

# 5

## TYPE, NUMBER, AND DIMENSIONAL SYNTHESIS

### 5-1 KINEMATIC SYNTHESIS

The study of motions in machines may be considered from the two different points of view generally identified as kinematic analysis and kinematic synthesis. Kinematic analysis is the determination of the motion inherent in a given machine or mechanism. Formerly displacement analysis was of paramount interest, and it still may be. However, increases in rotational speeds have made a knowledge of velocity and acceleration characteristics critical factors in the design of the many elements comprising the complete machine. Inertia forces deriving from the accelerations may be several times as large as the static forces. In consequence, cross-section dimensions of links and bearing selection are contingent upon acceleration magnitudes and directions. Kinematic synthesis is the reverse problem: it is the determination of mechanisms that are to fulfill certain motion specifications. Synthesis is the very fundamental of design, for it represents the creation of new hardware to meet particular needs in motion—displacement, velocity, or acceleration—singly or in combination. Some typical examples of kinematic synthesis are given in the following:

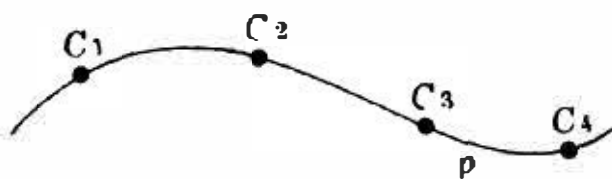


FIGURE 5-1 Point  $C$  is guided along curve  $p$  to successively occupy positions  $C_1, C_2, C_3, C_4$ .

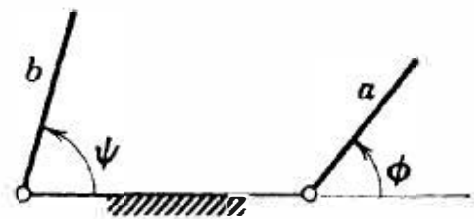


FIGURE 5-2 Correlation of the angular positions of two cranks.

*Guiding a point along a specified curve.* Point  $C$  (Fig. 5-1) is to be guided along the path  $p$  while successively occupying the positions  $C_1, C_2, C_3, C_4$ . An early application of this problem is found in Watt's design of his double-acting steam engine (Fig. 2-10). Other applications are found, for example, in the posthole borer (Fig. 2-11) and the baler (Fig. 2-18).

*Correlation of the angular positions of two cranks.* Two cranks  $a$  and  $b$  (Fig. 5-2) are to have a specified relationship between their crank angles  $\phi$  and  $\psi$ . An application of this problem is found in the logarithmic-scale converter (Fig. 2-8), in which a linearly divided scale in  $\phi$  presents  $\psi$  as a linearly displayed logarithmic function.

*Correlation of the angular position of a crank with the positions of a point along a curve.* Point  $C$  (Fig. 5-3) is to be guided along the curve  $p$  and to occupy the successive positions  $C_1, C_2, C_3, C_4$  as the crank occupies the positions  $a_1, a_2, a_3, a_4$ . An application of this problem is found in the film-feed mechanism (Fig. 2-14), where point  $C$  must move from  $C_1$  to  $C_3$  through  $C_2$  for a  $180^\circ$  rotation of the crank and return to  $C_1$  through  $C_4$  during the remaining  $180^\circ$  of crank rotation.

*Correlation of the angular positions of three cranks.* Three cranks  $a, b,$  and  $c$  (Fig. 5-4) are to have a specified relationship between the crank angles  $\phi_1$  and  $\phi_2$  and the crank angle  $\psi$ . An application of this problem is found in a fire-control computer giving the elevation angle as a function of range and relative altitude.

As remarked in Chap. 1, the overall problem of synthesis may be

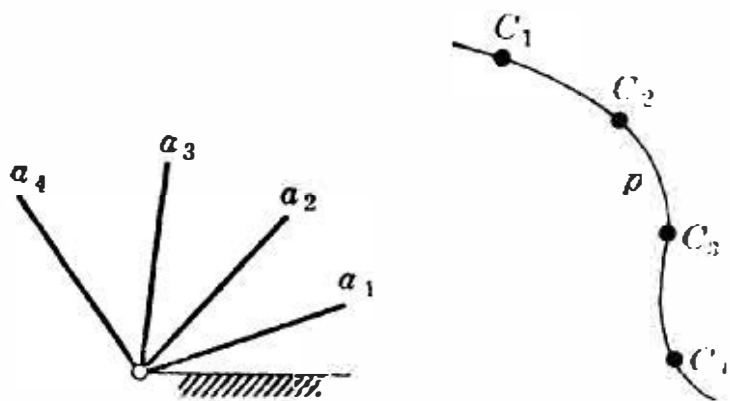


FIGURE 5-3 Correlation of the angular positions of a crank with the positions of a point along a curve.

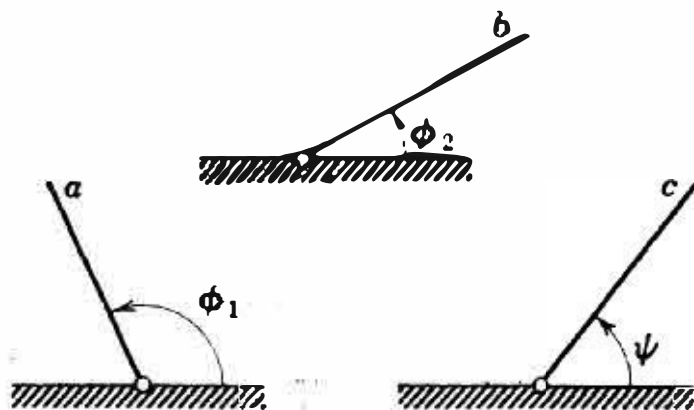


FIGURE 5-4 Correlation of the angular positions of three cranks.

approached in three sometimes interrelated phases. It is necessary to reach decisions on (1) the form or type of mechanism, (2) the number of links and the nature of the connections needed to permit the required movability, and (3) the proportions (lengths) of the links necessary to accomplish the specified motion transformation.

The first phase is called *type synthesis*. Here the choice of the kind of links or constructional units is determined, as linkwork, gears, cams, belts, etc. The second phase, called *number synthesis*, deals with the number of links and the number of pairs of a given type required to obtain a given number of degrees of freedom, i.e., a given number of independent inputs to the mechanism. The third phase is called *dimensional synthesis*. This last phase of the problem, which is relatively straightforward in most cases of cam and gear mechanisms, presents challenging problems in the case of linkages, and it is to these problems that the greater part of this book is addressed.

## 5-2 TYPE SYNTHESIS

The selection of the type of mechanism needed to accomplish a given purpose depends to a great extent on considerations of usage, materials available, manufacturing processes, etc., which lie outside the field of kinematics. With so many factors, there can be no scheme whereby a mechanism may be uniquely determined on naming desired motion specifications. It will be necessary to settle for a line-up of possible combinations that could do the job, from which the "best one" for the particular application in view is chosen. For example, many quite different steam-engine configurations have existed side by side, each arrangement having been dictated by nonkinematic requirements such as location of rotating shaft, headroom, ease of maintenance, overall size, critical material, manufacturing facilities, and so on, to name only some features the designer must consider.

For type synthesis, the classification of mechanisms, and in particular Reuleaux's six groups and their extensions, are often helpful for a systematic consideration of various possibilities, such as the choice in the use of a cam or a linkage. It may be impossible, however, to reach a decision as to which is the optimum solution until the best possible linkage design is compared with the best possible cam design.

## 5-3 NUMBER SYNTHESIS

A collection of connected links must meet certain requirements in order to be called a mechanism. If links are connected in a manner resulting in a configuration such that relative motion of the links is impossible, the assembly is called a structure. Three bars, pin-connected to form a triangle, represent a well-known structural unit. Four

pin-connected bars of proper relative proportions constitute the familiar chain known as the four-bar linkage, said to be movable if relative motion of the links is possible. Further, after fixing one link, designating another as the input link and still another as the output, the motion of the output (or for that matter the relative motions between any two links) depends on only (1) the link lengths (or, more properly, relative lengths) as parameters and (2) one variable, usually the position of the input link. All motions repeat themselves identically with each cycle. Such a mechanism is thus not only movable but is also said to be constrained; i.e., an exact duplication of the motions may be counted upon each time the input link assumes the same position.

When a projected mechanism (or kinematic chain) has more than four links, the existence of constrained movability may not be immediately apparent and some sort of movability criterion is desirable.

The most obvious external characteristics of a kinematic chain are the number of parts and the number of connections. Movability studies based on only these two factors—the number of links and the number of joints—have acquired the name of *number synthesis*. The oldest and still useful, although incomplete, estimate of movability is known as the Grübler criterion. Number synthesis is presently based on a choice of parameters limited to the above two, but the situation is not that simple. Link lengths and special configurations of axis directions and locations, positions of instantaneous centers of velocity, complexity of connections, and perhaps other factors contribute to movability or lack of it. The interaction of all factors has not been summarized in one comprehensive and all-revealing relation. As we shall see, a simple relation exists based on a count of links and joints but exceptions have to be noted when particular geometries are present in the chain.

The number of degrees of freedom of a system is the number of independent variables that must be specified to define completely the condition of the system. In the case of kinematic chains, it is the number  $F$  of independent pair variables needed to completely define the relative positions of all links. For example, the truss shown in Fig. 5-5a has *zero* degrees of freedom ( $F = 0$ ); here the relative positions of the links result from their lengths, and no pair variable can be specified.

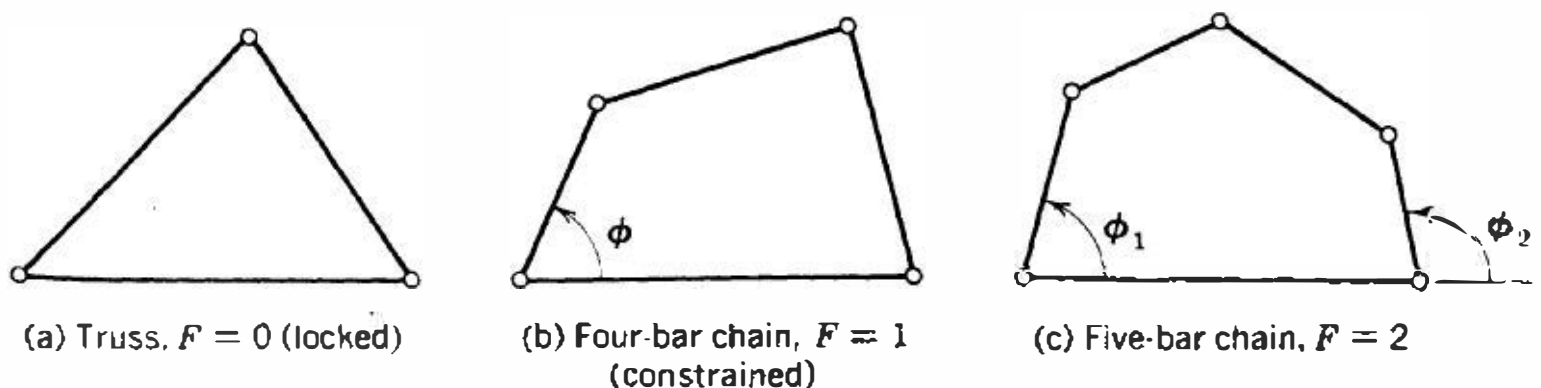


FIGURE 5-5 Examples of kinematic chains with  $F = 0, 1,$  and  $2.$

The number of degrees of freedom of the four-bar chain of Fig. 5-5b is unity ( $F = 1$ ); here one variable such as  $\phi$  is needed to define the relative positions of all links. A five-bar chain (Fig. 5-5c) has two degrees of freedom ( $F = 2$ ), for two angles such as  $\phi_1$  and  $\phi_2$  are needed to define the relative positions of all links.

A kinematic chain is said to be *movable* when its number of degrees of freedom is one or greater ( $F \geq 1$ ); it is otherwise *locked* ( $F < 1$ ). If the number of degrees of freedom is equal to unity ( $F = 1$ ), the chain is said to be *constrained*. Most mechanisms used in machinery have constrained motion; however, in computer or control mechanisms, which may have several inputs, the number of degrees of freedom will be the same as the number of inputs.

The four-bar linkage has been subjected to many studies, including movability! One such is that of Grashof (1883), whose criterion for the identification of a four-bar linkage possessing at least one link able to rotate continuously was discussed in Chap. 3. Grashof, it will be remembered, was concerned with the kind of mechanism—crank rocker, double crank, or double rocker—and noted that the link proportions, in addition to the fixing of a particular link, played a dominant role in establishing the kind of mechanism.

Grashof began his deductive study with a closed four-bar chain, which he took to consist of the links  $a$ ,  $b$  (coupler),  $c$ , and  $d$  (frame). Furthermore, he specified  $c > a$ ,  $d \geq a$ , and  $0 < b < a + c + d$  and excluded the special cases in which two links were of the same length. In words, he started from a general and movable four-bar mechanism.

If the lengths  $a$ ,  $c$ , and  $d$  are given, and if  $b$  is allowed to increase from its lower limit of zero, or if  $b$  is allowed to decrease from its upper limit of  $a + c + d$ , the movability of the mechanism increases gradually in the sense that links  $a$  and  $c$  will oscillate over increasing arcs. The mechanism, locked at the limits  $b = 0$  and  $b = a + c + d$ , becomes a double rocker with the onset of each of the indicated changes in  $b$ . From here Grashof went on to show the relations needed to delineate the three kinds of mechanisms inherent in the four-bar linkage. With even simple chains the influence of link lengths is seen to be a powerful one.

Another approach, and one suited to planar chains of more than four links, is that taken by Grübler, also in 1883. In this the development proceeds from the number of degrees of freedom allowed by the kinematic pairs connecting successive links and leads to the degree of freedom of the chain. The deductions appear as a rule expressed in the form of an equation which may be called a criterion of movability. The planar criterion was rediscovered in 1928.<sup>1</sup> The Grübler criterion for spatial linkages bears the date of 1917.

<sup>1</sup> W. J. Walker, The Relation between Kinematic Pairs and Links in a Mechanism, *Phil. Mag.*, ser. 7, vol. 6, no. 37, p. 631, October, 1928.

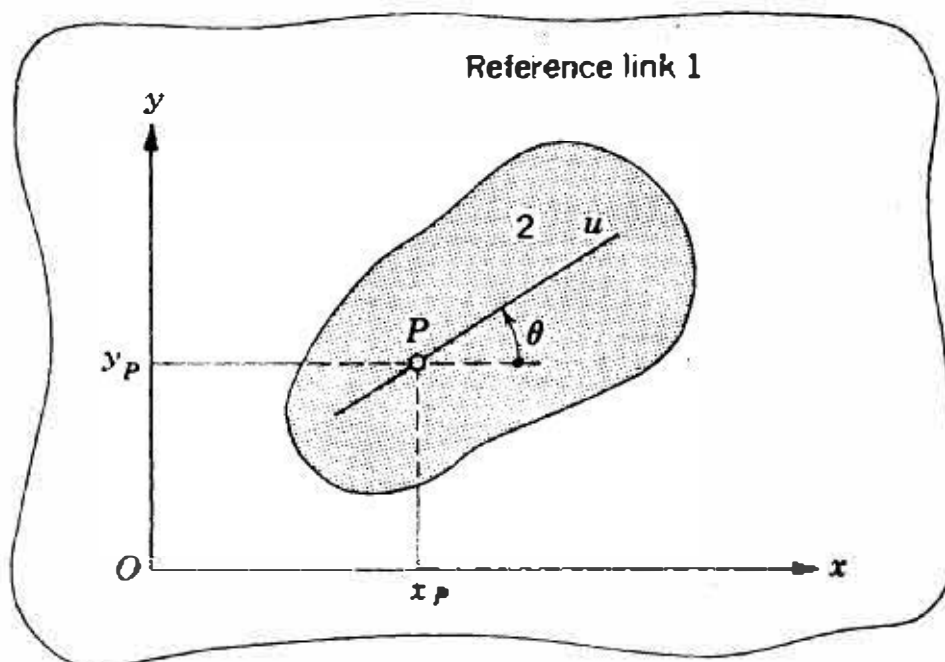


FIGURE 5-6 Rigid link 2 in planar motion with respect to reference link 1.

Grübler's criterion for planar mechanisms of lower pairs (1883) incorporated the number of links  $n$  and the number of joints  $j$  into an equation valid for constrained motion; that is,  $F = 1$  for the chain. The Grübler equation is  $2j - 3n + 4 = 0$ . It is possible, however, to present a criterion that predicts more than just the fulfillment of the single condition  $F = 1$ . Accordingly, we depart from Grübler's approach although following it in spirit.

Consider two links in planar motion (Fig. 5-6). Link 1, with coordinate system  $Oxy$ , is taken as the reference plane. The moving plane, or link 2, carries a line  $u$  containing a point  $P$ . The position of link 2 with respect to link 1 is determined from the specification of three variables, the coordinates  $x_P$  and  $y_P$  of the point  $P$  and the inclination  $\theta$  of the line  $u$ . These three variables are further identified as the degrees of freedom of a body in planar motion; for such a body  $f_{\max} = 3$ . If  $x_P$  and  $y_P$  are made invariant by a revolute connection at  $P$ , two restraints  $r$  to motion are imposed on link 2 and its degree of freedom drops to  $f = 1$ ; its position is now decreed by only a single variable, here  $\theta$ . A revolute connection is thus characterized by either  $f = 1$  or  $r = 2$ .

Suppose that we have a closed chain of  $n$  links, connected by  $j$  revolute pairs, that is to have a degree of freedom of  $F$ . Each link initially possesses three degrees of freedom before connection to any other link, whence the total-degrees-of-freedom number  $3n$ . On choosing one link as a reference for all others, i.e., fixing one link to create a mechanism,  $n - 1$  moving links remain, and the degrees of freedom now total only  $3(n - 1)$ . Each revolute connection means the loss of two degrees of freedom; with  $j$  connections there is a loss of  $2j$  degrees of freedom. We may summarize our situation by writing

$$F = 3(n - 1) - 2j \quad (5-1)$$

A mechanism with constrained motion has  $F = 1$ , whence

$$1 = 3(n - 1) - 2j$$

Rearranging the terms produces

$$2j - 3n + 4 = 0 \quad (5-2)$$

This last is the classical form of the Grübler criterion.<sup>1</sup> We remark that it involves only the number of links  $n$  and the number of revolute connections  $j$  and pays no attention to link dimensions or other geometric features.

Further discussion of the movability criterion is helped by considering Eq. (5-1). This equation is especially useful in that it has a form that can be remembered easily, even derived on the spot. Having two unknown quantities, we are forced to solve for them by comparing trial and error with experience. Application of this equation to various closed kinematic chains will allow certain deductions:

1. Two links, two joints: two overlapped links

$$n = 2 \quad j = 2 \quad F = 3(2 - 1) - 2 \times 2 = -1$$

We may call this a statically indeterminate structure.

2. Three links, three joints: truss

$$n = 3 \quad j = 3 \quad F = 3(3 - 1) - 2 \times 3 = 0$$

We recognize this as a statically determinate structure.

3. Four links, four joints as a movable four-bar chain

$$n = 4 \quad j = 4 \quad F = 3(4 - 1) - 2 \times 4 = 1$$

This is constrained motion, and one input is required. However, we note that the criterion fails if one link is as long as the sum of the other three (Grashof), for the linkage is then immovable and statically indeterminate.

4. Five links, five joints

$$n = 5 \quad j = 5 \quad F = 3(5 - 1) - 2 \times 5 = 2$$

This is unconstrained motion unless two inputs are provided. Also, the criterion is again insensitive to the case in which one link's length is equal to that of the sum of the other link lengths (extension of Grashof), such a linkage being immovable and statically indeterminate.

Two additional examples might be considered. In Fig. 5-7a one diagonal link, number 5, has been added to a four-bar chain, forming two

<sup>1</sup> Writing in both *Civilingenieur*, vol. 29, p. 187, 1883, and "Encyklopädie der mathematischen Wissenschaften," vol. IV, p. 127, 1901, Grübler recognized that both Chebyshev (1869) and Sylvester (1874) were aware of a relation equivalent to our Eq. (5-2).

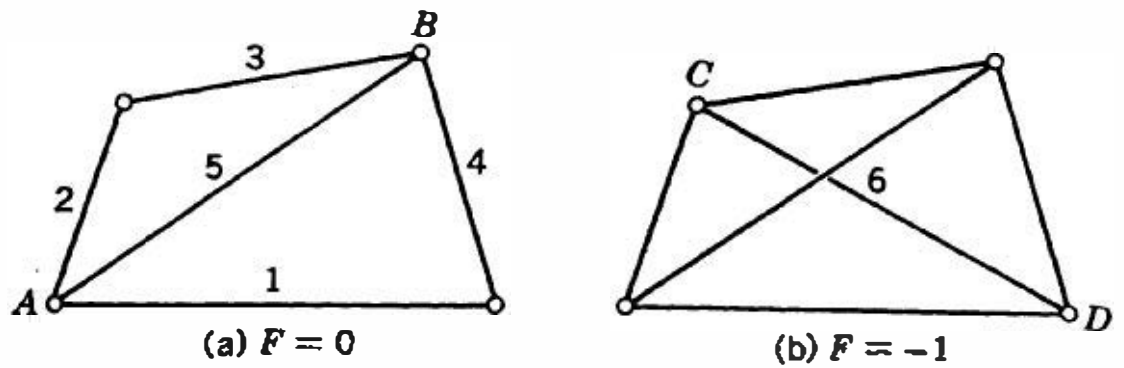


FIGURE 5-7 Locked and statically indeterminate chains.

immovable trusses. Clearly,  $n = 5$ . The number of joints is six, however, for the revolutes at  $A$  and  $B$  serve also as connections of and to link 5 and hence must be counted for this service. Applying the criterion,

$$F = 3(5 - 1) - 2 \times 6 = 0$$

and we agree that the linkage represents a statically determinate truss. The addition of a second diagonal, link 6 (Fig. 5-7b), means  $n = 6$ ,  $j = 8$ , and

$$F = 3(6 - 1) - 2 \times 8 = -1$$

The structure is statically indeterminate.

Grübler noted also that prismatic pairs, like revolutes, possess one degree of freedom and hence might be included as joints. In Fig. 5-8 the moving link 2 is connected to the reference plane (link 1) by a prismatic guide enforcing the direction of motion  $\theta$  between the links. This guide could be anywhere; two physical possibilities are shown. However, some fixed point such as  $P$  is needed as reference from which to measure the translation  $s$ . This  $s$  is then the pair variable or degree of

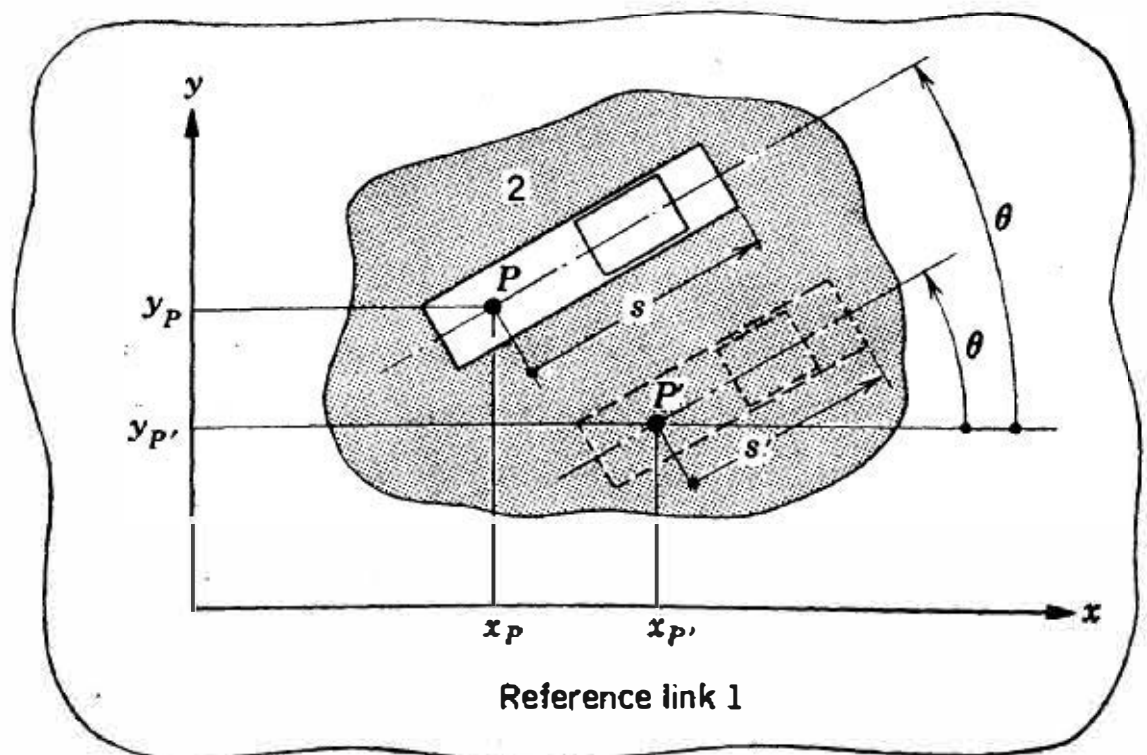


FIGURE 5-8 Rigid link 2 connected to reference link 1 by a prismatic pair.

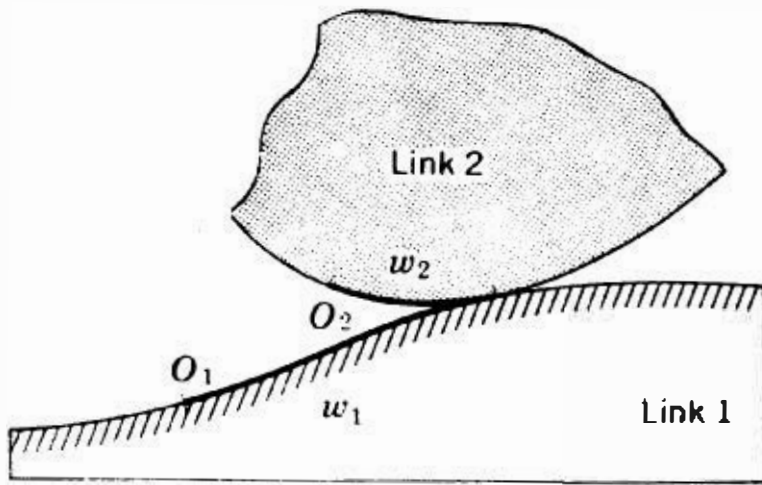


FIGURE 5-9 Higher-pair connection between links 1 and 2 allowing simultaneous rolling and sliding between elements. Origins  $O_1$  and  $O_2$  are fixed on their respective elements; the pair variables are  $w_1$  and  $w_2$ , whence  $f = 2$ .

freedom characteristic of the prismatic pair; we may also say that two restraints are offered.

Prismatic joints need to be regarded with more circumspection than revolute connections when used in the Grübler criterion, for geometric singularities make themselves felt much sooner. In general, several parallel prismatic connections should be viewed with suspicion, and they certainly should not be on the same link. A closed three-link chain with nonparallel prismatic connectors is constrained and movable ( $F = 1$ ), while a four-link affair evidences two degrees of freedom, even with nonparallel connections. Further discussion of prismatic pairs would be somewhat academic, for it is unlikely that they will be the sole constituents of chains found in practice, and they will certainly not appear in profusion in the synthesis problems to be considered.

Higher pairs found in planar mechanisms may also be included in the Grübler criterion if the two degrees of freedom they provide are recognized. The higher pair of Fig. 5-9 allows both roll and slide between pair elements, whence  $f = 2$  and  $r = 1$ . It would be counted as two joints, and another (binary) link would have to be included in the count, since a higher pair is the equivalent of two lower pairs, or, when replaced by an equivalent linkage, picks up another link (see Fig. 5-10).

#### 5-4 DIMENSIONAL SYNTHESIS

By dimensional synthesis we understand the determination of the dimensions of parts—lengths and angles—necessary to create a mechanism that will effect a desired motion transformation. Laying out a cam to meet certain specifications is dimensional synthesis. The cam provides a functional relationship between shaft rotation and follower displacement, velocity and acceleration: the cam is a function generator. A linkage is another type of function generator, but the paths to the goal are not as direct as with cams.

In considering dimensional synthesis, it is recognized that it has two aspects, called approximate and exact. We shall meet this distinc-

tion in terms of the Watt approximate straight-line mechanism and the Hart and Peaucellier devices able to produce a true straight line (Sec. 6-6). Were we to ask a Watt linkage to generate the function  $y = mx$ , a true straight line, it could do so only approximately in the sense that the coupler point's trajectory would be a wavy line intersecting the true line every now and then. The two curves, that generated mechanically and the one desired, have in common only their points of intersection: these are aptly called the *accuracy* (or *precision*) *points* of the generated function. For a given length of line we should like to have (1) as many accuracy points as possible and (2) a minimum deviation (or error) between the curves.

Exact synthesis is limited, since few arbitrary functions can be handled. For example, it takes at least a six-bar linkage such as the

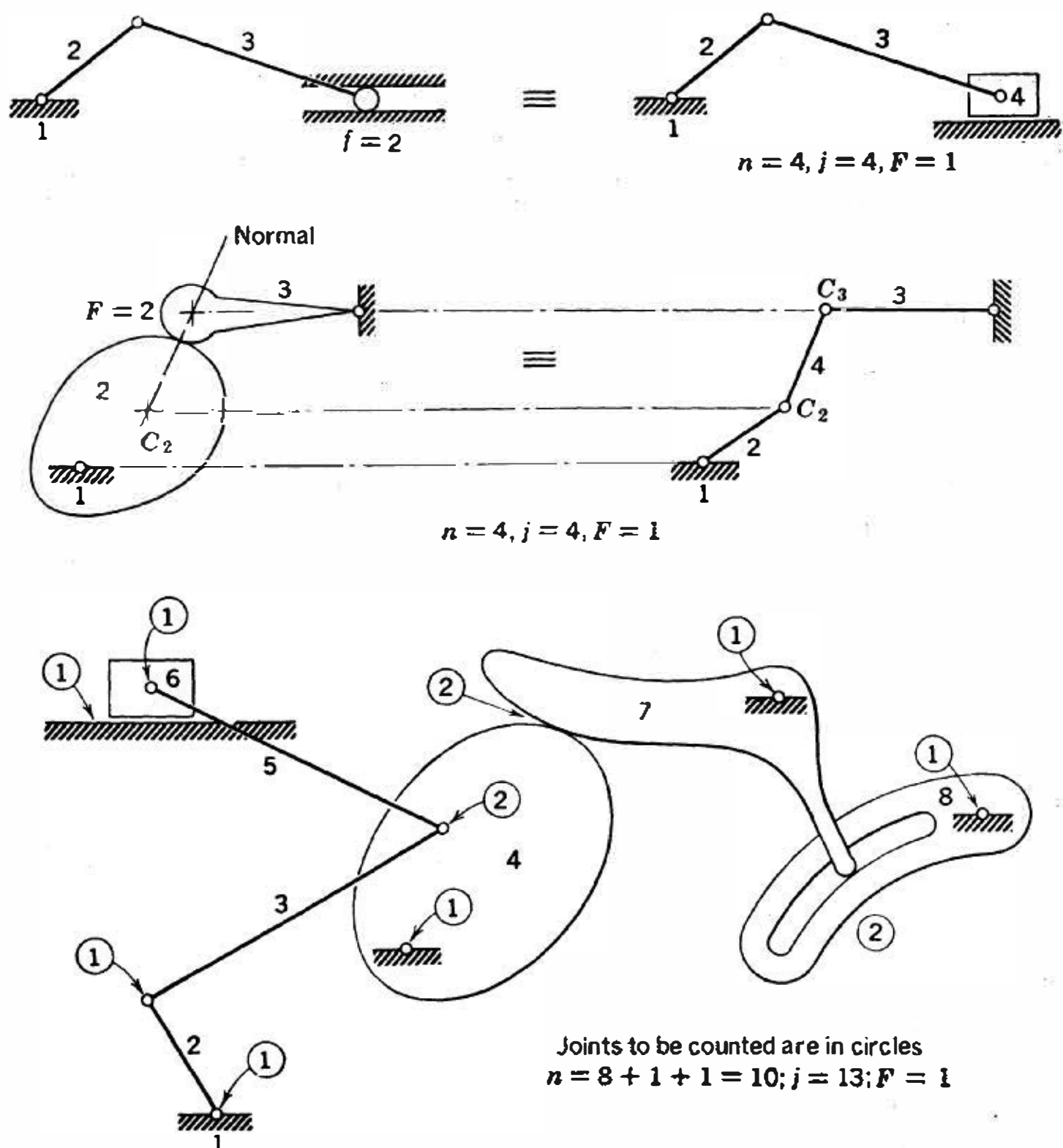


FIGURE 5-10 Higher pairs and the Grübler criterion.

Hart to generate a true straight line. On the other side of the coin there is a notable exception: only a trivial mechanism is required to describe a circle, while a four-bar is needed to produce circular arcs that are only approximate. Nevertheless, exact, or precision, synthesis is limited to certain "nice" functions, whereas approximate synthesis can do a job within a limited range on almost any function.

The concern in this text is with approximate synthesis. There are two approaches—the geometric, or graphic, and the analytic, or algebraic. Geometry is, of course, analytical, but the term analytic has become associated with algebraic methods of computation in comparison with graphical constructions. Speaking in rather broad terms, it may be said that the geometric methods have been developed by the German school, with emphasis on planar linkages. These methods are considered in Chaps. 6 to 9. The Russian effort, once heavily geometric, has now a strong bias toward the algebraic attack, well suited to spatial mechanisms. In America—and one can only speak of recent years—the principal developments have been in algebraic treatments, with the inclusion of spatial linkages. The algebraic methods, whose rise roughly parallels the coming of age of digital computers, are considered in Chaps. 10 to 12.

The geometric methods can furnish, with reasonable accuracy, quick and dependable solutions to a number of problems. They give direct feeling for mechanical details which will be important in reducing a given solution to hardware and which may be obtained on the drawing board without making use of the sometimes unfamiliar or unavailable techniques of automatic computation. In the solution of some problems, as in the case of spatial mechanisms or when the requirements for accuracy are more demanding, geometric methods may, however, become cumbersome, lengthy, and undependable. For such problems, analytic methods, dependent on the use of automatic computation techniques, may yield practical and economical solutions.

### 5-5 SPACING OF ACCURACY POINTS

It was pointed out earlier that, when a mechanism is designed to generate a given function or trace a given curve, it is not possible in general to obtain a mathematically exact solution but that the mechanism fits the function or curve at only a finite number of points, the accuracy points. The number of these accuracy points is equal to the number of fixed parameters that may be used in the synthesis and varies in general between three and six. The problem considered in this section is that of spacing the accuracy points within the interval of function generation to minimize the errors between accuracy points.

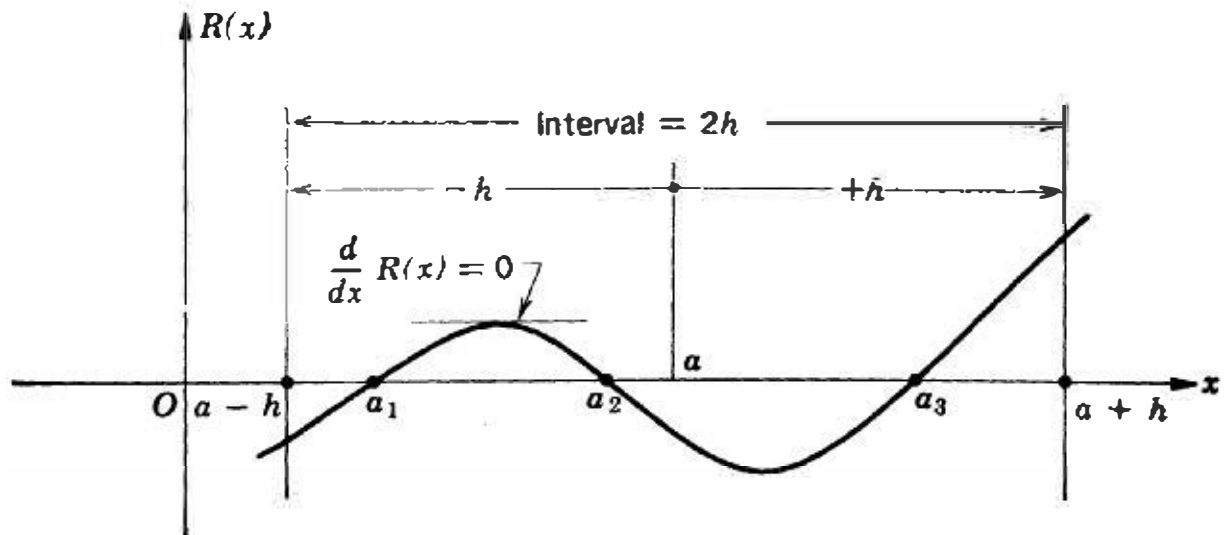


FIGURE 5-11 Error curve with three accuracy points.

Consider the function  $f(x)$  to be approximated in a given interval of variation of  $x$  by means of a mechanism which generates the function  $F(x; q_1, \dots, q_n) = F(x; q_k)$ , where  $q_1, \dots, q_n$  are the values of the design parameters in the mechanism. The function  $f(x)$  describes the desired feature of the motion; this is to be approximated by, say, the path of a coupler point or a relation between crank and follower rotations in a four-bar linkage. The function  $F(x; q_k)$  defines the feature of the motion as it is actually performed by the mechanism with design parameters set to  $q_1, \dots, q_n$ . The difference between these two functions is the *structural error*,

$$R(x) = f(x) - F(x; q_k)$$

The general appearance of the structural error when plotted against  $x$  is shown in Fig. 5-11, where the center of the interval has been taken at  $x = a$  and its width as  $2h$ , thus,

$$a - h \leq x \leq a + h$$

The error is zero at each accuracy point (points  $a_1$ ,  $a_2$ , and  $a_3$ ) and reaches a series of maxima and minima between accuracy points.

If the magnitude of the error  $R(x)$  is to be kept to a minimum throughout the whole interval of variation of  $x$ , the optimum spacing of accuracy points will be that for which the error curve takes the form shown in Fig. 5-12, where all maxima and minima as well as the values of the error at the extremities of the interval are of the same magnitude. The exact spacing of accuracy points corresponding to this optimum situation depends, of course, on the function to be generated as well as on the mechanism, and its exact determination is a complex problem. It may be shown, however, that the structural error  $R(x)$  may be expressed as

$$R(x) = KP_n(x) \tag{5-3}$$

with

$$P_n(x) = (x - a_1)(x - a_2) \cdots (x - a_n)$$

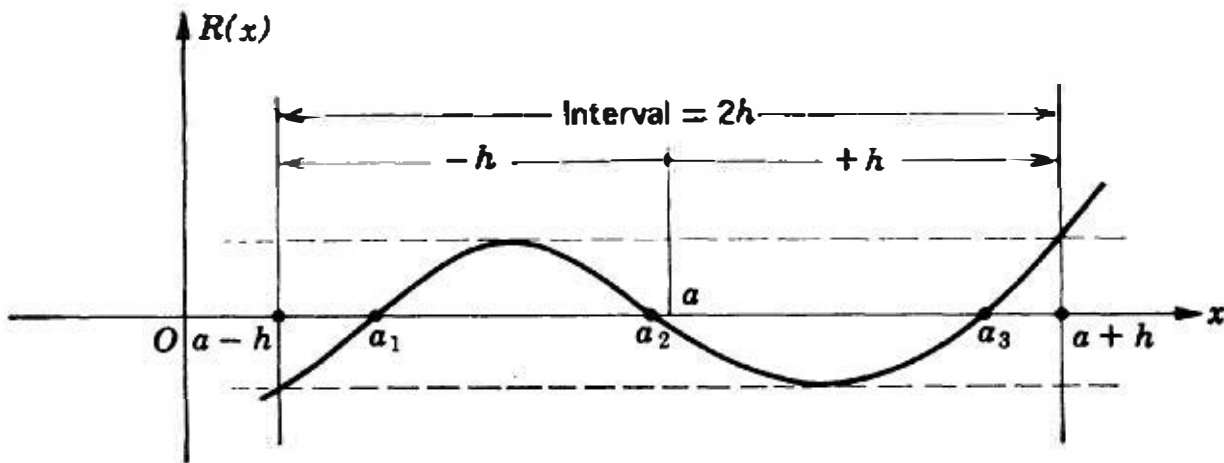


FIGURE 5-12 (Optimum error curve.)

where  $a_1, a_2, \dots, a_n$  are the accuracy points and  $K$  may be considered constant as a first approximation. Within this first approximation, the problem of optimum spacing is reduced to finding a polynomial  $P_n(x)$  of degree  $n$  and leading coefficient (the coefficient of  $x^n$ ) equal to unity that deviates least from zero. The solution to this problem is given by the Chebyshev polynomials (see Sec. 5-6),

$$T_2(x) = (x - a)^2 - \frac{h^2}{2} \quad \text{when } n = 2$$

$$T_3(x) = (x - a)^3 - \frac{3h^2}{4}(x - a) \quad \text{when } n = 3$$

or in general<sup>1</sup> 
$$T_n(x) = \frac{h^n}{2^{n-1}} \cos \left( n \arccos \frac{x - a}{h} \right)$$

The accuracy points are the roots of the corresponding Chebyshev polynomial, and their values may be conveniently found as follows. From the center of the interval ( $a - h, a + h$ ) taken along an axis  $x$ , draw a circle of radius equal to one-half this interval, and inscribe a regular polygon of  $2n$  sides in it in such a way that two of its sides are perpendicular to the  $x$  axis. The projections of the vertices of this polygon onto the  $x$  axis determine the accuracy points. The construction for  $n = 4$  is shown in Fig. 5-13.

For some problems it is desirable to minimize the first derivative of the structural error with respect to  $x$  rather than minimize the structural error itself. Such a problem would be the synthesis of a mechanism in which the velocity ratio between input and output links is to vary in a prescribed manner throughout the range of operation. Examination of Fig. 5-11 shows that the derivative of the error, i.e., the slope of the error curve, generally becomes zero for one value of  $x$  between two accuracy points and that it is comparatively large at the accuracy points themselves. This shows that, if a function is generated with  $n$

<sup>1</sup> The hardly obvious general expression is developed in Sec. 5-6.

accuracy points, its derivative will ordinarily be exact at only  $n - 1$  points, whence less accuracy must be expected in the design of a mechanism for prescribed velocity relations than in a design for prescribed displacement relations.

A first approximation to optimum spacing of accuracy points for the velocity case may be obtained by taking the derivative of the structural error as expressed in Eq. (5-3),

$$\frac{d}{dx} R(x) = K \frac{d}{dx} P_n(x)$$

and remarking that  $dR/dx$  deviates least from zero when  $dP_n/dx$  itself deviates least from zero,  $K$  being assumed constant. The polynomial  $P_n(x)$  must therefore be of degree  $n$  and have a leading coefficient equal to unity, and its derivative must deviate least from zero. Thus  $dP_n/dx$  is taken proportional to the Chebyshev polynomial  $T_{n-1}$ , and

$$P_n(x) = b[\int T_{n-1}(x) dx + C]$$

where  $b$  is a constant of proportionality and  $C$  a constant of integration.

The determination of the accuracy points and the meaning of the constants will be demonstrated by an example. Let it be required to establish four accuracy points  $a_1, a_2, a_3, a_4$  for the interval  $-1 \leq x \leq +1$ , for which (within the usual first approximation) the derivative of the error is minimized. Since  $n = 4$ ,  $h = 1$ ,  $a = 0$ , we find

$$P_4(x) = b\left[\int (x^3 - \frac{3}{4}x) dx + C\right] = b\left(\frac{x^4}{4} - \frac{3}{8}x^2 + C\right)$$

The value of  $b$  must be found such that the coefficient of  $x^4$  in  $P_4(x)$  is unity, whence  $b = 4$  and

$$P_4(x) = x^4 - \frac{3}{2}x^2 + 4C$$

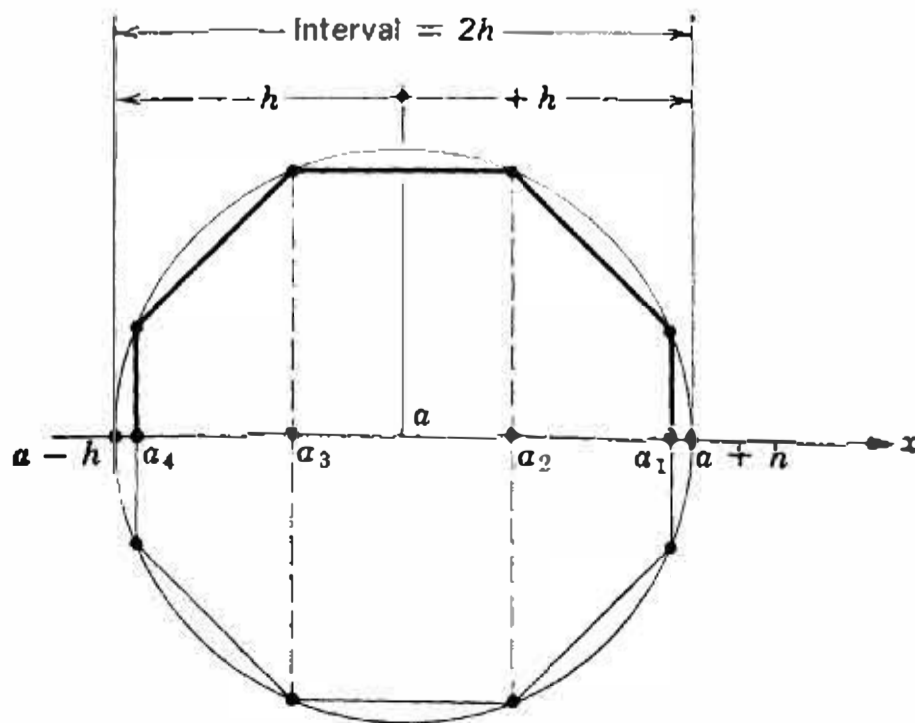


FIGURE 5-13 Determination of four accuracy points with Chebyshev spacing.

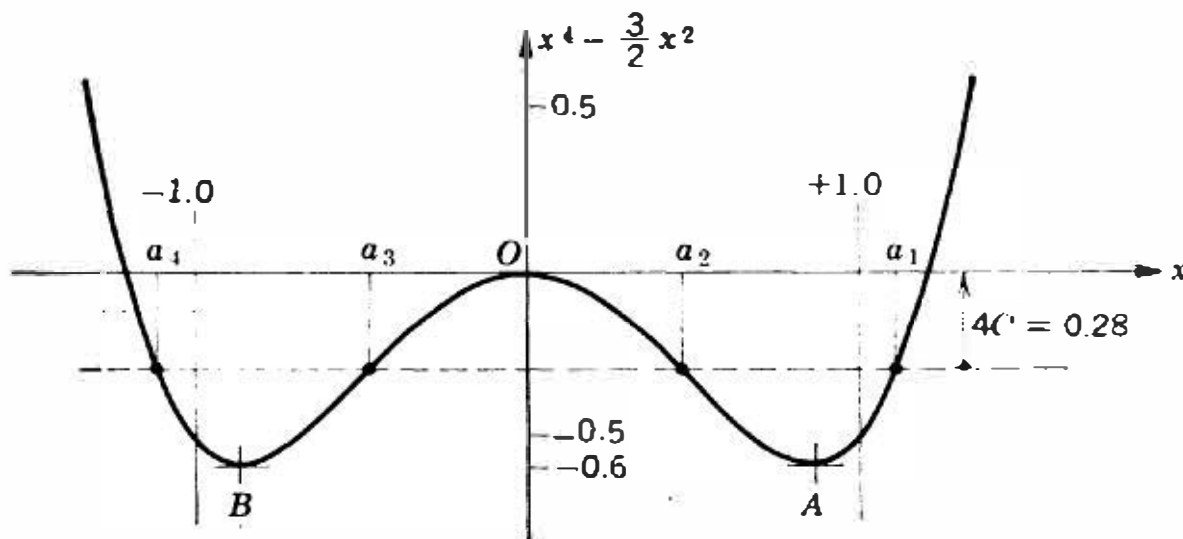


FIGURE 5-14 Determination of four accuracy points to minimize the derivative error according to the first approximation of Chebyshev.

The velocity error, which depends on the derivative  $dP_4/dx$ , is independent of the constant term  $4C$  related to the displacement error. This constant term may therefore be adjusted to reduce the displacement error. The variable portion of  $P_n(x)$ , that is,  $x^4 - \frac{3}{2}x^2$  is plotted against  $x$ , as shown in Fig. 5-14, and the accuracy points  $a_1, a_2, a_3, a_4$  are found by the intersection of this plot with a parallel to the  $x$  axis equidistant from the maxima and minima of the plot. Thus,

$$\begin{aligned} a_1 &= 1.12 \\ a_2 &= 0.47 \\ a_3 &= 0.47 \\ a_4 &= 1.12 \end{aligned}$$

The constant term  $4C$  represents the maximum displacement error. It is noted that two of the accuracy points,  $a_1$  and  $a_4$ , lie outside the interval of generation. This is not peculiar to the example but will occur in all spacing of accuracy points for which the derivative of the error is minimized. It is to be expected, since the slope of the error curve (Fig. 5-14) becomes larger for values of  $x$  beyond the minima at  $A$  and  $B$ .

## 5-6 CHEBYSHEV POLYNOMIALS<sup>1</sup>

The use of Chebyshev polynomials for the choice of accuracy points as presented in the preceding section would give the optimum choice only if the factor  $K$  in Eq. (5-3) were precisely a constant. This factor, however, is truly a constant only for the case in which the function  $f(x)$  is a polynomial of degree  $n$  and is to be approximated by the function  $F(x; q_k)$ , the latter being also a polynomial but of (lesser) degree  $n - 1$ . Here  $n$  is the number of accuracy points, or the number of design param-

<sup>1</sup> This section may be omitted on first reading.



degree  $n$  with leading coefficient equal to unity deviating least from zero in the interval  $a - h, a + h$ . This is known as a Chebyshev polynomial.

The polynomial  $P(x)$  of degree  $n$  has  $n$  zeros at  $x = a_1, \dots, a_n$  and has a leading coefficient equal to unity. Its graphical representation would exhibit a maximum or a minimum between each of the zeros, and the problem is to reduce to a minimum the magnitude of the largest maximum or minimum. It is reasonable to assume (as Chebyshev did) that this situation will occur when all maxima and minima are equal, but this succession of equal maxima and minima suggests that such a polynomial, by suitable changes of variables, could be represented by trigonometric functions. The interval of approximation  $a - h, a + h$  is first reduced to the interval  $-1 \leq z \leq +1$  by the change of variable

$$x - a = zh$$

Consider now the function  $\cos n\theta$ , where  $\cos \theta = z$ . This function has the desired features of  $P(x)$ , that is,  $n$  zeros in the interval  $a - h, a + h$  or  $-1 \leq z \leq +1$  with equal maxima and minima between all successive zeros. This function  $\cos n\theta$  may be expanded in terms of  $\cos \theta = z$ ; this is trivial for  $n = 0$  and  $n = 1$ . For  $n = 2$ ,

$$\cos 2\theta = 2 \cos^2 \theta - 1 = 2z^2 - 1$$

and for  $n > 2$ , the expansion may be deduced from the recurrence formula

$$\cos n\theta = 2 \cos (n - 1)\theta \cos \theta - \cos (n - 2)\theta$$

derived from elementary trigonometric identities. Thus,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta = 4z^3 - 3z$$

$$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1 = 8z^4 - 8z^2 + 1$$

Note that in these polynomials the leading coefficients are  $2^{n-1}$ . Since in the polynomial  $P(x)$  the leading coefficient must be unity,  $P(x)$  must be of the form

$$T_n(z) = 2^{1-n} \cos (n \arccos z)$$

or

$$T_n(x) = h^n 2^{1-n} \cos \left( n \arccos \frac{x - a}{h} \right)$$

and the zeros must correspond to

$$n \arccos \frac{x - a}{h} = \frac{(2k - 1)\pi}{2} \quad \text{or} \quad x = a + h \cos \frac{(2k - 1)\pi}{2n}$$

The polynomials  $T_n$ , introduced by Chebyshev in a memoir (1853) devoted primarily to approximate straight-line mechanisms, are called Chebyshev polynomials. It may be shown<sup>1</sup> that of all polynomials of

<sup>1</sup> R. Courant and D. Hilbert, "Methods of Mathematical Physics," vol. I, pp. 88-90, Interscience Publishers, Inc., New York, 1953.

same degree and the same leading coefficient they deviate least from zero. If the  $n$  accuracy points  $a_1, \dots, a_n$  of the interval are set equal to the roots of the polynomial  $T_n$ , that is,

$$a_k = a + h \cos \frac{(2k - 1)\pi}{2n} \quad k = 1, \dots, n$$

then  $P(x) = T_n(x)$  and the deviation  $R(x)$  for the resulting approximating polynomial  $F(x; q_k)$  will be smaller than for any other distribution of accuracy points.

Although the functions  $f(x)$  and  $F(x; q_k)$  in an actual case of dimensional synthesis would not be polynomials of degree  $n$  and  $n - 1$ , respectively, these functions may be expanded in Taylor series. As a first approximation, the terms of order higher than  $n$  in  $f(x)$  and higher than  $n - 1$  in  $F(x; q_k)$  may be neglected for the purpose of locating the accuracy points. The problem would then reduce to the case considered above, and a Chebyshev spacing would result. The Chebyshev spacing therefore appears as a convenient first choice, but it should not be expected to yield optimum design. A spacing of accuracy points reducing the structural error to a minimum can be obtained only by repeated trials: starting with a Chebyshev spacing, a dimensional synthesis is carried out and followed by an analysis of the mechanism to compare the functions  $f(x)$  and  $F(x; q_k)$ . The structural error is plotted as a function of  $x$ , and the accuracy points are moved close together where the maximum structural error occurs. The process is then repeated until all maxima and minima of the structural error between accuracy points and at the ends of the interval have been equalized. An example of this procedure, which requires the use of automatic computation to be feasible, is discussed in Sec. 10-8, where the synthesis of four-bar linkage with five accuracy points is considered.

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