PROBABILISTIC MODELS FOR POPULATION DYNAMICS

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Two interacting particle systems that serve as probabilistic models for population dynamics are studied in this work. The quadratic contact process is a stochastic spatial model for a population in which each individual has two parents and the dynamics are governed by random birth and death rates and an offspring distribution kernel. Another population model, due to Bolker and Pacala, models competition of different species in a forest. In both cases, we are interested in proving the existence of nontrivial stationary distributions.

BIOGRAPHICAL SKETCH

Mariya Bessonov was born in Belorussia. Following graduation from high school in North Carolina in 2002, Mariya entered North Carolina State University. She graduated in August of 2005 with a B.S. in mathematics, and earned an M.S. in mathematics in December of 2006. Mariya's interest in probability and stochastic processes were highly influenced by Min Kang and Stanislav Molchanov, when she was studying at North Carolina State University.

In the Fall of 2007, Mariya entered graduate school in mathematics at Cornell University, where she worked with Richard Durrett. She spent the academic years 2010-2012 at Duke University, which gave her an opportunity to work with Stanislav Molchanov and Joseph Whitmeyer on the Bolker-Pacala model. Mariya continued working in the field under the supervision of Richard Durrett. After receiving her Ph.D. from Cornell University, Mariya will become an Assistant Professor of Mathematics at the New York City College of Technology of the City University of New York.

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CHAPTER 1

INTRODUCTION

1.1 Interacting particle systems and population models

This work contains a study of two stochastic spatial models with important biological interpretations. We begin by describing the general setting for interacting particle systems and an overview of important results that apply to the models studied here. The two models, the quadratic contact process and the Bolker–Pacala process, will be introduced in the following two sections of the introduction and then analyzed in Chapters 2 and 3, respectively.

We consider models in which the set of spatial locations is the lattice $\mathbb{Z}^d$. There is a countable set of states S, which represents the number of individuals or *particles* at each spatial site, and the state of the system at time t is given by a function $\xi_t : \mathbb{Z}^d \to S$. There is also an *interaction neighborhood*, $\mathcal{N} \subset \mathbb{Z}^d$. Roughly speaking, the state of $x \in \mathbb{Z}^d$ may be influenced in the immediate future by the states of the sites in $x + \mathcal{N}$.

1.1.1 The contact process

First consider the continuous time contact process, ξ_t on $\mathbb{Z}^d$, introduced by Harris in 1974 [13]. Each vertex $x \in \mathbb{Z}^d$ is a site that may be occupied by a particle ($\xi_t(x) = 1$) or vacant ($\xi_t(x) = 0$) at time t. No more than 1 particle is allowed per site, thus, there are two states: $S = \{0, 1\}$.

The evolution is: if x is vacant, it becomes occupied at rate $\lambda|\mathcal{N}_x|$, $\lambda > 0$, and a particle at x dies with rate 1, independently of other particles. Here, $\mathcal{N}_x = \{y + x : y \in \mathcal{N}\}$, where $\mathcal{N}$ is a finite interaction neighborhood (the process is said to be finite range). The contact process is also translation invariant, because the rules that apply to any site x are just a translation of the rules at 0. Biologically, the contact process may be interpreted as the spread of a disease, where sites are individuals and 1's are infected individuals, or as the growth of a population where sites are patches that can be occupied (1) or vacant (0).

In particular, the rates have the following meaning: for $x \in \mathbb{Z}^d$,

$$\frac{P\left(\xi_{t+h}(x) = 0 \mid \xi_t(x) = 1\right)}{h} \to 1 \text{ as } h \to 0$$

and if $n_1(x, t)$ is the number of occupied neighbors of x at time t,

$$\frac{P\left(\xi_{t+h}(x) = 1 \mid \xi_t(x) = 0\right)}{h} \to \lambda n_1(x, t) \text{ as } h \to 0$$

Recall that 1 is the death rate for the process, the same for each particle.

Theorem 1.1.1. *[21] The specified rates for the contact process do determine a unique Markov process.*

The state of the process at time t is alternatively denoted by

$$\xi_t = \{x \in \mathbb{R}^d : \xi_t(x) = 1\},$$

the set of locations of all living particles. ξ_t^A will denote the state at time t if the process starts with initial configuration $\xi_0^A = A \subset \mathbb{R}$.

The contact process and the quadratic contact process that will be described in the next section are both *attractive processes*. That is, they have increasing birth rates and decreasing death rates: if the configurations ξ and ζ are such that

$\xi \subset \zeta \subset \mathbb{Z}^2$, for each x, the birth rate at x for the process in configuration η is at least the birth rate at x for the process in configuration ξ. That is, the birth rates are increasing. For the death rates to be decreasing, the inequality is reversed.

An important question to resolve is under which conditions the process has a nontrivial stationary distribution. To define this, start with the state space for the process, which is $S^{\mathbb{Z}^d}$, and take the product σ-algebra, $\mathcal{P}$ generated by the cylinder sets:

$$\{\xi(x_1) = i_1, \xi(x_2) = i_2, \ldots, \xi(x_n) = i_n\}.$$

The probability distribution $\pi_t : \mathcal{P} \to [0,1]$, with $\pi_t(A) = P(\xi_t \in A)$, is then determined by the finite dimensional distributions

$$\pi\left(\xi(x_1) = i_1, \xi(x_2) = i_2, \ldots, \xi(x_n) = i_n\right)$$

π is a *stationary distribution* if $\pi = \pi_t$ for all $t \geq 0$. It is nontrivial if $\pi \neq \delta_\emptyset$, the pointmass at all $0's$, which is an absorbing state, as there are no spontaneous births [8].

Let ξ_t^1 be a process starting from all sites initially occupied, $\xi_0^1(x) = 1$ for all $x \in \mathbb{Z}^d$. It is known that [8]:

Theorem 1.1.2. *If ξ_t is an attractive process, then ξ_t^1 converges in distribution to ξ_∞^1, a stationary distribution which is stochastically larger than any other stationary distribution for the process.*

If ξ_0 has distribution ξ_∞^1 and ζ_0 has distribution π, a stationary distribution, then ξ_∞^1 is *stochastically larger* than π if

$$E(f(\xi_\infty^1)) \geq \int_{S^{\mathbb{Z}^d}} f(\xi)\pi(d\xi)$$

for any increasing function f depending on finitely many coordinates [8].

Thus, if an attractive process is run from all sites initially occupied and $P(\xi_t^1(0) = 1) \to 0$ as $t \to \infty$, its only stationary distribution is the trivial one: the process dies out. We are interested in the phase transition from dying out to survival and if there are other intermediate phases. In the case of the contact process,

Theorem 1.1.3. *[8] There exists a critical value, λ_c, such that the process eventually dies out if $\lambda < \lambda_c$ and has a nontrivial stationary distribution for $\lambda > \lambda_c$.*

A useful tool for studying a large class of interacting particle systems, including attractive models and thus the contact process and quadratic contact process (discussed in the next section), is Harris' *graphical representation* or *percolation substructure* (see [15], also described in [8, 22]). To picture this for the contact process in $d = 1$, one can imagine the sites laid out on the horizontal axis and a vertical "pipe" emanating up from each site, the vertical direction representing time. Independent exponential random variables determine the temporal location of "dams" for each site, representing the death of a particle at that site. For each set of neighbors, x and $x + 1$, independent exponentials determine the times that x looks at $x + 1$ and becomes occupied if $x + 1$ is. In this case, an arrow ("bridge") is drawn from $x + 1$ to x. Then, one can input an initial configuration of open and closed sites. For the configuration at time t, it is enough to let water flow up through the open pipes, flowing across bridges (directionally) and blocked by dams. The set of wet sites at time t is the desired configuration. Thus, one can construct the process on the same probability space starting from any initial configuration. Versions of the graphical representation for discrete time processes were also considered in [26, 27] and will be used in Chapter 2.

Duality is another important tool in the study of contact processes, and it can be viewed through the graphical representation by reversing time. The contact process is self-dual [14]. With the dual, it can be seen that the state of a site x at time t is only influenced by the initial states of finitely many sites. It is also useful for proving survival of the contact process. Starting the process from all sites occupied,

A continuous version of the contact process has been introduced in [18], in which the set of spatial locations is $\mathbb{R}^d$. Here, the restriction of 1 particle per site is no longer necessary, as that will be the case with probability 1. Kondratiev, Kutoviy, and Pirogov [17] have shown the existence of a critical value, below which the process dies out. However, above the critical value, the density of particles approaches ∞ as $t \to \infty$. At the critical value, they have demonstrated the convergence of the process to a stationary measure, which uniquely depends on the initial density, in $d \geq 3$ but the lack of such behavior for $d = 2$.

1.2 The quadratic contact process

The quadratic contact process has generated interest in both the physics and mathematics literature. In contrast to the contact process, in the quadratic contact process, two particles (parents) are required in order to produce a new particle (offspring). Thus, it is also referred to as the contact process with sexual reproduction.

Durrett and Gray considered this model, introduced by Toom in 1974, with the rule that if x is vacant, a birth occurs at rate β if both of the nearest neighbors above and to right of x are occupied. Chen [5, 6] studied versions of the model where the interaction neighborhood is a subset of the set of nearest neighbors. He studied the stability of the trivial stationary distribution under certain perturbations.

Neuhauser considered the contact process with sexual reproduction in $d = 1$ with long-range interaction in continuous time (see [24]). In that version, the spatial locations are $\epsilon\mathbb{Z}$ and $\xi_t^\epsilon : \epsilon\mathbb{Z} \to \{0, 1\}$ has the following dynamics:

i) Particles die at rate 1, independently of others.

ii) A pair of adjacent particles at x and $x + \epsilon$ produces an offspring with rate λ, which is sent to a location y with probability $k_\epsilon(x - y)$. k_ϵ is the offspring distribution kernel, derived from an exponentially decaying, symmetric probability kernel k on $\mathbb{R}$.

iii) The birth at y is suppressed if y is already occupied by a particle.

Neuhauser has shown that:

Theorem 1.2.1. *[24] In the limit as $\epsilon \to 0$, starting from product measure, the density of particles, u, evolves as a solution to the integro-differential equation*

$$\frac{\partial u}{\partial t} = -u + \lambda(1 - u)(k * u^2) \tag{1.1}$$

In addition, (1.1) *admits traveling wave solutions, and there is a nondecreasing function* $c_k : (0, \infty) \to \mathbb{R} \cup \{\pm\infty\}$ *giving the wave speed corresponding to λ and k.*

Theorem 1.2.2. *[24] If $c_k(\lambda) > 0$, then for small enough ϵ, there is a nontrivial stationary distribution. Additionally, there is a constant λ^* above which the wave speed is indeed strictly positive.*

This is good news for the biological interpretation: a sufficiently long-range interaction and high enough birth rate allows for the possibility of survival. It contrasts with the case of nearest-neighbor interaction in Chen [5]: the process dies out for any $\lambda > 0$, if the initial configuration is product measure with low enough density.

Here, the version of the quadratic contact process that will be studied is based on Neuhauser's version, only in dimension $d = 2$ (the results will apply to higher dimensions as well) and in discrete time, with birth and death probabilities β and η, respectively. In the limit, it will be shown that as in Neuhauser, the system evolves according to an integro-difference equation analogous to (1.1) with traveling wave solutions.

Due to the existence of wave speeds for different directions in $d = 2$ as opposed to a single wave speed in $d = 1$, more interesting conclusions are reached regarding the stationary distributions: when all of the wave speeds are positive, the process has a nontrivial stationary distributions, and, moreover, can survive starting from a finite initial configuration. It is possible to have a nontrivial stationary distribution even if a negative wave speed exists (unlike in $d = 1$).

1.3 The Bolker–Pacala process

An important problem in population dynamics is to seek models which accurately incorporate the dynamic processes inherent in populations such as birth, death, migration, and competition for resources. The analysis of such a model should in turn yield realistic conclusions confirming existing knowledge and leading to new insight.

As a simple example, consider the stochastic process $\{Z_n\}_{\mathbb{N}\cup\{0\}}$ with $Z_0 = 1$ and

$$Z_{n+1} = \sum_{k=1}^{Z_n} X_k^{(n)},$$

where $X_k^{(n)}$ are independent and identically distributed random variables taking nonnegative integer values. The interpretation is that Z_n represents the number of individuals in the n^{th} generation, where $X_k^{(n)}$ is the number of offspring of the k^{th} individual from generation n. Let $\lambda = E(X_i^{(j)})$. To begin, the first question of interest is: under which conditions can the process survive? That is, when is the probability of extinction, $\lim_{n\to\infty} P(Z_n = 0) < 1$?

Theorem 1.3.1. *I. (subcritical and critical cases) If $\lambda \leq 1$, the population degenerates with probability* 1.

II. (supercritical case) If $\lambda > 1$, the population has a positive probability of survival, and if it does survive, it grows exponentially.

This is the Galton–Watson process, one of the earliest probabilistic models for population dynamics. While a good starting point, there are two major defects in the formulation alone:

i. A lack of spatial dynamics, that is, no information about the locations of individuals in relation to each other.

ii. The absence of interaction. For example, there is no mechanism to suppress the birth of new particles if resources become stressed.

The defect of no space can be removed if one considers an underlying graph (e.g. $\mathbb{Z}^d$, $\mathbb{R}^d$, a manifold, etc.) and the continuous time version of this process in which an individual particle dies with rate μ and gives birth to a new particle with rate b, independently of other particles. In addition, each particle performs a random walk, also independently of other particles.

Theorem 1.3.2. *If the underlying random walk is recurrent, in the critical case* $b = \mu$*, there is clustering.*

Namely, the particles form larger and larger clusters. The second moment tends to ∞ as $t \to \infty$. The particles form larger and larger colonies at increasing distances from each other, and as time progresses, the population consists of highly sparse colonies with a high population density. This effect is undesirable, as empirical evidence contradicts such phenomena in many populations. This exists because there is a lack of interaction in such a model.

The Bolker–Pacala process fixes both drawbacks and incorporates birth, death, migration, and an inter-species competition factor in a natural way into the dynamics of a population ([1], see also [7, 23]). The general model on $\mathbb{Z}^d$ is formulated below.

The parameters for the model are

$$\begin{aligned} b &= \text{birth rate} \\ \mu &= \text{natural death rate} \\ \kappa &= \text{rate of the random walk} \\ \gamma &= \text{competition rate} \\ a^+(z) &: \text{displacement kernel}\,, z \in \mathbb{Z}^d \\ a^-(z) &: \text{competition kernel}\,, z \in \mathbb{Z}^d \end{aligned}$$

The dynamics are that particles duplicate with rate b and die with rate μ, independently of other particles. In addition, they perform independent random walks with rate κ and displacement kernel a^+. With rate $\gamma a^-(x-y)$, a particle at x is annihilated if there is a particle present at y. No restriction is made on the number of particles per site. So the interaction is due to the competition between particles.

Let $n(t,x)$ denote the number of particles at time t, site x. Then

$$n(t+dt,z) = \begin{cases} n(t,z)+1 & : \text{ w. pr. } n(t,x)bdt + \kappa \sum_{y:y\neq x} a^+(x-y)n(t,y)dt \\ n(t,z)-1 & : \text{ w. pr. } n(t,x)\mu dt + \kappa n(t,x)dt + \gamma \sum_{y\in\mathbb{Z}^d} a^-(x-y)n(t,x)n(t,y)dt \end{cases}$$

The biological interpretation in $d = 2$ is that land has been divided into square plots. The number of particles at some $z \in \mathbb{Z}^2$ represents the number of species in the corresponding plot of land (we can suppose that z is the center of that square 1×1 plot). This is an simple model for the forest environment.

The main conjecture (which has been proven under some less than optimal conditions) is that

Conjecture 1.3.3. *[10, 11] If $\gamma > 0$, $b > \mu$, for any a^+, a^-, there exists a limiting point field $n(\infty, \gamma)$.*

Heuristically, if the population is small at each site, the competition will also be very small, but $b > \mu$ will allow the population to grow. On the other hand, if the population is very large, competition is very large, and death is proportional to $n^2(t,x)$, due to the competition term, while birth is proportional to $n(t,x)$, thus the population should decrease. In full generality, the above conjecture has not been proven.

In this work, a mean-field model derived from the general Bolker–Pacala model is studied. It will be analyzed via a logistic Markov chain. Results, which include a local central limit theorem, a large deviations result, and a functional law of large numbers and central limit theorem, are precisely stated in Chapter 3.

CHAPTER 2

THE QUADRATIC CONTACT PROCESS

We consider the two-dimensional contact process with sexual reproduction in discrete time (the results generalize to all higher dimensions as well). At time $n \in \{0, 1, 2, \ldots\}$ the state of a site x on the lattice $\mathbb{Z}^2/L$ is given by $\xi_n^L(x)$, which can take on the values 1 (there is a particle at site x) or 0 (site x is vacant). Starting with an initial configuration of particles, ξ_0^L on Z^2/L, the process evolves in the following manner:

i) At time n, given the configuration at the previous time $n-1$, with probability β, a vacant site x on the lattice will choose a pair of adjacent sites according to a probability kernel, described below. If both of the chosen sites are occupied, x will also become occupied. We will consider this a birth at site x with the parents being the chosen pair.

ii) After all births have occurred, with probability η, each particle is killed, independently of the others.

To describe how the parents are chosen, start with a probability kernel $k : \mathbb{R}^2 \to [0, 1]$. Assume that k is piecewise continuous and has $\mathbb{Z}^2$ symmetry. That is, if T_θ is a counterclockwise rotation on $\mathbb{R}^2$ by angle θ, for any $x \in \mathbb{R}^2$,

$$k(x) = k(T_{\pi/2}(x)) = k(T_\pi(x)) = k(T_{3\pi/2}(x)).$$

Fix $\gamma \in (0, 1)$ and partition the lattice $\mathbb{Z}^2/L$ into small boxes of side length $L^{-\gamma}$ whose corners are at the points $\frac{(m,n)}{L^\gamma}$, with $m, n \in \mathbb{Z}$.

For any $x \in \mathbb{Z}^2/L$, let x^* be the left bottom corner of the box containing x.

Then the box containing x is

$$B^L(x) = \left\{ y \in \mathbb{Z}^2/L : y \in x^* + [0, L^{-\gamma})^2 \right\}.$$

Each such box contains

$$\left(\frac{L^{-\gamma}}{1/L}\right)^2 = L^{2-2\gamma}$$

points. Then for $x \in \mathbb{Z}^2/L$, let

$$k_L(x) = \int_{B^L(x)} k(y)dy, \tag{2.1}$$

For $y \in Z^2/L$, with probability $k_L(y - x)$, x will choose y as its first parent. The second parent will then be chosen at random among $y + (0, 1/L)$ and $y + (1/L, 0)$, with equal probability for each.

Before stating the first result, it is necessary to introduce some notation. Now define $S_n^L(x)$ to be the number of particles alive in $B^L(x)$ at time n (S for "sum"):

$$S_n^L(x) = \sum_{y \in B^L(x)} \xi_n^L(y).$$

Also, for $x \in \mathbb{Z}^2/L$, define

$$u_n^L(x) := \mathbb{P}_0\left(\xi_n^L(x) = 1\right)$$

and extend u_n^L to a function on $\mathbb{R}^2$ by setting

$$u_n^L(y) = u_n^L(x) \text{ for } y \in x + [0, 1/L)^2$$

Let $u_0 : \mathbb{R}^2 \to [0, 1]$ be continuous. Fix $n \in \{0, 1, 2, \ldots\}$. Define $\xi_n^L \sim u_n$ to mean that

$$\sup_{x \in [-K,K]^2 \cap \mathbb{Z}^2/L} \left| \frac{S_n^L(x)}{L^{2-2\gamma}} - u_n(x^*) \right| < \epsilon_L, \tag{2.2}$$

where $\epsilon_L \to 0$ as $L \to \infty$ for all K.

This says that the proportion of particles in each box is very close to the value of u_n at the lower left corner of the box, that is, ξ_n^L is asymptotically like u_n as $L \to \infty$.

Let $\zeta_n^L : \mathbb{Z}^2/L \times \mathbb{Z}^2/L \to \{0, 1\}$ be defined by

$$\zeta_n^L(x, y) = \begin{cases} 1 & \text{if } |x - y| = 1/L \text{ and } \xi_n^L(x) = \xi_n^L(y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\zeta_n^L \sim u_n^2$ is defined in a similar way. Let $P_n^L(x)$ = number of pairs of adjacent particles alive in $B^L(x)$ at time n. Each box $B^L(x)$ contains a total of $2L^{1-\gamma}(L^{1-\gamma} - 1)$ different pairs.

Then $\zeta_0^L \sim u_0^2$ if

$$\sup_{x \in [-K,K]^2 \cap \mathbb{Z}^2/L} \left| \frac{P_n^L(x)}{2L^{1-\gamma}(L^{1-\gamma} - 1)} - u_n^2(x^*) \right| < \epsilon_L, \tag{2.3}$$

where $\epsilon_L \to 0$ as $L \to \infty$ for all K.

Suppose from now on that L is a positive integer.

Theorem 2.0.4. *Suppose the sequence of initial configurations $\xi_0^L \sim u_0$, where $u_0(x) : \mathbb{R}^2 \to [0, 1]$ is continuous. Assume also that $\zeta_0^L \sim u_0^2$. Then as $L \to \infty$, $u_n^L(x) \to u_n(x)$, the solution of*

$$u_{n+1} = (1 - \eta) \left[u_n + \beta(1 - u_n) \left(k * u_n^2 \right) \right], \tag{2.4}$$

What this theorem says is that in the limit, the density of particles evolves as if neighboring sites are independent.

Wave speeds are defined for a class of operators that satisfy some natural assumptions. In this case, let $Q(u_n) := u_{n+1}$, where

$$u_{n+1} = (1-\eta)\left[u_n + \beta(1-u_n)\left(k * u_n^2\right)\right]$$

for $\beta > \frac{4\eta}{1-\eta}$.

Let $\theta \in S^1$ be a unit vector, then there exists a function $c^* : S^1 \to \mathbb{R} \cup \{-\infty, \infty\}$, and $c^*(\theta)$ can be thought of as the speed of plane waves whose normal is in the direction ξ.

Theorem 2.0.5. *If $c^*(\theta) > 0$ for all $\theta \in S^1$, there exists a nontrivial stationary distribution for the particle system for large enough L.*

This is shown by a block argument as in [3].

Theorem 2.0.6. *If there exists a $\theta \in S^1$ with $c^*(\theta) > 0$, the particle system has a nontrivial stationary distribution for large enough L.*

However, if the process starts from finitely many particles, that group of particles would be pushed to extinction from the directions corresponding to the negative speeds and

Conjecture 2.0.7. *If there also exists a $\theta \in S^1$ with $c^*(\theta) < 0$, the system cannot survive starting from an initial configuration with only a finite number of particles.*

This result 2.0.6 is proved using a "generic population model" [3] as a comparison process whose vacant region encloses all the regions in the quadratic contact process where the density of particles is too low.

Survival of the comparison process then implies the existence of a nontrivial stationary distribution for the quadratic contact process.

2.1 Proof of Theorem 2.0.4

In order to simplify the computation of expectations and variances, a slightly different process $\hat{\xi}_n^L$ on $\mathbb{Z}^2/L$ will now be defined. Let T_x be the translation by x.

i) At each time n, given the configuration at time $n-1$, with probability β, a vacant site x on the lattice will become occupied if a randomly chosen pair of parents, located at a pair of adjacent sites, are both alive at time $n-1$. The first parent will be chosen according to the probability kernel $T_{x^*}k_L$, instead of $T_x k_L$. The second parent is chosen at random, with equal probability, among the nearest neighbors to the left and above the first parent. We will consider this a birth at site x with the parents being the chosen pair.

ii) After all births have occurred, with probability η, each particle is killed, independently of the others.

At time 1 the two processes are close with high probability, which will allow us to work with $\hat{\xi}_n^L$ instead of ξ_n^L in the proof.

Claim 2.1.1. *Suppose $\xi_0^L(x) = \hat{\xi}_0^L(x)$ for each $x \in \mathbb{Z}^2/L$, then*

$$\mathbb{P}\left(\xi_1^L(x) \neq \hat{\xi}_1^L(x)\right) \to 0 \text{ as } L \to \infty \tag{2.5}$$

Proof. For $y \in \mathbb{Z}^2/L$, let $\alpha_L(y) = k_L(y-x)$ and $\beta_L(y) = k_L(y-x^*)$. And set

$$p_L = \sum_{y \in \mathbb{Z}^2/L} (\alpha_L(y) \wedge \beta_L(y)) .$$

Couple ξ_n^L and $\hat{\xi}_n^L$ as follows:

Both processes start with the same initial configuration.

The parents are chosen in the following manner. If a birth event occurs at site $x \in Z^2/L$ at time 1, flip a coin with probability p_L of heads. If heads comes up, then with probability

$$\frac{\alpha_L(y) \wedge \beta_L(y)}{p_L}$$

choose the same parents for x in both processes, with y being one of the parents and a randomly chosen nearest neighbor above or to the left of y the second parent. Otherwise, the particle in ξ_L chooses a parent y with probability

$$\frac{2}{1-p}\left(\alpha_L(y) - \beta_L(y)\right)^+$$

and the particle in $\hat{\xi}_L$ chooses a parent z with probability

$$\frac{2}{1-p}\left(\beta_L(z) - \alpha_L(z)\right)^+ .$$

If the particle at site x dies at time 1 in ξ^L and is present in $\hat{\xi}^L$, it also dies in $\hat{\xi}^L$.

Given a sequence $\{x_L : x_L \in Z^2/L\}$ that converges to a continuity point, x, of k, since

$$|(x_L)^* - x_L| \leq \sqrt{2}L^{-\gamma}, \quad (x_L)^* \to x$$

as well. So $k_L(x_L) \to k(x)$ and $k_L(x_L^*) \to k(x)$, which implies that $p_L \to 1$, so

$$P\left(\xi_1^L(x) \neq \hat{\xi}_1^L(x)\right) \to 0$$

as $L \to \infty$. □

It will only be necessary to show that Theorem 2.0.4 holds at time 1. That is, given the assumptions on the initial conditions, it will be shown that $u_1^L \to u_1$ pointwise, $\xi_1^L \sim u_1$, and $\zeta_1^L \sim u_1^2$. An induction argument completes the proof.

Starting with the initial distribution $\xi_0^L \equiv \hat{\xi}_0^L$, we can compute the expectation and variance of the state of any site at time 1. Given that, the expected value and variance of the proportion of occupied sites in each box, $B^L(x)$, will be computed. For now, fix L and let $\mathbb{P}_0$ denote the probability law for the process $\hat{\xi}_n^L$ with initial configuration $\hat{\xi}_0^L$. In order to write out the computations more compactly, introduce the notations

$$X_0 = \hat{\xi}_0^L(x),\ X_1 = \hat{\xi}_1^L(x),\ Y_0 = \hat{\xi}_0^L(y),\ Z_0 = \hat{\xi}_0^L(z),\ \text{etc.},$$

for $x, y, z \in \mathbb{Z}^2/L$, and set

$$K_L(x) = \sum_{y \in \mathbb{Z}^2/L} k_L(y - x^*)\hat{\xi}_0^L(y)\frac{1}{2}\left(\hat{\xi}_0^L(y + (1/L, 0)) + \hat{\xi}_0^L(y + (0, 1/L))\right).$$

Then

$$\mathbb{E}_0\left(X_1\right) = \mathbb{P}_0\left(X_1 = 1\right) = (1 - \eta)\left[X_0 + \beta\left(1 - X_0\right) \cdot K_L(x)\right] := p_x.$$

Since X_1 is Bernoulli,

$$\mathrm{Var}_0\left(X_1\right) = p_x - p_x^2.$$

The total number of particles alive in $B^L(x)$ at time 1 is

$$\hat{S}_1^L(x) := \sum_{y \in B^L(x)} \hat{\xi}_1^L(y),$$

so the expected proportion of occupied sites in $B^L(x)$ at time 1 is

$$\begin{aligned}\mathbb{E}_0\left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}}\right) &= \frac{1}{L^{2-2\gamma}}\sum_{y \in B^L(x)} \mathbb{E}_0\left(\hat{\xi}_1^L(y)\right) \\ &= (1 - \eta)\left[\frac{\hat{S}_0^L(x)}{L^{2-2\gamma}} + \beta\left(1 - \frac{\hat{S}_0^L(x)}{L^{2-2\gamma}}\right)K_L(x)\right]. \qquad (2.6)\end{aligned}$$

For $y \neq z$ in $B^L(x)$,

$$
\begin{aligned}
\mathrm{Cov}_0\left(Y_1, Z_1\right) &= \mathbb{E}_0\left(Y_1 Z_1\right) - \mathbb{E}_0\left(Y_1\right)\mathbb{E}_0\left(Z_1\right) \\
&= (1-\eta)^2\Big[Y_0 Z_0 + \beta \cdot K_L(x)\left((1-Y_0)Z_0 + Y_0(1-Z_0)\right) \\
&\qquad + \left(\beta \cdot K_L(x)\right)^2 (1-Y_0)(1-Z_0)\Big] - p_y p_z \\
&= 0. \qquad (2.7)
\end{aligned}
$$

So

$$
\begin{aligned}
\mathrm{Var}_0\left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}}\right) &= \frac{1}{L^{4-4\gamma}} \sum_{y \in B^L(x)} \mathrm{Var}\left(\hat{\xi}_1^L(y)\right) \\
&= \frac{1}{L^{4-4\gamma}} \sum_{y \in B^L(x)} \left(p_y - p_y^2\right) \\
&= \frac{1}{L^{4-4\gamma}}(1-\eta)\left[\hat{S}_0^L(x) + \beta\left(L^{2-2\gamma} - \hat{S}_0^L(x)\right) \cdot K_L(x)\right] \\
&\quad - \frac{1}{L^{4-4\gamma}}(1-\eta)^2\left[\hat{S}_0^L(x) + (\beta \cdot K_L(x))^2\left(L^{2-2\gamma} - \hat{S}_0^L(x)\right)\right] \\
&\leq C \cdot \frac{1}{L^{2-2\gamma}},
\end{aligned}
$$

where C is a constant that does not depend on L. Now by Chebyshev's inequality,

$$
\mathbb{P}_0\left(\left|\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} - \mathbb{E}_0\left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}}\right)\right| \geq \delta\right) \leq \frac{\mathrm{Var}\left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}}\right)}{\delta^2} \leq \frac{C}{\delta^2 L^{2-2\gamma}}.
$$

There are $4L^{2\gamma} \cdot K^2$ boxes of side length $L^{-\gamma}$ in each $[-K, K]^2$ box, so

$$
\mathbb{P}_0\left(\sup_{x \in [-K,K]^2 \cap \mathbb{Z}^2/L}\left|\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}} - \mathbb{E}_0\left(\frac{\hat{S}_1^L(x)}{L^{2-2\gamma}}\right)\right| \geq \delta\right) \leq \frac{CL^{2\gamma}}{\delta^2 L^{2-2\gamma}}. \qquad (2.8)
$$

For $\gamma \in (0, \frac{1}{2})$, this probability tends to 0 as $L \to \infty$. For $x \in \mathbb{R}^2$ and $x_L \in \mathbb{Z}^2/L$ such that $x_L \to x$ as $L \to \infty$,

$$
K_L(x_L) \to (k * u_0^2)(x).
$$

By the assumptions of Theorem 2.0.4,

$$\frac{\hat{S}_0^L(x)}{L^{2-2\gamma}} = \frac{S_0^L(x)}{L^{2-2\gamma}} \to u_0(x).$$

Together with (2.6), this implies that

$$\mathbb{E}_0\left(\frac{\hat{S}_1^L(x_L)}{L^{2-2\gamma}}\right) \to u_1(x) \quad \text{as} \quad L \to \infty.$$

Then by (2.8),

$$\frac{\hat{S}_1^L(x_L)}{L^{2-2\gamma}} \to u_1(x)$$

and thus $\hat{\xi}_1^L \sim u_1$, and from (2.5) it follows that $\xi_1^L \sim u_1$.

Next, it must be shown that $\hat{\zeta}_1^L \sim u_1^2$. Starting with some notation, let

$$P^L = 2L^{1-\gamma}\left(L^{1-\gamma} - 1\right)$$

be the total number of adjacent pairs of sites in a box $B^L(x)$. There are $P_0^L(x)$ pairs with both sites occupied at time 0,

$$4S_0^L(x) - 2P_0^L(x)$$

pairs with just one site occupied at time 0, and there are

$$P^L - 4S_0^L(x) + P_0^L$$

pairs with both sites vacant at time 0. So the expected number of pairs at time 1 is

$$\begin{aligned}\mathbb{E}_0\left(P_1^L(x)\right) = (1-\eta)^2[P_0^L(x) + \beta \cdot K(x^*)(4S_0^L(x) - 2P_0^L(x)) \\ + (\beta \cdot K(x^*))^2\left(P^L - 4S_0^L(x) + P_0^L\right)].\end{aligned}$$

By the assumptions of Theorem 2.0.4, if $x_L \in \mathbb{Z}^2/L$ such that $x_L \to x$,

$$\begin{aligned}\mathbb{E}_0\left(\frac{P_1^L(x)}{P^L}\right) &\to (1-\eta)^2[u_0^2(x) + (k * u_0^2)(2u_0(x) - 2u_0^2(x)) \\ &\quad + \left(k * u_0^2\right)^2\left(1 - 2u_0(x) + u_0^2(x)\right)] \\ &= u_1^2(x), \qquad \text{as } L \to \infty.\end{aligned}$$

To compute the variance, first consider

$$\mathrm{Var}_0\left(\hat{\xi}_1^L(x)\hat{\xi}_1^L(x')\right)$$

for x' a neighbor of x. Like earlier, the notation

$$X = \hat{\xi}_0^L(x),\ X' = \hat{\xi}_0^L(x'),\ X_1 = \hat{\xi}_1^L(x),\ \text{etc.}$$

will be used, and now set $c(x) = \beta \cdot K_L(x)$.

$$\begin{aligned}\mathbb{E}_0\left(X_1X_1'\right) &= \mathbb{P}_0(X_1X_1' = 1)\\ &= (1-\eta)^2[XX' + c(x)((1-X)X' + X(1-X')) + c^2(x)(1-X)(1-X')]\\ &:= p_{x,x'}\end{aligned}$$

And since X_1X_1' is Bernoulli,

$$\mathrm{Var}_0\left(X_1X_1'\right) = p_{x,x'} - p_{x,x'}^2.$$

Now consider the covariance of two different pairs. By a similar computation to the one done in (2.7), it is seen that

$$\mathrm{Cov}_0\left(X_1X_1', Y_1Y_1'\right) = 0$$

for any pairs that do not share points in common. If two pairs do share a common point, we have the following:

$$\begin{aligned}\mathrm{Cov}_0\left(X_1X_1', X_1X_1''\right) &= \mathbb{E}_0\left(X_1X_1'X_1X_1''\right) - \mathbb{E}_0\left(X_1X_1'\right)\mathbb{E}_0\left(X_1X_1''\right)\\ &= \mathbb{E}_0\left(X_1X_1'X_1''\right) - \mathbb{E}_0\left(X_1X_1'\right)\mathbb{E}_0\left(X_1X_1''\right)\\ &= (1-\eta)^3\Big[XX'X'' + c(x)\left((1-X)X'X'' + X(1-X')X'' + XX'(1-X'')\right)\\ &+\ c^2(x)\left((1-X)(1-X')X'' + X(1-X')(1-X'') + (1-X)X'(1-X'')\right)\\ &+\ c^3(x)(1-X)(1-X')(1-X'')\Big] - p_{x,x'}p_{x,x''}\end{aligned}$$

So the variance of the proportion of occupied pairs in $B^L(x)$ is

$$\mathrm{Var}_0\left(\frac{P_1^L(x)}{P^L}\right) = \frac{1}{(P^L)^2}\left(\sum_{\text{pairs}\in B^L(x)} \mathrm{Var}_0\left(Y_1Y_1'\right) + \sum_{\text{pairs of pairs}\in B^L(x)} \mathrm{Cov}_0\left(Y_1Y_1', Y_1Y_1''\right)\right)$$

The first summation is

$$\begin{aligned}\sum_{\text{pairs}\in B^L(x)} \mathrm{Var}_0\left(Y_1Y_1'\right) &= \sum_{\text{pairs}\in B^L(x)} \left(p_{y,y'} - p_{y,y'}^2\right)\\ &= (1-\eta)^2\left[P_0^L(x) + c(x)(4S_0^L(x) - 2P_0^L(x)) + c^2(x)\left(P^L - 4S_0^L(x) + P_0^L\right)\right]\\ &- (1-\eta)^4\left[P_0^L(x) + c^2(x)(4S_0^L(x) - 2P_0^L(x)) + c^4(x)\left(P^L - 4S_0^L(x) + P_0^L\right)\right]\\ &\le C_1 \cdot P^L,\end{aligned}$$

where C_1 is a constant independent of L.

For the second summation, note that a single pair of particles may share a point in common with at most 6 other pairs. Therefore,

$$\sum_{\text{pairs of pairs}\in B^L(x)} \mathrm{Cov}\left(Y_1Y_1', Y_1Y_1''\right) \le C_2 \cdot P^L$$

(when summing over terms such as $XX'X''$, the result is proportional to 6 times the total number of pairs). Therefore,

$$\mathrm{Var}_0\left(\frac{P_1^L(x)}{P^L}\right) \le \frac{C}{P^L} \le \frac{C}{L^{2-2\gamma}}.$$

According to Chebyshev's inequality,

$$\mathbb{P}_0\left(\left|\frac{P_1^L(x)}{P^L} - \mathbb{E}_0\left(\frac{P_1^L(x)}{P^L}\right)\right| \ge \delta\right) \le \frac{\mathrm{Var}\left(\frac{P_1^L(x)}{2L^{1-\gamma}(L^{1-\gamma}-1)}\right)}{\delta^2} \le \frac{C}{\delta^2 L^{2-2\gamma}}$$

There are $4L^{2\gamma}\cdot K^2$ little boxes of side length $L^{-\gamma}$ in each $[-K,K]^2$ box, so

$$\mathbb{P}_0\left(\sup_{x\in[-K,K]^2\cap\mathbb{Z}^2/L}\left|\frac{P_1^L(x)}{P^L} - \mathbb{E}_0\left(\frac{P_1^L(x)}{P^L}\right)\right| \ge \delta\right) \le \frac{C'L^{2\gamma}}{L^{2-2\gamma}}$$

For $\gamma \in (0, \frac{1}{2})$, this probability tends to 0 as $L \to \infty$. Thus $\hat{\zeta}_1^L \sim u_1^2$ and by (2.5), $\zeta_1^L \sim u_1^2$.

2.2 When all of the wave speeds are positive

When all of the wave speeds are positive, the particle system has a nontrivial stationary distribution. This is shown by using Weinberger's shape theorem [28] together with a block argument, a rescaling technique of Bramson and Durrett [2]. The proof of theorem (2.0.5) is nearly identical to the proof provided in [24] for the 1-dimensional quadratic contact process in continuous time and is therefore only outlined below.

Consider first spatially homogeneous solutions to the integro-difference equation (2.4) of Theorem 2.0.4,

$$u_{n+1} = (1-\eta)\left[u_n + \beta(1-u_n)\left(k * u_n^2\right)\right]. \tag{2.4}$$

Given a constant initial density u_0, any spatially homogeneous solution u_n of (2.4) satisfies

$$u_{n+1} = (1-\eta)u_n + \beta(1-u_n)u_n^2. \tag{2.9}$$

When $\beta > \frac{4\eta}{1-\eta}$, there are three equilibrium solutions of (2.9): $0, \rho_u$, and ρ_s, where 0 and ρ_s are stable and ρ_u is unstable, and

$$\rho_u = \frac{1-\sqrt{1-\frac{4\eta}{\beta(1-\eta)}}}{2} \qquad \text{and} \qquad \rho_s = \frac{1+\sqrt{1-\frac{4\eta}{\beta(1-\eta)}}}{2}.$$

Thus, $0 < \rho_u < 1/2 < \rho_s < 1$.

There are three main steps in the proof of Theorem 2.0.5. Let $\delta > 0$ be small and $K > 0$ be large. First, Weinberger's shape theorem implies that

Lemma 2.2.1. *There is an $N < \infty$ such that $u_0(x) > \rho_u + \delta$ on $[-K,K]^2$ implies that $u_N(x) > \rho_s - \delta$ on $[-4K,4K] \times [-K,K]$.*

The second step is to show that the behavior of the quadratic contact process for large L is close to that of the limiting system with high probability. More precisely,

Lemma 2.2.2. *If ξ_0^L is such that (i) the density of particles in $B^L(x)$ is $> \rho_u + 3\delta$ and (ii) the density of pairs of particles is $> (\rho_u+2\delta)^2$, for all $x \in [-K,K]^2 \cap \mathbb{Z}^2/L$, then (i) and (ii) hold for ξ_N^L for all $x \in [-3K,-K] \times [-K,K] \cap \mathbb{Z}^2/L$ and all $x \in [K,3K] \times [-K,K] \cap \mathbb{Z}^2/L$.*

The details of this proof are found in [24] for the 1-dimensional continuous time process, with only slight modifications being necessary for the 2-dimensional discrete time version. To summarize, item (i) of the lemma is first shown to be true for the process starting from product measure. The dual process, which allows one to work backwards in time to determine the state of a site x at a time n given the configuration at some time $m < n$, is used here. As in the continuous time case, the dual process at site x at time $n-m$, $n > m$, consists of finitely many subsets of sites representing the possible ancestors of x. If any single one of the subsets is totally occupied at time m in the contact process, x will be occupied at time n. It is shown that for the dual processes of two sites $x \neq y \in \mathbb{Z}^2/L$ are asymptotically independent. This fact can be used to bound the covariance of $\xi_N^L(x)\xi_N^L(y)$ and an application of Chebyshev's inequality yields that with high probability, the proportion of occupied sites in $[-3K,-K]\times[-K,K]$ and $[K,3K] \times [-K,K]$ is close to ρ_s.

The procedure of creating new "high-density blocks" should be iterated, however, the configuration after one iteration is not product measure. Thus, Neuhauser has shown that with high probability, the dual processes for two different sites do not collide, and with this an estimation of the expectation and variance of the densities or particles and densities of pairs at time N together with Chebyshev's inequality provides the result of Lemma 2.2.2. With very slight modifications, the same can be shown for the 2-dimensional discrete time process.

The third and final step is to note that the particle system dominates a weakly dependent oriented percolation process, in which there is positive probability of percolation.

An important consequence of this proof is that

Lemma 2.2.3. *If $\xi_0^L(x) = 1$ on $[-K, K]^2 \cap \mathbb{Z}^2/L$ and L is sufficiently large, the quadratic contact process has a positive probability of survival.*

Thus, the process can survive starting from an initial configuration with only finitely many particles.

2.3 Negative and positive wave speeds

Theorem 2.0.6 states that if there is at least one direction which has a positive wave speed, then a nontrivial stationary distribution for the process exists – even if there are also negative wave speeds. This section is entirely dedicated to the proof of Theorem 2.0.6.

The result will be proved using Bramson and Gray's comparison model [3]. The goal is to show that the contact process and the comparison process can be coupled so that regions in the quadratic contact process with a low density of particles are entirely contained in the vacant region of the comparison model. Then the existence of a nontrivial stationary distribution for the comparison model will imply the existence of a nontrivial stationary distribution for the quadratic contact model.

The comparison process of Bramson and Gray takes place on $\mathbb{R}^2$, in continuous time. At each time, the plane is divided into two regions: a vacant and a nonvacant region. Let A_t denote the vacant region of the comparison process at time t and take $A_0 = \emptyset$. $\mathcal{P}$ is a Poisson process on $\mathbb{R}^2 \times [0, \infty)$ with intensity ϵ. If $(x, t) \in \mathcal{P}$, a triangular vacant region centered at $x \in \mathbb{R}^2$ of a certain fixed size is created at time t. The edges of this region will be perpendicular to the fixed unit orientation vectors n_1, n_2, n_3 and will move inward with rates a_1, a_2, a_3, respectively. In case two or more vacant regions overlap or collide, a new vacant region is formed with the same geometry, whose edges move outward at rate $b > 0$ (the *interaction rate*) until each edge catches up to the corresponding edges of all of the regions which produced the original overlap or collision. At that point, the edges of the overlap or collision region will begin to move inward with rates a_i. If $a_i > 0$, for $i = 1, 2, 3$, Bramson and Gray have shown that the process has a nontrivial stationary distribution.

Coming back to the quadratic contact process, let ξ_i be the directions with positive wave speeds $\alpha_i > 0$, $i = 1, 2, 3$. Since the offspring distribution kernel has $\mathbb{Z}^2$ symmetry, the existence of a positive speed in one direction implies the existence of a positive speed in three directions, perpendicular to the edges of an acute triangle in the plane.

We shall say that the box $B^L(x)$ is *bad* at time n if the density of the particles inside $B^L(x)$ falls below an appropriate threshold $\alpha \in (\rho_u, \rho_s)$, to be specified below. Recall that $0 < \rho_u < \rho_s < 1$ are the nonzero fixed points of the operator

$$Q[u] = (1-\eta)\left[u + \beta(1-u)\left(k * u^2\right)\right], \tag{2.10}$$

which exist when $\beta > \frac{4\eta}{1-\eta}$.

The collection of bad boxes at time n will be contained in the vacant region of the comparison process. Let $\tilde{A}_n$ be the bad region (the collection of all bad boxes) of the contact process on $\mathbb{Z}^2/L$ at time n. The process initially has all sites occupied, so that $\tilde{A}_0 = \emptyset$.

From the definition of the wave speed in [28], there is a function $\phi : \mathbb{R} \to \mathbb{R}$ with the following properties:

i) ϕ is continuous,

ii) ϕ is nonincreasing,

iii) $\phi(-\infty) \in (\rho_u, \rho_s)$ and $\phi(x) = 0$ for $x \geq k_0 > 0$.

In addition, $Q[\phi]$ dominates certain translates of ϕ:

$$\phi(s-c) \leq Q[\phi](s) \tag{2.11}$$

for all $s \in \mathbb{R}$, where one can take $c = \min\{\alpha_i/2 : i = 1, 2, 3\}$. Now, set $\alpha = \phi(-\infty)$ and let

$$m = \sup\{s : Q[\phi](s) = \alpha\} \qquad \text{and} \qquad M = \inf\{s : Q[\phi](s) = 0\}.$$

Set $l = M - m$.

If an *error* occurs in the contact process such as a box $B^L(x)$ that is good at time $n-1$ becomes bad at time n, in the comparison process this box is covered by a triangular region, $D(x,n,t)$ (the first n indicating that the region forms at time n and the t indicating the shape of the region at time t), of an appropriate size, so that the center of the box is at the incenter of the triangle whose inscribed circle has radius

$$r = \lceil l + d(B) + c + d(k) \rceil,$$

initially at time n, where $d(B) = L^{-\gamma}\sqrt{2}$ is the diameter of the box and

$$d(k) = \sup\{|x-y| : x, y \in \mathbb{R}^2, k(x) \neq 0, k(y) \neq 0\}$$

is the diameter of the kernel. Let R be the radius of the circumscribed circle of $D(x,n,n)$. Once such a region is created, the edges move inward, each with linear rate c.

Set the *interaction rate* $b := 2d(k)$. In the comparison process, should two or more triangles collide or overlap, a new *collision* or *overlap* region is formed at that time, which is the intersection of the maximal collection of the colliding or overlapping regions that has nonempty intersection. The edges of the new collision of overlap region then move outward at rate b in each direction until they have caught up to all of the respective edges of the regions that initiated the overlap or collision in each respective direction, and after that the edge of the overlap or collision region will again move inward with rate c.

Bramson and Gray use a Poisson point process in their comparison model. Instead, we will consider a different point process of errors, $\mathcal{P}$, derived from the quadratic contact process, which will be described below after introducing some definitions. In fact, the results of Bramson and Gray still hold with this underlying point process instead of the Poisson.

$\mathcal{P}$ will be a point process of *errors* in the quadratic contact process. Two different types of errors are possible:

Type I Error: Suppose that

$$B^L(x) \cap \tilde{A}_{n-1} = \emptyset.$$

Thus, $B^L(x)$ that was good at time $n-1$ and all other boxes within distance $d(k)$ are also good at $n-1$ (since each triangle has a type of buffer region distance $d(k)$ away from the edges with good boxes). If $B^L(x)$ is bad at n, this spontaneous error will be considered a *Type I* error.

Type II Error: If

$$B^L(x) \cap \tilde{A}_{n-1} \neq \emptyset,$$

let

$$\{R_i(y_i, m_i, n-1)\}_{i \in I},$$

I a countable index set, be the maximal collection of triangular regions such that

$$B^L(x) \subset \bigcap_{i \in I} R_i(y_i, m_i, n-1),$$

where $R_i(y_i, m_i, n-1)$ is a triangular region centered at $y_i \in \mathbb{Z}^2/L$ that was created at time $m_i \in \mathbb{N}$ (it could be one of the regions D or an overlap or a collision region). For $j = 1, 2, 3$, let $H_{i,j}(t)$ be the halfspace in $\mathbb{R}^2$ containing $R(y_i, m_i, t)$ and with boundary containing the j^{th} side of the triangle. Then the results of Bramson and Gray imply that there exists a region $R(y, m, n-1)$ centered at y and created at time m such that

$$\bigcap_{i \in I} R_i(y_i, m_i, t) \subset R(y, m, t) \text{ for all times } t \in [m, n-1],$$

and $B^L(x) \subset R(y, m, n-1)$.

Furthermore, the j^{th} edge and corresponding halfspace containing R and its j^{th} edge at time t, $H_j(t)$ either:

i) moves outward with rate β, or

ii) moves inward with rate c and is such that

$$\bigcup_{i\in I} H_{i,j}(t) \subset H_j(t).$$

For $x \in \cup_{i\in I} R_i(y_i, m_i, n)$, let

$$h_n^{(i,j)}(x) = Q^{n-m_i}[\phi(\xi_j \cdot (x - y_i))],$$

where Q^{n-m_i} is the application of Q $(n - m_i)$ times. Let $h_n^{(j)}(x) = \inf_{i\in I} h_n^{(i,j)}(x)$ for $x \in \mathbb{R}^2$, and let $h_n(x) = \max_{j=1,2,3} h_n^{(j)}(x)$.

A *type II error* occurs at time n if for $x \in \cup_{i\in I} R_i(y_i, m_i, n)$, the density of particles in $B^L(x)$ is below $h_n(x)$ at time n.

Next, we describe the point process, $\mathcal{P}$ on $\mathbb{R}^2 \times [0, \infty)$ used in the comparison process. It is derived from the quadratic contact process on $\mathbb{Z}^2/L$. For each type I or type II error that occurs in the contact process, there is a single corresponding point in $\mathcal{P}$. If the error occurs in $B^L(x)$ at time n, let $(y, t) \in \mathcal{P}$ where $y \in x^* + [0, L^{-\gamma})^2$ is a single point in the box, chosen uniformly and at random from $x^* + [0, L^{-\gamma})^2$ and t is chosen uniformly and at random from $[n-1, n)$. Although $\mathcal{P}$ is not quite a Poisson point process, it shares two key properties with the Poisson process, the only two used in Bramson and Gray's proof, that are sufficient to demonstrate a nontrivial stationary distribution for large enough L. Namely (see [3], 2-1 and 2-2),

$$P(| B \cap \mathcal{P} | \geq 2) = O\left(\lambda(B)^2\right) \tag{2.12}$$

as $\lambda(B) \to 0$, where λ is Lebesgue measure on $\mathbb{R}^2 \times [0, \infty)$, for Borel sets B.

(2.12) is satisfied, as $\mathcal{P}$ is dominated by a Poisson process with parameter $\tilde{\epsilon}$ that satisfies

$$\epsilon L^{2\gamma} = 1 - e^{-\tilde{\epsilon}},$$

where ϵ is an upper bound on the probability of an error (type I or type II) in the contact process with $\epsilon L^{2\gamma} \to 0$ as $L \to \infty$ (see the proof of Theorem 2.3.1 below).

The second property is that for all small enough disjoint cubes $B_1, B_2, \ldots, B_m$ in $\mathbb{R}^2 \times [0, \infty)$,

$$P\left(\bigcap_{j=1}^{m} \{B_j \cap \mathcal{P} \neq \emptyset\}\right) \leq \prod_{j=1}^{m} \left(2\epsilon L^{2\gamma} \lambda(B_j)\right). \tag{2.13}$$

If $B_j = b_j \times [s_j, t_j]$, where b_j is a cube in $\mathbb{R}^2$, it is sufficient to assume that $\lambda(b_j) < L^{-2\gamma}$ for each j and $t_j - s_j < 1$ for $j = 1, 2, \ldots, m$ (the time interval may also be open or half open). To see that (2.13) is then satisfied, first consider the case when $n \in (t_j - s_j)$ for all $j = 1, \ldots, m$, and some $n \in \mathbb{N}$. Given the configuration of the quadratic contact process at time $n - 1$, births at different sites at time n are independent of each other. The same holds for deaths. For each j,

$$P\left(B_j \cap \mathcal{P} \neq \emptyset\right) \leq \epsilon L^{2\gamma} \lambda(B_j).$$

Since the densities of particles in different boxes are independent, (2.13) holds.

We still assume that that $B_j = b_j \times [s_j, t_j]$, where b_j is a cube in $\mathbb{R}^2$, $\lambda(b_j) < L^{-2\gamma}$ for each j and $t_j - s_j < 1$. If $n_j \in (t_j - s_j)$ for some $n_j \in \mathbb{N}$, split B_j into two cubes:

$$B_{j,1} = b_j \times [s_j, n_j) \quad \text{and} \quad B_{j,2} = b_j \times [n_j, t_j].$$

We note that

$$P\left(\{B_{j,1} \cap \mathcal{P} \neq \emptyset\} \cap \{B_{j,2} \cap \mathcal{P} \neq \emptyset\}\right) \leq 2\epsilon L^{2\gamma} \max\{\lambda(B_{j,1}), \lambda(B_{j,2})\}$$

So we can suppose this does not occur and let $n_j = \lfloor s_j \rfloor$. Also suppose that the cubes are ordered in a way that $n_1 \leq n_2 \leq \ldots \leq n_m$. (2.13) can be shown by induction on m. It clearly holds for $m = 1$. Let

$$A_j = \{B_j \cap \mathcal{P} \neq \emptyset\}.$$

Then

$$\begin{aligned} P\left(\bigcap_{j=1}^{m} A_j\right) &= P\left(A_m \middle| \bigcap_{j=1}^{m-1} A_j\right) P\left(\bigcap_{j=1}^{m-1} A_j\right) \\ &\leq \epsilon L^{2\gamma} \lambda(B_m) \prod_{j=1}^{m-1} \left(2\epsilon L^{2\gamma} \lambda(B_j)\right). \end{aligned}$$

The second factor comes from the induction assumption and the first factor is from the upper bound on the error probability.

Theorem 2.3.1. *Let $\tilde{A}_n$ be the bad region of the contact process at time n with death probability η, birth probability β, and finite offspring distribution kernel k such that there is at least one direction with a positive wave speed.*

Let A_t be the vacant region of the comparison process with point process $\mathcal{P}$, orientation vectors $n_i = \xi_i$, speeds $a_i = c$, and interaction rate $b = 2d(k)$. Then the processes A_n and $\tilde{A}_n$ can be jointly coupled so that

$$\tilde{A}_n \subset A_n$$

for all $n \in \{0, 1, 2, \ldots\}$. Hence, for large enough L, the quadratic contact process has a nontrivial stationary distribution.

Proof. Run the contact process from an initial configuration with every site occupied: with $\tilde{A}_0 = \emptyset$. Since all boxes are then good, Bramson and Gray's comparison process has $A_0 = \emptyset$ as well.

We will start by estimating the probability of each error occurring in the transition from time $n-1$ to n.

First, suppose that a type I error occurs. Let $x \in \mathbb{Z}^2/L$ and $B^L(x)$ the box containing x. Then if the box is good,

$$S_{n-1}^{L}(x) > \alpha \cdot m,$$

where $m = L^{2-2\gamma}$ is the number of points in $B(x)$.

From previous calculations,

$$\mathbb{E}\left(S_n^L(x)\right) = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y)$$

and

$$\mathbb{E}\left[S_n^L(x) \,\middle|\, B(x) \text{ is good at time } n-1\right] = \sum_{y \in B^L(x)} Q[u_{n-1}^L](y) \geq Q(\alpha) \cdot m > \alpha \cdot m$$

We also have

$$\operatorname{Var}\left(S_n^L(x)\right) \leq cL^{2-2\gamma}.$$

Now fix any

$$\delta_1 \in (0, Q(\alpha) - \alpha).$$

By Chebyshev's inequality,

$$\mathbb{P}_{\hat{\xi}_{n-1}^L}\left(\left|S_n^L(x) - \sum_{y \in B^L(x)} Q(u_{n-1}^L)(y)\right| \geq \delta_1 m\right) \leq \frac{cm}{\delta_1^2 m^2} = C_1 L^{2\gamma-2} \tag{2.14}$$

This gives an upper bound on the probability of a good box spontaneously going bad, for a good box has density of particles $> \alpha$ and is therefore expected at the next time step to have density of particles $> Q(\alpha)$, so

$$Q(\alpha) - \delta_1 > \alpha.$$

Next, consider the other way that an error can occur: if an already existing bad region's behavior is too different from what is expected, that is, a Type II error. Notice that

$$\phi(\xi_i \cdot x - c) \leq Q[\phi(\xi_i \cdot x)] \leq Q[u_{n-1}^L(x)] = \mathbb{E}\left(\hat{\xi}_n^L(x)\right)$$

Again, for the probability of an error, we use Chebyshev's inequality to obtain, for any x in the bad region,

$$\mathbb{P}_{\hat{\xi}_{n-1}^L}\left(\left|S_n^L(x) - \sum_{y \in B^L(x)} Q(u_{n-1}^L)(y)\right| \geq \delta_2 m\right) \leq \frac{cm}{\delta_2^2 m^2} = C_2 L^{2\gamma-2}$$

Taking $\delta = \min\{\delta_1, \delta_2\}$ and $C = \max\{C_1, C_2\}$, we obtain an upper bound on the probability of an error occurring in any single box at a single time step, $\tilde{\epsilon} := CL^{2\gamma-2}$.

The two processes are already coupled in the following way: if in the contact process, an error occurs at time n in some box, place a single point in the cube

$$Q = B^L \times [n-1, n) \in \mathbb{R}^2 \times [0, \infty)$$

uniformly at random. If no error occurs, leave the corresponding box empty in $\mathcal{P}$.

It follows directly from our definitions of the parameters that $\tilde{A}_n \subset A_n$ for each n. The rates have been set up so that all boxes with density $< \alpha$ in the contact process are covered by some triangular region. A triangular region automatically covers type I and type II errors. All other low density boxes can only be in the vicinity of the type I and II errors: within distance $d(K)$ times the age of the error. The interaction rate b ensures that if there is a large cluster of errors, the overlap/collision region grows faster than the surrounding bad boxes can spread (only by $d(k)$ units per single time step), and from the perimeter of the bad regions, the positive wave speeds "propagate" the high density boxes into the former bad regions. The comparison process A_n has a nontrivial stationary distribution. Hence, so does the contact process. □

CHAPTER 3

THE BOLKER–PACALA PROCESS

3.1 Mean field approximation and the logistic Markov chain

As a first step to analyzing the general Bolker–Pacala model, here, we begin with a mean field approximation. Let $Q_L \subset \mathbb{Z}^d$ be a box with $|Q_L| = L$, and impose the condition that no particles are allowed to exist outside of Q_L. Time, t, is continuous. Let b be the birth rate and μ be the natural mortality rate. Define the displacement kernel and the competition kernel, respectively, as

$$a^{+}(x) = \frac{\kappa}{L}, x \in Q_L,$$
$$a^{-}(x) = \frac{\gamma}{L^2}, x \in Q_L,$$

and both functions are 0 outside of Q_L, where $\kappa, \gamma > 0$ are constants.

$n(t, x)$ denotes the number of particles at site $x \in Q_L$ at time t ($n(t, x) = 0$ for $x \in \mathbb{Z}^d \cap Q_L^c$ for all t). Unlike for the quadratic contact process, there is no restriction on the number of particles per site (in Q_L). The dynamics are the same as those for the general process.

The total number of particles is a Markov chain (since the kernels, a^+ and a^-, are independent on x). This chain will be denoted by

$$N_L(t) = \sum_{x \in Q_L} n(t, x).$$

$N_L(t)$ will be called the *logistic* Markov chain. It has the transition probabilities

$$P\left(N_L(t+h) = j \mid N_L(t) = n\right) = \begin{cases} nbh + o(h^2), & j = n+1 \\ \left(n\mu + \frac{\gamma n^2}{L}\right) h + o(h^2), & j = n-1 \\ o(h^2), & \text{otherwise.} \end{cases}$$

This chain has an equilibrium point n_L^* when

$$bn_L^* = \mu n_L^* + \frac{\gamma n_L^{*2}}{L},$$

Thus,

$$n_L^* = \left\lfloor \frac{L(b-\mu)}{\gamma} \right\rfloor.$$

If $N_L(t) > n_L^*$,

$$bN_L(t) < \mu N_L(t) + \frac{\gamma N_L(t)^2}{L},$$

thus there is a left drift, and

$$bN_L(t) > \mu N_L(t) + \frac{\gamma N_L(t)^2}{L}$$

when $N_L(t) < n_L^*$, so in that case there is a right drift.

In the case of the logistic Markov chain, the birth and death rates are

$$\begin{aligned} \beta_i &= bi, && i = 1, 2, \ldots, \\ \beta_0 &= 1, && \\ \alpha_i &= \mu i + \frac{\gamma i^2}{L}, && i = 1, 2, \ldots \end{aligned}$$

It turns out to be more convenient to study a slightly modified logistic chain with the rates

$$\begin{aligned} \beta_i &= b(i+1), \quad i = 0, 1, 2, \ldots, \\ \alpha_i &= \mu i + \frac{\gamma i^2}{L}, \quad i = 1, 2, \ldots. \end{aligned} \tag{3.1}$$

The equilibrium point, $\tilde{n}_L^*$, then becomes

$$\tilde{n}_L^* = \left\lfloor \frac{(b-\mu) + \sqrt{(b-\mu)^2 + 4\gamma/L}}{2\gamma/L} \right\rfloor,$$

and it is equal to the old equilibrium point n_L^* for large enough L. For convenience, assume that L is such that $(b-\mu)L/\gamma$ is an integer. Thus,

$$\tilde{n}_L^* = \frac{(b-\mu)L}{\gamma}.$$

There are two reasons for this change. First, the modified chain now has no absorbing state at $i = 0$. Second, calculations for the modified logistic chain are simpler. As $L \to \infty$, the asymptotic properties of both chains are identical.

3.2 Hypergeometric functions

In this section, the asymptotic properties of the confluent hypergeometric function are studied. For a particular choice of parameters, such functions will play an important role in the analysis of the logistic Markov chain with transition rates (3.1).

The confluent hypergeometric function depends on parameters α, γ and is given by the power series

$$\Xi(\alpha,\gamma,z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots + \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)}\frac{z^n}{n!} + \ldots$$

(see [12], 9.21). Ξ is an entire function of order 1. For the case of the Bolker Pacala process, the special case when $\alpha = 1$ and $\gamma = A$, $A \gg 0$ will be useful. Let

$$\mathcal{F}(A,z) = 1 + \frac{z}{A} + \ldots + \frac{z^n}{A(A+1)\ldots(A+n-1)} + \ldots$$

The general integral representation of $\Xi(\alpha, \gamma, z)$ ([12], 9.211,1) leads to the formula

$$\mathcal{F}(A, z) = \frac{2^{1-A} e^{z/2} \Gamma(A)}{\Gamma(A-1)} \int_{-1}^{1} (1-t)^{A-2} e^{zt/2}\, dt$$
$$= 2^{1-A} e^{z/2} (A-1) \int_{-1}^{1} (1-t)^{A-2} e^{zt/2}\, dt$$

First substituting $t = 1 - s$, one obtains

$$\mathcal{F}(A, z) = 2^{1-A}\, e^{z/2}\, (A-1) \int_{0}^{2} s^{A-2} e^{z(1-s)/2}\, ds$$
$$= 2^{1-A}\, e^{z}\, (A-1) \int_{0}^{2} s^{A-2} e^{-zs/2}\, ds$$
$$= \frac{e^{z}\, (A-1)}{z^{A-1}} \int_{0}^{z} t^{A-2} e^{-t}\, dt, \tag{3.2}$$

where, in the last step, the substitution $sz/2 = t$ is used. The integral factor in (3.2) is the incomplete Γ-function:

$$\gamma(A-1, z) = \int_{0}^{z} t^{A-2} e^{-t}\, dt.$$

Set

$$\Gamma(A-1, z) = \int_{z}^{\infty} t^{A-2} e^{-t}\, dt.$$

Note that

$$\gamma(A-1, z) = \Gamma(A-1) - \Gamma(A-1, z). \tag{3.3}$$

If $A, z \gg 1$, one can use the Laplace method to obtain asymptotics for $\mathcal{F}(A, z)$. Putting $t = As$,

$$\gamma(A-1, z) = A^{A-1} \int_{0}^{z/A} e^{-A\left(s - \frac{A-2}{A} \ln s\right)}\, ds \tag{3.4}$$

The critical point of the phase function

$$\Psi(s) = s - \frac{A-2}{A} \ln s, \tag{3.5}$$

(where $\Psi'(s_0) = 0$) is $s_0 = (A-2)/A$ and the asymptotic behavior of $\gamma(A-1, z)$ depends on the relationship between z/A and s_0.

Theorem 3.2.1. *Consider*

$$\mathcal{F}(A, z) = 1 + \frac{z}{A} + \ldots + \frac{z^n}{A(A+1)\ldots(A+n-1)} + \ldots.$$

If $A, z \to \infty$ *and*

I. $z < A,\ \sqrt{A}/(A-z) \to 0$*, then*

$$\mathcal{F}(A, z) \sim \frac{A}{A-z} = o(\sqrt{A}) \tag{3.6}$$

II. $z > A$ *and* $\sqrt{A}/(z-A) \to 0$*, then*

$$\mathcal{F}(A, z) \sim \frac{e^z \Gamma(A)}{z^{A-1}} \sim e^{z-A+1} \left(\frac{A-1}{z}\right)^{A-1} \sqrt{2\pi A} \tag{3.7}$$

III. $z = A + h\sqrt{A}, \quad h > 0$ *is a constant, then*

$$\mathcal{F}(A, z) \sim e^{-h^2/2} \Phi(h) \sqrt{2\pi A}, \qquad \Phi(h) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{h} e^{-y^2/2} dy$$

IV. $z = A - h\sqrt{A}, \quad h > 0$ *is a constant, then*

$$\mathcal{F}(A, z) \sim e^{h^2/2} \Phi(-h) \sqrt{2\pi A}$$

where $f(A, z) \sim g(A, z)$ *means that* $\frac{f(A,z)}{g(A,z)} \to 1$ *as* $A, z \to \infty$ *such that the conditions stated in each item are satisfied.*

Proof. We prove the first two items. The remaining two are obtained similarly. Apply the Laplace method: the value of the integral (3.4) is estimated by the Taylor approximation to the phase function (3.5). The first and second derivatives of (3.5) are needed for this.

$$\Psi'(s) = 1 - \frac{A-2}{As}, \quad \text{and} \quad \Psi''(s) = \frac{A-2}{As^2}.$$

I. Ψ has a minimum on $[0, z/A]$ at the endpoint, z/A. Thus the first order Taylor expansion can be used:

$$\begin{aligned}\Psi(s) &= \left(\frac{z}{A} - \frac{A-2}{A}\ln\frac{z}{A}\right) + \left(1 - \frac{A-2}{z}\right)\left(s - \frac{z}{A}\right) + O\left((s-s_0)^2\right) \\ &= \frac{A-2}{A}\left(1 - \ln\frac{z}{A}\right) + s\left(1 - \frac{A-2}{z}\right) + O\left((s-s_0)^2\right).\end{aligned}$$

By the Laplace method,

$$\begin{aligned}\gamma(A-1,z) &\sim A^{A-1}\int_0^{z/A} e^{(2-A)\left(1-\ln\frac{z}{A}\right)-sA\left(1-\frac{A-2}{z}\right)}\,ds \\ &= A^{A-1}e^{2-A}\left(\frac{z}{A}\right)^{A-2}\int_0^{z/A} e^{-s\left(A-\frac{A(A-2)}{z}\right)}\,ds \\ &= \frac{e^{2-A}z^{A-1}(e^{A-2-z}-1)}{(A-2)-z} \\ &\sim \frac{z^{A-1}e^{-z}}{(A-2)-z}.\end{aligned}$$

Thus by (3.2),

$$\mathcal{F}(A,z) \sim \frac{e^z(A-1)}{z^{A-1}}\frac{z^{A-1}e^{-z}}{(A-2)-z} \sim \frac{A}{A-z}.$$

II. Apply the Laplace method to

$$\Gamma(A-1,z) = A^{A-1}\int_{z/A}^{\infty} e^{-A\left(s-\frac{A-2}{A}\ln s\right)}\,ds.$$

Ψ has a minimum on $[z/A, \infty]$ at z/A. Therefore the first order Taylor expansion is used, as in I.

$$\begin{aligned}
\Gamma(A-1,z) &\sim A^{A-1} \int_{z/A}^{\infty} e^{(2-A)\left(1-\ln \frac{z}{A}\right)-sA\left(1-\frac{A-2}{z}\right)}\, ds \\
&= A^{A-1} e^{2-A} \left(\frac{z}{A}\right)^{A-2} \int_{z/A}^{\infty} e^{-s\left(A-\frac{A(A-2)}{z}\right)}\, ds \\
&= \frac{e^{2-A} z^{A-1} (e^{A-2-z})}{z-(A-2)} \\
&\sim \frac{z^{A-1} e^{-z}}{z-(A-2)}.
\end{aligned}$$

By (3.3) and (3.2),

$$\begin{aligned}
\mathcal{F}(A,z) &\sim \frac{e^z(A-1)}{z^{A-1}} \left(\Gamma(A-1) - \frac{z^{A-1}e^{-z}}{z-(A-2)}\right) \\
&\sim \frac{e^z \Gamma(A)}{z^{A-1}} - \frac{A}{z-A} \\
&\sim \frac{e^z \Gamma(A)}{z^{A-1}}.
\end{aligned}$$

□

3.3 General results for random walks

The logistic Markov chain is a particular case of a birth and death chain (e.g. [9, 16, 19]). Consider a birth death random walk on $\mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ with transition rates α_i from i to $i-1$ and β_i from i to $i+1$, with $\beta_i > 0, i \geq 0, \alpha_i > 0, i \geq 1$, and $\alpha_0 = 0$.

The generator for a birth and death chain with birth rates $\{\beta_i\}_{i\in\mathbb{N}\cup\{0\}}$ and death rates $\{\alpha_i\}_{i\in\mathbb{N}\cup\{0\}}$ is given by

$$\mathcal{L}\psi(i) = \alpha_i \psi(i-1) - (\alpha_i + \beta_i)\psi(i) + \beta_i \psi(i+1)$$

for $i = 1, 2, \ldots$ and

$$\mathcal{L}\psi(0) = \beta_0\psi(1) - \beta_0\psi(0)$$

for $i = 0$. The chain is ergodic if and only if the following series converges

$$S = 1 + \frac{\beta_0}{\alpha_1} + \frac{\beta_0\beta_1}{\alpha_1\alpha_2} + \ldots + \frac{\beta_0 \ldots \beta_n}{\alpha_1 \ldots \alpha_{n+1}} + \ldots . \tag{3.8}$$

In this case, the invariant distribution is given by

$$\pi(i) = \lim_{t\to\infty} p(t, \cdot, i) = \begin{cases} S^{-1}, & i = 0 \\ S^{-1}\frac{\beta_0 \ldots \beta_{i-1}}{\alpha_1 \ldots \alpha_i}, & i > 0 \end{cases} \tag{3.9}$$

Let $X(t)$ be an ergodic birth and death chain with transition rates $\{\beta_i\}$ to the right and $\{\alpha_i\}$ to the left, and let $\tau_j = \min\{t : X(t) = j\}$. By ergodicity, it follows that for any $i, j \in \mathbb{N} \cup \{0\}, i \neq j$, $E_i\tau_j$ is finite. We will also use the notations $E_i\tau_j = E\tau_{i\to j}$ and $E_i\tau_j = u(i, j)$.

For fixed j and $i > j$, $u(i, j)$ satisfies

$$\begin{cases} \mathcal{L}u(i, \cdot) = -1, & i > j \\ u(j, \cdot) = 0 \end{cases}$$

One can understand $u(i, \cdot)$ as $\lim_{L\to\infty} u_L(i, j), i \in [j, L]$, where

$$\begin{cases} \mathcal{L}u_L(i, \cdot) = -1, \\ u_L(j, \cdot) = u_L(L, \cdot) = 0 \end{cases}$$

Remark 3.3.1. *For $i > j$,*

$$\tau_{i\to j} = \tau_{i\to i-1} + \tau_{i-1\to i-2} + \ldots \tau_{j+1\to j}.$$

Lemma 3.3.2. *Assume that $X(t)$ is ergodic. Then*

$$\mathrm{E}\tau_{j+1\to j} = u(j+1, j) = \frac{1}{\alpha_{j+1}}\left[1 + \frac{\beta_{j+1}}{\alpha_{j+2}} + \frac{\beta_{j+1}\beta_{j+2}}{\alpha_{j+2}\alpha_{j+3}} + \ldots\right] \tag{3.10}$$

The second factor has the following interpretation. Let

$$S_j := 1 + \frac{\beta_j}{\alpha_{j+1}} + \frac{\beta_j\beta_{j+1}}{\alpha_{j+1}\alpha_{j+2}} + \dots$$

($S = S_0 = 1 + \frac{\beta_0}{\alpha_1} + \frac{\beta_0\beta_1}{\alpha_1\alpha_2} + \dots$). Then,

$$\pi(i) = \lim_{t\to\infty} p(t, \cdot, i) = S_j^{-1}\frac{\beta_0 \dots \beta_{i-1}}{\alpha_1 \dots \alpha_i}. \tag{3.11}$$

If $X^j(t)$ is the chain with the same transition rates but restricted to $[j, \infty)$ with the reflection rate α_j at j and π^j is the invariant distribution for this chain, then $S_j^{-1} = \pi^j(j)$ and

$$\mathrm{E}\tau_{j\to j} = \alpha_j \mathrm{E}\tau_{j\to j} + \beta_j \mathrm{E}\tau_{j+1\to j} + 1.$$

Therefore,

$$\pi^j(j) = \frac{1/\beta_j}{1/\beta_j + \mathrm{E}\tau_{j+1\to j}},$$

and so

$$S_j = 1 + \beta_j \mathrm{E}\tau_{j+1\to j}.$$

Lemma 3.3.3. $\mathrm{E}\tau_{j+1\to j} = \frac{1}{\alpha_{j+1}} S_{j+1}$.

Proof. Let j be fixed. Set

$$v(i - j) := \mathrm{E}\tau_{i\to j}, \quad i \geq j.$$

Then $v(0) = 0$ and

$$\alpha_i v(i-1) - (\alpha_i + \beta_i)v(i) + \beta_i v(i+1) = -1. \tag{3.12}$$

Set

$$\Delta(i) = v(i) - v(i-1), i \geq 1.$$

Then

$$\Delta(i+1) = \frac{\alpha_i}{\beta_i}\Delta(i) - \frac{1}{\beta_i} \tag{3.13}$$

Consider a continuous random walk $\tilde{X}_n(t)$ with the same transition rates but on the finite interval $[0, N]$. We are interested in the first entrance to N. Let

$$\tau_{0,N} = \min\{t : \tilde{X}_N(t) = 0 \text{ or } N\}, \quad v_N(i) := \mathrm{E}_i(\tau_{0,N}), \quad \text{and}$$

$$\Delta_N(i) := v_N(i) - v_N(i-1).$$

Then, u_N satisfies (3.12) with boundary conditions

$$v_N(0) = v_N(N) = 0$$

and $\Delta_N(i)$ satisfies (3.13) as well as $\lim_{N\to\infty} \Delta_N(i) = \Delta(i)$, pointwise.

For the finite chain,

$$\Delta_N(1) + \Delta_N(2) + \ldots + \Delta_N(N) = 0 \tag{3.14}$$

From (3.13) and (3.14),

$$\Delta_N(k) = \frac{\alpha_1\alpha_2\ldots\alpha_{k-1}}{\beta_1\beta_2\ldots\beta_{k-1}}\Delta_N(1) - \frac{\alpha_2\ldots\alpha_{k-1}}{\beta_1\beta_2\ldots\beta_{k-1}} - \frac{\alpha_3\ldots\alpha_{k-1}}{\beta_2\ldots\beta_{k-1}} - \ldots - \frac{1}{\beta_{k-1}}$$

Note that

$$\Delta_N(1) = v_N(1) - v_N(0) = v_N(1).$$

We now multiply through by $\frac{\beta_1\beta_2\ldots\beta_{k-1}}{\alpha_1\alpha_2\ldots\alpha_{k-1}}$ to obtain equations for $\Delta_N(1)$

$$\Delta_N(1) = \frac{\beta_1\beta_2\ldots\beta_{k-1}}{\alpha_1\alpha_2\ldots\alpha_{k-1}}\Delta_N(k) + \frac{\beta_1\beta_2\ldots\beta_{k-2}}{\alpha_1\alpha_2\ldots\alpha_{k-1}} + \frac{\beta_1\beta_2\ldots\beta_{k-3}}{\alpha_1\alpha_2\ldots\alpha_{k-2}} + \ldots + \frac{1}{\alpha_1}$$

This is true for all k, including $k = N$, and so

$$\Delta(1) := \lim_{N\to\infty} \Delta_N(1) = \frac{1}{\alpha_1}\left(1 + \frac{\beta_1}{\alpha_2} + \frac{\beta_1\beta_2}{\alpha_2\alpha_3} + \ldots\right).$$

Therefore, $E\tau_{j+1\to j}$ is given by

$$E\tau_{j+1\to j} = \Delta(j+1) = \frac{1}{\alpha_{j+1}}\left(1 + \frac{\beta_{j+1}}{\alpha_{j+2}} + \frac{\beta_{j+1}\beta_{j+2}}{\alpha_{j+2}\alpha_{j+3}} + \ldots\right). \qquad \square$$

For the logistic chain with

$$\beta_i = b\,(i+1)\,, \quad \alpha_i = i\left(\mu + \frac{\gamma i}{L}\right),$$

represent S_j, $j \geq 0$ in terms of the hypergeometric function $F(A, z)$.

$$\begin{aligned}
S_j &= 1 + \frac{\beta_j}{\alpha_{j+1}} + \frac{\beta_j\beta_{j+1}}{\alpha_{j+1}\alpha_{j+2}} + \ldots \\
&= 1 + \frac{b}{\mu + \frac{\gamma(j+1)}{L}} + \frac{b^2}{\left(\mu + \frac{\gamma(j+1)}{L}\right)\left(\mu + \frac{\gamma(j+2)}{L}\right)} + \ldots \\
&= 1 + \frac{bL/\gamma}{\frac{\mu L}{\gamma} + j + 1} + \frac{(bL/\gamma)^2}{\left(\frac{\mu L}{\gamma} + j + 1\right)\left(\frac{\mu L}{\gamma} + j + 2\right)} + \ldots \\
&= F\left(\mu L/\gamma + j, bL/\gamma\right).
\end{aligned} \tag{3.15}$$

This formula, asymptotics from Theorem 3.2.1, and the relation

$$\mathrm{E}\tau_{i\to j} = \mathrm{E}\tau_{i\to i-1} + \mathrm{E}\tau_{i-1\to i-2} + \cdots + \mathrm{E}\tau_{j+1\to j}, \qquad \text{for } i > j$$

will be the basis for the analysis of the first passage times.

Also note that there is a recurrence formula connecting $\mathrm{E}\tau_{j+1\to j}$ and $\mathrm{E}\tau_{j+2\to j+1}$. From (3.10),

$$\begin{aligned}
\mathrm{E}\tau_{j+1\to j} &= \frac{1}{\alpha_{j+1}} + \frac{\beta_{j+1}}{\alpha_{j+1}}\left[\frac{1}{\alpha_{j+2}}\left(1 + \frac{\beta_{j+2}}{\alpha_{j+3}} + \frac{\beta_{j+2}\beta_{j+3}}{\alpha_{j+3}\alpha_{j+4}} + \ldots\right)\right] \\
&= \frac{1}{\alpha_{j+1}} + \frac{\beta_{j+1}}{\alpha_{j+1}}\mathrm{E}\tau_{j+2\to j+1}.
\end{aligned}$$

3.3.1 Limit theorems for the invariant distribution of the logistic Markov chain

Applying the general results on the 1D ergodic random walk on $\mathbb{N}$ to the particular case of the modified logistic Markov chain, one obtains a local Central Limit Theorem.

Theorem 3.3.4 (Local CLT)**.** *Let $b > \mu$. If $k = O(L^{2/3})$, then, the invariant distribution π_L satisfies*

$$\pi_L(n_L^* + k) \sim \frac{e^{-k^2/2\sigma_L^2}}{\sqrt{2\pi\sigma_L^2}},$$

where $\sigma_L^2 = Lb/\gamma$.

Recall that $f(L) \sim g(L)$ means that $\frac{f(L)}{g(L)} \to 1$ as $L \to \infty$.

Proof. From (3.8),

$$S = 1 + \frac{b}{\mu + \gamma/L} + \frac{b^2}{(\mu + \gamma/L)(\mu + 2\gamma/L)} + \ldots + \frac{b^n}{(\mu + \gamma/L)\ldots(\mu + n\gamma/L)} + \ldots \quad (3.16)$$

And for $n \in \mathbb{Z}_+^1$, from (3.9)

$$\pi_L(n) = S^{-1}\frac{b^n}{(\mu + \gamma/L)\ldots(\mu + n\gamma/L)}. \quad (3.17)$$

To analyze this series, consider the position, n_L^*, such that the ratio of the terms $\frac{\pi_L(n)}{\pi_L(n-1)}$ is closest to 1. (If there is more than one such position, take n_L^* to be the maximum.) Taking this ratio,

$$\frac{(b-\mu)L}{\gamma} - 1 \le n_L^* \le \frac{(b-\mu)L}{\gamma}.$$

We obtain the formula

$$\pi_L(n_L^* + k) = \pi_L(n_L^*) \cdot \frac{b^k}{A^k} \cdot \frac{1}{(1 + \frac{\gamma}{LA})\ldots(1 + \frac{k\gamma}{LA})},$$

where $A = \mu + \frac{n_L^*\gamma}{L} = b + O(1/L)$. Now, consider

$$\sum_{i=0}^{k} \ln\left(1 + \frac{\gamma i}{AL}\right) = \int_0^k \ln\left(1 + \frac{\gamma x}{AL}\right) dx + O\left(\ln\left(1 + \gamma k/AL\right)\right).$$

Integrate the series $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$, and take $k = O\left(L^{2/3}\right)$.

$$\begin{aligned}\int_0^k \ln\left(1+\frac{\gamma x}{AL}\right) dx &= \int_0^k \frac{\gamma x}{AL}\,dx - \frac{1}{2}\int_0^k \left(\frac{\gamma x}{AL}\right)^2 dx \\ &= \frac{\gamma k^2}{2AL} - \frac{1}{6}\cdot\frac{\gamma^2 k^3}{A^2L^2} + \dots \\ &= \frac{\gamma k^2}{2AL} + O(1).\end{aligned}$$

Thus,

$$\pi_L(n_L^* + k) = \pi_L(n_L^*)\left(\frac{b}{b+O(1/L)}\right)^k e^{-\gamma k^2/2LA}$$

It remains to calculate $\pi_L(n_L^*)$, using (3.17).

Applying (3.7) to (3.16) to calculate S,

$$\begin{aligned}S_L &= 1 + \frac{(bL/\gamma)}{(1+\mu L/\gamma)} + \frac{(bL/\gamma)^2}{(1+\mu L/\gamma)(2+\mu L/\gamma)} + \dots \\ &= \mathcal{F}\left(\mu L/\gamma, bL/\gamma\right) \\ &\sim e^{n_L^*}\left(\frac{\mu}{b}\right)^{\mu L/\gamma}\sqrt{2\pi\left(\frac{\mu L}{\gamma}\right)}.\end{aligned}$$

The Euler-Maclaurin formula can be used on the product in the denominator (see [29]). First,

$$\begin{aligned}\pi_{n_L^*} := (\mu+\gamma/L)\cdots(\mu+n\gamma/L) &= \mu^{n_L^*}\left(1+\frac{\gamma}{\mu L}\right)\cdots\left(1+\frac{n_L^*\gamma}{\mu L}\right) \\ &= \mu^{n_L^*}\exp\left(\sum_{k=0}^{n_L^*}\ln(1+k\omega)\right),\end{aligned}$$

where $\omega := \frac{\gamma}{\mu L}$. By the Euler-Maclaurin formula,

$$\begin{aligned}\sum_{k=0}^{r}\ln(1+\omega k) &= \frac{1}{\omega}\int_1^{1+r\omega}\ln x\,dx + \frac{1}{2}\ln(1+r\omega) + O(\omega) \\ &= \left(\frac{1}{\omega} + r + \frac{1}{2}\right)\ln(1+r\omega) - r + O(\omega).\end{aligned} \tag{3.18}$$

Thus

$$\pi_{n_L^*} \sim \mu^{n_L^*} \exp\left[\left(\frac{1}{\omega} + r + \frac{1}{2}\right)\ln(1 + r\omega) - r\right]$$
$$= \frac{\mu^{n_L^*}\left(\frac{b}{\mu}\right)^{Lb/\gamma+1/2}}{e^{L(b-\mu)/\gamma}}.$$

And so, finally,

$$\pi_L(n_L^*) = \frac{b^{n_L^*}}{S_L \pi_{n_L^*}} \sim \frac{1}{\sqrt{2\pi\left(\frac{bL}{\gamma}\right)}}. \tag{3.19}$$

□

For deviations of order L, there is a large deviations result:

Theorem 3.3.5. *For $\delta > 0$,*

$$\pi_L(n_L^* + \delta L) \asymp \frac{1}{\sqrt{L}} e^{-Lf(\delta\gamma/b)b/\gamma},$$

where $f(z) := \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} z^n = \int_0^z \ln(1+x)dx$ and $g(x) \asymp f(x)$ means that there are positive constants a, b such that $af(x) \leq g(x) \leq bf(x)$ for all x.

Proof. Set

$$N := n_L^*(1+\delta) \quad \text{and} \quad k := \delta n_L^*.$$

Applying the formula for general Markov chains (3.9), the calculated invariant probability $\pi(n_L^*)$, and the Euler-Maclaurin formula with $\omega = \frac{\gamma}{bL}$,

$$\begin{aligned}
\pi(N) &= \pi(n_L^*)\frac{\beta_{n_L^*}\cdots\beta_{n_L^*+k-1}}{\alpha_{n_L^*+1}\cdots\alpha_{n_L^*+k}} \\
&\sim \frac{1}{\sqrt{2\pi\left(\frac{bL}{\gamma}\right)}}\frac{b^k}{\left(\mu+(n_L^*+1)\frac{\gamma}{L}\right)\cdots\left(\mu+(n_L^*+k)\frac{\gamma}{L}\right)} \\
&= \frac{1}{\sqrt{2\pi\left(\frac{bL}{\gamma}\right)}}\frac{1}{(1+\omega)\cdots(1+k\omega)} \\
&= \frac{1}{\sqrt{2\pi\left(\frac{bL}{\gamma}\right)}}\exp\left(-\sum_{j=0}^{k}\ln(1+j\omega)\right) \\
&\sim \frac{1}{\sqrt{2\pi\left(\frac{bL}{\gamma}\right)}}\exp\left(-\left[\left(\frac{1}{\omega}+k+\frac{1}{2}\right)\ln(1+k\omega)-k\right]\right). \qquad (3.20)
\end{aligned}$$

Using the expansion $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$, substituting for ω, k, and n_L^*, and multiplying through,

$$\left(\frac{1}{\omega}+k+\frac{1}{2}\right)\ln(1+k\omega)-k = L\frac{b}{\gamma}\sum_{n=2}^{\infty}\frac{(-1)^n}{n(n-1)}\left(\frac{\delta(b-\mu)}{b}\right)^n.$$

□

3.3.2 Global limit theorems

A functional Law of Large Numbers and Central Limit Theorem for the logistic Markov chain follow directly from theorems of Kurtz [19, 20]. Thus, they are stated here without proof.

Define a new stochastic process for the population density, $Z_L(t) = \frac{N_L(t)}{L}$. Set

$$z^* = \frac{n_L^*}{L} = \frac{b-\mu}{\gamma}.$$

Define the transition function,

$$f_L\left(\frac{N_L}{L}, j\right) = \frac{p(N_L, N_L+j)}{L}.$$

Thus,

$$f_L(z,j) = \begin{cases} \frac{bN_L}{L} = bz & j = 1 \\ \frac{\mu N_L + \frac{\gamma}{L} N_L^2}{L} = \mu z + \gamma z^2 & j = -1 \\ \dots & j = 0 \end{cases}$$

Note that $f_L(z, j)$ is independent of L and we can write $f(z, j)$ instead of $f_L(z, j)$.

Theorem 3.3.6 (Functional LLN). *As $L \to \infty$, $Z_L(t) \to Z(t)$ uniformly in probability, where $Z(t)$ is a deterministic process, the solution of*

$$\frac{dZ(t)}{dt} = F(Z(t)), \qquad Z(0) = z_0. \tag{3.21}$$

where

$$F(z) = \sum_j j f(z, j) = bz - \mu z - \gamma z^2 = \gamma z(z^* - z).$$

Equation (3.21) is in fact that of the stochastic logistic model, see [25]. It has the solution

$$Z(t, z) = \frac{z^* z}{z + (z^* - z)e^{-\gamma z^* t}}, \ t \geq 0.$$

Next, define

$$G_L(z) = \sum_j j^2 f_L(z, j) = (b + \mu)z + \gamma z^2.$$

$G_L(z)$ does not depend on L because f_L does not, and we will write $G(z)$ instead.

Theorem 3.3.7 (Functional CLT). *If $\sqrt{L}\,(Z_L(0) - z^*) = \zeta_0$, the processes*

$$\zeta_L(t) := \sqrt{L}(Z_L(t) - Z(t))$$

converge weakly in the space of cadlag functions on any finite time interval $[0, T]$ to a Gaussian diffusion $\zeta(t)$ with

1) *initial value $\zeta(0) = \zeta_0$,*

2) mean

$$E\zeta(s) = \zeta_0 L_s := \zeta_0 e^{\int_0^s F'(Z(u))du},$$

3) variance

$$\mathrm{Var}(\zeta(s)) = L_s^2 \int_0^s L_u^{-2} G(Z(u))du.$$

Suppose, moreover, that $F(z_0) = 0$, i.e., $z_0 = z^*$, the equilibrium point. Then, $Z(t) \equiv z_0$ and $\zeta(t)$ is an Ornstein-Uhlenbeck process (OUP) with initial value ζ_0, infinitesimal drift

$$q := F'(z_0) = \mu - b$$

and infinitesimal variance

$$a = G(z_0) = \frac{2b}{\gamma}(b - \mu).$$

Thus, $\zeta(t)$ is normally distributed with mean

$$\zeta_0 e^{qt} = \zeta_0 e^{-(b-\mu)t}$$

and variance

$$-\frac{a}{2q}\left(1 - e^{2qt}\right) = \frac{b}{\gamma}\left(1 - e^{-2(b-\mu)t}\right).$$

Finally, as $L \to \infty$, the first exit time τ_A from n_L^* to $n_L^* \pm A\sqrt{L}$ will be approximately the first exit time τ_A for the OUP starting at 0 from $\pm A$. The distribution of τ_A has been shown in [4] to be

$$P(\tau_A > t) = \alpha e^{-2\nu(A)t} + O\left(e^{-2(\nu(A)+\delta)t}\right), \quad (3.22)$$

such that

(i) $\lim_{A\to\infty} \nu(A) = 0$,

(ii) $\lim_{A\to 0} \nu(A) = \infty$, and

(iii) if A^2 is the smallest positive root of

$$0 = \sum_{k=0}^{m} \frac{(-2A^2)^k}{(2k)!} \frac{m!}{(m-k)!},$$

then $\nu(A) = m$.

3.3.3 First passage times

Recall that the original logistic Markov chain was modified by shifting the birth rates one unit to the left. One reason for this was to eliminate the possibility of absorption at 0 and thus, having the new chain be ergodic. We now calculate the first passage time to 0 and find out how much time is expected to pass until absorption at 0. As $L \to \infty$,

$$\begin{aligned} \mathrm{E}\tau_{n_L^* \to 0} &= \sum_{k=1}^{n_L^*} \tau_{k \to k-1} = \sum_{k=1}^{n_L^*} \frac{S_k}{\alpha_k} = S_1 \sum_{k=1}^{n_L^*} \frac{1}{\alpha_k} \frac{S_k}{S_1} \\ &\sim S_1 \frac{1}{\mu} \left(1 + \frac{1}{2} \frac{\mu}{b} + \frac{1}{3} \left(\frac{\mu}{b} \right)^2 + \dots \right) \\ &= \frac{b}{\mu^2} \ln \left(\frac{b}{b-\mu} \right) S_1. \end{aligned}$$

Let $n_1 = (1-\delta_1)n_L^*$, $0 < \delta_1 < 1$. Then

$$\begin{aligned} \mathrm{E}\tau_{n_L^* \to n_1} &= \sum_{k=n_1+1}^{n_L^*} \tau_{k \to k-1} = \sum_{k=n_1+1}^{n_L^*} \frac{S_k}{\alpha_k} \\ &\sim \frac{S_{n_1+1}}{\alpha_{n_1+1}} \left(1 + \frac{n_1+1}{n_1+2} \rho_1 + \frac{n_1+1}{n_1+3} \rho_1^2 + \dots \right) \\ &\sim \frac{S_{n_1+1}}{\alpha_{n_1+1}} \cdot \frac{1}{1-\rho_1}, \end{aligned} \tag{3.23}$$

where $\rho_1 := 1 - (1 - \frac{\mu}{b})\delta_1$.

Next, we analyze the first passage time $u(x)$ to one of the boundaries $\{n_1, n_2\}$

for initial $x \in [n_1, n_2]$, where $n_1 = (1-\delta_1)n_L^*$, $n_2 = (1+\delta_2)n_L^*$, $0 < \delta_1, \delta_2 < 1$. u satisfies

$$\mathcal{L}u = -1, \tag{3.24}$$

as well as the boundary conditions $u(n_1) = u(n_2) = 0$. A particular solution to (3.24) is $\psi_1(x) = \mathrm{E}\tau_{x\to 0}$. Then,

$$u = \psi_1 + c_1 + c_2\psi_2,$$

where c_1 and c_2 are constants and ψ_2 satisfies $\mathcal{L}\psi_2 = 0$.

Choose ψ_2 so that $\psi_2(n_L^*) = 0$ and $\psi_2(n_L^* + 1) = 1$. This gives

$$\psi_2(x) = \begin{cases} -\left(\frac{\beta_{n_L^*}}{\alpha_{n_L^*}} + \frac{\beta_{n_L^*}\beta_{n_L^*-1}}{\alpha_{n_L^*}\alpha_{n_L^*-1}} + \ldots + \frac{\beta_{n_L^*}\cdots\beta_{x+1}}{\alpha_{n_L^*}\cdots\alpha_{x+1}}\right) & \text{if } x < n_L^* \\ 0 & \text{if } x = 0 \\ 1 + \frac{\alpha_{n_L^*+1}}{\beta_{n_L^*+1}} + \ldots + \frac{\alpha_{n_L^*+1}\cdots\alpha_{x-1}}{\beta_{n_L^*+1}\cdots\beta_{x-1}} & \text{if } x > n_L^* \end{cases} \tag{3.25}$$

Next we determine the asymptotics of $\psi_2(x)$ for

$$x = n_1 = n_L^*(1-\delta_1), \quad x = n_2 = n_L^*(1+\delta_2).$$

Calculations will be performed up to a constant factor and the notation, $a(L) \asymp b(L)$ is used when there are constants c_1 and c_2 satisfying

$$0 < c_1 \leq \frac{a(L)}{b(L)} \leq c_2 < \infty.$$

The last term in (3.25)

$$A_{n_2-1} := \frac{\alpha_{n^*+1}\cdots\alpha_{n_2-1}}{\beta_{n^*+1}\cdots\beta_{n_2-1}}$$

forms the main contribution to $\psi_2(n_2)$. The terms A_{n_2-k} asymptotically form a geometric progression with common ratio

$$\rho_2 = \frac{1}{1+\delta_2\left(1-\frac{\mu}{b}\right)} < 1$$

(because

$$\frac{\alpha_{n_2-k}}{\beta_{n_2-k}} = \frac{\mu(n_2-k)+\frac{\gamma}{L}(n_2-k)^2}{b(n_2-k+1)} = \frac{\mu}{b} + \frac{\gamma}{bL}n_2 + o(1)$$
$$= 1+\delta_2\left(1-\frac{\mu}{b}\right)+o(1).)$$

Thus, for $\omega := \frac{\gamma}{bL}$,

$$\psi_2(n_2) \sim A_{n_2-1}(1+\rho_2+\rho_2^2+\ldots) = \frac{A_{n_2-1}}{1-\rho_2} = O\left(A_{n_2-1}\right).$$

Moreover, from the Euler-Maclaurin formula (compare to (3.18)),

$$A_{n_2-1} \sim \prod_{k=0}^{\delta_2 n_L^*}(1+k\omega) = \exp\left(\sum_{k=0}^{\delta_2 n_L^*}\ln(1+k\omega)\right) \sim \exp\left(\frac{1}{\omega}\int_0^{\delta_2(1-\frac{\mu}{b})}\ln(1+x)dx\right),$$

and so

$$\psi_2(n_2) \asymp \exp\left(\frac{1}{\omega}\int_0^{\delta_2(1-\frac{\mu}{b})}\ln(1+x)dx\right).$$

Similar calculations for $\psi_2(n_1)$ yield

$$\psi_2(n_1) \asymp -\exp\left(-\frac{1}{\omega}\int_0^{\delta_1(1-\frac{\mu}{b})}\ln(1-x)dx\right).$$

It is convenient to introduce a certain symmetry to the logistic Markov chain with respect to the equilibrium point $n_L^* = \left\lfloor \frac{(b-\mu)L}{\gamma} \right\rfloor$. One way to achieve this is to assume that

$$\mathrm{P}_{n_L^*}\{N_L(\tau_{[n_1,n_2]}) = n_2\} \approx \frac{1}{2}.$$

This means that $\psi_2(n_2) \approx -\psi_2(n_1)$, that is,

$$-\int_0^{\delta_1(1-\frac{\mu}{b})}\ln(1-x)dx = \int_0^{\delta_2(1-\frac{\mu}{b})}\ln(1+x)dx.$$

The last equation uniquely determines δ_2 as a function of δ_1.

With this symmetry, we can determine the expected escape time from the interval $[n_1, n_2]$: $\mathrm{E}\tau_{n_L^* \to \{n_1, n_2\}}$. Consider again the problem

$$\mathcal{L}u = -1, \quad u(n_1) = u(n_2) = 0. \tag{3.26}$$

Modify the previous particular solution and define $\tilde{\psi}_1(x) := \mathrm{E}\tau_{x \to n_1}$ for $x > n_1$. Then,

$$u(x) := \tilde{\psi}_1(x) + c_1 + c_2\psi_2(x)$$

satisfies (3.26) for constants c_1 and c_2. Using

$$\psi_2(n_2) \asymp -\psi_2(n_1) \quad \text{and} \quad \tilde{\psi}_1(n_1) = 0,$$

one obtains

$$c_1 = -\frac{1}{2}\tilde{\psi}_1(n_2).$$

From (3.23) and (3.6),

$$\begin{aligned} \tilde{\psi}_1(n_2) &= \mathrm{E}\tau_{n_2 \to n_1} \\ &= \frac{S_{n_1+1}}{\alpha_{n_1+1}(1-\rho_1)} \\ &\sim \frac{\sqrt{2\pi\gamma}\exp\left(\frac{bL}{\gamma}\ln\rho_1 + \delta_1(1-\ln\rho_1)n_L^*\right)}{(b-\mu)(1-\delta_1)(1-\rho_1)\sqrt{b\rho_1 L}} \\ &\asymp \frac{\exp\left(\frac{bL}{\gamma}\ln\rho_1 + \delta_1(1-\ln\rho_1)n_L^*\right)}{\sqrt{L}}. \end{aligned}$$

And so, finally

$$\begin{aligned} \mathrm{E}\tau_{n_L^* \to \{n_1, n_2\}} &= u(n_L^*) = \tilde{\psi}_1(n_L^*) - \frac{1}{2}\tilde{\psi}_1(n_2) \\ &\sim \frac{1}{2}\tilde{\psi}_1(n_2) \asymp \frac{\exp\left(\frac{b}{\gamma}L\ln\rho_1 + \delta_1(1-\ln\rho_1)n_L^*\right)}{\sqrt{L}}. \end{aligned} \tag{3.27}$$

This result shows that the expected time to arrival at 0 is exponentially large. For a model representing a relatively short time scale, it is therefore not necessary to adjust the rates as in the modified logistic chain in order to prevent absorption at 0. In addition, the expected recurrence time to $k \in \mathbb{N}$, $\tau_k \sim \pi(k)^{-1}$, with $\pi(k)$ the invariant probability of state k. In particular, the expected recurrence time to the equilibrium point n_L^* is $O(\sqrt{L})$ and to $n_L^* + \delta L$ for $\delta > 0$, it is $\sqrt{L}e^{O(L)}$.

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