

by the property of inner product, we have $\widehat{P}_t\Lambda^* = \Lambda^*\widehat{\mathbb{Q}}_t$.

From Theorem 4.3.1(2), for discrete CBI, there exist a q-invariant function $\widehat{\mathbb{H}}_q$, i.e. for any $q > 0$, $t \geq 0$.

$$e^{-qt}\widehat{\mathbb{Q}}_t\widehat{\mathbb{H}}_q = \widehat{\mathbb{H}}_q.$$

Therefore, we can compute

$$e^{-qt}\widehat{P}_t\widehat{H}_q \quad = \quad e^{-qt}\widehat{P}_t\Lambda^*\widehat{\mathbb{H}}_q = e^{-qt}\Lambda^*\widehat{\mathbb{Q}}_t\widehat{H}_q = \Lambda^*\widehat{\mathbb{H}}_q = \widehat{H}_q,$$

therefore, $\widehat{H}_q$ is a q-invariant function. In addition, $\widehat{H}_q(y)$ is non-negative because

$$\widehat{H}_q(y) = \Lambda^*\widehat{\mathbb{H}}_q(y) = e^{-y}\sum_{n=0}^{\infty} y^n\widehat{\mathbb{H}}_q(n)\pi_\alpha(n)\frac{1}{n!} \geq 0,$$

where we get the last identity as $\widehat{\mathbb{H}}_q(n) \geq 0$ and $\pi_\alpha(n) \geq 0$ for all $n \geq 0$.

For $0 \leq x \leq y$, we have

$$\begin{aligned}
\widehat{H}_q(y) \quad &= \quad e^{-y}\sum_{n=0}^{\infty} y^n\widehat{\mathbb{H}}_q(n)\pi_\alpha(n)\frac{1}{n!} = \mathbb{E}\left[\pi_\alpha\mathbb{H}_q\left(N(y)\right)\right] = \mathbb{E}\left[\pi_\alpha\mathbb{H}_q\left(N(x) + N(y-x)\right)\right] \\
&\geq \quad \mathbb{E}\left[\pi_\alpha\mathbb{H}_q\left(N(x)\right)\right] = \widehat{H}_q(x),
\end{aligned}$$

where we get the second last identity because $\widehat{\mathbb{H}}_q$ is non-decreasing. Therefore $\widehat{H}_q$ is a non-decreasing and q-invariant function for $\widehat{P}_t$.

Next we we find the explicit expression of $\widehat{H}_q(y)$ by computing its Laplace transform, for $\lambda > 0$ and positive functions $f \in C_0(\mathbb{R}^+)$, let

$$\mathcal{L}_f(\lambda) = \int_0^\infty e^{-\lambda y} f(y)dy.$$

Apply equation (4.31) and Tonelli theorem, we get

$$
\begin{aligned}
\mathcal{L}_{\widehat{H}_q}(\lambda) &= \int_0^\infty e^{-\lambda y}\widehat{H}_q(y)dy = \int_0^\infty e^{-\lambda y}\sum_{n=0}^\infty \widehat{\mathbb{H}}_q(n)\pi_\alpha(n)\frac{y^n}{n!}e^{-y}dy && (4.34)\\
&= \sum_{n=0}^\infty \frac{\widehat{\mathbb{H}}_q(n)\pi_\alpha(n)}{n!}\int_0^\infty e^{-\lambda y}y^n e^{-y}dy = \sum_{n=0}^\infty \frac{\widehat{\mathbb{H}}_q(n)\pi_\alpha(n)}{n!}\int_0^\infty e^{-(\lambda+1)y}y^n dy && (4.35)\\
&= \sum_{n=0}^\infty \frac{\widehat{\mathbb{H}}_q(n)\pi_\alpha(n)}{n!}n!(1+\lambda)^{-(n+1)} = \sum_{n=0}^\infty \widehat{\mathbb{H}}_q(n)\pi_\alpha(n)\left(\frac{1}{1+\lambda}\right)^{n+1} && (4.36)\\
&= \frac{1}{1+\lambda}\mathcal{G}_{\pi_\alpha \widehat{\mathbb{H}}_q}\left(\frac{1}{1+\lambda}\right) = \frac{1}{1+\lambda}\exp\left(\int_0^{\frac{1}{1+\lambda}}\left(\frac{b\,(1-g(r))+q-\alpha}{F_a(r)}\right)dr\right) && (4.37)\\
&= \frac{1}{1+\lambda}\exp\left(\int_0^{\frac{1}{1+\lambda}}\left(\frac{b\phi(\frac{1-r}{r})+q-\alpha}{r^2\psi(\frac{1-r}{r})}\right)dr\right) && (4.38)
\end{aligned}
$$

where we get the last identity using the (4.22) and (4.23).

Let $u = \frac{1-r}{r}$ and use change of variable, we get

$$
\begin{aligned}
\mathcal{L}_{\widehat{H}_q}(\lambda) &= \frac{1}{1+\lambda}\exp\left(\int_\infty^\lambda \frac{b\phi(u)+q-\alpha}{(\frac{1}{(1+u)^2}\psi(u)}\frac{-1}{(1+u)^2}dv\right) && (4.39)\\
&= \frac{1}{1+\lambda}\exp\left(\int_\lambda^\infty \frac{b\phi(u)+q-\alpha}{\psi(u)}du\right) && (4.40)
\end{aligned}
$$

(3) Define $\mathbf{H}_q^{[y} = (\Lambda)^{-1}\mathbb{H}_q^{[y}$. As $\mathbb{H}_q^{[y}$ has the same property as $\mathbb{H}_q$ for discrete CBI, by the same arguments above for H_q, $\mathbf{H}_q^{[y}$ in continuous CBI is also decreasing and q-excessive for $P_t^{[b}$.

□

Next we have the theorem of first hitting time for continuous CBI.

Theorem 4.3.3. *1. For any $x, y \in E$, $x \le y$,*

$$\mathbb{E}_x[e^{-qT_y}] = \frac{H_q(x)}{H_q(y)}$$

2. *Let $b \in E$ and choose a reference point $\mathfrak{d}^{[b} \in E^{[b}$. Then, for all $q > 0$, the function $H_q^{[b}(x) = K_q^{[b}\delta_{\mathfrak{l}}(x)$ defined on $E^{[b}$ is positive on $E^{[b}$, minimal, increasing q-harmonic for $P^{[b}$ with $H_q^{[b}(\mathfrak{d}^{[b}) = 1$. Moreover, for any $b \leq x \leq a$,*

$$\mathbb{E}_x^{[b}[e^{-qT_a}] = \mathbb{E}_x[e^{-qT_a}\mathbb{1}_{\{T_a<T_{[b}\}}] = \frac{H_q^{[b}(x)}{H_q^{[b}(a)}.$$

3. *For any $x, y \in E$ and $q > 0$, we have*

$$\mathbb{E}_x[e^{-qT_y}] = \frac{H_q(x) - \mathbf{H}_q^{y]}(x)}{H_q(y)}.$$

4. *For all $q > 0$ and any $x \geq y$,*

$$\widehat{\mathbb{E}}_y[e^{-qT_x}] \quad = \frac{\widehat{H}_q(y)}{\widehat{H}_q(x)}.$$

From (4.24), $U_q\Lambda f = \Lambda V_q f$ for all positive $f \in C_0(\mathbb{R}^+)$. Let u_q be the q-potential kernel of Y (or its Green function), that is, for any $q > 0$, and $x, y \in \mathbb{N}$, .

$$u_q(x, y) = \int_0^\infty e^{-qt}\mathbb{P}\left(Y_t = y|Y_0 = x\right)dt = \int_0^\infty e^{-qt}\mathbb{P}_t(x, y)dt.$$

From the result in section (2.1) in [40], for f a bounded function on $\mathbb{N}$, we have

$$U_q f(x) = \int_0^\infty e^{-qt}\mathbb{Q}_t f(x)dt = \sum_{y\in\mathbb{N}} f(y)u_q(x, y) \tag{4.41}$$

Similarly, for continuous CBI, let v_q be the q-potential kernel of X , that is, for any $q > 0$, and $x, y \in \mathbb{R}^+$, .

$$v_q(x, y) = \int_0^\infty e^{-qt}p_t(x, y)dt,$$

where $p_t(x, y)$ is the density of X_t given $X_0 = x$. For f a bounded function on $\mathbb{R}^+$, using Tonelli theorem, we get

$$\begin{aligned} V_q f(x) &= \int_0^\infty e^{-qt}\mathbb{P}_t f(x)dt = \int_0^\infty e^{-qt}\mathbb{E}_x\left[f(X_t)\right]dt && (4.42)\\ &= \int_0^\infty e^{-qt}\int_0^\infty f(y)p_t(x, y)dydt = \int_0^\infty f(y)\int_0^\infty e^{-qt}p_t(x, y)dtdy && (4.43)\\ &= \int_0^\infty f(y)v_q(x, y)dy && (4.44)\end{aligned}$$

For fixed $x \geq 0$ and $y \leq x$, let $f(y) = \delta_x(y)$, where δ_x is the Dirac mass at x. Using the property of Dirac mass that $\int_0^\infty \delta_x(y) f(y) dy = f(x)$, and (4.41), we have, on the one hand,

$$U_q \Lambda \delta_x(y) = \int_0^\infty e^{-qt} \mathbb{Q}_t \Lambda \delta_x(y) dt = \int_0^\infty e^{-qt} \mathbb{Q}_t \int_0^\infty \delta_x(r) e^{-r} r^y \frac{dr}{\Gamma(y+1)} dt \tag{4.45}$$

$$= \int_0^\infty e^{-qt} \mathbb{Q}_t e^{-x} \frac{x^y}{\Gamma(y+1)} dt = \sum_{z \in \mathbb{N}} \frac{e^{-x} x^z}{\Gamma(z+1)} u_q(y, z) = \sum_{z \in \mathbb{N}} u_q(y, z) \Gamma_x(z), \tag{4.46}$$

where Γ_x is the discrete measure on $\mathbb{N}$ such that $\Gamma_x(z) = \frac{e^{-x} x^z}{\Gamma(z+1)}$ for $z \in \mathbb{N}$. On the other hand, using (4.44) and property of Dirac mass, we have

$$\Lambda V_q \delta_x(y) = \Lambda \int_0^\infty \delta_x(z) v_q(y, z) dz = \Lambda v_q(y, x) \tag{4.47}$$

$$= \int_0^\infty v_q(r, x) e^{-r} r^y \frac{dr}{\Gamma(y+1)} = \int_0^\infty v_q(r, x) \gamma_y(dr), \tag{4.48}$$

where $\gamma_y(dr)$ is a continuous measure on $\mathbb{R}^+$ such that $\gamma_y(dr) = e^{-r} r^y \frac{dr}{\Gamma(y+1)}$.

Use (4.24) and let $f = \delta_x(y)$, combining equation (4.46) and (4.48), we have

$$\sum_{z \in \mathbb{N}} u_q(y, z) \Gamma_x(z) = \int_0^\infty v_q(r, x) \gamma_y(dr) \tag{4.49}$$

We aim to find the expression for $v_q(y, x)$. From [40] Theorem 3.1(3), for $x, y \in \mathbb{N}$ and fix a reference point $\mathfrak{o} = 0$, we have

$$u_q(x, y) = C_q \widehat{\mathbb{H}}_q(y) \left(\mathbb{H}_q(x) - \mathbb{H}_q^{[y}(x) \right) \tag{4.50}$$

Combing (4.46) and (4.47), we have

$$\Lambda v_q(y, x) = U_q \Lambda \delta_x(y) = \sum_{z \in \mathbb{N}} u_q(y, z) \Gamma_x(z)$$

Use the injectivity and linearity of Λ and plug in the expression of $u_q(x, y)$ above, we have

$$v_q(y,x) = \Lambda^{-1}\left(\sum_{z\in\mathbb{N}} u_q(y,z)\Gamma_x(z)\right) = \Lambda^{-1}\left(\sum_{z\in\mathbb{N}} C_q\widehat{\mathbb{H}}_q(z)\left(\mathbb{H}_q(y) - \mathbb{H}_q^{[z}(y)\right)\Gamma_x(z)\right) \quad (4.51)$$

$$= \sum_{z\in\mathbb{N}} C_q\widehat{\mathbb{H}}_q(z)\left(\Lambda^{-1}\mathbb{H}_q(y) - \Lambda^{-1}\mathbb{H}_q^{[z}(y)\right)\Gamma_x(z) \quad (4.52)$$

$$= C_q\sum_{z\in\mathbb{N}}\widehat{\mathbb{H}}_q(z)H_q(y)\Gamma_x(z) - C_q\sum_{z\in\mathbb{N}}\widehat{\mathbb{H}}_q(z)\left(\Lambda^{-1}\mathbb{H}_q^{[z}(y)\right)\Gamma_x(z) \quad (4.53)$$

$$= C_qH_q(y)\sum_{z\in\mathbb{N}}\widehat{\mathbb{H}}_q(z)\frac{e^{-x}x^z}{z!} - C_q\sum_{z\in\mathbb{N}}\Lambda^{-1}\left(\widehat{\mathbb{H}}_q(z)\mathbb{H}_q^{[z}(y)\right)\frac{e^{-x}x^z}{z!} \quad (4.54)$$

$$= C_qH_q(y)\left(\Lambda^*\widehat{\mathbb{H}}_q(x)\right) - C_q\left(\Lambda^*\Lambda^{-1}\widehat{\mathbb{H}}_q(x)\mathbb{H}_q^{[x}(y)\right) \quad (4.55)$$

$$= C_qH_q(y)\widehat{H}_q(x) - C_q\widehat{H}_q(x)\mathbf{H}_q^{[x}(y) = C_q\widehat{H}_q(x)\left(H_q(y) - \mathbf{H}_q^{[x}(y)\right), \quad (4.56)$$

where the last line comes from the definition of $\mathbf{H}_q^{[x}(y)$.

4.4 Spectral

4.4.1 preliminary results on discrete CBI

For continuous time discrete state space branching process without immigration Z_t with initial condition $Z_0 = x$, define the infinitesimal generating function $F_a(s) = a(f(s) - s), 0 < s < 1$, where a is the rate constant and $f(s)$ is the p.g.f. of number of offspring. Denote $\phi_t^{(x)}(s)$ as generating function of Z, i.e.

$$\phi_t^{(x)}(s) = P_t p_s(Z_t)$$

and we have

$$\phi_t^{(x)}(s) = \phi_t^x(s) \quad (4.57)$$

where $\phi_t(s)$ is satisfy the classical Recatti Equation.

We will concern ourselves with the case (mean one) when $f(0) > 0, f(1) = f'(1) = 1$, or equivalently,

$$F_a(0) > 0, F_a(1) = F_a'(1) = 0 \tag{4.58}$$

$F_a(s)$ has no zeros in$[0, 1)$, and we define for s on $(0, 1)$.

For a real valued function f on $\mathbb{N}$, we write its generating function by

$$\mathcal{G}_f(s) = \sum_{y=0}^{\infty} s^y f(y)$$

Recall the Proposition in Patie, Wang,

Proposition 4.4.1. *Let Z be a continuous-time branching process, let s_0 be the smallest root of $F_a(s)$ on the interval $(0, 1]$, as defined before.*

(1) For $0 \leq s \leq 1$,

$$\lim_{t\to\infty} \phi_t(s) = s_0.$$

(2) There exist a non-negative stationary measure π on $\mathbb{N}$, i.e. $\pi P_t = \pi$, whose generating function $\mathcal{G}_\pi$ is given by

$$\mathcal{G}_\pi(s) = \int_0^s \frac{dr}{F_a(r)}, \quad s < s_0. \tag{4.59}$$

Moreover, $\mathcal{G}_\pi$ satisfies the functional equation, for any $t \geq 0$ and $0 < s < s_0$,

$$\mathcal{G}_\pi(\phi_t(s)) = \mathcal{G}_\pi(s) + t. \tag{4.60}$$

(3) ϕ_t has the following representation

$$\phi_t(s) = \mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s) + t) \tag{4.61}$$

where $\mathcal{G}_\pi^{-1}$ is the inverse function of $\mathcal{G}_\pi$, showing that it has the semigroup property.

Remark 4.4.2. $\mathcal{G}_\pi$ maps $(0, s_0)$ onto $(0, \infty)$ monotonously and there exists a well-defined inverse function $\mathcal{G}_\pi^{-1}$ mapping $(0, \infty)$ onto $(0, s_0)$ monotonously.

Karlin proved the following representation theorem

Theorem 4.4.3. *When $F_a(s)$ is analytic at $s = 1$, the function $\mathcal{G}_\pi^{-1}(\omega)$ is regular in some right halp-plane $Re(\omega) \geq u_0$ and there is a representation*

$$1 - \mathcal{G}_\pi^{-1}(\omega) = \int_0^\infty e^{-\omega\xi} e^{u_0} \beta(\xi) d\xi, \quad Re(\omega) \geq u_0 \tag{4.62}$$

where β is square integrable on $[0, \infty)$

Combing equation (4.62) and (4.61), and the definition of $\phi_t(s)$, the transition probability have the following representation

$$P_t(i, j) = \int_0^\infty e^{-t\xi} Q_j(\xi) \psi_i(d\xi), \quad i, j = 0, 1, 2, ...; \quad t \geq 0 \tag{4.63}$$

where Q_j is a polynomial of degree j and ψ_t is a real measure on $[0, \infty)$ with finite total variation.

Now consider continuous time discrete state space branching process with immigration (Y_t), with initial condition $Y_0 = x$. Let g be the p.g.f. of the immigration size, and constant $b > 0$ as immigration rate. See detailed notations and results in Patie, Jian. The generating function of Y_t can be expressed as

$$\Phi_t(s; x) = \phi_t^x(s) \psi_t(s) = \phi_t^x(s) e^{-b \int_0^t (1 - g(\phi_s(s))) du}$$

Recall Proposition 5.3 in Patie, Wang

Proposition 4.4.4. *let Y be a CBI process and let $\alpha = b(1 - g(s_0)) \geq 0$.*

(1) A nonnegative measure π_α on $\mathbb{N}_0$ is an α-stationary measure, i.e. $\pi_\alpha P_t = e^{-\alpha t}\pi_\alpha$ for all $t \geq 0$, if and only if its generating function $\mathcal{G}_{\pi_\alpha}(s) < \infty$ for all $0 < s < s_0$ and

$$\mathcal{G}_{\pi_\alpha}(\phi_t(s))\psi_t(s) = e^{-\alpha t}\mathcal{G}_{\pi_\alpha}(s). \tag{4.64}$$

(2) There exists a unique, up to a constant multiple, α-stationary measure whose generating function $\mathcal{G}_{\pi_\alpha}$ has the following expression, for $s_1 < s < s_0$,

$$\mathcal{G}_{\pi_\alpha}(s) = e^{\int_{s_1}^{s} \frac{b(1-g(r))-\alpha}{Fa(r)} dr} \tag{4.65}$$

where s_1 is some constant between s_0 and 1.

(3) For $0 < s_1 < s < s_0$, $t \geq 0$ the generating function of Y can be represented as

$$\Phi_t(s; x) = \phi_t^x(s)\psi_t(s)$$

where $\phi_t(s) = \mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s) + t))$ and $\psi_t(s) = \frac{e^{-\alpha t}\mathcal{G}_{\pi_\alpha}(s)}{\mathcal{G}_{\pi_\alpha}(\mathcal{G}_\pi^{-1}(\mathcal{G}_\pi(s)+t)))}$.

Remark 4.4.5. Without lose of generality, we can assume that $s_0 = 1$. Otherwise there is an equivalent transformation as stated in Proposition 1.9 in [18] to transfer s_0 to 1. Therefore here we can set $\alpha = b(1 - g(1)) = 0$.

Note when the mean of offspring is 1, $\alpha = b(1 - g(1)) = 0$. Use the idea in the proof of Theorem 7 in Karlin-4, picking out the coefficient of s^j on the right-hand side we can get the following result

Theorem 4.4.6. *Under some regularity condition on f and g, the transit matrix have the representation*

$$\mathbb{P}_t(i, j) = \int_0^\infty e^{-t\xi} Q_j(\xi) d\psi_i(\xi) \tag{4.66}$$

where Q_j is a polynomial of degree j and ψ_i is a signed measure of bounded variation.

4.4.2 Spectral representation of continuous CBI

Combining Theorem 2.2 (2) and Theorem 3.6, use Tonelli Theorem, we have for $\lambda > 0$,

$$\begin{aligned}\int_0^\infty e^{-\lambda y} P_t(x, dy) &= \mathbb{E}_x\left(e^{\lambda X_t}\right) = \Lambda^{-1}\mathbb{E}_x\left(s^{Y_t}\right) = \Lambda^{-1}\sum_{y=0}^{\infty} s^y \mathbb{P}_t(x, y) \\ &= \Lambda^{-1}\sum_{y=0}^{\infty}(1+\lambda)^{-y}\mathbb{P}_t(x, y) = \Lambda^{-1}\sum_{y=0}^{\infty}(1+\lambda)^{-y}\int_0^\infty e^{-t\xi} Q_y(\xi) d\psi_x(\xi) \\ &= \Lambda^{-1}\int_0^\infty e^{-t\xi}\left(\sum_{y=0}^{\infty}(1+\lambda)^{-y} Q_y(\xi)\right) d\psi_x(\xi)\end{aligned}$$

from which we get

$$\int_0^\infty e^{-\lambda y} P_t(x, dy) = \Lambda^{-1}\int_0^\infty e^{-t\xi}\left(\sum_{y=0}^{\infty}(1+\lambda)^{-y} Q_y(\xi)\right) d\psi_x(\xi) \tag{4.67}$$

CHAPTER 5

SCALED LIMIT APPROACH

5.1 Introduction

Continuous-state branching (CB) processes are the continuous time and space version of GaltonWatson processes. They were introduced in different levels of generality by Jirina (1958), Lamperti (1967b) and Silverstein (1967/1968). They are Feller processes with state-space $[0, \infty]$ (with any metric that makes it homeomorphic to $[0, 1]$) satisfying the following branching property: the sum of two independent copies started at x and y has the law of the process started at $x + y$. The states 0 and ∞ are absorbing. The branching property can be recast by stating that the logarithm of the Laplace transform of the transition semigroup is given by a linear transformation of the initial state.

Continuous-state branching processes with immigration (or CBI processes) are the continuous time and space version of GaltonWatson processes with immigration and were introduced by Kawazu and Watanabe (1971). They are Feller processes with state-space $[0, \infty]$ such that the logarithm of the Laplace of the transition semigroup is given by an affine transformation of the initial state. [They thus form part of the affine processes studied by Dawson and Li (2006).] As shown by Kawazu and Watanabe (1971), they are characterized by the Laplace exponents of a spectrally positive Levy process (spLp) and of a subordinator: the logarithmic derivative of the semigroup of a CB process at zero applied to the function $x \to e^{-\lambda x}$ exists and is equal to the function

$$x \to x\Psi(\lambda) - \Phi(\lambda)$$

where Ψ is the Laplace exponent of a SPLP and Φ is the Laplace exponent of a subordinator. We will give a probabilistic explanation of this characterization, similar to the Lamperti representation.

5.1.1 Continuous State Space Continuous Time Branching Process

The continuous state space Branching process (C.B.) X_t will be defined in terms of its transition function. A one parameter family $\{Q_t(x, E); t \geq 0\}$ of functions will be called a continuous branching function if it satisfies the following conditions:

(1) $\mathbb{Q}_t(x, E)$ is defined for $t \geq 0, x \geq 0$, and E a Borel subset of the half line $[0, \infty]$.

(2) For fixed t and x, $P_t(x, E)$, as a function of E, is a probability measure; and for fixed E, $Q_t(x, E)$ is jointly measurable in x and t.

(3) The Chapman-Kolmogorov equation holds:

$$\int_0^\infty Q_t(u, E) Q_s(x, du) = Q_{t+s}(x, E).$$

(4) For any $x, y, t \geq 0$, $\{Q_t\}$ satisfies

$$P_t(x + y, E) = \int P_t(x, E - u) P_t(y, du)$$

for each E

(5) There exist $t > 0$ and $x > 0$ such that $P_t(x, \{0\}) < 1$.

.

The branching mechanism function $\phi(\lambda)$ can be specified by Levy-Khinchin formula

$$\phi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) m(dr) \quad \lambda \geq 0 \tag{5.1}$$

where $\beta \geq 0$ denotes the Gaussian coefficient, and m is a positive measure on $(0, \infty)$ such that $\int_0^\infty (r^2 \wedge r) m(dr) < \infty$

Define $v_t(\lambda)$ by the Laplace transforms

$$\mathbb{E}(e^{-\lambda X_t}) = \int_0^\infty e^{-\lambda y} Q_t(x, dy) = e^{-x v_t(\lambda)} \tag{5.2}$$

then we have $v_{t+s}(\lambda) = v_t(v_s(\lambda))$ from the Chapman-Kolmogorov equation and $\lim_{h\to 0+} \psi_h(v_t(\lambda)) = v_t(\lambda)$

$t \to v_t(\lambda)$ is the unique nonnegative solution of the integral equation

$$v(t) + \int_0^\infty \phi(v(s)) ds = \lambda \tag{5.3}$$

5.1.2 Scaling limits of Continuous time discrete state space process

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Let $\{Y_t^{(k)}\}$ be a sequence of continuous time discrete state space without immigration discussed before and $\mathbb{E}(s^{Y_t}) = \phi_t^x(s)$ with $Y_0 = x$. Let$X_t^{(k)} = \frac{Y_{t\gamma_k}^{(k)}}{k}$ for $\{\gamma_k\}$ be a positive sequence going to infinite. $\{X_t^{\{k\}} : t \geq 0\}$ is a Markov process with state space $E_k := \{0, 1/k, 2/k, ...\}$. We are interested in the asymptotic behavior of the sequence of continuous time process $\{X_t^{\{k\}}\}$. We have

$$\int_{E_k} e^{\lambda_y} P_{t\gamma_k}^{(k)}(x, dy) = \exp\{-x v_t^{(k)}(\lambda)\} \tag{5.4}$$

$$\tag{5.5}$$

where

$$v_t^{(k)}(\lambda) = -k \log \phi_{k\gamma_k}^{(k)}(e^{\frac{-\lambda}{k}}) \tag{5.6}$$

Clearly, if $X_0^{(k)} = x \in E_k$, then the probability $P_{t\gamma_k}^{(k)}(x, .)$ gives the distribution of $X_t^{(k)}$ on $\mathbb{R}^+$. Let us consider the function sequences

$$G_k = k\gamma_k \left(F_a^{(k)}(e^{-z/k}) \right), \quad z \geq 0, \tag{5.7}$$

and

$$\phi^{(k)}(z) = k\gamma_k \left(F_a^{(k)}(1 - z/k) \right), \quad z \geq 0, \tag{5.8}$$

Proposition 5.1.1. *The sequence $\{G_k\}$ is uniformly Lipschitz on each bounded interval if and only if so is $\{\phi_k\}$. In this case we have $\lim_{k\to\infty} |\phi_k(z) - G_k(z)| = 0$ uniformly on each bounded interval.*

By the above proposition, if either $\{G_k\}$ or$\{\phi_k\}$ is uniformly Lipschitz on each bounded interval, then they converge or diverge simultaneously and in the convergent case they have the same limit. For convenience of statement of the results, we formulate the following conditions:

Condition 5.1.1. The sequence $\{G_k\}$ is uniformly Lapschitz on $[0, a]$ for every $a \geq 0$ and there is a function ϕ on $[0, \infty)$ so that $\{G_k\} \to \{\phi_k\}$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.

Proposition 5.1.2. *Suppose that condition 1.1 is satisfied. then the function ϕ has representation*

$$\phi(z) = bz + cz^2 + \int_0^\infty (e^{-zu} - 1 + zu)m(du), \quad z \geq 0, \tag{5.9}$$

where $c \geq 0$ and b are constants and $(u \wedge u^2)md(u)$ is a finite measure on $(0, \infty)$.

Proposition 5.1.3. *For any function ϕ with representation (1.11) there is s sequence $\{G_k\}$ in the form (1.9) satisfying Condition 1.1.*

Theorem 5.1.4. *Suppose that Condition 1.1 holds. Then for every $a \geq 0$ we have $v_t^{(k)}(\lambda) \to$ some $v_t(\lambda)$ uniformly on $[0, a]^2$ as $k \to \infty$ and the limit function solves the integral equation*

$$v(\lambda) = \lambda - \int_0^\infty \phi(v_s(\lambda))ds, \quad \lambda, t \geq 0 \tag{5.10}$$

Theorem 5.1.5. *Suppose that ϕ is given by (1.11). Then there is a Feller transition semigroup $(Q_t)_{t\geq 0}$ on $\mathbb{R}+$ defined by*

$$\int_0^\infty e^{-\lambda y} Q_t(x, dy) = e^{-xv_t(\lambda)}, \quad \lambda \geq 0, x \geq 0 \tag{5.11}$$

Moreover, if $x_k \in E_k$, $x_k \to x \geq 0$, we have $P_{t\gamma_k}^{(k)}(x_k, .) \to Q_t(x, .)$ weakly.

Theorem 5.1.6. *Suppose Condition 1.1 holds. Let X_t be a CB-process with transition semigroup Q_t defined above.If $X_0^{(k)}$ converge to X_0 in distribution, then $\{X_t^{(k)} : t \geq 0\}$ converges to $\{X_t : t \geq 0\}$ in distribution on $D([0, \infty), \mathbb{R})$*

5.1.3 Formulation of immigration process

Let $(Q_t)_{t\geq 0}$ be the transition semigroup defined in 1.1. Let $(\gamma_t)_{t\geq 0}$ be a family of probability measure on $\mathbb{R}_+$. We call $(\gamma_t)_{t\geq 0}$ a skew convolution semigroup (SC-semigroup) associated with $(Q_t)_{t\geq 0}$ provided

$$\gamma_{r+t} = (\gamma_r Q_t) \star \gamma_t, \qquad r, t \geq 0 \tag{5.12}$$

It is easy to show (1.12) holds if and only if

$$u_{r+t}(\lambda) = u_t(\lambda) + u_r(v_t(\lambda)) \tag{5.13}$$

where

$$u_t(\lambda) = -\log \int_0^\infty e^{-y\lambda} \gamma_t(dy) \tag{5.14}$$

The concept of SC-semigroup is of interest because of the following

Theorem 5.1.7. *The family of probability measure $(\gamma_t)_{t\geq 0}$ on $\mathbb{R}_+$ is an SC-semigroup if and only if*

$$Q_t^\gamma(x, \cdot) := Q_t(x, \cdot) \star \gamma_t, \qquad t, x \geq 0 \tag{5.15}$$

defines a Markov semigroup $(\gamma_t^\gamma)_{t\geq 0}$ on $\mathbb{R}_+$

If $\{Y_t : t \geq 0\}$ is a positive Markov process with transition semigroup $(Q_t^\gamma)_{t\ geq 0}$ given by (1.15), we call it an immigration process or a CBI-process associated with $(Q_t)_{t\ geq 0}$. The intuitive meaning of the model is clear in view of (5.12) and (5.15). From (5.15) we see that the population at any time $t \geq 0$ is made up of two parts; the native part generated by the mass $x \geq 0$ has distribution $Q_t(x, \cdot)$ and the immigration in the time interval $(0, t]$ gives the distribution γ_t. In a similar way, the equation (5.12) decomposes the mass immigrating to the population during the time interval $(0, r + t]$ into two parts; the immigration in the interval $(r, r + t]$ gives the distribution γ_t while the immigration in the interval $(0, r]$ generates the distribution γ_t at time r and gives the distribution $\gamma_t Q_t$ at time $r + t$. It is not hard to understand that (5.15) gives a general formulation of the immigration independent of the state of the population.

Theorem 5.1.8. *The family of probability measures $(\gamma_t)_{t\geq 0}$ on $\mathbb{R}_+$ is an SC-semigroup of and only if there exists $\psi \in \mathcal{F}$ so that*

$$\int_0^\infty e^{-\lambda y}\gamma_t(dy) = \exp\left\{-\int_0^t \psi(v_s(\lambda))ds\right\} \quad t, \lambda \geq 0 \tag{5.16}$$

Note: If an immigration process has transition semigroup $(Q_t^\gamma)_{t\geq 0}$, we say it has branching mechanism ϕ and immigration mechanism ψ. It it easy to see that

$$\int_0^\infty e^{-\lambda y} Q_t^\gamma(x, dy) = \exp\left\{-xv_t(\lambda) - \int_0^t \psi(v_s(\lambda))ds\right\} \tag{5.17}$$

The function $\psi \in \mathcal{F}$ has the representation

$$\psi(\lambda) = \beta\lambda + \int_0^\infty (1 - e^{-\lambda u})n(du), \quad z \geq 0 \tag{5.18}$$

where $\beta \geq 0$ is a constant that $(1 \wedge u)n(du)$ is a finite measure on $(0, \infty)$.

5.1.4 Scaling limits of discrete immigration models

.

Let $\{Y_t^{(k)}\}$ be a sequence of continuous time discrete state space with immigration discussed before and $\mathbb{E}(s^{Y_t^{(k)}}) = \Phi_t^{(k)}(s; x) = (\phi_t^{(k)}(s))^x \psi_t^{(k)}(s) = (\phi_t^{(k)}(s))^x e^{-b\int_0^t \left(1-g^{(k)}(\phi_r^{(k)}(s))\right)dr}$ with $Y_0^{(k)} = x$. Let$X_t^{(k)} = \frac{Y_{t\gamma_k}^{(k)}}{k}$ for $\{\gamma_k\}$ be a positive sequence going to infinite. $\{X_t^{\{k\}} : t \geq 0\}$ is a Markov process with state space $E_k := \{0, 1/k, 2/k, ...\}$. We are interested in the asymptotic behavior of the sequence of continuous time process $\{X_t^{\{k\}}\}$. We have

$$\int_{E_k} e^{-\lambda y} Q_{t\gamma_k}^{(k)}(x, y) = \exp\{-xv_t^{(k)}(\lambda) - \int_0^t \psi^{(k)}(v_r^{(k)}(\lambda))dr\} \tag{5.19}$$

where $v_t^{(k)}(\lambda)$ is given by (5.6) and

$$\psi^{(k)}(\lambda) = \gamma_k b\left[1 - g^{(k)}(e^{-\lambda/k})\right] \tag{5.20}$$

Condition 5.1.2. There is a function ψ on $[0,\infty)$ such that $\psi^{(k)}(\lambda) \to \psi(\lambda)$ uniformly on $[0, a]$ for every $a \geq 0$ as $k \to \infty$.

Note if the above condition is satisfied, the limit function ψ has the representation (5.18).

Theorem 5.1.9. *Suppose that Condition 1.1 and Condition 1.2 are satisfied. Let* $\{X_t : t \geq 0\}$ *be a CBI-process with transition semigroup* $(Q_t^\gamma)_{t\geq 0}$ *defined by* (5.17). *If* X_0^k *converge to* X_0 *in distribution, then* X_t^k *converges to* X_t *in distribution on* $D([0,\infty), \mathbb{R}_+)$

5.2 Expression of q-excessive functions and first hitting time

Following the construction above, for CBI-process define $g_q(x,\cdot) = \int_0^\infty e^{-qt} Q_t(x,\cdot)/\pi(\cdot)dt$ and for continuous time discrete state space branching process define $g_q^{(k)}(x_k, y) = \int_0^\infty e^{-qt} P_{t\gamma_k}^{(k)}(x_k, y)/\pi_k(y)dt$ for , $x_k, y \in E_k$ and some γ_k going to infinite as k goes to infinity. We have the following proposition

Proposition 5.2.1. *For* $E_k \ni x_k \to x \geq 0$ *we have* $g_q^{(k)}(x_k,\cdot) \to g_q(x,\cdot)$ *weakly.*

Proof. Define $U_q^{(k)} f(x_k) = \sum_{y\in E_K} f(y) g_q^{(k)}(x_k, y)\pi_k(y)$ for $f \in C_0(\mathbb{R}_+)$

$$U_q^{(k)} f(x_k) = \int_{E_k} f(y) g_q^{(k)}(x_k, y)\pi_k(y) = \int_{E_k} f(y) \int_0^\infty e^{-qt} P_{t\gamma_k}^{(k)}(x_k, y)dt \quad (5.21)$$

$$= \int_0^\infty e^{-qt} \int_{E_k} f(y) P_{t\gamma_k}^{(k)}(x_k, y)\, dt = \int_0^\infty e^{-qt} P_{t\gamma_k}^{(k)} f(x_k)\, dt \quad (5.22)$$

$$\to \int_0^\infty e^{-qt} Q_t f(x)dt = \int_0^\infty f(y)\bar{g}_q(x, dy) = U_q f(x) \quad (5.23)$$

for some measure $\bar{g}_q(x, dy)$, where the last line holds because

$$\lim_{k\to\infty} \sup_{x\in E_k} |P_t^{(k)} f(x) - Q_t f(x)| = 0, \quad f \in C_0(\mathbb{R}_+)$$

Next take $f(x) = e^{-\lambda x}$, $\lambda \geq 0$. $U_q f(x)$ defines the Laplace transform of measure $\bar{g}_q(x, dy)$.

$$\int_0^\infty f(y)\bar{g}_q(x, dy) = \int_0^\infty e^{-qt} Q_t f(x) dt = \int_0^\infty e^{-qt} \mathbb{E}_x[e^{-\lambda X_t}] dt \tag{5.24}$$

is finite because $0 \leq \mathbb{E}_x[e^{-\lambda X_t}] < 1$. Similarly we have

$$\int_0^\infty f(y) g_q^{(k)}(x, y)\pi_k(y) = \int_0^\infty e^{-qt} P_t^{(k)} f(x_k) dt = \int_0^\infty e^{-qt} \mathbb{E}_{x_k}[e^{-\lambda X_t^{(k)}}] dt \tag{5.25}$$

is finite for each k. Therefore the measure $g_q^{(k)}(x_k, y)\pi_k(y)$ on E_k converge to $\bar{g}_q(x, dy)$ if $x_k \to x$. Since $\pi_k(\cdot) \to \pi(\cdot)$ we have $g_q^{(k)}(x_k, \cdot) \to g_q(x, \cdot)$.

□

From Choi and Patie's result,for the process $\{X_t^{(k)}\}$

$$g_q^{(k)}(x_k, y) = C_q^{(k)} \widehat{H}_q^{(k)}(y) \left(H_q^{(k)}(x_k) - \mathbf{H}_q^{[y^{(k)}}(x_k) \right) \tag{5.26}$$

for $x_k, y \in E_k$. The proposition above indicates that if $x_k \to x$, $g_q^{(k)}(x_k, y) \to g_q(x, dy)$. Next we'll show $H_q^{(k)}(x_k), \widehat{H}_q^{(k)}(y), \mathbf{H}_q^{[y^{(k)}}(x_k)$ will converge to the corresponding terms given $x_k \to x$.

5.2.1 Expression of C_q

We know that

$$C_q^{(k)} = g_q^{(k)}(0,0) = \int_0^\infty e^{-qt} \Phi(0;0) dt = \int_0^\infty \exp\{-qt - \int_0^t \psi^{(k)}(v_r^{(k)}(\infty)) dr\} dt \tag{5.27}$$

$$\to \int_0^\infty \exp\{-qt - \int_0^t \psi^{(k)}(v_r(\infty)) dr\} dt \quad \textit{by the convergence of } v_r^{(k)} \tag{5.28}$$

$$\to \int_0^\infty \exp\{-qt - \int_0^t \psi(v_r(\infty)) dr\} dt \tag{5.29}$$

as $k \to \infty$ by Theorem 1.1 and Condition 1.2. and because $v_r(x)$ is converge to a finite number as $x \to \infty$??????

5.2.2 Expression of $H_q(x)$

From the previous main result, for $x_k \in E_k$, $x_k \to x$

$$
\begin{aligned}
H_q^{(k)}(x_k) &= \frac{1}{C_q^{(k)}} \int_0^\infty e^{-qt} \Phi_t^{(k)}(0; x_k) dt \\
&= \frac{1}{C_q^{(k)}} \int_0^\infty \exp\left\{-qt - x_k v_t^{(k)}(\infty) - \int_0^t \psi^{(k)}(v_r^{(k)}(\infty))\right\} dt \\
&\to \frac{1}{C_q} \int_0^\infty \exp\left\{-qt - x v_t(\infty) - \int_0^t \psi^{(k)}(v_r(\infty))\right\} dt \quad \textit{by the convergence of } v_r^{(k)} \\
&\to \frac{1}{C_q} \int_0^\infty \exp\left\{-qt - x v_t(\infty) - \int_0^t \psi(v_r(\infty))\right\} dt := H_q(x)
\end{aligned}
$$

?? as $k \to \infty$ by Theorem 1.1, Condition 1.2 and convergence of $C_q^{(k)}$.

5.2.3 Expression of $\widehat{H}_q(dy)$

From the main result we know for $y \in E_k$

$$
\widehat{H}_q^{(k)}(y) = \lim_{x \to \infty} \frac{g_q^{(k)}(x, y)}{g_q^{(k)}(x, 0)}
$$

and the generating function with respect to π_k is

$$
\pi_k \widehat{H}_q^{(k)}(s) = e^{q \widehat{\pi}^{(k)}(s)} \widehat{\pi}_0^{(k)}(s) \tag{5.30}
$$

where

$$
\widehat{\pi}(\lambda) = \int_\lambda^\infty \frac{1}{\phi(\lambda)}
$$

and

$$\widehat{\pi}_0(\lambda) = \exp\left(\int_\lambda^{\lambda_0} \frac{\psi(\mathrm{r})}{\phi(\mathrm{r})} d\mathrm{r}\right)$$

As $k \to \infty$, by replacing s by $e^{-\lambda}$ the $\widehat{\pi}^{(k)}(s)$ will converge to $\widehat{\pi}(\lambda)$ and $\widehat{\pi}_0^{(k)}(s)$ converges to $\widehat{\pi}_0(\lambda)$ where λ_0 is a constant with $\lambda > 0$. Then we can define $\widehat{H}_q(dy)$ by its Laplace Transform

$$\int_0^\infty e^{-\lambda y} \widehat{H}_q(dy) = e^{q\widehat{\pi}(\lambda)} \widehat{\pi}_0(\lambda) \tag{5.31}$$

5.2.4 Expression of $\widehat{\mathbf{H}}_q^{[y}(x)$

From the previous main result we know for $x_k, y \in E_k$,

$$\begin{aligned}
\pi \widehat{\mathbf{H}}_{\mathbf{q};\mathbf{x_k}}^{[(\mathbf{k})}(s) &= \sum_{y \in E_k} \widehat{H}_q^{(k)}(y) (\mathbf{H}_q^{[y)})^{(k)}(x_k) \pi_k(y) && (5.32)\\
&= H_q^{(k)}(x_k) \pi_k \widehat{H}_q^{(k)}(s) - \frac{1}{C_q^{(k)}} \int_0^\infty e^{-qr} \Phi_r^{(k)}(s; x_k) dr && (5.33)\\
&\to H_q(x) e^{q\widehat{\pi}(\lambda)} \widehat{\pi}_0(\lambda) - H_q(\lambda; x) && (5.34)
\end{aligned}$$

where

$$H_q(\lambda; x) = \frac{1}{C_q} \int_0^\infty \exp\left\{-qt - x v_t(\infty) - \int_0^t \psi(v_r(\infty))\right\} dt \tag{5.35}$$

So we define $\widehat{\mathbf{H}}_q^{[y}(x) \widehat{H}_q(y)$ by its Laplace Transform

$$\int_0^\infty e^{-\lambda y} \widehat{\mathbf{H}}_q^{[y}(x) \widehat{H}_q(y) = H_q(x) e^{q\widehat{\pi}(\lambda)} \widehat{\pi}_0(\lambda) - H_q(\lambda; x) \tag{5.36}$$

BIBLIOGRAPHY

[1] D. Aldous and J. A. Fill. Reversible Markov Chains and Random Walks on Graphs, 2002. Unfinished monograph, recompiled 2014, available at `http://www.stat.berkeley.edu/ aldous/RWG/book.html`.

[2] S. Asmussen. *Applied probability and queues*, volume 51 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, second edition, 2003. Stochastic Modelling and Applied Probability.

[3] M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.*, 29(1):165–184, 1997.

[4] D. K. Cope and H.C. Tuckwell. Firing rates of neurons with random excitation and inhibition, *J. Theoret. Biol.* 80(1):1–14, 1979.

[5] X. Duhalde, C. Foucart, and C. Ma. On the hitting times of continuous-state branching processes with immigration. *Stochastic Process. Appl.*, 124(12):4182–4201, 2014.

[6] T. Huillet On Mittag-Leffler distributions and related stochastic processes *J. Comput. Appl. Math.*, 296: 181–211, 2016.

[7] Karlin, S. and Tavarè, S. Linear birth and death processes with killing. *J. Appl. Probab.* 19(3): 477–487, 1982.

[8] H. Kunita and T. Watanabe. Markov processes and Martin boundaries. I, *Illinois J. Math.*, 9: 485–526, 1965.

[9] M.-P. Malice and C. Lefèvre. On the general epidemic model in discrete time, *Mathematics in biology and medicine (Bari, 1983), Lecture Notes in Biomath.*, 57, 164–170, Springer, Berlin, 1985.

[10] P. Picard and C. Lefèvre. The moments of ruin time in the classical risk model with discrete claim size distribution, *Insurance Math. Econom.*, 23(2): 157–172, 1998.

[11] W. Ledermann and G. E. H. Reuter. Spectral theory for the differential equations of simple birth and death processes. *Philos. Trans. Roy. Soc. London. Ser. A.*, 246: 321–369, 1954.

[12] S. Redner. A Guide to First-Passage Processes. *Cambridge University Press*, 2001.

[13] L.C.G. Rogers and D. Williams. Diffusions, Markov processes, and martingales. Vol. 1. *Cambridge University Press, Cambridge,* 2000.

[14] C. Van Weverberg. Contributions to the Study Affine Processes with Applications in Insurance. PhD thesis, Université Libre de Bruxelles, 2015.

[15] S. Sagitov. Tail generating functions for extendable branching processes. *Stochastic Process. Appl.*, 127(5):1649–1675, 2017.

[16] K. B. Athreya and P. E. Ney. Branching processes, Springer-Verlag, New York-Heidelberg, 1972.

[17] F. Avram, P. Patie and J. Wang, Purely excessive functions and hitting times of continuous-time branching processes, *Methodol. Comput. Appl. Probab.*, 21:391-399, 2019.

[18] M Chazal, R.Loeffen, P. Patie. Smoothness of continuous state branching with immigration semigroups, *Journal of Mathematical Analysis and Application*, 459(2) 2018.

[19] M. Chesney, M. Jeanblanc-Picqué, and M. Yor. Brownian excursions and Parisian barrier options. *Adv. in Appl. Probab.*, 29(1):165–184, 1997.

[20] M. Choi and P. Patie. Skip-free Markov chains. *Trans. Amer. Math. Soc.*, 371(10), 7301-7342, 2019.

[21] K. L. Chung and J. B. Walsh. Markov processes, Brownian motion, and time symmetry. *Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), Springer, New York, second edition*, V249, 2005.

[22] Xan Duhalde, Clément Foucart, and Chunhua Ma. On the hitting times of continuous-state branching processes with immigration. *Stochastic Processes and their Applications*, 124(12):4182–4201, 2014.

[23] R. A. Doney. *Fluctuation Theory for Lévy Processes.* Ecole d'Eté de Probabilités de Saint-Flour XXXV-2005. Springer, 2007.

[24] R. A. Doney and M. Savov. The asymptotic behavior of densities related to the supremum of a stable process, *Ann. Probab.* 38, no. 1, 316326, 2010.

[25] Xan Duhalde, Clément Foucart, and Chunhua Ma. On the hitting times of continuous-state branching processes with immigration. *Stochastic Processes and their Applications*, 124(12):4182–4201, 2014.

[26] E.B. Dynkin The boundary theory of Markov processes. *Uspehi Mat. Nauk*, 24(2 (146)):3-42, 1969

[27] Fill, J. A. On hitting times and fastest strong stationary times for skip-free and more general chains. *J. Theoret. Probab.*, 22: 587600, 2009.

[28] T. Huillet. On Mittag-Leffler distributions and related stochastic processes *Journal of Computational and Applied Mathematics*, 296(C), 181-211, 2016.

[29] J. G. Kemeny, J. L. Snell, and A. W. Knapp. *Denumerable Markov chains. Springer-Verlag, New York- Heidelberg-Berlin, second edition, 1976. With a chapter on Markov random fields, by David Griffeath, Grad- uate Texts in Mathematics*, No. 40.

[30] H. Kunita and T. Watanabe. Markov processes and Martin boundaries. I, *Illinois J. Math.*, 9, 485–526, 1965.

[31] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

[32] Y. Ogura. Spectral Representation for Branching Processes on the Real Half Line *Publications of the Research Institute for Mathematical Sciences*, 5, 423-441, 1970.

[33] Y. Ogura. Spectral Representation for Branching Processes with Immigration on the Real Half Line *Publications of the Research Institute for Mathematical Sciences*, 6(2), 307-321, 1970.

[34] F. Spitzer. Principles of random walk. *The University Series in Higher Mathematics. D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London*, 1964.

[35] C. Van Weverberg. Contributions to the Study Affine Processes with Applications in Insurance. PhD thesis, Université Libre de Bruxelles, 2015.

[36] M. Vidmar. Some harmonic functions for killed Markov branching processes with immigration and culling, *arXiv 1908.04714*, 2019.

[37] W. Woess Denumerable Markov chains *EMS Textbook in Mathematics, European Mathematical Society (EMS)*, Zurich, 2009

[38] A. Yakovlev and N, Yanev. Branching stochastic processes with immigration in analysis of renewing cell populations *Mathematical Biosciences*, 203(1), 37-63, 2006.

[39] K. B. Athreya and P. E. Ney. Branching processes, Springer-Verlag, New York-Heidelberg, 1972.

[40] C. Constantinescu, R. Loeffen, P. Patie, J. Wang. Fluctuation theory of continuous-time Skip-free Markov chains with applications to branching processes with immigration

[41] X. Duhaldea, C. Foucartb, C. Ma On the hitting times of continuous-state branching processes with immigration, Stochastic Processes and their Applications Volume 124, Issue 12, 2014.

[42] L. Miclo, P. Patie On a gateway between continuous and discrete Bessel and Laguerre processes, Annales Henri Lebesgue, 2019.

[43] A. Kyprianou Fluctuations Theory of Levy Processes with Applications, Springer, 2014.