PORTFOLIO OPTIMIZATION IN INCOMPLETE MARKETS IN THE PRESENCE OF ASSET PRICE BUBBLES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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May 2014

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Cornell University 2014

In this work, the effect an asset price bubble has on optimal portfolio allocations is investigated. A price bubble is an economic phenomenon that occurs when the observed market price of an asset does not coincide with its value in an objective sense. Advancements have recently been made in the mathematical modeling of price bubbles and allow us to investigate the effect the presence of a bubble has on portfolio optimization. A duality viewpoint allows us to gain insight in our investigation and the tools from the Malliavin Calculus are used to characterize the investor's optimal holdings. A simulation framework is developed and the results are analyzed. From this investigation, it is concluded that the presence of asset price bubbles cause investors to reduce the number of shares they trade of the asset.

BIOGRAPHICAL SKETCH

Alexander Mull-Osborn was born in Salt Lake City, Utah and graduated from Park City High School in 2005. He completed his Bachelors of Science summa cum laude in Physics, Mathematics, and Theoretical Physics and Applied Mathematics from Loyola University Chicago in 2009. He also received a Master of Science in Mathematics from Loyola University Chicago in 2009. He joined the Center for Applied Mathematics at Cornell University to pursue his PhD and received his Masters of Science in Applied Mathematics in August of 2013.

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor Robert Jarrow for his guidance. I would also like to thank Adrian Lewis and Laurent Saloff-Coste for serving on my special committee. Most importantly, I would like to thank my family and friends, especially Tuyen and my parents. This could not have been acheived without your love and support.

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CHAPTER 1
INTRODUCTION

The portfolio optimization problem is a central problem in mathematical finance and has been well studied since Markowitz famously developed modern portfolio theory [25] in 1952. Markowitz's mean-variance criterion was further generalized by Merton [26] who introduced von Neumann and Morgenstern's utility functions [36] to represent the investor's risk preferences in the problem. To solve the problem in this expected utility maximization form, Merton made use of dynamic programming. Merton's method of solving the portfolio optimization problem via dynamic programming, as well as techniques for solving more general stochastic control problems, is described in Pham's textbook [31], as well as in Yong and Zhou [38]. As described in these references, the method of dynamic programming transforms the stochastic optimization into a partial differential equation (the Hamilton-Jacobi-Bellman equation) and then uses PDE techniques to find the maximum expected utility the investor can attain. As modern asset pricing theory has developed, the tools of this theory, namely Equivalent Local Martingale Measures, have since been integrated into the theory of portfolio optimization.

The Equivalent Local Martingale Measures have elucidated a duality viewpoint for the portfolio optimization problem, as seen for complete markets in Pliska [32], Karatzas et al. [20], and Cox and Huang [1]; then expanded to incomplete markets in Karatzas et al. [21] and He and Pearson [12]; and now in its current state of generality in Kramkov and Schachermayer [22]. The duality viewpoint has several major advantages over the dynamic programming approach. Duality better aligns the problem to other theories and results of mathematical finance, it provides theoretical results in a general semimartingale framework rather than the Markovian framework required by the dynamic programming technique, and it transforms the dynamic optimization problem into a static convex optimization problem. The duality theory will be described in detail in a subsequent chapter.

Asset price bubbles have long been observed in markets dating back to Holland's Tulip Mania (1634-1637) [8]. Recent advances in the mathematical modeling of this phenomenon ([18],[19],[16],[34]) prompt the question: in what way do asset price bubbles effect the trading strategies of an optimal portfolio? It is the objective of this work to provide answers to that question.

The economic phenomenon commonly refered to as a bubble is characterized by a sharp price increase followed by a price collapse. Intuitively, an asset price bubble occurs when the observed market price of an asset does not coincide with its value in an objective sense. The assets "value" is precisely defined in Jarrow et al., [18] and [19], and is known as the assets **fundamental value**. An asset price bubble is defined as the difference between the asset's price and its fundamental value. The difficulty in creating a mathematical model of bubbles is to construct the model in a way that preserves desirable, intuitive, and reasonable structure in the market, such as arbitrage-free prices and put-call parity. The works of Jarrow et al. [18] and [19], deal with all these issues and the model presented there will be described later. Additionlly, Jarrow, Kchia, and Protter have developed bubble-detection techniques in [16]. A comprehensive review of these advancements in modelling asset-price bubbles can be found in Protter [34]. The tools utilized in this investigation of the effect of bubbles on optimal portfolio holdings include: the current model of asset price bubbles, Malliavin Calculus, dynamic programming, duality theory, and Monte-Carlo Simulation. The necessary tools used from these theories will be reviewed where appropriate.

CHAPTER 2

PORTFOLIO OPTIMIZATION, DUALITY, AND STOCHASTIC VOLATILITY

This chapter describes the relevant background theory including asset price bubble theory, Malliavin Calculus, the portfolio optimization problem, and duality theory. In reviewing these theories, the financial model used for the remainder of the work is also set-up and described.

Assume there is a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ where $\mathbb{F}$ satisfies the usual conditions (consult Protter's textbook on Stochastic Differential Equations for an explanation of the usual conditions [33]). Further assume the space contains two standard, independent Brownian motions $B^{(S)} \in \mathbb{R}^m$ and $B^{(V)} \in \mathbb{R}^d$. The symbol B is used to denote the $(m+d)$-dimensional standard Brownian motion $[B^{(S)}\ B^{(V)}]^T$.

Now, define the two stochastic processes S and V in terms of the following stochastic differential equations.

$$\begin{aligned} dS_t &= \mu_t dt + \sigma(t, S_t, V_t) dB_t^{(S)} \\ dV_t &= \eta_t dt + \rho(t, S_t, V_t) dB_t^{(S)} + \gamma(t, S_t, V_t) dB_t^{(V)} \end{aligned} \tag{2.1}$$

S is the **price process** and V is the **volatility process**. This is the stochastic volatility model as discussed in Sin [35], Lions [23], Hobson [13], Hull and White [14], and elsewhere in the literature. The model describes an economy with n risky assets and 1 money market account[1]. The processes μ and η are adapted to $\mathbb{F}$ and the functions $\sigma(t, s, v)$, $\rho(t, s, v)$, and $\gamma(t, s, v)$ are Borel measurable mappings. Take $m \geq n$ and assume that $\sigma(t, s, v)$ is full rank for all (t, s, v).

[1]The Money Market Account is the locally riskless asset $R(t) = \exp(\int_0^t r_s ds)$. The MMA begins with one unit of account, $R(0) = 1$, and all interest is continuously reinvested into the account at (instantaneously riskless) spot interest rate r_t, which is positive and adapted to $\mathbb{F}$.

For convenience, S is assumed to be the price process with the money market account taken as numéraire[2]. That is to say that rather than quote prices relative to a currency, such as dollars, the prices of the risky assets are quoted relative to the value of the money market account and the money market account maintains the constant value 1. This is a standard technique and is expanded upon in Geman [9], where it is shown that the optimal portfolio allocations are invariant to this transformation.

It is easy to see that this is the most general model for stochastic volatility, at least in terms of the Brownian motions. Suppose the term $\tilde{\sigma}(t, S_t, V_t)dB_t^{(V)}$ is added to the dS_t differential in 2.1. Then the new system can be expressed in terms of correlated Brownian motions as follows. Let $H(t, S_t, V_t) = \begin{bmatrix} \sigma & \tilde{\sigma} \\ \rho & \gamma \end{bmatrix}$. Then,

$$\begin{bmatrix} dS_t \\ dV_t \end{bmatrix} = \begin{bmatrix} \mu_t \\ \eta_t \end{bmatrix} dt + H(t, S_t, V_t) \begin{bmatrix} dB_t^{(S)} \\ dB_t^{(V)} \end{bmatrix} \tag{2.2}$$

The coordinates of $\tilde{B} = H(t, S_t, V_t)B_t$ are correlated Brownian motions with correlation matrix $\rho_t = HH^T$. Now, $d\tilde{B}_t$ can be substituted for the right-most summand above to describe the dynamics of the system. This gives,

$$\begin{bmatrix} dS_t \\ dV_t \end{bmatrix} = \begin{bmatrix} \mu_t \\ \eta_t \end{bmatrix} dt + d\tilde{B}_t. \tag{2.3}$$

Since for each t, ρ_t is symmetric positive definite, ρ_t has a Cholesky decomposition. This means there exists a unique lower triangular matrix L_t, with strictly positive diagonal entries, such that $\rho_t = L_t L_t^T$. Then $Z_t = L_t^{-1}\tilde{B}_t$ is a standard

[2]Let $\tilde{S}$ denote the price process before the transformation, then S, the process with MMA as numéraire, is $S_t = \exp\left(-\int_0^t r_s ds\right) \cdot \tilde{S}_t$. The transformation is applied to all assets, including the MMA, and the transformed value of the MMA is $\exp\left(-\int_0^t r_s ds\right) \cdot R(t) = 1,\ \forall t$, making the effective interest rate of the transformed asset prices zero for all time.

Brownian motion and the price and volatility satisfy the following system of SDEs,

$$\begin{bmatrix} dS_t \\ dV_t \end{bmatrix} = \begin{bmatrix} \mu_t \\ \eta_t \end{bmatrix} dt + L_t dZ_t. \tag{2.4}$$

Therefore, without loss of generality, the model assummed in (2.1) can describe the new system and the extra flexibility provided by adding the $\tilde{\sigma}$ submatrix to H in equation 2.2 is redundant.

2.1 Martingale Measures and Asset-Price Bubbles

2.1.1 Martingale Measures

Let $\alpha \in L^1(S)$, the set of progressively measurable processes integrable with respect to S (see Protter [33] for a detailed explanation), denote a trading strategy for the investor. The i'th component of α at time t, $\alpha_t^{(i)}$, is the number of shares the investor holds of asset i at time t. Then, starting with an initial capital of $x \in \mathbb{R}_+\{x \in \mathbb{R} : x \geq 0\}$, define the investors **Wealth Process** by,

$$X_t^{\alpha,x} = x + \int_0^t \alpha_s dS_s \tag{2.5}$$

From this definition of the investor's wealth, it is seen that the condition $\alpha \in L^1(S)$ is necessary simply to ensure the wealth process is well-defined, in particular that the stochastic integral with respect to S is well-defined. The set of trading strategies available to the investor is further restricted to those that are **admissible**, denoted $\alpha \in \mathcal{A}(S)$. An admissible strategy's defining characteristic is that $\int \alpha dS$ is lower-bounded. This condition prevents doubling strategies (as described in Harrison and Pliska [11]) and has the economic interpretation that the amount of capital the investor is able to borrow is finite and bounded over her trading period $[0, T]$.

Suppose, for some $\alpha' \in \mathcal{A}(S)$ and some $x \in \mathbb{R}_+$, there is a nonnegative $\mathcal{F}_T$ measurable random variable X_T, that satisfies the inequality

$$x + \int_0^t \alpha'_s dS_s \geq X_T, \text{ a.s.} \tag{2.6}$$

Then, viewing X_T as a contingent claim at time T, this claim is said to be **superreplicable** at initial wealth x and the control α' that satisfies the inequality is called a superreplication portfolio strategy. The idea here is that, the random cash flow that X_T represents at time T, can be approximated by our investor if she begins with initial endowed wealth x and follows the trading strategy α'. At time T she can then discard any possible excess in wealth above the value X_T (from equation 2.6) and has superreplicated this contingent claim. Taking $x = 0$ and $\alpha = 0$ will always superreplicate a negative contingent claim. Therefore, the analysis will be restriced to X_T that are nonnegative valued. Henceforth, L^0_+ will be used to denote the nonnegative, $(\Omega, \mathcal{F}, P)$-measurable random variables.

In constructing the duality viewpoint, the following set will prove useful. Take some $x \in \mathbb{R}_+$, and denote the set of superreplicable contingent claims, the subset of nonnegative and $\mathcal{F}_T$-measurable random variables that can be superreplicated with wealth x, as

$$\mathcal{C}(x) = \left\{ X_T \in \mathcal{F}_T \cap L^0_+ : \exists \alpha \in \mathcal{A}(S),\ x + \int_0^T \alpha_t dS_t \geq X_T, \text{ a.s.} \right\} \tag{2.7}$$

When considering the risky assets, the following wealth process is important enough to deserve its own name, W,

$$W_t = X_t^{\vec{1},0} = 0 + \int_0^t \vec{1} \cdot dS_s \tag{2.8}$$

This wealth process corresponds to having no initial wealth and maintaining a constant portfolio consiting of one share of each of the risky assets.[3]

Using this wealth process, define on the measurable space $(\Omega, \mathcal{F}_T)$ the following set of probability measures,

$$\mathcal{M}_{\text{loc}}(W) = \{Q \sim P^4 : W \text{ is a } Q \text{ Local Martingale}\} \tag{2.9}$$

These are the **Equivalent Local Martingale Measures** (ELMM), the set of probability measures equivalent to P and under which the process W is a Q-Local Martingale, see Protter for definitions[33], and are the crucial tool used in Asset Pricing Theory as well as in the duality theory currently being described.

Assumption 1 $\mathcal{M}_{loc}(W) \neq \emptyset$.

By the first fundamental theorem of asset pricing, see Delbaen and Schachermayer [2], this assumption is equivalent to *No Free Lunch with Vanishing Risk (NFLVR)*, the standard no arbitrage condition in modern Mathematical Finance. As a corollary of the optional decomposition theorem, (see [31]), the following relation holds between the sets $\mathcal{M}_{\text{loc}}(W)$ and $\mathcal{C}(x)$,

$$\mathcal{C}(x) = \left\{ X_T \in \mathcal{F}_T \cap L^0_+ : \sup_{Q \in \mathcal{M}_{\text{loc}}(W)} \mathbb{E}^Q[X_T] \leq x \right\} \tag{2.10}$$

Comparing the two characterizations of $\mathcal{C}(x)$, 2.7 and 2.10, it is apparent that for a given initial wealth x, there is a duality between superreplication portfolio strategies, the α's, and equivalent local martingale measures, the Q's. This duality allows the Portfolio Optimization Problem to be transformed and simplified, and will be described later in this chapter. Before introducting the portfolio optimization problem, the definitions necessary to describe Asset Price Bubbles

[3]In this case $W \equiv S$ and making this distinction is seemingly unimportant. However, in general when assets are modeled as paying dividends and having a random lifetime this distinction becomes important, and is reflected in the standard notation which is adopted here.

will be reviewed. This theory also relies crucially on ELMM's, connecting the theory of bubbles with the duality viewpoint of portfolio optimization.

2.1.2 Bubbles

Markets are said to be **complete** if for any contingent claim $X_T \in \mathcal{C}(x)$ there exists an $\alpha \in \mathcal{A}(S)$ and an $x \in \mathbb{R}_+$, such that $X_T = x + \int_0^T \alpha_s dS_s$. This is stronger than the superreplication described by 2.6. Completeness says that all contingent claims can be exactly replicated using admissible trading strategies and all contingent claims are therefore redundant securities in the market. It has been shown in Jarrow et al. [19] that bubbles cannot exist in a complete economy. Therefore, the market considered must be incomplete.

In incomplete markets, by the second fundamental theorem of asset pricing, [33], $|\mathcal{M}_{\text{loc}}(S)| > 1$. In words, markets are incomplete if and only if there exists more than one ELMM. So, one must choose the valuation measure used to price assets from the set of ELMM's. There have been multiple schemes suggested for dealing with the issue of choosing "the measure" among the set of ELMM's, and the one adopted here will be to let the market choose the measure using option prices, as described by Jacod and Protter [15].

In subsequent sections, when the valuation measure Q is mentioned, it should be interpreted as the measure "chosen by the market."

Given the valuation measure Q, the **fundamental value** of S is defined to be the risky assets' discounted expected cash flows under Q. In this stochastic volatility setting the fundamental value[5] as defined in [18] is,

$$S_t^* = \mathbb{E}^Q[S_T \mid \mathcal{F}_t] \tag{2.11}$$

Given the asset price process S and the fundamental value process S^*, the **bubble process** β is defined as their difference

$$\beta_t = S_t - S_t^* \geq 0 \tag{2.12}$$

The nonnegativity of β holds because S is a nonnegative Q local martingale, and therefore a Q supermartingale; and, the inequality follows directly from the definition of a supermartingale.
As noted in Jarrow et al.'s paper [18], under the assumption of NFLVR, it is not necessarily true that the market price equals the fundamental value. Under NFLVR the market price always equals the arbitrage free price, but this need not equal the fundamental value. This difference occurs precisely when W is only a strict Q local martingale, rather than a true Q martingale. This is the very fact that motivated the use of the stochastic volatility model.

2.2 Malliavin Calculus and the Clark-Ocone formula

The theory of Malliavin Calculus has its beginnings in the 1978 work by Paul Malliavin [24]. The theory's use was limited until 1984 when Ocone [28] developed an explicit formula of the Clark representation formula, in terms of

[5][18] develops the fundamental value under much more general assumptions on the dynamics of the asset prices. In that case, the form of the fundamental value uses a random stopping time representing the life of the asset and the cumulative dividend process of the asset. Here the assets are assumed to pay no dividends and only expire at $t = T$ in the sense that the investment horizon has been reached.

the Malliavin Derivative. In 1991 Ocone and Karatzas [29] applied this Clark-Ocone formula in finance by using it to find optimal trading strategies. It is for this same application that the theory will be briefly reviewed here. For brevity, proofs will be omitted. Please consult the monograph by Nunnan and Øksendal [5] on Malliavin Calculus for further information.

2.2.1 Malliavin Derivative Definition and Properties

There are several ways to define the Malliavin Derivative, the one used here will be via the Wiener-Itô chaos expansion. This will be done both because of its expositional clarity and because it is the natural definition when extending from the continuous Brownian case to general Lévy-Jump processes.

To begin, consider a function $f : [0,T]^n \to \mathbb{R}$. Such a function is **symmetric** if given any $\sigma \in S_n$ (the symmetric group of order n),

$$f(t_{\sigma_1}, \ldots, t_{\sigma_n}) = f(t_1, \ldots, t_n) \tag{2.13}$$

Of all the symmetric functions, the subset considered here will be the symmetric functions that are square integrable Borel functions on $[0,T]^n$, denoted $\tilde{L}^2\left([0,T]^n\right) \subset L^2\left([0,T]^n\right)$.

Given a function $f \in \tilde{L}^2\left([0,T]^n\right)$, define the **n-fold iterated Itô Integral** $I_n(f)$,

$$I_n(f) = \int_{[0,T]^n} f(t_1, \ldots, t_n) dB(t_1) \ldots dB(t_n) \tag{2.14}$$

Where B is a standard Brownian motion. Thanks to results in Functional Analysis and Stochastic Analysis, these Iterated Itô Integrals can be used to characterize the space of $L^2(P)$ random variables, as the following result shows.

Wiener-Itô Chaos Expansion

Given an $L^2(P)$ random variable $F \in \mathcal{F}_T$, there exists a unique sequence $\{f_n\}_{n=0}^{\infty}$ of function in $\tilde{L}^2\left([0,T]^n\right)$ such that,

$$F = \sum_{n=0}^{\infty} I_n\left(f_n\right) \tag{2.15}$$

where convergence occurs in the $L^2(P)$ sense.

The proof of this expansion relies on the Itô representation theorem and Hilbert Space techniques and can be found in Øksendal's monograph [5].

Given $F \in L^2(P)$ and $\mathcal{F}_T$-measurable with chaos expansion $F = \sum_{n=0}^{\infty} I_n\left(f_n\right)$, $F \in \mathbb{D}_{1,2}$ if,

$$\|F\|_{\mathbb{D}_{1,2}}^2 = \sum_{n=0}^{\infty} nn! \left\|f_n\right\|_{L^2([0,T]^n)}^2 < \infty \tag{2.16}$$

And for $F \in \mathbb{D}_{1,2}$ the **Malliavin Derivative** D_tF of F at time t is defined to be the following,

$$D_tF = \sum_{n=0}^{\infty} nI_{n-1}\left(f_n(\cdot,t)\right) \tag{2.17}$$

where the iterated integrals are taken with respect to the first $(n-1)$ variables and the n-th variable is left as a parameter. Notice the resemblance of this definition to the standard derivative of the monomial x^n.

The following basic properties for the Malliavin Derivative hold.

Product Rule Given $F, G \in \mathbb{D}_{1,2}$,

$$D_t(FG) = FD_t(G) + GD_t(F) \tag{2.18}$$

Chain Rule Given $G \in \mathbb{D}_{1,2}$ and $g \in \mathcal{C}^1\left(\mathbb{R}\right)$,

$$D_tg\left(G\right) = g'\left(G\right)D_tG \tag{2.19}$$

In the simulation section, the following result will also prove useful. It is found in the appendix of [3]. Suppose a diffusion process Y has dynamics described by the following SDE,

$$dY_s = \mu(Y_s)ds + \sigma(Y_s)dB_s$$
$$Y_0 = y. \tag{2.20}$$

Then, as seen in section 2.2 of Nualart's textbook [27], if $\mu(\cdot)$ and $\sigma(\cdot)$ are measurable, Lipschitz, and $\mathcal{C}^1$ functions, the chain rule and Itô's theorem give the following dynamics for the processes $\{D_t Y_\cdot : t \in [0,T]\}$.

$$d\left(D_t Y_s\right) = \frac{\partial \mu(Y_s)}{\partial Y} D_t Y_s ds + \frac{\partial \sigma(Y_s)}{\partial Y} D_t Y_s dB_s,\ s \geq t$$
$$D_t Y_t = \sigma(Y_t) \tag{2.21}$$

It should be emphasized that this is a collection of processes, one for each $t \in [0,T]$ representing a Malliavin Derivative at time t. So, in equation 2.21 t is a fixed parameter and s is the time variable.

2.2.2 Generalized Clark-Ocone Formula

Let u_t be a progessively measurable process adapted to $\mathbb{F}$ that satisfies the Novikov condition,

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T u_s^2 ds\right)\right] < \infty \tag{2.22}$$

Then by the Girsanov Theorem (cf. Protter [33]), The measure Q defined by,

$$\frac{dQ}{dP} = Z(T) = \exp\left(-\int_0^T u_s dB_s - \frac{1}{2}\int_0^T u_s^2 ds\right) \tag{2.23}$$

is a probability measure equivalent to P. Furthermore, the Girsanov Theorem states that

$$\tilde{B}_t = \int_0^t u_s ds + B_t \tag{2.24}$$

is a Q-Brownian motion. In subsequent sections $\mathbb{E}^Q$ will denote expectation with respect to this new probability measure Q.

The process Z_t found in equation 2.23 is known as the Doléans-Dade or Stochastic Exponential [33] and is often denoted,

$$\mathcal{E}\left(\int_0^\cdot u_s dB_s\right)_t = \exp\left(-\int_0^t u_s dB_s - \frac{1}{2}\int_0^t u_s^2 ds\right). \tag{2.25}$$

The Clark-Ocone Formula under a change of measure

Let $F \in \mathbb{D}_{1,2}$ be $\mathbb{F}_T$-measurable. Then under mild conditions on u and $D_t F$,

$$F = \mathbb{E}^Q[F] + \int_0^T \mathbb{E}^Q \left[\left(D_t F - F \int_0^T D_t u_s d\tilde{B}_s \right) \mid \mathcal{F}_t \right] d\tilde{B}_t \tag{2.26}$$

Equation 2.26 is the crucial tool from Malliavin Calculus that will be used to analyze the effects of asset price bubbles on optimal portfolio holdings.

2.3 The Portfolio Optimization Problem: A Duality Viewpoint

2.3.1 The Primal Formulation

The portfolio optimization problem will now be set up and solved. Assume the investor's risk preferences are described by a **utility function** $U(x)$.[6] That is,

Definition: The function $U : \mathbb{R} \to \mathbb{R}$ is strictly increasing, strictly concave, and of class $\mathcal{C}^1$ (continuously differentiable).

Kramkov and Schachermayer have shown in [22] that it is both necessary and sufficient that U satisfies the condition of reasonable asymptotic elasticity in order for there to exist a solution to the portfolio optimization problem.

Reasonable Asymptotic Elasticity: The Asymptotic Elasticity of U satisfies,

$$AE(U) = \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1 \tag{2.27}$$

[6]The implications of this assumption are discussed in [6].

Investment occurs in the time interval $[0, T]$ for some $T \in [0, \infty)$. The investor's goal is to choose an **admissible trading policy**, $\alpha \in \mathcal{A}(S)$ that maximizes her expected utility. That is, starting from wealth $x \geq 0$ at time $t \in [0, T]$ she seeks to solve,

$$v(t, x) = \sup_{\alpha \in \mathcal{A}(S)} \mathbb{E}^P[U(x + \int_t^T \alpha dS)]. \tag{2.28}$$

This optimization problem has been well studied. It can be solved directly, under suitable conditions, by using the Hamilton-Jacobi-Bellman equations. As shown in, for example, Pham [31] and Yong and Zhou [38] the value function $v(t, x)$ can be obtained by solving the following PDE[7],

$$\begin{aligned}
&w_t + H(t, x, D_x w(t, x), D_x^2 w(t, x)) = 0, \ \forall (t, x) \in [0, T) \times \mathbb{R}^2, \text{ where} \\
&H = \sup_{a \in A} \left\{ \frac{1}{2} \text{trace} \left(\begin{bmatrix} a^{(1)}\sigma & 0 \\ a^{(2)}\rho & a^{(2)}\gamma \end{bmatrix} \begin{bmatrix} a^{(1)}\sigma & 0 \\ a^{(2)}\rho & a^{(2)}\gamma \end{bmatrix}^T D_x^2 w \right) + D_x w \cdot (a^{(1)}\mu, a^{(2)}\eta)^T \right\} \\
&w(T, x) = x
\end{aligned} \tag{2.29}$$

Where H is the **Hamiltonian** of the system. When casting the stochastic volatility dynamics 2.1 in the HJB framework of [38], V is considered to be one of the state variables. Because of this, the set $A \subset \mathbb{R}^2$ of controls is constrained to live on the x-axis. This is because volatility is considered a state variable only so that it can be fit into this framework, but it is not a risky asset and the investor has no ability to hold shares of volatility. No explicit constraints are placed on the $a^{(1)}$-axis, so the set of controls for the investor is $A = \mathbb{R} \times \{0\}$.

The classical solution of this PDE requires the restrictive assumption that $v \in \mathcal{C}^{1,2}$, continuously differentiable one in its t argument and twice in its x argument. The more modern method of viscosity solutions has weakened this assumption, but its proof still imposes conditions on the dynamics of S and V that

[7] Again, the exposition in the references allow much more general dynamics of the asset prices and, as such, appear in a simplified form in this work.

can be relaxed by using different techniques. As stated in, for example, theorem 5.2 of Yong and Zhou [38], a sufficient condition for the existence of a viscosity solution is that the processes σ, ρ, γ, μ, and η are functions of (S_t, V_t, t) and satisfy a uniform Lipschitz condition in (S, V). The duality formulation, however, allows μ and η to be progressively measurable processes in $L^1(S)$. Viscosity solutions, while useful and of theoretical interest, do not acheive the highest level of generality. Furthermore, as previously stated, the tools of duality theory are in better alignment with the tools used to describe asset price bubbles. Therefore, the remainder of the paper will take a duality viewpoint.

2.3.2 The Duality Formulation

The martingale duality framework has been developed in its current state of generality by Kramkov and Schachermayer [22]. As their work shows, duality theory can be used in this setting and the **Optimal Wealth Process** $\hat{X}$ can be easily recovered in terms of the investor's utility function and the Radon-Nikodym density of the market's chosen risk-neutral measure Q. Kramkov and Schachermayer, [22], show the Optimal Wealth Process satisfies the following,

$$\hat{X}_T^x = I(\mathcal{Y}(x) Z_T), \tag{2.30}$$

where $I = (U')^{-1}$ (the inverse of the derivative of the utility function), Z_T is the density of Q with respect to P at time T, (as in 2.23) and $\mathcal{Y}(x)$ is a normalizing function defined such that $\mathbb{E}^{(Q)}[\hat{X}_T^x] = x$.

The well developed duality theory gives a nice, clean characterization of the optimal wealth process. However, it is of both theoretical interest and practical importance to understand not only the wealth process obtained from optimal investment, but the trading strategy itself. This is only found implicitly in the full semimartingale generality of Kramkov and Schachermayer via the Galtchouk-Kunita-Watanabe decomposition. This theorem only ensures the existence of an optimal trading strategy, but no closed-form expression for how to create such a strategy.

To calculate the trading strategy more or less explicitely requires the theory of the Malliavin Calculus and specifically the generalized Clark-Ocone formula. The form of this theory used here is adapted from Øksendal and Nullan's monograph on the subject [5]. Given valuation measure Q with Radon Nikodym density $Z_t = \mathcal{E}(-u \cdot (B^{(S)}, B^{(V)}))_t$, the Girsanov Theorem says $\tilde{B}_t := B_t + \int_0^t u(s)ds$ is a Q Brownian motion.

By duality,

$$X_T = I(\mathcal{Y}(x)Z_T) = x + \int_0^T \alpha_s dS_s \tag{2.31}$$

The dynamics of the wealth process X with respect to $\tilde{B}$ the Q Brownian motion will be needed to apply the Clark-Ocone formula. The definition of $\tilde{B}$ gives,

$$\begin{aligned} dX_t &= \alpha_t \cdot dS_t = \alpha_t \cdot \left(\mu_t dt + \sigma(t, S_t, V_t) dB_t^{(S)}\right) = \\ &= \alpha_t \cdot \left(\mu_t dt + \sigma(t, S_t, V_t)(d\tilde{B}_t - u(t)dt)\right) = \\ &= \alpha_t \cdot \sigma(t, S_t, V_t) d\tilde{B}_t + \alpha_t \cdot \left(\mu_t - \sigma(t, S_t, V_t)u(t)\right)dt \end{aligned}$$

And, since S is a Q local martingale, it follows that X is a Q local martingale and so the drift term above is zero.

This requires, by the optional decomposition theorem, that

$$u_t = \sigma(t, S_t, V_t)^T (\sigma(t, S_t, V_t)\sigma(t, S_t, V_t)^T)^{-1} \mu_t. \tag{2.32}$$

Define a new process π_t such that $dX_t = \pi_t d\tilde{B}_t$. Then, from the above equation it is easily seen that the optimal trading strategy α is related to the new process π, by,

$$\alpha_t = (\sigma(t, S_t, V_t)\sigma(t, S_t, V_t)^T)^{-1} \sigma(t, S_t, V_t)\pi_t \tag{2.33}$$

The Generalized Clark-Ocone formula 2.26 states that the process π must be the following 2.26,

$$\pi_t = \mathbb{E}^Q\big[D_t X_T - X_T \int_t^T D_t u_s \cdot d\tilde{B}_s \mid \mathcal{F}_t\big] \tag{2.34}$$

Where, D_t is the Malliavin derivative operator.

This formula can be simplified by explicit calculation of $D_t X_T$ and by using Bayes' Theorem to write π as a conditional expectation in terms of the market measure P.

To simplify $D_t X_T$, recall that $X_T = I(\mathcal{Y}(x) Z_T)$ and make use of the Chain Rule for Malliavin derivative $D_t g(Y) = g'(Y) D_t Y$ as given in 2.19.

$$D_t X_T = D_t I(\mathcal{Y}(x) Z_T) = I'(\mathcal{Y}(x) Z_T) D_t\big(\mathcal{Y}(x) Z_T\big) \tag{2.35}$$

$$= I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) D_t Z_T \tag{2.36}$$

The Malliavin Derivative $D_t Z_T$ of the density of measure Q with respect to P is given in corollary 3.19 of Øksendal and yields,

$$D_t X_T = -I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) Z_T\big\{u(t) + \int_t^T D_t u_s d\tilde{B}_s\big\} \tag{2.37}$$

Now, substituting this into equation (2.34) gives,

$$\begin{aligned}\pi_t &= \mathbb{E}^Q\big[-I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) Z_T\big\{u(t) + \int_t^T D_t u_s \cdot d\tilde{B}_s\big\} - X_T \int_t^T D_t u_s \cdot d\tilde{B}_s \mid \mathcal{F}_t\big] = \\ &= -\mathbb{E}^Q[I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) Z_T u(t) + \big\{I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) Z_T + X_T\big\} \int_t^T D_t u_s \cdot d\tilde{B}_s \mid \mathcal{F}_t]\end{aligned} \tag{2.38}$$

Remark: Considering the form of the above equation, it is clear that the second term dissappears precisely when the coefficient $I'(\mathcal{Y}(x) Z_T)\mathcal{Y}(x) Z_T + X_T = 0$. This occurs when $X_T = I(\mathcal{Y}(x) Z_T) = \frac{1}{\mathcal{Y}(x) Z_T}$ i.e. precisely when log-utility is assumed. This expresses that log-utility is particularly simple, in part, because the $D_t u_s$ term has no influence on the optimal trading strategy.

CHAPTER 3

PRELIMINARY RESULTS IN A CONSTRUCTED EXAMPLE

In this chapter, a simplified and specialized model is constructed, differing slightly from the stochastic volatility model introduced earlier, to build intuition for the more general setting which will appear in the subsequent chapter.

The market is modeled following the example (ex. 5.6) in Jarrow et al. [18] and using two independent one dimensional P-Brownian motions denoted $B^{(1)}$ and $B^{(2)}$.

Fix a constant $k > 1$ and define the stopping time $\tau = \inf\{t : \mathcal{E}(B^{(2)})_t = k\}$, recalling the notation in 2.25. This stopping time is crucial in creating a bubble in this modified Black-Scholes economy.

Define the processes E and S as follows,

$$E_t = \mathcal{E}(B^{(2)})_{t\wedge\tau} \qquad\qquad S_t = \mathcal{E}(B^{(1)})_{t\wedge\tau} \tag{3.1}$$

The process S represents the market price of a stock with zero dividends and terminal payoff at time τ of $X_\tau = S_\tau$. Therefore, the wealth process W associated with holding one unit of the risky asset is given by $W_t = S_t$. Notice that on $[t < \tau]$, S is identical to the standard geometric Brownian motion in the standard Black-Scholes model.

The stopping time is what makes this model differ from the previously described stochastic volatility model (2.1). In particular, the fact that the price is fixed after τ means that $\sigma(\cdot, \cdot, k)$ is not full rank on the (stochastic) time interval $[\tau, T]$.

Define the ELMM Q on the probability space $(\Omega, \mathcal{F}, P)$ in terms of the following Radon-Nikodym derivatives,

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = E_t. \tag{3.2}$$

As shown in Jarrow [18] lemma 2.6, under the market measure P, S is a non-uniformly integrable martingale, and ES is a uniformly integrable martingale. Therefore, $P \in \mathcal{M}_{NUI}(W)$, the set of ELMM's that make the wealth process W a non-UI (local) martingale, and $Q \in \mathcal{M}_{UI}(W)$, the set of ELMM's under which W is UI.

So, there can be a price bubble under P corresponding to its fundamental price process, S^*,

$$\begin{aligned} S_t^* &= \mathbb{E}_P[X_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_t] \\ &= \mathbb{E}_P[S_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_t] \\ &= \mathbb{E}_P[S_\tau 1_{\{\tau\le t\}} + 1_{\{\tau>t\}}\mathbb{E}_P[S_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_\tau] \mid \mathcal{F}_t] \\ &= S_\tau 1_{\{\tau\le t\}} + 1_{\{\tau>t\}} \int_t^\infty \mathbb{E}_P[S_u \mid \mathcal{F}_t] P[\tau \in du \mid \mathcal{F}_t] \\ &= S_\tau 1_{\{\tau\le t\}} + 1_{\{\tau>t\}} \int_t^\infty S_t P[\tau \in du \mid \mathcal{F}_t] \\ &= S_{\tau\wedge t}\mathbb{E}_P[1_{\{\tau<\infty\}} \mid \mathcal{F}_t] \\ &= S_t\mathbb{E}_P[1_{\{\tau<\infty\}} \mid \mathcal{F}_t]. \end{aligned} \tag{3.3}$$

Where the equality of lines 4 and 5 follow from the independence of $B^{(1)}$ and $B^{(2)}$. Given S and the fundamental price S^* the bubble process, β, is

$$\beta_t := S_t - S_t^* = S_t(1 - P[\{\tau < \infty\} \mid \mathcal{F}_t]).$$

The expression for β can be simplified and written in terms of the process E. Notice that E is bounded, by k, and hence a uniformly integrable P martingale, and that it is also nonnegative. By the Burkholder-Davis-Gundy inequality, there exists a constant $c > 0$ such that,

$$c \cdot \mathbb{E}_P[(\langle E \rangle_t)^{1/2}] \leq \mathbb{E}_P[\bigvee_{0 \leq s \leq t} E_s] \leq k. \text{ So,}$$

$$\mathbb{E}_P[(\langle E \rangle_t)^{1/2}] = \mathbb{E}_P[(E_t^2 t)^{1/2}] \leq \frac{k}{c}. \text{ Hence,}$$

$$\mathbb{E}_P[E_t] \leq \frac{k}{c\sqrt{t}}.$$

Taking a limit as $t \to \infty$, $\mathbb{E}_P[E_\infty] = 0$. And since E_∞ is nonnegative, it follows that $E_\infty = 0$ a.s.

Combining this with an application of Doob's optional sampling theorem to E gives,

$$\begin{aligned} E_t =& \mathbb{E}_P[E_\tau \mid \mathcal{F}_t] = \mathbb{E}_P[E_\tau 1_{\{\tau = \infty\}} \mid \mathcal{F}_t] + \mathbb{E}_P[E_\tau 1_{\{\tau < \infty\}} \mid \mathcal{F}_t] = \\ =& 0 \cdot P[\{\tau = \infty\} \mid \mathcal{F}_t] + k \cdot P[\{\tau < \infty\} \mid \mathcal{F}_t]. \end{aligned} \tag{3.4}$$

Hence, $P[\{\tau < \infty\} | \mathcal{F}_t] = \frac{1}{k} E_t$. And therefore,

$$\beta_t = S_t(1 - \frac{1}{k} E_t) \tag{3.5}$$

3.1 Optimal Portfolio

The Lemma in this section will show that under this model, and actually much more generally, it is optimal to not invest in the risky asset.

Lemma 1

Given a concave, nondecreasing utility function and a probability measure P such that S is a P local martingale, $\alpha \equiv 0$ always solves the portfolio optimization problem.

Proof:

$x + \int_0^T \alpha_s dS_s$ is an integrable random variable and since S is a P local martingale and $\int \alpha dS$ is a lower bounded P local martingale, the process $x + \int_0^t \alpha_s dS_s$ is a P supermartingale and $x + \int_0^T \alpha_s dS_s$ has expectation $\mathbb{E}_P[x + \int_0^T \alpha_s dS_s] \leq x$. Since U is concave, and $x + \int_0^T \alpha_s dS_s$ is integrable, Jensen's inequality may be applied and gives,

$\mathbb{E}_P[U(x + \int_0^T \alpha_s dS_s)] \leq U\big(\mathbb{E}_P[x + \int_0^T \alpha_s dS_s]\big) \leq U(x)$.

Where the second inequality holds because U is nondecreasing.

This inequality must hold for all admissible α. Therefore, $\alpha \equiv 0$ solves the portfolio optimization problem and achieves the maximum utility $U(x)$. $\square$

This result is obvious in Markowitz's mean-variance criterion. Indeed, if S is a nonnegative P local martingale, it is a P supermartingale. Therefore, the expected return at any time t of $\mathbb{E}[\int_0^t dS_s] \leq \mathbb{E}[0] = 0$. So, the assets expected return is less than or equal to 0 and it is optimal to minimize variance over portfolio's with zero return and simply invest in the money market account.

3.2 Modifications to achieve a nontrivial result

The preceeding lemma shows that a necessary condition for investment is that the dynamics of S with respect to P have a non-zero drift term. A drift term is now introduced into the model.

Let μ and σ be strictly positive constants, k and τ be defined as before, and modify S to be the following process,

$$\begin{aligned} S_t &= \mathcal{E}(\mu + \sigma B^{(1)})_{t\wedge\tau} = \exp\big(\int_0^t \sigma dB^{(1)}_{s\wedge\tau} + \int_0^t \mu d(s\wedge\tau) - \frac{1}{2}\int_0^t |\sigma|^2 d(s\wedge\tau)\big) = \\ &= \exp\big(\sigma B^{(1)}_{t\wedge\tau} + (\mu - \frac{1}{2}\sigma^2)(t\wedge\tau)\big). \\ E_t &= \mathcal{E}(B^{(2)})_{t\wedge\tau} \end{aligned} \tag{3.6}$$

Now, define the ELMM's R and Q on the probability space in terms of the following Radon-Nikodym derivatives,

$$
\begin{aligned}
\frac{dR}{dP}\Big|_{\mathcal{F}_t} &= \mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t \\
\frac{dQ}{dR}\Big|_{\mathcal{F}_t} &= E_t
\end{aligned}
\tag{3.7}
$$

Then, on the time interval when the asset exists, $[0,\tau]$, $S\frac{dR}{dP} = \mathcal{E}\big((\sigma - \frac{\mu}{\sigma})B^{(1)}\big)$ is a geometric Brownian motion. Therefore, up to a constant, the uniform integrability results coincide with the previous model and by Jarrow [18] lemma 2.6, $R \in \mathcal{M}_{NUI}(W)$ and $Q \in \mathcal{M}_{UI}(W)$.

There is a price bubble under R corresponding to its fundamental price process. Denote the P-GBM $\bar{S}_t = \mathcal{E}\big((\sigma - \frac{\mu}{\sigma})B^{(1)}\big)_t$, then the fundamental price process for S under R is,

$$
\begin{aligned}
S_t^* &= \mathbb{E}_R[X_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_t] \\
&= \mathbb{E}_P[S_\tau \frac{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_\tau}{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t} 1_{\{\tau<\infty\}} \mid \mathcal{F}_t] \\
&= \frac{1}{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t}\mathbb{E}_P[\bar{S}_\tau 1_{\{\tau\leq t\}} + 1_{\{\tau>t\}}\mathbb{E}_P[\bar{S}_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_\tau] \mid \mathcal{F}_t] \\
&= \frac{1}{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t}\big(\bar{S}_\tau 1_{\{\tau\leq t\}} + 1_{\{\tau>t\}}\int_t^\infty \mathbb{E}_P[\bar{S}_u \mid \mathcal{F}_t]P[\tau \in du \mid \mathcal{F}_t]\big) \\
&= \frac{1}{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t}\big(\bar{S}_\tau 1_{\{\tau\leq t\}} + 1_{\{\tau>t\}}\int_t^\infty \bar{S}_t P[\tau \in du \mid \mathcal{F}_t]\big) \\
&= \frac{1}{\mathcal{E}(-\frac{\mu}{\sigma}B^{(1)})_t}\bar{S}_{\tau\wedge t}\mathbb{E}_P[1_{\{\tau<\infty\}} \mid \mathcal{F}_t]
\end{aligned}
\tag{3.8}
$$

and on $[0,\tau]$, this gives $S_t^* = S_t P[\{\tau < \infty\}|\mathcal{F}_t]$.

Given S and the fundamental price S^* the bubble process, β, is the following,

$$
\beta_t = S_t - S_t^* = S_t(1 - P[\{\tau < \infty\} \mid \mathcal{F}_t]).
$$

This can be written in terms of the process E, as noted in the previous case, by noticing,

$$E_t = \mathbb{E}_P[E_\tau \mid \mathcal{F}_t] = \mathbb{E}_P[E_\tau 1_{\{\tau=\infty\}} \mid \mathcal{F}_t] + \mathbb{E}_P[E_\tau 1_{\{\tau<\infty\}} \mid \mathcal{F}_t] =$$
$$= 0 \cdot P[\{\tau = \infty\} \mid \mathcal{F}_t] + k \cdot P[\{\tau < \infty\} \mid \mathcal{F}_t]. \tag{3.9}$$

Hence,

$$P[\{\tau < \infty\} \mid \mathcal{F}_t] = \frac{1}{k} E_t.$$

And therefore,

$$\beta_t = S_t(1 - \frac{E_t}{k}) \tag{3.10}$$

3.2.1 Portfolio Selection Problem

Now that the existence of a bubble under ELMM R has been established, the investor's problem will be considered. This will be analyzed by an artificial market completion. It will be assumed that the set of ELMM's $\mathcal{M}_{\text{loc}}(W)$ is a singleton (equivalent to the market being complete by the second fundamental theorem of asset pricing). First the case $\mathcal{M}_{\text{loc}}(W) = \{Q\}$ is considered, followed by assuming that $\mathcal{M}_{\text{loc}}(W) = \{R\}$.

The investor is assumed to possess the utility function $U(x) = \log(x)$, describing her preferences, and is investing over the interval $[0, T]$ for some $T \in [0, \infty)$. Recall, her goal is to choose an admissible trading policy $\alpha \in \mathcal{A}(S) := \{\alpha \in L^1(S) : \int \alpha dS$ is lower-bounded$\}$ that maximizes her expected utility. Starting at $t = 0$ with initial wealth $x \geq 0$ she seeks to solve,

$$v(x) = \sup_{\alpha \in \mathcal{A}(S)} \mathbb{E}_P[U(x + \int_0^T \alpha_t dS_t)]. \tag{3.11}$$

Solution under ELMM Q

Notice that the problem is identical to the following problem,

$$\tilde{v}(x) = \sup_{\alpha \in \mathcal{A}(\tilde{S})} \mathbb{E}_P[U(x + \int_0^{T\wedge\tau} \alpha_t 1_{\{T\leq\tau\}} d\tilde{S}_t]$$

where $\tilde{S} := \mathcal{E}(\mu + \sigma B^{(1)})$ is the unstopped GBM.

The problem will be easier to analyze by first defining a new set of controls $\tilde{\alpha}$ in terms of the original controls in $\mathcal{A}(\tilde{S})$. Let $\tilde{\alpha}_t = P[\{T \leq \tau\}]\tilde{S}_t\alpha_t/X_t$.

Then, for a given control process $\tilde{\alpha}$, the investor's wealth process satisfies,

$$dX_t = \frac{X_t}{P[\{T\leq\tau\}]\tilde{S}_t}\tilde{\alpha}_t d\tilde{S}_t = X_t\frac{\tilde{\alpha}_t}{P[\{T\leq\tau\}]}(\mu dt + \sigma dB_t^{(1)})$$

Furthermore, the infinitesimal generator corresponding to this process is given by,

$$Af(x) = P[T\leq\tau]\{a\mu x\frac{df}{dx} + \frac{1}{2}a^2\sigma^2x^2\frac{d^2f}{dx^2}\}.$$

The localized version of Dynkin's formula (see theorem 1.24 in Øksendal [30]) may be applied to this problem. For a given control $\tilde{\alpha}$ it gives,

$$\mathbb{E}_P[U(X_{(T\wedge\tau)})] = U(x) + \mathbb{E}_P[\int_0^{T\wedge\tau} (\tilde{\alpha}_t\mu - \tilde{\alpha}_t^2\sigma^2/2)dt] \quad (3.12)$$

The integral on the right is maximized by making the integrand as large as possible and that value is $\tilde{\alpha}_t^* \equiv \mu/\sigma^2$ for all t. This corresponds to $\alpha^* \in \mathcal{A}(S)$ of,

$$\alpha_t^* = \frac{\mu}{\sigma^2}\frac{X_t^*}{S_tP[T\leq\tau]}1_{\{t\leq\tau\}}. \quad (3.13)$$

To show that this is the solution under Q, it needs to be shown that $X_T^* = x + \int_0^T \alpha^* dS$ is in $\mathcal{C}^{(Q)}(x) := \{X_T \in (\Omega, \mathcal{F}_T, P) : \mathbb{E}_Q[X_T] \leq x\}$. By the optional decomposition theorem (see Pham [31]), this is true if and only if X^* is a Q supermartingale. X^* is a Q supermartingale since S is a Q local martingale and hence X^* is a non-negative Q local martingale and therefore a Q supermartingale.

Solution under ELMM R

Again, the problem can be solved via Dynkin's formula as above. Now, the only delicacy is to notice the following lemma.

Lemma 2

For any $X_T \in \mathcal{C}^{(R)}(x)$ the random variable $\mathbb{E}_P[X_T \mid \mathcal{F}^{B^{(1)}}]$ is in $\mathcal{C}^{(R)}(x)$ and its expected utility is an upper bound for the expected utility of X_T.

Proof

Notice that $\frac{dR}{dP}$ is $\mathcal{F}^{B^{(1)}}$-measurable. Then, given $X \in \mathcal{C}^{(R)}(x)$,

$$\mathbb{E}_R[\mathbb{E}_P[X_T \mid \mathcal{F}^{B^{(1)}}]] = \mathbb{E}_P[\mathbb{E}_P[X_t \frac{dR}{dP} \mid \mathcal{F}^{B^{(1)}}]] = \mathbb{E}_P[X_T \frac{dR}{dP}] = \mathbb{E}_R[X_T] \leq x$$

So, the random variable is in $\mathcal{C}^{(R)}(x)$ and it remains to be shown that its expected utility dominates that of the original X.

$$\mathbb{E}_P[U(X_T)] = \mathbb{E}_P[\mathbb{E}_P[U(X_T) \mid \sigma(B^{(1)})]] \leq \mathbb{E}_P[U(\mathbb{E}_P[X_T \mid \sigma(B^{(1)})])]$$

Where the last inequality holds by Jensen's inequality since the utility function is concave. □

It follows from this lemma that the problem can be restricted to one over $\mathcal{F}^{B^{(1)}}$-measureable elements in $\mathcal{C}^{(R)}(x)$. So, the result from Q needs to be projected onto this sigma-field. The result is that under R,
$\alpha_t^{*,(R)} = \frac{\mu}{\sigma^2}\frac{X_t}{S_t}1_{\{t \leq \tau\}}$. And, therefore, $\alpha_t^{*,(R)} < \alpha_t^{*,(Q)}$.
What is found here is that the effect of the bubble, which occurs under R and not Q, is to invest proportionately less in the risky asset. The proportion is $\frac{1}{P[T \leq \tau]}$. Intuitively, what is happening here is that Q's relation to the process E gives the investor information about $B^{(2)}$, which is related to τ. This information makes the expiration of S at time τ less "surprising" so W is a uniformly integrable Q martingale and S does not have a bubble under Q. However, this information makes S appear more risky, so less of it is held in the optimal portfolio. In contrast, under R the investor has no knowledge of $B^{(2)}$, as demonstrated in Lemma 2. Under R, the investor behaves as if she is unaware of τ's potential to "kill" the asset before the investment horizon T is reached. This is reflected also in W being a non-UI R martingale.

CHAPTER 4

SIMULATING THE OPTIMAL TRADING STRATEGY

With the insights gained from last chapter's special example using logarithmic utility, this chapter now investigates the effect of bubbles on optimal portfolios for general utility funtions. When an investor's preferences are described by a general utility function the $D_t u_s$ term in equation (2.38) might affect the optimal trading strategy. This means the optimal trading strategy has a very complex dependency on asset price bubbles. Therefore, theoretical analysis of the trading strategy's dependence on a price bubble is intractable in general, and a simulation analysis is required to investigate this dependency. The simulation model used in this work is described below, followed by the simulation results and analysis in the subsequent chapter.

4.1 Simulation Model

For simulation purposes, a simplified version of the stochastic volatility model 2.1 will be assumed for the dynamics of the asset price process S and the volatility process V. Specifically, the following dynamics are assumed,

$$dS_t = \text{Diag}(S_t)[\mu dt + \text{Diag}(V_t)^{\beta^{(S)}} \sigma dB_t^{(S)}] \tag{4.1}$$

$$dV_t = \text{Diag}(V_t)[\eta dt + \text{Diag}(V_t)^{\beta^{(V)}} \rho dB_t^{(S)} + \text{Diag}(V_t)^{\beta^{(V)}} \gamma dB_t^{(V)}] \tag{4.2}$$

Where for vectors v and β in $\mathbb{R}^n$, the $n \times n$ matrix $Diag(v)^\beta$ is,

$$\text{Diag}(v)^\beta = \begin{pmatrix} v_1^{\beta_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & v_n^{\beta_n} \end{pmatrix}$$

Also, μ, η, σ, ρ, γ, $\beta^{(S)}$, and $\beta^{(V)}$ are constant valued and of appropriate dimension.

To simplify notation, let $A_t = \text{Diag}(S_t)\text{Diag}(V_t)^{\beta^{(S)}}\sigma$.

Then under the simulation model dynamics (4.1) equations (2.32) and (2.33) give,

$$u_t = A_t^T (A_t A_t^T)^{-1} (\text{Diag}(S_t)\mu)$$
$$\alpha_t = (A_t A_t^T)^{-1} A_t \pi_t \tag{4.3}$$

Malliavin Derivatives

Let $G_t = \text{Diag}(V_t)^{\beta^{(V)}}[\rho \ \gamma]$. Then, using the result in 2.21 from the Malliavin Calculus review, the following dynamics hold for $D_t V$. the Malliavin Derivative at time t of the volatility process.

$$d\left(D_t V_s\right) = [\frac{\partial(\text{Diag}(V_s)\eta)}{\partial V} ds + \sum_{j=1}^{m+d} \frac{\partial G_s^{(j)}}{\partial V} dB_s^{(j)}] D_t V_s, \ s \geq t$$
$$D_t V_t = G_t. \tag{4.4}$$

Where for a vector-valued function f depending on V,

$$\frac{\partial f(V)}{\partial V} = \begin{pmatrix} \partial_1 f^{(1)}(V) & \cdots & \partial_m f^{(1)}(V) \\ \vdots & \ddots & \vdots \\ \partial_1 f^{(m)}(V) & \cdots & \partial_m f^{(m)}(V) \end{pmatrix}.$$

Using the chain rule for the Malliavin derivative, $D_t u_s$, the Malliavin derivative at time t of u, can be found in terms of the process $D_t V_s$.

$$D_t u_s = \sigma^T \left(\sigma\sigma^T\right)^{-1} \left[-(\beta^{(S)} \cdot \mu^T)\text{Diag}(V_s)^{-\beta^{(S)}-1}\right] D_t V_s, \ \ s \geq t \tag{4.5}$$

Now that all the dynamics of the processes needed for the simulation have been derived, the method of simulating them will be addressed. Computers are inherently discrete in nature, so the continuous dynamics derived here need to be discretized and the scheme used to do this will now be described.

4.2 Discretization Scheme

To simulate this system on a computer requires making a discrete approximation of the processes. To effect this approximation, the so-called Euler Scheme is used, as described in Glasserman's textbook [10].

In an effort to enforce the nonnegativity of the S, V, and Z processes in the approximation, the logarithm of these processes are approximated, using Itô's formula, and then transformed via exponentiation. Discretizing the processes logarithms and then exponentiating is a standard method used to enforce nonnegativity.

Given an investment horizon T and a discretization size N, denote by $\Delta = \frac{T}{N}$ the time step of the standard partition $\{0, \Delta, 2\Delta, ..., N\Delta\}$ of $[0, T]$. This partition is the set of time points at which the processes will be approximated.

The discrete approximation for $\log(S)$ is the following,

$$\log(S)_{i\Delta} = \log(S)_{(i-1)\Delta} + \mu\Delta + \big((V_{(i-1)\Delta})^{\beta^{(S)}}\big)\sigma(B^{(S)}_{i\Delta} - B^{(S)}_{(i-1)\Delta}) + \\ -\frac{\Delta}{2}\big((V_{i-1)\Delta})^{2\beta^{(S)}}\big)\text{diag}(\sigma\sigma^T). \tag{4.6}$$

Where, $\log(S)_0 = \log(s)$. Similarly, the following equations are used to discretize the other processes,

$$\log{(V)}_{i\Delta} = \log{(V)}_{(i-1)\Delta} + \eta\Delta + \big((V_{(i-1)\Delta})^{\beta^{(V)}}\big)[\rho\,\gamma](B_{i\Delta} - B_{(i-1)\Delta}) +$$
$$-\frac{\Delta}{2}\big((V_{(i-1)\Delta})^{2\beta^{(V)}}\big)\mathrm{diag}([\rho\,\gamma][\rho\,\gamma]^T)$$
$$\log(Z)_{i\Delta} = \log(Z)_{(i-1)\Delta} - u_{i\Delta}^T(B_{i\Delta}^{(S)} - B_{(i-1)\Delta}^{(S)}) - \frac{\Delta}{2}u_{i\Delta}^T u_{i\Delta}$$
$$D_{j\Delta}V_{i\Delta} = D_{j\Delta}V_{(i-1)\Delta} + \big\{\mathrm{Diag}(\beta^{(V)} + e)\mathrm{Diag}(V_{j\Delta}^{\beta^{(V)}})[\rho\,\gamma](B_{i\Delta} - B_{(i-1)\Delta}) +$$
$$+\mathrm{Diag}(\eta)\Delta\big\}D_{j\Delta}V_{(i-1)\Delta} \tag{4.7}$$

The Clark-Ocone formula 2.26 gives the optimal trading strategy in terms of a conditional expectation, as seen above in equation 2.34. Therefore, a numerical approximation of conditional expectation must be made.

This is performed by simulating the argument of the conditional expectation K-times, denoted here by $\Xi_t^{(i)}$ and then taking an average of the K simulations to get the approximation $\hat{\pi}_t$. Since there is conditioning on $\mathcal{F}_t$ in 2.34, care is taken to treat $\mathcal{F}_t$-measureable random variables as constants, so that their value is the same in each of the K-terms when perfoming the simulations.

$$\Xi_t^{(i)} = -I'(\mathcal{Y}(x)Z_T)\mathcal{Y}(x)Z_T\big\{u(t) + \int_t^T D_t u_s \cdot d\tilde{B}_s\big\} - X_T\int_t^T D_t u_s \cdot d\tilde{B}_s \tag{4.8}$$
$$\hat{\pi}_t = \frac{1}{K}\sum_1^K \Xi_t^{(i)} \tag{4.9}$$

The appendix A, contains the actual MATLAB code that was used in the simulations analyzed in the next chapter. This code implements the discretization scheme that has just been described and is included for the sake of completeness.

CHAPTER 5

SIMULATION RESULTS

This chapter will review the output of the simulation model for various parameter values using the model described in 4.1. Economies with a single risky asset and multiple risky assets will be considered. Assets with and without price bubbles will be compared, and the influence of price correlation will be examined. For simplicity, in all simulations analyzed, the parameters in 4.1 satisfied the following additional constraints,

$$\begin{aligned} &\mu = \eta \qquad \sigma = \rho \qquad \beta^{(V)} = \beta^{(S)} \\ &d = 0 \qquad v = s \qquad\quad x = 1 \end{aligned} \tag{5.1}$$

Recall that d is the dimension of Brownian motion $B^{(V)}$, 2.1. With these restrictions on the parameters, the model now gives the price process as a collection of correlated CEV processes. The CEV process has been well studied in, for example, [37], [17],[7] and elsewhere. As shown in Jarrow et al. [17], the vector of parameters $\beta^{(S)}$, will control the existence and "size" of the bubble.

Furthermore, in all simulations discussed, the investor is assumed to have a **Constant Relative Risk Aversion** (CRRA) utility function. That is, either the investor will have logarithmic utility, or a **power utility** $U(x) = \frac{x^p}{p},\ p < 1,\ p \neq 0.$ To simplify notation, CRRA utility functions will be identified via the exponent p; and, $p = 0$ will signify logarithmic utility.

In all the simulations, the bubbles are simulated in the following way: The optimal holdings under no bubble are simulated using the same underlying source of randomness as the optimal holdings under a price bubble. That is, the same realization of a standard Brownian motion is used to calculate both trading processes. The price process with no bubble S^{NB} is assumed to follow the same stochastic volatility model as the price with a bubble, but with parameter vector β set to zero. This method is slightly different than the asset pricing theory method of choosing a different Equivalent Local Martingale Measure under which S equals its fundamental value. However, as has been noted in [17], there is no systematic method of choosing the correct ELMM in the case of an incomplete market. Therefore, this heuristic method is reasonable.

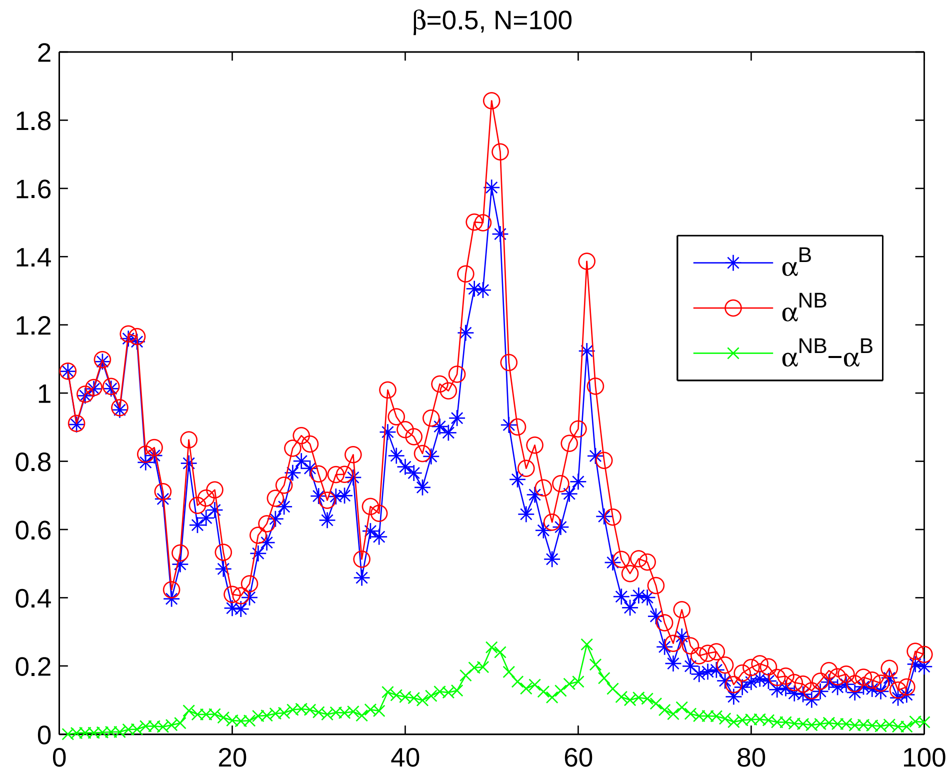
5.1 Single Risky Asset, n=1

5.1.1 One asset Logarithmic Utility, p=0

First, the case of optimally investing in a single asset with risk aversion described by logarithmic utility is analyzed. The optimal trading strategy appears in figure 5.1.

Recall that S is the price of the risky asset with the money market account taken as numéraire. In figure 5.1 the vertical axis represents the optimal number of shares to hold, and the horizontal axis is time. α^{B}, the blue line with asterisk markers, is the optimal holdings of the risky asset with a simulated price bubble. α^{NB}, the red line with the circle markers, is the optimal holdings of the risky asset without a price bubble. The green line with x markers is the difference between optimal holdings under no bubble and optimal holdings under a bubble. Under the simulated economy, the investor's optimal strategy under a price bubble versus her optimal strategy under no price bubble are very similar in appearance, and both paths have a similar shape. The existence of a bubble makes

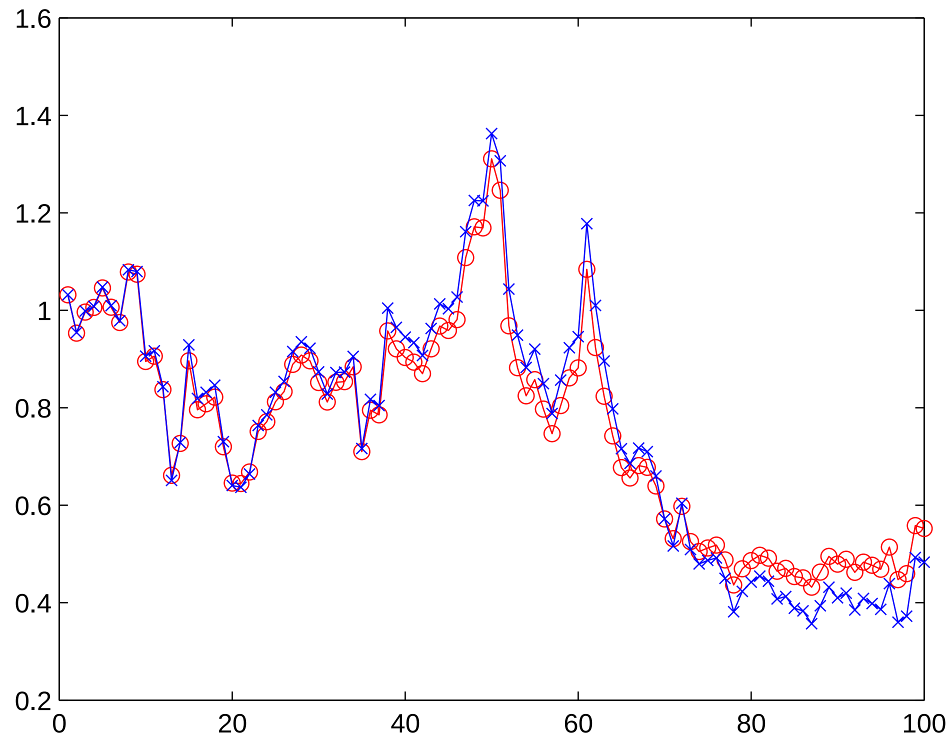
Figure 5.1: Trading Strategy under logarithmic utility for one risky asset with a bubble $\beta = 0.5$



it optimal for the investor to simply invest a smaller fraction of her wealth in the risky asset, but otherwise to follow a trading path with the same overall shape. The line plotting the difference between trading strategies is always above the x-axis and clearly expresses the fact that the existence of a bubble causes a reduction in optimal holdings of the risky asset. This result is consistent with the theoretical result under logarithmic utility found in chapter 3.

Consider the graph of the risky asset's price path in figure 5.2 as compared to the trading strategy 5.1. The trading strategies follow the same form as the asset prices. That is, when prices are high, more of the asset is held and when prices drop less of the asset is held. This is consistent with the results of chapter 3,

Figure 5.2: Risky Asset Price Processes Corresponding to figure 5.1

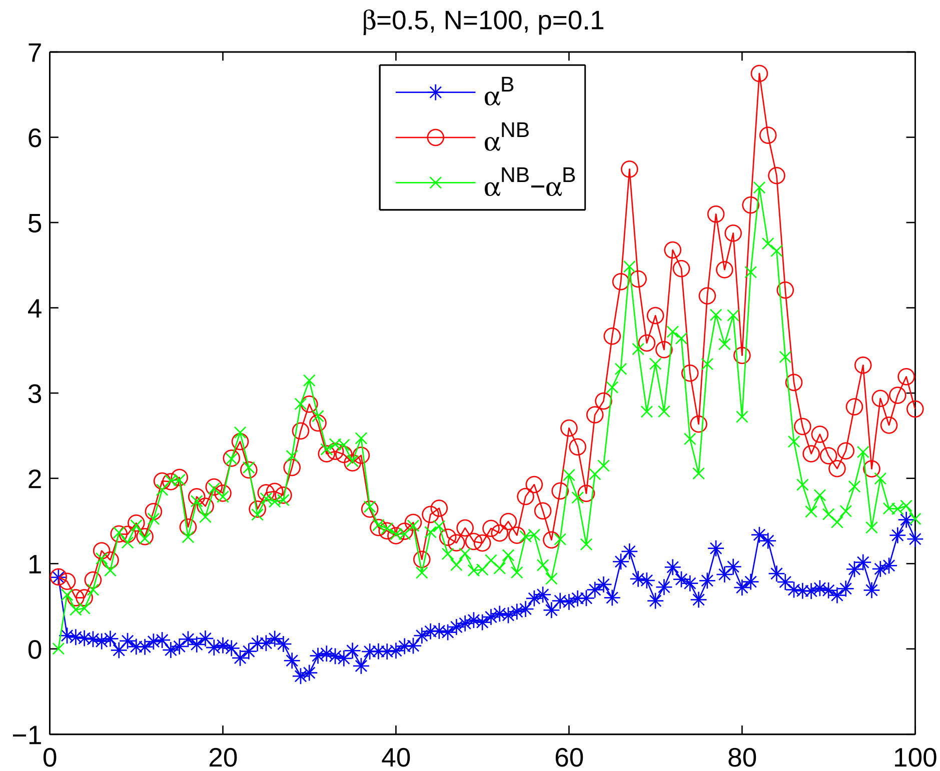


where the optimal trading strategy is found to be directly proportional to the asset price process 3.13.

5.1.2 One asset, Power Utility, p=0.1

Now consider the case of a different CRRA utility function, the power utility with $p = 0.1$, $U(x) = 10x^{0.1}$. In [4], DeTemple et al. have investigated optimal trading under these types of utility functions and have shown that they exhibit a phenomenon known as **intertemporal hedging**. Intertemporal hedging refers to any deviation from Markowitz's mean-variance investment rule.

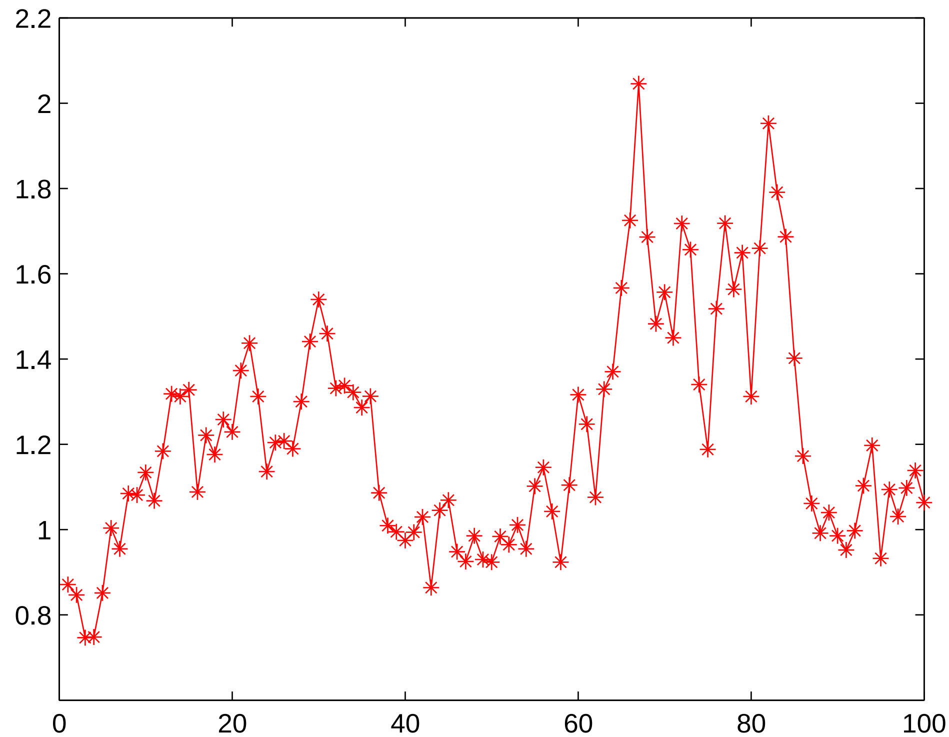
Figure 5.3: Trading Strategy under CRRA utility p=0.1 for one asset with a bubble $\beta = 0.5$



What has been shown in [4], is that when any utility function other than logarithmic utility is used, it is optimal to hedge against variations in means and variances. This type of hedging can potentially impart interesting influences on the trading strategies, and should therefore be investigated when considering the influence of asset price bubbles on optimal portfolio holdings. Indeed, this phenomenon has been a criticism of Markowitz's mean-variance rules since at least Merton [26].

Under this particular simulation of the economy, the optimal trading strategy under a bubble is much smaller in magnitude than that under no bubble, as seen in figure 5.3 where the green and red lines are nearly overlapping. The

Figure 5.4: Simulated Asset Price under CRRA utility with p=0.1



overall shape and timing of trading in either case remain similar, as before. It is also interesting to see that in the second half of the trading period the trades become much more erratic, fluctuating between large holdings and small holdings repeatedly over a small time period. This trading behavior can be attributed to the price process following this same erratic behavior, as seen in figure 5.4.

These two examples in the $n = 1$ case portray that in an economy with a single risky asset, optimal investment strategies under an asset price bubble cause an investor to reduce her holdings of the asset when compared to the case of no price bubble.

Adding more risky assets to the economy will complicate the investor's decision. More risky assets will potentially allow the investor to hedge the risk associated with the bubble by investing in other risky assets. Furthermore, there are a myriad of economies to investigate: when only one asset has a bubble, when all assets have a bubble, when assets are uncorrelated, when assets are negatively correlated, etc. Each of these situations could influence how the trader responds to the existence of a bubble, and will now be examined.

5.2 Multiple Risky Assets, n $>$1

5.2.1 Correlated assets, n=3, p=0

Figures 5.5 and 5.6 display the asset prices and optimal trading strategy results for an economy that includes three risky assets all with price bubbles with $\beta = 0.125[1\ 1\ 1]^T$. In this economy, asset prices are correlated: assets 1 and 2 are positively correlated, assets 2 and 3 are negatively correlated, and assets 1 and 3 are uncorrelated. Namely, the correlation matrix of the price process is,

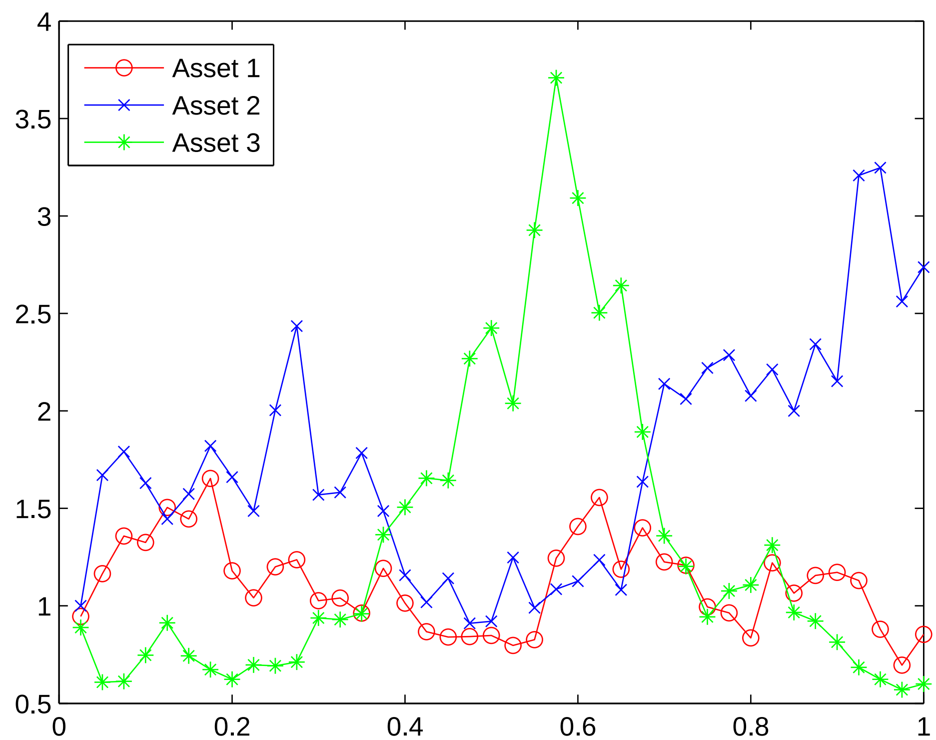
$$\rho = \begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 1 & -0.5 \\ 0 & -0.5 & 1 \end{bmatrix} \tag{5.2}$$

The investor's risk preferences follow logarithmic utility. In words this case can be described as having correlated assets, all assets having bubbles, and with no intertemporal hedging terms.

Here a very interesting result is found in subfigures 5.6a, 5.6c,and 5.6e, the first asset is shorted to raise additional capital for investing in the other two risky assets. Observing the asset prices in figure 5.5, it becomes apparent that this strategy and the correlation between the assets, as described previously makes it optimal to short the least expensive asset, asset 1, and hedge that risk by investing more heavily in asset 2. Furthermore, since asset 2 is negatively correlated with asset 3, the increased exposure to two's systematic risk can be partially hedged by investing in asset 3. The net effect of the correlations between the assets is that it is optimal to short asset 1 in order to leverage investment in assets 2 and 3.

It is also observed here that the optimal holdings with no bubble have higher magnitude than optimal holdings with a bubble, consistent with theoretical results in the special case of chapter 3. This simulation demonstrates that shorting can be optimal even under logarithic utility, when there are no intertemporal

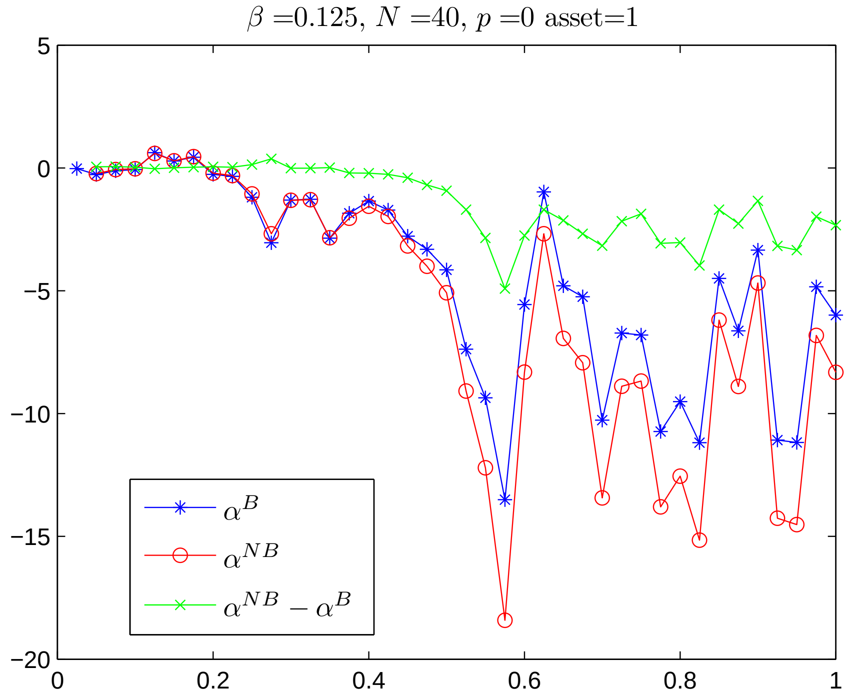
Figure 5.5: Asset Prices



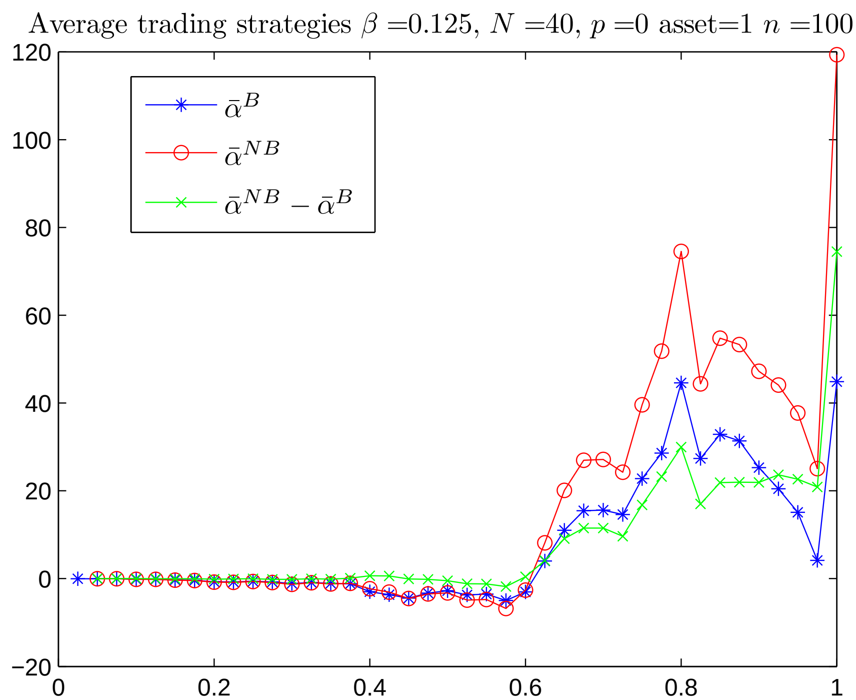
hedging considerations, if the price processes correlate amenably.

Subfigures 5.6b, 5.6d,and 5.6f demonstrate that when optimal trading strategies are simulated many times and averaged, the result becomes quite smooth as the number of simulations gets large. This is consistent with the Central Limit Theorem. Furthermore, it can be demonstrated that the average trend is related to the drift terms chosen (μ) and not on the diffusion parameters (σ). The averages of optimal trading strategies also display a reduction in holdings in the presence of an asset price bubble, as expected.

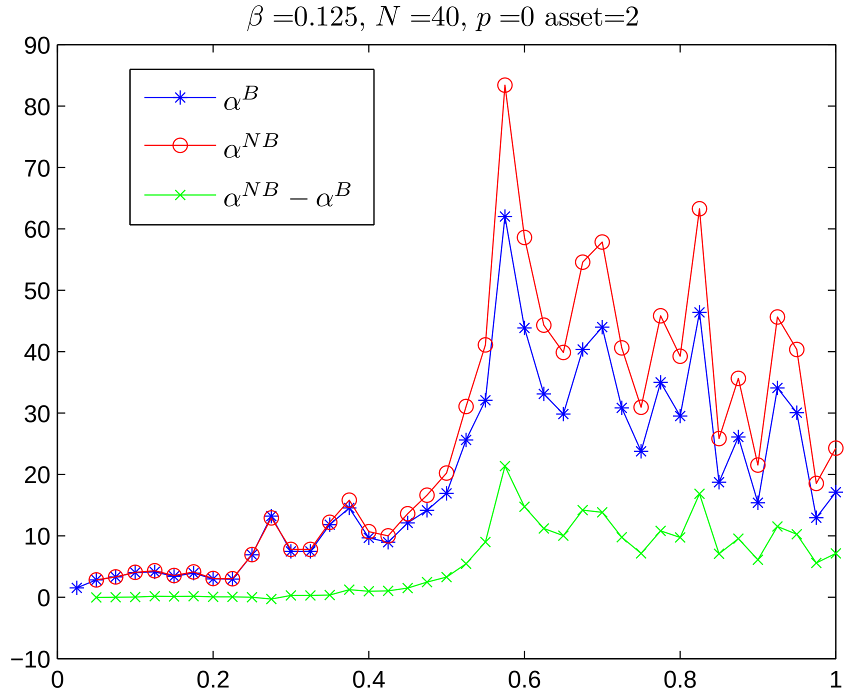
Figure 5.6: Three asset economy, all bubbles, assets correlated, log utility



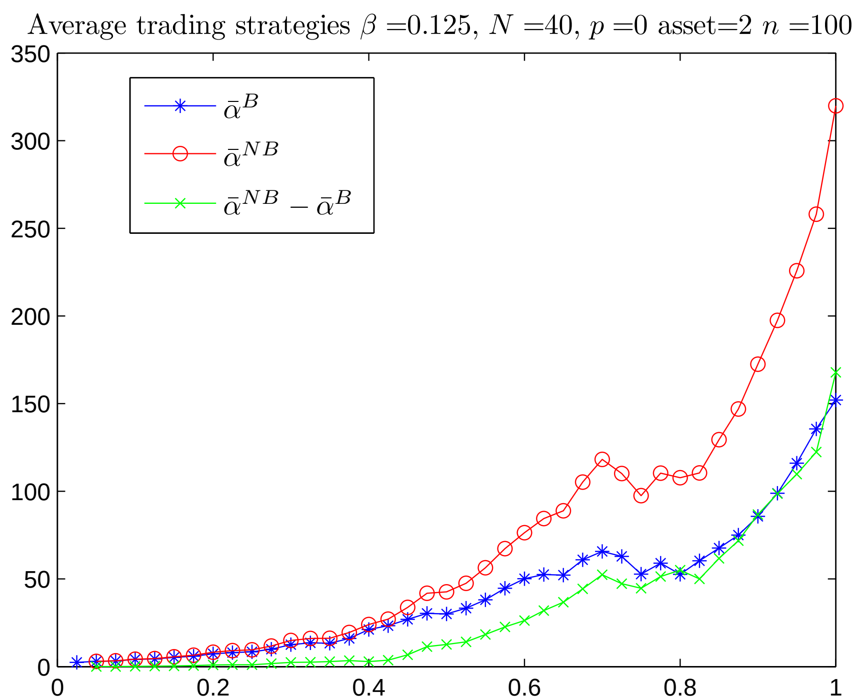
(a) Trading Strategy under log utility for three stock with a bubble Asset 1



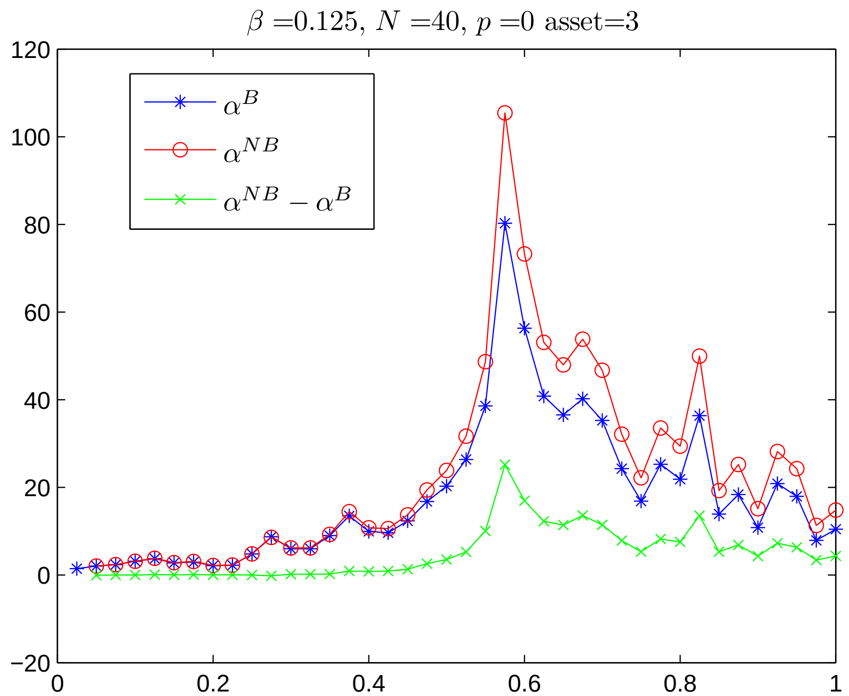
(b) Average Trading Strategy under log utility for three stock with a bubble Asset 1



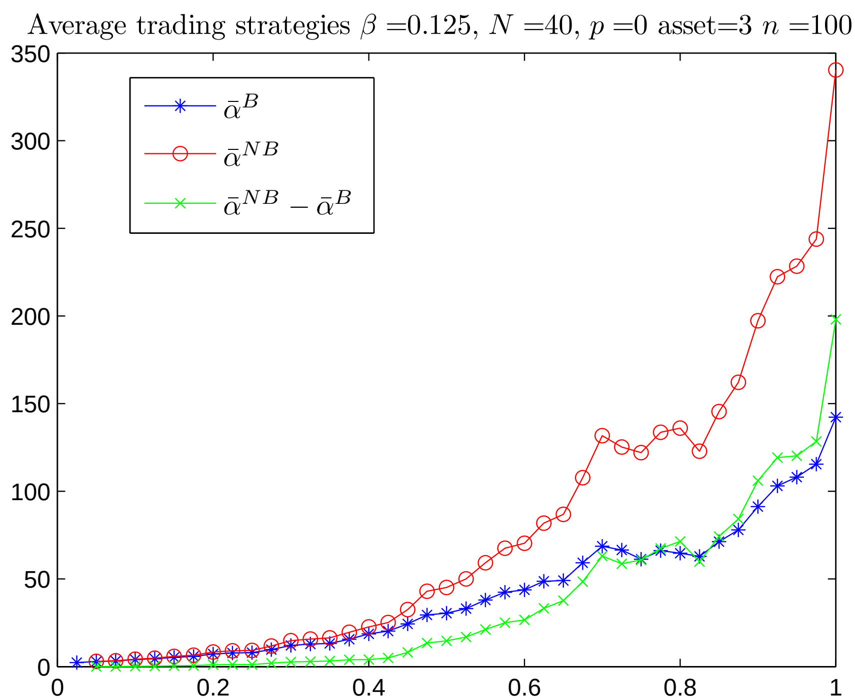
(c) Trading Strategy under log utility for three stock with a bubble Asset 2



(d) Average Trading Strategy under log utility for three stock with a bubble Asset 2



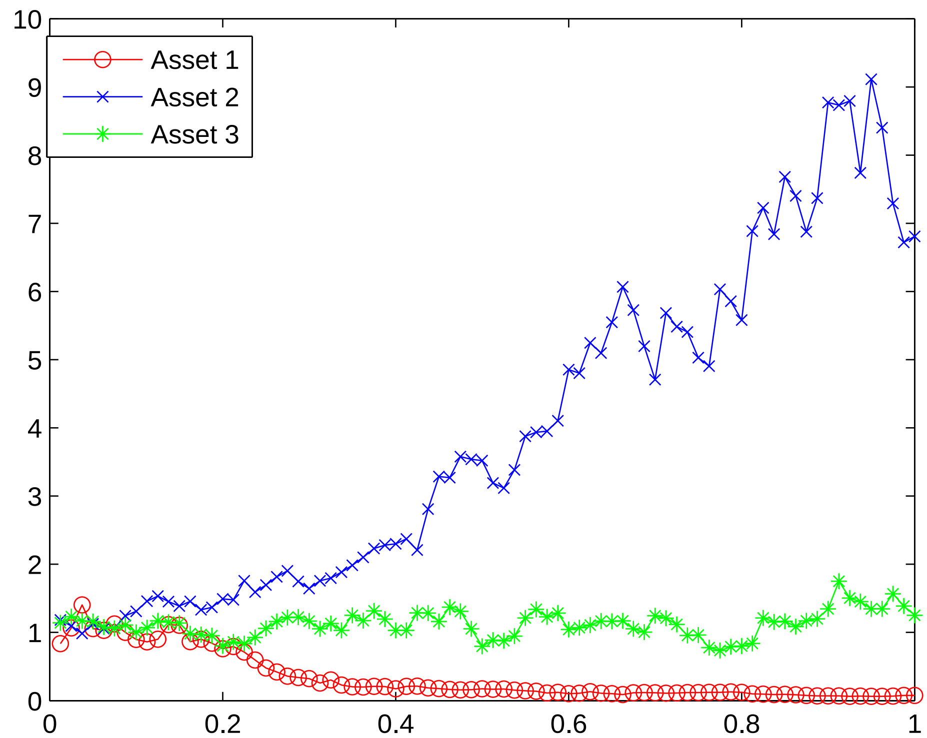
(e) Trading Strategy under log utility for three stock with a bubble Asset 3



(f) Average Trading Strategy under log utility for three stock with a bubble Asset 3

5.2.2 Correlated Assets, n=3, p=0.5

Figure 5.7: Asset Prices, n=3,p=0.5

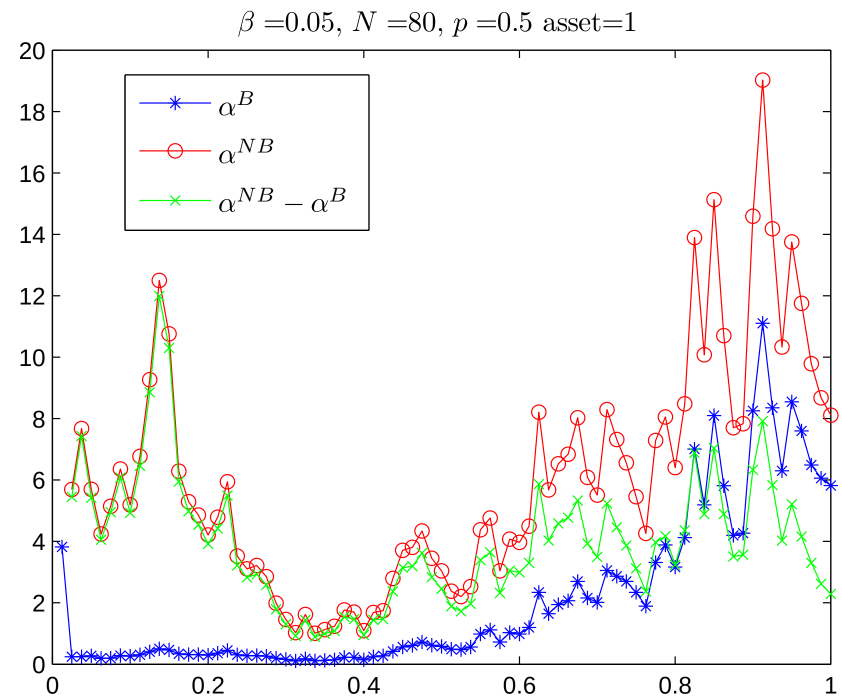


In this simulation, three assets all with bubbles with $\beta = 0.05[1\ 1\ 1]^T$, with the first two assets negatively correlated and independent from the third asset, and risk preferences described by a power utility function with $p = \frac{1}{2}$, $U(x) = 2\sqrt{x}$. Here, the correlation matrix is,

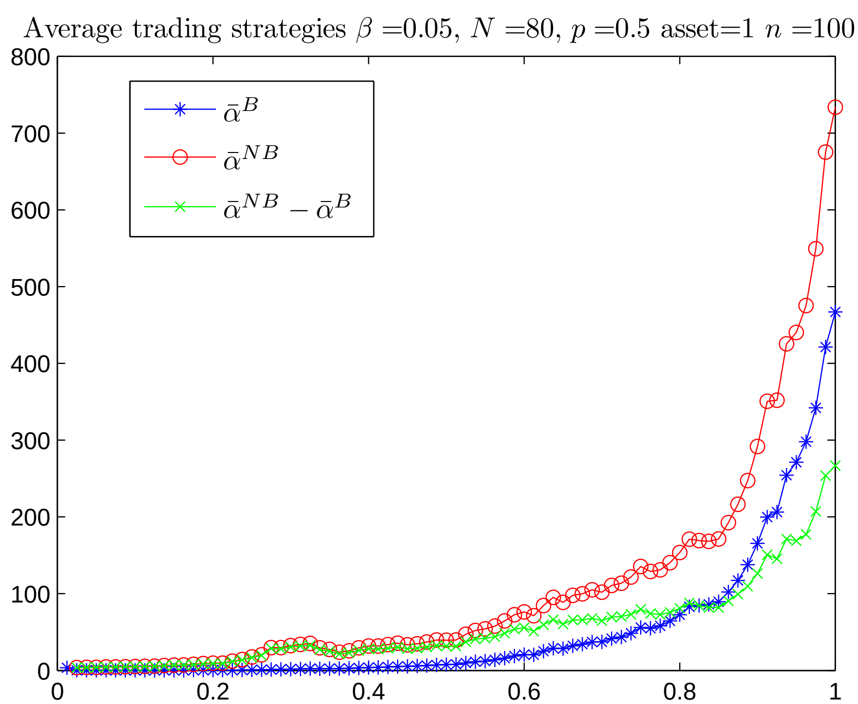
$$\rho = \begin{bmatrix} 1 & -0.5 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.3}$$

In all three assets, the bubble case has a significantly lower proportion of wealth invested in the first half of the trading period.

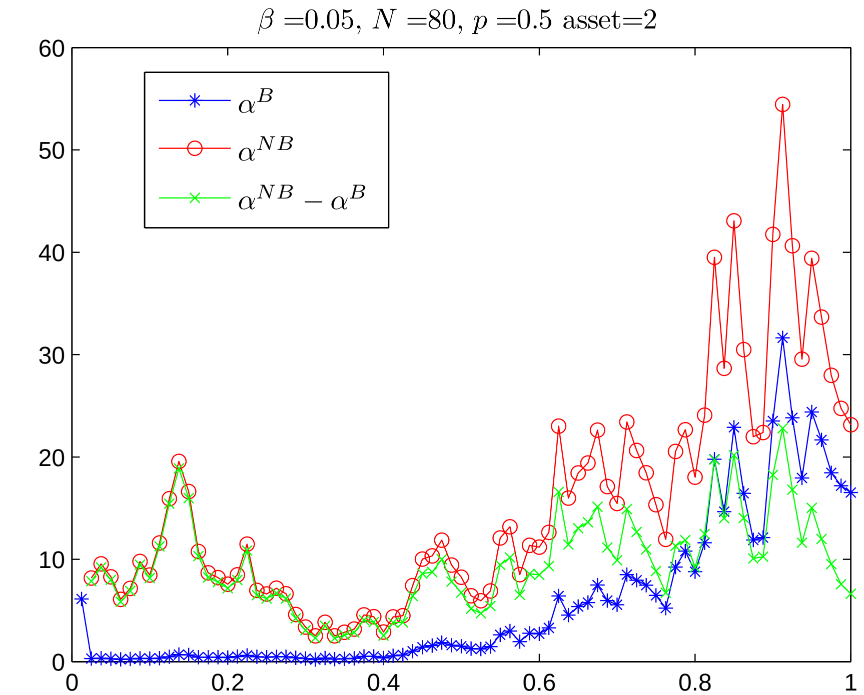
Figure 5.8: Three asset economy, all bubbles, assets correlated, power utility



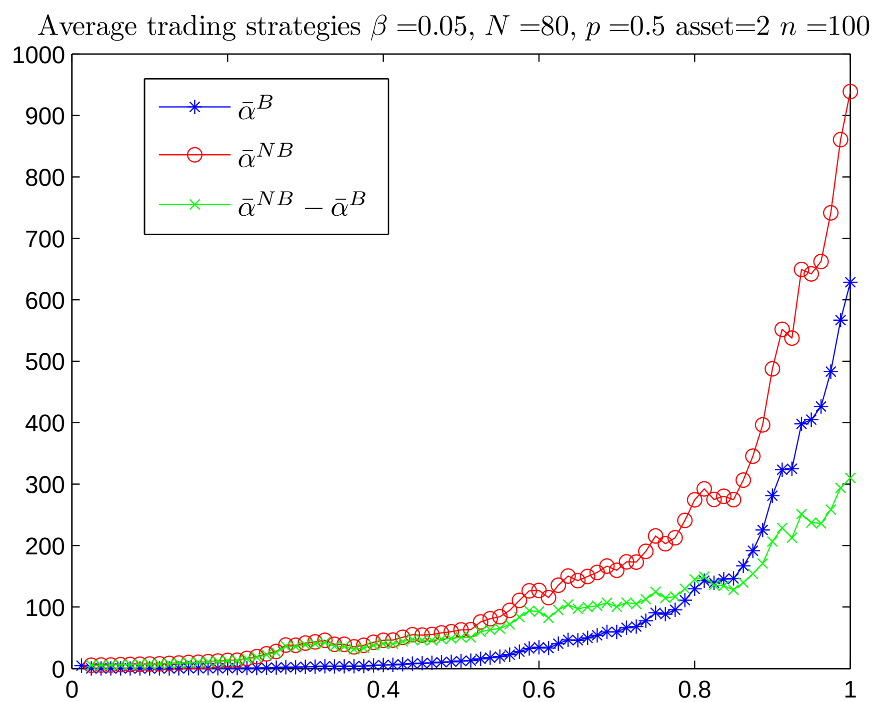
(a) Trading Strategy under CRRA utility for three stock with a bubble Asset 1



(b) Average Trading Strategy under CRRA utility for three stock with a bubble Asset 1



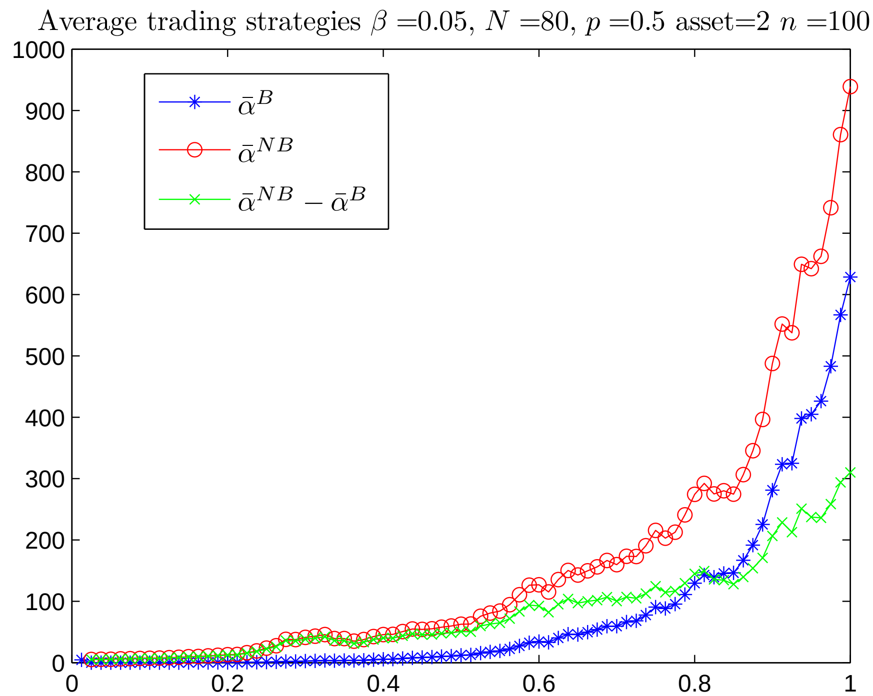
(c) Trading Strategy under CRRA utility for three stock with a bubble Asset 2



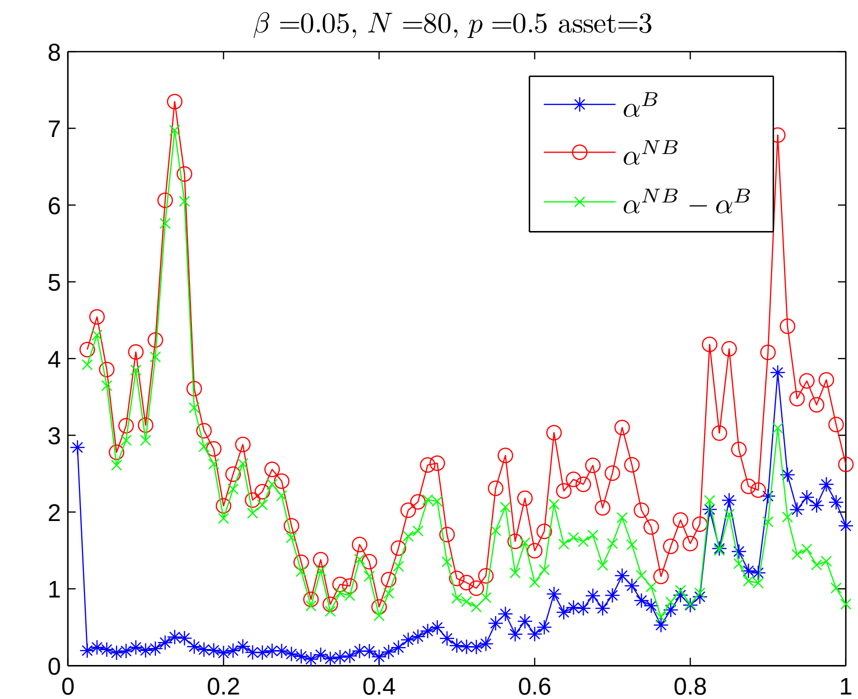
(d) Trading Strategy under CRRA utility for three stock with a bubble Asset 2

Noticing the price paths in this case in figure 5.7, it is observed that asset 3 has the most gains, asset 1 the most losses, and asset 2's price is relatively stable. However, due to the negative correlation between assets 1 and 2, the trading in these assets becomes much more chaotic after the price of asset 1 begins declining, but remain aligned with one another. In contrast, holdings of asset 3 maintain the same form throughout the duration of the trading period. It can also be noticed that in the first half of trading, when all the assets' prices remain relatively constant, the optimal holding of all three assets in the case of a bubble is much lower than the case of no bubble. Furthermore, the disparity between bubble and no bubble holdings decreases as the prices of the assets fluctuate and the end of trading approaches.

This phenomenon is consistent with intertemporal hedging. With a $p = \frac{1}{2}$ utility function, it is optimal for the investor to hedge the risk of the dynamic volatility even in the presence of an asset price bubble. This phenomenon occured in this particular case, when asset prices became rather volatile, but as the average of 100 simulations shows, the difference between bubble and no bubble holdings increases with time on average, consistent with mean-variance trading strategies. This occurs because the intertemporal hedging terms in any particular realization of the economy will differ, and on average these terms will cancel with each other and the mean-variance trading dominates in the average.



(e) Trading Strategy under CRRA utility for three stock with a bubble Asset 2



(f) Average Trading Strategy under CRRA utility for three stock with a bubble Asset 3

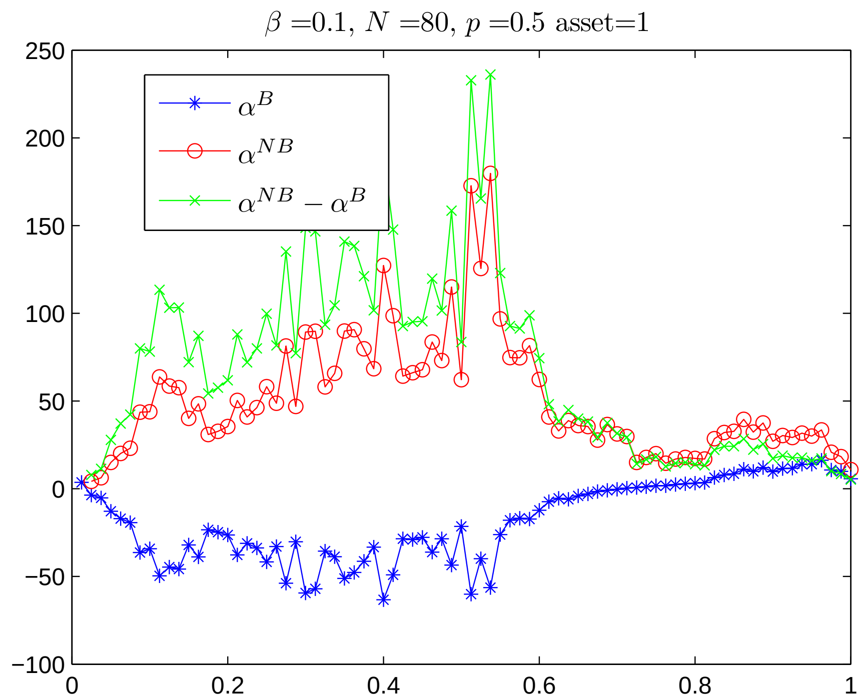
5.2.3 Correlated assets, n=3,p=0.5, larger bubble

In this simulation, the same parameters were used as in the last section, but the β was doubled to $\beta = 0.1[1\ 1\ 1]^T$. In this way, the effect of a more pronounced bubble can be compared to results from the previous simulation. A very interesting result occurs, namely that the severity of the bubble creates a situation in which it is optimal to short the assets in early trading in the case of a bubble, but to hold them in a long position in the case of no bubble. This occurs for all three assets, even among the two that are negatively correlated. The increased perceived risk of a bubble burst creates a scenario where the investor will short even negatively correlated assets rather than use them to hedge risk.

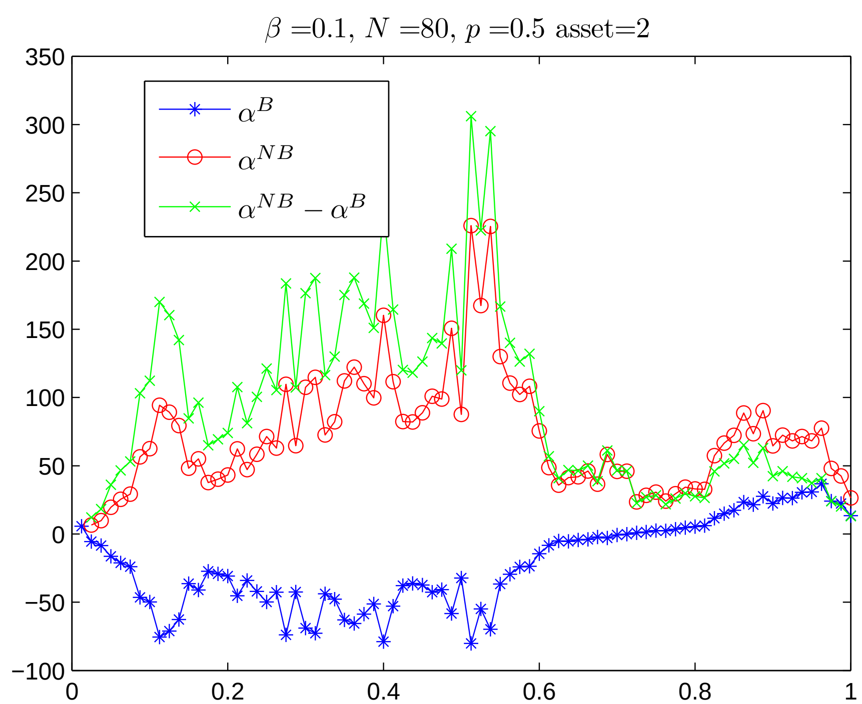
In this case, what is still observed is that $|\alpha^{NB}| > |\alpha^{B}|$, which in words says that the number of shares of the asset being traded in an economy with a bubble is less than an economy without a bubble, even if in one case the asset is held in a long position and in the other case the asset is shorted. This simulation demonstrates that when a bubble is "large" enough, that risk is the dominant consideration when trading, and correlation among asset prices cannot overcome the perceived risk of a bubble burst. This is consistent with observed trading behavior in markets that are suspected to have bubbles.

Looking at the average trading strategies, again it is observed that the intertemporal hedging concerns in any specific realization cancel with one another and the dominant term in the average is the mean-variance strategy. This is consistent with the Central Limit Theorem, as previously stated, and should be seen in every case.

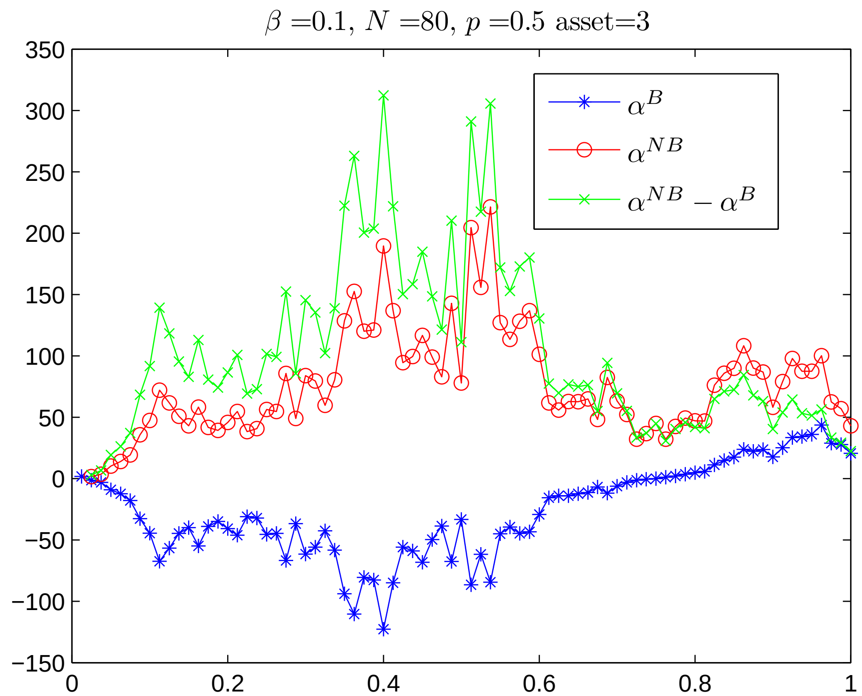
Figure 5.9: Three asset economy, severe bubbles, assets correlated, power utility



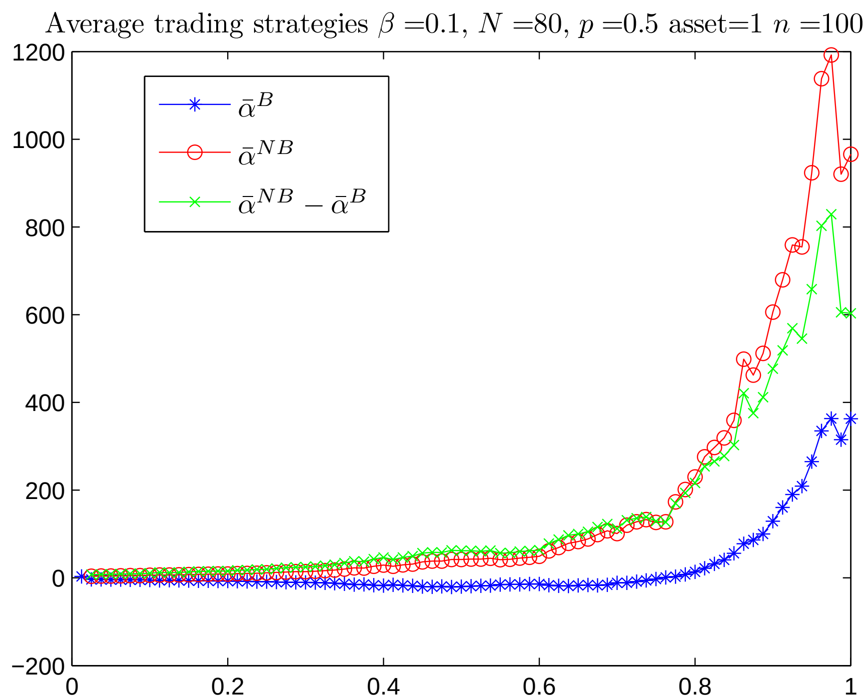
(a) Trading Strategy under CRRA utility for three stock with a bubble Asset 1



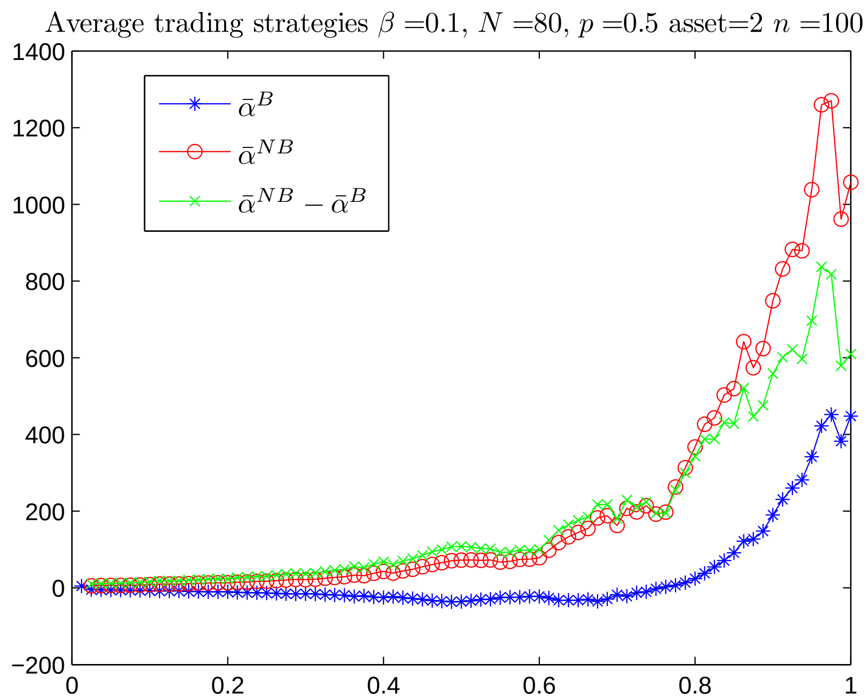
(b) Trading Strategy under CRRA utility for three stock with a bubble Asset 2



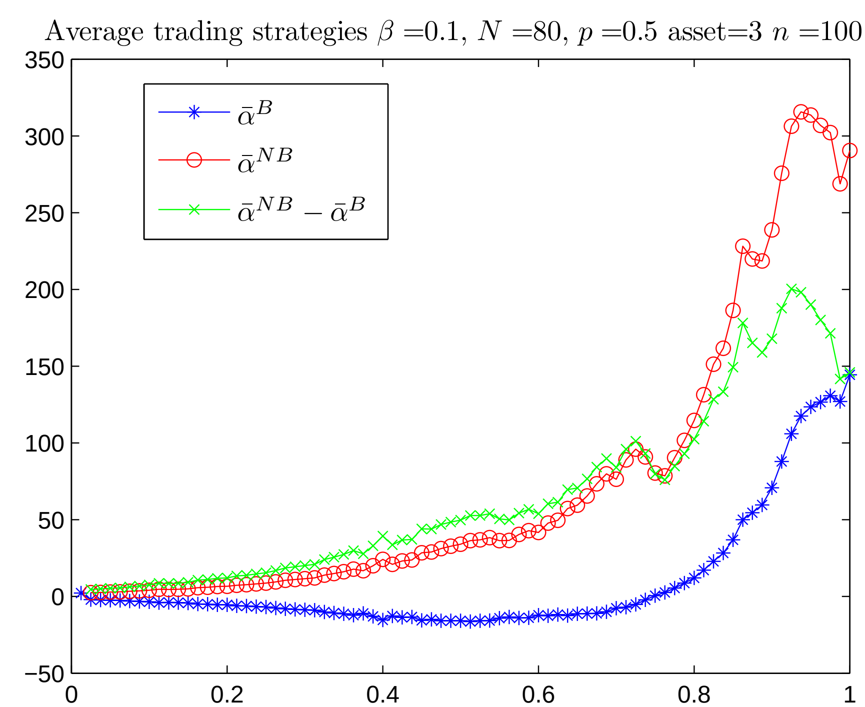
(c) Trading Strategy under CRRA utility for three stock with a bubble Asset 3



(d) Average Trading Strategy under CRRA utility for three stock with a bubble Asset 1



(e) Average Trading Strategy under CRRA utility for three stock with a bubble Asset 2



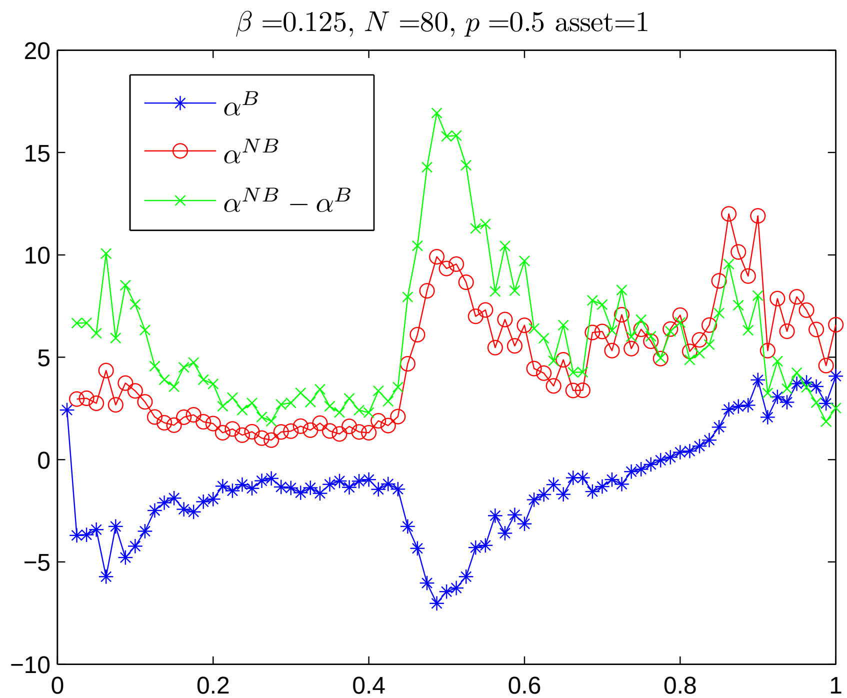
(f) Average Trading Strategy under CRRA utility for three stock with a bubble Asset 3

5.2.4 Uncorrelated Assets n=3,p=0.5

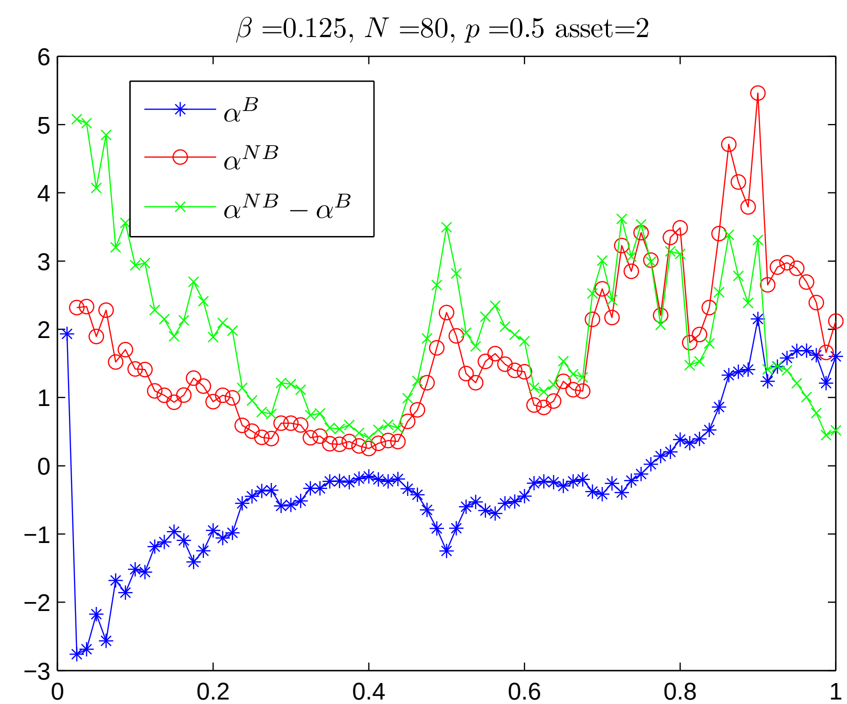
In this simulation, the assets are all uncorrelated and $\beta = \frac{1}{8}[1\ 1\ 1]^T$. Again, the effect of the bubble is to short the assets in the case of a bubble, but to hold them in a long position in the case of no bubble. Again $|\alpha^{NB}| > |\alpha^{B}|$ as expected. The results of these simulation provide further evidence that the effect of intertemporal hedging and the investor's percieved risk of a bubble are what make shorting in the case of a bubble optimal, as opposed to asset price correlation effects.

Again, the average trading figures portray that the mean-variance term dominates in the average, and the average holdings of any asset stabilizes.

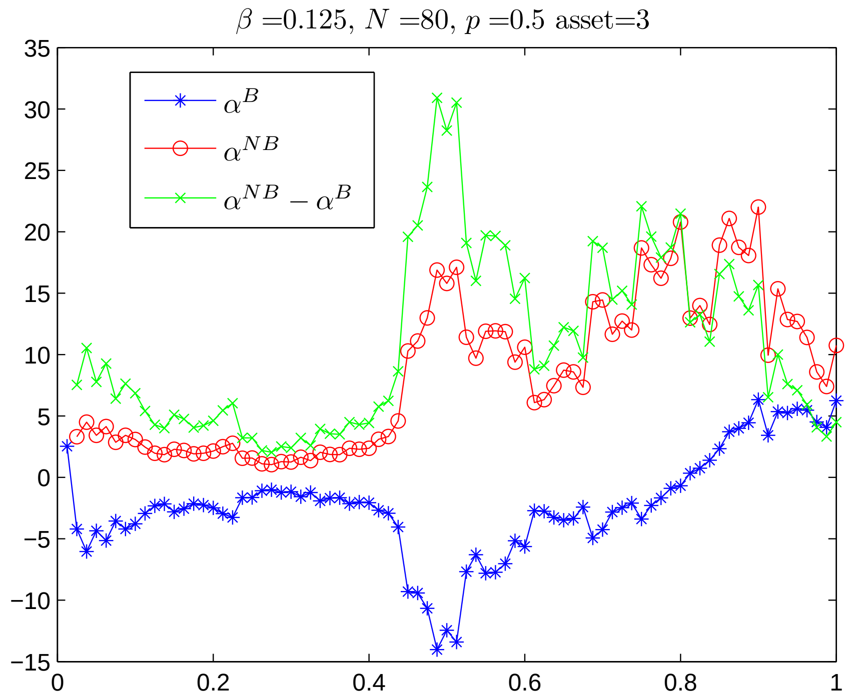
Figure 5.10: Three asset economy, price bubbles, assets uncorrelated, power utility



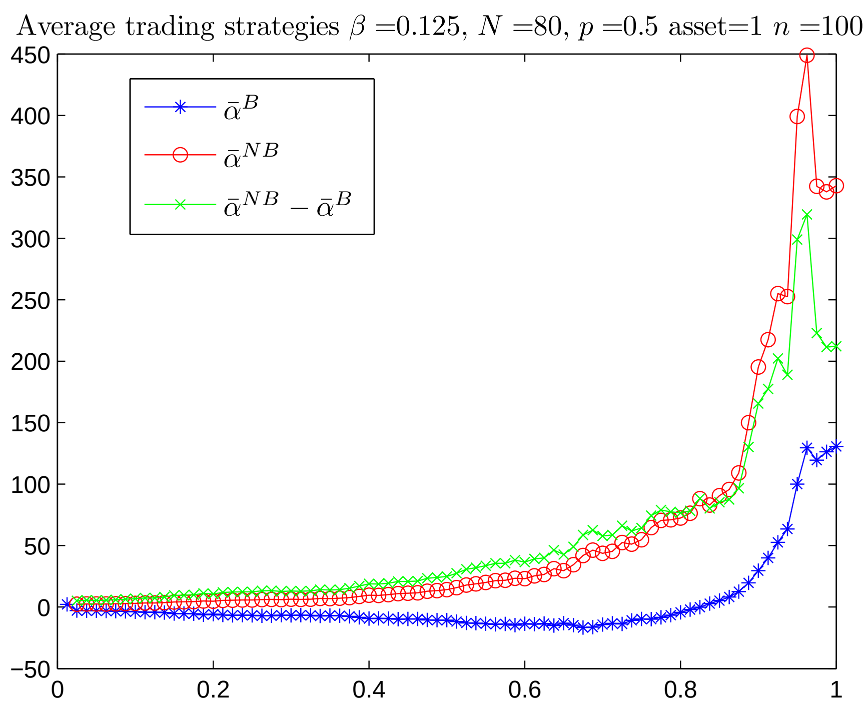
(a) Trading Strategy Asset 1



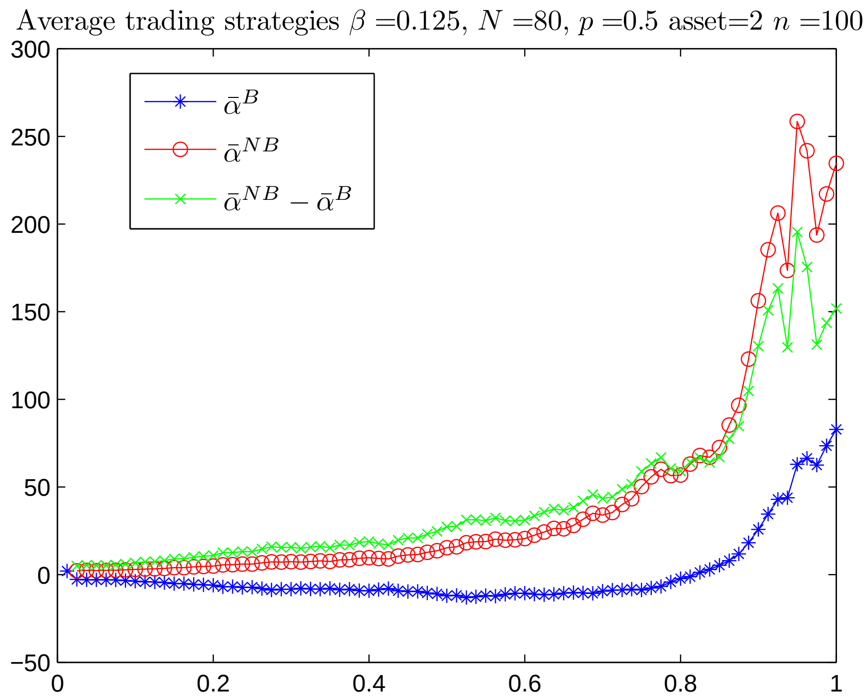
(b) Trading Strategy Asset 2



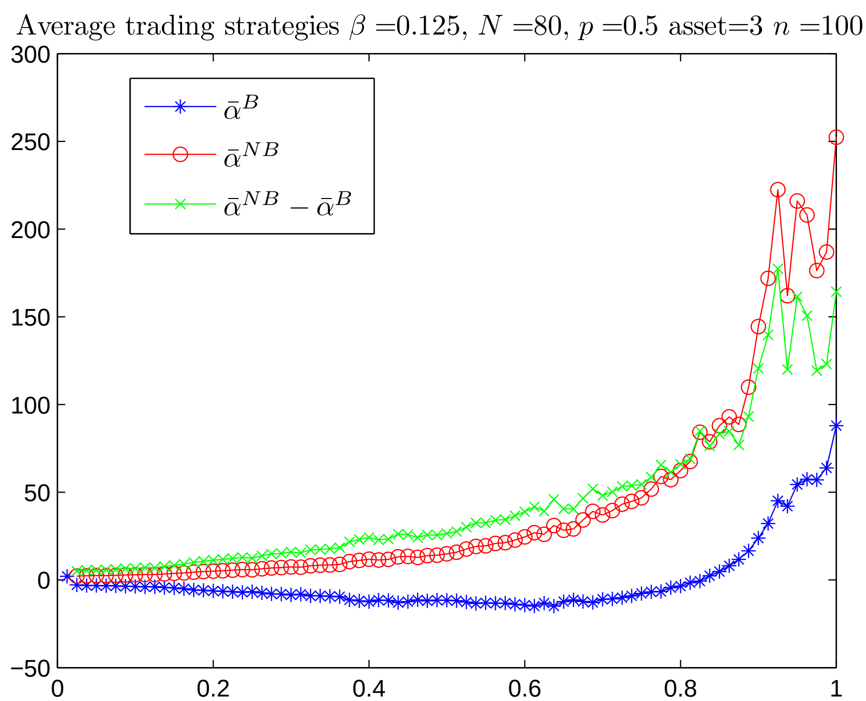
(c) Trading Strategy Asset 3



(d) Average Trading Strategy Asset 1



(e) Average Trading Strategy Asset 2



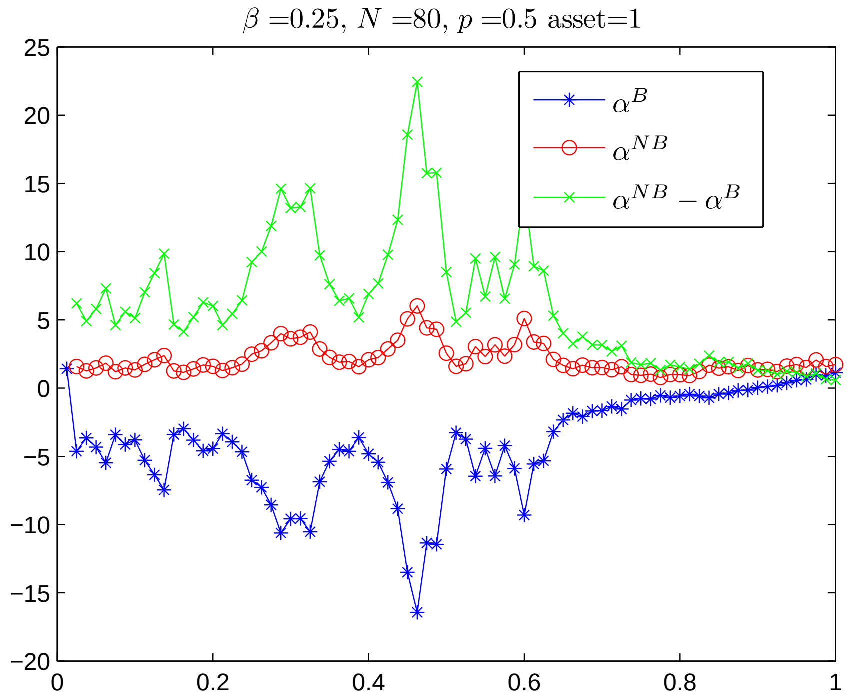
(f) Average Trading Strategy Asset 3

Figure 5.10

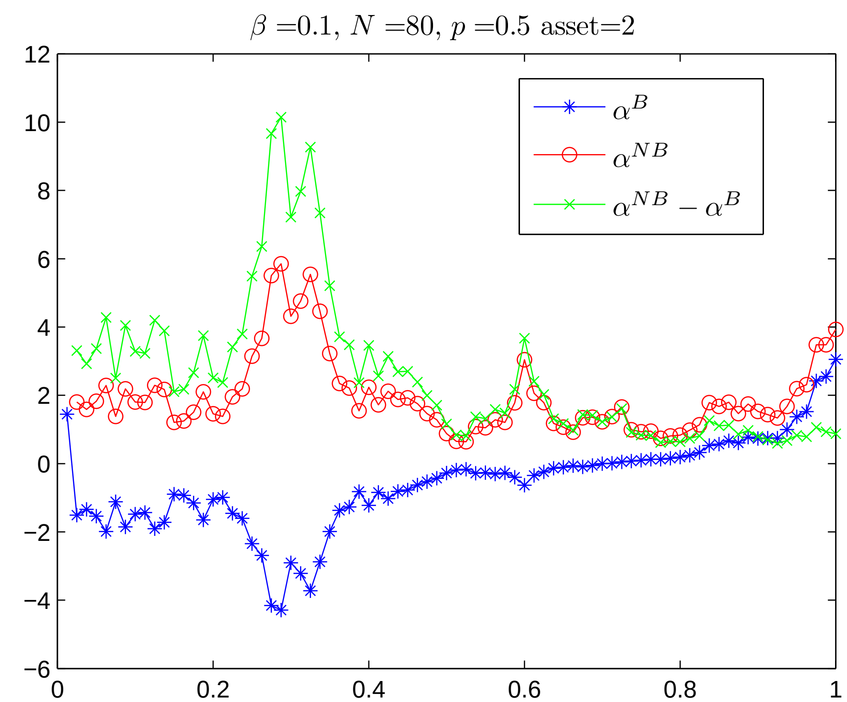
5.2.5 Uncorrelated, n=3, p=0.5

In this simulation, three uncorrelated assets are given with different bubble parameters. $\beta = \begin{bmatrix} 0.25 & 0.1 & 0 \end{bmatrix}^T$. Here it is apparent again that the bubble can cause shorting when the no bubble situation would not. Furthermore, the larger β corresponds to a longer period of shorting. In the asset with no bubble, the asset is never shorted. In this economy, even the asset with no bubble is affected by the other assets with bubbles. Again, $|\alpha^{NB}| > |\alpha^{B}|$.

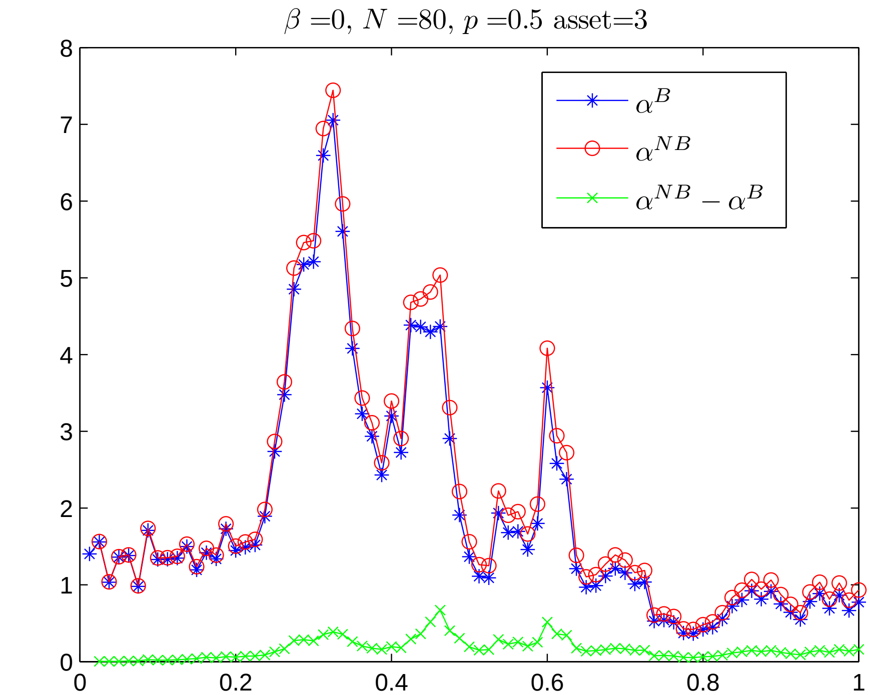
Figure 5.11: Three asset economy, different price bubbles, assets uncorrelated, power utility



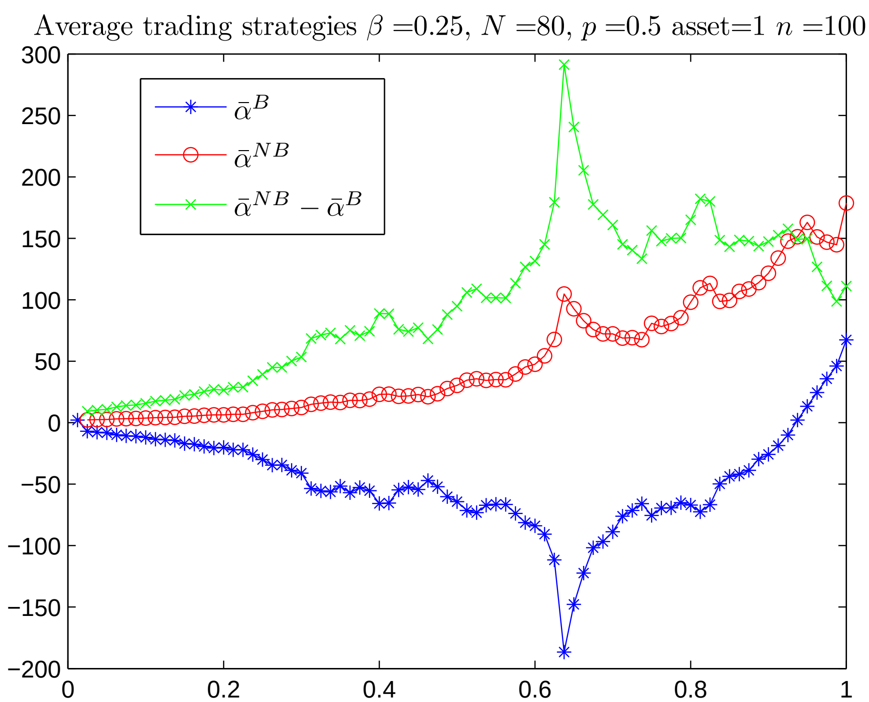
(a) Asset 1 holdings



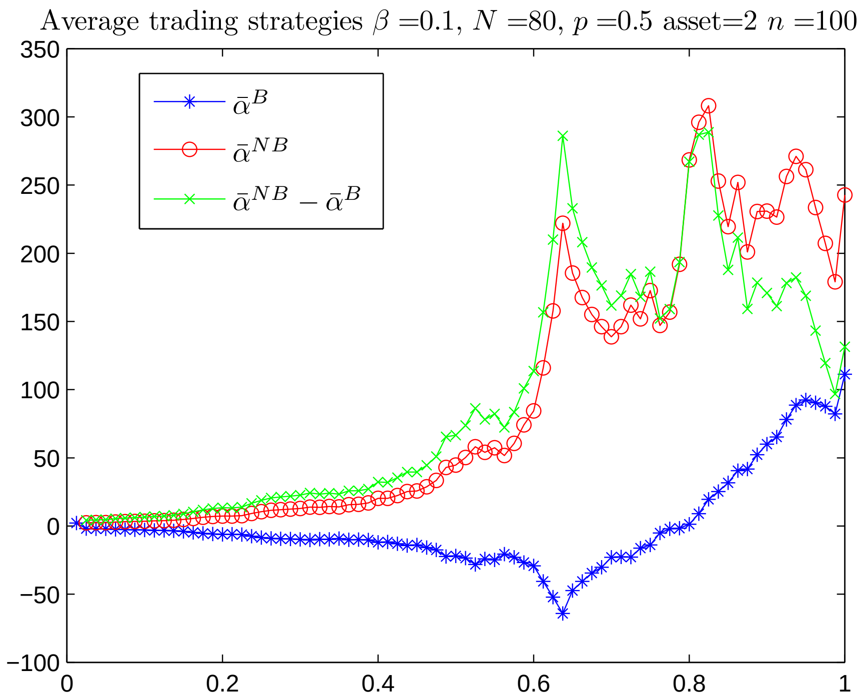
(b) Asset 2 holdings



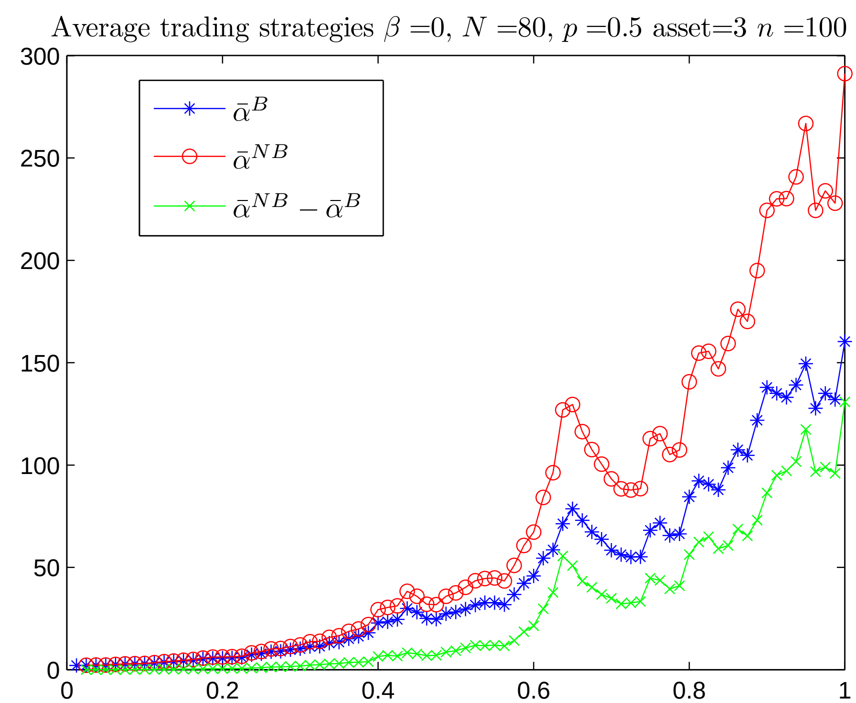
(c) Asset 3 holdings



(d) Ave Asset 1 holdings



(e) Average Asset 2 holdings

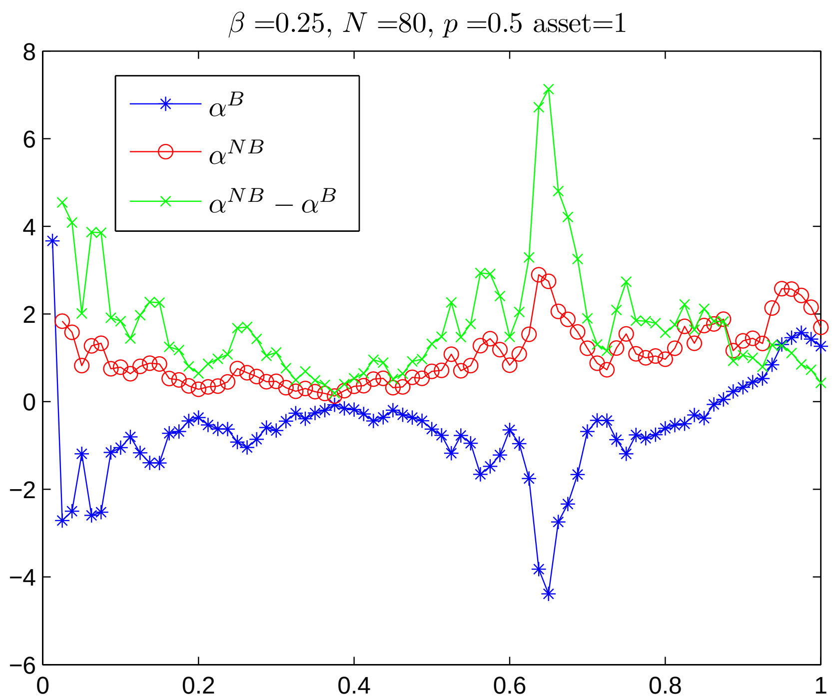


(f) Average Asset 3 holdings

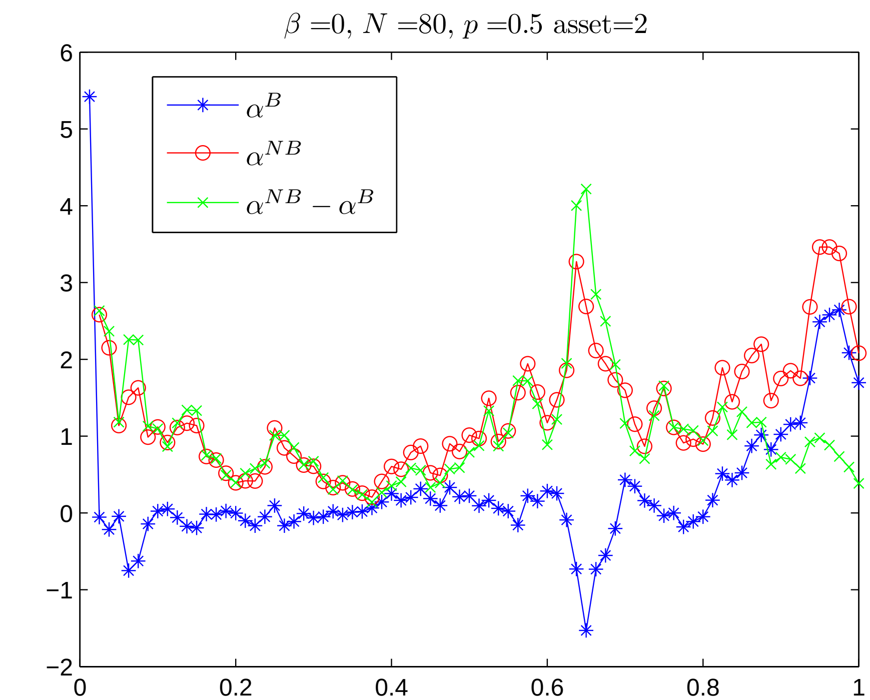
Figure 5.11: Trading Strategies for differing β values

5.2.6 correlated n=2, p=0.5

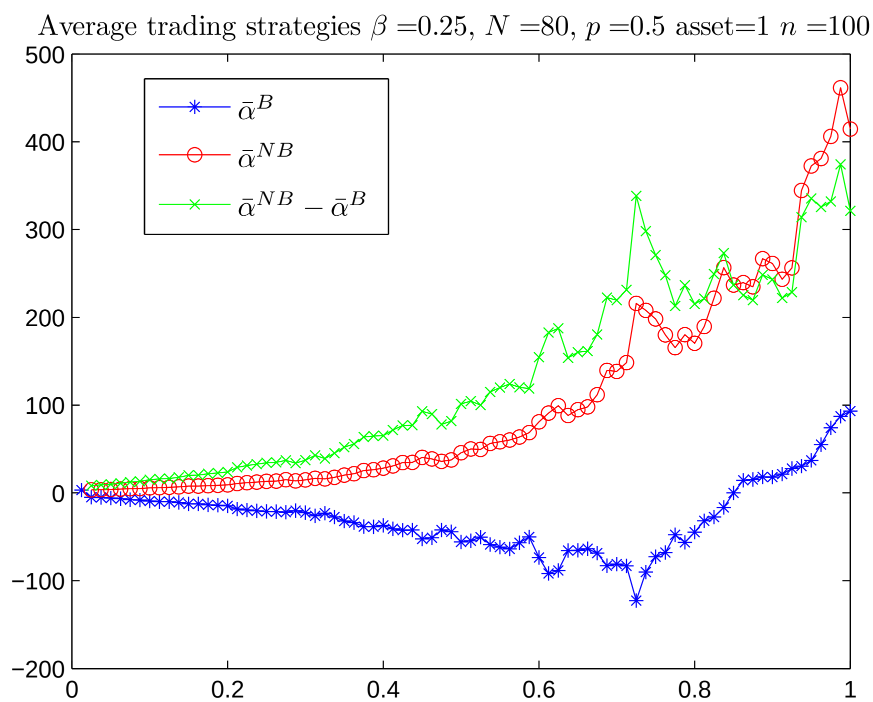
Figure 5.12: Two asset economy, different price bubbles, assets correlated, power utility



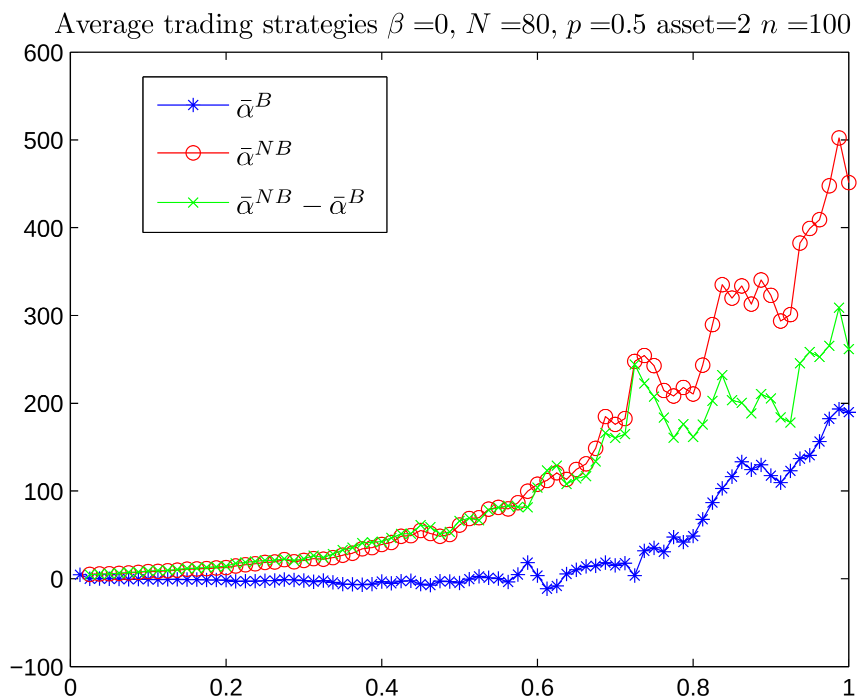
(a) Asset 1 holdings



(b) Asset 2 holdings



(c) Ave Asset 1 holdings



(d) Average Asset 2 holdings

Figure 5.12: Trading Strategies for 2 assets differing β values

In this example, $\beta = \begin{bmatrix} 0.25 & 0 \end{bmatrix}^T$. The assets are negatively correlated with correlation $\rho(1,2) = -0.5$. Here the shorting occurs in both assets when one has a bubble. This is due to the perceived risk of the bubble and the correlation that will allow for heding of risk. Again, $|\alpha^{NB}| > |\alpha^B|$.

CHAPTER 6

CONCLUDING REMARKS AND OPEN QUESTIONS

The objective of this work was to investigate the effect of asset price bubbles on optimal portfolio allocations. The result found both in a theoretical example of chapter 3 and in the simulation output of chapter 5 is that $|\alpha^{NB}| > |\alpha^{B}|$. So, while it is possible for a bubble to alter the direction of a trade (short instead of long), what remains true is that an optimal investor will always hold fewer shares of an asset with a bubble in her portfolio compared to if that asset did not have a price bubble.

The effect of a bubble birth on optimal holdings remains to be investigated. Modeling the birth of an asset price bubble was one of the central hurdles that was overcome in Jarrow et al. [18]. The method developed in [18] relied on regime shifting in what was termed a "dynamic market." The valuation measure chosen by the market was allowed to change over time, and at time epochs when the market changed from an ELMM with no bubble to an ELMM with a bubble, a bubble was born.

It would also be of interest to consider the effect to the investor's terminal wealth of ignoring the bubble and choosing the α^{NB} trading strategy. Suppose there was in fact a bubble, but the investor overlooked this, is there something definitive that can be said about how this will affect her wealth process? How large does the β need to be before this has a significant impact on her wealth? This question seems particularly relevant when considering the dramatic increase of computation time involved in calculating optimal holdings when modeling a bubble. If incorporating a bubble into asset price models does not significantly decrease the terminal wealth, then it may not be worth the extra computation time in an industrial setting.

Finally, it would be interesting to consider the effect, if any, of adding a random time of expiry, as seen in the special example of 3. As seen in theorem 4.1 of [18], and mentioned in the review of asset price bubbles, such an addition would allow for a more general types of bubbles to be considered. Here only type 3 bubbles, bubbles that are strict Q local martingales, have been investigated. This is because the investor has a finite time horizon, so this places a finite and bounded expiration time on all the assets, as far as the investor is concerned. Therefore, adding this random expiry to the model would require not only adding it in the model of the asset prices, but also modifying the portfolio optimization problem to an infinite horizon problem.

APPENDIX A

MATLAB CODE

A.1 Price Simulator

```
function [Sout,SoutNB,Vout,VoutNB,Xout,XoutNB,piout,pioutNB,uout,uoutNB,QzPout,QzPoutNB]=PriceSimulator(mu,eta,sigma,rho,gamma,T,delta,s,v,beta,I,x,maxiter)
  n=maxiter;
  m=length(s); d=length(v); n_s=size(rho,2);n_v=size(gamma,2);
  N=floor(T/delta);
  V=zeros(d,N); VNB=V;
  S=zeros(m,N); SNB=S;
  u=zeros(n_s,N);
  Du=zeros(n_s,n_s+n_v,N,N);
  DV=zeros(d,n_s+n_v,N,N);
  QzP=zeros(N,1); QzPNB=QzP;
  CEarg=zeros(n_s,N,n);
  CEargNB=CEarg;
  syms t;

  for iter=1:n
      Z=randn(n_s+n_v,N);
      b=Z*sqrt(delta);
      tildeb=zeros(size(b));tildebNB=tildeb;

```

```
LNV=log(v)+eta*delta+v.^beta.*([rho,gamma]*b(:,1))-.5*v.^(2*beta).*diag([rho,gamma]*[rho';gamma'])*delta;
LNVNB=log(v)+eta*delta+([rho,gamma])*b(:,1)-.5*diag([rho,gamma]*[rho';gamma'])*delta;

V(:,1)=exp(LNV);
VNB(:,1)=exp(LNVNB);

LNS=log(s)+mu*delta+v.^beta.*(sigma*b(1:n_s,1))-.5*v.^(2*beta).*diag(sigma*sigma')*delta;
LNSNB=log(s)+mu*delta+sigma*b(1:n_s,1)-.5*diag(sigma*sigma')*delta;

S(:,1)=exp(LNS);
SNB(:,1)=exp(LNSNB);

u(:,1)=diag(V(:,1).^-beta)*sigma'*((sigma*sigma')\mu);
uNB=sigma'*((sigma*sigma')\mu)*ones(1,size(u,2));
DV(:,:,1,1)=diag(V(:,1).^(ones(size(beta))+beta))*[rho,gamma];

Du(:,:,1,1)=sigma'*((sigma*sigma')\(diag(-beta.*mu)*diag(V(:,1).^(-beta-ones(size(beta)))))*squeeze(DV(:,:,1,1)));
```

```
LNQzP=-u(:,1)'*b(1:n_s,1)-.5*delta*u(:,1)'*u(:,1);
LNQzPNB=-uNB(:,1)'*b(1:n_s,1)-.5*delta*uNB(:,1)'*
    uNB(:,1);

QzP(1)=exp(LNQzP(1));

tildeb(:,1)=b(:,1)+delta*u(:,1);
tildebNB(:,1)=b(:,1)+delta*uNB(:,1);

for i=2:N
    LNVlast=LNV;
    LNV=LNVlast+eta*delta+(V(:,i-1).^beta).*([rho,
        gamma]*b(:,i))-.5*(V(:,i-1).^(2*beta)).*
        diag([rho,gamma]*[rho';gamma'])*delta;

    LNVlastNB=LNVNB;
    LNVNB=LNVlastNB+eta*delta+([rho,gamma])*b(:,i)
        -.5*diag([rho,gamma]*[rho';gamma'])*delta;

    V(:,i)=exp(LNV);
    VNB(:,i)=exp(LNVNB);

    LNSlast=LNS;
    LNS=LNSlast+mu*delta+(V(:,i-1).^beta).*(sigma*
        b(1:n_s,i))-.5*(V(:,i-1).^(2*beta)).*diag(
```

```
            sigma*sigma')*delta;
        S(:,i)=exp(LNS);

        LNSlastNB=LNSNB;
        LNSNB=LNSlastNB+mu*delta+sigma*b(1:n_s,i)-.5*diag(sigma*sigma')*delta;
        SNB(:,i)=exp(LNSNB);

        u(:,i)=diag(V(:,i).^(-beta))*(sigma'*((sigma*sigma')\mu));
        DV(:,:,i,i)=diag(V(:,i).^(ones(size(beta))+beta))*[rho,gamma];
        Du(:,:,i,i)=sigma'*((sigma*sigma')\(diag(-beta.*mu)*diag(V(:,i).^(-beta-ones(size(beta))))*squeeze(DV(:,:,i,i))));

        LNQzPlast=LNQzP;
        LNQzP=LNQzPlast-u(:,i)'*b(1:n_s,i)-.5*delta*u(:,i)'*u(:,i);

        LNQzPlastNB=LNQzPNB;
        LNQzPNB=LNQzPlastNB-uNB(:,i)'*b(1:n_s,i)-.5*delta*uNB(:,i)'*uNB(:,i);

        QzP(i)=exp(LNQzP);
        tildeb(:,i)=b(:,i)+sum(delta*u(:,1:i),2);
```

```
        QzPNB(i)=exp(LNQzPNB);
        tildebNB(:,i)=b(:,i)+sum(delta*uNB(:,1:i),2);
    end
    for i=2:N
        for j=i+1:N
            DV(:,:,i,j)=DV(:,:,i,j-1)+(diag(beta+ones(
                size(beta)))*diag(V(:,i).^(beta))*diag
                ([rho,gamma]*b(:,j))+diag(eta)*delta)*
                DV(:,:,i,j-1);
            Du(:,:,i,j)=sigma'*((sigma*sigma')\(diag(-
                beta.*mu)*diag(V(:,j).^(-beta-ones(size
                (beta))))))*squeeze(DV(:,:,i,j));
        end
    end
    Y=fsolve(@(y)x-QzP(N)*I(y*QzP(N)),1,optimset('
        display','off'));
    YNB=fsolve(@(y)x-QzPNB(N)*I(y*QzPNB(N)),1,optimset
        ('display','off'));

    X=I(Y*QzP);
    XNB=I(YNB*QzPNB);

    if iter==1
        tildebout=tildeb;
        Sout=S;
```

```
            Vout=V;
            Xout=X;
            uout=u;
            Duout=Du;
            QzPout=QzP;

            SoutNB=SNB;
            VoutNB=VNB;
            XoutNB=XNB;
            uoutNB=uNB;
            QzPoutNB=QzPNB;
        end
        a=QzP(N)*(subs(t.*diff(I(t),t),Y*QzP(N)));
        b=QzP(N)*(subs(t.*diff(I(t),t)+I(t),Y*QzP(N)));

        aNB=QzPNB(N)*(subs(t.*diff(I(t),t),YNB*QzPNB(N)));
        for i=1:N
            stochint=zeros(n_s,1);

            for k=i+1:N
                stochint=stochint+Du(:,:,i,k)*tildeb(:,k);
            end
            CEarg(:,i,iter)=-a/QzPout(i)*uout(:,i)-b/
               QzPout(i)*(Duout(:,:,i,i)*tildebout(:,i)+
               stochint);
```

```
                CEargNB(:,i,iter)=-aNB/QzPoutNB(i)*uoutNB(:,i)
                    ;
            end
        end
        piout=squeeze(mean(CEarg,3));

        pioutNB=squeeze(mean(CEargNB,3));
end
```

A.2 MDMC

```
function [S,SNB,V,VNB,X,XNB,pi,piNB,Save,Vave,Xave,piave,
    SaveNB,VaveNB,XaveNB,piaveNB,u,uNB,alpha,alphaNB,QzP,
    QzPNB,supnorm,simtime]=MDMC(n,mu,eta,sigma,rho,gamma,T,
    delta,s,v,beta,I,x,maxiter)
      N=floor(T/delta);
      S=zeros(length(s),N,n); V=zeros(length(v),N,n); X=
          zeros(N,n);pi=zeros(size(rho,2),N,n);u=zeros(size(
          rho,2),N,n);
      SNB=S;VNB=V;XNB=X;piNB=pi;uNB=u;alpha=zeros(length(s),
          N,n);alphaNB=alpha;
      QzP=zeros(N,n);QzPNB=QzP;
      supnorm=zeros(size(s));
      tic;
      for t=1:n
```

```
        [S(:,:,t),SNB(:,:,t),V(:,:,t),VNB(:,:,t),X(:,t),XNB(:,t),pi(:,:,t),piNB(:,:,t),u(:,:,t),uNB(:,:,t),QzP(:,t),QzPNB(:,t)]=PriceSimulator(mu,eta,sigma,rho,gamma,T,delta,s,v,beta,I,x,maxiter);
    end
    simtime=toc;
    Save=mean(S,3);
    Vave=mean(V,3);
    Xave=mean(X,2);
    piave=mean(pi,3);

    SaveNB=mean(SNB,3);
    VaveNB=mean(VNB,3);
    XaveNB=mean(XNB,2);
    piaveNB=mean(piNB,3);

    for i=1:N
        for j=1:n
            alpha(:,i,j)=diag(S(:,i,j).^(-1-beta))*(sigma*sigma')\(sigma*pi(:,i,j));
            alphaNB(:,i,j)=diag(SNB(:,i,j).^-1)*(sigma*sigma')\(sigma*piNB(:,i,j));
        end
    end
    alphaave=mean(alpha,3);
```

```
    alphaaveNB=mean(alphaNB,3);

    for i=1:length(s)
        supnorm(i)=max(max((alphaNB(i,:,:)-alpha(i,:,:))));
        figure;
        plot(delta:delta:N*delta,alpha(i,:,1),'bl-*',delta:delta:N*delta,alphaNB(i,:,1),'r-o',delta:delta:N*delta,alphaNB(i,:,1)-alpha(i,:,1),'g-x');
        title(strcat('$\beta=$',num2str(beta(i)),', $N=$',num2str(N),', $p=$',num2str(log(2)/log(I(2))+1),' asset=',num2str(i)),'fontsize',12,'Interpreter','latex');
        h = legend('$\alpha^B$','$\alpha^{NB}$','$\alpha^{NB}-\alpha^B$','location','Best');
        set(h,'fontsize',12,'Interpreter','latex');
        figure;
        plot(delta:delta:N*delta,alphaave(i,:),'bl-*',delta:delta:N*delta,alphaaveNB(i,:),'r-o',delta:delta:N*delta,alphaaveNB(i,:)-alphaave(i,:),'g-x');
        title(strcat('Average trading strategies $\beta=$',num2str(beta(i)),', $N=$',num2str(N),', $p=$',num2str(log(2)/log(I(2))+1),' asset=',num2str(i),' $n=$',num2str(n)),'fontsize',12,'Interpreter','latex');
```

```
        h = legend('$\bar{\alpha}^B$','$\bar{\alpha}^{NB}$','$\bar{\alpha}^{NB}-\bar{\alpha}^B$','location','Best');
        set(h,'fontsize',12,'Interpreter','latex');
    end
    savefile=strcat('BubbleSimulationData',num2str(fix(clock)));
    save(savefile);
end
```

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