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COMPUTING THE VOLUME  
IS DIFFICULT

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COMPUTING THE VOLUME  
IS DIFFICULT

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Abstract. For every polynomial time algorithm which gives an upper bound  $\overline{\text{vol}}(K)$  and a lower bound  $\underline{\text{vol}}(K)$  for the volume of a convex set  $K \subset \mathbb{R}^d$ , the ratio  $\overline{\text{vol}}(K)/\underline{\text{vol}}(K)$  is at least  $(d/\log d)^d$  for some convex set  $K \subset \mathbb{R}^d$ .

1. Introduction

The problem addressed in this paper is the behaviour of algorithms that compute the volume of convex sets. We prove a negative result. For any polynomial time algorithm which gives a lower bound  $\underline{\text{vol}}(K)$  and an upper bound  $\overline{\text{vol}}(K)$  for the volume of a convex set  $K \subset \mathbb{R}^d$ , the ratio  $\overline{\text{vol}}(K)/\underline{\text{vol}}(K)$  is at least  $(cd/\log d)^d$  for some convex body  $K \subset \mathbb{R}^d$  where  $c$  is a constant independent of  $d$ .

Our model of a convex set coincides with that of Lovász [1985] and Grötschel-Lovász-Schrijver [1985]. In this model a convex set  $K \subset \mathbb{R}^d$  is black box that answers questions of the following

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type. Given a point  $x \in Q^d$ , is  $x \in K$ ? In this case we say that the black box (or the convex set) is given by a membership oracle. The convex set  $K$  may be given by a separation oracle as well. This is again a black box which, given a point  $x \in Q^d$ , decides whether  $x \in K$  and if not, then it gives a hyperplane separating  $x$  and  $K$ .

A moment's meditation shows that one needs some further information on the convex set given by the black box. So the black box will have to wear an additional guarantee: The convex set described by this box is contained in  $RB^d$  and contains  $rB^d$ , where  $B^d$  is the Euclidean unit ball around the origin and  $R > r > 0$ . In this case we say that the oracle describing the convex body is well-guaranteed. For technical reasons we assume that  $R = 2^{\ell_1}$  and  $r = 2^{-\ell_2}$  where  $\ell_1$  and  $\ell_2$  are non-negative integers, then the input size of the oracle is  $d + (1 + \ell_1) + (1 + \ell_2)$ . So we assume that convex sets are given by a separation oracle which is well-guaranteed. Using a version of the ellipsoid method Lovász [1985] gave an algorithm that determines a lower bound  $\underline{\text{vol}}(K)$  and an upper bound  $\overline{\text{vol}}(K)$  for the volume of the convex set  $K$ . This algorithm is polynomial in the input size of the oracle and has the following property:

$$\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq d^{d/2} (d+1)^d.$$

Moreover, if the convex set described by the oracle is centrally symmetric, then the result is better:

$$\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq d^d.$$

Both estimations seem to be very poor, but the following result of Elekes [1982] (see Lovasz [1985]) shows that any polynomial time algorithm must leave a huge gap between  $\overline{\text{vol}}(K)$  and  $\underline{\text{vol}}(K)$ . He proved, in fact, that there is no polynomial time algorithm which would compute a lower bound and an upper bound for  $\text{vol}(K)$  with

$$\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq 1.999^d.$$

Lovasz [1983] thought that even  $(\overline{\text{vol}}(K)/\underline{\text{vol}}(K))^{1/d}$  cannot be bounded. We prove this in the following stronger form:

Theorem 1. There is no polynomial time algorithm which would compute a lower and an upper bound for  $\text{vol}(K)$  with

$$\frac{\overline{\text{vol}}(K)}{\underline{\text{vol}}(K)} \leq \left(c \frac{d}{\log d}\right)^d$$

where the constant  $c$  does not depend on  $d$ .

Theorem 1 shows that Lovasz' algorithm is very close to being optimal when the oracle contains centrally symmetric convex bodies only.

Let  $V(d,n)$  denote the maximum volume of the convex hull of  $n$  points from  $B^d$ . Theorem 1 will follow from

Theorem 2. If  $n = d^a$ , then, for sufficiently large  $d$

$$\frac{V(d,n)}{\text{vol}(B^d)} \leq \left(\frac{2ae \log d}{d}\right)^{d/2}.$$

The estimation in Theorem 2 is fairly good. This can be seen from

Theorem 3. If  $n = d^a$ , then, for sufficiently large  $d$

$$\frac{V(d,n)}{\text{vol}(B^d)} \geq \left( \frac{(2a-3)\log d}{d} \right)^{d/2}.$$

Theorem 2 and 3 may be written as

$$c_1 \left( \frac{a \log d}{d} \right)^{1/2} < \left[ \frac{V(d,d^a)}{\text{vol}(B^d)} \right]^{1/d} < c_2 \left( \frac{a \log d}{d} \right)^{1/2},$$

and these inequalities are about the approximability of the ball by polytopes with "few" vertices. We have some other results in this direction and are going to publish them in a forthcoming paper (Barány, Füredi [1986]).

We will use a beautiful new result of Bourgain and Milman [1985] which we now describe. Let  $\mathcal{K}$  be the family of all centrally symmetric (with respect to the origin), convex, compact,  $d$ -dimensional bodies in  $R^d$ . The polar,  $K^*$  of  $K \in \mathcal{K}$  is defined as

$$K^* = \{x \in R^d: \langle x,y \rangle \leq 1 \quad \forall y \in K\}$$

where  $\langle x,y \rangle$  denotes the scalar product. An old conjecture says that for all  $K \in \mathcal{K}$

$$\text{vol}(K)\text{vol}(K^*) \geq \frac{4^d}{d!}$$

Now Bourgain and Milman [1985] proved this in a slightly weaker form: for all  $K \in \mathcal{K}$

$$\text{vol}(K)\text{vol}(K^*) \geq \frac{c_0^d}{d!}$$

where  $c_0 > 0$  is a universal constant.

We will see from the proofs that the constant  $c$  in Theorem 1 can be taken for  $(2aec_0)^{-1}$  when the algorithm considered tests the membership on  $n = d^a$  points.

In the last section we give some results about the complexity of computing the width of a convex body.

## 2. Proof of Theorem 1

We are going to use a well-guaranteed separation oracle with some additional properties. The first is that the oracle discloses (as a first step, say) that  $e_i e_i \in K$  and  $K \subset \{x \in \mathbb{R}^d: \langle x, e_i e_i \rangle \leq \epsilon_i\}$  for each  $\epsilon_i \in \{-1, 1\}$  and  $i = 1, \dots, d$  where  $e_1, \dots, e_d$  form an orthonormal basis in  $\mathbb{R}^d$ . This property simply means that  $K$  is contained in the cube of side length 2 and contains the cross polytope of diameter 2. In accordance with this  $\ell_1 = \lceil \frac{1}{2} \log d \rceil$  and  $\ell_2 = \lfloor -\frac{1}{2} \log d \rfloor$ . Thus the input size of the oracle is  $d+1+\ell_1+1+\ell_2 < 2d$  if  $d$  is large enough.

We need some notation. For  $x \in \mathbb{R}^d$  ( $x \neq 0$ ) define  $x^0 = x/\|x\|$  and  $H^+(x^0) = \{z \in \mathbb{R}^d: \langle z, x^0 \rangle \leq 1\}$  and  $H^-(x^0) = \{z \in \mathbb{R}^d: \langle z, x^0 \rangle \geq -1\}$ . The second additional property of the oracle is that for the question "is  $x \in K$ " it answers " $x^0 \in K$  and  $-x^0 \in K$  and  $K \subset H^+(x^0)$  and  $K \subset H^-(x^0)$ ". So the oracle gives the endpoints of the line segment  $\{\lambda x: \lambda \in \mathbb{R}\} \cap K$  and also the supporting hyperplanes at the endpoints. We mention

that this information (with any prescribed precision) can be obtained from a separation oracle in polynomial time. So our oracle is just a little bit stronger than a usual separation oracle on centrally symmetric convex bodies.

Now we begin with the proof. Assume that we have an algorithm that gives an upper bound and a lower bound for the volume of a convex body given by the above separation oracle. Let us run this algorithm with  $K = B^d$  first, the points whose membership has been asked by the algorithm are  $x_1, x_2, \dots, x_n$  with  $n = d^a$  ( $a > 1$ ). Assume the algorithm produced  $\overline{\text{vol}}(B^d)$  and  $\underline{\text{vol}}(B^d)$ .

Set now  $C = \text{conv}\{\pm e_1, \dots, \pm e_d, \pm x_1^0, \dots, \pm x_n^0\}$ . It is clear that when running the algorithm with  $C$  or with  $C^*$  (the polar of  $C$ ), the questions and the answers are the same as with  $B^d$ , so

$$\begin{aligned}\overline{\text{vol}}(B^d) &= \overline{\text{vol}}(C^*) \geq \text{vol}(C^*), \quad \text{and} \\ \underline{\text{vol}}(B^d) &= \underline{\text{vol}}(C) \leq \text{vol}(C).\end{aligned}$$

Then

$$\frac{\overline{\text{vol}}(B^d)}{\underline{\text{vol}}(B^d)} \geq \frac{\text{vol}(C^*)}{\text{vol}(C)} \geq \text{vol}(C^*)\text{vol}(C) \left(\frac{1}{\text{vol}(C)}\right)^2.$$

From the result of Bourgain and Milman [1985] we infer

$$\frac{\overline{\text{vol}}(B^d)}{\underline{\text{vol}}(B^d)} \geq c_0^d \left(\frac{\text{vol}(B^d)}{\text{vol}(C)}\right)^2.$$

Now the number of vertices of  $C$  is  $2(n+d) \approx d^a$ , so from Theorem 2 we have

$$\frac{\overline{\text{vol}}(B^d)}{\underline{\text{vol}}(B^d)} \geq \left( \frac{d}{2c_0 e a \log d} \right)^d. \quad \square$$

Remark. It may seem strange that the volume of the unit ball (when it is given by a separation oracle) cannot be determined within a large factor. However this is not so surprising when one thinks of the fact that among all convex bodies the ellipsoids admit the worst approximation by polytopes. (See Macbeath [1951] for an exact statement.)

### 3. The proof of Theorem 2

Some preparation is needed. Given a convex set  $C \subset \mathbb{R}^d$  with  $L = \text{aff}(C)$ , define  $L^\perp$  as the maximal subspace of  $\mathbb{R}^d$  orthogonal to  $L$ . Further, for  $\rho > 0$  let

$$C^\rho = C + (L^\perp \cap \rho B^d),$$

i.e.,  $C^\rho$  is the set of points  $x \in \mathbb{R}^d$  such that if  $x'$  is the nearest point to  $x$  in  $C$ , then  $\|x-x'\| < \rho$  and  $x-x'$  is orthogonal to  $L$ . Define  $\rho(d,1) = 1$ ,  $\rho(d,d) = d^{-1}$  and for  $1 < k < d$

$$\rho(d,k) = \left( \frac{d-k+1}{d(k-1)} \right)^{1/2}.$$

We need a lemma which says that any point of a simplex in  $B^d$  is "near" and "orthogonal" to some  $(k-1)$ -face of the simplex.



Lemma. Given a simplex  $F$  in  $B^d$  and  $k \in \{1, 2, \dots, d\}$  and a point  $x \in F$ , there is a  $(k-1)$ -face  $F_k$  of  $F$  with  $x \in F_k^{\rho(d,k)}$ .

Proof. An easy calculation shows that the statement of the Lemma is true when  $k = 1$ . The case  $k = d$  is equivalent to the following well-known fact (see Fejes Toth [1964]). The ratio of the radii of the circumscribed and inscribed balls of a simplex in  $R^d$  is at least  $d$ . We prove the Lemma using this fact for the cases  $k = 2, 3, \dots, d-1$ . Rename  $x$  as  $x_{d+1}$  and  $F$  as  $F_{d+1}$ . By the above fact there is a facet  $F_d$  such that if  $x_d$  denotes the projection of  $x_{d+1}$  to  $F_d$ , then  $\|x_{d+1} - x_d\| \leq d^{-1}$  and  $x_{d+1} - x_d$  is orthogonal to  $\text{aff}(F_d) = H_d$ . Now  $F_d$  lies in  $H_d \cap B^d$ , so  $F_d$  lies in  $B^{d-1}$  if we choose the origin in  $H_d$  properly. On applying the same argument to  $F_d \subset B^{d-1}$  and  $x_d$  we get a point  $x_{d-1}$  in a facet  $F_{d-1}$  of  $F_d$  such that  $\|x_d - x_{d-1}\| \leq 1/(d-1)$  and  $x_d - x_{d-1}$  is orthogonal to  $\text{aff}(F_{d-1}) = H_{d-1}$ . And so on. We stop with  $x_k \in F_k$ . The vectors  $x_{j+1} - x_j$  ( $j = d, \dots, k$ ) are pairwise orthogonal and all of them orthogonal to  $F_k$ . Consequently  $x_{d+1} - x_k$  is orthogonal to  $F_k$ . By the Pythagoras theorem  $\|x_{d+1} - x_k\|^2 = \|x_{d+1} - x_d\|^2 + \|x_d - x_{d-1}\|^2 + \dots + \|x_{k+1} - x_k\|^2 \leq 1/d^2 + 1/(d-1)^2 + \dots + 1/k^2 < 1/(d(d-1)) + 1/((d-1)(d-2)) + \dots + 1/(k(k-1)) = (d-k+1)/(d(k-1))$ , as claimed.

Remark. It is very likely that the smallest value of  $\rho(d,k)$  for which the Lemma holds is  $((d-k+1)/(dk))^{1/2}$ . This is

the value of  $\rho(d,k)$  when  $F$  is a regular simplex with its vertices in  $S^d$ . However for our purposes the  $\rho(d,k)$  from the Lemma will do and we could gain nothing in Theorem 2 with the best value of  $\rho$ .

Now we prove Theorem 2. Let  $x_1, \dots, x_n \in B^d$ . By Caratheodory's theorem (see Danzer-Grünbaum-Klee [1963]) every point  $x \in \text{conv}\{x_1, \dots, x_n\}$  belongs to some simplex with vertices from  $\{x_1, \dots, x_n\}$ , i.e.,  $x \in \text{conv}\{x_{i_0}, \dots, x_{i_d}\} = F$  for some indices  $1 \leq i_0 < i_1 < \dots < i_d \leq n$ . By the Lemma,  $F$  has a  $(k-1)$ -dimensional face  $F_k$  with  $x \in F_k^{\rho(d,k)}$ . This implies that  $\text{conv}\{x_1, \dots, x_n\} \subseteq \bigcup \{C^{\rho(d,k)} : C = \text{conv}\{x_{j_1}, \dots, x_{j_k}\}\}$  where the union is taken over all  $k$ -tuples from  $\{x_1, \dots, x_n\}$ . This shows that

$$\begin{aligned} \text{vol}(\text{conv}\{x_1, \dots, x_n\}) &\leq \binom{n}{k} \max\{\text{vol}(C^{\rho(d,k)}) : \\ &C = \text{conv}\{a_1, \dots, a_k\}, a_1, \dots, a_k \in B^d\}. \end{aligned}$$

It is easy to see now that

$$\begin{aligned} &\max\{\text{vol}(C^{\rho(d,k)}) : C = \text{conv}\{a_1, \dots, a_k\} \subseteq B^d\} \\ &= \max\{\text{vol}(\text{conv}\{a_1, \dots, a_k\}) : a_1, \dots, a_k \in B^d\} \\ &\quad \cdot \text{vol}_{d-k+1}(B^{d-k+1})[\rho(d,k)]^{d-k+1} \\ &= \binom{k}{k-1}^{(k-1)/2} \frac{k^{1/2} \pi^{(d-k+1)/2}}{(k-1)! \Gamma(\frac{d-k+1}{2} + 1)} [\rho(d,k)]^{d-k+1}. \end{aligned}$$

This implies that

$$V(d,n) \leq \binom{n}{k} \left(\frac{k}{k-1}\right)^{(k-1)/2} \frac{k^{1/2} \pi^{-(d-k+1)/2}}{(k-1)! \Gamma\left(\frac{d-k+1}{2} + 1\right)} \\ \cdot [\rho(d,k)]^{d-k+1}.$$

This holds for every  $k = 1, 2, \dots, d$ . Now we choose  $k = d(2 \log n)^{-1} = d(2\alpha \log d)^{-1}$ . This gives, after a tiresome calculation that

$$\frac{V(d,n)}{\text{vol}(B^d)} < \frac{e^{d(1/2-1/\alpha+\epsilon)} 2^{d/2} (\alpha \log d)^{d/2}}{d^{d/2}}$$

for every  $\epsilon > 0$  if  $d$  is large enough.  $\square$

#### 4. The proof of Theorem 3

We would like to compute the expected volume of the convex hull of  $n$  points chosen uniformly and independently from  $S^d$ . Unfortunately there is no known formula for this. We use instead an integral formula due to Buchta, Müller and Tichy [1985] which gives the expected surface area  $E(d,n)$  of the convex hull of  $n$  points chosen uniformly and independently from  $S^d$ :

$$E(d,n) = \binom{n}{d} \frac{dw_{d-1}}{(d-1)^{d-1}} \left(\frac{w_{d-1}}{w_d}\right)^{d-1} \\ \cdot \int_{-1}^1 \left(\frac{w_{d-1}}{w_d}\right) \int_p^1 (1-q^2)^{(d-3)/2} dq)^{n-d} (1-p^2)^{(d^2-d-2)/2} dp$$

where  $w_d = \text{Area}(S^d)$  denotes the surface area of  $S^d$ . In order to use this formula we choose  $n-d$  points  $x_1, \dots, x_{n-d}$  uniformly and independently from  $S^d$ . Then we take  $d$  points  $y_1, \dots, y_d \in S^d$  in such a way that  $x_1, y_1, \dots, y_d$  form the

vertices of a regular simplex. Denote by  $L_1, \dots, L_m$  the facets of  $C = \text{conv}\{x_1, \dots, x_{n-d}, y_1, \dots, y_d\}$ .  $C$  contains  $d^{-1}B^d$  hence

$$\text{vol}(C) \geq d^{-2} \sum_{i=1}^m \text{vol}_{d-1}(L_i) = d^{-2} \text{Area}(C).$$

Moreover  $C \supset C_0 = \text{conv}\{x_1, \dots, x_{n-d}\}$ . Thus

$$\text{vol}(C) \geq d^{-2} \text{Area}(C) \geq d^{-2} \text{Area}(C_0).$$

This clearly implies that

$$V(d, n) \geq d^{-2} E(d, n-d).$$

After a lengthy computation (we omit the details) we get for  $d$  large enough that

$$\frac{V(d, n)}{\text{vol}(B^d)} \geq \left(\frac{2(a-1)\log d}{d}\right)^{d-2} (\log d)^{-d/\log d}.$$

##### 5. The error in computing the width

Lovasz [1985] gives a polynomial time algorithm which computes a lower bound  $\underline{w}(K)$  and an upper bound  $\overline{w}(K)$  for the width  $w(K)$  of a convex body  $K \subset \mathbb{R}^d$  with  $\overline{w}(K)/\underline{w}(K) \leq d^{1/2}(d+1)$ . The convex sets are again given by a well-guaranteed separation oracle. Elekes [1982] proved that there is no polynomial time algorithm which would compute  $\overline{w}(K)$  and  $\underline{w}(K)$  with  $\overline{w}(K)/\underline{w}(K) \leq 2$ . We improve on this result.

Theorem 4. There is no polynomial time algorithm which would compute an upper bound  $\overline{w}(K)$  and a lower bound  $\underline{w}(K)$  for the width of convex bodies  $K \subset \mathbb{R}^d$  with

$$\frac{\overline{w(K)}}{\underline{w(K)}} \leq (d/\log d)^{1/2}.$$

Proof. We consider the same model as in the proof of Theorem 1. Then

$$\begin{aligned} \overline{w(B^d)} &= \overline{w(C^*)} \geq w(C^*) = 2, \quad \text{and} \\ \underline{w(B^d)} &= \underline{w(C)} \leq w(C). \end{aligned}$$

So the theorem will follow if we can show that

$$(1) \quad w(C) \leq 2 \left( \frac{2a \log d}{d} \right)^d$$

when  $C \subset B^d$  is a centrally symmetric polytope with  $n = 2d^a$  vertices.

There are two ways to see this. The first is that we find a spherical cap  $S \subset S^d$  with

$$\begin{aligned} S \cap \{\pm e_1, \dots, \pm e_d, \pm x_1^0, \dots, \pm x_n^0\} &= \emptyset \quad \text{and} \\ \text{dist}(0, \text{conv } S) &= (2a \log d/d)^{1/2}. \end{aligned}$$

This can be shown by a simple averaging argument.

The second way to see that (1) holds is to use Theorem 2.

It follows from there that  $C$  cannot contain the ball  $\rho B^d$  with  $\rho > (2a(\log d)/d)^{1/2}$ . So there is a point  $z$  on the boundary of  $C$  with  $\|z\| \leq (2a(\log d)/d)^{1/2}$ . Taking the supporting hyperplanes to  $C$  at  $z$  and at  $-z$  one concludes with (1).  $\square$

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