

On Irregular  $\frac{1}{2^n}$  Fractions of a  $2^m$  Factorial

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ABSTRACT

A rigorous definition of regular and irregular fractional replicates from an  $s^m$  factorial is presented. This definition is based on geometric concepts and is in agreement with group theory. Defining contrasts and aliasing structures for any  $2^{-n}$  fraction of a  $2^m$  factorial are developed utilizing two theorems presented in the paper. An algorithm for obtaining the defining contrast is presented and illustrated with specific examples.

# On Irregular $\frac{1}{2^n}$ Fractions of a $2^m$ Factorial

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## 0. Summary

A rigorous definition of regular and irregular fractional replicates from an  $s^m$  factorial is presented. This definition is based on geometric concepts and is in agreement with group theory. Defining contrasts and aliasing structures for any  $2^{-n}$  fraction of a  $2^m$  factorial are developed utilizing two theorems presented in the paper. An algorithm for obtaining the defining contrast is presented and illustrated with specific examples.

## 1. Introduction

Consider the full replicate of a  $s^m$  factorial ( $s$  is a prime or power of a prime), then we know that the  $s^m$  treatment combinations form the points of the finite Euclidean geometry  $EG(m, s)$  and that the  $\frac{s^m - 1}{s - 1}$  effects are in 1:1 correspondence with the points of the finite geometry  $PG(m-1, s)$ . Both of these geometries are based on the Galois field  $GF(s)$ . A fraction or fractional replicate is then simply a subset of points of  $EG(m, s)$  at which a response is observed. Regular fractions have been defined in the literature in various ways. For example, Addelman [1961] defined a regular fraction as a fraction which leads to orthogonal estimates of effects, i.e. the covariance matrix of the estimated effects is a diagonal matrix and Banerjee and Federer [1963] call regular fractions those obtained by completely confounding one or more effects with the mean. Both these definitions are not completely satisfactory and in

the section below, we will furnish a definition, which is geometric in nature and hence is in complete agreement with group theory and confounding.

## 2. Some basic equations and definitions

When a full replicate of a  $s^m$  factorial is observed, then it is well known that the set of normal equations corresponding to this is:

$$(1) \quad XB = Y$$

where  $X$  is an  $s^m \times s^m$  orthogonal matrix in the sense that  $X'X$  is a diagonal matrix,  $\beta$  is the  $s^m \times 1$  column vector of single degree of freedom parameters with  $\mu$  as its first element and  $Y$  is the  $s^m \times 1$  column vector of observations. Note that when  $s=2$ ,  $X$  is the Hadamard matrix  $H_{2^m} = H_2 \otimes H_2 \otimes \dots \otimes H_2$  with  $H_2 = \begin{bmatrix} + & - \\ + & + \end{bmatrix}$  and  $\otimes$  denoting Kronecker product. Let  $Y_R$  denote a fraction consisting of  $t$  observations, then the set of normal equations corresponding to this fraction is:

$$(2) \quad X_R \beta = Y_R$$

where  $X_R$  is simply read off from  $X$ . Next consider the partitioned system

$$(3) \quad [X_{R1} : X_{R2}] \begin{bmatrix} \beta_R \\ \beta_0 \end{bmatrix} = [Y_R]$$

where  $\beta_R$  is the  $t \times 1$  vector of parameters (with  $\mu$  as its first element) such that  $X_{R1}$  is invertible. Solving this system results in:

$$(4) \quad \beta_R + X_{R1}^{-1} X_{R2} \beta_0 = X_{R1}^{-1} Y_R$$

In the usual case of fractional replication  $\beta_R$  refers to the set of parameters to be estimated from the observation vector  $Y_R$ . We now are ready for the following definitions:

Define the first element of  $\beta_R + X_{R1}^{-1}X_{R2}\beta_0$  as a defining contrast of the fraction  $Y_R$  and the whole vector  $\beta_R + X_{R1}^{-1}X_{R2}\beta_0$  as a aliasing scheme or structure of  $Y_R$ . Now, we know that an  $(m-n)$ -flat,  $n \leq m$  of  $EG(m,s)$  consists of  $s^{m-n}$  points. Hence, define a fraction  $Y_R$  to be regular if it is of type  $\frac{1}{s^n}$  and if the corresponding subset forms an  $(m-n)$ -flat of  $EG(m,s)$ . [More simply  $Y_R$  is regular if it is observed at an  $(m-n)$ -flat of  $EG(m,s)$ ].

The definition given above implies that a  $\frac{1}{s^n}$  fraction will be regular if the corresponding subset in  $EG(m,s)$  forms all the points of a combination of levels of the generators of a  $(n-1)$ -flat of  $PG(m-1,2)$ . This implication gives us immediately the group theoretic confounding associated with the fraction, in agreement with lattice or incomplete block design theory. It is now clear that not every  $\frac{1}{s^n}$  fraction is regular. Consider the  $\frac{1}{2}$  fraction  $\{000,100,010,001\}$  of the  $2^3$  factorial; this fraction is not a 2-flat of  $EG(3,2)$  and hence it is irregular. This example shows also that the definition given by Addelman [1961] is not complete, since the following solution:

$$(5) \quad \begin{bmatrix} \mu & -A/4 & -B/4 & -C/4 & +ABC/4 \\ AB/2 & -A/4 & -B/4 & +C/4 & -ABC/4 \\ AC/2 & -A/4 & +B/4 & -C/4 & -ABC/4 \\ BC/2 & +A/4 & -B/4 & -C/4 & -ABC/4 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} Y_{000} \\ Y_{100} \\ Y_{010} \\ Y_{001} \end{bmatrix}$$

$\beta_R + X_{R1}^{-1}X_{R2}\beta_0$ 
 $X_{R1}^{-1}$ 
 $Y_R$

has a covariance matrix which is obviously diagonal, but the fraction is irregular. In this same example, had we made another choice of  $X_{R1}$  or equivalently of  $\beta_R$ , we would not achieve a solution with a diagonal covariance matrix. As another example consider the fraction  $\{0000,1001,1101,1111\}$  from a  $2^4$  factorial. This fraction is irregular, since it does not form all the points of  $(AD)_0$  nor does it form a 2-flat of  $EG(4,2)$ . In the following section we will investigate the irregular fractions of type  $\frac{1}{2^n}$  of  $2^m$  factorials, thereby establishing their defining contrasts.

2. Two theorems concerning irregular fractions of type  $\frac{1}{2^n}$

Before we go to the main results of this section, let us recall certain aspects of the  $2^n$  series by going through a specific example. Consider, therefore, the  $2^3$  factorial with the system:

$$(6) \quad \begin{bmatrix} + & - & - & + & - & + & + & - \\ + & + & - & - & - & - & + & + \\ + & - & + & - & - & + & - & + \\ + & + & + & + & - & - & - & - \\ + & - & - & + & + & - & - & + \\ + & + & - & - & + & + & - & - \\ + & - & + & - & + & - & + & - \\ + & + & + & + & + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} \mu \\ A/2 \\ B/2 \\ AB/2 \\ C/2 \\ AC/2 \\ BC/2 \\ ABC/2 \end{bmatrix} = \begin{bmatrix} Y_{000} \\ Y_{100} \\ Y_{010} \\ Y_{110} \\ Y_{001} \\ Y_{101} \\ Y_{011} \\ Y_{111} \end{bmatrix}$$

$X \quad \cdot \quad \beta \quad = \quad Y$

It is clear that each row vector of  $X$  times  $\beta$  is equal to an element of  $Y$ , for example,  $Y_{000} = (+ - - + - + + -)\beta$ . In fact, the defining contrast of any observation is given by such an inner product. Since two observations

always form an  $1$ -flat of  $EG(m,2)$  and hence always form a regular fraction, we know that its defining contrast is obtained by averaging the defining contrasts of two observations involved. Thus for example, the defining contrast for  $\{000,100\}$  is  $\frac{1}{2}\{(+ - - + - + + -) + (+ + - - - - + +)\}\beta = \frac{1}{2}(2 0 - 2 0 - 2 0 2 0)\beta$ , i.e.  $\mu - B/2 - C/2 + BC/2$ . This process of averaging can be carried out for any regular fraction, so that, for example, for the four observations  $\{100,010,001,111\}$  one obtains the defining contrast  $\frac{1}{4}\{(+ + - - - - + +) + (+ - + - - + - +) + (+ - - + + - - +) + (+ + + + + + + +)\}\beta = \frac{1}{4}(4 0 0 0 0 0 0 4)\beta = \mu + ABC/2$ . More generally, the defining contrast for any regular fraction is merely  $\frac{\beta^r}{2^{m-r}} \cdot \sum_{j \in J} X_j$ , where  $X_j^r$  refers to  $j^{th}$  row vector in  $X$  corresponding to the  $j^{th}$  observation of the fraction indexed by  $J$ .

In section 2 we presented an example of a  $\frac{1}{2^n}$  fraction, which was irregular. For the  $2^3$  case the only irregular fraction of type  $\frac{1}{2^n}$  is that consisting of 4 observations, selected in such a manner that they do not form a 2-flat or plane of  $EG(3,2)$ . For the  $2^4$  factorial, irregular fractions of type  $\frac{1}{2^n}$  can occur only for the 4 and 8 case and for a general  $2^m$  factorial we will have  $(m-2)$  irregular cases of type  $\frac{1}{2^n}$ . Now, the question naturally arises as to whether these irregular fractions can be characterized by their defining contrasts. This question can be answered in the affirmative, if certain conditions are taken into account. The following two theorems will be needed in the development of defining contrasts for irregular fractions of type  $\frac{1}{2^n}$ :

Theorem 1:

For irregular fractions of type  $\frac{1}{2^n}$ ,  $n \leq m-2$ , it is always possible to choose  $X_{R1}$  in the equation system  $[X_{R1}:X_{R2}] \begin{bmatrix} \beta_R \\ \beta_0 \end{bmatrix} = Y_R$  as a Hadamard matrix and the resulting solution has then a diagonal covariance matrix  $\frac{\sigma^2}{2^{m-n}} I_{2^{m-n}}$ .

Proof:

The proof of this theorem follows immediately from the following result in Hadamard matrix theory: "If  $2^{m-n}$  rows of a Hadamard matrix of order  $2^m$  are taken, then it is always possible to form a Hadamard matrix of order  $2^{m-n}$ ." The proof is straightforward. Applying this result by taking  $X_{R1}$  as the Hadamard matrix  $G_{2^{m-n}}$  we have the solution:

$$(7) \quad \beta_R + \frac{1}{2^{m-n}} G'_{2^{m-n}} X_{R2} \beta_0 = \frac{1}{2^{m-n}} G'_{2^{m-n}} Y_{2^{m-n}}$$

with covariance matrix:

$$(8) \quad \left( \frac{1}{2^{m-n}} G'_{2^{m-n}} \right) \sigma^2 I_{2^{m-n}} \left( \frac{1}{2^{m-n}} G'_{2^{m-n}} \right)' = \frac{1}{(2^{m-n})^2} 2^{m-n} \sigma^2 I = \frac{\sigma^2}{2^{m-n}} I_{2^{m-n}}$$

which proves the theorem. Note that when the covariance matrix is diagonal, we need not have a regular fraction.

Theorem 2:

If we are given an irregular fraction of type  $\frac{1}{2^n}$  of an  $2^m$  factorial and if  $X_{R1}$  is selected as in Theorem 1, then the defining contrast of the fraction is  $\frac{\beta'}{2^{m-n}} \cdot \sum_{j \in J} X_j$ , where  $X_j'$  refers to the row of  $X$  corresponding to the  $j^{th}$  observation in the fraction as indexed by  $J$ .

Proof:

If  $X_{R1}$  is chosen as the Hadamard matrix  $G_{2^{m-n}}$  in the partitioned system  $[X_{R1}:X_{R2}] \begin{bmatrix} \beta_R \\ \beta_0 \end{bmatrix} = Y_R$  then the resulting solution is as given by Theorem 1, namely:

$$\beta_R + \frac{1}{2^{m-n}} G'_{2^{m-n}} X_{R2} \beta_0 = \frac{1}{2^{m-n}} G'_{2^{m-n}} Y_{2^{m-n}}$$

Now, since  $\mu$  is the first element of  $\beta_R$ , we know that the first row (and first column) of  $G_{2^{m-n}}$  consists of one's only, i.e. the first row is  $(1 \ 1 \ \dots \ 1)_{2^{m-n}}$ . Hence, the defining contrast is:

$$\begin{aligned}
 & \mu + \frac{1}{2^{m-n}} (1 \ 1 \ \cdots \ 1) X_{R2} \beta_0 \\
 &= \frac{1}{2^{m-n}} [2^{m-n} \mu + (1 \ 1 \ \cdots \ 1) X_{R2} \beta_0] \\
 &= \frac{1}{2^{m-n}} [\mu (1 \ 1 \ \cdots \ 1) (1 \ 1 \ \cdots \ 1)' + (1 \ 1 \ \cdots \ 1) X_{R2} \beta_0] \\
 &= \frac{1}{2^{m-n}} [(1 \ 1 \ \cdots \ 1) G_{2^{m-n}} \beta_R + (1 \ 1 \ \cdots \ 1) X_{R2} \beta_0] \\
 &= \frac{1}{2^{m-n}} (1 \ 1 \ \cdots \ 1) X_{R1} \beta_R + (1 \ 1 \ \cdots \ 1) X_{R2} \beta_0 \\
 &= \frac{1}{2^{m-n}} (1 \ 1 \ \cdots \ 1) X_R \beta \\
 &= \frac{\beta'}{2^{m-n}} \sum_{j \in J} X_j
 \end{aligned}$$

which proves the theorem.

4. Some examples of defining contrasts for irregular  $\frac{1}{2^n}$  fractions

Consider irregular fractions from the  $2^4$  factorial; these consist either of 4 or 8 observations. Taking, for example, the observations {0000, 1001, 1101, 1111} for the  $\frac{1}{2^2}$  irregular fraction, we see that the defining contrast according to theorem 2 is:

$$\frac{\beta'}{2^2} \sum_{j \in J} X_j = \frac{1}{4} (1 \ 1 \ 1 \ 1) \begin{bmatrix} + & - & + & - & + & + & - & - & + & + & - & + & - & - & + \\ + & + & - & - & - & - & + & + & + & + & - & - & - & - & + \\ + & + & + & + & - & - & - & - & + & + & + & + & - & - & - \\ + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \end{bmatrix} \begin{bmatrix} \beta \end{bmatrix}$$



$$\begin{aligned}
 &= \frac{1}{4}(4 \ 2 \ 0 \ 2 \ -2 \ 0 \ 2 \ 0 \ 2 \ 4 \ 2 \ 0 \ 0 \ -2 \ 0 \ 2)\beta \\
 &= \mu + \frac{1}{4}A + \frac{1}{4}AB - \frac{1}{4}C + \frac{1}{4}BC + \frac{1}{4}D + \frac{1}{2}AD + \frac{1}{4}BD - \frac{1}{4}ACD + \frac{1}{4}ABCD .
 \end{aligned}$$

The aliasing or confounding structure is obtained in the usual way by using group theoretic multiplication of the defining contrast, i.e.

$$\begin{aligned}
 \mu &+ \frac{1}{4}A + \frac{1}{4}AB - \frac{1}{4}C + \frac{1}{4}BC + \frac{1}{4}D + \frac{1}{2}AD + \frac{1}{4}BD - \frac{1}{4}ACD + \frac{1}{4}ABCD \\
 B/2 &+ \frac{1}{4}A + \frac{1}{4}AB + \frac{1}{4}C - \frac{1}{4}BC + \frac{1}{4}D + \frac{1}{2}ABD + \frac{1}{4}BD + \frac{1}{4}ACD - \frac{1}{4}ABCD \\
 AC/2 &- \frac{1}{4}A + \frac{1}{4}AB + \frac{1}{4}C + \frac{1}{4}BC - \frac{1}{4}D + \frac{1}{2}CD + \frac{1}{4}BD + \frac{1}{4}ACD + \frac{1}{4}ABCD \\
 ABC/2 &+ \frac{1}{4}A - \frac{1}{4}AB + \frac{1}{4}C + \frac{1}{4}BC + \frac{1}{4}D + \frac{1}{2}BCD - \frac{1}{4}BD + \frac{1}{4}ACD + \frac{1}{4}ABCD
 \end{aligned}$$

Since this fraction forms a half of  $(AD)_0$ , we know immediately that the mean will be completely confounded with AD, B with ABD, AC with CD and ABC = BCD, i.e. in conventional notation:

$$\begin{aligned}
 I &= AD \\
 B &= ABD \\
 AC &= CD \\
 ABC &= BCD
 \end{aligned}$$

This fact is reflected in the above aliasing structure by noting that the coefficients of these effects are  $\frac{1}{2}$ . Coefficients other than  $\frac{1}{2}$  denote partial confounding or fractional aliasing (Federer (1964)). Going back to the original matrix  $X_R$  we see that complete confounding is reflected by the fact that columns corresponding to  $\mu$  and AD, B and ABD, AC and CD, ABC and BCD are identical. Hence, we may conclude that complete confounding in irregular fractions

will occur if and only if columns in  $X_R$  coincide. Also, it is evident that irregular fractions with complete confounding can occur if and only if  $n \neq 1$ , when  $X_{R1}$  is chosen as in Theorem 1.

Finally consider the  $\frac{1}{2}$  fraction  $\{0000, 1000, 0100, 1100, 0010, 1010, 0110, 0001\}$ . Obviously, this fraction is irregular and complete confounding will not occur here. The defining contrast according to Theorem 2 is:

$$\frac{\beta'}{2^3} \sum_{j \in J} X_j = \frac{1}{8} (1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1)$$

+	-	-	+	-	+	+	-	-	+	+	-	+	-	-	+
+	+	-	-	-	-	+	+	-	-	+	+	+	+	-	-
+	-	+	-	-	+	-	+	-	+	-	+	+	-	+	-
+	+	+	+	-	-	-	-	-	-	-	-	+	+	+	+
+	-	-	+	+	-	-	+	-	+	+	-	-	+	+	-
+	+	-	-	+	+	-	-	-	-	+	+	-	-	+	+
+	-	+	-	+	-	-	-	+	-	+	-	-	+	-	+
+	-	-	+	-	+	+	-	-	+	-	-	+	-	+	-

$$= \frac{1}{8} (8 \ -2 \ -2 \ 0 \ -2 \ 0 \ 0 \ -2 \ -6 \ 0 \ 0 \ 2 \ 0 \ 2 \ 2 \ 0) \beta$$

$$= \mu - \frac{1}{8}A - \frac{1}{8}B - \frac{1}{8}C - \frac{1}{8}ABC - \frac{3}{8}D + \frac{1}{8}ABD + \frac{1}{8}ACD + \frac{1}{8}BCD$$

The aliasing or confounding structure is then found by group theoretic multiplication as above.

From the above examples one has the impression that when  $m$  is large and  $n$  small the problem of addition of the rows of  $X_R$  becomes quite formidable. But this is not the case if one recognizes the relationship between the treatments and the effects as illustrated for the above example.

1. Always use 1 as coefficient for  $\mu$  in the defining contrast.
2. The coefficients of the main effects are obtained as follows; add the eight subscripts for each factor, thus  $(0000 + 1000 + 0100 + 1100 + 0010$

+ 1010 + 0110 + 0001) = (-2 -2 -2 -6), where we use the function 0 = -1 and 1 = 1. Hence, the coefficients for the main effects are  $\frac{1}{16} (-2 -2 -2 -6)$ .

3. Using the function in 2, we have as coefficient for AB  $\frac{1}{16} \{(-1)(-1) + 1(-1) + (-1)1 + 1(1) + (-1)(-1) + 1(-1) + (-1)1 + (-1)(-1)\} = \frac{1}{16}(0) = 0$ . In the same fashion we derive the coefficients of the remaining two factor interactions, three factor interactions and the four factor interactions. All we have to do is use the proper indicator function.

#### References

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