COMMUTATIVE AND HOMOLOGICAL ALGEBRA OF INCOMPLETE TAMBARA FUNCTORS

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COMMUTATIVE AND HOMOLOGICAL ALGEBRA OF INCOMPLETE TAMBARA FUNCTORS

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Equivariant algebra is the study of algebra in the category of Mackey functors. In this setting, Mackey functors play the role of abelian groups and incomplete Tambara functors play the role of commutative rings. The study of incomplete Tambara functors parallels the study of classical commutative algebra in many ways, but there are some striking differences as well. We develop aspects of the homological and commutative algebra of incomplete Tambara functors in this thesis.

One of the most notable differences is the fact that free incomplete Tambara functors often fail to be free as Mackey functors, even though free algebras (polynomial rings) are always free as modules. We provide conditions under which a free incomplete Tambara functor is flat as a Mackey functor. When the group is solvable, we show that a free incomplete Tambara functor is flat precisely when these conditions hold. Our results imply that free incomplete Tambara functors are almost never flat as Mackey functors. However, we show that after suitable localizations, free incomplete Tambara functors are always free as Mackey functors.

BIOGRAPHICAL SKETCH

David Mehrle was born Columbus, Ohio on January 12, 1993. He graduated in 2015 from Carnegie Mellon University with both a Master of Science and Bachelor of Science in Mathematics. David earned a Master of Advanced Study from Cambridge University in 2016, and a Master of Science in Mathematics from Cornell University in 2017.

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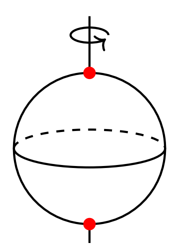
CHAPTER 1

INTRODUCTION

Equivariant homotopy theory is the study of algebraic invariants of spaces or spectra equipped with a group action. It has been instrumental in several recent advances in algebraic topology, such as Hill–Hopkins–Ravenel's solution to the Kervaire invariant one problem [HHR16] and calculations of algebraic K-theory via trace methods [BoHM93, HM03, NS18]. Here, we will consider only finite groups G.

Invariants of G-spaces should distinguish between different group actions on the same space. For example, the cyclic group of order three can act on a 2-sphere in two different ways. Either it does nothing, or it rotates the sphere like a globe through an angle of 120°. The homotopy groups of the sphere can't tell the difference between the different actions, but the homotopy groups of the fixed points can. The fixed points of the trivial action are the whole sphere, while the fixed points of rotation are the north and south poles.



The cyclic group C_3 acts on S^2 by rotation with fixed points S^0 (in red).

Here's some intuition for why the fixed points show up. Non-equivariantly, picking a point in a space X can be done by choosing a map from a point into X. There is a homeomorphism between the mapping space of all such maps and the space itself:

$$\mathrm{Map}(\{*\}, X) \cong X.$$

Equivariantly, whenever you pick a point in a G-space X, you must say how the group acts on that point. Instead of specifying a point in X, this amounts to specifying an entire G-orbit. We say that "orbits are the points of equivariant homotopy theory." Any G-orbit in X is homeomorphic to a coset space G/H for some subgroup $H \subseteq G$, and can be picked out by a G-equivariant map from the coset space G/H to our space X. The mapping space of all such G-equivariant maps is homeomorphic to the H-fixed points of X:

$$\mathrm{Map}^G(G/H, X) \cong X^H. \tag{1.0.1}$$

Since orbits are the basic building blocks of equivariant homotopy theory, we should take care to keep track of the morphisms between them. To do so, we form the G-*orbit category* $\mathcal{O}\mathrm{rb}_G$, whose objects are the coset spaces G/H and whose morphisms are all G-equivariant maps between them. The collection of fixed point spaces for a G-space X can be encoded in the functor

$$\mathrm{Map}^G(-, X)\colon \mathcal{O}\mathrm{rb}_G^{\mathrm{op}} \to \mathbf{Top}.$$

On objects, this functor takes the orbit G/H to the H-fixed points of the space X by (1.0.1). A morphism $G/H \to G/K$ in $\mathcal{O}\mathrm{rb}_G$ becomes an inclusion of fixed points $X^H \hookrightarrow X^K$. In fact, all of the homotopy theory of G-spaces is contained in such functors: Elmendorf's Theorem [Elm83] says that the category of G-spaces and the category of contravariant functors from the orbit category to topological spaces have the same homotopy theory.

Post-composing one of these functors $\mathrm{Map}^G(-, X)$ with a homotopy group functor π_n gives us the correct invariant for equivariant homotopy theory, called a *homotopy coefficient system* of X. It consists of the homotopy groups of all of the fixed point spaces of X, as well as the homomorphisms induced on homotopy groups by the inclusions of fixed point spaces. Generically, a *coefficient system* is a contravariant functor from the orbit category to abelian groups. The name comes from the fact that coefficient systems are the coefficients for equivariant cohomology theories.

Another example of a coefficient system is the fixed points of a G-module M. We will call this coefficient system $\underline{M}$; as a functor, $\underline{M}$ takes an orbit G/H to the H-fixed points of M and takes morphisms of $\mathcal{O}rb_G$ to inclusions of fixed points. There is more structure here than just the data of a coefficient system. From the additive structure of M, we get homomorphisms in the other direction to the inclusions. For subgroups $K \subseteq H \subseteq G$, a K-fixed point $m \in M^K$ becomes an H-fixed point by summing over the action by the K-cosets of H:

$$\begin{aligned} M^K &\longrightarrow M^H \\ m &\longmapsto \sum_{gK \in H/K} g \cdot m. \end{aligned} \tag{1.0.2}$$

This is our first example of a *Mackey functor*, called the *fixed point functor* of the G-module M.

We define Mackey functors in Section 2.1, but here is another equivalent definition. A *Mackey functor* [Dre73] is a pair of functors $\underline{M} = (\underline{M}_*, \underline{M}^*)$ subject to the following conditions. Both $\underline{M}_*$ and $\underline{M}^*$ are functors from the orbit category to abelian groups, but $\underline{M}_*$ is covariant and $\underline{M}^*$ is contravariant. These two functors must agree on objects; we write $\underline{M}(U)$ for the common value $\underline{M}^*(U) = \underline{M}_*(U)$. This must satisfy a formula analogous to the double coset formula. In a Mackey functor, the contravariant morphisms $\underline{M}^*(f)$ are called *restrictions*, and the covariant morphisms $\underline{M}_*(g)$ are called *transfers.* A morphism of Mackey functors from $\underline{M}$ to $\underline{N}$ is a natural transformation that works for both the covariant and contravariant functors simultaneously. We write $\mathcal{M}ack_G$ for the category of G-Mackey functors.

Examples of Mackey functors are abundant. Group cohomology, homology, and Tate cohomology can be expressed as Mackey functors. The representation ring of any finite group is a Mackey functor in which transfers are induction and restriction is restriction of representations to a subgroup. The earlier example of the fixed points of any G-module is the Mackey functor for the zeroth group cohomology. A class of particularly important examples are the homotopy Mackey functors of a G-spectrum.

Because Mackey functors play the role of abelian groups in equivariant stable homotopy theory, we tend to think of them as algebraic objects instead of functors. This is borne out in the properties of the category of G-Mackey functors. The category $\mathcal{M}\mathrm{ack}_G$ is abelian, so it makes sense to do homological algebra and to talk about free, projective, and flat Mackey functors. There is a symmetric monoidal product on $\mathcal{M}\mathrm{ack}_G$, called the *box product*. When G is the trivial group, the category of G-Mackey functors is equivalent to the category of abelian groups, and the box product is the tensor product over $\mathbb{Z}$. In this sense, the category of Mackey functors generalizes the category of abelian groups.

Commutative monoids for the box product are called *Green functors*. Green functors are in many ways analogous to commutative rings. Loosely speaking, a Mackey functor $\underline{M}$ is a Green functor when $\underline{M}(U)$ is a commutative ring for all finite G-sets U, suitably compatible with the other data of $\underline{M}$. If R is a commutative ring with an action of G by ring homomorphisms, then the fixed point Mackey functor of R is a Green functor. Write $\underline{R}$ for this fixed point Green functor.

This fixed point functor has more structure. In the fixed point Mackey functor for a G-module M, we obtained transfer homomorphisms (1.0.2) from the additive structure of the G-module. In the fixed point Green functor $\underline{R}$, there is also a morphism that comes from the multiplicative structure. For subgroups

$K \subseteq H \subseteq G$, a K-fixed point $r \in R^K$ becomes an H-fixed point by multiplying over the action by K-cosets of H:

$$\begin{aligned} R^K &\longrightarrow R^H \\ r &\longmapsto \prod_{gK \in H/K} g \cdot r. \end{aligned}$$

We call this these the *norm morphisms*. They are extra structure that is not present in the definition of Green functors as commutative monoids for the box product.

Just as cohomology rings are more powerful invariants than cohomology groups, the extra structure of norms makes analysis easier. A Green functor $\underline{R}$ together with norm morphisms $nm_K^H \colon \underline{R}(G/K) \to \underline{R}(G/H)$ compatible with the rest of the data of $\underline{R}$ is called a *Tambara functor* [Tam93]. A Tambara functor is another kind of commutative ring in the category of Mackey functors. Tambara functors are the more honestly equivariant object, insofar as they are the monoids for an equivariant symmetric monoidal structure on the category of Mackey functors [HM19].

Sometimes we don't have all of the norm morphisms, but only some of them. An example of this situation is Bousfield localization of genuine equivariant ring spectra, which does not necessarily preserve all of the norms [HH16]. It is useful to remember which ones we do have. A Green functor with some, but not all, of the norms is called an *incomplete Tambara functor* [BH18]. The norms that are present in an incomplete Tambara functor are parameterized by an N_∞*-operad* [BH15] – the N is for "norms".

N_∞-operads for a finite group G are equivariant generalizations of E_∞-operads (commutative operads), in the sense that an N_∞-operad is a G-operad whose underlying non-equivariant operad is an E_∞-operad. However, while there is a unique homotopy class of E_∞-operads, there is a finite lattice of homotopy classes of N_∞-operads ordered by which norms are present. This lattice has several combinatorial interpretations which we explore in Section 2.2. The greatest element of this lattice is the N_∞-operad corresponding to Tambara functors, with all norms. The least element is the operad corresponding to Green functors, with no norms. In other words, a Green functor is the most naïve way to give a Mackey functor equivariant multiplicative structure.

Any incomplete Tambara functor $\underline{R}$ is a commutative ring-like object in the category of Mackey functors – it is an N_∞-algebra in $\mathcal{M}ack_G$. It makes sense to talk about modules over $\underline{R}$, prime and maximal ideals of $\underline{R}$ [Nak12a], and to localize $\underline{R}$ at a multiplicatively closed system [Nak12b]. However, the commutative and homological algebra of incomplete Tambara functors has not been systematically pursued.

This thesis develops aspects of the algebra of incomplete Tambara functors, which we term *equivariant algebra*. A large portion is devoted to understanding the free objects in the category of incomplete Tambara functors, which are analogous to polynomial rings in ordinary commutative algebra. Here, intuition from commutative algebra does not translate well to the equivariant setting – Lewis calls the free Green functors "strange and beautiful beasts full of mystery" [Lew81]. We are inclined to agree.

CHAPTER 2

EQUIVARIANT ALGEBRA

Equivariant algebra is the study of algebra in the category of Mackey functors. We think of Mackey functors as the equivariant counterpart to abelian groups, and the various N_∞-algebras in this category – Green functors, Tambara functors, and incomplete Tambara functors – are the equivariant counterpart to commutative rings.

2.1 Mackey functors

In equivariant algebra, Mackey functors play the role of abelian groups.

Definition 2.1.1. The *Burnside category* $\mathcal{A}^G$ is the category of spans of finite G-sets. Objects are finite G-sets, and a morphism in $\mathcal{A}^G(X, Y)$ is an isomorphism class of spans

$$[X \leftarrow A \rightarrow Y],$$

where two spans $X \leftarrow A \rightarrow Y$ and $X \leftarrow B \rightarrow Y$ are isomorphic if there is a bijection $A \rightarrow B$ such that the diagram below commutes:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & \downarrow\cong & \searrow & \\ X & & & & Y. \\ & \nwarrow & & \nearrow & \\ & & B & & \end{array}$$

Composition is given by pullback.

The Burnside category $\mathcal{A}^G$ is a pre-additive category with direct sums given by disjoint union of finite G-sets. In particular, each hom-set $\mathcal{A}^G(X, Y)$ is a commutative monoid.

Any morphism in $\mathcal{A}^G$ can be written as a composite $T_h \circ R_f$ where T_h and R_f are the morphisms

$$R_f := [X \xleftarrow{f} Y \xrightarrow{id} Y] \quad \text{and} \quad T_h := [Y \xleftarrow{id} Y \xrightarrow{h} Z].$$

Definition 2.1.2. A *semi-Mackey functor* is a product-preserving functor

$$\underline{M}\colon \mathcal{A}^G \to \mathcal{S}\mathrm{et}.$$

A semi-Mackey functor $\underline{M}$ is a *Mackey functor* if $\underline{M}(T)$ is an abelian group for each finite G-set T.

A *morphism of semi-Mackey functors* is a natural transformation. We write $\mathcal{M}\mathrm{ack}_G$ for the full subcategory of semi-Mackey functors spanned by the Mackey functors for the group G. In particular, a *morphism of Mackey functors* is a natural transformation.

In equivariant algebra, semi-Mackey functors are analogous to commutative monoids while Mackey functors play the role of abelian groups.

A Mackey functor $\underline{M}$ is a product-preserving functor, so its value on any finite G-set X is determined by its values on the orbits of X. Moreover, because any morphism in $\mathcal{A}^G$ may be written as a composite $T_h \circ R_f$, it suffices to define a Mackey functor on morphisms of the form T_h and R_f.

Definition 2.1.3. If $\underline{M}$ is a Mackey functor, we write $\mathrm{tr}_h := \underline{M}(T_h)$ and $\mathrm{res}_f := \underline{M}(R_f)$. These are the *transfer* and *restriction* morphisms of $\underline{M}$, respectively. In the case when $\pi\colon G/K \to G/H$ is the canonical projection, we write $\mathrm{tr}_K^H := \mathrm{tr}_\pi$ and $\mathrm{res}_K^H := \mathrm{res}_\pi$.

Proposition 2.1.4 ([Maz13, Definition 1.1.2])**.** *If* G *is an abelian group, the data of a Mackey functor is equivalent to the following:*

- *A collection of* $W_G(H)$*-modules* $\underline{M}(G/H)$*, one for each orbit* G/H*;*
- *restriction homomorphisms (of abelian groups)*

$$\mathrm{res}_K^H\colon \underline{M}(G/H) \to \underline{M}(G/K);$$

- *and transfer homomorphisms (of abelian groups)*

$$\mathrm{tr}_K^H\colon \underline{M}(G/K) \to \underline{M}(G/H).$$

These data are subject to the following conditions:

- *transfers are equivariant for the Weyl group actions in the sense that*

$$\mathrm{tr}_K^H(g \cdot x) = \mathrm{tr}_K^H(x)$$

  for all $K < H \leq G$, $g \in W_H(K) \leq W_G(K)$, *and* $x \in \underline{M}(G/K)$*;*

- *restrictions are equivariant for the Weyl group actions in the sense that*

$$g \cdot \mathrm{res}_K^H(y) = \mathrm{res}_K^H(y)$$

  for all $K < H \leq G$, $g \in W_H(K) \leq W_G(K)$, *and* $y \in \underline{M}(G/H)$*;*

- *for all subgroup inclusions* $K \leq H$, $L \leq H$,

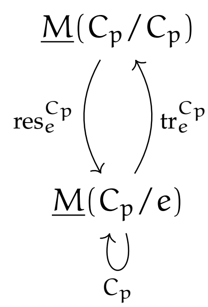
$$\mathrm{res}_L^H \mathrm{tr}_K^H(x) = \sum_{g \in W_H(L)} g \cdot \mathrm{tr}_{K \cap L}^L(x)$$

  for all $x \in \underline{M}(G/(K \cap L))$.

If G is not abelian, there is a similarly explicit definition of Mackey functors. See [Web00, Section 2].

Example 2.1.5. When G is the trivial group, a Mackey functor is an abelian group and the category of Mackey functors is equivalent to the category of abelian groups.

Example 2.1.6. When $G = C_p$ is cyclic of prime order p, a Mackey functor $\underline{M}$ is determined by the data of an abelian group $\underline{M}(C_p/C_p)$, a C_p-module $\underline{M}(C_p/e)$, and the transfer and restriction homomorphisms between them. We capture these data in a *Lewis diagram*, after [Lew88].



Example 2.1.7. The *Burnside Mackey functor* is the Mackey functor $\underline{A}$ whose value on a finite G-set X is the additive group completion of the monoid of isomorphism classes of G-sets Y over X, with monoid operation given by disjoint union.

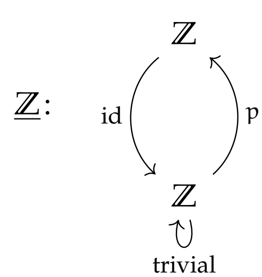
When $X = G/H$, $\underline{A}(G/H)$ is the group completion of finite H-sets under disjoint union. The restriction res_K^H is induced by the inclusion of finite K-sets into finite H-sets and the transfer tr_K^H is induction from K-sets to H-sets:

$$\mathrm{res}_K^H([Y]) = [i_K^* Y]$$
$$\mathrm{tr}_K^H([X]) = [H \times_K X].$$

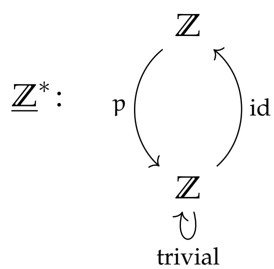
Example 2.1.8. For $G = C_p$, with p a prime, we consider the Mackey functor $\underline{\mathbb{Z}}$ defined by

$$\underline{\mathbb{Z}}(G/H) = \mathbb{Z},$$

with the restriction given by the identity, the transfer given by multiplication by p, and the Weyl group acting trivially. This is the *constant Mackey functor on* $\mathbb{Z}$, written $\underline{\mathbb{Z}}$.



We also consider the Mackey functor $\underline{\mathbb{Z}}^*$ defined identically except the transfer and identity are swapped. This is the *dual of the constant Mackey functor on* $\mathbb{Z}$, written $\underline{\mathbb{Z}}^*$.



2.1.1 Properties of $\mathcal{M}\mathrm{ack}_G$

We aim to think of Mackey functors as a kind of equivariant abelian group, so the category of Mackey functors had better be well-behaved. Fortunately, this is the case.

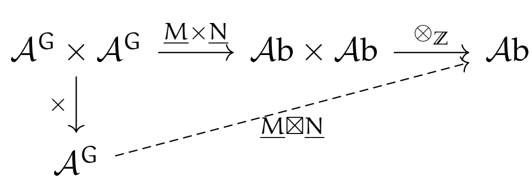
Proposition 2.1.9 ([Lew81]). *The category of Mackey functors is an abelian category.*

Not only is the category of Mackey functors abelian, but it also has a symmetric monoidal product that resembles the tensor product of abelian groups.

Definition 2.1.10 ([Lew81]). Let $\underline{M}$ and $\underline{N}$ be Mackey functors. The *box product* $\underline{M} \boxtimes \underline{N}$ is the Mackey functor obtained by left Kan extending the tensor product of abelian groups along the functor

$$\times \colon \mathcal{A}^G \times \mathcal{A}^G \to \mathcal{A}^G$$

given on objects by $(T, T') \mapsto T \times T'$.



Theorem 2.1.11 ([Lew81]). *The box product makes the category of Mackey functors into a closed symmetric monoidal category with unit the Burnside Mackey functor $\underline{A}$.*

This leads us to our first ring-like object in the category of Mackey functors.

Definition 2.1.12. A *Green functor* $\underline{R}$ is a commutative monoid in the symmetric monoidal category $(\mathcal{M}\mathrm{ack}_G, \boxtimes, \underline{A})$.

We study Green functors in more detail in Section 2.3.2.

Green functors are not the only kind of ring in the category of Mackey functors. There is also an equivariant symmetric monoidal structure on the category of Mackey functors, and monoids for the equivariant symmetric monoidal structure are called *Tambara functors* [HM19]. Tambara functors have more structure than Green functors, but both Green and Tambara functors are instances of algebras for commutative G-operads, or N_∞-operads. In general, algebras over these operads in the category of Mackey functors are called *incomplete Tambara functor*. So to study rings in the category of Mackey functors, we take a detour to study N_∞-operads.

2.2 Combinatorics of N_∞-operads

The N_∞-operads for a finite group G are equivariant generalizations of E_∞-operads (commutative operads). However, while there is a single homotopy class of E_∞-operads, there are many inequivalent homotopy classes of N_∞-operads. This leads to many different notions of commutative ring in the equivariant setting, all of which reduce to ordinary commutative rings when $G = e$. In the category of Mackey functors, algebras over N_∞-operads are called *incomplete*

Tambara functors. To study incomplete Tambara functors, we must first analyze N_∞-operads and their various combinatorial incarnations.

Fix a finite group G, and write Σ_n for the symmetric group on n letters.

Definition 2.2.1 ([LMS86, VII.§1 1.1])**.** A G*-operad* $\mathcal{O}$ is a sequence $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots$ such that $\mathcal{O}_n$ is a $G \times \Sigma_n$-space, and:

(a) there is an identity element $1 \in \mathcal{O}_1$ which is a fixed-point for the G-action on $\mathcal{O}_1$;

(b) there are G-equivariant composition maps

$$\mathcal{O}_k \times \mathcal{O}_{n_1} \times \mathcal{O}_{n_2} \times \cdots \times \mathcal{O}_{n_k} \to \mathcal{O}_{n_1+n_2+\ldots+n_k}$$

that satisfy the usual identity, associativity, and symmetry conditions for symmetric operads.

A *morphism of* G*-operads* $f\colon \mathcal{O} \to \mathcal{O}'$ is a morphism of operads such that each $f_n\colon \mathcal{O}_n \to \mathcal{O}'_n$ is $G \times \Sigma_n$-equivariant.

Definition 2.2.2 ([BH15, Definition 3.9])**.** A *weak-equivalence of* G*-operads* is a morphism $f\colon \mathcal{O} \to \mathcal{O}'$ of G-operads such that the induced morphisms of fixed point spaces $f^\Gamma\colon \mathcal{O}_n^\Gamma \to (\mathcal{O}'_n)^\Gamma$ are weak equivalences of spaces for all subgroups Γ of $G \times \Sigma_n$.

Examples of G-operads include the linear isometries operad, the equivariant little disks operad, and the Steiner operad [GM17], or the E_σ-operad that shows up in discussions of Real hochschild homology [AGH21, Section 3].

We are mostly interested in N_∞-operads for G, which are equivariant analogues of E_∞-operads (commutative operads). To define N_∞-operads, we must first define families of subgroups.

Definition 2.2.3. A *family* for a group G is a set of subgroups of G that is closed under taking subgroups and conjugation.

Definition 2.2.4. If $\mathcal{F}$ is a family for G, then a *universal space for $\mathcal{F}$* is a G-space $E\mathcal{F}$ such that for all subgroups $H \leq G$,

$$(E\mathcal{F})^H \simeq \begin{cases} * & H \in \mathcal{F}, \\ \emptyset & H \notin \mathcal{F}. \end{cases}$$

Definition 2.2.5 ([BH15, Definition 3.7])**.** An N_∞*-operad* is a G-operad $\mathcal{O}$ such that:

(a) the space $\mathcal{O}_0$ is G-contractible;

(b) the $\{e\} \times \Sigma_n \subseteq G \times \Sigma_n$-action on $\mathcal{O}_n$ is free; and

(c) for each n, $\mathcal{O}_n \simeq E\mathcal{F}_n(\mathcal{O})$ for a family $\mathcal{F}_n(\mathcal{O})$ of subgroups of $G \times \Sigma_n$ which contains all of the subgroups of the form $\{H\} \times 1$.

Let $N_\infty(G)$ be the full subcategory of the category of G-operads spanned by the N_∞-operads.

The N in the phrase "N_∞"-operad stands for "norms."

If we forget the G-action on a G-operad $\mathcal{O}$, we end up with a normal symmetric operad. When this forgetful functor is applied to an N_∞-operad, you are left with an E_∞-operad.

Proposition 2.2.6 ([BH15, Lemma 3.8])**.** *The underlying symmetric operad of an* N_∞*-operad is an* E_∞*-operad.*

Examples of N_∞-operads include the linear isometries operad, the equivariant little disks operad, and the Steiner operad [BH15, Corollary 3.14 and Lemma 3.15].

If we are to consider algebras over N_∞-operads, it will be useful to understand the homotopy category of N_∞-operads for G. This category, it turns out, can be described combinatorially in several different ways. We outline these combinatorial incarnations of N_∞-operads in the sections below.

2.2.1 Indexing systems

The first structure that is used to describe N_∞-operads is called an *indexing system*. We recall the definition of an indexing system from [BH15, BH18] before describing the equivalence between indexing systems and homotopy classes of N_∞-operads.

Definition 2.2.7. The *orbit category* of G, denoted $\mathcal{O}\mathrm{rb}_G$, is the category whose objects are transitive G-sets and morphisms are G-equivariant functions.

Definition 2.2.8 ([BH18, Definition 1.1])**.** A *symmetric monoidal coefficient system* is a contravariant functor

$$\underline{\mathcal{C}} : \mathcal{O}\mathrm{rb}_G^{op} \to \mathcal{S}\mathrm{ym}\mathcal{C}\mathrm{at}$$

from the opposite of the orbit category of G to the category of symmetric monoidal categories and strong symmetric monoidal functors.

Notation 2.2.9. Let $\mathcal{S}\mathrm{et}^G$ denote the category of finite G-sets.

Example 2.2.10. The functor

$$\underline{\mathcal{S}\mathrm{et}} : \mathcal{O}\mathrm{rb}_G^{op} \to \mathcal{S}\mathrm{ym}\mathcal{C}\mathrm{at}$$

defined by $\underline{\mathcal{S}\mathrm{et}}(G/H) = (\mathcal{S}\mathrm{et}^H, \sqcup, \emptyset)$ is a symmetric monoidal coefficient system. Note that $\mathcal{S}\mathrm{et}^H$ is equivalent to the category $\mathcal{S}\mathrm{et}^G_{/(G/H)}$ of finite G sets over G/H, so a morphism $G/H \to G/K$ in $\mathcal{O}\mathrm{rb}_G^{op}$ naturally determines a (strong symmetric monoidal) functor $\mathcal{S}\mathrm{et}^H \to \mathcal{S}\mathrm{et}^K$.

Definition 2.2.11 ([BH18, Definition 1.2]). An *indexing system* for G is a full symmetric monoidal sub-coefficient system $\underline{\mathcal{C}}$ of $\underline{\mathcal{S}\text{et}}$ that contains all trivial sets and is closed under

(a) finite limits and

(b) self-induction: if $H/K \in \underline{\mathcal{C}}(G/H)$ and $T \in \underline{\mathcal{C}}(G/K)$, then $H \times_K T \in \underline{\mathcal{C}}(G/H)$.

If $\mathcal{I}$ is an indexing system, then we say that an H-set T is *admissible for* $\mathcal{I}$ if $T \in \mathcal{I}(G/H)$.

Remark 2.2.12. A full subcategory $\mathcal{C} \subset \mathcal{D}$ is called a truncation subcategory of $\mathcal{C}$ if whenever $X \to Y$ is monic in $\mathcal{D}$ and Y is in $\mathcal{C}$, then X is also in $\mathcal{C}$. A truncation sub-coefficient system of a symmetric monoidal coefficient system $\underline{\mathcal{D}}$ is a sub-coefficient system which is levelwise a truncation subcategory.

In [BH15, Definition 3.16], an indexing system is defined as a truncation sub symmetric monoidal coefficient system of $\underline{\mathcal{S}\text{et}}$ that is closed under self induction and Cartesian product. This is equivalent to the definition given above.

Indexing systems for G form a poset under inclusion. For an arbitrary finite group G, it is difficult to describe this poset. Nevertheless, it can be done in some cases. We give a few examples of this poset below.

Example 2.2.13. Let $G = C_p$, where p is prime. Then $\mathcal{O}\text{rb}_{C_p}$ has two objects, C_p/e and C_p/C_p. We define two distinct indexing systems for C_p:

(a) The *trivial indexing system* $\mathcal{I}^{\text{triv}}$ defined by:

$$\mathcal{I}^{\text{triv}}(C_p/e) = (\mathcal{S}\text{et}, \sqcup, \varnothing),$$

$$\mathcal{I}^{\text{triv}}(C_p/C_p) = (\mathcal{S}\text{et}, \sqcup, \varnothing).$$

(b) The *complete indexing system* $\mathcal{I}^{\mathrm{cplt}}$ defined by $\mathcal{I}^{\mathrm{cplt}} = \underline{\mathcal{S}\mathrm{et}}$:

$$\mathcal{I}^{\mathrm{cplt}}(C_p/e) = (\mathcal{S}\mathrm{et}, \sqcup, \emptyset),$$

$$\mathcal{I}^{\mathrm{cplt}}(C_p/C_p) = (\mathcal{S}\mathrm{et}^{C_p}, \sqcup, \emptyset).$$

Proposition 2.2.14. *The only indexing systems for* C_p *are* $\mathcal{I}^{\mathrm{triv}}$ *and* $\mathcal{I}^{\mathrm{cplt}}$.

Proof. Let $\underline{\mathcal{C}}$ be an indexing system for C_p. If $\underline{\mathcal{C}}(C_p/e)$ is not $\mathcal{I}^{\mathrm{triv}}$, then there is a proper inclusion

$$\mathcal{I}^{\mathrm{triv}}(C_p/C_p) = (\mathcal{S}\mathrm{et}, \sqcup, \emptyset) \subset \underline{\mathcal{C}}(C_p/C_p)$$

This implies that $\underline{\mathcal{C}}(C_p/C_p)$ contains a C_p-set S with nontrivial action. Any such S decomposes as a disjoint union of transitive C_p-sets, and must contain at least one nontrivial orbit

$$S = S' \sqcup C_p/e.$$

As $\underline{\mathcal{C}}(C_p/C_p)$ is a truncation subcategory of $\underline{\mathcal{S}\mathrm{et}}(C_p/C_p) = (\mathcal{S}\mathrm{et}^{C_p}, \sqcup, \emptyset)$, it follows that C_p/e is in $\underline{\mathcal{C}}(C_p/C_p)$ by considering the (monic) inclusion

$$C_p/e \to S' \sqcup C_p/e$$

in $\mathcal{S}\mathrm{et}^{C_p}$.

Note that $\underline{\mathcal{C}}(C_p/C_p)$ also contains the trivial C_p-set C_p/C_p, so $\underline{\mathcal{C}}(C_p/C_p)$ contains all finite transitive C_p-sets. As any finite C_p-set can be expressed as a disjoint union of finite transitive C_p-sets, we conclude that $\underline{\mathcal{C}}(C_p/C_p) = (\mathcal{S}\mathrm{et}^{C_p}, \sqcup, \emptyset)$.

So any indexing system for C_p that is not $\mathcal{I}^{\mathrm{triv}}$ is $\mathcal{I}^{\mathrm{cplt}}$. □

The poset of indexing systems for C_p is thus given by

$$\mathcal{I}^{\mathrm{triv}} < \mathcal{I}^{\mathrm{cplt}}.$$

Definition 2.2.15. For any finite group G, we define the *trivial indexing system* $\mathcal{I}^{\mathrm{triv}}$ to be the constant functor

$$\mathcal{I}^{\mathrm{triv}}(G/H) = (\mathcal{S}\mathrm{et}, \sqcup, \emptyset).$$

We define the *complete indexing system* $\mathcal{I}^{\mathrm{cplt}}$ by

$$\mathcal{I}^{\mathrm{cplt}} = \underline{\mathcal{S}\mathrm{et}}.$$

The following definitions allow us to take an N_∞-operad and construct an indexing system from it.

Definition 2.2.16 ([BH15, Definition 4.3])**.** Let T be a finite H-set, and let Γ_T be the graph of the homomorphism $H \to \Sigma_{|T|}$ defining the H-action on T. If $\mathcal{O}$ is an N_∞-operad, we say that T is an *admissible* H*-set for* $\mathcal{O}$ if $\Gamma_T \in \mathcal{F}_{|T|}(\mathcal{O})$.

One can check that the class of admissible H-sets for an N_∞-operad $\mathcal{O}$ is closed under conjugation and if T is an admissible H-set for $\mathcal{O}$, then i_K^*T is an admissible K-set for any subgroup $K \leq H$. Here, $i_K^* \colon \mathcal{S}\mathrm{et}^H \to \mathcal{S}\mathrm{et}^K$ is the forgetful functor.

Definition 2.2.17 ([BH15, Definition 4.5])**.** Given an N_∞-operad $\mathcal{O}$, define a coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ to be the full sub-coefficient system of $\underline{\mathcal{S}\mathrm{et}}$ where $\underline{\mathcal{C}}(\mathcal{O})(G/H)$ is the full subcategory of $\mathcal{S}\mathrm{et}^H$ spanned by the admissible H-sets for $\mathcal{O}$.

By [BH15, Theorem 4.17], the coefficient system $\underline{\mathcal{C}}(\mathcal{O})$ is an indexing system for any N_∞-operad $\mathcal{O}$. The following theorem describes how this functor relates N_∞-operads and indexing systems.

Theorem 2.2.18 ([BP21, Rub21a, GW18])**.** *The functor $\underline{\mathcal{C}}$ descends to an equivalence between the homotopy category of N_∞-operads and the poset of indexing systems.*

Remark 2.2.19. Indexing systems and the functor taking an N_∞-operad to an indexing system were first defined in [BH15], where it was also shown that this functor is fully faithful on the homotopy category of N_∞-operads. The fact that this functor is an equivalence was proven in three different ways in [BP21], [GW18], and [Rub21a].

2.2.2 Indexing categories

Another object that describes homotopy classes of N_∞-operads is an *indexing category*. We recall the definition of an indexing category from [BH18], and describe how to translate between an indexing category and indexing system.

Definition 2.2.20. Let $\mathcal{C}$ be a category. We say a subcategory $\mathcal{D}$ of $\mathcal{C}$ is:

(a) *wide* if it contains all objects;

(b) *finite coproduct complete* if $\mathcal{D}$ has all finite coproducts and coproducts are created in $\mathcal{C}$;

(c) *pullback stable* if $\mathcal{C}$ admits pullbacks and whenever

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C & \longrightarrow & D \end{array}$$

is a pullback diagram in $\mathcal{C}$ with $g \in \mathcal{D}$, the morphism f is also in $\mathcal{D}$.

Definition 2.2.21 (cf. [BH18, Section 3])**.** An *indexing category* $\mathcal{C}$ for G is a wide, pullback stable, finite coproduct complete subcategory of the category $\mathcal{S}\mathrm{et}^G$ of finite G-sets.

Indexing categories for G form a poset under inclusion. This poset is finite whenever G is finite. The least element of this poset is the *trivial indexing category* $\mathcal{C}^{\mathrm{triv}}$, and the greatest element is the *complete indexing category* $\mathcal{C}^{\mathrm{cplt}}$.

Definition 2.2.22. The *trivial indexing category* $\mathcal{C}^{\text{triv}}$ is the wide subcategory of $\mathcal{S}\text{et}^{G}$ containing all morphisms $g: X \to Y$ that preserve isotropy, i.e. for all $x \in X$, the stabilizer of $g(x)$ is also the stabilizer of x.

We also define the *complete indexing category* $\mathcal{C}^{\text{cplt}} := \mathcal{S}\text{et}^{G}$.

Example 2.2.23. In this example we describe the poset of indexing categories for C_p. If an indexing category $\mathcal{C}$ for C_p contains the projection $C_p/e \to C_p/C_p$, then it contains all morphisms of finite transitive C_p-sets and, because it is finite coproduct complete, contains all morphisms of finite C_p-sets. In this case, the indexing category is the complete indexing category $\mathcal{C}^{\text{cplt}} = \mathcal{S}\text{et}^{C_p}$.

If an indexing category does not contain the morphism $C_p/e \to C_p/C_p$, then the only morphisms on transitive C_p-sets are the identities. Under finite coproducts, this yields all morphisms of finite C_p-sets that preserve isotropy, but no more. Thus, the indexing category must be the trivial indexing category $\mathcal{C}^{\text{triv}}$.

We can also vary the group: if H is a subgroup of G, then for any indexing category $\mathcal{C}$ for G, there is an associated indexing category $i_H^*\mathcal{C}$.

Proposition 2.2.24 ([BH18, Proposition 6.3])**.** *Let $i_H^*: \mathcal{S}\text{et}^{G} \to \mathcal{S}\text{et}^{H}$ be the restriction (forgetful) functor. If $\mathcal{C}$ is an indexing category for G, the image of the restriction of i_H^* to $\mathcal{C}$ lands in $i_H^*\mathcal{C}$.*

Because we will frequently use it later, we record one more definition here.

Definition 2.2.25. Let $\mathcal{C}$ be an indexing category for G. We say that H/K is an *admissible H-set for $\mathcal{C}$* if $\mathcal{C}$ contains a morphism $G/K \to G/H$.

Example 2.2.26. In Example 2.2.23, C_p/e is an admissible C_p-set for $\mathcal{C}^{\text{cplt}}$ but not for $\mathcal{C}^{\text{triv}}$.

We now describe how to construct an indexing category from an indexing system.

Definition 2.2.27 ([BH18, Definition 3.8]). Given an indexing system $\mathcal{I}$ for G, let $\mathcal{S}\mathrm{et}^G_{\mathcal{I}}$ denote the wide subgraph of $\mathcal{S}\mathrm{et}^G$ containing morphisms $f\colon S \to T$ if and only if

$$\mathrm{Stab}_G(f(s)) \cdot s \in \mathcal{I}(\mathrm{Stab}_G(f(s)))$$

for all $s \in S$. Here $\mathrm{Stab}_G(x)$ is the stabilizer subgroup of x.

The subgraph $\mathcal{S}\mathrm{et}^G_{\mathcal{I}}$ is a subcategory of $\mathcal{S}\mathrm{et}^G$ by [BH18, Theorem 3.10] and an indexing category by [BH18, Theorem 3.13]. Moreover, the assignment $\mathcal{I} \mapsto \mathcal{S}\mathrm{et}^G_{\mathcal{I}}$ sends inclusions of indexing systems to inclusions of indexing categories; it is a morphism of posets.

Example 2.2.28. For $G = C_p$, we have

$$\mathcal{S}\mathrm{et}^{C_p}_{\mathcal{I}^{\mathrm{triv}}} \simeq \mathcal{C}^{\mathrm{triv}} \quad \text{and} \quad \mathcal{S}\mathrm{et}^{C_p}_{\mathcal{I}^{\mathrm{cplt}}} \simeq \mathcal{C}^{\mathrm{cplt}}.$$

Conversely, there is a functor taking indexing categories to indexing systems.

Definition 2.2.29. Given an indexing category $\mathcal{C}$ for G, define a symmetric monoidal coefficient system

$$\mathcal{I}_{\mathcal{C}}\colon \mathcal{O}\mathrm{rb}_G^{\mathrm{op}} \to \mathcal{S}\mathrm{ym}\mathcal{C}\mathrm{at}$$

on objects by $G/H \mapsto \mathcal{C}_{/(G/H)}$ and on morphisms $f\colon G/H \to G/K$ by pullback $f^*\colon \mathcal{C}_{/(G/K)} \to \mathcal{C}_{/(G/H)}$. Here $\mathcal{C}_{/T}$ is the category of objects over T in $\mathcal{C}$.

This is an indexing system by [BH18, Lemma 3.24], and the assignment $\mathcal{C} \mapsto \mathcal{I}_{\mathcal{C}}$ respects inclusion of indexing categories. Hence, it is a morphism of posets.

Theorem 2.2.30 ([BH18, Theorem 3.17]). *The functors $\mathcal{I} \mapsto \mathcal{S}\mathrm{et}^G_{\mathcal{I}}$ and $\mathcal{C} \mapsto \mathcal{I}_{\mathcal{C}}$ are inverse and give an isomorphism between the poset of indexing systems for G and the poset of indexing categories for G.*

Corollary 2.2.31. *The homotopy category of N_∞-operads is equivalent to the poset of indexing categories.*

2.2.3 Transfer systems

The final avatar for homotopy classes of N_∞-operads is the poset of transfer systems. Transfer systems are more overtly combinatorial than either indexing systems or indexing categories, and at first glance the furthest removed from N_∞-operads. Nevertheless, we will prove in this section that the poset of transfer systems is equivalent to either of the two previous posets, and therefore equivalent to the homotopy category of N_∞-operads.

Definition 2.2.32 ([Rub21b, Section 3][BBR21, Lemma 1])**.** Let $\mathrm{Sub}(G)$ denote the set of subgroups of G. A *transfer system* $\mathcal{T}$ is a partial order on $\mathrm{Sub}(G)$, written $\to_{\mathcal{T}}$, which refines the subset relation and is closed under

- conjugation: if $K \to_{\mathcal{T}} H$, then $(gKg^{-1}) \to_{\mathcal{T}} (gHg^{-1})$ for every $g \in G$; and
- restriction: if $K \to_{\mathcal{T}} H$ and $L \subseteq H$, then $(K \cap L) \to_{\mathcal{T}} L$.

We depict transfer systems as graphs with vertex set $\mathrm{Sub}(G)$ whose edges define a partial order (hence the choice of the arrow notation). When the transfer system is clear from context, we will omit the subscript on the arrow. We refer to a relation $K \to H$ in the transfer system as an *edge*.

Transfer systems for G form a poset under inclusion: we say that $\mathcal{T} \subseteq \mathcal{T}'$ if $K \to_{\mathcal{T}} H$ implies that $K \to_{\mathcal{T}'} H$. For general finite groups it may be difficult to describe this poset, but it can be done for small finite groups. A complete classification of transfer systems for cyclic groups of prime power order is given in [BBR21]. Transfer systems for other finite groups are analyzed in [BBPR20, Rub21b].

Example 2.2.33 ([BBR21, Rub21b])**.** Consider the group $G = C_p$, where p is prime. There are two transfer systems on $\mathrm{Sub}(C_p)$, which we term the trivial transfer system $\mathcal{T}^{\text{triv}}$ and the complete transfer system $\mathcal{T}^{\text{cplt}}$. They are pictured below.

$$\mathcal{T}^{\text{triv}}: \quad e \qquad C_p \qquad\qquad \mathcal{T}^{\text{cplt}}: \quad e \longrightarrow C_p$$

Example 2.2.34 ([BBR21, Example 3])**.** For C_{p^2}, there are 2^3 choices for partial orders on $\mathrm{Sub}(C_{p^2})$. There are five that obey the rules outlined in Definition 2.2.32, pictured below. The first is the trivial transfer system, and the last is the complete transfer system.

$$\mathcal{T}^{\text{triv}}: \quad e \qquad C_p \qquad C_{p^2}$$

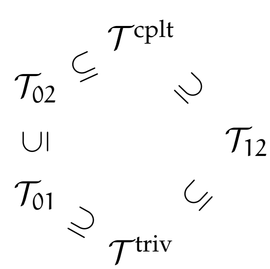
$$\mathcal{T}_{01}: \quad e \longrightarrow C_p \qquad C_{p^2}$$

$$\begin{tikzcd} \mathcal{T}_{02}: & e \arrow[r] \arrow[rr, bend left] & C_p & C_{p^2} \end{tikzcd}$$

$$\mathcal{T}_{12}: \quad e \qquad C_p \longrightarrow C_{p^2}$$

$$\begin{tikzcd} \mathcal{T}^{\text{cplt}}: & e \arrow[r] \arrow[rr, bend left] & C_p \arrow[r] & C_{p^2} \end{tikzcd}$$

These form a poset under inclusion, pictured below:



For C_{p^3}, there are 14 transfer systems. The pattern $1, 2, 5, 14, \ldots$ in the number of transfer systems for C_{p^n} is the Catalan numbers [BBR21, Theorem 20].

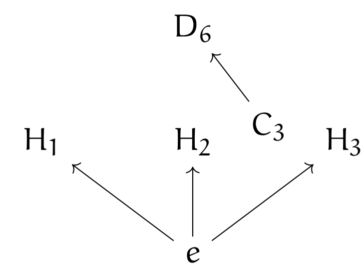
Definition 2.2.35. For any group G, we define the *trivial transfer system* $\mathcal{T}^{\mathrm{triv}}$ by $K \to_{\mathcal{T}^{\mathrm{triv}}} H$ if and only if $K = H$.

We also define the *complete transfer system* $\mathcal{T}^{\mathrm{cplt}}$ to be the subset partial order on $\mathrm{Sub}(G)$.

We define how to restrict a transfer system to a subgroup.

Definition 2.2.36. Let $\mathcal{T}$ be a transfer system for G. For any subgroup $H \leq G$, let $i_H^*\mathcal{T}$ denote the partial order on $\mathrm{Sub}(H)$ such that $K_1 \to_{i_H^*\mathcal{T}} K_2$ if and only if $K_1 \to_{\mathcal{T}} K_2$.

Example 2.2.37. Let $G = D_6$, C_3 its unique subgroup of order three, and H_1, H_2, and H_3 its subgroups of order two. Let $\mathcal{T}$ be the transfer system described by the graph below.



The transfer system $i_{C_3}^*\mathcal{T}$ is the trivial transfer system for C_3, while $i_{H_i}^*\mathcal{T}$ is the complete transfer system for C_2 for each $i \in \{1, 2, 3\}$.

Definition 2.2.38. Let $\mathcal{T}$ be a transfer system for G. We say that H/K is an *admissible* H*-set for* $\mathcal{T}$ if $K \to_{\mathcal{T}} H$.

We now relate transfer systems and indexing categories following [Rub21b, Section 3.1]

Definition 2.2.39. Given an indexing category $\mathcal{C}$ for G, define a transfer system $(\to_{\mathcal{C}})$ by $K \to_{\mathcal{C}} H$ if and only if $K \subseteq H$ and the canonical projection $G/K \to G/H$ is a morphism in $\mathcal{C}$.

Definition 2.2.40. Given a transfer system $\mathcal{T}$ for G, define an indexing category $\mathcal{S}et^G_{\mathcal{T}}$ to be the wide subcategory of $\mathcal{S}et^G$ with morphisms $f\colon S \to T$ such that $\mathrm{Stab}_G(s) \to_{\mathcal{T}} \mathrm{Stab}_G(f(s))$ for all $s \in S$.

Example 2.2.41. For $G = C_p$, we have

$$\mathcal{S}et^{C_p}_{\mathcal{T}^{\mathrm{triv}}} \simeq \mathcal{C}^{\mathrm{triv}} \quad \text{and} \quad \mathcal{S}et^{C_p}_{\mathcal{T}^{\mathrm{cplt}}} \simeq \mathcal{C}^{\mathrm{cplt}}.$$

Similarly, the transfer system associated to $\mathcal{C}^{\mathrm{triv}}$ is $\mathcal{T}^{\mathrm{triv}}$ and the transfer system associated to $\mathcal{C}^{\mathrm{cplt}}$ is $\mathcal{T}^{\mathrm{cplt}}$.

Both of the assignments $\mathcal{T} \mapsto \mathcal{S}et^G_{\mathcal{T}}$ and $\mathcal{C} \mapsto (\to_{\mathcal{C}})$ respect inclusion, and therefore define morphisms of posets between the poset of indexing categories and the poset of transfer systems. In fact, they are inverse, as the next theorem shows.

Theorem 2.2.42 ([Rub21b, Corollary 3.9])**.** *The poset morphisms $\mathcal{T} \mapsto \mathcal{S}et^G_{\mathcal{T}}$ and $\mathcal{C} \mapsto (\to_{\mathcal{C}})$ are inverse and give an isomorphism between the poset of indexing categories for G and the poset of transfer systems for G.*

Corollary 2.2.43. *The posets of indexing systems for G and transfer systems for G are isomorphic.*

Corollary 2.2.44. *The homotopy category of N_∞-operads is equivalent to the poset of transfer systems.*

In light of these equivalences, we may use indexing categories, transfer systems, and homotopy classes of N_∞-operads interchangeably. Most of what follows will be phrased in terms of indexing categories, with an occasional foray into transfer systems. We will typically use the letter $\mathcal{O}$ to refer to a homotopy class of N_∞-operads, or its associated indexing system, indexing category, or transfer system.

Notation 2.2.45. Let $\mathcal{O}^{\mathrm{triv}}$ be the initial object of the homotopy category of N_∞-operads, and let $\mathcal{O}^{\mathrm{cplt}}$ be its terminal object. By abuse of notation, we will also use $\mathcal{O}^{\mathrm{triv}}$ for the least element of the poset of indexing categories, and $\mathcal{O}^{\mathrm{cplt}}$ for the greatest element.

2.3 Incomplete tambara functors

With our newfound understanding of N_∞-operads, we turn to incomplete Tambara functors. Like Mackey functors, incomplete Tambara functors are functors out of a category whose objects are finite G-sets.

2.3.1 Categories of polynomials

To each indexing category $\mathcal{O}$, we associate a category of polynomials (or *bispans*) with exponents in $\mathcal{O}$. These categories are the domains of incomplete Tambara functors.

Definition 2.3.1. Let $\mathcal{D}$ be a wide, pullback stable subcategory of $\mathcal{S}\mathrm{et}^G$. Let $\mathcal{P}^G_\mathcal{D}$ denote the *category of polynomials with exponents* in $\mathcal{D}$. Objects are finite G-sets and morphisms $\mathcal{P}^G_\mathcal{D}(X, Y)$ are *polynomials with exponents in* $\mathcal{D}$, i.e. equivalence classes of diagrams

$$\left[X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y\right]$$

with $g \in \mathcal{D}$. Two such diagrams are equivalent if there is a diagram of the form

$$\begin{array}{ccccccc} & & A & \xrightarrow{g} & B & & \\ & \overset{f}{\swarrow} & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & \overset{h}{\searrow} & \\ X & & & & & & Y \\ & \underset{f'}{\nwarrow} & & & & \underset{h'}{\nearrow} & \\ & & A' & \xrightarrow[g']{} & B' & & \end{array}$$

Composition of polynomials is given by [Tam93, Proposition 7.1].

Below, we will describe how to work with the morphisms in this category in a more practical way. First, we give a few important examples.

Example 2.3.2. Any wide, pullback stable subcategory of $\mathcal{S}\mathrm{et}^G$ contains the subcategory $\mathcal{S}\mathrm{et}^G_{\mathrm{iso}}$ of finite G-sets and isomorphisms. This subcategory is wide and pullback stable, so yields a category of polynomials $\mathcal{P}^G_{\mathrm{iso}}$. Any polynomial of the form $X \leftarrow A \cong B \rightarrow Y$ is canonically isomorphic to one of the form $X \leftarrow A \xrightarrow{\mathrm{id}} A \rightarrow Y$, and thus the category $\mathcal{P}^G_{\mathrm{iso}}$ is isomorphic to the category of spans of finite G-sets.

Example 2.3.3. If $\mathcal{O}$ is any indexing category, then we get a category $\mathcal{P}^G_{\mathcal{O}}$ of polynomials with exponents in $\mathcal{O}$.

The fact that $\mathcal{P}^G_{\mathcal{D}}$ is a category is not obvious, and composition can be messy. We define a generating set of morphisms and describe how to compose them following [BH18].

Definition 2.3.4. Let $f\colon X \rightarrow Y$ be a morphism of finite G-sets. Define three morphisms in $\mathcal{P}^G_{\mathcal{O}}(X, Y)$

$$R_f := \left[X \xleftarrow{f} Y \xrightarrow{\mathrm{id}} Y \xrightarrow{\mathrm{id}} Y\right]$$

$$N_f := \left[X \xleftarrow{\mathrm{id}} X \xrightarrow{f} Y \xrightarrow{\mathrm{id}} Y\right]$$

$$T_f := \left[X \xleftarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X \xrightarrow{f} Y\right]$$

Theorem 2.3.5 (cf. [BH18, Section 2.1])**.**

(a) R, N, T *give functors from* $\mathcal{S}\mathrm{et}^G$ *to* $\mathcal{P}^G_{\mathcal{O}}$*.* R *is contravariant;* N *and* T *are covariant.*

(b) Any morphism in $\mathcal{P}^G_{\mathcal{O}}$ *can be written as a composite*

$$T_h \circ N_g \circ R_f = \left[X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y\right].$$

(c) *Given a pullback diagram of finite* G*-sets*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

then we have

$$R_g \circ N_f = N_{f'} \circ R_{g'},$$

$$R_g \circ T_f = T_{f'} \circ R_{g'}.$$

(d) *Given any diagram isomorphic to one of the form (an* exponential diagram*)*

$$\begin{array}{ccccc} S & \xleftarrow{g} & A & \xleftarrow{f'} & S \times_T \Pi_h A \\ \downarrow{\scriptstyle h} & & & & \downarrow{\scriptstyle g'} \\ T & & \xleftarrow[h']{} & & \Pi_h A \end{array}$$

then

$$N_h \circ T_g = T_{h'} \circ N_{g'} \circ R_{f'}.$$

Here, $\Pi_h A$ *is the* dependent product*, right adjoint to the pullback*

$$h^*\colon \mathcal{S}\mathrm{et}^G_{/T} \to \mathcal{S}\mathrm{et}^G_{/S}.$$

There is more structure on the morphism sets of $\mathcal{P}^G_{\mathcal{O}}$. Recall that a *semiring* is a ring where elements need not have additive inverses.

Theorem 2.3.6 ([Tam93, Proposition 7.6])**.** *The hom-set* $\mathcal{P}^G_{\mathcal{O}}(X, Y)$ *becomes a commutative semiring, with units*

$$0 = [X \leftarrow \varnothing \to \varnothing \to Y],$$

$$1 = [X \leftarrow \varnothing \to Y \xrightarrow{\mathrm{id}} Y].$$

The addition and multiplication operations are defined on polynomials

$$\Sigma = \left[X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y\right] \quad \textit{and}$$
$$\Sigma' = \left[X \xleftarrow{f'} A' \xrightarrow{g'} B' \xrightarrow{h'} Y\right]$$

by

$$\Sigma + \Sigma' = \left[X \xleftarrow{\nabla\circ(f\sqcup f')} A \sqcup A' \xrightarrow{g\sqcup g'} B \sqcup B' \xrightarrow{\nabla\circ(h\sqcup h')} Y\right], \quad \textit{and}$$
$$\Sigma \cdot \Sigma' = \left[X \xleftarrow{\widehat{f}} (A \times_Y B') \sqcup (B \times_Y A') \xrightarrow{\widehat{g}} B \times_Y B' \xrightarrow{\widehat{h}} Y\right],$$

where:

- $\nabla\colon S \sqcup S \to S$ *is the codiagonal (fold) morphism;*
- $\widehat{f}$ *is the morphism given by projecting onto* A *or* A' *then applying* f *or* f'*;*
- $\widehat{g}$ *is the morphism given by applying* g *to* A *or* g' *to* A'*;*
- $\widehat{h}$ *is the morphism given by either of the two equivalent morphisms* $h \circ \mathrm{pr}_1$ *or* $h' \circ \mathrm{pr}_2$.

Remark 2.3.7. Note that, although $\mathcal{P}^G_{\mathcal{O}}(X, Y)$ is a commutative semiring, the additive completion of $\mathcal{P}^G_{\mathcal{O}}$ is *not* enriched over rings. This fails because the multiplicative unit

$$1 = \left[X \leftarrow \varnothing \to X \xrightarrow{\mathrm{id}} X\right]$$

in $\mathcal{P}^G_{\mathcal{O}}(X, X)$ is not the identity morphism

$$\mathrm{id}_X = \left[X \leftarrow X \to X \to X\right].$$

The hom-sets $\mathcal{P}^G_{\mathcal{O}}(X, Y)$ also carry a group action when X and Y are transitive G-sets.

Definition 2.3.8. Let $H \leq G$. Recall that the *Weyl group* $W_G(H)$ of H in G is the quotient of the normalizer of H in G by H:

$$W_G(H) := N_G(H)/H.$$

Remark 2.3.9. The Weyl group $W_G(K)$ acts on G/K by conjugation, so each $\gamma \in W_G(K)$ defines a G-equivariant function

$$c_\gamma : G/K \to G/K.$$

This extends to a $W_G(K)$-action on $\mathcal{P}^G_{\mathcal{O}}(G/H, G/K)$ by transferring along c_γ, i.e.

$$\gamma \cdot \left[G/H \leftarrow A \to B \xrightarrow{h} G/K\right] = \left[G/H \leftarrow A \to B \xrightarrow{c_\gamma \circ h} G/K\right].$$

2.3.2 Incomplete tambara functors

With the categories $\mathcal{P}^G_{\mathcal{O}}$, we can finally define incomplete Tambara functors. These are the equivariant analogue of commutative rings.

Definition 2.3.10 ([BH18, Definition 4.1])**.** Let $\mathcal{O}$ be an indexing category. An *$\mathcal{O}$-Tambara functor* for G is a product-preserving functor $\underline{R}\colon \mathcal{P}^G_{\mathcal{O}} \to \mathcal{S}et$ such that $\underline{R}(X)$ is an abelian group for every finite G-set X. An *incomplete Tambara functor* is an $\mathcal{O}$-Tambara functor for some $\mathcal{O}$.

A *morphism of $\mathcal{O}$-Tambara functors* is a natural transformation. We write $\mathcal{O}$-$\mathcal{T}\mathrm{amb}_G$ for the category of $\mathcal{O}$-Tambara functors for the group G.

Although we only insist that the functor $\underline{R}\colon \mathcal{P}^G_{\mathcal{O}} \to \mathcal{S}et$ takes values in abelian groups, each $\underline{R}(X)$ becomes a commutative ring from the semiring structure on the hom-sets in $\mathcal{P}^G_{\mathcal{O}}$ from Theorem 2.3.6. In particular, an $\mathcal{O}$-Tambara functor is a Green functor with some extra structure [BH18, Theorem 4.13], as we explain below.

Because any Tambara functor $\underline{R}$ is product-preserving, and disjoint union is the categorical product in $\mathcal{P}^G_{\mathcal{O}}$, it suffices to give the value of $\underline{R}$ on each transitive finite G-set G/K. Moreover, because any morphism in the category $\mathcal{P}^G_{\mathcal{O}}$ may be written as a composite $T_h \circ N_g \circ R_f$, it suffices to define the Tambara functor on morphisms of the form T_h, N_g, and R_f.

Definition 2.3.11. If $\underline{R}$ is an incomplete Tambara functor, we write $\mathrm{tr}_h := \underline{R}(T_h)$, $\mathrm{nm}_g := \underline{R}(N_g)$, and $\mathrm{res}_f := \underline{R}(R_f)$. These are the *transfer*, *norm*, and *restriction* of $\underline{R}$, respectively. In the case when $\pi\colon G/K \to G/H$ is the canonical projection, we write $\mathrm{tr}_K^H := \mathrm{tr}_\pi$, $\mathrm{nm}_K^H := \mathrm{nm}_\pi$ and $\mathrm{res}_K^H := \mathrm{res}_\pi$.

Proposition 2.3.12 ([Tam93, Section 2])**.** *The data of an $\mathcal{O}$-Tambara functor is equivalent to the following:*

- *A collection of commutative $W_G H$-rings $\underline{R}(G/H)$, one orbit G/H;*
- *restriction homomorphisms (of non-unital commutative rings)*

$$\mathrm{res}_K^H\colon \underline{R}(G/H) \to \underline{R}(G/K);$$

- *transfer homomorphisms (of abelian groups)*

$$\mathrm{tr}_K^H\colon \underline{R}(G/K) \to \underline{R}(G/H);$$

- *and multiplicative norm morphisms*

$$\mathrm{nm}_K^H\colon \underline{R}(G/K) \to \underline{R}(G/H)$$

  whenever H/K is an admissible H-set for $\mathcal{O}$.

These data are subject to some conditions. See, for example, [Tam93, Section 2], [BH18, Section 4], or [Maz13, Section 1.4].

Recall the indexing categories $\mathcal{O}^{\mathrm{triv}}$ and $\mathcal{O}^{\mathrm{cplt}}$ (Notation 2.2.45).

Example 2.3.13 ([BH18, Section 4])**.** There are two special classes of incomplete Tambara functors which appear in equivariant algebra:

(a) An $\mathcal{O}^{\mathrm{triv}}$-Tambara functor is a Green functor.

(b) An $\mathcal{O}^{\mathrm{cplt}}$-Tambara functor is a *Tambara functor*.

We give another definition of Mackey functors that emphasizes the connection with incomplete Tambara functors defined below. Recall from Example 2.3.2 that the category $\mathcal{P}^G_{\text{iso}}$ is equivalent to the Burnside category.

Proposition 2.3.14 ([BH18, Proposition 4.3])**.** *A Mackey functor for* G *is equivalently a product-preserving functor* $\underline{M}\colon \mathcal{P}^G_{\text{iso}} \to \mathcal{S}\text{et}$ *such that* $\underline{M}(X)$ *is an abelian group for every finite* G*-set* X*.*

Proposition 2.3.15 ([BH18, Corollary 4.4])**.** *Any* $\mathcal{O}$*-Tambara functor* $\underline{R}\colon \mathcal{P}^G_{\mathcal{O}} \to \mathcal{S}\text{et}$ *has an underlying Mackey functor given by restricting the domain of* $\underline{R}$ *to the category* $\mathcal{P}^G_{\text{iso}}$*.*

Proposition 2.3.16 ([BH18, Proposition 5.14])**.** *Let* $\mathcal{O}$ *and* $\mathcal{O}'$ *be indexing categories such that* $\mathcal{O}' \subseteq \mathcal{O}$*. Then every* $\mathcal{O}$*-Tambara functor* $\underline{R}\colon \mathcal{P}^G_{\mathcal{O}} \to \mathcal{S}\text{et}$ *has an underlying* $\mathcal{O}'$*-Tambara functor given by restricting the domain of* $\underline{R}$ *to the category* $\mathcal{P}^G_{\mathcal{O}'}$*.*

In particular, any $\mathcal{O}$-Tambara functor has an underlying Green functor by restricting the domain to $\mathcal{P}^G_{\mathcal{O}^{\text{triv}}}$.

There are several fundamental Tambara functors we will use in this thesis. Each can be regarded as an incomplete Tambara functor by forgetting structure.

Definition 2.3.17 ([Tam93, Example 3.2])**.** The Burnside Mackey functor $\underline{A}$ from Example 2.1.7 becomes a Tambara functor when we endow the monoid of isomorphism classes of finite G-sets over X with a multiplication given by Cartesian product. That is, the *Burnside Tambara functor* is the Tambara functor $\underline{A}$ whose value on a G-set X is the additive group completion of the semiring of isomorphism classes of G-sets Y over X, with semiring operations given by Cartesian product and disjoint union.

When $X = G/H$, $\underline{A}(G/H)$ is the group completion of the monoid of isomorphism classes of finite H-sets under disjoint union. The restriction res^H_K is given by pullback, transfer tr^H_K is induction, and norm nm^H_K is coinduction.

$$\begin{aligned}\mathrm{res}^H_K([Y]) &= [i_K^* Y]\\ \mathrm{tr}^H_K([X]) &= [H \times_K X]\\ \mathrm{nm}^H_K([X]) &= \mathcal{S}\mathrm{et}^K(H, X).\end{aligned}$$

Remark 2.3.18. The Burnside Tambara functor is the initial incomplete Tambara functor, just like $\mathbb{Z}$ is the initial commutative ring. Modules over $\underline{A}$, or Mackey functors, are the equivariant analog of abelian groups, while algebras over $\underline{A}$ are incomplete Tambara functors.

Example 2.3.19. We examine the structure of the C_p-Burnside functor.

At the underlying level, $\underline{A}(C_p/e)$ is generated by isomorphism classes of finite C_p-sets over C_p/e. Such a C_p-set must be a disjoint union of orbits isomorphic to C_p/e. Hence,

$$\underline{A}(C_p/e) \cong \mathbb{Z}$$

At level $\underline{A}(C_p/C_p)$, generators may be any finite C_p-set since $C_p/C_p = *$ is terminal. Any C_p-set is a disjoint union of the transitive C_p-sets, C_p/e and C_p/C_p. Let t be the class of C_p/e. We have a relation

$$t \cdot t = [C_p/e \times C_p/e] = \left[\coprod_{i=1}^{p} C_p/e\right] = p\,t.$$

Therefore,

$$\underline{A}(C_p/C_p) \cong \mathbb{Z}[t]/\langle t^2 - pt\rangle$$

The restriction sends the class of C_p/e to its underlying finite set, and is therefore determined by $t \mapsto p$. The transfer is induction and is given by multiplication by t. When $\underline{A}$ is considered as an incomplete Tambara functor, norms that exist are coinduction and are given by [Maz13, Example 1.4.6]:

$$\mathrm{nm}_e^{C_p}(a) = a + \left(\frac{a^p - a}{p}\right) t. \tag{2.3.20}$$

For $G = C_p$, the Burnside Tambara functor is given by the Lewis diagram

$$\underline{A}\colon \quad \begin{array}{c} \mathbb{Z}[t]/\langle t^2 - pt\rangle \\ t\mapsto p \left(\; \uparrow \mathrm{nm}_e^{C_p} \; \right) \cdot t \\ \mathbb{Z} \\ \circlearrowleft \\ \text{trivial} \end{array}$$

2.3.3 Modules over incomplete Tambara functors

Recall the definition of the box product from Definition 2.1.10. For Mackey functors $\underline{M}$ and $\underline{N}$, their box product $\underline{M} \boxtimes \underline{N}$ is the Mackey functor obtained by left Kan extending the tensor product of abelian groups along the functor $\times\colon \mathcal{P}^G_{\mathrm{iso}} \times \mathcal{P}^G_{\mathrm{iso}} \to \mathcal{P}^G_{\mathrm{iso}}$.

$$\begin{array}{ccccc} \mathcal{P}^G_{\mathrm{iso}} \times \mathcal{P}^G_{\mathrm{iso}} & \xrightarrow{\underline{M}\times\underline{N}} & \mathcal{A}b \times \mathcal{A}b & \xrightarrow{\otimes_{\mathbb{Z}}} & \mathcal{A}b \\ \times\big\downarrow & & & \underline{M}\boxtimes\underline{N} \nearrow & \\ \mathcal{P}^G_{\mathrm{iso}} & & & & \end{array}$$

As in Section 2.1.1, this makes $\mathcal{M}\mathrm{ack}_G$ into a closed symmetric monoidal category whose unit is the Burnside Mackey functor $\underline{A}$, and Green functors are monoids for this symmetric monoidal structure.

The box product also plays a special role for incomplete Tambara functors.

Theorem 2.3.21 ([Str12, BH18]). *If $\underline{R}$ and $\underline{R}'$ are $\mathcal{O}$-Tambara functors, then $\underline{R} \boxtimes \underline{R}'$ has a natural structure as an $\mathcal{O}$-Tambara functor.*

The natural morphisms $\underline{R} \to \underline{R} \boxtimes \underline{R}' \leftarrow \underline{R}'$ are morphisms of $\mathcal{O}$-Tambara functors and witness the box product as the coproduct.

Definition 2.3.22. If $\underline{R}$ is a Green functor with unit $\eta : \underline{A} \to \underline{R}$ and multiplication $\mu : \underline{R} \boxtimes \underline{R} \to \underline{R}$, then an $\underline{R}$*-module* $\underline{M}$ is a Mackey functor $\underline{M}$ together with a homomorphism of Mackey functors $\nu : \underline{R} \boxtimes \underline{M} \to \underline{M}$ such that the the following diagram commutes:

$$\begin{array}{ccccc} \underline{R} \boxtimes \underline{R} \boxtimes \underline{M} & \xrightarrow{\mu \boxtimes 1} & \underline{R} \boxtimes \underline{M} & \xleftarrow{\eta \boxtimes 1} & \underline{M} \\ \downarrow{\scriptstyle 1 \boxtimes \nu} & & \downarrow{\scriptstyle \nu} & \swarrow{\scriptstyle 1} & \\ \underline{R} \boxtimes \underline{M} & \xrightarrow{\nu} & \underline{M}. & & \end{array}$$

A *morphism of* $\underline{R}$*-modules* $f \colon \underline{M} \to \underline{N}$ is a homomorphism of Mackey functors such that the following diagram commutes:

$$\begin{array}{ccc} \underline{R} \boxtimes \underline{M} & \xrightarrow{\nu_{\underline{M}}} & \underline{M} \\ \downarrow{\scriptstyle \mathrm{id} \boxtimes f} & & \downarrow{\scriptstyle f} \\ \underline{R} \boxtimes \underline{N} & \xrightarrow{\nu_{\underline{N}}} & \underline{N}. \end{array}$$

If $\underline{R}$ is an $\mathcal{O}$-Tambara functor, an $\underline{R}$*-module* $\underline{M}$ is a module over the underlying Green functor of $\underline{R}$.

Denote the category of $\underline{R}$-modules by $\underline{R}$-$\mathcal{M}$od.

Remark 2.3.23. This notion of module is the extension of Strickland's notion of *naive module* [Str12, Definition 14.1] to incomplete Tambara functors. Strickland points out that this definition only uses the underlying Green functor structure of $\underline{R}$ and proposes a new notion of module in [Str12, Definition 14.3] which incorporates extra Tambara structure. Hill showed in [Hil17] that Strickland's genuine modules are the abelian group objects in the category of $\underline{R}$-Tambara functors, whereas modules as defined above are *Mackey functor objects* in the category of $\underline{R}$-Tambara functors. We will not consider these more refined notions of modules here and refer the reader to [Hil17] for further discussion.

Definition 2.3.24. If $\underline{M}$ and $\underline{N}$ are $\underline{R}$-modules, we define the *relative box product:*

$$\underline{M} \boxtimes_{\underline{R}} \underline{N} := \mathrm{coeq}\left(\underline{M} \boxtimes \underline{R} \boxtimes \underline{N} \underset{\mathrm{id} \boxtimes \nu_{\underline{N}}}{\overset{\nu_{\underline{M}} \boxtimes \mathrm{id}_{\underline{N}}}{\rightrightarrows}} \underline{M} \boxtimes \underline{N} \right).$$

Proposition 2.3.25 ([Lew81]).

(a) The relative box product $\boxtimes_{\underline{R}}$ *makes the category* $\underline{R}$-$\mathcal{M}$od *into a closed symmetric monoidal category with unit* $\underline{R}$.

(b) $\underline{R}$-$\mathcal{M}$od *is an abelian category.*

Remark 2.3.26. The internal Hom of $\underline{R}$-modules $\underline{M}$ and $\underline{N}$ is the $\underline{R}$-module $\underline{\mathrm{Hom}}_{\underline{R}}(\underline{M}, \underline{N})$ whose value on a finite G-set T is given by

$$\underline{\mathrm{Hom}}_{\underline{R}}(\underline{M}, \underline{N})(T) := \mathrm{Hom}_{\underline{R}}(\underline{M}, \underline{N}\{x_T\}) = \mathrm{Hom}_{\underline{R}}(\underline{M}\{x_T\}, \underline{N}),$$

where $\mathrm{Hom}_{\underline{R}}$ without an underline denotes the ordinary abelian group of morphisms between $\underline{R}$-modules and $\underline{M}\{x_T\}$ is the free $\underline{R}$-module described below in Definition 2.3.29. For details, see [Lew81] or [Lee19, Section 2.1].

The category $\underline{R}$-$\mathcal{M}$od is closed symmetric monoidal, so for any $\underline{R}$-module $\underline{M}$, the functor $\underline{M} \boxtimes_{\underline{R}} (-)$ has a right adjoint and is therefore right exact. Therefore, it makes sense to consider the derived functors of the box product.

Proposition 2.3.27 ([Zen18, Proposition 3.2])**.** *The category of* $\underline{R}$*-modules has enough projectives.*

Definition 2.3.28. Let $\underline{\mathrm{Tor}}_i^{\underline{R}}(\underline{M}, -)$ denote the i-th derived functor of $\underline{M} \boxtimes_{\underline{R}} (-)$. This is the $\underline{R}$-module-valued Tor functor.

2.3.4 Free, projective, and flat modules

Definition 2.3.29. Let T be a finite G-set. The *free Mackey functor on a generator at level* T, denoted $\underline{A}\{x_T\}$, is the Mackey functor defined by

$$\underline{A}\{x_T\} := \mathcal{P}^G_{\mathrm{iso}}(T, -)^+,$$

where the superscript $+$ denotes group completion.

Let $\underline{R}$ be an incomplete Tambara functor. The *free* $\underline{R}$*-module on a generator at level* T, denoted $\underline{R}\{x_T\}$, is defined by

$$\underline{R}\{x_T\} := \underline{R} \boxtimes \underline{A}\{x_T\}.$$

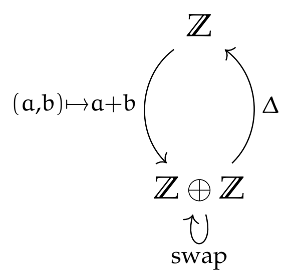
Example 2.3.30. The free $\underline{R}$-module on a single generator at level G/G is $\underline{R}$ itself:

$$\underline{R}\{x_{G/G}\} = \underline{R}.$$

A free $\underline{R}$-module on a single generator at level T can be understood by means of the formula

$$\underline{R}\{x_T\}(U) = \underline{R}(T \times U).$$

Example 2.3.31. The free $\underline{A}$-module $\underline{A}\{x_{C_2/e}\}$ is the C_2-Mackey functor



Definition 2.3.32. A *free* $\underline{R}$*-module* is any $\underline{R}$-module $\underline{M}$ with

$$\underline{M} \cong \bigoplus_{i \in I} \underline{R}\{x_{T_i}\}$$

where each T_i is a finite G-set.

Remark 2.3.33. As $\underline{R}\{x_T\} \oplus \underline{R}\{x_{T'}\} \cong \underline{R}\{x_{T \sqcup T'}\}$, any finitely generated free $\underline{R}$-module has the form $\underline{R}\{x_U\}$ for a single finite G-set U. Importantly, this is not the case if the free $\underline{R}$-module is infinitely generated. Nevertheless, if a free $\underline{R}$-module is infinitely generated, it is a direct limit of finitely generated free $\underline{R}$-modules.

Free modules enjoy the following universal property.

Lemma 2.3.34. *Free $\underline{R}$-modules on a single generator represent evaluation:*

$$\underline{R}\text{-}\mathcal{M}\mathrm{od}(\underline{R}\{x_T\}, \underline{M}) \cong \underline{M}(T).$$

Definition 2.3.35. An $\underline{R}$-module $\underline{M}$ is *projective* if it is a summand of a free $\underline{R}$-module.

An $\underline{R}$-module $\underline{M}$ is *flat* if the functor $- \boxtimes_{\underline{R}} \underline{M} : \underline{R}\text{-}\mathcal{M}\mathrm{od} \to \underline{R}\text{-}\mathcal{M}\mathrm{od}$ is exact.

Free, projective, and flat modules interact in the way one would expect from classical homological algebra.

Proposition 2.3.36 ([Lee19, Proposition 2.2.13])**.** *Projective $\underline{R}$-modules are flat.*

Proposition 2.3.37. *Let $f \colon \underline{R} \to \underline{S}$ be a morphism of $\mathcal{O}$-Tambara functors. The base change functor*

$$\underline{S} \boxtimes_{\underline{R}} (-) \colon \underline{R}\text{-}\mathcal{M}\mathrm{od} \to \underline{S}\text{-}\mathcal{M}\mathrm{od}$$

takes free $\underline{R}$-modules to free $\underline{S}$-modules, projective $\underline{R}$-modules to projective $\underline{S}$-modules, and flat $\underline{R}$-modules to flat $\underline{S}$-modules.

Proof. The functor $\underline{S} \boxtimes_{\underline{R}} (-)$ commutes with direct sums and direct limits, so $\underline{S} \boxtimes_{\underline{R}} (-)$ takes free $\underline{R}$-modules to free $\underline{S}$-modules.

Alternatively, let $\underline{M}$ be an $\underline{S}$-module. We have

$$\underline{S}\text{-}\mathcal{M}\mathrm{od}(\underline{S} \boxtimes_{\underline{R}} \underline{R}\{x_T\}, \underline{M}) \cong \underline{R}\text{-}\mathcal{M}\mathrm{od}(\underline{R}\{x_T\}, \underline{M}) \cong \underline{M}(T),$$

so $\underline{S} \boxtimes_{\underline{R}} \underline{R}\{x_T\}$ represents evaluation at T.

If $\underline{P}$ is projective over $\underline{R}$, then there is some $\underline{Q}$ such that $\underline{P} \oplus \underline{Q}$ is free over $\underline{R}$. Then $(\underline{S} \boxtimes_{\underline{R}} \underline{P}) \oplus (\underline{S} \boxtimes_{\underline{R}} \underline{Q})$ is free over $\underline{S}$, and $\underline{S} \boxtimes_{\underline{R}} \underline{P}$ is projective over $\underline{S}$.

If $\underline{M}$ is flat over $\underline{R}$, then $(\underline{S} \boxtimes_{\underline{R}} \underline{M}) \boxtimes_{\underline{S}} (-)$ is naturally isomorphic to $\underline{M} \boxtimes_{\underline{R}} (-)$, and therefore exact. □

Flatness is a surprisingly subtle concept for Mackey functors. The following example will be used often.

Lemma 2.3.38. *For* $G = C_p$ *with* p *a prime, neither* $\underline{\mathbb{Z}}$ *nor* $\underline{\mathbb{Z}}^*$ *is flat.*

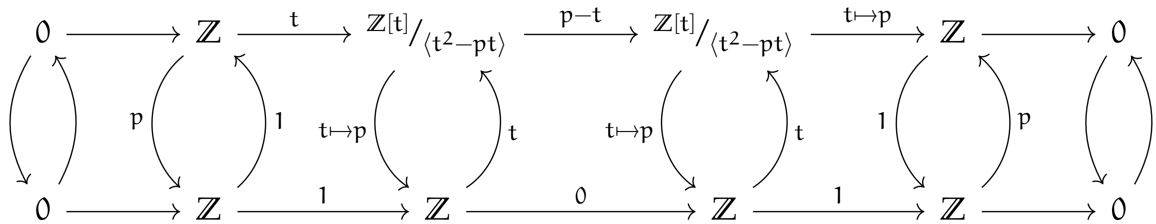
Proof. We have two exact sequences

$$0 \to \underline{\mathbb{Z}}^* \to \underline{A} \xrightarrow{p-t} \underline{A} \to \underline{\mathbb{Z}} \to 0$$

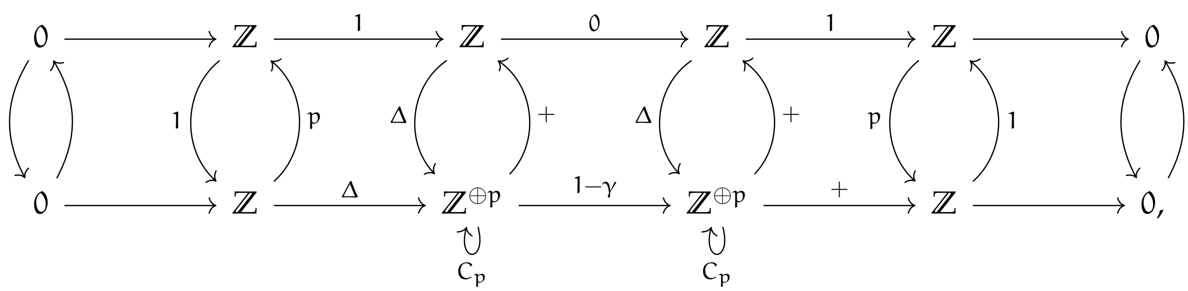
and

$$0 \to \underline{\mathbb{Z}} \to \underline{A}\{x_{C_p/e}\} \xrightarrow{1-\gamma} \underline{A}\{x_{C_p/e}\} \to \underline{\mathbb{Z}}^* \to 0,$$

where γ is a generator of C_p. Drawn out as Lewis diagrams, these sequences look like:



and

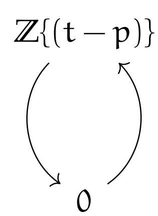


where $\Delta\colon \mathbb{Z} \to \mathbb{Z}^{\oplus p}$ is the diagonal and $+\colon \mathbb{Z}^{\oplus p} \to \mathbb{Z}$ takes a p-tuple of integers and adds them together.

Splicing these exact sequences together gives a projective (even free) resolution of either $\underline{\mathbb{Z}}$ or $\underline{\mathbb{Z}}^*$.

Consider the augmentation ideal $\underline{I}$, the kernel of the Mackey functor homomorphism $\underline{A} \to \underline{\mathbb{Z}}$ that sends a finite H-set to its cardinality.

$\underline{I}$:



Since the underlying level of $\underline{I}$ is zero, we have $\underline{I} \boxtimes \underline{A}\{x_{C_p/e}\} = 0$. This together with the projective resolution shows that

$$\underline{\mathrm{Tor}}_3(\underline{\mathbb{Z}}^*, \underline{I}) \cong \underline{I} \quad \text{and} \quad \underline{\mathrm{Tor}}_1(\underline{\mathbb{Z}}, \underline{I}) \cong \underline{I}$$

In particular, neither are zero, so $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}^*$ cannot be flat. □

Remark 2.3.39. In fact, the argument above shows that

$$\underline{\mathrm{Tor}}_n(\underline{\mathbb{Z}}^*, \underline{I}) \cong \begin{cases} \underline{I} & (n \equiv 3 \mod 4), \\ \underline{I}/p & (n \equiv 2 \mod 4), \\ 0 & (\text{otherwise}), \end{cases}$$

and

$$\underline{\mathrm{Tor}}_n(\underline{\mathbb{Z}}, \underline{I}) \cong \begin{cases} \underline{I} & (n \equiv 1 \mod 4), \\ \underline{I}/p & (n \equiv 0 \mod 4), \\ 0 & (\text{otherwise}). \end{cases}$$

In particular, $\underline{\mathbb{Z}}^*$ and $\underline{\mathbb{Z}}$ (and $\underline{I}$) have infinite Tor-dimension.

2.3.5 $\underline{\mathbb{Z}}$-modules and cohomological Mackey functors

In this section, we discuss an extended example: modules over the constant Mackey functor $\underline{\mathbb{Z}}$. We will see that $\underline{\mathbb{Z}}$-modules are a particularly nice class of Mackey functors called *cohomological* Mackey functors, and free $\underline{\mathbb{Z}}$-modules are nicer still: they are fixed point Mackey functors.

Definition 2.3.40 ([Gre71, Section 1.4])**.** A Mackey functor $\underline{M}$ is *cohomological* if

$$\mathrm{tr}_K^H \mathrm{res}_K^H(x) = [H : K]x$$

for all $x \in \underline{M}(G/H)$ and all subgroups $K \leq H \leq G$.

The name "cohomological" comes from the analogous relations satisfied by restriction and transfer in group cohomology. Cohomological Mackey functors have a coordinate free description as well.

Proposition 2.3.41 ([TW95, Proposition 16.3])**.** *There is an equivalence between the category of $\underline{\mathbb{Z}}$-modules and the full subcategory of the category of Mackey functors whose objects are cohomological Mackey functors.*

In particular, being a $\underline{\mathbb{Z}}$-module is a property of a Mackey functor, rather than extra structure. A Mackey functor $\underline{M}$ is a $\underline{\mathbb{Z}}$-module if and only if the homomorphism

$$M \cong \underline{A} \boxtimes \underline{M} \to \underline{\mathbb{Z}} \boxtimes \underline{M}$$

induced by the unit $\eta : \underline{A} \to \underline{\mathbb{Z}}$ is an isomorphism.

Remark 2.3.42. Base change $\underline{\mathbb{Z}} \boxtimes -$ from $\underline{A}$ to $\underline{\mathbb{Z}}$ may be calculated levelwise by the formula

$$(\underline{\mathbb{Z}} \boxtimes \underline{M})(G/H) \cong {}^{\underline{M}(G/H)}\Big/_{\left\langle [H:K]x - \mathrm{tr}_K^H \mathrm{res}_K^H x \right\rangle} .$$

This follows from the fact that $\underline{\mathbb{Z}}$ is a quotient of $\underline{A}$ and the box product commutes with colimits in each variable.

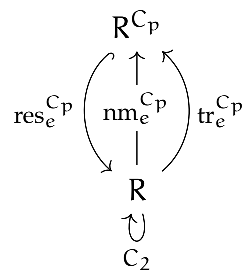
We now aim to characterize the free cohomological Mackey functors. To do so, we must first introduce another class of Mackey functors.

Definition 2.3.43. Let FP be the right adjoint to the forgetful functor from Mackey functors to G-modules which sends $\underline{M}$ to $\underline{M}(G/e)$ with its Weyl group action. If V is a G-module, then $FP(V)$ is called its *fixed point Mackey functor*.

If k is a G-ring, then $FP(k)$ is a Tambara functor, and we call it a *fixed point Tambara functor* of k. Forgetting norms, we obtain *fixed point $\mathcal{O}$-Tambara functors*.

Explicitly, if V is a G-module, its fixed point Mackey functor $FP(V)$ is given at level G/H by $FP(V)(G/H) = V^H$. Restriction is given by inclusion of fixed points, transfer is the sum over orbits, and norm is the product over orbits.

Example 2.3.44. Let R be a C_p-ring. The fixed point Tambara functor $FP(R)$ is described by the Lewis diagram



where

$$\mathrm{res}_e^{C_p}(y) = y,$$

$$\mathrm{tr}_e^{C_p}(x) = \sum_{g \in C_p} g \cdot x,$$

$$\mathrm{nm}_e^{C_p}(x) = \prod_{g \in C_p} g \cdot x.$$

Example 2.3.45. If R is a G-ring, the transfer tr_K^H and norm nm_K^H of the fixed point Tambara functor $FP(R)$ are given by:

$$\begin{aligned} \mathrm{tr}_K^H \colon R^K &\longrightarrow R^H \\ x &\longmapsto \sum_{g \in W_H(K)} g \cdot x, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{nm}_K^H \colon R^K &\longrightarrow R^H \\ x &\longmapsto \prod_{g \in W_H(K)} g \cdot x. \end{aligned}$$

Example 2.3.46. Consider $\mathbb{Z}$ equipped with a trivial G-action. The fixed point Tambara functor of this G-ring is the constant Tambara functor $\underline{\mathbb{Z}}$.

Example 2.3.47. Let $H \leq G$. There is an isomorphism of $\underline{\mathbb{Z}}$-modules

$$\underline{\mathbb{Z}}\{x_{G/H}\} \cong FP(\mathbb{Z}[G/H])$$

between the free $\underline{\mathbb{Z}}$-module on a generator at level G/H and the fixed point Mackey functor (Definition 2.3.43) of the permutation G-module $\mathbb{Z}[G/H]$.

Remark 2.3.48. Any Mackey functor $\underline{M}$ has a canonical homomorphism

$$\underline{M} \to \mathrm{FP}(\underline{M}(G/e)).$$

The right-hand side is a $\underline{\mathbb{Z}}$-module, so this homomorphism factors through the base change to $\underline{\mathbb{Z}}$ to give another homomorphism.

$$\underline{\mathbb{Z}} \boxtimes \underline{M} \to \mathrm{FP}(\underline{M}(G/e)).$$

Recall that a *permutation* G*-module* is a G-module V where G acts by permutation of the basis vectors. For example, if T is a finite G-set, then $\mathbb{Z}[T]$ is a permutation module.

Proposition 2.3.49. *A free cohomological Mackey functor is of the form* $\mathrm{FP}(V)$ *where* V *is a permutation* G*-module.*

Proof. Any free cohomological Mackey functor $\underline{M}$ is of the form

$$\underline{M} = \bigoplus_{i \in I} \underline{\mathbb{Z}}\{x_{T_i}\}$$

where each T_i is a finite G-set. Decomposing these finite G-sets into orbits, we may write $\underline{M}$ as a sum of free $\underline{\mathbb{Z}}$-modules on transitive finite G-sets:

$$\underline{M} \cong \bigoplus_{j \in J} \underline{\mathbb{Z}}\{x_{G/H_j}\}.$$

From Example 2.3.47, each summand $\underline{\mathbb{Z}}\{x_{G/H_j}\}$ is a fixed point functor $\mathrm{FP}(\mathbb{Z}[G/H_j])$ of a permutation module. The functor FP commutes with direct sums [TW95, Proposition 2.3], and direct sums of permutation modules are again permutation modules. □

Remark 2.3.50. Thevenaz–Webb prove in [TW95, Theorem 16.5] that every cohomological Mackey functor is a quotient of a fixed point functor $\mathrm{FP}(V)$ for some permutation G-module V, but they do not explicitly describe the free objects.

2.4 Commutative algebra of incomplete Tambara functors

Incomplete Tambara functors are analogous to commutative rings in equivariant algebra, and this analogy runs deeper than the surface-level comparison. Like commutative rings, incomplete Tambara functors have a theory of ideals and a theory of localization, but as with most things in equivariant algebra there are some subtle differences from what you might expect.

2.4.1 Free incomplete Tambara functors

We begin by discussing free incomplete Tambara functors – the polynomial rings of equivariant algebra.

Definition 2.4.1 ([BH18, Definition 5.4])**.** For a finite G-set T, define

$$\underline{A}^{\mathcal{O}}[x_T] := \mathcal{P}^G_{\mathcal{O}}(T, -)^+$$

be the group completion of the functor represented by T. We will refer to this as the *free $\mathcal{O}$-Tambara functor on a generator at level* T.

More generally, if $\underline{R}$ is an $\mathcal{O}$-Tambara functor, the *free $\underline{R}$-algebra on a generator at level* T, denoted $\underline{R}[x_T]$, is defined by

$$\underline{R}[x_T] := \underline{R} \boxtimes \underline{A}^{\mathcal{O}}[x_T].$$

Remark 2.4.2. The free $\mathcal{O}$-Tambara functor $\underline{A}^{\mathcal{O}}[x_T]$ represents evaluation at T:

$$\mathcal{O}\text{-}\mathcal{T}\text{amb}(\underline{A}^{\mathcal{O}}[x_T], \underline{R}) \cong \underline{R}(T).$$

Remark 2.4.3. Since $\underline{R}[x_T] \boxtimes \underline{R}[x_S] \cong \underline{R}[x_{T \sqcup S}]$, any free incomplete Tambara functor on a finite number of generators is of the form $\underline{R}[x_U]$ for a single finite G-set U. When a free incomplete Tambara functor has infinitely many generators, then it is a direct limit of ones with finitely many generators.

Example 2.4.4. We describe the free Green functor $\underline{A}^{\mathcal{O}^{\text{triv}}}[x_{G/G}]$ on a generator at level G/G. By [BH19, Corollary 2.11], there is an isomorphism of Mackey functors

$$\underline{A}^{\mathcal{O}^{\text{triv}}}[x_{G/G}] \cong \mathbb{Z}[x] \otimes \underline{A},$$

where $E \otimes \underline{A}$ is the Mackey functor which sends a G-set T to $E \otimes \underline{A}(T)$, for an abelian group E.

The norm, transfer, and restriction of a free incomplete Tambara functor are given by post-composition, i.e. applying $T_\phi \circ -$, $N_\phi \circ -$, or $R_\psi \circ -$ to a polynomial $\Sigma = T_h \circ N_g \circ R_f$. For the transfer and restriction, simple descriptions of this operation are possible.

Proposition 2.4.5. *Let*

$$\Sigma = [G/H \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y] \in \underline{A}^{\mathcal{O}}[x_{G/H}](Y) = \mathcal{P}^G_{\mathcal{O}}(G/H, Y)^+$$

be a generator of $\underline{A}^{\mathcal{O}}[x_{G/H}](Y)$*. If* $j : Y \to Z$ *is a function of* G*-sets, then*

$$\text{tr}_j(\Sigma) = [G/H \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{j \circ h} Z].$$

If $k : Z \to Y$ *is a function of* G*-sets, then*

$$\text{res}_k(\Sigma) = [G/H \xleftarrow{f \circ \pi_A} A \times_Y Z \xrightarrow{g \times \text{id}} B \times_Y Z \xrightarrow{\pi_Z} Z].$$

Proof. The transfer along j is pre-composition with T_j:

$$\text{tr}_j(\Sigma) = T_j \circ \Sigma = T_j \circ (T_h \circ N_g \circ R_f) = T_{j \circ h} \circ N_g \circ R_f.$$

Restriction along k is pre-composition with R_k:

$$\text{res}_k(\Sigma) = R_k \circ (T_h \circ N_g \circ R_f).$$

Using Theorem 2.3.5, we may commute R_k with T_h and N_g in turn. The diagram

$$\begin{array}{ccc} B \times_Y Z & \xrightarrow{\pi_Z} & Z \\ \downarrow{\scriptstyle \pi_B} & & \downarrow{\scriptstyle k} \\ B & \xrightarrow{h} & Y \end{array}$$

demonstrates that $R_k \circ T_h = T_{\pi_Z} \circ R_{\pi_B}$, and the diagram

$$\begin{array}{ccc} A \times_Y Z & \xrightarrow{g \times \mathrm{id}} & B \times_Y Z \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{g} & B \end{array}$$

demonstrates that $R_{\pi_B} \circ N_g = N_{g\times \mathrm{id}} \circ R_{\pi_A}$. Therefore,

$$\begin{aligned} \mathrm{res}_k(\Sigma) &= R_k \circ (T_h \circ N_g \circ R_f) \\ &= T_{\pi_Z} \circ R_{\pi_B} \circ N_g \circ R_f \\ &= T_{\pi_Z} \circ N_{g\times \mathrm{id}} \circ R_{\pi_A} \circ R_f \\ &= T_{\pi_Z} \circ N_{g\times \mathrm{id}} \circ R_{f \circ \pi_A}. \end{aligned}$$

□

Proposition 2.4.6. *For any orbits G/H and G/K, and an indexing category $\mathcal{O}$, we have that*

$$\mathcal{P}^G_{\mathcal{O}}(G/H, G/K)^+$$

is the free abelian group with basis given by isomorphism classes of polynomials of the form

$$G/H \xleftarrow{f} S \xrightarrow{g} G/J \xrightarrow{h} G/K$$

where $g \in \mathcal{O}$.

Proof. Let $\Sigma = [G/H \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} G/K] \in \mathcal{P}^G_{\mathcal{O}}(G/H, G/K)$. We will explain how to obtain Σ via addition and multiplication of polynomials in $\mathcal{P}^G_{\mathcal{O}}(G/H, G/K)$ of the form above. Recall the semiring operations in $\mathcal{P}^G_{\mathcal{O}}(G/H, G/K)$ from Theorem 2.3.6.

First, observe that g has the form

$$g : A \to \mathrm{im}(g) \sqcup B'$$

and we can write

$$\Sigma = [G/H \xleftarrow{f} A \xrightarrow{g} \mathrm{im}(g) \xrightarrow{h} G/K] + [G/H \leftarrow \emptyset \to B' \xrightarrow{h} G/K].$$

Note that the right summand above may be written as a sum of polynomials $[G/H \leftarrow \emptyset \to B_i' \xrightarrow{h} G/K]$, where $B' = \bigsqcup_{i\in I} B_i'$ with each B_i' transitive. Thus we may reduce to the case where $g\colon A \to B$ is surjective.

Second, we may decompose

$$B = \bigsqcup_{i\in I} B_i, \quad A = \bigsqcup_{i\in I} g^{-1}(B_i)$$

where B_i is transitive for all $i \in I$. Then

$$\Sigma = \sum_{i\in I} [G/H \xleftarrow{f} g^{-1}(B_i) \xrightarrow{g} B_i \xrightarrow{h} G/K],$$

so we may further reduce to the case where $B = G/J$ is transitive. This leaves us with a polynomial of the desired form:

$$[G/H \leftarrow S \to G/J \to G/K]. \qquad \square$$

Corollary 2.4.7. *The free incomplete Tambara functor* $\underline{A}^{\mathcal{O}}[x_{G/H}]$ *is generated as a Green functor by polynomials of the form*

$$[G/H \xleftarrow{f} G/L \xrightarrow{g} G/K \xrightarrow{=} G/K]$$

with $g \in \mathcal{O}$.

Proof. At level G/K, the abelian group $\underline{A}^{\mathcal{O}}[x_{G/H}](G/K) = \mathcal{P}_G^{\mathcal{O}}(G/H, G/K)^+$ is generated by isomorphism classes of polynomials of the form:

$$G/H \xleftarrow{f} S \xrightarrow{g} G/J \xrightarrow{h} G/K.$$

If J is subconjugate to K, then this polynomial is isomorphic to one of the form

$$G/X \xleftarrow{f'} S \xrightarrow{g'} G/J \xrightarrow{\pi} G/K,$$

where $\pi\colon G/J \to G/K$ is the canonical projection. This is the transfer from J to K applied to the polynomial

$$G/X \xleftarrow{f'} S \xrightarrow{g'} G/J \xrightarrow{=} G/J,$$

that is, a transfer of an element from $\underline{A}^{\mathcal{O}}[x_{G/H}](G/J)$.

If J is conjugate to K, then this polynomial is isomorphic to one of the form

$$G/X \xleftarrow{f'} S \xrightarrow{g'} G/K \xrightarrow{=} G/K.$$

Let Σ denote the isomorphism class of the polynomial above. Write $S = \coprod_{j \in J} G/L_j$. Then the result follows by observing that

$$\Sigma = \prod_{j \in J} [G/H \leftarrow G/L_j \to G/K \xrightarrow{\mathrm{id}} G/K]. \qquad \square$$

Remark 2.4.8. In fact, if we include polynomials $[G/H \leftarrow S \to G/J \to G/K]$ where S is not necessarily transitive, then the proof above also shows that $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is generated as a Mackey functor by this collection.

We can use Corollary 2.4.7 to explicitly describe some free incomplete Tambara functors. Below, we record the four free incomplete Tambara functors for cyclic groups of prime order. For $p = 2$, these free incomplete Tambara functors were described in [BH19, Section 3], and the situation here is completely analogous. As in [BH19], we adopt the convention that the level G/G is the *fixed* level (because the Weyl action is trivial there) and G/e is the *underlying* level.

These examples are tedious if straightforward applications of Proposition 2.4.6.

Example 2.4.9. The free Green functor on a fixed generator is described by the Lewis diagram

$$\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{C_p/C_p}]: \qquad \begin{array}{c} \mathbb{Z}[x,t]/\langle t^2 - pt\rangle \\ t\mapsto p \ \big\downarrow \ \big\uparrow \ \cdot t \\ \mathbb{Z}[x] \\ \circlearrowleft \\ \text{trivial} \end{array}$$

This is just an instance of Example 2.4.4.

Example 2.4.10. Let γ be a generator of the cyclic group C_p. The free Green functor on an underlying generator is described by the Lewis diagram

$$\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{C_p/e}]: \qquad \begin{array}{c} \mathbb{Z}\left[\{t_{\vec{v}}\}_{\vec{v}\in\mathbb{N}^{\times p}}\right]/I \\ \mathrm{res}_e^{C_p} \ \big\downarrow \ \big\uparrow \ \mathrm{tr}_e^{C_p} \\ \mathbb{Z}[x_0,\dots,x_{p-1}], \\ \circlearrowleft \\ \gamma \end{array}$$

where I is the ideal of $\mathbb{Z}\left[\{t_{\vec{v}}\}_{\vec{v}\in\mathbb{N}^{\times p}}\right]$ generated by relations

$$t_{\vec{0}}^2 - pt_{\vec{0}}, \qquad t_{\vec{v}} - t_{\gamma\cdot\vec{v}}, \qquad \text{and} \qquad t_{\vec{v}}\cdot t_{\vec{w}} - \sum_{i=0}^{p-1} t_{\vec{v}+\gamma\cdot\vec{w}},$$

where $\gamma\cdot\vec{v}$ is shorthand for the cyclic permutation of indices:

$$\gamma\cdot(v_0,\dots,v_{p-1}) = (v_1,\dots,v_{p-1},v_0).$$

The Weyl action on $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{C_p/e}](C_p/e)$ is given by permuting the variables:

$$\gamma^i\cdot x_j = x_{i+j}$$

with indices taken mod p.

For $\vec{v} = (v_0,\dots,v_{p-1}) \in \mathbb{N}^{\times p}$, write $x^{\vec{v}}$ as shorthand for

$$x^{\vec{v}} := x_0^{v_0}x_1^{v_1}\cdots x_{p-1}^{v_{p-1}}.$$

The restriction and transfer of $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{C_p/e}]$ are then determined by the formulae:

$$\mathrm{res}_e^{C_p}(t_{\vec{v}}) = \sum_{i=0}^{p-1} x^{\gamma^i \cdot \vec{v}},$$
$$\mathrm{tr}_e^{C_p}(x^{\vec{v}}) = t_{\vec{v}}.$$

Example 2.4.11. The free Tambara functor on a fixed generator is described by the Lewis diagram

$$\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{C_p/C_p}]: \qquad \begin{array}{c} \mathbb{Z}[x,t,n]/\langle t^2 - pt, tn - tx^p \rangle \\ \mathrm{res}_e^{C_p} \downarrow \quad \uparrow \mathrm{nm}_e^{C_p} \quad \uparrow \mathrm{tr}_e^{C_p} \\ \mathbb{Z}[x], \\ \circlearrowleft \\ \text{trivial} \end{array}$$

The restriction, and transfer are determined by

$$\mathrm{res}_e^{C_p}(t) = p \quad \mathrm{res}_e^{C_p}(x) = x, \quad \mathrm{res}_c^{C_p}(n) = x^p \quad \text{and} \quad \mathrm{tr}_e^{C_p}(f) = tf.$$

The norm is determined by the norm of $\underline{A}$ (2.3.20), $\mathrm{nm}_e^{C_p}(x) = n$, and the formula for the norm of a sum [Tam93, Proposition 4.1], [Maz13, Section 1.4.1].

Remark 2.4.12 (cf. [BH18, Warning 5.5])**.** The free Tambara functor on a fixed generator is not flat: as a Mackey functor, it is isomorphic to

$$\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{C_p/C_p}] \cong \bigoplus_{n \geq 0} \underline{A}\{x^n\} \oplus \bigoplus_{j \in J} \underline{I},$$

where $\underline{I}$ is the augmentation ideal and J is a $\mathbb{Z}$-basis for the ideal generated by $n - x^2$ in $\mathbb{Z}[n,x]$. By Lemma 2.3.38, the augmentation ideal $\underline{I}$ is not flat.

Example 2.4.13. The free Tambara functor on an underlying generator is described by the Lewis diagram

$$\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{C_p/e}]: \qquad \begin{array}{c} \mathbb{Z}[n]\left[\{t_{\vec{v}}\}_{\vec{v}\in\mathbb{N}^{\times p}}\right]/I \\ \mathrm{res}_e^{C_p}\Big\downarrow \; \Big\uparrow \mathrm{nm}_e^{C_p} \; \Big\uparrow \mathrm{tr}_e^{C_p} \\ \mathbb{Z}[x_0, x_1, \ldots, x_{p-1}], \\ \circlearrowleft \\ \gamma \end{array}$$

where I is the ideal of $\mathbb{Z}[n]\left[\{t_{\vec{v}}\}_{\vec{v}\in\mathbb{N}^{\times p}}\right]$ generated by relations

$$t_{\vec{0}}^2 - pt_{\vec{0}}, \qquad t_{\vec{v}} - t_{\gamma\cdot\vec{v}}, \qquad t_{\vec{v}}\cdot t_{\vec{w}} - \sum_{i=0}^{p-1} t_{\vec{v}+\gamma\cdot\vec{w}}, \qquad \text{and} \qquad nt_{\vec{v}} - t_{\vec{v}+\vec{1}},$$

where $\gamma \cdot \vec{v}$ is shorthand for the cyclic permutation of indices:

$$\gamma\cdot(v_0, \ldots, v_{p-1}) = (v_1, \ldots, v_{p-1}, v_0)$$

and $\vec{1} = (1, 1, \ldots, 1) \in \mathbb{N}^{\times p}$.

The Weyl action on $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{C_p/e}](C_p/e)$ is given by permuting the variables:

$$\gamma^i \cdot x_j = x_{i+j}$$

with indices taken mod p.

For $\vec{v} = (v_0, \ldots, v_{p-1}) \in \mathbb{N}^{\times p}$, write $x^{\vec{v}}$ as shorthand for

$$x^{\vec{v}} := x_0^{v_0} x_1^{v_1} \cdots x_{p-1}^{v_{p-1}}.$$

The restriction and transfer of $\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{C_p/e}]$ are then determined by the formulae:

$$\begin{aligned} \mathrm{res}_e^{C_p}(t_{\vec{v}}) &= \sum_{i=0}^{p-1} x^{\gamma^i\cdot\vec{v}}, \\ \mathrm{res}_e^{C_p}(n) &= x^{\vec{1}} = x_0 x_1 \cdots x^{p-1}, \\ \mathrm{tr}_e^{C_p}(x^{\vec{v}}) &= t_{\vec{v}}. \end{aligned}$$

The norm is determined by the norm of $\underline{A}$ (2.3.20), $\mathrm{nm}_e^{C_p}(x_i) = n$ for all $i \in \{0, 1, \ldots, p-1\}$, and the formula for the norm of a sum [Tam93, Proposition 4.1], [Maz13, Section 1.4.1].

2.4.2 Ideals of incomplete Tambara functors

There is a robust theory of ideals for incomplete Tambara functors, but care must be taken when talking about prime ideals.

Definition 2.4.14 ([BH18, Definition 5.6]). If $\underline{R}$ is an $\mathcal{O}$-Tambara functor, then an *$\mathcal{O}$-ideal* is a sub-Mackey functor $\underline{J}$ such that:

(a) the multiplication on $\underline{R}$ makes $\underline{J}$ an $\underline{R}$-module;

(b) if $f : S \to T$ is in $\mathcal{O}$ and is surjective, then $\underline{J}$ is closed under nm_f, where nm_f is the norm along f in $\underline{R}$ (see Definition 2.3.11).

The surjective condition allows for ideals that don't contain units $1 \in \underline{R}(T)$ for each T, because 1 is the norm associated to the unique morphism of G-sets $\emptyset \to T$. See [BH18, Remark 5.7].

Proposition 2.4.15 ([BH18, Example 5.10]). *The data of an $\mathcal{O}$-ideal $\underline{J}$ of $\underline{R}$ is equivalent to a collection of ideals $\underline{J}(G/K)$ of $\underline{R}(G/H)$ for all G-orbits G/H such that*

(a) $res^H_K(\underline{J}(G/H)) \subseteq \underline{J}(G/K)$*,*

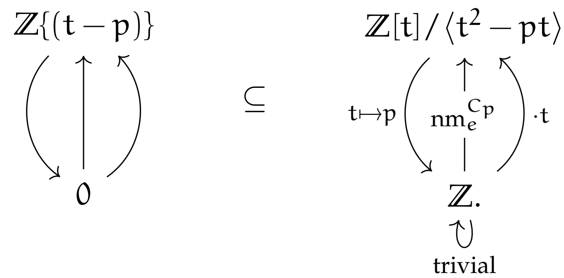
(b) $tr^H_K(\underline{J}(G/K)) \subseteq \underline{J}(G/H)$*,*

(c) $nm^H_K(\underline{J}(G/K)) \subseteq \underline{J}(G/H)$ *whenever H/K is an admissible H-set for $\mathcal{O}$.*

Example 2.4.16. When $\mathcal{O}$ is the complete indexing system, an $\mathcal{O}^{cplt}$-ideal of $\underline{R}$ is a *Tambara ideal* in the sense of Nakaoka [Nak12a, Definition 2.1].

Example 2.4.17. We have already seen one example of an ideal of a Tambara functor: the *augmentation ideal* of the Burnside $\mathcal{O}$-Tambara functor. This is the kernel of the morphism of $\mathcal{O}$-Tambara functors $\underline{A} \to \underline{\mathbb{Z}}$ that sends the class of a finite H-set in $\underline{A}(G/H)$ to its cardinality.

For $G = C_p$, the augmentation ideal has the following Lewis diagram:



Because an $\mathcal{O}$-ideal $\underline{I}$ is in particular a submodule of $\underline{R}$, the quotient $\underline{R}/\underline{I}$ is a Green functor, and it receives a homomorphism from the underlying Green functor of $\underline{R}$. Fortunately, when $\underline{I}$ is an $\mathcal{O}$-ideal, then this Green functor becomes a Tambara functor.

Proposition 2.4.18 ([BH18, Proposition 2.11])**.** *Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor with $\mathcal{O}$-ideal $\underline{I}$. The Green functor $\underline{R}/\underline{I}$ has the structure of an $\mathcal{O}$-Tambara functor such that the natural homomorphism $\underline{R} \to \underline{R}/\underline{I}$ is a homomorphism of $\mathcal{O}$-Tambara functors.*

Definition 2.4.19. Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $B \subseteq \underline{R}$ be a collection of elements of $\underline{R}$ at various levels. The *$\mathcal{O}$-ideal generated by* B is the smallest $\mathcal{O}$-ideal containing B.

If B consists of a single element $b \in \underline{R}(U)$, then the ideal generated by B is *principal* and we write $\langle b \rangle_{\mathcal{O}}$ for this ideal.

Proposition 2.4.20 (cf. [Nak12a, Proposition 3.6])**.** *Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $\underline{I}$ be an $\mathcal{O}$-ideal of $\underline{R}$. If $\underline{I}$ is generated by a finite collection of elements of $\underline{R}$, then $\underline{I}$ is a principal ideal.*

One nice consequence of this proposition is that to describe finitely generated ideals, it suffices to describe the principal ones. Nakaoka does exactly that for Tambara ideals, and his proof also works for $\mathcal{O}$-ideals provided one restricts to only those norms which are admissible for $\mathcal{O}$.

Proposition 2.4.21 (cf. [Nak12a, Proposition 3.4]). *Let $a \in \underline{R}(U)$. At the level of a finite G-set V, a principal $\mathcal{O}$-ideal $\langle a \rangle_{\mathcal{O}}$ is given by*

$$\langle a \rangle_{\mathcal{O}}(V) = \left\{ \mathrm{tr}_f(b \cdot \mathrm{nm}_g \, \mathrm{res}_h(a)) \middle| U \xleftarrow{h} X \xrightarrow{g} Y \xrightarrow{f} V \in \mathcal{P}^G_{\mathcal{O}}(U, V), \; b \in \underline{R}(x) \right\}$$

The usual operations on ideals can be performed on $\mathcal{O}$-ideals as well.

Definition 2.4.22 (cf. [Nak12a, Section 3]). Let $\underline{I}$ and $\underline{J}$ be $\mathcal{O}$-ideals of an $\mathcal{O}$-Tambara functor $\underline{R}$.

(a) the *intersection* $\underline{I} \cap \underline{J}$ is the $\mathcal{O}$-ideal with $(\underline{I} \cap \underline{J})(G/H) = \underline{I}(G/H) \cap \underline{J}(G/H)$

(b) the *sum* $\underline{I} + \underline{J}$ is the $\mathcal{O}$-ideal with $(\underline{I} + \underline{J})(G/H) = \underline{I}(G/H) + \underline{J}(G/H)$

(c) the *product* of $\underline{I}$ and $\underline{J}$, denoted $\underline{I}\underline{J}$, is the $\mathcal{O}$-ideal generated by products ij with $i \in \underline{I}(G/H)$, $j \in \underline{J}(G/H)$ for $H \leq G$.

With the product of ideals, we can define what it means for an $\mathcal{O}$-ideal to be prime.

Definition 2.4.23 (cf. [Nak12a, Definition 4.1]). An $\mathcal{O}$-ideal $\underline{I}$ of an $\mathcal{O}$-Tambara functor $\underline{R}$ is *prime* if for any $a \in \underline{R}(G/H)$ and any $b \in \underline{R}(G/K)$, whenever $\langle a \rangle_{\mathcal{O}} \langle b \rangle_{\mathcal{O}} \subseteq \underline{I}$, then either $a \in \underline{I}(G/H)$ or $b \in \underline{I}(G/K)$.

Lewis gives another definition of prime for Green ideals using the external product of elements in a Green functor. To compare Lewis's definition of prime with Nakaoka's, we first define the external product.

Recall that the free Mackey functor $\underline{A}\{x_U\}$ is the representable functor $\mathcal{P}^G_{\mathrm{iso}}(U, -)$. By the Yoneda lemma, an element $a \in \underline{R}(U)$ yields a homomorphism of Mackey functors (a natural transformation) $\underline{A}\{x_U\} \to \underline{R}$ which is given on level V by

$$\begin{aligned} \underline{A}\{x_U\}(V) &\longrightarrow \underline{R}(V) \\ \Sigma &\longmapsto \underline{R}(\Sigma)(a), \end{aligned} \tag{2.4.24}$$

where $\Sigma \in \underline{A}\{x_U\}(W) = \mathcal{P}^G_{\text{iso}}(U, V)$ is an isomorphism class of spans. If we write

$$\Sigma = T_f \circ R_g = \left[U \xleftarrow{f} W \xrightarrow{g} V\right]$$

then this homomorphism is given on level V by $\underline{R}(\Sigma)(a) = \text{tr}_f(\text{res}_g(a))$.

Definition 2.4.25 ([Lew81, Section 3])**.** Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $a \in \underline{R}(U)$ and $b \in \underline{R}(V)$. Consider $a \colon \underline{A}\{x_U\} \to \underline{R}$ and $b \colon \underline{A}\{x_V\} \to \underline{R}$ as homomorphisms of Mackey functors by the Yoneda Lemma. The *external product* $a \times b$ is the element of $\underline{R}(U \times V)$ that corresponds to the homomorphism of Mackey functors

$$a \times b \colon \underline{A}\{x_{U \times V}\} \cong \underline{A}\{x_U\} \boxtimes \underline{A}\{x_V\} \xrightarrow{a \boxtimes b} \underline{R} \boxtimes \underline{R} \to \underline{R}.$$

The external product allows us to treat $\mathcal{O}$-Tambara functors as a kind of $\mathcal{S}et^G$-graded commutative ring. We can also realize the external product in terms of restrictions and transfers.

Lemma 2.4.26. *Let* $\underline{R}$ *be an* $\mathcal{O}$*-Tambara functor and let* $a \in \underline{R}(U)$*,* $b \in \underline{R}(V)$*. Then*

$$a \times b = \text{res}_{\pi_U}(a) \cdot \text{res}_{\pi_V}(b) \in \underline{R}(U \times V),$$

where $\pi_U \colon U \times V \to U$ *and* $\pi_V \colon U \times V \to Y$ *are the canonical projections.*

Proof. The external product $a \times b$ is the element of $\underline{R}(U \times V)$ that corresponds to the natural transformation

$$a \times b \colon \underline{A}\{x_{U \times V}\} \cong \underline{A}\{x_U\} \boxtimes \underline{A}\{x_V\} \xrightarrow{a \boxtimes b} \underline{R} \boxtimes \underline{R} \xrightarrow{\mu} \underline{R}.$$

The Yoneda lemma tells us that this element is $(a \times b)(\text{id}_{U \times V})$, where $\text{id}_{U \times V}$ is isomorphism class of spans

$$\left[U \times V \xleftarrow{\text{id}} U \times V \xrightarrow{\text{id}} U \times V\right] \in \underline{A}\{x_{U \times V}\}(U \times V).$$

Under the isomorphism $\underline{A}\{x_{U\times V}\} \cong \underline{A}\{x_U\} \boxtimes \underline{A}\{x_V\}$, this element corresponds to the tensor product of span classes

$$R_{\pi_U} \otimes R_{\pi_V} \in \underline{A}\{x_U\} \boxtimes \underline{A}\{x_V\})(U \times V),$$

where

$$R_{\pi_U} = \left[U \xleftarrow{\pi_U} U \times V \xrightarrow{\text{id}} U \times V\right], \text{ and}$$
$$R_{\pi_V} = \left[V \xleftarrow{\pi_V} U \times V \xrightarrow{\text{id}} U \times V\right].$$

(Recall the notation R_f from Definition 2.3.4).

Applying the natural transformation $a \boxtimes b$ to $R_{\pi_U} \otimes R_{\pi_V}$, we get by (2.4.24)

$$\underline{R}(R_{\pi_U})(a) \otimes \underline{R}(R_{\pi_V}(b)) = \text{res}_{\pi_U}(a) \otimes \text{res}_{\pi_V}(b).$$

Therefore, the external product corresponds to the element

$$(a \times b)(\text{id}_{U\times V}) = (\mu \circ (a \boxtimes b))(\text{id}_{U\times V}) = \text{res}_{\pi_U}(a) \cdot \text{res}_{\pi_V}(b) \qquad \square$$

Lewis uses the external product to define prime ideals.

Definition 2.4.27 (cf. [Lew81, Definition 3.1(f)])**.** An ideal $\underline{I}$ of an $\mathcal{O}$-Tambara functor is *externally prime* if for all $a \in \underline{R}(G/H)$ and $b \in \underline{R}(G/K)$,

$$a \times b \in \underline{I}(G/H \times G/K) \implies a \in \underline{I}(G/H) \text{ or } b \in \underline{I}(G/K).$$

We begin by noting that if an ideal is externally prime, then it is prime.

Proposition 2.4.28. *If $\underline{I} \subseteq \underline{R}$ is an externally prime $\mathcal{O}$-ideal, then $\underline{I}$ is prime.*

Proof. Suppose that $a \in \underline{R}(U)$, $b \in \underline{R}(V)$ such that $\langle a\rangle_{\mathcal{O}}\langle b\rangle_{\mathcal{O}} \subseteq \underline{I}$. In particular, the product ideal $\langle a\rangle_{\mathcal{O}}\langle b\rangle_{\mathcal{O}}$ contains the external product $a \times b$, and therefore either $a \in \underline{I}(U)$ or $b \in \underline{I}(V)$. Hence, either $\langle a\rangle_{\mathcal{O}} \subseteq \underline{I}$ or $\langle b\rangle_{\mathcal{O}} \subseteq \underline{I}$. $\square$

However, not all prime ideals are externally prime.

Example 2.4.29. Consider the Burnside C_2-Tambara functor $\underline{A}$, and the elements $3 \in \underline{A}(C_2/e)$ and $(t-2) \in \underline{A}(C_2/C_2)$. The external product of these elements is

$$3 \times (t-2) = \operatorname{res}_{\pi_1}(3)\operatorname{res}_{\pi_2}(t-2) = 3 \cdot \operatorname{res}_e^{C_2}(t-2) = 3 \cdot (2-2) = 0,$$

where π_1 and π_2 are the projections

$$C_2/e \xleftarrow{\pi_1} C_2/e \times C_2/C_2 \xrightarrow{\pi_2} C_2/C_2.$$

Hence, any prime Tambara ideal of $\underline{A}$ that contains neither 3 nor $t-2$ is not externally prime. For example, zero ideal of $\underline{A}$ is prime by [Nak12a, Theorem 4.40]. The zero ideal contains neither of these elements, but it does contain their external product.

After considering prime ideals, the next natural thing to consider is maximal ideals.

Definition 2.4.30. An $\mathcal{O}$-ideal of $\underline{R}$ is *maximal* if it is a maximal element of the poset of proper $\mathcal{O}$-ideals of $\underline{R}$.

The standard argument shows that maximal $\mathcal{O}$-ideals are prime, so we won't repeat it here.

Proposition 2.4.31. *A maximal $\mathcal{O}$-ideal is prime.*

Example 2.4.32. Consider the Burnside C_2-Tambara functor $\underline{A}$. The Tambara ideal generated by $2 \in \underline{A}(C_2/e)$ is the only maximal ideal of $\underline{A}$ by [Nak14, Example 6.14]. The quotient by this maximal ideal is the constant C_2-Tambara functor $\underline{\mathbb{Z}/2}$.

2.4.3 Localization

We now discuss localization for incomplete Tambara functors. The theory of localization is not nearly as well-behaved as that of ideals, but it is still a useful tool in the equivariant setting.

Definition 2.4.33 ([BH18, Definition 5.21])**.** Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $\underline{S} = \{(a_i, T_i) \mid a_i \in \underline{R}(T_i),\ i \in I\}$ be a collection of elements in the values of $\underline{R}$ at various finite G-sets.

A morphism $\phi : \underline{R} \to \underline{R}'$ of $\mathcal{O}$-Tambara functors *inverts* $\underline{S}$ if $\phi(a_i)$ is a unit in $\underline{R}'(T_i)$ for all $i \in I$.

Let $\phi : \underline{R} \to \underline{S}^{-1}\underline{R}$ be the initial homomorphism of $\mathcal{O}$-Tambara functors which inverts $\underline{S}$. We will refer to $\underline{S}^{-1}\underline{R}$ as the *localization of* $\underline{R}$ *at* $\underline{S}$.

If $\underline{M}$ is an $\underline{R}$-module, define $S^{-1}\underline{M} := \underline{S}^{-1}\underline{R} \boxtimes_{\underline{R}} \underline{M}$.

The initial homomorphism of $\mathcal{O}$-Tambara functors which inverts $\underline{S}$ exists by [BH18, Theorem 5.23]. There is also an explicit construction of the localization $\underline{S}^{-1}\underline{R}$ for Tambara functors due to Nakaoka [Nak12b]. Below, we expand Nakoka's construction to include incomplete Tambara functors, and prove that it has the universal property of the localization above.

Recall that a semi-Mackey functor is a product-preserving functor $\mathcal{A}^G \to \mathcal{S}et$. While Mackey functors are valued in abelian groups, semi-Mackey functors take values in commutative monoids (this follows from the product-preserving condition and the fact that $\mathcal{A}^G$ is pre-additive).

Construction 2.4.34. Any Tambara functor $\underline{R}$ has a multiplicative semi-Mackey functor $\underline{R}^{\mu}$ defined as follows. Define a functor $\underline{A}^G \to \mathcal{P}^G_{\mathcal{O}^{cplt}}$ which is the identity on objects and takes the isomorphism class of a span $X \leftarrow A \to Y$ to the class of the polynomial

$$X \leftarrow A \to Y \xrightarrow{\text{id}} Y.$$

By precomposition, this yields a forgetful functor $(-)^{\mu}$ from $\mathcal{T}amb_G$ to the category of semi-Mackey functors which is distinct from the forgetful functor in

Proposition 2.3.15. For a finite G-set T, $\underline{R}^{\mu}(T)$ is the multiplicative monoid of the commutative ring $\underline{R}(T)$, and the transfers in the resulting semi-Mackey functor $\underline{R}^{\mu}$ are the norms of $\underline{R}$.

We think of semi-Mackey sub-functors of the multiplicative semi-Mackey functor of a Tambara functor as multiplicatively closed sets in a commutative ring, and we may localize at any such semi-Mackey sub-functor [Nak12b, Proposition 4.4].

However, if we are to localize incomplete Tambara functors, we must modify the above construction. If an incomplete Tambara functor doesn't contain all norms, then the resulting multiplicative semi-Mackey functor will not have all of its transfers. We must consider incomplete semi-Mackey functors – semi-Mackey functors whose transfers are parameterized by an indexing category.

Definition 2.4.35 ([BH21, Definition 2.23]). For any indexing category $\mathcal{O}$, define the category $\mathcal{A}^G_{\mathcal{O}}$ whose objects are finite G-sets and whose morphisms are isomorphism classes of spans of finite G-sets

$$X \xleftarrow{f} A \xrightarrow{g} Y$$

such that f is any morphism of finite G-sets and $g \in \mathcal{O}$. This is a subcategory of the Burnside category $\mathcal{A}^G$.

Definition 2.4.36 (cf. [BH21, Definition 2.24]). For any indexing category $\mathcal{O}$, an *$\mathcal{O}$-semi-Mackey functor* is a product preserving functor $\underline{M}\colon \mathcal{A}^G_{\mathcal{O}} \to \mathcal{S}et$. An *incomplete semi-Mackey functor* is an $\mathcal{O}$-semi-Mackey functor for some $\mathcal{O}$. An incomplete semi-Mackey functor is an *incomplete Mackey functor* if $\underline{M}(T)$ is an abelian group for each finite G-set T.

A *morphism of semi-Mackey functors* is a natural transformation. We write $\mathcal{O}\text{-}\mathcal{M}\mathrm{ack}_G$ for the full subcategory of incomplete semi-Mackey functors spanned by the incomplete Mackey functors.

An incomplete semi-Mackey functor is a semi-Mackey functor that might be missing some of its transfers, much as an incomplete Tambara functor is a Tambara functor missing some norms. Those transfers that do exist are parameterized by the indexing category $\mathcal{O}$. Explicitly, an incomplete semi-Mackey functor has a transfer $\mathrm{tr}_K^H \colon \underline{M}(G/K) \to \underline{M}(G/H)$ whenever H/K is an admissible H-set for $\mathcal{O}$.

Example 2.4.37. A $\mathcal{O}^{\mathrm{cplt}}$-Mackey functor is a Mackey functor, while a $\mathcal{O}^{\mathrm{triv}}$-Mackey functor is a *coefficient system* – a presheaf on the orbit category of G.

Incomplete semi-Mackey functors allow us to construct the multiplicative semi-Mackey functor for incomplete Tambara functors, analogous to the multiplicative monoid of a commutative ring.

Definition 2.4.38. The *multiplicative $\mathcal{O}$-semi-Mackey functor* of an $\mathcal{O}$-Tambara functor $\underline{R}$ is the $\mathcal{O}$-semi-Mackey functor $\underline{R}^{\mu}$ given by precomposition with the functor $\mathcal{A}_{\mathcal{O}}^G \to \mathcal{P}_{\mathcal{O}}^G$ which is the identity on objects and sends the isomorphism class of a span $X \xleftarrow{f} A \xrightarrow{g} Y$ with $g \in \mathcal{O}$ to the isomorphism class of the polynomial

$$X \xleftarrow{f} A \xrightarrow{g} Y \xrightarrow{\mathrm{id}} Y.$$

Explicitly, $\underline{R}^{\mu}$ is the $\mathcal{O}$-semi-Mackey functor whose value $\underline{R}^{\mu}(T)$ at a finite G-set T is the multiplicative monoid of the commutative ring $\underline{R}(T)$. Those transfers that exist in $\underline{R}^{\mu}$ are the norms that exist in $\underline{R}$.

We now have the ingredients necessary to construct localizations of $\mathcal{O}$-Tambara functors.

Definition 2.4.39. Recall that for commutative monoids $N \subseteq M$, we say that N is *saturated* if N contains all divisors of elements in N. If N is not saturated, its *saturation* $\widetilde{N} \subseteq M$ is the monoid of divisors of elements of N:

$$\widetilde{N} := \{m \in M \mid a \cdot m \in N \text{ for some } a \in M\}.$$

Definition 2.4.40. Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $\underline{S} \subseteq \underline{R}^{\mu}$ be an $\mathcal{O}$-semi-Mackey subfunctor of the multiplicative $\mathcal{O}$-semi-Mackey functor of $\underline{R}$. We say that $\underline{S}$ is *restriction saturated* if

$$\underline{S}(X) \subseteq (\mathrm{res}_f \underline{S}(Y))^{\sim},$$

for each morphism $f\colon X \to Y$ of finite G-sets, where the superscript tilde denotes saturation of the monoid $\mathrm{res}_f \underline{S}(Y) \subseteq \underline{R}^{\mu}(X)$.

Remark 2.4.41. This condition asks that for each element $x \in \underline{S}(X)$, there is $y \in \underline{S}(Y)$ such that x divides $\mathrm{res}_f(y)$. Such a condition is necessary to define the transfer in the localization. For Tambara functors, any semi-Mackey subfunctor of $\underline{R}^{\mu}$ is automatically restriction saturated [Nak12b, Proposition 3.1]. Indeed, if $\underline{R}$ is an $\mathcal{O}$-Tambara functor, then $\underline{S} \subseteq \underline{R}^{\mu}$ satisfies

$$\underline{S}(X) \subseteq (\mathrm{res}_f \underline{S}(Y))^{\sim}$$

whenever $\underline{R}$ has a norm nm_f, that is, whenever $f \in \mathcal{O}$.

Example 2.4.42. In [The91, Proposition 14.8], Thévenaz describes how to localize a Green functor $\underline{R}$ at the coefficient system generated by a multiplicatively closed subset of $U \subseteq \underline{R}(G/G)$. The coefficient system generated by U is an example of a restriction saturated subfunctor of $\underline{R}^{\mu}$, but it is far from the only kind of restriction saturated subfunctor for Green functors.

Proposition 2.4.43 (cf. [Nak12b, Proposition 4.4])**.** *Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and let $\underline{S} \subseteq \underline{R}^{\mu}$ be restriction-saturated. Then the collection of rings*

$$\underline{S}^{-1}\underline{R}(T) := \underline{S}(T)^{-1}\underline{R}(T)$$

becomes an $\mathcal{O}$-Tambara functor and the ring homomorphisms

$$\begin{aligned} \underline{R}(T) &\longrightarrow \underline{S}^{-1}\underline{R}(T) \\ x &\longmapsto \frac{x}{1} \end{aligned}$$

become a morphism of $\mathcal{O}$-Tambara functors $\phi\colon \underline{R} \to \underline{S}^{-1}\underline{R}$.

Proof. We define restriction and norms (where they exist in $\underline{R}$) in $\underline{S}^{-1}\underline{R}$ by applying restriction and norm to both numerator and denominator:

$$\mathrm{res}_f\left(\frac{y}{t}\right) := \frac{\mathrm{res}_f(y)}{\mathrm{res}_f(t)}$$
$$\mathrm{nm}_f\left(\frac{x}{s}\right) := \frac{\mathrm{nm}_f(x)}{\mathrm{nm}_f(s)}$$

whenever $f\colon X \to Y$, $\frac{y}{t} \in \underline{S}^{-1}\underline{R}(Y)$, and $\frac{x}{s} \in \underline{S}^{-1}\underline{R}(X)$. Note that the norm above is only defined when $f \in \mathcal{O}$, so that $\underline{R}$ has a norm along f.

Transfers are slightly trickier to define. Suppose that $\frac{x}{s} \in \underline{S}^{-1}\underline{R}(X)$. Since $\underline{S}$ is restriction saturated, choose $t \in \underline{S}(Y)$ such that $\mathrm{res}_f(t) = as$ for some $a \in \underline{R}(X)$. Then define

$$\mathrm{tr}_f\left(\frac{x}{s}\right) := \frac{\mathrm{tr}_f(ax)}{t}.$$

When $f \in \mathcal{O}$, we may choose $t = \mathrm{nm}_f(s)$. The proof of [Nak12b, Proposition 4.4] shows that this is well-defined, and the remainder of the argument proceeds exactly as Nakaoka's proof. □

Corollary 2.4.44. *If $\underline{S} \subseteq \underline{R}^{\mu}$ is a restriction saturated $\mathcal{O}$-semi-Mackey subfunctor such that $\underline{S}(X)$ consists entirely of units of $\underline{R}(X)$, then the morphism of $\mathcal{O}$-Tambara functors $\underline{R} \to \underline{S}^{-1}\underline{R}$ is an isomorphism.*

Proof. In this case, the ring homomorphisms $\underline{R}(X) \to \underline{S}^{-1}\underline{R}(X) = \underline{S}(X)^{-1}\underline{R}(X)$ are isomorphisms for all finite G-sets X. Hence, $\underline{R} \to \underline{S}^{-1}\underline{R}$ is an isomorphism of $\mathcal{O}$-Tambara functors. □

Theorem 2.4.45 (cf. [Nak12b, Corollary 4.7])**.** *Let $\underline{R}$ be an $\mathcal{O}$-Tambara functor and $\underline{S} \subseteq \underline{R}^{\mu}$ restriction saturated. Then $\phi\colon \underline{R} \to \underline{S}^{-1}\underline{R}$ is the initial homomorphism of $\mathcal{O}$-Tambara functors that inverts S.*

Proof. Suppose that $\psi\colon \underline{R} \to \underline{T}$ is any homomorphism of $\mathcal{O}$-Tambara functors that inverts $\underline{S}$. For each finite G-set X, $\phi_X\colon \underline{R}(X) \to \underline{S}^{-1}\underline{R}(X)$ is the initial ring homomorphism that inverts $\underline{S}(X)$, witnessed by homomorphisms $\overline{\phi}_X\colon \underline{S}^{-1}\underline{R}(X) \to \underline{T}(X)$ of rings that make the diagram below commute:

$$\begin{array}{ccc} \underline{R}(X) & \xrightarrow{\psi_X} & \underline{T}(X). \\ {\scriptstyle \phi_X}\big\downarrow & \nearrow_{\overline{\phi}_X} & \\ \underline{S}^{-1}\underline{R}(X) & & \end{array}$$

We will show that the ring homomorphisms $\overline{\phi}_X$ assemble into a homomorphism $\overline{\phi}$ of $\mathcal{O}$-Tambara functors. To do so, it suffices to show that the $\overline{\phi}_X$ commute with restrictions, norms, and transfers.

It is quick to check that these commute with restriction and norm, since norm and restriction are homomorphisms of the multiplicative monoids of the commutative rings.

It remains to check that these ring homomorphisms commute with transfers. Observe that if $\underline{T}^{\times} \subseteq \underline{T}^{\mu}$ is the $\mathcal{O}$-Mackey subfunctor such that $\underline{T}^{\times}(X)$ is the abelian group of units of the ring $\underline{T}(X)$, then $\underline{T} \to (\underline{T}^{\times})^{-1}\underline{T}$ is an isomorphism. Importantly, this tells us something about the structure of the transfer in $\underline{T}$; if $u, z \in \underline{T}(X)$ such that a is a unit, then the transfer of $u^{-1}z$ along $f\colon X \to Y$ is:

$$\mathrm{tr}_f(u^{-1}z) = v^{-1}\,\mathrm{tr}_f(cz), \tag{2.4.46}$$

where $v \in T(Y)$ is a unit such that $\mathrm{res}_f(v) = cu$. The transfer is independent of the choice of v, as in the proof of Proposition 2.4.43.

Let $f\colon X \to Y$ be a morphism of finite G-sets. We aim to show that the diagram below commutes:

$$\begin{array}{ccc} \underline{S}^{-1}\underline{R}(X) & \xrightarrow{\overline{\phi}_X} & \underline{T}(X) \\ {\scriptstyle \mathrm{tr}_f}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{tr}_f} \\ \underline{S}^{-1}\underline{R}(Y) & \xrightarrow{\overline{\phi}_Y} & \underline{T}(Y). \end{array}$$

To that end, let $\frac{x}{s} \in \underline{S}^{-1}\underline{R}(X)$. We have

$$\operatorname{tr}_f\left(\frac{x}{s}\right) = \frac{\operatorname{tr}_f(ax)}{t}$$

for some $t \in \underline{S}^{-1}\underline{R}(Y)$ such that $\operatorname{res}_f(t) = as$. Then

$$\overline{\phi}_Y\left(\operatorname{tr}_f\left(\frac{x}{s}\right)\right) = \overline{\phi}_Y(t)^{-1}\overline{\phi}_Y(\operatorname{tr}_f(ax)) \tag{2.4.47}$$

On the other hand,

$$\operatorname{tr}_f\left(\overline{\phi}_X\left(\frac{x}{s}\right)\right) = \operatorname{tr}_f\left(\overline{\phi}_X(s)^{-1}\overline{\phi}_X(x)\right).$$

By (2.4.46), the right-hand side of the above is equal to $\nu^{-1}\operatorname{tr}_f(c\overline{\phi}_X(x))$ for some $\nu \in \underline{T}(Y)$ such that $\operatorname{res}_f(\nu) = c\overline{\phi}_X(x)$. We may as well choose $\nu = \overline{\phi}_Y(t)$, so that

$$\operatorname{res}_f(\overline{\phi}_Y(t)) = \overline{\phi}_X(\operatorname{res}_f(t)) = \overline{\phi}_X(as) = \overline{\phi}_X(a)\overline{\phi}_X(x).$$

In particular, $c = \overline{\phi}_X(a)$, so we have

$$\operatorname{tr}_f\left(\overline{\phi}_X\left(\frac{x}{s}\right)\right) = \operatorname{tr}_f\left(\overline{\phi}_X(s)^{-1}\overline{\phi}_X(x)\right) = \overline{\phi}_Y(t)^{-1}\operatorname{tr}_f(\overline{\phi}_X(a)\overline{\phi}_X(x)).$$

This agrees with (2.4.47), so the ring homomorphisms $\overline{\phi}_X$ commute with the transfer. □

We conclude this section with an example to show that the compliment of a prime ideal need not be a multiplicative semi-Mackey subfunctor of $\underline{R}^{\mu}$.

Example 2.4.48. Consider the C_p-Burnside Tambara functor $\underline{A}$, and consider the zero Tambara ideal. This is a prime ideal of $\underline{A}$ by [Nak12a, Theorem 4.40]. Nevertheless, the compliment of the zero ideal contains both $t \in \underline{A}(C_p/C_p)$ and $(t-p) \in \underline{A}(C_p/C_p)$, and $t(t-p) = t^2 - pt = 0$. Hence, the compliment of 0 is not a multiplicative semi-Mackey subfunctor of $\underline{A}^{\mu}$ – the fixed level fails to be a monoid.

CHAPTER 3

FREE INCOMPLETE TAMBARA FUNCTORS ARE ALMOST NEVER FLAT

Let k be a commutative ring. The free k-algebra on a single generator is the polynomial algebra $k[x]$. This is also a free k-module with basis $1, x, x^2, \ldots$. We take this fact for granted in commutative algebra, but it is not always true in the equivariant setting! Notably, the free C_2-Tambara functor on a fixed is not flat Remark 2.4.12.

In this chapter, we ask when a free incomplete Tambara functor is free as a Mackey functor. If it is not free, we might ask whether it is projective or flat instead. Surprisingly, we find that this is almost never the case. We provide conditions under which a free incomplete Tambara functor will be flat as a Mackey functor (Theorem 3.3.26), and show that such examples are exceedingly rare (Theorem 3.4.7). Finally, we prove that after localization, *all* free incomplete Tambara functors are flat (Theorem 3.5.19).

This chapter is an exposition of results that originally appeared in [HMQ22].

3.1 Sufficient conditions for freeness

To find sufficient conditions for a free incomplete Tambara functor to be free as a Mackey functor, we must first discuss norms of Mackey functors and incomplete Tambara functors. The norm is a kind of coinduction functor that takes H-Mackey functors to G-Mackey functors, or $i_H^*\mathcal{O}$-Tambara functors to $\mathcal{O}$-Tambara functors.

For any subgroup H, the G-set G/H is in the image of the induction functor on H-sets, and this functor extends to polynomials with exponents in various indexing categories. Precomposition with the induction functor

$$G \times_H (-)\colon \mathcal{P}^H_{i_H^*\mathcal{O}} \to \mathcal{P}^G_{\mathcal{O}}$$

yields the restriction functor

$$i_H^*\colon \mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_G \to i_H^*\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_H.$$

This functor has a left-adjoint, given by left Kan extension.

Definition 3.1.1 ([BH18, Definition 6.8])**.** Let

$$n_H^G\colon i_H^*\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_H \to \mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_G$$

be the left adjoint to the restriction i_H^* to H.

Proposition 3.1.2. *There is an isomorphism of $\mathcal{O}$-Tambara functors*

$$\underline{A}^{\mathcal{O}}[x_{G/K}] \cong n_H^G \underline{A}^{i_H^*\mathcal{O}}[x_{H/K}].$$

Proof. It suffices to show that $n_H^G \underline{A}^{i_H^*\mathcal{O}}[x_{H/K}]$ represents evaluation at G/K in $\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_G$. We compute:

$$\begin{aligned}
\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_G\big(n_H^G \underline{A}^{i_H^*\mathcal{O}}[x_{H/H}], \underline{R}\big) &\cong i_H^*\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_H\big(\underline{A}^{i_H^*\mathcal{O}}[x_{H/K}], i_H^*\underline{R}\big) \\
&\cong i_H^*\underline{R}(H/K) \\
&\cong \underline{R}(G \times_H H/K) \\
&\cong \underline{R}(G/H).
\end{aligned}$$

□

Example 3.1.3. When $H = e$, we have $n_e^G(\mathbb{Z}[x]) \cong \underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{G/e}]$.

When G/H is an admissible G-set, then the underlying Mackey functor for $n_H^G\underline{R}$ can be computed as a functor of the underlying Mackey functor of $\underline{R}$. This is a key step in forming the "external" or "G-symmetric monoidal" description of Tambara functors.

Definition 3.1.4 ([Hoy14, Section 2.3])**.** The *norm*

$$N_H^G : \mathcal{M}\mathrm{ack}_H \to \mathcal{M}\mathrm{ack}_G,$$

is defined by left Kan extension along the coinduction functor

$$\mathcal{S}et^{H}(G,-)\colon \mathcal{P}^{H}_{iso} \to \mathcal{P}^{G}_{iso}.$$

Proposition 3.1.5. *The norm*

$$N_H^G : \mathcal{M}ack_H \to \mathcal{M}ack_G$$

takes free H*-Mackey functors to free* G*-Mackey functors.*

Proof. If a free H-Mackey functor is finitely generated it is of the form $\underline{A}\{x_T\}$ for some finite G-set T (see Remark 2.3.33). We have

$$N_H^G\underline{A}\{x_T\} \cong \underline{A}\{x_{\mathcal{S}et^H(G,T)}\}$$

by definition. If it is not finitely generated, it can be written as a direct limit of finitely generated ones. The norm commutes with direct limits, so again the norm is free. □

Theorem 3.1.6 ([Hoy14, Theorem 2.3.3], [BH18, Theorem 6.15])**.** *If* G/H *is an admissible* G*-set for* $\mathcal{O}$*, then we have a natural isomorphism of functors from* $i_H^*\mathcal{O}$*-Tambara functors to* G*-Mackey functors:*

$$U \circ n_H^G \cong N_H^G \circ U,$$

where U *is the forgetful functor from incomplete Tambara functors to Mackey functors.*

Corollary 3.1.7. *If* G/H *is an admissible* G*-set for* $\mathcal{O}$*, then we have a natural isomorphism of Mackey functors*

$$U\big(\underline{A}^{\mathcal{O}}[x_{G/K}]\big) \cong N_H^G\big(U\big(\underline{A}^{i_H^*\mathcal{O}}[x_{H/K}]\big)\big).$$

Proof. This follows by applying the theorem to the isomorphism of Proposition 3.1.2. □

Combined with Proposition 3.1.5, we deduce some guarantees for free underlying Mackey functors.

Corollary 3.1.8. *If* G/H *is an admissible* G*-set for* $\mathcal{O}$*, and if the Mackey functor underlying* $\underline{A}^{i_H^*\mathcal{O}}[x_{H/H}]$ *is free, then the Mackey functor underlying* $\underline{A}^{\mathcal{O}}[x_{G/H}]$ *is free.*

There is one case where we can easily show that the underlying Mackey functor is free: the free Green functor on a fixed generator.

Proposition 3.1.9. *The free Green functor on a fixed generator,* $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{G/G}]$*, is free as a Mackey functor.*

Proof. By Example 2.4.4, $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{G/G}]$ is the direct sum of infinitely many copies of $\underline{A}$, so $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{G/G}]$ is free as a Mackey functor. □

Applying this to Corollary 3.1.8, we deduce a class of free incomplete Tambara functors that are free as $\underline{A}$-modules.

Theorem 3.1.10. *Let* $\mathcal{O}$ *be an indexing category for a finite group* G *and let* H *be a subgroup of* G*. If* $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$ *and* G/H *is admissible for* $\mathcal{O}$*, then* $\underline{A}^{\mathcal{O}}[x_{G/H}]$ *is free as an* $\underline{A}$*-module.*

3.2 Geometric fixed points

Studying the converse to Theorem 3.1.10 requires us to also be able to "restrict" along quotient homomorphisms. Additively, this is the geometric fixed points functor, but we also need to understand this on Tambara functors.

Throughout this section, N is a normal subgroup of G and $Q := G/N$ is the quotient group.

3.2.1 Cleaving indexing system

Recall that a family of subgroups is a set of subgroups of G closed under taking subgroups and conjugation (Definition 2.2.3).

Notation 3.2.1. If N is a normal subgroup of G, define

$$\mathcal{F}_N := \{H \le G \mid N \not\le H\}.$$

This is the family of subgroups of G which do not contain N. It is closed under conjugation because N is normal in G.

Associated to $\mathcal{F}_N$, we have a universal indexing category. As is often the case with indexing categories, we first define a wide subgraph of $\mathcal{S}et^G$ and then show that it is a subcategory that is pullback stable and finite coproduct complete.

Definition 3.2.2. Let $\mathcal{O}^N_{\text{gen}}$ be the wide subgraph of $\mathcal{S}et^G$ such that $f\colon S \to T$ is in $\mathcal{O}^N_{\text{gen}}$ if and only if the canonical morphism

$$S^N \to S \times_T T^N$$

is an isomorphism.

We record two useful reformulations of $\mathcal{O}^N_{\text{gen}}$.

Proposition 3.2.3.

(a) A morphism $f\colon S \to T$ is in $\mathcal{O}^N_{\text{gen}}$ if and only if the inclusion $S^N \subseteq f^{-1}(T^N)$ is the identity.

(b) A morphism of orbits $G/K \to G/H$ is in $\mathcal{O}^N_{\text{gen}}$ if and only if one of two things hold: either

(i) K contains N; or

(ii) H does not contain N.

Proof. To prove part (a), observe that the preimage $f^{-1}(T^N)$ has the universal property of the pullback $S \times_T T^N$, so they are in bijection. The canonical morphism in Definition 3.2.2 is then the natural inclusion

$$S^N \hookrightarrow f^{-1}(T^N).$$

Part (b) follows from the identifications

$$(G/H)^N \cong \begin{cases} \varnothing & N \not\subset H \\ G/H & N \subset H, \end{cases} \tag{3.2.4}$$

and part (a). □

Thinking of an indexing category as parameterizing transfers, we can reinterpret this as saying that we have no transfers from subgroups in the family to those not in the family. We think of this as a chasm, cleaving the groups that contain N from those that do not. In particular, intersecting any indexing category with this one gives a universal way to remove any transfers that bridge from the family to the complementary cofamily.

Theorem 3.2.5. *The wide subgraph $\mathcal{O}^N_{gen}$ is an indexing subcategory of $\mathcal{S}et^G$.*

Proof. We first show that it is a subcategory. It is clear that for any finite G-set T, the identity morphism on T satisfies the conditions of Definition 3.2.2. Now let

$$U \xrightarrow{f} S \xrightarrow{g} T$$

be morphisms in $\mathcal{O}^N_{gen}$. By assumption, we then have

$$U^N = f^{-1}(S^N) = f^{-1}\big(g^{-1}(T^N)\big) = (g \circ f)^{-1}(T^N),$$

as desired. Thus $\mathcal{O}^N_{gen}$ is a subcategory. By assumption, this is a wide subcategory.

To show that this subcategory is an indexing category, we apply [BH18, Lemma 3.2], which says that a subcategory $\mathcal{D}$ of $\mathcal{C}$ is finite coproduct complete if and only if $\mathcal{D}$ is symmetric monoidal and contains the morphisms $\varnothing \to *$ and $* \amalg * \to *$. The morphisms $\varnothing \to *$ and $* \amalg * \to *$ visibly satisfy the conditions of Definition 3.2.2. So $\mathcal{O}^N_{\text{gen}}$ is finite coproduct complete.

It remains to verify that $\mathcal{O}^N_{\text{gen}}$ is pullback stable. Let

$$\begin{array}{ccc} S_1 & \longrightarrow & S \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} \\ T_1 & \longrightarrow & T \end{array} \tag{3.2.6}$$

be a pullback diagram in $\mathcal{S}et^G$ with g in $\mathcal{O}^N_{\text{gen}}$. Since fixed points are a limit, this gives a pullback diagram

$$\begin{array}{ccc} S_1^N & \longrightarrow & S^N \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle g} \\ T_1^N & \longrightarrow & T^N \end{array} \tag{3.2.7}$$

Consider now the natural morphism

$$S_1^N \to S_1 \times_{T_1} T_1^N.$$

By assumption on the diagram and associativity, we have natural isomorphisms

$$S_1 \times_{T_1} T_1^N \cong (S \times_T T_1) \times_{T_1} T_1^N \cong S \times_T T_1^N \cong S \times_T (T^N \times_{T^N} T_1^N).$$

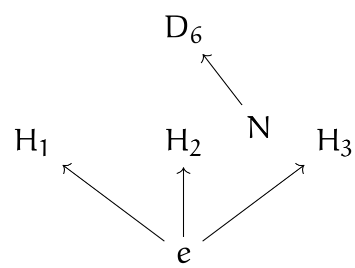
By assumption on g, we have a further natural isomorphism

$$(S \times_T T^N) \times_{T^N} T_1^N \cong S^N \times_{T^N} T_1^N.$$

(3.2.7) then shows this to be isomorphic to S_1^N via the natural morphisms, as desired. $\square$

Example 3.2.8. The dihedral group of order 6 has a unique normal subgroup N of order 3 and three conjugate subgroups of order 2, which we label H_1, H_2, and

H_3. The transfer system $\mathcal{O}^N_{gen}$ is depicted below.



3.2.2 Incomplete Tambara functors and nullifications

In equivariant homotopy theory, the N-geometric fixed points are the composite of two functors:

(1) the nullification which annihilates anything induced from those H in $\mathcal{F}_N$;

(2) the "restriction" along the surjection $q\colon G \to Q$.

Both of these can be realized in Mackey functors, cf. [BGHL19, Section 5.2].

Definition 3.2.9. Let $\underline{E\mathcal{F}_N}$ be the sub-Mackey functor of $\underline{A}$ generated by $\underline{A}(G/H)$ for all H in $\mathcal{F}_N$. Let $\underline{\widetilde{E}\mathcal{F}_N}$ be the quotient of $\underline{A}$ by $\underline{E\mathcal{F}_N}$.

Since we are working with the Burnside Mackey functor, we can be more explicit. Recall that the Burnside Mackey functor at level G/H is Grothendieck group of finite H-sets under disjoint union. In particular, it is generated by the classes of transitive H sets $[H/K]$ for $K \leq H$.

Proposition 3.2.10. *For all* $H \leq G$*, the subfunctor* $\underline{E\mathcal{F}_N}(G/H)$ *is generated as an abelian group by the set*

$$\{[H/K] \mid K \in \mathcal{F}_N\}.$$

In particular, $\underline{E\mathcal{F}_N}(G/H)$ *is a direct summand of* $\underline{A}(G/H)$*.*

Proof. If $H \in \mathcal{F}_N$ and $K \leq H$, then $K \in \mathcal{F}_N$ as well. At level G/H, $\underline{E\mathcal{F}_N}(G/H) = \underline{A}(G/H)$, and any $[H/K] \in \underline{A}(G/H)$ satisfies $K \in \mathcal{F}_N$.

If $H \notin \mathcal{F}_N$, then $L \notin \mathcal{F}_N$ for any $L \geq H$. Although $\underline{E\mathcal{F}_N}(G/H)$ is *a priori* generated by restrictions and transfers of the other levels, any level of $\underline{E\mathcal{F}_N}$ above G/H does not have any generators of $\underline{E\mathcal{F}_N}$ to restrict to G/H. Therefore, $\underline{E\mathcal{F}_N}(G/H)$ consists entirely of transfers of generators of $\underline{A}(G/L)$ for $L \leq H$ with $L \in \mathcal{F}_N$. These transfers are of the form

$$\mathrm{tr}_L^H([L/K]) = [H \times_L L/K] = [H/K],$$

with $K \in \mathcal{F}_N$.

Finally, since $\underline{E\mathcal{F}_N}(G/H)$ is generated by a subset of the generators of $\underline{A}(G/H)$, it is a direct summand. □

Note in particular that if $H \notin \mathcal{F}_N$, then every generator of $\underline{E\mathcal{F}_N}(G/H)$ is the transfer of an element (in fact, a generator) from $\underline{E\mathcal{F}_N}(G/K)$ with $K \in \mathcal{F}_N$.

Recall the definition of an $\mathcal{O}$-ideal of an $\mathcal{O}$-Tambara functor from Definition 2.4.14. We want to show that $\underline{E\mathcal{F}_N}$ is an $\mathcal{O}^N_{gen}$-ideal of $\underline{A}$, so that the quotient $\underline{\widetilde{E}\mathcal{F}_N} = \underline{A}/\underline{E\mathcal{F}_N}$ is an $\mathcal{O}^N_{gen}$-Tambara functor. First, we need a lemma.

Lemma 3.2.11. *An element* $x \in \underline{A}(G/H)$ *is actually in* $\underline{E\mathcal{F}_N}(G/H)$ *if and only if there is a finite* G*-set* T *with two properties:*

(a) the N*-fixed points of* T *are empty;*

(b) there is a morphism $h\colon T \to G/H$ *such that* x *is in the image of the transfer along* h*.*

Proof. Decomposing a finite G-set T into a disjoint union of transitive G-sets, we see by (3.2.4) that T^N is empty if and only if T is a disjoint union of transitive G-sets G/K_i with $K_i \in \mathcal{F}_N$.

If $x \in \underline{E\mathcal{F}_N}(G/H)$, with $H \in \mathcal{F}_N$, then we may choose $T = G/H$ and $h\colon G/H \to G/H$ the identity morphism. If $H \notin \mathcal{F}_N$, then $x \in \underline{E\mathcal{F}_N}(G/H)$ is a linear combination of transfers from levels G/K_i with $K_i \in \mathcal{F}_N$ by the previous proposition. We may take T to be the disjoint union of the G/K_i and $h\colon T \to G/H$ the composite of $\coprod_i \mathrm{tr}_{K_i}^H$ with the fold morphism.

Conversely, let $x = \mathrm{tr}_h(y)$ with $y \in \underline{A}(T)$ where T is a finite G-set with no N-fixed points. Write $T = \coprod_i G/K_i$ with $K_i \in \mathcal{F}_N$. Then x is a linear combination of transfers from $\underline{A}(G/K_i)$, so $x \in \underline{E\mathcal{F}_N}$. □

Theorem 3.2.12. *For any normal subgroup N of G, the sub-Mackey functor $\underline{E\mathcal{F}_N}$ of $\underline{A}$ is an $\mathcal{O}_{gen}^N$-ideal.*

Proof. Because $\underline{E\mathcal{F}_N}$ is a sub-Mackey functor, it is closed under restriction and transfer. Moreover, any level $\underline{E\mathcal{F}_N}(G/H)$ is either all of $\underline{A}(G/H)$ or consists entirely of transfers, which is an ideal by the Frobenius relation:

$$\mathrm{tr}(a)b = \mathrm{tr}(a\,\mathrm{res}(b)).$$

Hence, $\underline{E\mathcal{F}_N}$ is a Green ideal.

It remains to show that this is closed under norms parameterized by $\mathcal{O}_{gen}^N$: if $x \in \underline{E\mathcal{F}_N}(G/K)$, then any norm parameterized by $\mathcal{O}_{gen}^N$ applied to x is again in $\underline{E\mathcal{F}_N}$.

If $K \in \mathcal{F}_N$, then the only norms from G/K parameterized by $\mathcal{O}_{gen}^N$ are those along $G/K \to G/H$ with H also in $\mathcal{F}_N$ by Proposition 3.2.3. The values of $\underline{E\mathcal{F}_N}$ and $\underline{A}$ agree at these groups, so there is nothing to check.

Now if H contains N, then admissibility says that K must as well by Proposition 3.2.3. We are therefore reduced to understanding the composite

$$\mathrm{nm}_g \circ \mathrm{tr}_h,$$

where $g\colon G/K \to G/H$ is arbitrary and $h\colon T \to G/K$ is the morphism from Lemma 3.2.11 describing some element $x \in \underline{E\mathcal{F}}_N(G/K)$. We have an exponential diagram

$$\begin{array}{ccccc} G/K & \xleftarrow{h} & T & \xleftarrow{f'} & (h')^*G/H \\ {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle g'} \\ G/H & & \xleftarrow[h']{} & & G \times_H \mathcal{S}et^K(H, T'), \end{array}$$

where T' is the K-set $h^{-1}(eK)$. This diagram allows us to write

$$\mathrm{nm}_g \circ \mathrm{tr}_h = \mathrm{tr}_{h'} \circ \mathrm{nm}_{g'} \circ \mathrm{res}_{f'}.$$

Closure under the norm associated to g is then equivalent to

$$\big(G \times_H \mathcal{S}et^K(H, T')\big)^N = \varnothing,$$

again by Lemma 3.2.11.

Since H and K both contain N, the N-fixed points commute with the induction $G \times_H (-)$ and it suffices to check

$$\big(\mathcal{S}et^K(H, T')\big)^N = \varnothing.$$

Both N-fixed points and coinduction are right adjoints, and since the corresponding left adjoints commute, we can swap these:

$$\big(\mathcal{S}et^K(H, T')\big)^N \cong \mathcal{S}et^{K/N}\big(H/N, (T')^N\big).$$

By assumption T, and hence T', has no N fixed points. $\square$

Applying Proposition 2.4.18, we obtain:

Corollary 3.2.13. *If $\mathcal{O}$ is any indexing category contained in $\mathcal{O}^N_{\mathrm{gen}}$, then for any $\mathcal{O}$-Tambara functor $\underline{R}$, $\underline{\widetilde{E}\mathcal{F}}_N \boxtimes \underline{R}$ is an $\mathcal{O}$-Tambara functor and the natural homomorphism*

$$\underline{R} \to \underline{\widetilde{E}\mathcal{F}}_N \boxtimes \underline{R}$$

is a homomorphism of $\mathcal{O}$-Tambara functors.

3.2.3 Changing groups

Let N be a normal subgroup of G and $q\colon G \to Q = G/N$ the quotient homomorphism. This yields a fully-faithful functor on finite sets with Q-action:

$$q^*\colon \mathcal{S}\mathrm{et}^Q \to \mathcal{S}\mathrm{et}^G.$$

The essential image of q^* is the full subcategory of G-sets T for which $T = T^N$.

The functor q^* actually gives us a morphism the other way on indexing subcategories.

Proposition 3.2.14. *Given an indexing subcategory $\mathcal{O}$ of $\mathcal{S}\mathrm{et}^G$, the intersection with $\mathcal{S}\mathrm{et}^Q$ gives an indexing subcategory of $\mathcal{S}\mathrm{et}^Q$.*

Proof. Since $\mathcal{S}\mathrm{et}^Q$ is fully-faithfully embedded in $\mathcal{S}\mathrm{et}^G$ via q^* and since $\mathcal{O}$ is wide and finite-coproduct complete, so is the intersection with $\mathcal{S}\mathrm{et}^Q$. Pullback stability follows from noting that we can compute pullbacks via the fully-faithful embedding, where this is immediate. □

Definition 3.2.15. If $\mathcal{O}$ is an indexing category for G, then let $q_*\mathcal{O}$ be the corresponding indexing category for Q.

We can extend this to functors on Burnside categories and categories of polynomials.

Proposition 3.2.16. *The natural inclusion $\mathcal{S}\mathrm{et}^Q \to \mathcal{S}\mathrm{et}^G$ extends to a faithful, but not full, product-preserving embedding*

$$q^*\colon \mathcal{A}^Q \to \mathcal{A}^G.$$

If $\mathcal{O}$ is an indexing category for Q and if $\mathcal{O}'$ is any indexing category for G such that $\mathcal{O} \subseteq q_*\mathcal{O}'$, then we have a faithful product preserving embedding

$$q^*\colon \mathcal{P}^Q_{\mathcal{O}} \to \mathcal{P}^G_{\mathcal{O}'}.$$

In both cases, we take a diagram in $\mathcal{S}\mathrm{et}^Q$ *to itself, viewed as a diagram in* $\mathcal{S}\mathrm{et}^G$ *via* q^*.

Proof. To show that q^* extends to a functor on the Burnside category, we must show that it preserves composition of spans. This follows because $q^* \colon \mathcal{S}\mathrm{et}^Q \to \mathcal{S}\mathrm{et}^G$ preserves pullback diagrams. As a functor $\mathcal{A}^Q \to \mathcal{A}^G$, q^* is faithful because it is faithful as a functor $\mathcal{S}\mathrm{et}^Q \to \mathcal{S}\mathrm{et}^G$.

To show that q^* extends to a functor on categories of polynomials, we need slightly more to show that q^* respects composition of polynomials: we need that q^* preserves not only pullback diagrams but also exponential diagrams. Equivalently, we must verify that q^* preserves dependent products. This follows from computing the N-fixed points. □

We give an example to show that $q^* \colon \mathcal{S}\mathrm{et}^Q \to \mathcal{S}\mathrm{et}^G$ is not full.

Example 3.2.17. If $Q = e$, then q^* is the usual embedding of $\mathcal{S}\mathrm{et}$ into $\mathcal{S}\mathrm{et}^G$. For any non-trivial G, note that $\underline{A}(G/G)$ is the Burnside ring of G. We have:

$$\underline{A}(G/G) = \mathcal{A}^G(*,*) \neq \mathcal{A}^e(*,*) = \mathbb{Z}.$$

Since the functor q^* is product preserving, precomposition with it gives a functor on Mackey functors and appropriate incomplete Tambara functors.

Definition 3.2.18. If $\mathcal{O}$ is an indexing category for G, then let

$$q_* \colon \mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_G \to q_*\mathcal{O}\text{-}\mathcal{T}\mathrm{amb}_Q$$

be the functor given by precomposition with q^*, and similarly for Mackey functors.

Remark 3.2.19. Unpacking the functor q_*, we see that it is really formalizing two procedures:

(a) forget $\underline{R}(G/H)$ for any H that does not contain N;

(b) forget any norms or transfers up from finite G-sets which do not contain N.

The heart of Proposition 3.2.16 is that this actually gives an incomplete Tambara functor on Q.

This is the final piece of our geometric fixed points.

Definition 3.2.20 (cf. [BGHL19, Section 5.2])**.** Let $\mathcal{O}$ be an indexing category for G such that $\mathcal{O} \subseteq \mathcal{O}^N_{gen}$. The *Tambara geometric fixed points functor*

$$\widetilde{\Phi}^N \colon \mathcal{O}\text{-}\mathcal{T}amb_G \to q_*\mathcal{O}\text{-}\mathcal{T}amb_Q$$

is the composite

$$q_* \circ \big(\underline{\widetilde{E}\mathcal{F}_N} \boxtimes (\text{-})\big).$$

The *Mackey geometric fixed points functor*

$$\Phi^N \colon \mathcal{M}ack_G \to \mathcal{M}ack_Q$$

is the composite

$$q_* \circ \big(\underline{\widetilde{E}\mathcal{F}_N} \boxtimes (\text{-})\big).$$

Remark 3.2.21. Note that for any Mackey functor $\underline{M}$, the Mackey functor $\underline{\widetilde{E}\mathcal{F}_N} \boxtimes \underline{M}$ vanishes when evaluated on G/H with $N \not\subset H$. In particular, we see that on the essential image of the localization functor given by boxing with $\underline{\widetilde{E}\mathcal{F}_N}$, q_* throws away no real information.

Note that the embedding

$$\mathcal{P}^G_{iso} \hookrightarrow \mathcal{P}^G_{\mathcal{O}}$$

is compatible with q^*: for any G indexing category $\mathcal{O}$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{P}^{Q}_{\text{iso}} & \longrightarrow & \mathcal{P}^{Q}_{q_*\mathcal{O}} \\ {\scriptstyle q^*}\downarrow & & \downarrow{\scriptstyle q^*} \\ \mathcal{P}^{G}_{\text{iso}} & \longrightarrow & \mathcal{P}^{G}_{\mathcal{O}}. \end{array}$$

Precomposition with the inclusion $\mathcal{P}^{G}_{\text{iso}} \hookrightarrow \mathcal{P}^{G}_{\mathcal{O}}$ gives the underlying Mackey functor of a Tambara functor (Example 2.3.2).

Proposition 3.2.22. *The underlying Mackey functor of the Tambara geometric fixed points is the Mackey geometric fixed points of the underlying Mackey functor, i.e.*

$$U \circ \widetilde{\Phi}^N \cong \Phi^N \circ U,$$

where U *is the forgetful functor from incomplete Tambara functors to Mackey functors.*

3.2.4 Geometric fixed points of frees and flats

Let T be a finite G-set. We now compute the geometric fixed points of the free Mackey or $\mathcal{O}$-Tambara functor on T. The key feature here is that since N is a normal subgroup, T^N is actually a G-equivariant summand of T, and hence we have a natural inclusion of G-sets $T^N \to T$.

Theorem 3.2.23. *The homomorphism*

$$\underline{A}^Q\{x_{T^N}\} \to \Phi^N\big(\underline{A}\{x_T\}\big)$$

corresponding to the element

$$[T \leftarrow T^N \to T^N] \in \big(\underline{\widetilde{E}\mathcal{F}_N} \boxtimes \underline{A}\{x_T\}\big)(T^N)$$

is an isomorphism.

Proof. We analyze more directly the quotient $\widetilde{\underline{E\mathcal{F}}}_N \boxtimes \underline{A}\{x_T\}$. A generic element of $\underline{A}\{x_T\}(G/H)$ is a span of the form

$$T \leftarrow S \rightarrow G/H.$$

If $N \not\subset H$, then this span was automatically set to zero by the quotient, so it suffices to study $N \subset H$. Breaking S into orbits $\coprod G/K_i$, we can write the element as the sum of elements

$$T \leftarrow G/K_i \rightarrow G/H.$$

If $N \not\subset K_i$, then this element is in the image of the transfer from $\underline{A}\{x_T\}(G/K_i)$ with $K_i \in \mathcal{F}_N$, and hence is set equal to zero in the quotient. Thus the image under the quotient morphism is

$$T \leftarrow S^N \rightarrow G/H.$$

In particular, we see that a basis for the free abelian group $\widetilde{\underline{E\mathcal{F}}}_N \boxtimes \underline{A}\{x_T\}(G/H)$ is given by isomorphism classes of spans

$$T \leftarrow G/K \rightarrow G/H$$

with $N \subset K$. Since $(G/K)^N = G/K$, the morphism $G/K \rightarrow T$ actually factors through $T^N \hookrightarrow T$.

By naturality, the homomorphism $\underline{A}^Q\{x_{T^N}\} \rightarrow \Phi^N\left(\underline{A}\{x_T\}\right)$, when evaluated at G/H with $N \subset H$, takes a diagram

$$T^N \xleftarrow{f} S \xrightarrow{g} G/H = T_g \circ R_f\big(T^N \leftarrow T^N \rightarrow T^N\big)$$

with $S = S^N$ to the diagram

$$T_g \circ R_f(T \leftarrow T^N \rightarrow T^N) = T \xleftarrow{f} S \xrightarrow{g} G/H.$$

In particular, this takes a basis to a basis, and hence is an isomorphism. $\square$

Corollary 3.2.24. *The* N*-geometric fixed points functor preserves projective Mackey functors.*

Proof. Recall that any projective is a retract of a free. Since any functor preserves retract diagrams, and since geometric fixed points of free Mackey functors are free, the geometric fixed points of a projective Mackey functor are projective. □

The argument for incomplete Tambara functors is almost identical.

Theorem 3.2.25. *Let* $\mathcal{O}$ *be an indexing category for* G *for which* $\mathcal{O} \subseteq \mathcal{O}^N_{\text{gen}}$*. The homomorphism of* $q_*\mathcal{O}$*-Tambara functors*

$$\underline{A}^{q_*\mathcal{O}}[x_{T^N}] \to \widetilde{\Phi}^N \underline{A}^{\mathcal{O}}[x_T]$$

corresponding to the element

$$[T \leftarrow T^N \to T^N \to T^N] \in \big(\underline{\widetilde{E}\mathcal{F}_N} \boxtimes \underline{A}^{q_*\mathcal{O}}[x_T]\big)(T^N),$$

is an isomorphism.

Proof. The proof proceeds almost identically to the Mackey functor case. Using the same reductions as before, we find that a basis for the quotient $\underline{\widetilde{E}\mathcal{F}_N} \boxtimes \underline{A}^{\mathcal{O}}[x_T]$ at G/H is given by isomorphism classes of polynomials of the form

$$T \leftarrow A \xrightarrow{g} G/K \to G/H,$$

with again $N \leq K$, $N \leq H$ and now $g \in \mathcal{O}$. Here our assumption on $\mathcal{O}$ enters: we have no morphisms $G/J \to G/K$ in $\mathcal{O}^N_{\text{gen}}$ if J does not contain N. In particular, we deduce that $A = A^N$, and hence it is in the image of q^*. The rest of the proof follows identically. □

More generally, the geometric fixed points functor preserves flat objects. We begin with an observation linking this to another well-studied functor: inflation.

Definition 3.2.26 ([TW95, Section 5]). Given a Q-Mackey functor $\underline{M}$, we define a G-Mackey functor $\operatorname{Inf}_Q^G \underline{M}$ called the *inflation of* $\underline{M}$ *from* Q *to* G by

$$\operatorname{Inf}_Q^G \underline{M}(G/H) = \begin{cases} 0 & \text{if } N \not\subseteq H, \\ \underline{M}(H/N) & \text{if } N \subseteq H. \end{cases}$$

The transfers and restrictions are zero unless $N \subseteq K \subseteq H$, in which case the transfer tr_K^H and restriction res_K^H between K and H in the inflation are the corresponding transfer $\operatorname{tr}_{K/N}^{H/N}$ and restriction $\operatorname{res}_{K/N}^{H/N}$ between K/N and H/N in $\underline{M}$.

Thévanaz–Webb show also that this functor has both adjoints. Unpacking our definition of geometric fixed points shows that it agrees with the left adjoint of inflation.

Proposition 3.2.27. *The* N*-geometric fixed points on Mackey functors is left adjoint to the inflation functor* Inf_Q^G.

In fact, we can do better. Inflation actually gives a section of the N-geometric fixed points.

Proposition 3.2.28. *The canonical natural transformation*

$$\operatorname{Inf}_Q^G(\text{-}) \cong \underline{A} \boxtimes \operatorname{Inf}_Q^G(\text{-}) \Rightarrow \underline{\widetilde{E}\mathcal{F}_N} \boxtimes \operatorname{Inf}_Q^G(\text{-})$$

is an isomorphism, and the composite $\Phi^N \circ \operatorname{Inf}_Q^G$ *is naturally isomorphic to the identity.*

Proof. For any G-Mackey functor $\underline{M}$, the canonical quotient homomorphism $\underline{A} \to \underline{\widetilde{E}\mathcal{F}_N}$ allows us to identify

$$\underline{\widetilde{E}\mathcal{F}_N} \boxtimes \underline{M}$$

with the quotient of $\underline{M}$ by the sub-Mackey functor generated by $\underline{M}(G/H)$ for all $H \in \mathcal{F}_N$. By definition, if $H \in \mathcal{F}_N$, then $\operatorname{Inf}_Q^G \underline{M}(G/H) = 0$, so we are forming the quotient by the zero Mackey functor. $\square$

Identifying the geometric fixed points as the left-adjoint to inflation gives another reformulation of it.

Definition 3.2.29. Let $\mathrm{Fix}_N \colon \mathcal{S}\mathrm{et}^G \to \mathcal{S}\mathrm{et}^Q$ denote the N-fixed point functor.

Proposition 3.2.30. *The functor* Fix_N *extends to a product-preserving functor*

$$\mathcal{A}^G \to \mathcal{A}^Q$$

which commutes with Cartesian products.

Proof. The composition in $\mathcal{A}^G$ is given by pullback, and since Fix_N is a limit, it commutes with pullbacks. The compatibility with the Cartesian product is identical. For the categorical product, we observe that the fixed points of a disjoint union are the disjoint union of the fixed points. □

The following is immediate from the definition of inflation and (3.2.4).

Proposition 3.2.31. *The functor* Inf_Q^G *is naturally isomorphic to the precomposition with* Fix_N*:*

$$\mathrm{Inf}_Q^G(\underline{M}) \cong \underline{M} \circ \mathrm{Fix}_N \,.$$

Corollary 3.2.32. *The* N*-geometric fixed points are given by the left Kan extension along* Fix_N*.*

Corollary 3.2.33. *The* N*-geometric fixed points functor is strong symmetric monoidal.*

Proof. Since both the box product and the N-geometric fixed points are given by left Kan extensions, it suffices to show that the underlying diagram

$$\begin{array}{ccc} \mathcal{A}^G \times \mathcal{A}^G & \xrightarrow{\times} & \mathcal{A}^G \\ {\scriptstyle \mathrm{Fix}_N \times \mathrm{Fix}_N}\downarrow & & \downarrow{\scriptstyle \mathrm{Fix}_N} \\ \mathcal{A}^Q \times \mathcal{A}^Q & \xrightarrow[\times]{} & \mathcal{A}^Q, \end{array}$$

expressing the ways we can take the iterated Kan extensions, commutes. It commutes because Fix_N is strong symmetric monoidal for the Cartesian product. □

Theorem 3.2.34. *The* N*-geometric fixed points preserves flat Mackey functors.*

Proof. We must show that if $\underline{M}$ is a flat Mackey functor, then the box product with $\Phi^N\underline{M}$ is an exact functor on Q-Mackey functors. We show this by rewriting it several ways. Let $\underline{N}$ be a Q-Mackey functor. Then using Proposition 3.2.28 and Corollary 3.2.33, we have a natural (in $\underline{N}$) isomorphism

$$\underline{N} \boxtimes \Phi^N\underline{M} \cong \Phi^N\left(\mathrm{Inf}_Q^G \underline{N} \boxtimes \underline{M}\right).$$

The definition of geometric fixed points allows us to further rewrite this, now using the first clause of Proposition 3.2.28:

$$\Phi^N\left(\mathrm{Inf}_Q^G \underline{N} \boxtimes \underline{M}\right) \cong q_*\left(\mathrm{Inf}_Q^G \underline{N} \boxtimes \underline{M}\right).$$

Now, the inflation functor and q_* are both exact, since both are given by precomposition with an additive functor and exactness is checked objectwise. We deduce that if $\underline{M}$ is flat, then the functor

$$\underline{N} \mapsto q_*\left(\mathrm{Inf}_Q^G \underline{N} \boxtimes \underline{M}\right) \cong \underline{N} \boxtimes \Phi^N(\underline{M})$$

is exact. □

3.3 Necessary conditions for freeness

To give necessary conditions for flatness, we must first establish some properties of the restriction functor.

3.3.1 Restriction from G-Mackey functors to H-Mackey functors

Recall that $i_H^*\colon \mathcal{M}\mathrm{ack}_G \to \mathcal{M}\mathrm{ack}_H$ for $H \leq G$ is defined by precomposition with the induction functor

$$G \times_H (-)\colon \mathcal{A}^H \to \mathcal{A}^G.$$

We first introduce its adjoints.

Definition 3.3.1 ([TW95, Section 4])**.** Given an H-Mackey functor $\underline{N}$, we define a G-Mackey functor $\mathrm{Ind}_H^G(\underline{N})$, the *induction* of $\underline{N}$, by

$$\mathrm{Ind}_H^G \underline{N} = \underline{N} \circ i_H^*,$$

where i_H^* is the restriction functor from finite G-sets to finite H-sets.

Lemma 3.3.2 ([TW95, Proposition 4.2])**.** *The restriction functor is both left and right adjoint to the induction functor.*

Both restriction and induction are exact functors, as again, exactness is checked objectwise.

Proposition 3.3.3. *For any subgroup H, both restriction i_H^* and induction Ind_H^G preserve projective objects.*

Proof. A functor that has an exact right adjoint preserves projective objects. Both i_H^* and Ind_H^G have an exact right adjoints. □

We can moreover identify the restriction of free Mackey functors.

Proposition 3.3.4. *There is an isomorphism of Mackey functors*

$$i_H^* \underline{A}^G\{x_T\} \cong \underline{A}^H\{x_{i_H^* T}\}.$$

Moreover, i_H^* sends free Mackey functors to free Mackey functors.

Proof. It suffices to show that $i_H^* \underline{A}^G\{x_T\}$ represents evaluation at $i_H^* T$ in H-Mackey functors. We compute:

$$\begin{aligned} \mathcal{M}\mathrm{ack}^H(i_H^* \underline{A}^G\{x_T\}, \underline{N}) &\cong \mathcal{M}\mathrm{ack}^G(\underline{A}^G\{x_T\}, \mathrm{Ind}_H^G \underline{N}) \\ &\cong \mathrm{Ind}_H^G \underline{N}(T) \\ &\cong \underline{N}(i_H^* T). \end{aligned}$$

If a free Mackey functor is finitely generated, the calculation above shows that its restriction is again free. Otherwise, it is a direct limit of finitely generated ones, and i_H^* preserves direct limits because it is a left adjoint. □

As with geometric fixed points, it is a bit more work to show that this functor preserves flat objects.

Lemma 3.3.5. *The restriction functor i_H^* is strong symmetric monoidal.*

Proof. Because restriction is left adjoint to a functor Ind_H^G given by precomposition with $i_H^* \colon \mathcal{S}\mathrm{et}^G \to \mathcal{S}\mathrm{et}^H$, it is given by a left Kan extension along the restriction of finite G-sets $i_H^* \colon \mathcal{S}\mathrm{et}^G \to \mathcal{S}\mathrm{et}^H$. Then this follows nearly identically to Corollary 3.2.33, replacing instances of Fix_N by i_H^*. Note that $i_H^* \colon \mathcal{S}\mathrm{et}^G \to \mathcal{S}\mathrm{et}^H$ is a right adjoint and so preserves limits, and so extends to a product-preserving functor $\mathcal{A}^G \to \mathcal{A}^H$ which commutes with Cartesian products as Fix_N does in Proposition 3.2.30. □

Lemma 3.3.6. *The unit of the restriction-induction adjunction*

$$\underline{N} \mapsto i_H^* \mathrm{Ind}_H^G \underline{N}$$

is a split inclusion, natural in $\underline{N}$.

Proof. For any finite H-set T, the restriction of the induction of $\underline{N}$ is the functor

$$T \mapsto \underline{N}(i_H^*(G \times_H T)).$$

On a transitive H-set H/K,

$$i_H^*(G \times_H H/K) = i_H^*(G/K) = \coprod_{KgH \in K\backslash G/H} {}^{H}/_{H \cap gKg^{-1}}.$$

In particular, H/K is the term of the disjoint union corresponding to the identity double coset KeH. By decomposing into orbits, this shows that T includes into $i_H^*(G \times_H T)$. Hence, $\underline{N}(T)$ is a summand of $i_H^* \operatorname{Ind}_H^G \underline{N}(T)$. □

Theorem 3.3.7. *If $\underline{M}$ is a flat G-Mackey functor, then $i_H^*\underline{M}$ is a flat H-Mackey functor.*

Proof. Since induction and restriction are exact functors, if $\underline{M}$ is a flat G-Mackey functor, then the functor

$$\underline{N} \mapsto i_H^*\big(\operatorname{Ind}_H^G(\underline{N}) \boxtimes \underline{M}\big) \tag{3.3.8}$$

is exact. By the Frobenius relation, we have an isomorphism

$$\operatorname{Ind}_H^G(\underline{N}) \boxtimes \underline{M} \cong \operatorname{Ind}_H^G\left(\underline{N} \boxtimes i_H^*\underline{M}\right).$$

Hence by Lemma 3.3.6, the functor

$$\underline{N} \mapsto \underline{N} \boxtimes i_H^*\underline{M}$$

is a retract of the functor (3.3.8). In particular, it is a retract of an exact functor, and hence is exact. □

3.3.2 Necessity of triviality of the restriction

We begin by proving that $i_H^*\mathcal{O} \cong \mathcal{O}^{\mathrm{triv}}$ is a necessary condition for freeness. We start here with a technical observation:

Lemma 3.3.9. *If $\mathcal{O}$ is an indexing category for G, then there is a smallest subgroup N such that*

(a) G/N *is admissible.*

(b) *If* G/H *is admissible, then* $N \leq H$.

(c) *If* H/K *is an admissible* H*-set with* $N \leq H$*, then* $N \leq K$.

(d) N *is normal in* G.

Proof. Let $S = \{H \leq G \mid G/H \text{ is admissible}\}$. Let $N = \bigcap_{H \in S} H$. Then N satisfies (a) and (b):

(a) G/N is admissible since $G/(H \cap H')$ is admissible whenever G/H and G/H' are admissible.

(b) By definition, $N \leq H$ for every H such that G/H is admissible.

Now, suppose H/K is an admissible H-set and $N \leq H$. By restriction, $N/(N \cap K)$ is an admissible N-set, and since admissibles are closed under self-induction, $G/(N \cap K)$ is an admissible G-set. Minimality of N implies that $N \cap K = N$, and hence $N \leq K$.

Finally, the closure under conjugacy of admissibles implies by minimality that N is actually normal. □

As in Section 3.2, we write Q for G/N.

Lemma 3.3.10. *If* N *is the smallest normal subgroup associated to* $\mathcal{O}$ *from Lemma 3.3.9, then*

$$\mathcal{O} \cap \mathcal{O}_{\text{gen}}^{N} = \mathcal{O}.$$

Proof. It suffices to show that if H/K is admissible for $\mathcal{O}$, then it is admissible for $\mathcal{O}_{\text{gen}}^{N}$. Recall that H/K is admissible for an indexing category $\mathcal{O}$ if $\mathcal{O}$ contains the morphism $G/K \to G/H$ (Definition 2.2.25). If $\mathcal{O}$ contains the morphism of orbits $G/K \to G/H$, and H does not contain N, then the morphism of orbits is in $\mathcal{O}_{\text{gen}}^{N}$ as well by Proposition 3.2.3. Otherwise, if H contains N, then K contains

N as well by Lemma 3.3.9, and again the morphism of orbits is in $\mathcal{O}^{N}_{gen}$ by Proposition 3.2.3. □

In particular, we can take the N-geometric fixed points and not lose information we might have wanted to preserve.

Proposition 3.3.11. *Let $\mathcal{O}$ be an indexing category. If $\underline{A}^{\mathcal{O}}[x_{G/G}]$ is flat, then $\mathcal{O} = \mathcal{O}^{triv}$.*

Proof. Let N be the (normal) subgroup from Lemma 3.3.9, and take the N-geometric fixed points. By Theorem 3.2.25, we have

$$\Phi^{N}\left(\underline{A}^{\mathcal{O}}[x_{G/G}]\right) \cong \underline{A}^{q_*\mathcal{O}}[x_{Q/Q}],$$

which is flat by Theorem 3.2.34. Note that now we have norms from $N/N = \{e\}$ to all larger subgroups of Q.

If $N \neq G$, then we can choose a C_p in Q for some prime p. Restricting down to C_p, we get

$$i^*_{C_p}\underline{A}^{q_*\mathcal{O}}[x_{Q/Q}] \cong \underline{A}^{\mathcal{O}^{cplt}}[x_{C_p/C_p}],$$

since there are only two indexing categories for C_p and we have a non-trivial norm. This is not flat by [BH18, Lemma 3.6] – it contains the augmentation ideal of $\underline{A} \to \underline{\mathbb{Z}}$ (which is not flat by Lemma 2.3.38) as a summand. We conclude that $N = G$ and hence G/G is the only admissible transitive G-set for $\mathcal{O}$.

To show that the admissible H-sets are also all trivial, we note that we have a natural isomorphism

$$i^*_H\underline{A}^{\mathcal{O}}[x_{G/G}] \cong \underline{A}^{i^*_H\mathcal{O}}[x_{H/H}].$$

By Theorem 3.3.7 restriction preserves flats, so the only admissible transitive H-set for $\mathcal{O}$ is H/H, by the same argument as above. Induction on the subgroup lattice then shows then that $\mathcal{O} = \mathcal{O}^{triv}$. □

Corollary 3.3.12. *Let $\mathcal{O}$ be an indexing category and let $H \leq G$ be a subgroup. If $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat, then $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$.*

Proof. Write $i_H^* G/H = H/H \sqcup T$ for some H-set T. If $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat, then so is the restriction to H. This is given by

$$i_H^*\underline{A}[x_{G/H}] \cong \underline{A}^{i_H^*\mathcal{O}}[x_{i_H^* G/H}] \cong \underline{A}^{i_H^*\mathcal{O}}[x_{H/H}] \boxtimes \underline{A}^{i_H^*\mathcal{O}}[x_T].$$

Because this is flat, so are both box factors since they are also summands. The result then follows from Proposition 3.3.11. □

In the next section, we will see that the decomposition in the proof of Corollary 3.3.12 implies that H is a normal subgroup of G. Consequently, all of the factors in that decomposition are the same (and the Weyl action permutes them). These statements take some work to prove, however.

3.3.3 Necessity that H is normal in G

To show that H must be normal in G, we first analyze exactly when Green functors are free.

Theorem 3.3.13. *If $\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{G/H}]$ is flat, then $H = G$.*

The free Green functor on a class at level G/H can be equivalently described as the ordinary symmetric algebra on $\underline{A}\{x_{G/H}\}$:

$$\underline{A}^{\mathcal{O}^{\mathrm{triv}}}[x_{G/H}] \cong \bigoplus_{n\geq 0} \mathrm{Sym}^n(\underline{A}\{x_{G/H}\}).$$

For a general finite G-set T, the n-th symmetric power on $\underline{A}\{x_T\}$ is a quotient of a free:

$$\mathrm{Sym}^n(\underline{A}\{x_T\}) \cong \big(\underline{A}\{x_{T^{\times n}}\}\big)/\Sigma_n.$$

Here we use that $T^{\times n}$ is a $G \times \Sigma_n$-set, or equivalently, a Σ_n-object in G-sets.

Theorem 3.3.13 will follow immediately from a more precise statement.

Theorem 3.3.14. *If* $n = [G : H]$*, then* $\mathrm{Sym}^n(\underline{A}\{x_{G/H}\})$ *is not flat.*

Proof. A decomposition of $(G/H)^{\times n}$ into $G \times \Sigma_n$-sets gives a direct sum decomposition of $\mathrm{Sym}^n(\underline{A}\{x_{G/H}\})$. We single out some particular summands.

Choose a non-equivariant isomorphism

$$\big(f\colon \{1, \ldots, n\} \to G/H\big) \in (G/H)^{\times n}.$$

Using f to identify G/H with $\{1, \ldots, n\}$, the G-action on G/H is classified by a homomorphism

$$\phi\colon G \to \Sigma_n.$$

Thinking of ϕ as defining the G-set structure on G/H, we see that the kernel of ϕ is the normal subgroup

$$N_H = \bigcap_{g \in G} gHg^{-1}.$$

Let $\widetilde{\phi}$ be the inclusion of G/N_H into Σ_n induced by ϕ.

We have to understand the summand

$$\big(\underline{A}\{x_{G \times \Sigma_n \cdot f}\}\big)/\Sigma_n.$$

of $\mathrm{Sym}^n(\underline{A}\{x_{G/H}\})$, where $G \times \Sigma_n \cdot f$ is the orbit of $f \in (G/H)^{\times n}$ under the $G \times \Sigma_n$-action.

By the orbit-stabilizer theorem, as a $G \times \Sigma_n$-set, we have

$$G \times \Sigma_n \cdot f \cong G \times \Sigma_n/\Gamma_\phi,$$

where Γ_ϕ is the graph of ϕ. The underlying Mackey functor is the free Mackey functor

$$\underline{A}\{x_{i_G^*(G \times \Sigma_n \cdot f)}\},$$

and this has a residual Σ_n-action.

Restricting to G, we use the double-coset formula

$$i_G^*\big(G \times \Sigma_n/\Gamma_\phi\big) \cong \coprod_{G(g,\sigma)\Gamma_\phi \in G\backslash G\times\Sigma_n/\Gamma_\phi} G/G \cap (g,\sigma)\Gamma_\phi(g,\sigma)^{-1}.$$

The group $G \cong G \times \{e\}$ is a normal subgroup of $G \times \Sigma_n$, so it intersects all conjugates of a fixed group in conjugates:

$$G \times \{e\} \cap ((g,\sigma)\Gamma_\phi(g,\sigma)^{-1}) = (g,\sigma)\big(G \times \{e\} \cap \Gamma_\phi\big)(g,\sigma)^{-1}.$$

By definition of the graph, the intersection

$$G \times \{e\} \cap \Gamma_\phi = \{(g,e) \mid \phi(g) = e\}$$

is the graph of ϕ restricted to its kernel. Since the kernel is normal in G, this is normal in $G \times \Sigma_n$:

$$\big(G \times \{e\} \cap \Gamma_\phi\big)^{(g,\sigma)} = N_H \times \{e\}.$$

Therefore we have

$$i_G^*(G \times \Sigma_n/\Gamma_\phi) \cong \coprod_{G(g,\sigma)\Gamma_\phi \in G\backslash G\times\Sigma_n/\Gamma_\phi} G/N_H.$$

For the residual Σ_n-action, we recall that Σ_n acted freely on $G \times \Sigma_n/\Gamma_\phi$. The quotient

$$G \times \Sigma_n \to G \times \Sigma_n/\Gamma_\phi$$

takes G first to G/N_H and then identifies it with the subgroup $\widetilde{\phi}(G/N_H)$. This identifies the double cosets: we have

$$G\backslash G \times \Sigma_n/\Gamma_\phi = \Sigma_n/\widetilde{\phi}(G/N_H).$$

Putting this together, the restriction to G of $G \times \Sigma_n/\Gamma_\phi$ is just the decomposition of Σ_n into G/N_H-cosets. As a consequence, we have

$$\big(\underline{A}\{x_{G\times\Sigma_n\cdot f}\}\big)/\Sigma_n \cong \big(\underline{A}\{x_{G/N_H}\}\big)/(G/N_H),$$

where G/N_H acts on itself via the right action (which is then an isomorphism of left G-sets).

If this summand is flat, then applying N_H-geometric fixed points (and letting $Q = G/N_H$) shows that the Q-Mackey functor

$$\left(\underline{A}^{Q}\{x_Q\}\right)/Q$$

is also flat. In other words, without loss of generality, we reduce to the case that $N_H = \{e\}$ and $Q = G$.

Since G is finite and $G \neq \{e\}$, we can find an element of order p for some prime p. This gives us a subgroup $K \cong C_p$. Restricting to K, we have an isomorphism

$$i_K^*\left(\underline{A}\{x_G\}\right)/G \cong \left(\underline{A}\{x_K\}\right)/K.$$

The latter we can compute directly: it is the dual to the constant Mackey functor $\underline{\mathbb{Z}}$. Since this is not flat, by Lemma 2.3.38, we deduce that the summand we started with could not be either. □

This has a surprising consequence.

Proposition 3.3.15. *Let $\mathcal{O}$ be an indexing category, and let $H \subset G$. If $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat, then H is normal in G.*

Proof. Let us again write $i_H^* G/H = H/H \sqcup T$ for some finite H-set T. We have an isomorphism of $i_H^*\mathcal{O}$-Tambara functors

$$i_H^*\underline{A}^{\mathcal{O}}[x_{G/H}] \cong \underline{A}^{i_H^*\mathcal{O}}[x_{H/H}] \boxtimes \underline{A}^{i_H^*\mathcal{O}}[x_T],$$

as in Corollary 3.3.12, and that corollary also shows that $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$. Since all of the box factors are also direct summands, Theorem 3.3.13 shows that they are only free if the set T is a trivial H-set. This is equivalent to H being normal in G □

3.3.4 Connecting freeness to admissibility of G/H

We begin with a further constraint on the indexing category.

Proposition 3.3.16. *Let $\mathcal{O}$ be an indexing category and N a normal subgroup of G such that $i_N^*\mathcal{O}$ is trivial. Then*

$$\mathcal{O} \cap \mathcal{O}_{gen}^N = \mathcal{O}.$$

Proof. If a morphism of orbits $G/K \to G/H$ is in $\mathcal{O}$, then $K \not\subseteq N$ since $i_N^*\mathcal{O}$ is trivial. To show that $G/K \to G/H$ is in $\mathcal{O}_{gen}^N$, it suffices to show that $N \not\subseteq H$ by Proposition 3.2.3. If N were contained in H, then pullback of $G/K \to G/H$ along $G/N \to G/H$ is the G-set

$$G/K \times_{G/H} G/N = \coprod_{KhN \in K\backslash H/N} {}^{G}\!/\!{}_{K \cap hNh^{-1}}.$$

Since N is normal in G, all of the terms in the direct sum are $G/(K \cap N)$, and the canonical projection from the pullback to G/N yields $G/(K \cap N) \to G/N$ in $\mathcal{O}$ with $K \not\subseteq N$, which does not exist in $\mathcal{O}$ because $i_N^*\mathcal{O}$ is trivial. □

If $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat, then we have already seen that H is normal in G and that $i_H^*\mathcal{O} = \mathcal{O}^{triv}$. We want to study the constraints on $\mathcal{O}$, so we lose no information if we pass to the H-geometric fixed points. Put another way, we want to be able to assume without loss of generality that $H = \{e\}$.

To study the connection between flatness and admissibility, we will compare $\mathcal{O}$ with an indexing category where we know that we have flatness: the complete one. We need a small lemma on how the free incomplete Tambara functors compare.

Lemma 3.3.17. *If $\mathcal{O} \subseteq \mathcal{O}'$, then for all T, we have a natural inclusion of $\mathcal{O}$-Tambara functors*

$$\underline{A}^{\mathcal{O}}[x_T] \hookrightarrow \underline{A}^{\mathcal{O}'}[x_T]$$

corresponding to

$$[T \leftarrow T \rightarrow T \rightarrow T] \in \underline{A}^{\mathcal{O}'}[x_T](T).$$

Proof. By Corollary 2.4.7, the generators for $\underline{A}^{\mathcal{O}}[x_T]$ are a subset of the generators for $\underline{A}^{\mathcal{O}'}[x_T]$, so this is an inclusion of Green functors. It is a morphism of $\mathcal{O}$-Tambara functors by the Yoneda lemma:

$$\mathcal{O}\text{-}\mathcal{T}\text{amb}_G(\underline{A}^{\mathcal{O}}[x_T], \underline{A}^{\mathcal{O}'}[x_T]) \cong \underline{A}^{\mathcal{O}'}[x_T](T). \qquad \square$$

In effect, this inclusion takes a polynomial

$$\left[T \leftarrow A \xrightarrow{g} B \rightarrow S\right] \in \underline{A}^{\mathcal{O}}[x_T](S)$$

with $g \in \mathcal{O}$ and considers this as a polynomial in $\underline{A}^{\mathcal{O}'}[x_T](S)$, remembering that $\mathcal{O} \subseteq \mathcal{O}'$.

In particular, for any indexing category $\mathcal{O}$, we have an inclusion

$$\phi_{\mathcal{O}} : \underline{A}^{\mathcal{O}}[x_{G/e}] \hookrightarrow \underline{A}^{\mathcal{O}^{\text{cplt}}}[x_{G/e}] \tag{3.3.18}$$

into the free Tambara functor on an underlying generator. Recall that $\underline{A}^{\mathcal{O}^{\text{cplt}}}[x_{G/e}]$ is isomorphic to $N_e^G(\mathbb{Z}[x])$ as a Mackey functor by Corollary 3.1.7, so (3.3.18) expresses an inclusion of any free incomplete Tambara functor on an underlying generator into $N_e^G(\mathbb{Z}[x])$. We will use this homomorphism to identify summands of $\underline{A}^{\mathcal{O}}[x_{G/e}]$ by studying their image in $\underline{A}^{\mathcal{O}^{\text{cplt}}}[x_{G/e}]$.

Definition 3.3.19. Let

$$\mathcal{F}_{\mathcal{O}} := \{H \mid G/e \xrightarrow{\pi} G/H \in \mathcal{O}\},$$

be the collection of subgroups H for which H is an admissible H-set.

In other words, these are all of the subgroups for which we have norms from the trivial group to those subgroups.

Lemma 3.3.20. *The collection of subgroups $\mathcal{F}_{\mathcal{O}}$ of G is a family.*

Proof. This is most easily seen by reformulating the indexing category $\mathcal{O}$ as a transfer system. Let $\to_{\mathcal{O}}$ be the transfer system associated to $\mathcal{O}$. We may rewrite

$$\mathcal{F}_{\mathcal{O}} = \{H \mid e \to_{\mathcal{O}} H\}.$$

Then the conjugacy and restriction axioms of a transfer system (Definition 2.2.32) tell us $\mathcal{F}_{\mathcal{O}}$ is a family. □

We can define the sub-Mackey functor of $\underline{A}$ generated by the admissible H-sets using this family.

Definition 3.3.21. Let $\underline{E\mathcal{F}_{\mathcal{O}}}$ be the sub-Mackey functor of $\underline{A}$ generated by $\underline{A}(G/H)$ for all $H \in \mathcal{F}_{\mathcal{O}}$.

The sub-Mackey functor of $\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{G/e}] \cong N_e^G(\mathbb{Z}[x])$ generated by $\mathrm{nm}_e^G(x)$ forms a copy of $\underline{A}$ inside $N_e^G(\mathbb{Z}[x])$, which we write as $\underline{A}\{\mathrm{nm}_e^G(x)\}$. It is a Mackey-functor summand.

Proposition 3.3.22. *The image of*

$$\phi_{\mathcal{O}}\colon \underline{A}^{\mathcal{O}}[x_{G/e}] \to \underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{G/e}] \cong N_e^G(\mathbb{Z}[x])$$

in $\underline{A}\{\mathrm{nm}_e^G(x)\}$ is isomorphic to $\underline{E\mathcal{F}_{\mathcal{O}}}$.

Proof. The norm of the generator of $\underline{A}^{\mathcal{O}^{\mathrm{cplt}}}[x_{G/e}]$ is the class of the polynomial

$$\mathrm{nm}_e^G(x) = \left[G/e \leftarrow G/e \xrightarrow{\pi} G/G \to G/G\right].$$

The summand $\underline{A}\{\mathrm{nm}_e^G(x)\}$ is generated as a Mackey functor by this polynomial: all elements of this summand are linear combinations of transfers of restrictions of $\mathrm{nm}_e^G(x)$. By Proposition 2.4.5, these polynomials look like linear combinations of polynomials

$$\left[G/e \xleftarrow{p} G/e \times G/H \xrightarrow{g} G/H \to G/K\right], \tag{3.3.23}$$

where p and g are the canonical projections.

Since $\phi_{\mathcal{O}}$ takes a polynomial in $\mathcal{P}^G_{\mathcal{O}}(G/e, T)$ and considers it as a polynomial in $\mathcal{P}^G_{\mathcal{O}^{\text{cplt}}}(G/e, T)$, the elements of $\underline{A}\{\text{nm}^G_e(x)\}$ which lie in the image of $\phi_{\mathcal{O}}$ are linear combinations of the polynomials (3.3.23) with $g \in \mathcal{O}$.

But $g \in \mathcal{O}$ is equivalent to $H \in \mathcal{F}_{\mathcal{O}}$, so we see that the image is the sub-Mackey functor generated by $\underline{A}(G/H)$ with $H \in \mathcal{F}_{\mathcal{O}}$. □

Finally, we can connect admissibility of G/e to flatness.

Proposition 3.3.24. *If G is solvable and G is not admissible for $\mathcal{O}$, then $\underline{E\mathcal{F}_{\mathcal{O}}}$ is not flat.*

Proof. Assume to the contrary that $\underline{E\mathcal{F}_{\mathcal{O}}}$ is flat.

Let H be a minimal subgroup of G that is not in $\mathcal{F}_{\mathcal{O}}$. By assumption, all proper subgroups of H are in $\mathcal{F}_{\mathcal{O}}$, and since restriction preserves flat objects (Theorem 3.3.7), we may without loss of generality assume that $\mathcal{F}_{\mathcal{O}}$ is actually the family $\mathcal{P}$ of proper subgroups of G. It will never contain G itself, because G is not admissible for $\mathcal{O}$ by assumption.

Since G is solvable, there exists some normal subgroup N of G such that $G/N \cong C_p$ for some prime p. In particular, N is a maximal subgroup of G. We compute $\Phi^N(\underline{E\mathcal{P}})$. We have

$$\Phi^N(\underline{E\mathcal{P}})(C_p/C_p) = q_*(\underline{E\mathcal{P}} \boxtimes \underline{\widetilde{E}\mathcal{F}_N})(C_p/C_p) = (\underline{E\mathcal{P}}/\underline{E\mathcal{F}_N})(G/G)$$

and

$$\Phi^N(\underline{E\mathcal{P}})(C_p/e) = q_*(\underline{E\mathcal{P}} \boxtimes \underline{\widetilde{E}\mathcal{F}_N})(C_p/e) = (\underline{E\mathcal{P}}/\underline{E\mathcal{F}_N})(G/N)$$

To understand these abelian groups, we consider generators of $\underline{E\mathcal{P}}(T)$ and $\underline{E\mathcal{F}_N}(T)$. See Proposition 3.2.10 for the generators of $\underline{E\mathcal{F}_N}$. At the fixed level, there is only one generator of $\underline{E\mathcal{P}}(G/G)$ that does not lie in $\underline{E\mathcal{F}_N}(G/G)$, namely

$[G/N]$. Similarly, there is only one generator of the quotient at the level G/N, which is $[N/N]$. Hence, both the underlying and fixed level of the geometric fixed points are isomorphic to $\mathbb{Z}$. The restriction and transfer are inherited from $\underline{A}$, and are given on generators by

$$\mathrm{res}([G/N]) = [G : N][N/N] = p[N/N]$$
$$\mathrm{tr}([N/N]) = [G \times_N N/N] = [G/N].$$

Therefore,

$$\Phi^N(\underline{E\mathcal{P}}) \cong \underline{\mathbb{Z}}^*,$$

which is not flat by Lemma 2.3.38.

However, geometric fixed points Φ^N preserve flatness (Theorem 3.2.34), so $\Phi^N(\underline{E\mathcal{P}})$ should be flat, which contradicts the calculation above. □

Corollary 3.3.25. *Let G be a solvable group. If $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat, then G/H is admissible for $\mathcal{O}$.*

Proof. By Proposition 3.3.15, H is normal in G, and by Corollary 3.3.12, $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$. Hence, $\mathcal{O} \subseteq \mathcal{O}_{\mathrm{gen}}^H$ by Proposition 3.3.16, and we make take the H-geometric fixed points to reduce to the case $H = e$ (Theorem 3.2.25).

We demonstrate this by contrapositive: we show that if G/e is not admissible for $\mathcal{O}$, then $\underline{A}^{\mathcal{O}}[x_{G/e}]$ is not flat. All of this is operating under the assumption that G is solvable.

By Proposition 3.3.22, $\underline{A}^{\mathcal{O}}[x_{G/e}]$ contains $\underline{E\mathcal{F}_{\mathcal{O}}}$ as a summand, and by Proposition 3.3.24, this summand is not flat. Hence, $\underline{A}^{\mathcal{O}}[x_{G/e}]$ is not flat. □

We assemble all of the results here into the solvable case of our general theorem.

Theorem 3.3.26. *Let G be a solvable finite group, $\mathcal{O}$ an indexing system for G, and H a subgroup of G. The following are equivalent:*

(a) *The Mackey functor underlying the free $\mathcal{O}$-Tambara functor on a class at level* G/H *is flat.*

(b) *The Mackey functor underlying the free $\mathcal{O}$-Tambara functor on a class at level* G/H *is free.*

(c) H *is a normal subgroup of* G*,* G/H *is admissible for* $\mathcal{O}$*, and* $i_H^*\mathcal{O} = \mathcal{O}^{\text{triv}}$.

Proof. Clearly (b) implies (a). By Theorem 3.1.10, (c) implies (b). By Corollary 3.3.12, Proposition 3.3.15, and Corollary 3.3.25, (a) implies (c). □

3.4 Asymptotics

The conditions on H from Theorem 3.3.26 pin down H quite precisely: it must be the minimal normal subgroup $N \leq G$ such that G/N is admissible. In particular, it is uniquely determined by the indexing category.

Proposition 3.4.1. *If* G *is a solvable group, then for any indexing category* $\mathcal{O}$*, there is at most one subgroup* $H \leq G$ *such that* $\underline{A}^{\mathcal{O}}[x_{G/H}]$ *is flat as an* $\underline{A}$*-module.*

Proof. Let $\mathcal{O}$ be an indexing category for G, and let K, H be subgroups of G such that both $\underline{A}^{\mathcal{O}}[x_{G/H}]$ and $\underline{A}^{\mathcal{O}}[x_{G/K}]$ are flat. This is easiest to see using transfer systems, so let $\to_{\mathcal{O}}$ be the transfer system associated to $\mathcal{O}$. Because the two functors are flat, both $H \to_{\mathcal{O}} G$ and $K \to_{\mathcal{O}} G$ and $i_K^*\mathcal{O} = \mathcal{O}^{\text{triv}} = i_H^*\mathcal{O}$. This implies that $H \cap K \to_{\mathcal{O}} H$ as well by the restriction axiom for transfer systems (Definition 2.2.32). However, $i_H^*\mathcal{O}$ is trivial, so $H \cap K \to_{\mathcal{O}} H$ implies that $H = H \cap K$. Therefore, $K \supseteq H$. Symmetrically, $K \subseteq H$, so $K = H$. □

Suppose that $N \trianglelefteq G$ is the minimal normal subgroup of G such that G/N is admissible for $\mathcal{O}$. If $i_N^*\mathcal{O}$ is not trivial, then Theorem 3.3.26 shows that no free $\mathcal{O}$-Tambara functors are flat as Mackey functors.

These harsh conditions let us deduce tight upper bounds on the number of free incomplete Tambara functors that are flat as Mackey functors. Our strategy is to bound the proportion of free incomplete Tambara functors which are flat as Mackey functors in terms of the depth of a finite solvable group G. We then us a result from group theory to reduce from all finite groups to finite solvable groups.

Definition 3.4.2. Let G be a finite group.

(a) Let $d(G)$ denote the depth of the subgroup lattice of G, that is, $d(G)$ is the length of the longest chain of subgroups

$$\{e\} = H_0 \lneq H_1 \lneq \dots \lneq H_n = G.$$

(b) Let $I(G)$ denote the number of indexing categories for G.

(c) Let $T(G)$ be the number of pairs $(\mathcal{O}, H)$ of indexing category $\mathcal{O}$ for G and subgroup $H \leq G$.

(d) Let $P(G)$ denote the number of pairs $(\mathcal{O}, H)$ for which $\underline{A}^{\mathcal{O}}[x_{G/H}]$ is flat as a Mackey functor.

The number $P(G)/T(G)$ is the fraction of free incomplete Tambara functors for G which are flat as Mackey functors. The following proposition says that as the depth of a solvable group G increases, this fraction tends to zero.

Proposition 3.4.3. *Let G be a solvable group. Then*

$$\frac{P(G)}{T(G)} \leq \frac{1}{d(G)}.$$

Proof. Note that for any finite group G, we have $T(G) \geq d(G)I(G)$. And when G is solvable, if follows from Proposition 3.4.1 that $P(G) \leq I(G)$. Hence,

$$\frac{P(G)}{T(G)} \leq \frac{P(G)}{d(G)I(G)} \leq \frac{I(G)}{d(G)I(G)} = \frac{1}{d(G)}. \qquad \square$$

Example 3.4.4. For cyclic groups of prime power order, this inequality takes the form

$$\frac{P(C_{p^n})}{T(C_{p^n})} \leq \frac{1}{n+1},$$

because the cyclic group C_{p^n} has $n+1$ subgroups, linearly ordered.

Heuristically, the inequality $P(G)/T(G) \leq 1/d(G)$ says that as the depth of the subgroup lattice increases, the proportion of free incomplete Tambara functors which are flat as Mackey functors decreases. To extend this heuristic to a precise statement for all finite groups, we need a lemma:

Lemma 3.4.5. *Let S be a countable set and let $\{A_s\}_{s\in S}$ and $\{B_s\}_{s\in S}$ be sequences of positive integers. Write $S(-)\colon \mathbb{N} \to S$ for a bijection between S and the natural numbers. Then*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} A_{S(i)}}{\sum_{i=1}^{n} B_{S(i)}} \leq \lim_{n\to\infty} \max_{1\leq i\leq n} \frac{A_{S(i)}}{B_{S(i)}}$$

Proof. It suffices to show that

$$\frac{\sum_{i=1}^{n} A_{S(i)}}{\sum_{i=1}^{n} B_{S(i)}} \leq \max_{1\leq i\leq n} \frac{A_{S(i)}}{B_{S(i)}}$$

for all n. To that end, observe

$$A_{S(j)} = \frac{A_{S(j)}}{B_{S(j)}} B_{S(j)} \leq \left(\max_{1\leq i\leq n} \frac{A_{S(i)}}{B_{S(i)}} \right) B_{S(j)}.$$

Taking sums, we find

$$\sum_{j=1}^{n} A_{S(j)} \leq \left(\max_{1\leq i\leq n} \frac{A_{S(i)}}{B_{S(i)}} \right) \left(\sum_{j=1}^{n} B_{S(j)} \right).$$

Dividing through by $\sum_j B_{S(j)}$ yields the desired inequality. □

Definition 3.4.6.

(a) Let $\mathbb{G}$ denote the set of (isomorphism classes of) finite groups.

(b) Let $\mathbb{G}^s \subseteq \mathbb{G}$ denote the subset of finite solvable groups.

(c) Let $\mathbb{G}_{\leq d} \subseteq \mathbb{G}$ be the subset of finite groups of depth at most d.

(d) Let $\mathbb{G}_d \subseteq \mathbb{G}$ be the subset of finite groups of depth exactly d.

(e) Let $\mathbb{G}^s_{\leq d} = \mathbb{G}^s \cap \mathbb{G}_{\leq d}$ and $\mathbb{G}^s_d = \mathbb{G}^s \cap \mathbb{G}_d$.

Note that each of the sets in the previous definition is countable.

The next theorem establishes the slogan "free incomplete Tambara functors are almost never flat."

Theorem 3.4.7.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}(i))}{\sum_{i=1}^n T(\mathbb{G}(i))} = 0.$$

Proof. Evidently $P(G)$ and $T(G)$ are positive integers for all finite groups G, so

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}(i))}{\sum_{i=1}^n T(\mathbb{G}(i))} \geq 0.$$

We must show that this limit is less than zero.

By the main theorem of [CEG86], almost all finite groups are solvable. Therefore,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}(i))}{\sum_{i=1}^n T(\mathbb{G}(i))} = \lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s(i))}{\sum_{i=1}^n T(\mathbb{G}^s(i))},$$

so it suffices to prove the theorem for solvable groups. Now, $\mathbb{G}^s = \bigcup_{d=0}^{\infty} \mathbb{G}^s_d$ so

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s(i))}{\sum_{i=1}^n T(\mathbb{G}^s(i))} = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s_{\leq d}(i))}{\sum_{i=1}^n T(\mathbb{G}^s_{\leq d}(i))}.$$

Finally, observe that almost all solvable groups of depth at most d actually have depth precisely d, i.e. almost every element in $\mathbb{G}^s_{\leq d}$ is actually contained in the subset $\mathbb{G}^s_d$. Therefore,

$$\lim_{d\to\infty} \lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s_{\leq d}(i))}{\sum_{i=1}^n T(\mathbb{G}^s_{\leq d}(i))} = \lim_{d\to\infty} \lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s_d(i))}{\sum_{i=1}^n T(\mathbb{G}^s_d(i))}.$$

By Proposition 3.4.1 and Lemma 3.4.5, we have

$$\lim_{d\to\infty} \lim_{n\to\infty} \frac{\sum_{i=1}^n P(\mathbb{G}^s_d(i))}{\sum_{i=1}^n T(\mathbb{G}^s_d(i))} \leq \lim_{d\to\infty} \lim_{n\to\infty} \max_{1\leq i\leq n} \frac{P(\mathbb{G}^s_d(i))}{T(\mathbb{G}^s_d(i))} \leq \lim_{d\to\infty} \lim_{n\to\infty} \max_{1\leq i\leq n} \frac{1}{d} = 0,$$

which implies the result in view of the previous equalities. □

3.4.1 Explicit bounds for cyclic groups of prime power order

In the proof of Theorem 3.4.7, we used the fact that almost all finite groups are solvable to reduce our analysis to the solvable case. If one restricts further to cyclic groups of prime power order, one can obtain explicit bounds using the classification of indexing categories for such groups by Balchin–Barnes–Roitzheim [BBR21]. In this section, we give an explicit formula for the fraction of free incomplete C_{p^n}-Tambara functors which are flat.

The first step towards such a formula is to count the number of indexing systems for C_{p^n}.

Theorem 3.4.8 ([BBR21, Theorem 20])**.** *There are* $\mathrm{Cat}(n+1)$ *indexing systems for* C_{p^n}*, where* $\mathrm{Cat}(n)$ *is the* n*-th Catalan number*

$$\mathrm{Cat}(n) = \frac{(2n)!}{(n+1)!n!}.$$

Recall $T(G)$ and $P(G)$ from Definition 3.4.2. The number $T(G)$ counts the total number of free incomplete Tambara functors for G, while $P(G)$ counts those which are flat.

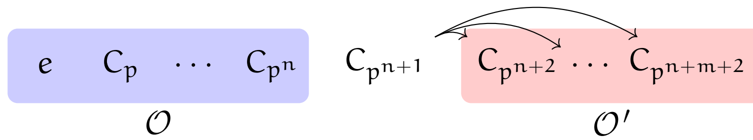
Lemma 3.4.9. *The total number of free incomplete* C_{p^n}*-Tambara functors is*

$$T(C_{p^n}) = (n+1)\,\mathrm{Cat}(n+1)$$

Proof. This number $T(C_{p^n})$ is the product of the number of subgroups $H \leq C_{p^n}$ and the number of indexing systems for C_{p^n}. The former is $(n+1)$, and the latter is $\mathrm{Cat}(n+1)$ by Theorem 3.4.8. □

To count the number of free incomplete C_{p^n}-Tambara functors, we recall the circle product of C_{p^n}-indexing systems from [BBR21].

Definition 3.4.10 ([BBR21, Section 3.2])**.** The *circle product* of a C_{p^n}-transfer system $\mathcal{O}$ and a C_{p^m}-transfer system $\mathcal{O}'$, denoted $\mathcal{O} \odot \mathcal{O}'$, is the $C_{p^{n+m+2}}$-transfer system depicted pictorially as follows (edges within $\mathcal{O}$ and $\mathcal{O}'$ are omitted for clarity):



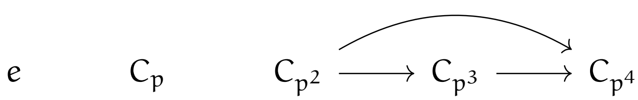
Explicitly, the edges in $\mathcal{O} \odot \mathcal{O}'$ consists of the union of the three following sets:

(1) The edges in $\mathcal{O}$

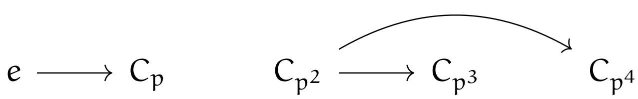
(2) An edge $C_{p^{i+n+2}} \to_{\mathcal{O}\odot\mathcal{O}'} C_{p^{j+n+2}}$ for each edge $C_{p^i} \to_{\mathcal{O}'} C_{p^j}$ in $\mathcal{O}'$

(3) An edge $C_{p^{n+1}} \to_{\mathcal{O}\odot\mathcal{O}'} C_{p^k}$ for $n+1 < k \leq n+m+2$

Example 3.4.11. For $m = n = 1$, the circle products of $\mathcal{O}^{\mathrm{triv}}$ and $\mathcal{O}^{\mathrm{cplt}}$ are below



$\mathcal{O}^{\mathrm{triv}} \odot \mathcal{O}^{\mathrm{cplt}}$



$\mathcal{O}^{\mathrm{cplt}} \odot \mathcal{O}^{\mathrm{triv}}$

Now that we have the circle product of C_{p^n}-indexing systems, we can count the number that satisfy the conditions of Theorem 3.3.26.

Lemma 3.4.12. *The number of free incomplete C_{p^n}-Tambara functors which are flat is*

$$P(C_{p^n}) = \sum_{i=0}^{n} \mathrm{Cat}(i).$$

Proof. We compute $P(C_{p^n})$ by determining the number of pairs $(\mathcal{O}, H)$ where C_{p^n}/H is admissible for $\mathcal{O}$ and $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$. Write $\to_{\mathcal{O}}$ for the transfer system corresponding to $\mathcal{O}$. The requirement that $i_H^*\mathcal{O} = \mathcal{O}^{\mathrm{triv}}$ is satisfied if and only if there are no edges $K_1 \to_{\mathcal{O}} K_2$ in the transfer system between subgroups $K_1 < K_2 \leq H$. The requirement that G/H is admissible for $\mathcal{O}$ is satisfied if and only if $H \to_{\mathcal{O}} C_{p^n}$ in the transfer system for $\mathcal{O}$.

If $H = C_{p^k}$, it follows that $\mathcal{O} = \mathcal{O}^{\text{triv}}_{k-1} \odot \mathcal{O}'$, where $\mathcal{O}^{\text{triv}}_{k-1}$ is the trivial indexing system for $C_{p^{k-1}}$ and $\mathcal{O}'$ is any indexing system for $C_{p^{n-k-1}}$. Therefore, by Theorem 3.4.8, there are $\mathrm{Cat}(n-k)$ indexing systems for which $\underline{A}^{\mathcal{O}}[x_{C_{p^n}/C_{p^k}}]$ is flat. Summing over all possible subgroups $H \leq C_{p^n}$, we have

$$P_n = \sum_{C_{p^i} \leq C_{p^n}} \mathrm{Cat}(n-i) = \sum_{i=0}^{n} \mathrm{Cat}(n-i) = \sum_{i=0}^{n} \mathrm{Cat}(i). \qquad \square$$

The previous two lemmas combine to give the fraction of free incomplete C_{p^n}-Tambara functors which are flat.

Proposition 3.4.13. *The fraction of free incomplete C_{p^n}-Tambara functors which are flat is*

$$\frac{P(C_{p^n})}{T(C_{p^n})} = \frac{\sum_{i=0}^{n} \mathrm{Cat}(i)}{(n+1)\,\mathrm{Cat}(n+1)}.$$

To compare this proposition to the bound from Example 3.4.4, note that the Catalan numbers satisfy the recurrence relation

$$\mathrm{Cat}(n+1) = \sum_{i=0}^{n} \mathrm{Cat}(i)\,\mathrm{Cat}(n-i).$$

Therefore, the sum of the first n Catalan numbers is bounded above by $\mathrm{Cat}(n+1)$, whence we recover the previous bound:

$$\frac{P(C_{p^n})}{T(C_{p^n})} = \frac{\sum_{i=0}^{n} \mathrm{Cat}(i)}{(n+1)\,\mathrm{Cat}(n+1)} < \frac{\mathrm{Cat}(n+1)}{(n+1)\,\mathrm{Cat}(n+1)} = \frac{1}{n+1}.$$

This calculation shows that the bound is not very tight. In fact, the the bound $\frac{1}{n+1}$ is already pretty far off for $n = 3$.

Example 3.4.14. Proposition 3.4.13 gives the actual fraction of free incomplete C_{p^3}-Tambara functors which are flat as $9/56 \approx 0.16$, but the bound from Example 3.4.4 gives $\frac{1}{4} = 0.25$. The nine free incomplete C_{p^3}-Tambara functors which are flat are listed in a table in Appendix A.

Example 3.4.15. Take $n = 4$. The true fraction is $\frac{23}{210} \approx 0.11$, whereas the bound gives $\frac{1}{5} = 0.2$.

3.5 Freeness after localization

In classical algebra, modules often become free when the base ring is enlarged. For example, base-change from $\mathbb{Z}$ to $\mathbb{Q}$ (or any field) forces any module to be free. In this section, we explore what happens to free incomplete Tambara functors after inverting various elements in the Burnside functor.

3.5.1 Localizations of the Burnside functor

We consider several important localizations of the Burnside $\mathcal{O}$-Tambara functor.

Example 3.5.1. Consider the localization of the Burnside Tambara functor $\underline{A}$ at $\underline{S} = \{(n, G/G) \mid n \in \mathbb{N}_{>0}\}$. The Tambara functor $\underline{S}^{-1}\underline{A}$ is the rational Burnside functor $\underline{A}_{\mathbb{Q}}$. On a finite G-set T, $\underline{A}_{\mathbb{Q}}(T) \cong \underline{A}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. A module over $\underline{A}_{\mathbb{Q}}$ is a *rational Mackey functor*, i.e. a Mackey functor $\underline{M}$ such that each $\underline{M}(T)$ is a rational vector space.

Example 3.5.2. Consider the Burnside Tambara functor $\underline{A}$. Let $\underline{S} = \{(|G|, G/G)\}$, and write $\underline{A}[\frac{1}{|G|}] := \underline{S}^{-1}\underline{A}$. On a finite G-set T, this Mackey functor is given by

$$\underline{A}[\tfrac{1}{|G|}](T) = \underline{A}(T)[\tfrac{1}{|G|}] = \underline{A}(T) \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{|G|}].$$

Modules over this Mackey functor are Mackey functors taking values in $\mathbb{Z}[\frac{1}{|G|}]$-modules rather than abelian groups.

Example 3.5.3. Consider the Burnside C_2-Tambara functor $\underline{A}$. Let $\underline{S} = \{(t, C_2/C_2)\}$. Inverting t means adding an element u such that $tu = 1 \in \underline{A}(C_2/C_2)$. By the relation $t^2 = 2t$, this implies that $t = 2$, so $u = \frac{1}{2}$. Since restriction is a ring homomorphism, this necessarily also inverts 2 at the underlying level. Hence, we have an isomorphism of Tambara functors

$$\underline{S}^{-1}\underline{A} \cong \underline{\mathbb{Z}}\left[\tfrac{1}{2}\right]$$

between the localization of the Burnside C_2-Tambara functor and the constant Tambara functor on $\mathbb{Z}\left[\frac{1}{2}\right]$.

In Example 3.5.2 we considered the localization of the Burnside functor at the element $|G|$. Instead of inverting the order of G, we could categorify and invert the class of G in the Burnside ring $\underline{A}(G/G)$ of finite G-sets.

Notation 3.5.4. Write $\underline{A}\left[\frac{1}{[G/e]}\right] := \underline{S}^{-1}\underline{A}$ for $\underline{S} = \{([G], G/G)\}$.

We describe this localization of the Burnside functor below.

Lemma 3.5.5. *For all* $K \leq H \leq G$*,* $[H/K]$ *and* $[H : K]$ *are invertible and equal in* $\underline{A}[\frac{1}{[G/e]}](G/H)$.

Proof. We show this first at level G/G. By the Frobenius relation

$$\mathrm{tr}_K^H(a) \cdot b = \mathrm{tr}_K^H(a \cdot \mathrm{res}_K^H(b)),$$

we obtain equations

$$[G/\{e\}] \cdot [G/H] = [G/\{e\}] \cdot [G : H].$$

in $\underline{A}(G/G)$. In the localization inverting $[G/\{e\}]$, we deduce that for all H,

$$[G/H] = [G : H].$$

When $H = \{e\}$, this implies that $[G/\{e\}] = |G|$ and hence that $|G|$ is a unit. Since $[G : H]$ divides $|G|$, it is a unit, and hence for all H, $[G/H]$ is a unit.

For an arbitrary level G/H, note that

$$i_H^* G/\{e\} = [G : H] H/\{e\},$$

and hence inverting $[G/\{e\}]$ at level G/G also inverts $[H/\{e\}]$. The result then follows from the analysis for G/G. □

Consider the homomorphism of Tambara functors $\underline{A} \to \underline{\mathbb{Z}}[\frac{1}{|G|}]$ given by the composite of $\underline{A} \to \underline{\mathbb{Z}}$ and localization. At level G/H, a finite H-set is sent to its cardinality. In particular, the image of $[G/e] \in \underline{A}(G/G)$ is a unit. Hence we obtain a homomorphism of Tambara functors

$$\alpha\colon \underline{A}[\tfrac{1}{[G/e]}] \to \underline{\mathbb{Z}}[\tfrac{1}{|G|}]$$

from the universal property of localization.

Theorem 3.5.6. *The homomorphism* $\alpha\colon \underline{A}[\frac{1}{[G/e]}] \to \underline{\mathbb{Z}}[\frac{1}{|G|}]$ *is an isomorphism of Tambara functors.*

Proof. Since $[G/\{e\}]$ being a unit implies that $|G|$ is a unit, and since all restrictions are ring homomorphisms, both the source and the target of the universal morphism are actually Mackey functors valued in $\mathbb{Z}[1/|G|]$-modules.

As a Mackey functor, the constant Mackey functor $\underline{\mathbb{Z}}[1/|G|]$ is generated by the element 1 at level G/G, and the element 1 is in the image of the homomorphism from $\underline{A}$, and hence the localization. This means that the natural homomorphism is surjective.

Lemma 3.5.5 shows that for any subgroup H, the homomorphism

$$\underline{A}\big[\tfrac{1}{|G|}\big] \to \underline{A}\big[\tfrac{1}{[G/e]}\big](G/H)$$

factors through the quotient by the ideal generated by $[H/K] - [H:K]$ for all K. This is the constant Mackey functor $\underline{\mathbb{Z}}[1/|G|]$, and hence the homomorphism is an isomorphism. □

3.5.2 Modules over $\underline{A}\left[\frac{1}{|G|}\right]$ and $\underline{A}\left[\frac{1}{[G/e]}\right]$

Before we consider free incomplete Tambara functors over localizations of the Burnside functor, we must describe their modules. By Theorem 3.5.6, modules over $\underline{A}[\frac{1}{[G/e]}]$ and modules over $\underline{\mathbb{Z}}[\frac{1}{|G|}]$ are equivalent. By this observation, $\underline{A}[\frac{1}{[G/e]}]$-modules are cohomological $\underline{A}[\frac{1}{|G|}]$-modules. We aim to characterize such modules.

We begin by discussing the splitting of the category of rational Mackey functors. Let $\mathrm{Sub}_c(G)$ denote a set of representatives for conjugacy classes of subgroups of G.

Work of Dress [Dre69] constructs a system of orthogonal idempotents for the rational Burnside ring. For each subgroup H of G, define a *mark homomorphism* $\phi_H\colon \underline{A}(G/G) \to \mathbb{Z}$ by $\phi_H([T]) = |T^H|$. Rationally, these assemble into an isomorphism of rings

$$\phi\colon \underline{A}_{\mathbb{Q}}(G/G) \xrightarrow{\cong} \prod_{H \in \mathrm{Sub}_c(G)} \mathbb{Q}.$$

The projections onto individual factors yield a system of orthogonal idempotents $\{e_H\}_{H\in \mathrm{Sub}_c(G)}$ for $\underline{A}_{\mathbb{Q}}(G/G)$ characterized by the property that

$$\phi_H(e_K) = \begin{cases} 1 & H \text{ and } K \text{ are conjugate in } G, \\ 0 & \text{otherwise.} \end{cases}$$

Example 3.5.7. For $H = e$, $e_{\{e\}} = \frac{1}{|G|}[G/e] \in \underline{A}_{\mathbb{Q}}(G/G)$ is the desired idempotent.

Indeed, one may verify that $e_{\{e\}}^2 = e_{\{e\}}$, and

$$\phi_H\left(\tfrac{1}{|G|}[G/e]\right) = \tfrac{1}{|G|}|G^H| = \begin{cases} 1 & H = e, \\ 0 & H \neq e. \end{cases}$$

$$e_{\{e\}}^2 = \tfrac{1}{|G|}[G/e] \cdot \tfrac{1}{|G|}[G/e] = \frac{1}{|G|^2}[G/e \times G/e] = \frac{1}{|G|^2}|G|[G/e] = e_{\{e\}}.$$

Since $\underline{A}(G/G) \cong \mathcal{M}\mathrm{ack}_G(\underline{A}, \underline{A})$, this system of orthogonal idempotents gives a splitting of the rational Burnside Mackey functor. We summarize these results in the following theorem:

Theorem 3.5.8 ([Dre69])**.** *There is a system of orthogonal idempotents* $\{e_H\}_{H \in \mathrm{Sub}_c(G)}$ *splitting the rational Burnside Green functor: there is an isomorphism of Green functors*

$$\underline{A}_{\mathbb{Q}} \cong \bigoplus_{H \in \mathrm{Sub}_c(G)} e_H \underline{A}_{\mathbb{Q}},$$

where $e_H\underline{A}_{\mathbb{Q}}$ *is the sub-Green functor of* $\underline{A}_{\mathbb{Q}}$ *with*

$$(e_H\underline{A}_{\mathbb{Q}})(G/K) = \mathrm{res}^G_K(e_H) \cdot \underline{A}_{\mathbb{Q}}(G/K).$$

This splitting of the monoidal unit gives a canonical splitting of the category of rational Mackey functors. Greenlees–May [GM95, Appendix A] then prove equivalences between $e_H\underline{A}_{\mathbb{Q}}$-modules and modules over a rational group-ring.

Theorem 3.5.9 ([GM95, Theorem A.9])**.** *For any* $H \leq G$ *there is an equivalence of categories*

$$U_H \colon e_H\underline{A}_{\mathbb{Q}}\text{-}\mathcal{M}\mathrm{od} \simeq \mathbb{Q}[W_G(H)]\text{-}\mathcal{M}\mathrm{od} \colon F_H$$

Both U_H *and* F_H *are exact functors.*

Together, the previous two theorems combine to give an equivalence of categories

$$\underline{A}_{\mathbb{Q}}\text{-}\mathcal{M}\mathrm{od} \simeq \prod_{H \in \mathrm{Sub}_c(G)} \mathbb{Q}[W_G(H)]\text{-}\mathcal{M}\mathrm{od}.$$

This equivalence was independently proven in [GM95, Appendix A] and [TW95] using very different methods – the former approaches the problem from the perspective of stable homotopy theory, whereas the latter uses algebraic techniques. A recent exposition of this result can be found in [BK21].

A consequence of this theorem is that all rational Mackey functors are projective. Indeed, by Maschke's theorem all $\mathbb{Q}[W_G(H)]$-modules are projective, and the equivalence of the previous theorem is by exact functors.

Corollary 3.5.10 ([GM95, Proposition A.2]). *All rational Mackey functors are projective.*

To prove Theorems 3.5.8 and 3.5.9, it turns out that it isn't necessary to work rationally, but merely to invert $|G|$.

Theorem 3.5.11. *There is an equivalence of categories*

$$U\colon \underline{A}[\tfrac{1}{|G|}]\text{-}\mathcal{M}od \simeq \prod_{H\in \mathrm{Sub}_c(G)} \mathbb{Z}[\tfrac{1}{|G|}][W_G(H)]\text{-}\mathcal{M}od\colon F. \tag{3.5.12}$$

Both U *and* F *are exact functors.*

Proof. By a careful analysis of the proofs of Theorems 3.5.8 and 3.5.9 in [BK21], there are only two places where it is necessary to invert an integer: in the construction of orthogonal idempotents splitting the Burnside ring in [BK21, Lemma 2.2] and the isomorphism between orbits and fixed points in [BK21, Example 2.9]. In both places, it suffices to invert the orders of all subgroups of G. Since inverting $|G|$ necessarily inverts all of its divisors, it is sufficient to work in $\mathbb{Z}[\frac{1}{|G|}]$ instead of $\mathbb{Q}$. □

From this theorem, we will discover that the summand of $\underline{A}[\frac{1}{|G|}]$ corresponding to the trivial subgroup is $\underline{A}[\frac{1}{[G/e]}]$, and therefore $\underline{A}[\frac{1}{[G/e]}]$-modules are equivalent to modules over a particular group-ring.

Lemma 3.5.13. *There is an isomorphism of Green functors* $e_{\{e\}}\underline{A}[\frac{1}{|G|}] \cong \underline{\mathbb{Z}}[\frac{1}{|G|}]$.

Proof. At level G/H, we have

$$\left(e_{\{e\}}\underline{A}[\tfrac{1}{|G|}]\right)(G/H) = \mathrm{res}^G_e(e_{\{e\}}) \cdot \underline{A}[\tfrac{1}{|G|}](G/H).$$

Recall from Example 3.5.7 that $e_{\{e\}} = \frac{1}{|G|}[G/e]$. Therefore,

$$\mathrm{res}^G_H(e_{\{e\}}) = \tfrac{1}{|H|}[H/e].$$

So at level G/H, $e_{\{e\}}\underline{A}[\frac{1}{|G|}]$ is the Green ideal of $\underline{A}[\frac{1}{|G|}](G/H)$ generated by $[H/e]$. In particular, at the underlying level

$$(e_{\{e\}}\underline{A}[\tfrac{1}{|G|}])(G/e) = \underline{A}[\tfrac{1}{|G|}](G/e) = \mathbb{Z}[\tfrac{1}{|G|}].$$

Therefore, every element in $e_{\{e\}}\underline{A}[\frac{1}{|G|}]$ at level G/H is a transfer of an element at level G/e. We conclude that $e_{\{e\}}\underline{A}[\frac{1}{|G|}]$ is the sub-Mackey functor of $\underline{A}[\frac{1}{|G|}]$ generated by the underlying level, that is,

$$e_{\{e\}}\underline{A}[\tfrac{1}{|G|}] \cong \underline{\mathbb{Z}}[\tfrac{1}{|G|}].$$

Since each level is a commutative ring, and we have levelwise isomorphisms of commutative rings, we conclude this is an isomorphism of Green functors. □

We also need a description of the functors U_H and F_H when H is the trivial subgroup.

Lemma 3.5.14 (cf. [BK21, Proposition 4.5])**.** *When* $H = \{e\}$*, the equivalence*

$$U_{\{e\}} \colon e_{\{e\}}\underline{A}_{\mathbb{Q}}\text{-}\mathcal{M}\mathrm{od} \simeq \mathbb{Q}[G]\text{-}\mathcal{M}\mathrm{od} \colon F_{\{e\}}$$

is given by functors

$$U_{\{e\}}(\underline{M}) := \underline{M}(G/\{e\}) \text{ and } F_{\{e\}}(V) := FP(G).$$

Corollary 3.5.15. *There is an equivalence of categories*

$$\underline{A}[\tfrac{1}{[G/e]}]\text{-}\mathcal{M}\text{od} \underset{\text{FP}}{\overset{U}{\rightleftarrows}} \mathbb{Z}[\tfrac{1}{|G|}][G]\text{-}\mathcal{M}\text{od},$$

where $U(\underline{M}) = \underline{M}(G/e)$ *and* FP *is the fixed point functor, such that:*

(a) both U *and* FP *are exact functors;*

(b) both U *and* FP *are strong symmetric monoidal for the box-product over* $\underline{A}[\frac{1}{[G/e]}]$ *on* $\underline{A}[\frac{1}{[G/e]}]$-$\mathcal{M}$od *and the tensor product over* $\mathbb{Z}[\frac{1}{|G|}]$ *on* $\mathbb{Z}[\frac{1}{|G|}][G]$-$\mathcal{M}$od.

Proof. By Theorem 3.5.11, there is an equivalence of categories

$$e_{\{e\}}\underline{A}[\tfrac{1}{|G|}]\text{-}\mathcal{M}\text{od} \simeq \mathbb{Z}[\tfrac{1}{|G|}][G]\text{-}\mathcal{M}\text{od},$$

which by Lemma 3.5.14 is given by functors U and FP as in the statement of the corollary. By Theorem 3.5.6 and Lemma 3.5.13 there are isomorphisms of Green functors

$$e_{\{e\}}\underline{A}[\tfrac{1}{|G|}] \cong \underline{\mathbb{Z}}[\tfrac{1}{|G|}] \cong \underline{A}[\tfrac{1}{[G/e]}],$$

yielding the desired equivalence of categories.

Exactness follows from exactness in Theorem 3.5.11. The strong symmetric monoidal property follows for U because

$$U(\underline{A}[\tfrac{1}{[G/e]}]) = \mathbb{Z}[\tfrac{1}{|G|}] \quad \text{and} \quad U(\underline{M} \boxtimes \underline{N}) = \underline{M}(G/e) \otimes_{\mathbb{Z}} \underline{N}(G/e);$$

the box product/tensor over the localization is the same as the ordinary box/tensor product. Then FP becomes strong symmetric monoidal as part of an equivalence. □

Corollary 3.5.16. *Any* $\underline{A}\left[\frac{1}{[G/e]}\right]$*-module* $\underline{M}$ *is a fixed point functor:* $\underline{M} \cong \text{FP}(\underline{M}(G/e))$.

Remark 3.5.17. In light of this corollary, every restriction in any $\underline{A}\left[\frac{1}{[G/e]}\right]$-module $\underline{M}$ is injective – it is the inclusion of fixed points. The condition that all restrictions are injective appears in several seemingly unrelated places. This is the *monomorphic restriction condition* of [Nak12a, Definition 4.19]. It is also the condition necessary for a Mackey functor to be a zero-slice of an equivariant spectrum [HHR16, Proposition 4.50]. This seemingly innocuous condition has many structural consequences for Mackey functors. In general any such functor satisfying the monomorphic restriction condition is a sub-functor of a fixed point functor, by [Nak12a, Proposition 4.21].

3.5.3 Underlying freeness after localization

We prove that all free incomplete Tambara functors over $\underline{A}[\frac{1}{[G/e]}]$ are free as $\underline{A}[\frac{1}{[G/e]}]$-modules. We will write $\underline{S}^{-1}\underline{A} = \underline{A}[\frac{1}{[G/e]}]$, with $\underline{S} = \{([G/e], G/G)\}$ to declutter notation.

Lemma 3.5.18. *Let $\mathcal{O}$ be any indexing category, and let $H \leq G$ be a subgroup. Then as a $\mathbb{Z}[\frac{1}{|G|}]$-algebra, $\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}](G/e)$ is polynomial on generators y_{gH} for cosets $gH \in G/H$:*

$$\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}](G/e) \cong \mathbb{Z}\left[\tfrac{1}{|G|}\right]\left[y_{gH} \mid gH \in G/H\right].$$

The group G acts on the generators by permuting the cosets. In particular, $\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}](G/e)$ is a permutation $\mathbb{Z}[\frac{1}{|G|}]$-module.

Proof. We have

$$\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}](G/e) = (\underline{S}^{-1}\underline{A} \boxtimes \underline{A}^{\mathcal{O}}[x_{G/H}])(G/e) = \mathcal{P}^G_{\mathcal{O}}(G/H, G/e)^+ \otimes_{\mathbb{Z}} \mathbb{Z}[\tfrac{1}{|G|}],$$

so it suffices to describe the ring $\mathcal{P}^G_{\mathcal{O}}(G/H, G/e)^+$. By Corollary 2.4.7, this ring is generated by polynomials of the form

$$y_g = [G/H \xleftarrow{\pi} G/e \xrightarrow{g} G/e \xrightarrow{\text{id}} G/e],$$

where π is the canonical projection and g is multiplication by $g \in G$. Note that the middle morphism is always admissible, for any indexing category $\mathcal{O}$. Two such polynomials are equivalent if the middle morphism differs by an element of H, so we have one such polynomial for each coset of G/H. Finally, this polynomial ring is a permutation module because the G-action permutes the monomials. □

Theorem 3.5.19. *Let $\underline{S}^{-1}\underline{A}$ be the Mackey functor obtained from $\underline{A}$ by inverting $[G/e] \in \underline{A}(G/G)$. For any indexing category $\mathcal{O}$ and subgroup $H \leq G$, the free $\mathcal{O}$-Tambara functor $\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}]$ is free as an $\underline{S}^{-1}\underline{A}$-module.*

Proof. By Lemma 3.5.18, the underlying level of $\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}]$ is a permutation module. By Corollary 3.5.16, $\underline{S}^{-1}\underline{A}^{\mathcal{O}}[x_{G/H}]$ is isomorphic to the fixed points of a permutation module as an $\underline{S}^{-1}\underline{A}$-module. In particular, it is a free module by Proposition 2.3.49. □

Lemma 3.5.18 also has another interesting consequence.

Proposition 3.5.20. *Any two free incomplete $\underline{S}^{-1}\underline{A}$-Tambara functors generated at the same level G/H are isomorphic as Green functors.*

Proof. By Lemma 3.5.18, they are both isomorphic to the fixed-point functor of the ring $\mathbb{Z}[\frac{1}{|G|}][y_{gH} \mid gH \in G/H]$. □

Essentially, the only difference between free incomplete $\underline{S}^{-1}\underline{A}$-Tambara functors is the norms.

APPENDIX A

TABLES OF INCOMPLETE TAMBARA FUNCTORS

The tables below describe the Mackey functors underlying free $\mathcal{O}$-Tambara functors generated by a single element at level G/H for various combinations G and H. The columns indicate the level of the generator and the rows indicate the N_∞-operad, in the guise of a transfer system. Within each cell of the table, the freeness of the underlying Mackey functor is stated, with the convention that an empty cell indicates that the Mackey functor underlying is not free.

The values in the tables are deduced from Theorem 3.3.26. The classifications of all of the transfer systems for cyclic groups below appear in [BBR21, Theorem 2] and [Rub21b, Section 3.2]. The classification of transfer systems for D_6 appears in [Rub21b, Section 3.2].

Table A.1: This table describes the underlying Mackey functor of $\underline{A}^{\mathcal{O}}[x_{C_p/H}]$. The table says, for example, that $\underline{A}^{\mathcal{O}^{\text{cplt}}}[x_{C_p/e}]$ is free as an $\underline{A}$-module, while $\underline{A}^{\mathcal{O}^{\text{cplt}}}[x_{C_p/C_p}]$ is not free as an $\underline{A}$-module.

	C_p/e	C_p/C_p
$\mathcal{O}^{\text{triv}}$	—	free
$\mathcal{O}^{\text{cplt}}$	free	—

Table A.2: The table below describes the Mackey functor underlying $\underline{A}^{\mathcal{O}}[x_{C_{p^2}/H}]$.

	C_{p^2}/e	C_{p^2}/C_p	C_{p^2}/C_{p^2}
$e \quad C_p \quad C_{p^2}$	—	—	free
$e \to C_p \quad C_{p^2}$	—	—	—
$e \to C_p \quad C_{p^2}$, $e \to C_{p^2}$	free	—	—
$e \quad C_p \to C_{p^2}$	—	free	—
$e \to C_p \to C_{p^2}$, $e \to C_{p^2}$	free	—	—

Table A.3: The table below describes the Mackey functor underlying $\underline{A}^{\mathcal{O}}[x_{C_{p^3}/H}]$.

	C_{p^3}/e	C_{p^3}/C_p	C_{p^3}/C_{p^2}	C_{p^3}/C_{p^3}
$e \quad C_p \quad C_{p^2} \quad C_{p^3}$	—	—	—	free
$e \to C_p \quad C_{p^2} \quad C_{p^3}$	—	—	—	
$e \quad C_p \to C_{p^2} \quad C_{p^3}$	—	—	—	
$e \quad C_p \quad C_{p^2} \to C_{p^3}$	—	—	free	—
$e \to C_p \quad C_{p^2} \to C_{p^3}$	—	—	—	—
$e \to C_p \quad C_{p^2} \quad C_{p^3}$; $e \to C_{p^2}$	—	—	—	—
$e \quad C_p \to C_{p^2} \quad C_{p^3}$; $C_p \to C_{p^3}$	—	free	—	—
$e \to C_p \quad C_{p^2} \quad C_{p^3}$; $e \to C_{p^2}$; $e \to C_{p^3}$	free	—	—	—
$e \to C_p \to C_{p^2} \quad C_{p^3}$; $e \to C_{p^2}$	—	—	—	—
$e \quad C_p \to C_{p^2} \to C_{p^3}$; $C_p \to C_{p^3}$	—	free	—	—
$e \to C_p \to C_{p^2} \quad C_{p^3}$; $e \to C_{p^2}$; $e \to C_{p^3}$	free	—	—	—
$e \to C_p \quad C_{p^2} \to C_{p^3}$; $e \to C_{p^2}$; $e \to C_{p^3}$	free	—	—	—
$e \to C_p \to C_{p^2} \quad C_{p^3}$; $e \to C_{p^2}$; $e \to C_{p^3}$; $C_p \to C_{p^3}$	free	—	—	—
$e \to C_p \to C_{p^2} \to C_{p^3}$; $e \to C_{p^2}$; $e \to C_{p^3}$; $C_p \to C_{p^3}$	free	—	—	—

Table A.4: Let p and q be distinct primes and consider the cyclic group C_{pq}. The transfer systems for C_{pq} are given in [Rub21b, Figure 2]. The table below describes the Mackey functor underlying $\underline{A}^{\mathcal{O}}[x_{C_{pq}/H}]$.

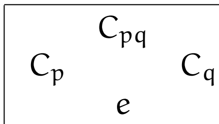
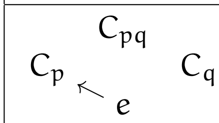
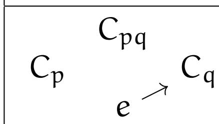
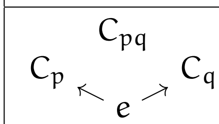
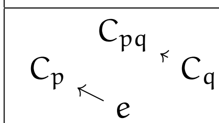
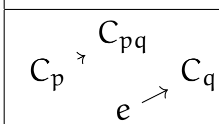
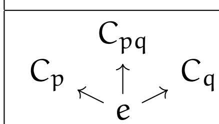
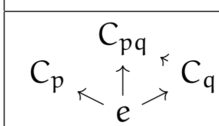
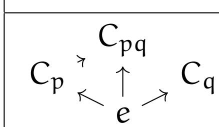
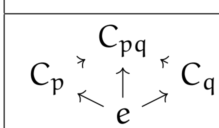
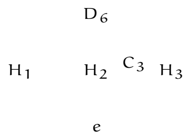
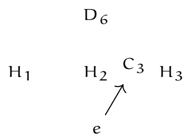
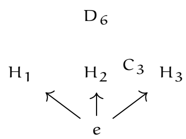
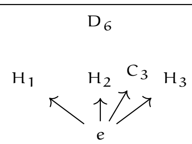
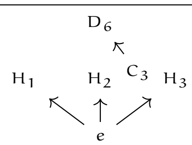
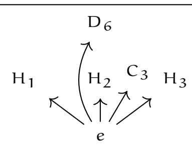
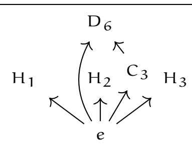
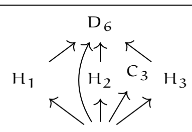
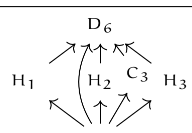
	C_{pq}/e	C_{pq}/C_p	C_{pq}/C_q	C_{pq}/C_{pq}
	—	—	—	free
	—	—	—	—
	—	—	—	—
	—	—	—	—
	—	—	free	—
	—	free	—	—
	free	—	—	—
	free	—	—	—
	free	—	—	—
	free	—	—	—

Table A.5: Consider the dihedral group D_6. This group has five proper subgroups: the trivial subgroup, three conjugate copies of C_2, and one copy of C_3. Write H_1, H_2, H_3 for its subgroups of order two and C_3 for its subgroup of order three. The transfer systems for D_6 are described in [Rub21b, Figure 4]. The table below describes the Mackey functor underlying $\underline{A}^{\mathcal{O}}[x_{D_6/H}]$.

	D_6/e	D_6/H_1	D_6/H_2	D_6/H_3	D_6/C_3	D_6/D_6
	—	—	—	—	—	free
	—	—	—	—	—	—
	—	—	—	—	—	—
	—	—	—	—	—	—
	—	—	—	—	free	—
	free	—	—	—	—	—
	free	—	—	—	—	—
	free	—	—	—	—	—
	free	—	—	—	—	—

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