DEGENERATE SERIES REPRESENTATIONS FOR SYMPLECTIC GROUPS

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Gautam Gopal Krishnan

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DEGENERATE SERIES REPRESENTATIONS FOR SYMPLECTIC GROUPS

Gautam Gopal Krishnan, Ph.D.

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We study the induced modules of real symplectic groups obtained by parabolic induction from the Siegel parabolic subgroup. The structure of the induced modules obtained from maximal parabolic subgroups of the general linear group were described by Barbasch, Sahi and Speh. We are interested in a similar description of the factors that are in the composition series of degenerate series representations of symplectic groups. More precisely, we use the $\tau-$invariant along with the $K-$types of the representations to restrict the factors that can occur in the induced modules with a given wave front set. We determine the factors that occur at singular infinitesimal characters using these algebraic techniques and then use a shift functor to describe the induced modules at all infinitesimal characters. We provide a general method for computing the Langlands parameters of the factors of the induced modules and explicitly illustrate these computations for $Sp(4,\mathbb{R})$ and $Sp(8,\mathbb{R})$.

BIOGRAPHICAL SKETCH

Gautam Gopal Krishnan was born in Chennai, India. He spent his undergraduate years at Chennai Mathematical Institute, where he majored in mathematics and computer science. He moved to Ithaca and began his graduate studies in mathematics in August 2014. He loves to teach and he has enjoyed his teaching experience as a graduate student. He will be moving to New Jersey for a postdoctoral position.

To my family, for the abundance of love and encouragement they have for me.

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TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

A fundamental problem in representation theory is to compute and classify irreducible unitary representations of real reductive groups. A standard technique to study representations is to build new representations from known representations and then decompose them into irreducible representations. Parabolic induction is a well-known procedure for building new representations from old. It is of natural interest to understand the structure of these induced modules by relating it to some combinatorial data.

Lie groups and their representations have been studied for a long time starting with work done by Sophus Lie with motivations coming from differential equations and mathematical physics. In more recent times, a systematic study of infinite dimensional representations of reductive Lie groups was developed by Harish-Chandra. When dealing with infinite dimensional continuous representations of real reductive Lie groups, representations are defined to be boundedly equivalent if there is an invertible intertwining operator between them. The notion of infinitesimal equivalence is provided by Harish-Chandra in [12] where the algebraic setting of $(\mathfrak{g}, K)-$modules is introduced. This reduces the question to understanding the structure of induced modules up to infinitesimal equivalence.

A close relationship between filtrations of induced representations by invariant subspaces and the orbit structure of a subgroup on a vector space was first observed by Jakobsen, Vergne in [17] and by Speh in [24] in the case of $SO(4, 2)$. A similar correspondence with an explicit construction was obtained by Barbasch, Sahi and Speh in [4] for $GL(2n, \mathbb{R})$, the general linear group of $2n \times 2n$ real matrices. In this work, they used Fourier analysis to obtain a natural analytic description of the lattice of invariant subspaces. In particular, they were able to use the parametrization of irreducible $(\mathfrak{g}, K)-$modules with a fixed infinitesimal character to get a priori control on possible composition factors of the induced modules with a given wave front set.

Following the theme of this body of work where we would like to relate induced modules to orbit structures, we provide a method to answer the following question:

Question 1. *What is the structure of the modules obtained by parabolic induction from the Siegel parabolic subgroup for $Sp(2n, \mathbb{R})$, the symplectic group of $2n \times 2n$ real matrices?*

The $p-$adic analog of this problem was considered in [11] by Gustafson and in [18] by Jantzen. Details regarding this problem for the orthogonal groups in the $p-$adic setting can be found in [2] by Ban and Jantzen, and for the real case in [16] by Howe and Tan.

Kashiwara and Vergne in [19] have found some factors in the induced modules for $Sp(2n, \mathbb{R})$. They find all the factors of the induced modules from the Siegel Parabolic subgroup (induced from one of the two one-dimensional characters of $GL(n, \mathbb{R})$ depending on whether n is even or odd) at the most singular infinitesimal character where the modules are not irreducible. They show that the factors are given by spaces of functions whose Fourier transform is supported on orbits of the action of the Levi part of the parabolic subgroup on its nilradical. Their motivation was to use the techniques described in detail in their work for Maxwell equations.

We provide a method to answer this question for all infinitesimal characters and for modules induced from both one dimensional characters of $GL(n,\mathbb{R})$ and then illustrate it by explicitly describing it in detail for $Sp(4,\mathbb{R})$ and $Sp(8,\mathbb{R})$. Furthermore, we find the precise Langlands parameters of the factors in the composition series of these induced modules with any given wave front set.

The thesis is organized as follows. Chapter 2 gives the basic information about symplectic groups and introduces the degenerate series representations. It also sets up the notation to be used for the rest of the thesis. This includes the definition of Langlands parameters and the notation we will use to denote them.

Chapter 3 follows [10] by Collingwood and McGovern for the most part and provides details about the (complex and real) nilpotent orbits and the combinatorial descriptions for the symplectic case. This chapter also explains the Lusztig symbol calculation parametrizing the double cell representations with illustration by means of examples.

Chapter 4 focuses on the coherent continuation representation with a description of the Harish-Chandra cells. We end this chapter with a method to obtain a list of possible candidates for the factors of the induced modules $I^{\pm}(\nu)$ defined in Definition 2.

Chapter 5 restricts the factors that can occur in these induced modules due to the $\tau-$invariant, which is introduced in this chapter. The lowest $K-$types of these factors are also computed here.

Chapter 6 summarizes the results and provides a conclusion to the thesis. The explicit results for the case of $Sp(4,\mathbb{R})$ and $Sp(8,\mathbb{R})$ are provided here.

In the appendix, we provide a detailed description of the theta correspondence and an outline of future work detailing how the composition series of the induced modules can be obtained.

CHAPTER 2

PRELIMINARIES

In this chapter, our goal is to introduce the degenerate series representations for symplectic groups and fix the notation. A standard reference for the theory of real reductive groups is [28] by Vogan. We will follow the set up and notation as introduced for $GL(2n, \mathbb{R})$ by Barbasch, Sahi and Speh in [4].

2.1 Symplectic Group

Let $G = Sp(2n, \mathbb{R})$ be a real symplectic group which can be identified as $Sp(V)$ where $V = \mathbb{R}^{2n}$ is a symplectic vector space of dimension $2n$ with a symplectic form defined by $< u, v >= u^t J_n v$ where $J_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. We write a typical element g in $Sp(2n, \mathbb{R})$ as

$$g = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

where $a, b, c, d \in M_n(\mathbb{R})$ and $g^t J_n g = J_n$. Let P_S be a Siegel parabolic subgroup. This is a maximal parabolic subgroup with Levi decomposition $P_S = LN$ where the Levi factor L is isomorphic to $GL(n, \mathbb{R})$ and $N = Sym_n(\mathbb{R})$ (the vector space of $n \times n$ symmetric matrices). Explicitly, we write the following for the indicated subgroups

$$P_S = \left\{ \begin{pmatrix} a & 0 \\ b & (a^{-1})^t \end{pmatrix} : a \in GL(n, \mathbb{R}), b \in M_n(\mathbb{R}), b = b^t \right\},$$

$$L = \left\{ \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} : a \in GL(n, \mathbb{R}) \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : b \in M_n(\mathbb{R}), b = b^t \right\},$$

$$\tilde{N} = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in M_n(\mathbb{R}), c = c^t \right\}$$

Observe that N and $\tilde{N}$ may be identified with $Sym_n(\mathbb{R})$.

2.2 Degenerate Series Representations

We will consider the representations $Ind_{P_S}^G(\chi)$ of G induced from an arbitrary one dimensional character χ of L with χ being trivial on N. One dimensional characters of $GL(n, \mathbb{R})$ are parametrized by elements in $\mathbb{Z}/2 \times \mathbb{C}$, where (ϵ, ν) corresponds to the character

$$a \mapsto (\text{sign det } a)^\epsilon \cdot |\text{det } a|^\nu$$

For each $\nu \in \mathbb{C}$, we let $\chi_\nu^\pm : P_S \to \mathbb{C}^\times$ be the character

$$\chi_\nu^\pm\left(\begin{pmatrix} a & 0 \\ b & (a^{-1})^t \end{pmatrix}\right) = \begin{cases} (\det a)^\nu & \text{if } \det a > 0 \\ \pm|\det a|^\nu & \text{if } \det a < 0 \end{cases}$$

The corresponding induced representations which we will now denote by $I^\pm(\nu)$ respectively will be realized in a space of functions on $\tilde{N}$ as defined below.

Definition 2. *The degenerate principal series representations $I^{\pm}(\nu)$ are realized on the space of $f \in C^{\infty}(G)$ respectively as*

$$I^{\pm}(\nu) = \{f : G \to \mathbb{C} : C^{\infty} | f(pg) = \delta(p)^{1/2}\chi^{\pm}_{\nu}(p)f(g), g \in G, p \in P_S\}$$

on which G acts by right translations, where δ is the modular function of P_S.

If we let $\rho_n = \frac{n+1}{2}$, then the modular function of P_S is given by

$$\delta(p) = |\det a|^{2\rho_n} = |\det a|^{n+1} = \chi^{+}_{n+1}(p)$$

where $p = \left(\begin{pmatrix} a & 0 \\ b & (a^{-1})^t \end{pmatrix}\right)$.

Since $Sym_n(\mathbb{R})$ is openly embedded into $P_S\backslash G$, a function $f \in Ind^G_{P_S}(\chi)$ is determined by the restriction f to the embedded image of $Sym_n(\mathbb{R})$ in $P_S\backslash G$. In our notation, the constant functions on $Sym_n(\mathbb{R})$ appear as a submodule of $I^{+}(-\frac{n+1}{2})$. Also, if $(I^{\pm}(\nu))^*$ is the Hermitian dual of $I^{\pm}(\nu)$, then

$$(I^{\pm}(\nu))^* = I^{\pm}(-\nu).$$

The representations given by ν and $-\nu$ are dual to each other.

Proposition 3. *$I^{\pm}(\nu)$ is irreducible if and only if $\nu + \rho_n \notin \mathbb{Z}$.*

The points of reducibility of $I^{\pm}(\nu)$ were first determined by Kudla and Rallis in [20], where they prove the above proposition by studying the action by the enveloping algebra. The proof of the above proposition also follows from the argument in the proof of Proposition II.1 in [4].

2.3 Langlands Parameters

We will use the same notation for the $(\mathfrak{g}, K)$−module associated to a representation as for the representation of G itself. Here $K \cong U(n)$ denotes a maximal compact subgroup of G and $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ denotes the complexification of the Lie algebra of G. We denote by $\mathfrak{k}_0$ the Lie algebra of K and by $\mathfrak{k}$ its complexification. $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R}) = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is its corresponding Cartan decomposition.

The Cartan subalgebra for $\mathfrak{g}$ is identified with $\mathbb{C}^n$ with standard basis $e_1, \cdots, e_n$. We fix a positive system where the positive roots for $\mathfrak{k}$ are given by $e_i - e_j$, $i < j$, while the non-compact positive roots for $\mathfrak{g}$ are $e_i + e_j$, $i < j$ and $2e_i$. The simple roots are given by $e_i - e_{i+1}$ and $2e_n$. This gives us

$$\rho = (n, \cdots, 1), \quad \rho_{\mathfrak{k}} = \left(\tfrac{n-1}{2}, \cdots, -\tfrac{n-1}{2}\right), \quad \rho_{\mathfrak{p}} = \left(\tfrac{n+1}{2}, \cdots, \tfrac{n+1}{2}\right)$$

where ρ is half the sum of positive roots, $\rho_{\mathfrak{k}}$ is half the sum of compact positive roots and $\rho_{\mathfrak{p}}$ is half the sum of non-compact positive roots.

The entries of ρ and $\rho_{\mathfrak{k}}$ decrease by one, while the entries of $\rho_{\mathfrak{p}} = \rho_{\mathfrak{g}} - \rho_{\mathfrak{k}}$ are constant. The Weyl group $W = W(\mathfrak{g}, \mathfrak{t})$ is a semidirect product of $(\mathbb{Z}/2)^n$ and S_n and it acts on the coordinates by sign changes and permutations.

We fix the Cartan involution to be

$$\theta(x) = (x^{-1})^t \text{ for } x \in G,$$

$$\theta(X) = -X^t \text{ for } X \in \mathfrak{g}.$$

A Cartan subgroup H of G is the centralizer in G of a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$. We will denote the complexification of $\mathfrak{h}_0$ by $\mathfrak{h}$. Any Cartan subgroup is conjugate under G to a θ-stable Cartan subgroup (that is, $\theta H = H$). As linear functionals on $\mathfrak{h}_0$, a root is real, imaginary or complex as its values on $\mathfrak{h}_0$ are.

We make θ act on the dual of a complex Cartan subalgebra $\mathfrak{h}^*$ by duality, and if α is a root, then

$$\alpha \text{ is real} \iff \theta(\alpha) = -\alpha,$$
$$\alpha \text{ is imaginary} \iff \theta(\alpha) = \alpha,$$
$$\alpha \text{ is complex} \iff \theta(\alpha) \neq \pm\alpha.$$

Complex conjugation acts on $\mathfrak{h}^*$, such that

$$\bar{\alpha} = -\theta(\alpha).$$

A θ-stable Cartan subgroup which is a representative for the conjugacy class of a Cartan subgroup in G can be specified by describing the Cartan involution on the simple roots. Irreducible representations for G have Langlands parameters associated to them which can be described by θ-stable Cartan subgroups and some additional data which is explained below.

Definition 4. *Suppose H is a θ-stable Cartan subgroup of G with Lie algebra $\mathfrak{h}_0$. A limit pseudocharacter (or limit character) of H is a triple*

$$\gamma = (\Psi, \Gamma, \bar{\gamma})$$

with the following properties.

1. *Ψ is a positive system for the imaginary roots of $\mathfrak{h}$ in $\mathfrak{g}$, $\Gamma \in \hat{H}$, and $\bar{\gamma} \in \mathfrak{h}^*$.*

2. *If $\alpha \in \Psi$, then $< \alpha, \bar{\gamma} > \geq 0$.*

3. *$d\Gamma = \bar{\gamma} + \rho(\Psi) - 2\rho_c(\Psi)$.*

A limit character γ is called final if it also satisfies

1. *Suppose α is a simple root of Ψ, and $< \alpha, \bar{\gamma} >= 0$, then α is noncompact.*

2. *Suppose α is a real root of $\mathfrak{h}$ in $\mathfrak{g}$ and $< \alpha, \bar{\gamma} >= 0$, then α does not satisfy the parity condition ([27], Definition 8.3.11).*

In the above definition, Ψ_c are the compact roots in Ψ and

$$\rho(\Psi) = \frac{1}{2} \sum_{\alpha \in \Psi} \alpha,$$

$$\rho_c(\Psi) = \frac{1}{2} \sum_{\alpha \in \Psi_c} \alpha.$$

Any γ which is a limit pseudocharacter that is also final will be called a Langlands parameter. We write $X(\gamma)$ to denote the standard module and $\bar{X}(\gamma)$ to denote the unique irreducible quotient corresponding to the parameter γ as descibed in [29] by Vogan.

Theorem 5. *(Langlands, Knapp-Zuckerman)*
(a) Suppose γ_1 and γ_2 are two Langlands parameters with θ-stable Cartan subgroups H_1 and H_2 resepectively. Then $\bar{X}(\gamma_1) \cong \bar{X}(\gamma_2)$ if and only if (H_1, γ_1) and (H_2, γ_2) are conjugate under K.
(b) Suppose X is an irreducible $(\mathfrak{g}, K)-$module. Then there is a θ-stable Cartan subgroup H, and a Langlands parameter γ, such that $X \cong \bar{X}(\gamma)$.

Thus, a Langlands parameter for an irreducible representation is a $K-$ conjugacy class of data

$$(H, \gamma) = (H, (\Psi, \Gamma, \bar{\gamma}))$$

with the notations as described above. In the special case when $\bar{\gamma}$ is regular, the inequalities in the definition of the Langlands parameters are strict and Ψ is uniquely determined by $\bar{\gamma}$. A more detailed description of Langlands parameters can be found in Chapter 11 of [1] by Adams, Barbasch and Vogan, where Γ is $\tilde{\Lambda}$ and $\bar{\gamma}$ is λ in their notation.

In our calculations, we will write the Langlands parameter γ in the following manner. The infinitesimal character $\bar{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ will be written in coordinates (where the simple roots and the positive system were fixed at the beginning of this section) with γ_i underlined to indicate that the root $2e_i^*$ is a real root; γ_j, γ_k underlined together to indicate $e_j^* + e_k^*$ is a real root and any γ_l that is not underlined will indicate that the root $2e_l^*$ is imaginary. This notation also implicitly describes what the θ-stable Cartan subalgebra associated to the Langlands parameter is, by describing the Cartan involution in terms of the real, imaginary and complex roots.

To take into account the parity condition in the definition of Langlands parameters, we will write an exponent of + or − for real roots and we fix the convention that $\underline{\gamma_i}^+$ means the real root $2e_i^*$ satisfies the parity condition if and only if γ_i is odd. For singular infinitesimal characters $\bar{\gamma}$, we will prefix γ_i with + or − to indicate the two different Langlands parameters that we can denote by it when γ_i is 0. In particular, these are the two limits of discrete series representations and we will make this precise in the example below.

Example 6. *For $G = Sp(4, \mathbb{R})$, with regular infinitesimal character $\rho = (2, 1)$, there are 18 Langlands parameters which, in the notation described above, are as follows.*

$$(2,1),(2,-1),(-2,1),(-2,-1),(2,\underline{1}^{+}),(2,\underline{1}^{-}),(-2,\underline{1}^{+}),(2,\underline{1}^{-}),(\underline{2}^{+},1),$$
$$(\underline{2}^{+},-1),(\underline{2}^{-},1),(\underline{2}^{-},-1),(\underline{2}^{+},\underline{1}^{+}),(\underline{2}^{-},\underline{1}^{-}),(\underline{2}^{+},\underline{1}^{-}),(\underline{2}^{-},\underline{1}^{+}),(\underline{2,-1}),(\underline{2,1}).$$

Example 7. *For $G = Sp(4, \mathbb{R})$, with singular infinitesimal character $\rho = (1, 0)$, there are 13 Langlands parameters which, in the notation described above, are as follows.*

$$(1,+0),(-1,+0),(1,\underline{0}^{+}),(1,-0),(-1,\underline{0}^{+}),(-1,-0),(\underline{1}^{+},+0),$$
$$(\underline{1}^{-},+0),(\underline{1}^{+},\underline{0}^{+}),(\underline{1}^{-},-0),(\underline{1}^{+},-0),(\underline{1}^{-},\underline{0}^{+}),(\underline{1,0}).$$

In this example, $(1, +0)$ denotes the limit of discrete series obtained from $(2, 1)$ when we make the parameter singular and $(1, -0)$ is the limit of discrete series obtained from $(2, -1)$ when we make the parameter singular. We will use this same convention everywhere.

In our notation, two Langlands parameters γ and γ' which differ in just one coordinate, with γ having $+\underline{\gamma_i}^{\epsilon}$ and γ' having $-\underline{\gamma_i}^{\epsilon}$ in the coordinate that differs, refer to the same Langlands parameter. Similarly, any occurrence of $\underline{\gamma_i, \gamma_j}$ can be replaced by $\underline{-\gamma_i, -\gamma_j}$ to denote the same parameter. The permutation of coordinates of any given Langlands parameter refers to the same given Langlands parameter because of the equivalence of the Langlands parameters under the real Weyl group. We describe this in the following examples.

Example 8. *Here is a list of examples that clarifies the above notation.*

- $(2, \underline{1}^{+})$ *and* $(2, -\underline{1}^{+})$ *refer to the same Langlands parameter.*
- $(\underline{2, -1})$ *and* $(\underline{-2, 1})$ *refer to the same Langlands parameter.*

- $(-2,-1)$ *and* $(-1,-2)$ *refer to the same Langlands parameter.*
- $(\underline{4,-1},3,2)$ *and* $(3,2,\underline{4,-1})$ *refer to the same Langlands parameter.*

We conclude this section with the notions of block equivalence and dual groups which will be used later in our calculations.

Definition 9. *Block equivalence of irreducible (or standard) representations of G is the equivalence relation generated by*

$$\gamma \sim \delta \text{ if } [X(\gamma) : \bar{X}(\delta)] \neq 0 \text{ or } [X(\delta) : \bar{X}(\gamma)] \neq 0.$$

where $[X(\gamma) : \bar{X}(\delta)]$ = *mutiplicity of* $\bar{X}(\delta)$ *in* $X(\gamma)$. *The equivalence classes are called blocks.*

This is Definition 1.14 in [28] and the following theorem on dual groups is Theorem 1.15 in [28] by Vogan.

Theorem 10. *Let* $G^{\mathbb{C}}$ *be a complex connected reductive Lie group, and G a real form of* $G^{\mathbb{C}}$. *Fix a block* $(\bar{\pi}_1, \cdots, \bar{\pi}_r)$ *of irreducible representations of G, having the same infinitesimal character as some finite dimensional representation of G. Write* $\check{G}^{\mathbb{C}}$ *for the complex simply connected semisimple group whose root system is dual to that of* $G^{\mathbb{C}}$. *Then there is a real form* $\check{G}$ *of* $\check{G}^{\mathbb{C}}$, *and a block* $(\bar{\rho}_1, \cdots, \bar{\rho}_r)$ *of irreducible representations of* $\check{G}$, *such that*

$$M(\pi_j, \bar{\pi}_i) = \epsilon_{ij} m(\bar{\rho}_i, \rho_j)$$
$$M(\rho_j, \bar{\rho}_i) = \epsilon_{ij} m(\bar{\pi}_i, \pi_j)$$

for all i, j; here $\epsilon_{ij} = \pm 1$. *In the correspondence* $\bar{\rho}_i \leftrightarrow \bar{\pi}_i$, *discrete series corresponds to Langlands quotients of principal series for split groups; and finite dimensional representations correspond to representations whose annihilator is a minimal primitive ideal.*

In the above theorem $M(,)$ and $m(,)$ denote multiplicities as understood in a Grothendieck group. For our calculations, G is $Sp(2n,\mathbb{R})$ and $\check{G}$ will be an indefinite special orthogonal group $SO(p,q)$ with $p+q=2n+1$, $p \geq q$.

CHAPTER 3

NILPOTENT ORBITS

This chapter gives a description of nilpotent orbits, describes the Springer correspondence and provides an exposition of the Lusztig symbol calculation parametrizing double cells. We will begin with describing the nilpotents orbits in $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$ and then provide examples of some of the real nilpotent orbits that arise in $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ which will be useful for later calculations. More details can be found in chapters 9 and 10 in [10] by Collingwood and McGovern.

3.1 Complex Nilpotent Orbits

For every $X \in \mathfrak{g}$, the adjoint representation $\mathrm{ad}_X : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ defined by $\mathrm{ad}_X(Y) = [X, Y] = XY - YX$ gives a Lie algebra homomorphism.

Definition 11. *$X \in \mathfrak{g}$ is nilpotent if $ad(X)$ is nilpotent in $End(\mathfrak{g})$.*

This is equivalent to saying that X is nilpotent if there exists some positive integer k such that $X^k = 0$. Nilpotent orbits refer to the orbit of a nilpotent element under the action of the connected complex adjoint group $G^{\mathbb{C}}_{ad}$. The adjoint group is the connected component of the automorphism group defined as follows:

$$\mathrm{Aut}(\mathfrak{g}) = \{\phi \in GL(\mathfrak{g}) | [\phi(X), \phi(Y)] = \phi([X, Y]), \text{for all } X, Y \in \mathfrak{g}\}.$$

Observe that for $\phi \in \mathrm{Aut}(\mathfrak{g}), X \in \mathfrak{g}$, we have $\phi \mathrm{ad}_X \phi^{-1} = \mathrm{ad}_{\phi(X)}$ which implies the nilpotence of X is equivalent to the nilpotence of all elements $\phi(X), \phi \in \mathrm{Aut}(\mathfrak{g})$.

A complex nilpotent orbit is an orbit of $\mathfrak{g}$ under the action of $G^{\mathbb{C}}_{ad}$, all of whose elements are nilpotent.

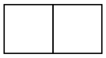
There is a partial ordering on the complex nilpotent orbits which depends on the Zariski closure operation. Every orbit is Zariski-open in its closure. Given two complex nilpotent orbits O, O', we say that $O \leq O'$ if $\bar{O} \subseteq \bar{O}'$.

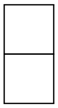
From the theory of Jordan normal forms, we know that there are finitely many nilpotent orbits. The nilpotent orbits in $\mathfrak{sl}(n, \mathbb{C})$ are given by partitions of n where each part of the partition corresponds to the size of a Jordan block. Similarly, the nilpotent orbits in $\mathfrak{sp}(2n, \mathbb{C})$ are parametrized by certain Young diagrams corresponding to some partitions of $2n$ as described by the following theorem.

Theorem 12. *Nilpotent orbits in $\mathfrak{sp}(2n, \mathbb{C})$ are in one-to-one correspondence with the set of partitions of $2n$ in which odd parts occur with even multiplicity.*

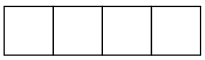
It is possible to view $\mathfrak{sp}(2n, \mathbb{C})$ as a subalgebra of $\mathfrak{sl}(2n, \mathbb{C})$ and obtain the correspondence stated above from the correspondence between the Jordan block decomposition in $\mathfrak{sl}(2n, \mathbb{C})$ and partitions of $2n$. The precise correspondence which associates a canonical representative of a nilpotent orbit to a given partition is elaborated in [10]. A Young diagram is a left-justified rows of empty boxes arranged so that no row is shorter than the one below it. Given a partition of $2n$, the corresponding Young diagram is obtained with row sizes equal to the parts of the partition. We use the notation $O_{[d_1, \cdots, d_r]}$ for the orbit attached to the partition $[d_1, \cdots, d_r]$.

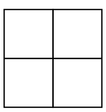
Example 13. *The following are all the nilpotent orbits in $\mathfrak{sp}(2, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C})$ for the adjoint action:*

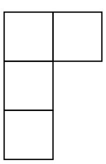
 *with partition* $[2]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

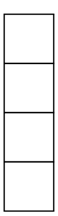
 *with partition* $[1,1]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 14. *The following are all the nilpotent orbits in* $\mathfrak{sp}(4,\mathbb{C})$ *for the adjoint action:*

 *with partition* $[4]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

 *with partition* $[2,2]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

 *with partition* $[2,1,1]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

 *with partition* $[1,1,1,1]$ *corresponds to nilpotent orbit of* $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Example 15. *There are* 8 *nilpotent orbits in* $\mathfrak{sp}(6,\mathbb{C})$ *given by*

$$\mathcal{O}_{[6]}, \mathcal{O}_{[4,2]}, \mathcal{O}_{[4,1,1]}, \mathcal{O}_{[3,3]}, \mathcal{O}_{[2,2,2]}, \mathcal{O}_{[2,2,1,1]}, \mathcal{O}_{[2,1,1,1,1]}, \mathcal{O}_{[1,1,1,1,1,1]}.$$

Given two partitions $\mathbf{d} = [d_1, d_2, \cdots, d_N]$ and $\mathbf{f} = [f_1, f_2, \cdots, f_N]$, we say that $\mathbf{d}$ dominates $\mathbf{f}$ and write $\mathbf{d} \geq \mathbf{f}$ if the following condition holds:

$$\sum_{1 \leq j \leq k} d_j \geq \sum_{1 \leq j \leq k} f_j \text{ for } 1 \leq k \leq N.$$

We append trailing zeros to the parts of a partition if neccesary so that $\mathbf{d}$ and $\mathbf{f}$ written above have the same number of parts N. We have the following theorem due to Gerstenhaber and Hesselink which is Theorem 6.2.5 in [10].

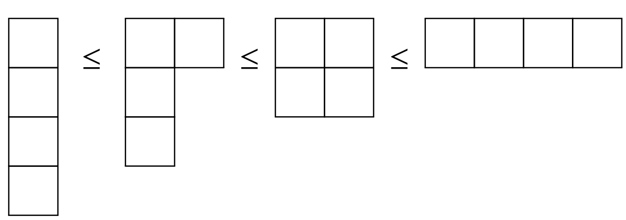
Theorem 16. *Let $\mathbf{d}$ and $\mathbf{f}$ be two partitions of $2n$ corresponding to nilpotent orbits $O_{\mathbf{d}}$ and $O_{\mathbf{f}}$ respectively. Then $O_{\mathbf{d}} \geq O_{\mathbf{f}}$ if and only if $\mathbf{d} \geq \mathbf{f}$.*

The boundary of a nilpotent orbit (i.e. the set difference $\bar{O} \backslash O$) has a smaller dimension than the orbit since it is a Zariski closed proper subvariety of its closure. Hence the boundary is a union of nilpotent orbits of smaller dimension. In particular, no two nilpotent orbits have the same closure. The dimensions of the nilpotent orbits can be calculated combinatorially from the partitions of $2n$ using the following theorem by Springer and Steinberg.

Theorem 17. *Given a nilpotent orbit $O_{[d_1,\cdots,d_N]}$ in $\mathfrak{sp}(2n, \mathbb{C})$ corresponding to a partition $[d_1, \cdots, d_N]$, put $r_i = |\{d_j | d_j = i\}|$ and $s_i = |\{d_j | d_j \geq i\}|$. Then*

$$dim(O_{[d_1,\cdots,d_N]}) = 2n^2 + n - \frac{1}{2} \sum_i s_i^2 - \frac{1}{2} \sum_{i \text{ odd}} r_i.$$

Example 18. *In $\mathfrak{sp}(4, \mathbb{C})$, the nilpotent orbits are ordered as follows:*



[1, 1, 1, 1] [2, 1, 1] [2, 2] [4]

dim = 0 4 6 8

3.2 Real Nilpotent Orbits

We will now describe the real nilpotent orbits in $\mathfrak{g}_0 = \mathfrak{sp}(2n, \mathbb{R})$ under the adjoint action of the real Lie group $G = Sp(2n, \mathbb{R})$. Similar to the case of complex nilpotent orbits, there are finitely many real nilpotent orbits and there is a classification in terms of combinatorial data. A signed Young diagram is a Young diagram in which every box is labelled with a + or - sign in such a way that signs alternate across rows. Two signed Young diagrams are regarded as equivalent if and only if one can be obtained from the other by interchanging rows of equal length. The signature of a signed Young diagram is an ordered pair (p, q), where p is the number of boxes labelled + and q is the number of boxes labelled -.

Theorem 19. *Nilpotent orbits in* $\mathfrak{sp}(2n, \mathbb{R})$ *are parametrized by signed Young diagrams of size* $2n$ *and signature* (n, n)*, up to permutations of rows with equal length.*

Similar to the case of complex nilpotent orbits, the correspondence can be made precise by writing out canonical representatives of nilpotent orbits starting with the Jordan normal form of the complexification of the real nilpotent orbit. The details regarding this parametrization with canonical representatives of nilpotent orbits can be found in chapter 9 of [10].

Example 20. *The real nilpotent orbits of* $\mathfrak{sp}(2, \mathbb{R}) = \mathfrak{sl}(2, \mathbb{R})$ *are parametrized by the following signed Young diagrams.*

$\begin{array}{|c|c|}\hline + & - \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}$ *with representatives* $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ *respectively.*

$\begin{array}{|c|}\hline + \\ \hline - \\ \hline\end{array}$ *with representative* $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 21. *The real nilpotent orbits of* $\mathfrak{sp}(4,\mathbb{R})$ *are parametrized by the following signed Young diagrams.*

$\begin{array}{|c|c|c|c|}\hline + & - & + & - \\ \hline\end{array}$ $\begin{array}{|c|c|c|c|}\hline - & + & - & + \\ \hline\end{array}$

$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline\end{array}$

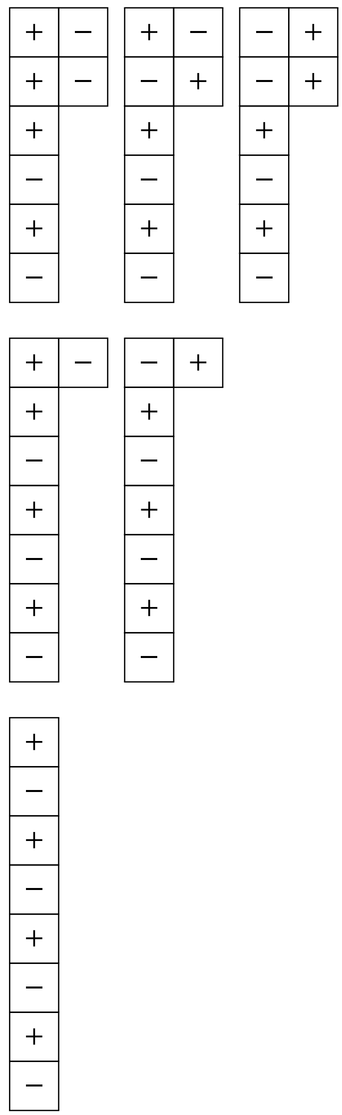
$\begin{array}{|c|c|}\hline + & - \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$

$\begin{array}{|c|}\hline + \\ \hline - \\ \hline + \\ \hline - \\ \hline\end{array}$

Example 22. *The real nilpotent orbits of* $\mathfrak{sp}(8,\mathbb{R})$ *whose complexification is contained in the closure of* $O_{[2,2,2,2]}$ *are parametrized by the following signed Young diagrams.*

$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + & - \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline - & + \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline - & + \\ \hline - & + \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline - & + \\ \hline - & + \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline - & + \\ \hline - & + \\ \hline\end{array}$

$\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline + & - \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline - & + \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline + & - \\ \hline - & + \\ \hline - & + \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline - & + \\ \hline + \\ \cline{1-1} - \\ \cline{1-1}\end{array}$



In Example 13 and Example 14 we have seen the complex nilpotent orbits of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sp}(4,\mathbb{C})$ respectively. Observe that we can now have multiple real nilpotent orbits that have the same complexification. For example, in $\mathfrak{sl}(2,\mathbb{R})$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ are representatives of the two non-zero nilpotent orbits, but they are elements of the same nilpotent orbit when viewed as elements of $\mathfrak{sl}(2,\mathbb{C})$.

3.3 Springer Correspondence

In this section, we describe the Springer correspondence relating representations of the Weyl group W (introduced in the previous chapter) to nilpotent orbits. We then describe the Lusztig symbol calculation parametrizing double cells and illustrate this for specific examples and show how to compute the Weyl group representation, given a nilpotent orbit in $\mathfrak{sp}(2n, \mathbb{C})$.

We begin by formally stating the Springer correspondence for $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$. Let $\mathcal{B}$ denote the flag variety of $\mathfrak{g}$ given by the set of all Borel subalgebras. For any nilpotent element $X \in \mathfrak{g}$, let $\mathcal{B}_X$ denote the Dynkin variety of all Borel subalgebras containing X. Let $A(O_X)$ denote the $G^{\mathbb{C}}_{ad}$-equivariant fundamental group defined by ${}^XG^{\mathbb{C}}_{ad}/({}^XG^{\mathbb{C}}_{ad})^o$ where ${}^XG^{\mathbb{C}}_{ad}$ is the stabilizer of X under the action of $G^{\mathbb{C}}_{ad}$ and $({}^XG^{\mathbb{C}}_{ad})^o$ is the connected component containing the identity.

Theorem 23. *1. There is a natural action of W on $H^*(\mathcal{B}_X, \mathbb{C})$ commuting with the action of $A = A(O_X)$.*

2. *The natural map $H^*(\mathcal{B}, \mathbb{C}) \to H^*(\mathcal{B}_X, \mathbb{C})$ is W-equivariant.*

3. *The top degree cohomology $H^{dim \mathcal{B}_X}(\mathcal{B}_X, \mathbb{C})$ decomposes as a direct sum $\oplus_{\mu \in \hat{A}}(\pi_\mu \otimes V_\mu)$, where π_μ is either 0 or an irreducible representation of W on which A acts trivially, while V_μ is a module on which A acts by μ and W acts trivially. Here "dim" denotes real dimension.*

4. *We have $\pi_{\mathbf{1}} \neq 0$ where **1** denotes the trivial representation.*

5. *Any irreducible W-module is isomorphic to π_μ for a unique complex nilpotent orbit O_X and a unique $\mu \in \hat{A}$.*

The Springer correspondence is the map sending (O_X, μ) to π_μ. Before we illustrate how this correspondence works in examples, we have the following theorem that parametrizes the irreducible representations of W. In the following theorem $\pi_{\mathbf{d}}$ and $\pi_{\mathbf{f}}$ are representations of the symmetric groups associated to partitions **d** and **f** respectively coming from the representation theory of symmetric groups. The precise statement is stated as Theorem 10.1.1 due to Young in [10].

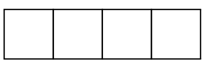
Theorem 24. *The elements of $\hat{W}$ are parametrized by ordered pairs $(\boldsymbol{d}, \boldsymbol{f})$ of partitions such that $|\boldsymbol{d}|+|\boldsymbol{f}| = n$. The representation $\pi_{(d,f)}$ is characterized by the following property. Let σ be the subspace of $\pi_{(d,f)}$ consisting of all vectors on which the first $|\boldsymbol{d}|$ copies of $\mathbb{Z}/2$ acts trivially while the remaining $|\boldsymbol{f}|$ copies act by -1. Then the product of symmetric group $S_{|\boldsymbol{d}|} \times S_{|\boldsymbol{f}|}$ acts on σ according to the representation $\pi_{\boldsymbol{d}} \times \pi_{\boldsymbol{f}}$.*

Harish-Chandra cells (or double cells as defined in [28] by Vogan) will be defined in the next chapter. To describe the computations for the correspondence between complex nilpotent orbits (parametrized by partitions of $2n$) and representations of the Weyl group (parametrized by pairs of partitions as described in the theorem above), we need Lusztig's notion of the symbol of a representation as described in [21] and [9].

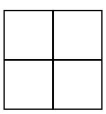
Given a partition $\mathbf{d} = [d_1, \cdots, d_N]$ of $2n$ denoting a complex nilpotent orbit, we compute its symbol as follows. If N is even, add a 0 at the end of **d**, otherwise leave it unchanged. Define an increasing sequence of integers $e_1 < e_2 < \cdots < e_N$ by $e_i = d_{N+1-i} + i - 1$. Enumerate the even e_i as $2f_1 < 2f_2 \cdots < 2f_a$ and the odd e_i as $2g_1 + 1 < 2g_2 + 1 < \cdots < 2g_b + 1$. It turns out that $a = b + 1$. Form the alternating sequence $(f_1, g_1, f_2, g_2, \cdots, g_b, f_a)$ and write it down so that all the f_i are in one line while the g_j are on a lower line. The resulting arrangement is called the symbol of **d**.

Given a symbol obtained from a partition corresponding to a complex nilpotent orbit, we can produce a pair of partitions which parametrizes an element of $\hat{W}$ as follows: subtract $(i-1)$ from the i^{th} element of the top row and the bottom row (counting from left). Then the new top and bottom row, when rearranged in non-increasing order forms the required pairs of partitions. This correspondence sends special nilpotent orbits (described in the next section) to special Weyl group representations. This coincides with the description of the Springer correspondence for $\mu = \mathbf{1}$ (the trivial representation of A), sending $(\mathcal{O}_X, \mathbf{1})$ to $\pi_{\mathbf{1}} \in \hat{W}$.

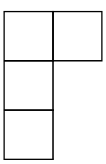
Example 25. *In $\mathfrak{sp}(4, \mathbb{C})$, here are the explicit Lusztig symbol calculations and the Weyl group representations corresponding to all the nilpotent orbits.*



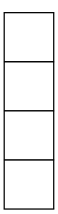
$\rightarrow \begin{pmatrix} 2 \end{pmatrix} \rightarrow ([2], [0])$



$\rightarrow \begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix} \rightarrow ([1], [1])$



$\rightarrow \begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix} \rightarrow ([1, 1], [0])$



$\rightarrow \begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix} \rightarrow ([0], [1, 1])$

Given a nilpotent orbit $\mathcal{O}$ with Lusztig symbol $(f_1, g_1, f_2, g_2, \cdots, g_b, f_a)$, let $(f_1', g_1', f_2', g_2', \cdots, g_b', f_a')$ be any permutation of it such that $f_1' < f_2' < \cdots < f_a'$ and $g_1' < g_2' < \cdots < g_b'$. Then we may construct an element of $\hat{W}$ from $(f_1', g_1', f_2', g_2', \cdots, g_b', f_a')$ by the process described above. The double cell representation parametrized by the Lusztig symbol of $\mathcal{O}$ is a sum of representations of W obtained in this manner.

Example 26. *In $\mathfrak{sp}(4, \mathbb{C})$, here are the calculations describing the double cell representations parametrized by some of the complex nilpotent orbits.*

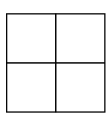
$$\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix} \to ([1], [1])$$

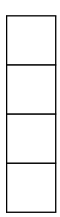
$$\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix} \to ([1, 1], [0])$$

$$\begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix} \to ([0], [2])$$

$$\begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix} \to ([0], [1, 1])$$

This tells us that the double cell representations parametrized by the nilpotent orbits $[2, 2]$ *and* $[1, 1, 1, 1]$ *are as follows.*


$\leftrightarrow ([1], [1]) + ([1, 1], [0]) + ([0], [2])$


$\leftrightarrow ([0], [1, 1])$

The multiplicities of these representations of W in the coherent continuation representation will be computed in the next chapter.

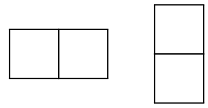
3.4 Special Nilpotent Orbits

The wave front set of a representation as defined in [13] by Howe gives us nilpotent orbits corresponding to the induced module representations for $G = Sp(2n, \mathbb{R})$ that we are interested in. The properties of the wave front set and the related notions of asymptotic support and associated cycle that we will use are described in detail by Barbasch and Vogan in [5], [6], [7] and [8]. We are interested in describing the composition factors of the induced modules $I^{\pm}(\nu)$ at the places where this induced module is not irreducible (i.e. when $\nu + \frac{n+1}{2} \in \mathbb{Z}$ following Proposition 3). The wave front set of $I^{\pm}(\nu)$ is the closure of $O_{[2,2,\cdots,2]}$.

Thus, the composition factors in the induced modules must correspond to nilpotent orbits in the closure of $[2, 2, \cdots, 2]$ which are given by $[2^k, 1^{2n-2k}]$ (i.e. partitions of $2n$ written in exponent notation where 2 occurs k times and 1 occurs $2n - 2k$ times). In addition to this, we have the constraint that these nilpotent orbits are special (as defined by Lusztig). The special nilpotent orbits in $\mathfrak{sp}(2n, \mathbb{C})$ are described combinatorially in terms of the partitions. A partition for $\mathfrak{sp}(2n, \mathbb{C})$ is special if it has an even number of even parts between any two consecutive odd parts and an even number of even parts greater than the largest odd part.

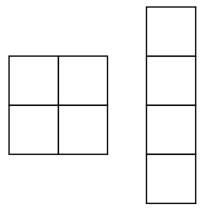
Since we are only interested in partitions of the form $[2^k, 1^{2n-2k}]$, the special nilpotent orbits are the partitions of $2n$ where both 2 and 1 occur an even number of times. The special nilpotent orbits can also be determined from the Lusztig symbol $(f_1, g_1, f_2, \cdots, g_b, f_a)$ associated with it by the condition that $f_1 \leq g_1 \leq f_2 \leq \cdots g_b \leq f_a$ as described in [9] and [10].

Example 27. *The special nilpotent orbits in* $\mathfrak{sl}(2, \mathbb{C})$ *are*

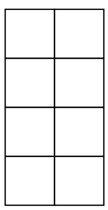


All nilpotent orbits in $\mathfrak{sl}(2n, \mathbb{C})$ *are special.*

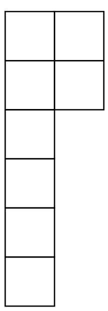
Example 28. *The relevant special nilpotent orbits in* $\mathfrak{sp}(4, \mathbb{C})$ *are*



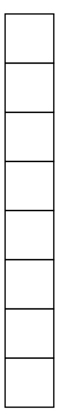
Example 29. *The relevant special nilpotent orbits in* $\mathfrak{sp}(8, \mathbb{C})$ *along with the Lusztig symbol calculations and corresponding special Weyl group representations are*



$$\rightarrow \begin{pmatrix} 0 & & 2 & & 3 \\ & 1 & & 2 & \end{pmatrix} \rightarrow ([1,1],[1,1])$$



$$\rightarrow \begin{pmatrix} 0 & & 1 & & 2 & & 4 \\ & 1 & & 2 & & 3 & \end{pmatrix} \rightarrow ([1],[1,1,1])$$



$$\rightarrow \begin{pmatrix} 0 & & 1 & & 2 & & 3 & & 4 \\ & 1 & & 2 & & 3 & & 4 & \end{pmatrix} \rightarrow ([0],[1,1,1,1])$$

These examples along with Example 20, Example 21 and Example 22 describe all the complex nilpotent orbits, the associated Lusztig symbols and Weyl groups representations, and the corresponding real nilpotent orbits that are relevant for the study of the induced modules $I^{\pm}(\nu)$ in $SL(2,\mathbb{R})$, $Sp(4,\mathbb{R})$ and $Sp(8,\mathbb{R})$. In general, the special Weyl group representation associated to the special nilpotent orbit $[2^{2k}, 1^{2n-4k}]$ is parametrized by $([1^k],[1^{n-k}])$ as observed in the above examples.

CHAPTER 4

WEYL GROUP REPRESENTATIONS

In this chapter, we will describe the tools we use to narrow down the potential factors of the composition series of the induced modules. We will describe the Harish-Chandra cells that are relevant for the study of the induced modules and describe its W-module structure. A general description of these Harish-Chandra cells along with the action of the Weyl group can be found in the works of McGovern in [22] and [23]. We will conclude this chapter with a general algorithm that produces a list of possible candidates that can occur in the induced modules at infinitesimal character ρ and enumerate this in the example of $Sp(4, \mathbb{R})$.

4.1 Coherent Continuation Representation

To find the possible candidates for the composition factors of the degenerate series representation, we will need the notion of coherent continuation representation which is the action of the Weyl group detailed in [25] by Speh and Vogan. We will state the properties of the representation as described in sections 12-14 in [28] by Vogan and illustrated in sections IV and VI in [4] by Barbasch, Sahi and Speh.

Given a regular integral infinitesimal character χ (which we will specialize to ρ), let $\mathcal{G}(\chi)$ be the Grothendieck group over $\mathbb{Q}$ generated by the characters of irreducible $(\mathfrak{g}, K)$−modules with infinitesimal character χ. This has as bases

$$\{\bar{X}(\gamma)\}_{\gamma \in \mathcal{P}(\chi)}, \{X(\gamma)\}_{\gamma \in \mathcal{P}(\chi)}$$

where $\mathcal{P}(\chi)$ is the set of parameters with infinitesimal character χ. We are using $X(\gamma)$ and $\bar{X}(\gamma)$ to denote the standard module and the unique irreducible quotient corresponding to γ respectively as described in Chapter 2.

The Weyl group W acts on $\mathcal{G}(\chi)$ by the coherent continuation representation $t(w)$ defined in sections 12-13 of [28]. Let $R^{+}(\gamma)$ be the positive system in $\Delta(\mathfrak{g}, \mathfrak{h})$ determined by $\bar{\gamma}$ and let $\Pi(\gamma) \subset R^{+}(\gamma)$ be the set of simple roots. Then the formulas for $t(s)$ with $s = s_{\alpha}$, $\alpha \in \Pi(\gamma)$ on the basis $X(\gamma)$ are given by the following formulas.

1. $t(s)X(\gamma) = -X(\gamma)$, if α is compact imaginary.
2. $t(s)X(\gamma) = \gamma$, if α is real not satisfying the parity condition.
3. $t(s)X(\gamma) = s \times X(\gamma)$, if α is complex, $\alpha \in R^{+}(\gamma)$.
4. $t(s)X(\gamma) = -s \times X(\gamma)$, if α is complex, $\alpha \notin R^{+}(\gamma)$.
5. $t(s)X(\gamma) = -s \times X(\gamma) + c^{\alpha}(X(\gamma))$, if α is noncompact imaginary type II.
6. $t(s)X(\gamma) = s \times X(\gamma)$, if α is real type II, satisfying the parity condition.
7. $t(s)X(\gamma) = -X(\gamma) + X(\gamma)^{\alpha}_{+} + X(\gamma)^{\alpha}_{-}$, if α is noncompact imaginary type I.
8. $t(s)X(\gamma) = X(\gamma)$, if α is real type I, satisfying the parity condition.

In these formulas, $\times$ is the cross action and $c^{\alpha}(X(\gamma)), X(\gamma)^{\alpha}_{+}, X(\gamma)^{\alpha}_{-}$ come from the Cayley transform of γ as defined in sections 4 and 7 of [28].

4.2 Harish-Chandra Cells

Fix a block B of representations with infinitesimal character χ (as defined in Definition 9). We will write $Z(B)$ to denote the free $\mathbb{Z}$-module with basis B. We identify $Z(B)$ with the Grothendieck group of the category of finite length admissible representations generated by the $X(\gamma)$, by sending γ to the class of $X(\gamma)$. The Weyl group acts on $Z(B)$ by the coherent continuation representation described above.

The LR preorder $<_{LR}$ on B is the smallest preorder relation with the following property. Fix $w \in W$, $\gamma \in B$, and write

$$t(w)\bar{X}(\gamma) = \sum_{\phi \in B} a_\phi \bar{X}(\phi).$$

Then,

$$a_\phi \neq 0 \Rightarrow \gamma <_{LR} \phi.$$

The LR equivalence $\approx_{LR}$ is defined by $\phi \approx_{LR} \gamma$ if and only if $\gamma <_{LR} \phi <_{LR} \gamma$. The cone over γ is defined as

$$\bar{\mathcal{C}}^{LR}(\gamma) = \{\phi \in B | \gamma <_{LR} \phi\}.$$

Define

$$\bar{\mathcal{V}}^{LR}(\gamma) = \text{span of } \{\bar{X}(\phi) | \gamma <_{LR} \phi\} \subseteq Z(B).$$

The Harish-Chandra cell of γ is

$$\mathcal{C}^{LR}(\gamma) = \{\phi \in B | \gamma \approx_{LR} \phi\}.$$

We define

$$\mathcal{C}_+^{LR}(\gamma) = \bar{\mathcal{C}}^{LR}(\gamma) - \mathcal{C}^{LR}(\gamma).$$

The corresponding objects in the Grothendieck group will be denoted by $\mathcal{V}^{LR}(\gamma), \mathcal{V}_+^{LR}(\gamma)$. Then $\mathcal{V}_+^{LR}(\gamma)$ is $t(W)$-invariant and the representation of W on $\mathcal{V}^{LR}(\gamma)$ defined by the natural isomorphism

$$\mathcal{V}^{LR}(\gamma) \cong \bar{\mathcal{V}}^{LR}(\gamma)/\mathcal{V}_+^{LR}(\gamma)$$

is called the double cell representation in section 14 of [28]. We have adopted the terminology Harish-Chandra cell for double cell.

Example 30. *For $G = Sp(4,\mathbb{R})$, with infinitesimal character $\rho = (2,1)$, here are the blocks:*

- $\{(2,-1), (-2,1), (2,1), (2,\underline{1}^+), (\underline{2}^-,1), (\underline{2,1}),$

  $(\underline{2,-1}), (\underline{2}^-,\underline{1}^-), (-2,-1), (-2,\underline{1}^+), (\underline{2}^-,-1), (\underline{2}^+,\underline{1}^+)\}$

  This corresponds to the dual group $SO(3,2)$.

- $\{(\underline{2}^+,1), (\underline{2}^+,-1), (2,\underline{1}^-), (-2,\underline{1}^-), (\underline{2}^-,\underline{1}^+)\}$

  This corresponds to the dual group $SO(4,1)$.

- $\{(\underline{2}^+,\underline{1}^-)\}$

  This corresponds to the dual group $SO(5,0)$.

4.3 Multiplicities of Representations

In this section, we calculate the dimensions and multiplicities of Weyl group representations that occur in the coherent continuation representation. We will follow the exposition in [23] by McGovern for $Sp(2n,\mathbb{R})$.

For $G = Sp(2n, \mathbb{R})$, conjugacy classes of Cartan subgroups are indexed by ordered pairs (p, q) of nonnegative integers with $p + 2q \leq n$. Cartan subgroups in the (p, q)th class are isomorphic to $\mathbb{T}^p \times (\mathbb{C}^*)^q \times (\mathbb{R}^*)^{n-p-2q}$, so that they have 2^{n-p-2q} components. Here $\mathbb{T}$ denotes the circle group. Their Weyl groups are isomorphic to $W(A_{p-1}) \times W_q \ltimes W(A_1)^q \times W_{n-p-2q}$, where the first and last terms act in the obvious way on the first p and last $n - p - 2q$ coordinates while the middle terms permute, interchange, and change the signs of the q remaining pairs of coordinates. Here there are $n + 1$ blocks and the coherent continuation representation becomes

$$\sum Ind_{W_{pqab}}^{W}(sgn \otimes triv \otimes sgn \otimes triv \otimes triv)$$

where W_{pqab} denotes $W(A_{p-1}) \times W_q \rtimes W(A_1)^q \times W_a \times W_b$, acting in the obvious way on $\mathbb{R}^n$, and the sum runs over all ordered quadruples (p, q, a, b) of nonnegative integers with $a + b = n - p - 2q$.

As stated in Theorem 24, the irreducible representations of W are given by pairs of Young diagrams. The multiplicity of an irreducible Weyl group representation that occurs in the coherent continuation representation is obtained by the Littlewood-Richardson rule. The Littlewood-Richardson rule in this case implies that the multiplicity of a pair of Young diagrams that occurs in the coherent continuation representation is given by the number of ways of filling the boxes of the Young diagrams (up to permutations) with r, c, c' and $\cdot \times \cdot$ such that no two r occur in the same row, no two c occur in the same column and no two c' occur in the same column.

Example 31. *For $G = Sp(4, \mathbb{R})$, the following Weyl group representations occur in the coherent continuation representation with the indicated multiplicities.*

$\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \quad \times \quad \phi \qquad \textit{Multiplicity} = 5$

$\begin{array}{|c|c|}\hline r & c \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|c|}\hline r & c' \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|c|}\hline c & c \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|c|}\hline c & c' \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|c|}\hline c' & c' \\ \hline\end{array}\times\phi$

$\begin{array}{|c|}\hline \ \\ \hline\end{array} \quad \times \quad \begin{array}{|c|}\hline \ \\ \hline\end{array} \qquad \textit{Multiplicity} = 4$

$\begin{array}{|c|}\hline \cdot \\ \hline\end{array}\times\begin{array}{|c|}\hline \cdot \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline r \\ \hline\end{array}\times\begin{array}{|c|}\hline r \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline c \\ \hline\end{array}\times\begin{array}{|c|}\hline r \\ \hline\end{array}\ ,\ \begin{array}{|c|}\hline c' \\ \hline\end{array}\times\begin{array}{|c|}\hline r \\ \hline\end{array}$

$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \quad \times \quad \phi \qquad \textit{Multiplicity} = 4$

$\begin{array}{|c|}\hline r \\ \hline r \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|}\hline r \\ \hline c \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|}\hline r \\ \hline c' \\ \hline\end{array}\times\phi\ ,\ \begin{array}{|c|}\hline c \\ \hline c' \\ \hline\end{array}\times\phi$

$\phi \quad \times \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} \qquad \textit{Multiplicity} = 1$

$\phi\times\begin{array}{|c|}\hline r \\ \hline r \\ \hline\end{array}$

In the above example, ϕ denotes the empty partition. This calculation tells us that in the double cell representation parametrized by $[2,2]$ in Example 26, the representation of W denoted by $\phi \times \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}$ does not appear.

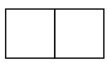
The length of a Harish-Chandra cell which is the number of Langlands parameters it contains, can be calculated by computing the dimensions of Weyl group representations using the hook length formula. The hook length formula computes the number of standard Young tableaux whose shape is a given Young diagram. The hook length formula of a standard Young diagram with shape λ corresponding to a partition of n is given by

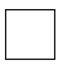
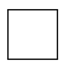
$$\frac{n!}{\Pi h_\lambda(i,j)}$$

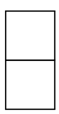
where $h_\lambda(i, j)$ is the hook length of cell (i, j) which is the number of cells (a, b) with $a = i, b \geq j$ or $a \geq i, b = j$.

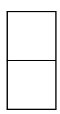
In particular, the hook length formula says that the dimension of a Weyl group representation given by $([1^k], [1^{n-k}])$ is $\frac{n!}{k!(n-k)!}$.

Example 32. *Here are the dimensions of the Weyl group representations from the previous example.*

 $\times$ ϕ *Dimension* = 1.

 $\times$  *Dimension* = 2.

 $\times$ ϕ *Dimension* = 1.

ϕ $\times$  *Dimension* = 1.

From the Lusztig symbol calculation parametrizing double cells, we see that $[2, 2]$ nilpotent orbit corresponds to $([1], [1]) + ([1, 1], [0])$ and $[1, 1, 1, 1]$ corresponds to $([0], [1, 1])$. Thus, the above computations of multiplicities and dimensions of the Weyl group representations say that, in the case of $Sp(4, \mathbb{R})$, the 18 Langlands parameters in Example 6 are divided as follows: five Harish-Chandra cells with 1 parameter each, four Harish-Chandra cells with 3 parameters each and one Harish-Chandra cell with 1 parameter corresponding to the trivial representation.

Example 33. *The* 18 *Langlands parameters in Example* 6 *are divided into Harish-Chandra cells as follows.*

- $\{(2, -1)\}$

- $\{(-2,1)\}$
- $\{(\underline{2}^{+},\underline{1}^{-})\}$
- $\{(\underline{2}^{+},1)\}$
- $\{(\underline{2}^{+},-1)\}$
- $\{(2,1),(2,\underline{1}^{+}),(\underline{2}^{-},1)\}$
- $\{(\underline{2,1}),(\underline{2,-1}),(\underline{2}^{-},\underline{1}^{-})\}$
- $\{(-2,-1),(-2,\underline{1}^{+}),(\underline{2}^{-},-1)\}$
- $\{(2,\underline{1}^{-}),(-2,\underline{1}^{-}),(\underline{2}^{-},\underline{1}^{+})\}$
- $\{(\underline{2}^{+},\underline{1}^{+})\}$

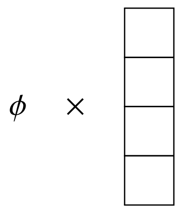
The possible candidates for the factors of the composition series in the induced modules for $Sp(2n,\mathbb{R})$ (when n is even), must correspond to the dual group $SO(n+1,n)$. This fact, along with the constraint that we are restricted to special nilpotent orbits contained in the closure of the wave front set of the induced module says that the only possible candidates for factors of $I^{\pm}(\frac{3}{2})$ when $G = Sp(4,\mathbb{R})$ are the three Harish-Chandra cells with 3 parameters corresponding to the dual group of $SO(3,2)$ and the one Harish-Chandra cell with the 1 parameter containing the trivial representation.

Example 34. *For $G = Sp(4,\mathbb{R})$, the Langlands parameters of the candidates for factors of $I^{\pm}(\frac{3}{2})$ are the following.*

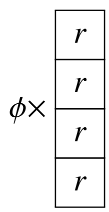
- $\{(2,1),(2,\underline{1}^{+}),(\underline{2}^{-},1)\}$
- $\{(\underline{2,1}),(\underline{2,-1}),(\underline{2}^{-},\underline{1}^{-})\}$
- $\{(-2,-1),(-2,\underline{1}^{+}),(\underline{2}^{-},-1)\}$

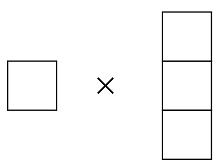
- $\{(\underline{2}^-, \underline{1}^-)\}$

Example 35. *For $G = Sp(8, \mathbb{R})$, the dimensions and multiplicities of the Weyl group representations corresponding to the relevant special nilpotent orbits are as follows.*

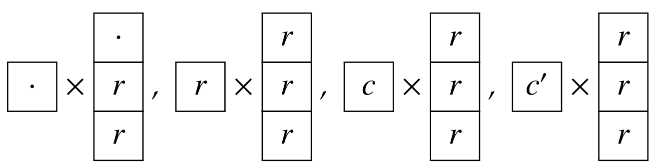


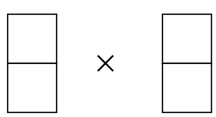
Multiplicity = 1, Dimension = 1



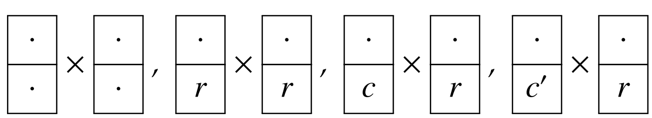


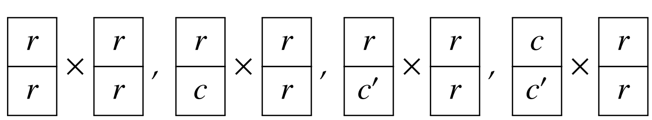
Multiplicity = 4, Dimension = 4

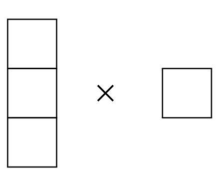




Multiplicity = 8, Dimension = 6







Multiplicity = 8, Dimension = 4

$$\begin{array}{|c|}\hline \cdot \\ \hline r \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|}\hline \cdot \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline \cdot \\ \hline r \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|}\hline \cdot \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline \cdot \\ \hline r \\ \hline c' \\ \hline \end{array} \times \begin{array}{|c|}\hline \cdot \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline \cdot \\ \hline c \\ \hline c' \\ \hline \end{array} \times \begin{array}{|c|}\hline \cdot \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline r \\ \hline r \\ \hline r \\ \hline \end{array} \times \begin{array}{|c|}\hline r \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline r \\ \hline r \\ \hline c \\ \hline \end{array} \times \begin{array}{|c|}\hline r \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline r \\ \hline r \\ \hline c' \\ \hline \end{array} \times \begin{array}{|c|}\hline r \\ \hline \end{array}\ , \quad \begin{array}{|c|}\hline r \\ \hline c \\ \hline c' \\ \hline \end{array} \times \begin{array}{|c|}\hline r \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline \ \\ \hline \end{array} \quad \times \quad \phi \qquad \textit{Multiplicity = 4, Dimension = 1}$$

$$\begin{array}{|c|}\hline r \\ \hline r \\ \hline r \\ \hline r \\ \hline \end{array} \times \phi\ , \quad \begin{array}{|c|}\hline r \\ \hline r \\ \hline r \\ \hline c \\ \hline \end{array} \times \phi\ , \quad \begin{array}{|c|}\hline r \\ \hline r \\ \hline r \\ \hline c' \\ \hline \end{array} \times \phi\ , \quad \begin{array}{|c|}\hline r \\ \hline r \\ \hline c \\ \hline c' \\ \hline \end{array} \times \phi$$

The other Weyl group representations not listed in the above example do not appear in the coherent continuation representation.

From the Lusztig symbol calculation parametrizing double cells, we see that $[2^4]$ nilpotent orbit corresponds to $([1,1],[1,1]) + ([1,1,1],[1])$, $[2,2,1^4]$ nilpotent orbit corresponds to $([1],[1,1,1]) + ([1,1,1,1],[0])$ and $[1^8]$ corresponds to $([0],[1,1,1,1])$. Thus, the above computations of multiplicities and dimensions of the Weyl group representations say that, in the case of $Sp(8,\mathbb{R})$, we are left with $8 \times (6+4) = 80$ possible candidates corresponding to $[2^4]$ nilpotent orbit, $4 \times (4+1) = 20$ possible candidates for $[2,2,1^4]$ nilpotent orbit and 1 candidate corresponding to $[1^8]$ nilpotent orbit (which we know is the Langlands parameter $(\underline{4}^+, \underline{3}^+, \underline{2}^+, \underline{1}^+)$ corresponding to the trivial representation.)

In general, the above calculations are available for any $Sp(2n,\mathbb{R})$. The multiplicities of the Weyl group representations can be computed using the Littlewood-Richardson rule and the dimensions of the representations can be computed by the hook length formula. By restricting to Weyl group representations corresponding to special nilpotent orbits contained in the closure of $[2^n]$ partition, and further restricting to Harish-Chandra cells in the blocks corresponding to the dual group $SO(n+1,n)$, we are left with a list of potential factors of the induced modules at infinitesimal character ρ.

CHAPTER 5

COMPOSITION FACTORS

We introduce the $\tau-$invariant in this chapter which is an important invariant that further restricts the potential candidates for the composition factors of the induced modules. The lowest $K-$type calculations will distinguish the possible occurrence of these factors between $I^{+}(\nu)$ and $I^{-}(\nu)$.

5.1 $\tau-$invariant

Let χ be a regular integral infinitesimal character. We will recall the definition of $\tau-$invariant of a Langlands parameter for an irreducible representation with infinitesimal character χ (as defined in [27]). We follow the notations from the previous chapters and use γ, $\bar{\gamma}$ and the Cartan involution θ as described before. Let $R^{+}(\gamma)$ be the positive system in $\Delta(\mathfrak{g}, \mathfrak{h})$ determined by $\bar{\gamma}$ and let $\Pi(\gamma) \subset R^{+}(\gamma)$ be the set of simple roots.

Definition 36. *The $\tau-$invariant of γ is*

$$\tau(\gamma) = \{\alpha \in \Pi(\gamma) | \alpha \textit{ is compact imaginary; or } \alpha \textit{ is complex, and } \theta(\alpha) \notin R^{+}(\gamma)\textit{; or } \alpha \textit{ real, satisfying the parity condition}\}.$$

For the action of the Weyl group W on $\mathcal{G}(\chi)$ by the coherent continuation representation $t(w)$, the formulas for $t(s)$ with $s = s_{\alpha}$, $\alpha \in \Pi(\gamma)$ on the basis $\bar{X}(\gamma)$ are given by

1. $t(s)\bar{X}(\gamma) = -\bar{X}(\gamma)$, if $\alpha \in \tau(\gamma)$.
2. $t(s)\bar{X}(\gamma) = \bar{X}(\gamma) + \sum_{\gamma \to \gamma'} \bar{X}(\gamma) + \sum_{\phi < \gamma, \alpha \in \tau(\phi)} \mu(\phi, \gamma)\bar{X}(\phi)$, if $\alpha \notin \tau(\gamma)$.

In the above formulas, the relation $\gamma \to \gamma'$ and the Bruhat order $<$ are as defined in sections 12-13 in [28].

In general, the $\tau-$invariant determines which irreducibles go to 0, and which ones don't, when we apply the Jantzen-Zuckerman translation functor ψ_α ([27], Definition 4.5.7) from a regular integral infinitesimal character to a singular infinitesimal character.

The induced module at infinitesimal character ρ becomes zero when applying the translation functor to singular parameter with a short simple root equal to 0. Therefore, the irreducible factors must have the short simple roots in their $\tau-$invariant. We will now calculate the $\tau-$invariant of the factors enumerated in Example 34. We will write + if the simple root is in the $\tau-$invariant and we will write − if the simple root is not in the $\tau-$invariant.

Example 37. *For $G = Sp(4, \mathbb{R})$, the following tabular column depicts the $\tau-$invariant of the Langlands parameters from Example 34.*

γ	α_1	α_2
$(2,1)$	$+$	$-$
$(2,\underline{1}^+)$	$-$	$+$
$(\underline{2}^-,1)$	$+$	$-$
$(\underline{2,-1})$	$+$	$-$
$(\underline{2,1})$	$-$	$+$
$(\underline{2}^-,\underline{1}^-)$	$+$	$-$
$(-2,-1)$	$+$	$-$
$(-2,\underline{1}^+)$	$-$	$+$
$(\underline{2}^-,-1)$	$+$	$-$
$(\underline{2}^+,\underline{1}^+)$	$+$	$+$

In this example, α_1 denotes the short simple root and α_2 denotes the long simple root in $\Pi(\gamma)$.

Hence, the possible candidates for the factors of $I^{\pm}(\frac{3}{2})$ are

$$\{\{(2,1),(\underline{2}^-,1)\}$$
$$\{(\underline{2,-1}),(\underline{2}^-,\underline{1}^-)\}$$
$$\{(-2,-1),(\underline{2}^-,-1)\}\}$$
$$\{(\underline{2}^+,\underline{1}^+)\}.$$

In general, for some Langlands parameter γ, if there are two adjacent simple roots α_1,α_2 with $\alpha_1 \in \tau(\gamma)$ and $\alpha_2 \notin \tau(\gamma)$, and if there exists some simple reflection s_β such that $\alpha_1 \notin \tau(s_\beta(\gamma))$ and $\alpha_2 \in \tau(s_\beta(\gamma))$, then γ and $s_\beta(\gamma)$ are in the same Harish-Chandra cell. This was used to compute the cells in Example 34.

Example 38. *For $G = Sp(8,\mathbb{R})$, with infinitesimal character $\chi = \rho = (4,3,2,1)$, the following tabular column contains the required list of Langlands parameters in the*

Harish-Chandra cells corresponding to the relevant special nilpotent orbits (Example 22, Example 29), with the appropriate dual group and $\tau-$invariant.

γ	α_1	α_2	α_3	α_4
$(4, 3, 2, 1)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, 3, 2, 1)$	$+$	$+$	$+$	$-$
$(\underline{4, -1}, 3, 2)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, \underline{3}^-, 2, 1)$	$+$	$+$	$+$	$-$
$(\underline{4, -1}, \underline{3, -2})$	$+$	$+$	$+$	$-$
$(\underline{4}^+, \underline{3}^+, \underline{2, -1})$	$+$	$+$	$+$	$-$
$(\underline{4, -1}, -3, -2)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, \underline{3}^-, -2, -1)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, -3, -2, -1)$	$+$	$+$	$+$	$-$
$(-4, -3, -2, -1)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, \underline{3}^-, \underline{2}^-, 1)$	$+$	$+$	$+$	$-$
$(\underline{4}^+, \underline{3}^+, 2, 1)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, \underline{3}^-, \underline{2}^-, \underline{1}^-)$	$+$	$+$	$+$	$-$
$(\underline{4}^-, \underline{3}^-, \underline{2, -1})$	$+$	$+$	$+$	$-$
$(\underline{4,}^-, \underline{3}^-, \underline{2}^-, -1)$	$+$	$+$	$+$	$-$
$(\underline{4}^+, \underline{3}^+, -2, -1)$	$+$	$+$	$+$	$-$
$(\underline{4}^+, \underline{3}^+, \underline{2}^+, \underline{1}^+)$	$+$	$+$	$+$	$+$

In this example, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the simple roots with α_i adjacent to α_{i+1} and α_4 is the long simple root in $\Pi(\gamma)$.

For $G = Sp(4, \mathbb{R})$, the infinitesimal character of the induced module at $\nu = \frac{1}{2}$ (i.e. $\chi = (1, 0)$) is singular for the long simple root. Therefore all its factors must be ψ_α applied to irreducible modules that do not have the long simple root in their $\tau-$invariant. This gives a constraint on the candidates for factors of $I^{\pm}(\frac{1}{2})$.

Example 39. *For $G = Sp(4, \mathbb{R})$, by Example 37 and the remark above, the Langlands parameters of the list of potential factors of $I^{\pm}(\frac{1}{2})$ are the following.*

$$(1, +0), (\underline{1}^-, +0), (\underline{1, 0}), (\underline{1}^+, \underline{0}^+), (-1, -0), (\underline{1}^-, -0)$$

Similarly, for $G = Sp(8, \mathbb{R})$, the Jantzen-Zuckerman translation functor (from $\nu = \frac{5}{2}$ to $\nu = \frac{3}{2}$) applied to the Langlands parameters at $\chi = \rho$ which do not contain the long simple root in the $\tau-$invariant will give us a list of potential candidates for factors of $I^{\pm}(\frac{3}{2})$ (i.e. $\chi = (3, 2, 1, 0)$).

Example 40. *For $G = Sp(8, \mathbb{R})$, the Langlands parameters of the list of potential factors of $I^{\pm}(\frac{3}{2})$ are the following.*

$$(3, 2, 1, +0), (\underline{3}^-, 2, 1, +0), (\underline{3, 0}, 2, 1), (\underline{3}^-, \underline{2}^-, 1, +0),$$
$$(\underline{3, 0}, \underline{2, -1}), (\underline{3}^-, \underline{2}^-, \underline{1, 0}), (\underline{3, 0}, -2, -1), (\underline{3}^-, \underline{2}^-, -1, -0),$$
$$(-3, -2, -1, -0), (\underline{3}^-, -2, -1, -0), (\underline{3}^-, \underline{2}^-, \underline{1}^-, +0), (\underline{3}^+, \underline{2}^+, 1, +0),$$
$$(\underline{3}^+, \underline{2}^+, \underline{1}^+, \underline{0}^+), (\underline{3}^+, \underline{2}^+, \underline{1, 0}), (\underline{3}^-, \underline{2}^-, \underline{1}^-, -0), (\underline{3}^+, \underline{2}^+, -1, -0)$$

For $G = Sp(8, \mathbb{R})$, the infinitesimal character at $\nu = \frac{1}{2}$ is $\chi = (2, 1, 1, 0)$. To find the candidates for factors of $I^{\pm}(\frac{1}{2})$, we find Langlands parameters from the relevant Harish-Chandra cells (as we did in example 38) at $\chi = \rho$ whose $\tau-$invariant does not contain α_2 and α_4 (in the notation of example 38) and apply the Jantzen-Zuckerman translation functor (from $\nu = \frac{5}{2}$ to $\nu = \frac{1}{2}$). The infinitesimal character $\chi = (2, 1, 1, 0)$ is singular for the short root α_2 given by $e_2 - e_3$ and the long root given by $2e_4$. For a parameter to not vanish, we require these two roots to not be in the $\tau-$invariant.

Example 41. *For $G = Sp(8,\mathbb{R})$, the Langlands parameters of the list of potential factors of $I^{\pm}(\frac{1}{2})$ are the following.*

$$(2, 1, \underline{1}^{-}, +0), (\underline{2}^{-}, \underline{1}^{-}, 1, +0), (\underline{2}^{-}, \underline{1}^{-}, \underline{1}^{-}, +0), (\underline{2}^{-}, 1, \underline{1, 0}), (\underline{2, 1}, \underline{1, 0}),$$
$$(\underline{2}^{+}, \underline{1}^{+}, \underline{1}^{+}, \underline{0}^{+}), (\underline{2}^{-}, \underline{1}^{-}, \underline{1}^{-}, -0), (\underline{2}^{-}, -1, \underline{1, 0}), (-2, -1, \underline{1}^{-}, -0), (\underline{2}^{-}, \underline{1}^{-}, -1, -0)$$

5.2 *K*-type Calculations

Similar to the parameterization of the $K-$types in [4], we use the parametrization of the irreducible representations of $U(n)$ given by non-increasing sequence of integers

$$\mu = (\mu_1, \cdots, \mu_n), \quad \mu_1 \geq \cdots \geq \mu_n$$

where $\mu_i \in \mathbb{Z}$ give the highest weight.

With this parametrization, by Helgason's theorem, if $(\mu_1, \cdots, \mu_n)$ is a $K-$type of $I^{+}(\nu)$, then every μ_i is an even integer. Similarly, if $(\mu_1, \cdots, \mu_n)$ is a $K-$type of $I^{-}(\nu)$, then every μ_i is an odd integer.

Given a Langlands parameter γ, we may compute the lowest $K-$types as

$$\mu = \lambda^{G} + 2\rho(\mathfrak{u} \cap \mathfrak{p}) - \rho(\mathfrak{u}) + \mu^{L}$$

following the formula in [27]. In this notation, $\lambda^{G} \in \mathfrak{h}^{*}$ is related to the Langlands parameter and a way to compute λ^{G} from γ is explained below. We write L to denote the centralizer of λ^{G} and $\mathfrak{u}$ is the nilradical of the parabolic subalgebra corresponding to λ^G i.e. the parabolic subalgebra of $\mathfrak{g}$ generated by roots α such that $< \lambda^G, \alpha > \geq 0$ with nilradical $\mathfrak{u}$ generated by roots α such that $< \lambda^G, \alpha >> 0$. As described in earlier chapters, $\mathfrak{p}$ is the -1 eigenspace of $\mathfrak{g}$ under the Cartan decomposition and ρ denotes half the sum of positive roots. In the formula above, μ^L denotes a fine $L \cap K$ type.

We can compute λ^G from γ by the following steps.

- Replace any occurrence of $\underline{\gamma_i, \gamma_j}$ by $(\gamma_i - \gamma_j)/2, (\gamma_i + \gamma_j)/2$. We have fixed our notation such that $\underline{\gamma_i, \gamma_j}$ means $e_i^* + e_j^*$ is a real root and $e_i^* - e_j^*$ is imaginary.
- Replace any occurrence of $\underline{\gamma_i^{\epsilon}}$ by 0.
- Reorder the entries in non-increasing order.

With these notations, we now compute the lowest $K-$types for some of the Langlands parameters for $Sp(4, \mathbb{R})$ and $Sp(8, \mathbb{R})$.

Example 42. *For $G = Sp(4, \mathbb{R})$, here are the lowest $K-$type computations for the set of Langlands parameters in Example 37.*

γ	λ^G	$2\rho(\mathfrak{u} \cap \mathfrak{p})$	$\rho(\mathfrak{u})$	μ
$(2, 1)$	$(2, 1)$	$(3, 3)$	$(2, 1)$	$(3, 3)$
$(\underline{2}^-, 1)$	$(1, 0)$	$(3, 1)$	$(2, 0)$	$(2, 2), (2, 0)$
$(-2, -1)$	$(-1, -2)$	$(-3, -3)$	$(-1, -2)$	$(-3, -3)$
$(\underline{2}^-, -1)$	$(0, -1)$	$(-1, -3)$	$(0, -2)$	$(-2, -2), (0, -2)$
$(\underline{2, -1})$	$(3/2, -3/2)$	$(2, -2)$	$(3/2, -3/2)$	$(2, -2)$
$(\underline{2}^-, \underline{1}^-)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(1, 1), (-1, -1)$
$(\underline{2}^+, \underline{1}^+)$	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$

In this example, μ^L is either 0 or ± 1 in the appropriate coordinate and the lowest $K-$types have been adjusted by ± 1 for parameters with $\underline{2}^-$ or $\underline{1}^-$ in the corresponding coordinate.

This computation shows that the Langlands parameters for the possible factors of $I^-(\frac{3}{2})$ are $(2,1), (-2,-1), (\underline{2}^-, \underline{1}^-)$ and the Langlands parameters for the possible factors of $I^+(\frac{3}{2})$ are $(\underline{2}^-, 1), (\underline{2}^-, -1), (\underline{2,-1})$ and $(\underline{2}^+, \underline{1}^+)$.

When a long root is 0, as seen in the following example, the lowest $K-$types will occur in different Langlands parameters coming from the various limits of discrete series. In particular, $(1,1)$ and $(-1,-1)$ are lowest $K-$types of the principal series with parameter $(\underline{1}^-, \underline{0}^-)$ but they occur in separate standard modules $(\underline{1}^-, \pm 0)$. In the standard module with parameter $(\underline{2}^-, \underline{1}^-)$, they both occur as lowest $K-$types of the corresponding Langlands quotient. The same holds for the induced modules $I^-(\frac{1}{2})$ and $I^-(\frac{3}{2})$. This is part of a general phenomenon of how the lowest $K-$types distribute among the standard modules.

Example 43. *For $G = Sp(4,\mathbb{R})$, here are the lowest $K-$type computations for the set of Langlands parameters in Example 39.*

γ	λ^G	$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$\rho(\mathfrak{u})$	μ
$(1,+0)$	$(1,0)$	$(3,1)$	$(2,0)$	$(2,2)$
$(\underline{1}^-,+0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(1,1)$
$(\underline{1,0})$	$(1/2,-1/2)$	$(2,-2)$	$(3/2,-3/2)$	$(1,-1)$
$(\underline{1}^+,\underline{0}^+)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(-1,-0)$	$(0,-1)$	$(-1,-3)$	$(0,-2)$	$(-2,-2)$
$(\underline{1}^-,-0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(-1,-1)$

In our notation, ± 0 is written instead of $\underline{0}^-$ to distinguish between the different limits of *principal series as explained in Example 7. Similar to the previous example, the lowest $K-$types have been adjusted by ± 1 for parameters with coordinates containing $\underline{1}^-$ or ± 0 to account for μ^L.*

This computation shows that the Langlands parameters for the possible factors of $I^+(\frac{1}{2})$ are $(1, +0), (-1, -0), (\underline{1}^+, \underline{0}^+)$ and the Langlands parameters for the possible factors of $I^-(\frac{1}{2})$ are $(\underline{1}^-, +0), (\underline{1}^-, -0), (\underline{1, 0})$. Since $(1, +0)$ is the limit of discrete series obtained from $(2, 1)$ when we make the parameter singular, its lowest $K-$type is $(2, 2)$ and not $(2, 0)$. The lowest $K-$type of $(1, -0)$ (which is a limit of $(2, -1)$) is $(2, 0)$. We have already ruled out $(2, -1)$ as a candidate, in the calculation in Example 34 because of the restriction imposed by the wave front set. Similarly, $(-2, -2)$ is the lowest $K-$type when $\gamma = (-1, -0)$ because $(-1, -0)$ is obtained as a limit from $(-2, -1)$. This can also be seen below in the lowest $K-$type calculations at singular infinitesimal characters when $G = Sp(8, \mathbb{R})$.

Example 44. *For $G = Sp(8, \mathbb{R})$, here are the lowest $K-$type computations for the Langlands parameters in Example 38.*

γ	$(4, 3, 2, 1)$	$(\underline{4}^-, 3, 2, 1)$	$(\underline{4, -1}, 3, 2)$	$(\underline{4}^-, \underline{3}^-, 2, 1)$
λ^G	$(4, 3, 2, 1)$	$(3, 2, 1, 0)$	$(3, 5/2, 2, -5/2)$	$(2, 1, 0, 0)$
$2\rho(\mathfrak{u} \cap \mathfrak{p})$	$(5, 5, 5, 5)$	$(5, 5, 5, 3)$	$(5, 4, 3, -2)$	$(5, 5, 2, 2)$
$\rho(\mathfrak{u})$	$(4, 3, 2, 1)$	$(4, 3, 2, 0)$	$(4, 5/2, 1, -5/2)$	$(4, 3, 0, 0)$
μ	$(5, 5, 5, 5)$	$(4, 4, 4, 4)$	$(4, 4, 4, -2)$	$(3, 3, 3, 3)$
		$(4, 4, 4, 2)$		$(3, 3, 1, 1)$

γ	$(\underline{4,-1},\underline{3,-2})$	$(\underline{4}^+,\underline{3}^+,\underline{2,-1})$	$(\underline{4,-1},-3,-2)$
λ^G	$(5/2,5/2,-5/2,-5/2)$	$(3/2,0,0,-3/2)$	$(5/2,-2,-5/2,-3)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(3,3,-3,-3)$	$(4,0,0,-4)$	$(2,-3,-4,-5)$
$\rho(\mathfrak{u})$	$(5/2,5/2,-5/2,-5/2)$	$(7/2,0,0,-7/2)$	$(5/2,-1,-5/2,-4)$
μ	$(3,3,-3,-3)$	$(2,0,0,-2)$	$(2,-4,-4,-4)$

γ	$(\underline{4}^-,\underline{3}^-,-2,-1)$	$(\underline{4}^-,-3,-2,-1)$	$(-4,-3,-2,-1)$
λ^G	$(0,0,-1,-2)$	$(0,-1,-2,-3)$	$(-1,-2,-3,-4)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(-2,-2,-5,-5)$	$(-3,-5,-5,-5)$	$(5,-5,-5,-5)$
$\rho(\mathfrak{u})$	$(0,0,-3,-4)$	$(0,-2,-3,-4)$	$(-1,-2,-3,-4)$
μ	$(-1,-1,-3,-3)$	$(-2,-4,-4,-4)$	$(-5,-5,-5,-5)$
	$(-3,-3,-3,-3)$	$(-4,-4,-4,-4)$	

γ	$(\underline{4}^+,\underline{3}^+,2,1)$	$(\underline{4}^-,\underline{3}^-,\underline{2,-1})$	$(\underline{4}^+,\underline{3}^+,-2,-1)$
λ^G	$(2,1,0,0)$	$(3/2,0,0,-3/2)$	$(0,0,-1,-2)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(5,5,2,2)$	$(4,0,0,-4)$	$(-2,-2,-5,-5)$
$\rho(\mathfrak{u})$	$(4,3,0,0)$	$(7/2,0,0,-7/2)$	$(0,0,-3,-4)$
μ	$(3,3,2,2)$	$(2,1,1,-2)$	$(-2,-2,-3,-3)$
		$(2,-1,-1,-2)$	

γ	$(\underline{4}^-,\underline{3}^-,\underline{2}^-,1)$	$(\underline{4}^-,\underline{3}^-,\underline{2}^-,\underline{1}^-)$	$(\underline{4}^-,\underline{3}^-,\underline{2}^-,-1)$	$(\underline{4}^+,\underline{3}^+,\underline{2}^+,\underline{1}^+)$
λ^G	$(1,0,0,0)$	$(0,0,0,0)$	$(0,0,0,-1)$	$(0,0,0,0)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(5,1,1,1)$	$(0,0,0,0)$	$(-1,-1,-1,-5)$	$(0,0,0,0)$
$\rho(\mathfrak{u})$	$(4,0,0,0)$	$(0,0,0,0)$	$(0,0,0,-4)$	$(0,0,0,0)$
μ	$(2,2,2,2)$	$(1,1,1,1)$	$(0,0,0,-2)$	$(0,0,0,0)$
	$(2,0,0,0)$	$(-1,-1,-1,-1)$	$(-2,-2,-2,-2)$	

This computation shows that the Langlands parameters for the possible factors of $I^-(\frac{5}{2})$ are

- $(4,3,2,1), (\underline{4}^-,\underline{3}^-,2,1), (\underline{4,-1},\underline{3,-2}), (\underline{4}^-,\underline{3}^-,-2,-1), (-4,-3,-2,-1)$
- $(\underline{4}^-,\underline{3}^-,\underline{2}^-,\underline{1}^-)$

and the Langlands parameters for the possible factors of $I^+(\frac{5}{2})$ are

- $(\underline{4}^-,3,2,1), (\underline{4,-1},3,2), (\underline{4}^+,\underline{3}^+,\underline{2,-1}), (\underline{4,-1},-3,-2), (\underline{4}^-,-3,-2,-1)$
- $(\underline{4}^-,\underline{3}^-,\underline{2}^-,1), (\underline{4}^-,\underline{3}^-,\underline{2}^-,-1)$
- $(\underline{4}^+,\underline{3}^+,\underline{2}^+,\underline{1}^+)$.

Example 45. *For $G = Sp(8,\mathbb{R})$, here are the lowest K−type computations for the Langlands parameters in Example 40.*

γ	$(3,2,1,+0)$	$(\underline{3}^-,2,1,+0)$	$(\underline{3,0},2,1)$	$(\underline{3}^-,\underline{2}^-,1,+0)$
λ^G	$(3,2,1,0)$	$(2,1,0,0)$	$(2,3/2,1,-3/2)$	$(1,0,0,0)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(5,5,5,3)$	$(5,5,2,2)$	$(5,4,3,-2)$	$(5,1,1,1)$
$\rho(\mathfrak{u})$	$(4,3,2,0)$	$(4,3,0,0)$	$(4,5/2,1,-5/2)$	$(4,0,0,0)$
μ	$(4,4,4,4)$	$(3,3,3,3)$	$(3,3,3,-1)$	$(2,2,2,2)$

γ	$(\underline{3,0},\underline{2,-1})$	$(\underline{3}^-,\underline{2}^-,\underline{1,0})$	$(\underline{3,0},-2,-1)$
λ^G	$(3/2,3/2,-3/2,-3/2)$	$(1/2,0,0,-1/2)$	$(3/2,-1,-3/2,-2)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(3,3,-3,-3)$	$(4,0,0,-4)$	$(2,-3,-4,-5)$
$\rho(\mathfrak{u})$	$(5/2,5/2,-5/2,-5/2)$	$(7/2,0,0,-7/2)$	$(5/2,-1,-5/2,-4)$
μ	$(2,2,-2,-2)$	$(1,1,1,-1)$ $(1,-1,-1,-1)$	$(1,-3,-3,-3)$

γ	$(\underline{3}^-,\underline{2}^-,-1,-0)$	$(\underline{3}^-,-2,-1,-0)$	$(-3,-2,-1,-0)$
λ^G	$(0,0,0,-1)$	$(0,0,-1,-2)$	$(0,-1,-2,-3)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(-1,-1,-1,-5)$	$(-2,-2,-5,-5)$	$(-3,-5,-5,-5)$
$\rho(\mathfrak{u})$	$(0,0,0,-4)$	$(0,0,-3,-4)$	$(0,-2,-3,-4)$
μ	$(-2,-2,-2,-2)$	$(-3,-3,-3,-3)$	$(-4,-4,-4,-4)$

γ	$(\underline{3}^-,\underline{2}^-,\underline{1}^-,+0)$	$(\underline{3}^+,\underline{2}^+,1,+0)$	$(\underline{3}^+,\underline{2}^+,\underline{1}^+,\underline{0}^+)$
λ^G	$(0,0,0,0)$	$(1,0,0,0)$	$(0,0,0,0)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(0,0,0,0)$	$(5,1,1,1)$	$(0,0,0,0)$
$\rho(\mathfrak{u})$	$(0,0,0,0)$	$(4,0,0,0)$	$(0,0,0,0)$
μ	$(1,1,1,1)$	$(2,2,1,1)$	$(0,0,0,0)$

γ	$(\underline{3}^+,\underline{2}^+,\underline{1,0})$	$(\underline{3}^-,\underline{2}^-,\underline{1}^-,-0)$	$(\underline{3}^+,\underline{2}^+,-1,-0)$
λ^G	$(1/2,0,0,-1/2)$	$(0,0,0,0)$	$(0,0,0,-1)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(4,0,0,-4)$	$(0,0,0,0)$	$(-1,-1,-1,-5)$
$\rho(\mathfrak{u})$	$(7/2,0,0,-7/2)$	$(0,0,0,0)$	$(0,0,0,-4)$
μ	$(1,0,0,-1)$	$(-1,-1,-1,-1)$	$(-1,-1,-1,-2)$

This computation shows that the Langlands parameters for the possible factors of $I^-(\frac{3}{2})$ are

- $(\underline{3}^-,2,1,+0)$, $(\underline{3,0},2,1)$, $(\underline{3}^-,\underline{2}^-,\underline{1,0})$, $(\underline{3,0},-2,-1)$, $(\underline{3}^-,-2,-1,-0)$
- $(\underline{3}^-,\underline{2}^-,\underline{1}^-,+0)$, $(\underline{3}^-,\underline{2}^-,\underline{1}^-,-0)$

and the Langlands parameters for the possible factors of $I^+(\frac{3}{2})$ are

- $(3,2,1,+0)$, $(\underline{3}^-,\underline{2}^-,1,+0)$, $(\underline{3,0},\underline{2,-1})$, $(\underline{3}^-,\underline{2}^-,-1,-0)$, $(-3,-2,-1,-0)$
- $(\underline{3}^+,\underline{2}^+,\underline{1}^+,\underline{0}^+)$.

Example 46. *For $G = Sp(8,\mathbb{R})$, here are the lowest K–type computations for the Langlands parameters in Example 41.*

γ	$(2,1,\underline{1}^-,+0)$	$(\underline{2}^-,\underline{1}^-,1,+0)$	$(\underline{2}^-,\underline{1}^-,\underline{1}^-,+0)$	$(\underline{2}^-,1,\underline{1,0})$
λ^G	$(2,1,0,0)$	$(1,0,0,0)$	$(0,0,0,0)$	$(1,1/2,0,-1/2)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(5,5,2,2)$	$(5,1,1,1)$	$(0,0,0,0)$	$(5,4,1,-2)$
$\rho(\mathfrak{u})$	$(4,3,0,0)$	$(4,0,0,0)$	$(0,0,0,0)$	$(4,5/2,0,-5/2)$
μ	$(3,3,3,3)$	$(2,2,2,2)$	$(1,1,1,1)$	$(2,2,2,0)$
				$(2,2,0,0)$

γ	$(\underline{2,1},\underline{1,0})$	$(\underline{2}^+,\underline{1}^+,\underline{1}^+,\underline{0}^+)$	$(\underline{2}^-,\underline{1}^-,\underline{1}^-,-0)$
λ^G	$(1/2,1/2,-1/2,-1/2)$	$(0,0,0,0)$	$(0,0,0,0)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(3,3,-3,-3)$	$(0,0,0,0)$	$(0,0,0,0)$
$\rho(\mathfrak{u})$	$(5/2,5/2,-5/2,-5/2)$	$(0,0,0,0)$	$(0,0,0,0)$
μ	$(1,1,-1,-1)$	$(0,0,0,0)$	$(-1,-1,-1,-1)$

γ	$(\underline{2}^-,-1,\underline{1,0})$	$(-2,-1,\underline{1}^-,-0)$	$(\underline{2}^-,\underline{1}^-,-1,-0)$
λ^G	$(1/2,0,-1/2,-1)$	$(0,0,-1,-2)$	$(0,0,0,-1)$
$2\rho(\mathfrak{u}\cap\mathfrak{p})$	$(2,-1,-4,-5)$	$(-2,-2,-5,-5)$	$(-1,-1,-1,-5)$
$\rho(\mathfrak{u})$	$(5/2,0,-5/2,-4)$	$(0,0,-3,-4)$	$(0,0,0,-4)$
μ	$(0,0,-2,-2)$	$(-3,-3,-3,-3)$	$(-2,-2,-2,-2)$
	$(0,-2,-2,-2)$		

This computation shows that the Langlands parameters for the possible factors of $I^-(\frac{1}{2})$ are

$$(2,1,\underline{1}^-,+0),(\underline{2}^-,\underline{1}^-,\underline{1}^-,+0),(\underline{2,1},\underline{1,0}),(\underline{2}^-,\underline{1}^-,\underline{1}^-,-0),(-2,-1,\underline{1}^-,-0)$$

and the Langlands parameters for the possible factors of $I^+(\frac{1}{2})$ are

$$(\underline{2}^-, \underline{1}^-, 1, +0), (\underline{2}^-, 1, \underline{1, 0}), (\underline{2}^+, \underline{1}^+, \underline{1}^+, \underline{0}^+), (\underline{2}^-, -1, \underline{1, 0}), (\underline{2}^-, \underline{1}^-, -1, -0)$$

From the list of possible candidates for composition factors that we have from the previous chapter, we have now reduced the list further using the constraints given by the $\tau-$invariant and the lowest $K-$type calculations.

CHAPTER 6

CONCLUSION

A general method to find factors of the induced modules $I^{\pm}(\nu)$ for $Sp(2n, \mathbb{R})$ has been detailed in the previous chapters. In this chapter, we illustrate it by providing the description of the irreducible subquotients of the induced modules for $Sp(4, \mathbb{R})$ and $Sp(8, \mathbb{R})$.

When n is even, $I^{\pm}(\nu)$ is reducible precisely when $\nu + \frac{1}{2} \in \mathbb{Z}$. This is the content of Proposition 3. When $\nu \geq \frac{n+1}{2}$, the infinitesimal character is regular and the Jantzen-Zuckerman translation functor from $\nu = \frac{n+1}{2}$ to $\nu' = \frac{2m+1}{2}$ (for any integer m such that $2m + 1 > n$) defines a one-to-one correspondence between the composition factors of $I^{\pm}(\nu)$ and the composition factors of $I^{\pm}(\nu')$. If $\nu < 0$, the structure of $I^{\pm}(\nu)$ can be deduced from the structure of $I^{\pm}(-\nu)$, since they are dual to each other. Hence, when n is even, it is enough to describe the induced modules at $\nu = \frac{1}{2}, \frac{3}{2}, \cdots, \frac{n+1}{2}$, to understand the structure of $I^{\pm}(\nu)$ at all values of ν.

Theorem 47. *For $G = Sp(4, \mathbb{R})$, the Langlands parameters of the irreducible subquotients of $I^{\pm}(\nu)$ at $\nu = \frac{1}{2}, \frac{3}{2}$ are as follows.*

$$I^{+}(\tfrac{1}{2}) : (1, +0), (-1, -0), (\underline{1}^{+}, \underline{0}^{+})$$
$$I^{-}(\tfrac{1}{2}) : (\underline{1}^{-}, +0), (\underline{1}^{-}, -0), (\underline{1, 0})$$
$$I^{+}(\tfrac{3}{2}) : (\underline{2}^{-}, 1), (\underline{2}^{-}, -1), (\underline{2, -1}), (\underline{2}^{+}, \underline{1}^{+})$$
$$I^{-}(\tfrac{3}{2}) : (2, 1), (-2, -1), (\underline{2}^{-}, \underline{1}^{-})$$

Proof. At $\nu = \frac{1}{2}, \frac{3}{2}$, we require Langlands parameters from the Harish-Chandra cells corresponding to the three real forms of the nilpotent orbit $[2, 2]$ (seen in

Example 21) to occur as irreducible subquotients of $I^{\pm}(\nu)$ because of the wave front set of $I^{\pm}(\nu)$.

We also know that $(\underline{2}^+, \underline{1}^+)$ occurs as an irreducible subquotient of $I^+(\frac{3}{2})$ because it is the Langlands parameter of the trivial representation.

This gives a complete list of irreducible subquotients at $\nu = \frac{1}{2}, \frac{3}{2}$ because of the remarks following Example 42 and Example 43. □

Theorem 48. *For $G = Sp(8, \mathbb{R})$, the Langlands parameters of the irreducible subquotients of $I^{\pm}(\nu)$ at $\nu = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ are as follows.*

$I^+(\frac{1}{2})$:

$(\underline{2}^-, \underline{1}^-, 1, +0), (\underline{2}^-, 1, \underline{1, 0}), (\underline{2}^+, \underline{1}^+, \underline{1}^+, \underline{0}^+), (\underline{2}^-, -1, \underline{1, 0}), (\underline{2}^-, \underline{1}^-, -1, -0)$

$I^-(\frac{1}{2})$:

$(2, 1, \underline{1}^-, +0), (\underline{2}^-, \underline{1}^-, \underline{1}^-, +0), (\underline{2, 1}, \underline{1, 0}), (\underline{2}^-, \underline{1}^-, \underline{1}^-, -0), (-2, -1, \underline{1}^-, -0)$

$I^+(\frac{3}{2})$:

$(3, 2, 1, +0), (\underline{3}^-, \underline{2}^-, 1, +0), (\underline{3, 0}, \underline{2, -1}), (\underline{3}^-, \underline{2}^-, -1, -0), (-3, -2, -1, -0)$

$(\underline{3}^+, \underline{2}^+, \underline{1}^+, \underline{0}^+)$

$I^-(\frac{3}{2})$:

$(\underline{3}^-, 2, 1, +0), (\underline{3, 0}, 2, 1), (\underline{3}^-, \underline{2}^-, \underline{1, 0}), (\underline{3, 0}, -2, -1), (\underline{3}^-, -2, -1, -0)$

$(\underline{3}^-, \underline{2}^-, \underline{1}^-, +0), (\underline{3}^-, \underline{2}^-, \underline{1}^-, -0)$

$I^+(\frac{5}{2})$:

$(\underline{4}^-, 3, 2, 1), (\underline{4, -1}, 3, 2), (\underline{4}^+, \underline{3}^+, \underline{2, -1}), (\underline{4, -1}, -3, -2), (\underline{4}^-, -3, -2, -1)$

$(\underline{4}^-, \underline{3}^-, \underline{2}^-, 1), (\underline{4}^-, \underline{3}^-, \underline{2}^-, -1)$

$(\underline{4}^+, \underline{3}^+, \underline{2}^+, \underline{1}^+)$

$I^-(\frac{5}{2})$:

$(4, 3, 2, 1), (\underline{4}^-, \underline{3}^-, 2, 1), (\underline{4, -1}, \underline{3, -2}), (\underline{4}^-, \underline{3}^-, -2, -1), (-4, -3, -2, -1),$

$(\underline{4}^-, \underline{3}^-, \underline{2}^-, \underline{1}^-)$

Proof. At $\nu = \frac{5}{2}$, we require Langlands parameters from the Harish-Chandra cells corresponding to the five real forms of the nilpotent orbit $[2,2,2,2]$ (seen in Example 22) to occur as irreducible subquotients of $I^{\pm}(\frac{5}{2})$ because of the wave front set of $I^{\pm}(\frac{5}{2})$. From the calculations in Example 38 and Example 44, we get lists of five Langlands parameters each that occur as irreducible subquotients of $I^{+}(\frac{5}{2})$ and $I^{-}(\frac{5}{2})$ respectively.

The Langlands parameter $(\underline{4}^-, \underline{3}^-, \underline{2}^-, \underline{1}^-)$ must occur as an irreducible subquotient of $I^{-}(\frac{5}{2})$ because of the lowest $K-$types. This gives a complete list of irreducible subquotients of $I^{-}(\frac{5}{2})$ because of the remarks following Example 44.

We know that $(\underline{4}^+, \underline{3}^+, \underline{2}^+, \underline{1}^+)$ occurs as an irreducible subquotient of $I^{+}(\frac{5}{2})$ and it is the Langlands parameter of the trivial representation. The $K-$types $\pm(2,2,2,2)$ occur in the induced module and therefore must be accounted for.

Here is a tabular column with the calculation of the lengths of the lowest $K-$types of the candidates for factors:

γ	μ	$\mu + 2\rho_{\mathfrak{k}}$	$< \mu + 2\rho_{\mathfrak{k}}, \mu + 2\rho_{\mathfrak{k}} >$
$(\underline{4}^-, 3, 2, 1)$	$(4,4,4,4)$	$(7,5,3,1)$	84
$(\underline{4}^-, 3, 2, 1)$	$(4,4,4,2)$	$(7,5,3,-1)$	84
$(\underline{4, -1}, 3, 2)$	$(4,4,4,-2)$	$(7,5,3,-5)$	108
$(\underline{4}^+, \underline{3}^+, \underline{2, -1})$	$(2,0,0,-2)$	$(5,1,-1,-5)$	52
$(\underline{4, -1}, -3, -2)$	$(2,-4,-4,-4)$	$(5,-3,-5,-7)$	108
$(\underline{4}^-, -3, -2, -1)$	$(-2,-4,-4,-4)$	$(1,-3,-5,-7)$	84
$(\underline{4}^-, -3, -2, -1)$	$(-4,-4,-4,-4)$	$(-1,-3,-5,-7)$	84
$(\underline{4}^-, \underline{3}^-, \underline{2}^-, 1)$	$(2,2,2,2)$	$(5,3,1,-1)$	36
$(\underline{4}^-, \underline{3}^-, \underline{2}^-, -1)$	$(-2,-2,-2,-2)$	$(1,-1,-3,-5)$	36

The 7 lowest $K-$types at the top of the table are accounted for by the five Langlands parameters we obtained from Harish-Chandra cells corresponding to the real forms of the nilpotent orbit $[2,2,2,2]$. The $K-$types $\pm(2,2,2,2)$ do not occur in the trivial representation nor do they occur in these five parameters, following the work of Vogan [26] on lowest $K-$types. Therefore, $(\underline{4}^-,\underline{3}^-,\underline{2}^-,1)$ and $(\underline{4}^-,\underline{3}^-,\underline{2}^-,-1)$ must occur. This gives a complete list of irreducible subquotients of $I^+(\frac{5}{2})$ because of the remarks following Example 44.

Applying the Jantzen-Zuckerman translation functor from $\nu = \frac{5}{2}$ to $\nu = \frac{3}{2}$ gives us a list of Langlands parameters of the irreducible subquotients of $I^\pm(\frac{3}{2})$. This gives a complete list of irreducible subquotients at $\nu = \frac{3}{2}$ because of the remarks following Example 45.

At $\nu = \frac{1}{2}$, we require Langlands parameters from the Harish-Chandra cells corresponding to the five real forms of the nilpotent orbit $[2,2,2,2]$ to occur as irreducible subquotients of $I^\pm(\frac{1}{2})$ because of the wave front set of $I^\pm(\frac{1}{2})$. From the remarks following Example 46, we get a complete list of irreducible subquotients at $\nu = \frac{1}{2}$. $\square$

APPENDIX A

THETA CORRESPONDENCE

In this section, we construct representations by the theta correspondence from one dimensional characters of $O(p,q)$. Our basic references for theta correspondence are [13], [14] and [15] by Howe. We will describe the precise construction and then construct intertwining operators from these representations to the induced modules $I^{\pm}(\nu)$, as described in [19] by Kashiwara and Vergne.

In [3], Barbasch and Pandžić give a realization of the theta correpondence from 1-dimensional representations of $O(p,q)$ to $Sp(2n,\mathbb{R})$ using the Fock model, which we describe here. Let $p+q=2k\leq n$. If $k=0$, we get a degenerate case in which we get the trivial representation from the theta correspondence, which we know is a subquotient of $I^{+}(\frac{n+1}{2})$. We will omit this case and assume $k>0$. Let $\epsilon,\eta\in\{0,1\}$. Denote by $\mathbb{C}_{\epsilon}$ the character det^{ϵ} of $O(p)$, and by $\mathbb{C}_{\eta}$ the character det^{η} of $O(q)$. If $p=0$, we require $\epsilon=0$, and if $q=0$, we require $\eta=0$.

Let $\mathbb{C}_{\epsilon,\eta}$ be the character of $O(p,q)$ with restriction to $O(p)\times O(q)$ equal to $\mathbb{C}_{\epsilon}\otimes\mathbb{C}_{\eta}$. The representation $X(p,q;\epsilon,\eta)$ of G is obtained by theta lifting the character $\mathbb{C}_{\epsilon,\eta}$ from $O(p,q)$ to G, i.e. $X(p,q;\epsilon,\eta)=\mathrm{Hom}_{G}[\Omega,\mathbb{C}_{\epsilon,\eta}]$, where Ω is the metaplectic representation of G. The $K-$types of $X(p,q;\epsilon,\eta)$ are of the form

$$\left(\tfrac{p-q}{2},\cdots,\tfrac{p-q}{2}\right)+\left(\epsilon'+2a_1,\cdots,\epsilon'+2a_p,0,\cdots,0,-\eta'-2b_q,\cdots,-\eta'-2b_1\right)$$

where $a_1\geq\cdots\geq a_p\geq 0$, $b_1\geq\cdots\geq b_q\geq 0$ are nonnegative integers, and $\epsilon',\eta'=\epsilon,\eta$ if k is even, $1-\epsilon,1-\eta$ if k is odd.

By Helgason's theorem, the $K-$types of $X(p,q;\epsilon,\eta)$ have to be comprised of even integers for it to be a candidate as a subquotient of $I^{+}(\nu)$, and comprised of odd integers for it to be a candidate as a subquotient of $I^-(\nu)$. Based on this Fock model, and the lowest $K-$type calculations for Langlands parameters that we have from the previous section, the following tabular columns list the Langlands parameters γ corresponding to each $X(p,q;\epsilon,\eta)$ that is a candidate for an irreducible subquotient of $I^{\pm}(\nu)$, for $Sp(4,\mathbb{R})$ and $Sp(8,\mathbb{R})$.

Example 49. $G = Sp(4,\mathbb{R}), n = 2, k = \frac{p+q}{2} = 1.$

p	q	ϵ	η	*$K-$types*	*Lowest $K-$type*	γ
2	0	0	0	$(2+2a_1, 2+2a_2)$	$(2,2)$	$(1,+0)$
2	0	1	0	$(1+2a_1, 1+2a_2)$	$(1,1)$	$(\underline{1}^-,+0)$
1	1	0	0	$(1+2a_1, -1-2b_1)$	$(1,-1)$	$(\underline{1,0})$
1	1	1	0	$(2a_1, -1+2b_1)$	$(0,-1)$	*Not a candidate*
1	1	0	1	$(1+2a_1, -2b_1)$	$(1,0)$	*Not a candidate*
1	1	1	1	$(2a_1, -2b_1)$	$(0,0)$	$(\underline{1}^+,\underline{0}^+)$
0	2	0	0	$(-2-2b_2, -2-2b_1)$	$(-2,-2)$	$(-1,-0)$
0	2	0	1	$(-1-2b_2, -1-2b_1)$	$(-1,-1)$	$(\underline{1}^-,-0)$

The $K-$type calculations for $X(p,q;\epsilon,\eta)$ in this example coincides with the $K-$type calculations in Example 43 (except for two cases which cannot appear in the induced modules we are considering, by Helgason's theorem).

Example 50. $G = Sp(8,\mathbb{R}), n = 4, k = \frac{p+q}{2} = 4.$

In this example, we compare the lowest $K-$type of $X(p,q;\epsilon,\eta)$ with the calculation in Example 46 to write γ.

p	q	ϵ	η	*Lowest* $K-$*type of* $X(p, q; \epsilon, \eta)$	γ
4	0	0	0	$(2, 2, 2, 2)$	$(\underline{2}^-, \underline{1}^-, 1, +0)$
4	0	1	0	$(3, 3, 3, 3)$	$(2, 1, \underline{1}^-, +0)$
3	1	0	0	$(1, 1, 1, 1)$	$(\underline{2}^-, \underline{1}^-, \underline{1}^-, +0)$
3	1	1	0	$(2, 2, 2, 1)$	*Not a candidate*
3	1	0	1	$(1, 1, 1, 0)$	*Not a candidate*
3	1	1	1	$(2, 2, 2, 0)$	$(\underline{2}^-, 1, \underline{1, 0})$
2	2	0	0	$(0, 0, 0, 0)$	$(\underline{2}^+, \underline{1}^+, \underline{1}^+, \underline{0}^+)$
2	2	1	0	$(1, 1, 0, 0)$	*Not a candidate*
2	2	0	1	$(0, 0, -1, -1)$	*Not a candidate*
2	2	1	1	$(1, 1, -1, -1)$	$(\underline{2, 1}, \underline{1, 0})$
1	3	0	0	$(-1, -1, -1, -1)$	$(\underline{2}^-, \underline{1}^-, \underline{1}^-, -0)$
1	3	1	0	$(0, -1, -1, -1)$	*Not a candidate*
1	3	0	1	$(-1, -2, -2, -2)$	*Not a candidate*
1	3	1	1	$(0, -2, -2, -2)$	$(\underline{2}^-, -1, \underline{1, 0})$
0	4	0	0	$(-2, -2, -2, -2)$	$(\underline{2}^-, \underline{1}^-, -1, -0)$
0	4	0	1	$(-3, -3, -3, -3)$	$(-2, -1, \underline{1}^-, -0)$

Example 51. $G = Sp(8, \mathbb{R}), n = 4, k = \frac{p+q}{2} = 2.$

In this example, we compare the lowest $K-$*type of* $X(p, q; \epsilon, \eta)$ *with the calculation in Example 45 to write* γ.

p	q	ϵ	η	*Lowest K−type of* $X(p,q;\epsilon,\eta)$	γ
2	0	0	0	$(2,2,1,1)$	*Not a candidate*
2	0	1	0	$(1,1,1,1)$	$(\underline{3}^-,\underline{2}^-,\underline{1}^-,+0)$
1	1	0	0	$(1,0,0,-1)$	*Not a candidate*
1	1	1	0	$(0,0,0,-1)$	*Not a candidate*
1	1	0	1	$(1,0,0,0)$	*Not a candidate*
1	1	1	1	$(0,0,0,0)$	$(\underline{3}^+,\underline{2}^+,\underline{1}^+,\underline{0}^+)$
0	2	0	0	$(-1,-1,-2,-2)$	*Not a candidate*
0	2	0	1	$(-1,-1,-1,-1)$	$(\underline{3}^-,\underline{2}^-,\underline{1}^-,-0)$

In [19], Kashiwara and Vergne give a realization of the theta correspondence and construct intertwining operators which intertwine the representations arising from theta correspondence with the induced modules $I^{\pm}(\nu)$.

We write $g(a), t(b)$ and σ to denote the following elements of $Sp(2n,\mathbb{R})$, which generates the group

$$g(a) = \begin{pmatrix} a & 0 \\ 0 & (a^{-1})^t \end{pmatrix} \text{ for } a \in GL(n,\mathbb{R}) \text{ ,}$$
$$t(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \text{ for } b \in Sym_n(\mathbb{R}) \text{ ,}$$
$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

When n is even, we give an explicit realization of a representation of $Sp(2n,\mathbb{R})$, which we denote by $L_{p,q}$. Here p,q,k denote nonnegative integers as described before, with $p+q=2k \le n$.

$L_{p,q}$ can be realized in $L^2(M_{n,p+q})$ as follows:

$$(L_{p,q}(g(a))f)(\xi) = (det\ a)^{-k} f(a^t\xi)$$
$$(L_{p,q}(t(b))f)(\xi) = e^{-\frac{i}{2}Tr(\xi Q\xi^t b)} f(\xi)$$
$$(L_{p,q}(\sigma)f)(\xi) = (\tfrac{i}{2\pi})^{\frac{p-q}{2}} \int_{M_{n,p+q}} e^{iTr(\xi Q\xi')} f(\xi'^t) d\xi'.$$

In this realization, Q denotes the canonical form on $\mathbb{R}^{p+q}$ of signature (p,q). Let $\lambda_{\epsilon,\eta}$ denote the one dimensional representation of $O(p,q)$ which we denoted by $\mathbb{C}_{\epsilon,\eta}$. On $L^2(M_{n,p+q})$, the action of the group $O(p,q)$ given by $(h \cdot f(\xi)) = \lambda^{-1}_{\epsilon,\eta}(h) f(\xi h)$ commutes with the representation $L_{p,q}$ of $Sp(2n,\mathbb{R})$. We write $L_{p,q;\epsilon,\eta}$ to denote the representation $L_{p,q}$ on the functions in $L^2(M_{n,p+q})$ invariant under this action of $O(p,q)$. The representation $L_{p,q;\epsilon,\eta}$ is isomorphic to $X(p,q;\epsilon,\eta)$ obtained from the Fock model.

In [19], Kashiwara and Vergne use T_k to denote the induced modules $I^{\pm}(k - \frac{n+1}{2})$ i.e. $\nu = k - \rho_n = k - \frac{n+1}{2}$. Explicitly, on the space of functions on $Sym_n(\mathbb{R})$, the representation T_k is realized as follows:

$$(T_k(g(a))\phi)(x) = (det\ a)^{-k} \phi(a^{-1}x(a^{-1})^t)$$
$$(T_k(t(b))\phi)(x) = \phi(x - b)$$
$$(T_k(\sigma)\phi)(x) = (det\ x)^{-k} \phi(-x^{-1}).$$

In the above expressions, these formulas are valid only if all terms are defined. In particular, $(T_k(\sigma)\phi)(x)$ is defined for symmetric matrices x which are invertible and it extends to an analytic function. This realization of the induced modules is obtained from the following equations:

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a^t \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^{-1}x(a^{-1})^t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a^t \end{pmatrix}$$

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x-b \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -x^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^{-1} & 0 \\ 1 & x \end{pmatrix}.$$

Let $\mathcal{S}(M_{n,p+q})$ denote the Schwartz space of rapidly decreasing functions on the vector space $M_{n,p+q}$. Consider the operator:

$$(\mathcal{J}_{p,q}\phi)(x) = \int_{M_{n,p+q}} e^{\frac{i}{2}Tr(\xi Q \xi^t x)}\phi(\xi)d\xi$$

for $\phi \in \mathcal{S}(M_{n,p+q})$. By Proposition 2.4 in [19], $\mathcal{J}_{p,q}$ intertwines $L_{p,q;\epsilon,\eta}$ with T_k.

From the lowest $K-$type calculations in Example 49, Example 50 and Example 51, the $L_{p,q;\epsilon,\eta}$ which we know cannot be candidates, map to 0 under the intertwiner $\mathcal{J}_{p,q}$. Proving that the other factors from these examples have a non-zero image under the intertwiner, tells us that the factors we found, occur as submodules of $I^{\pm}(k - \frac{n+1}{2})$ (or equivalently as subquotients of $I^{\pm}(\frac{n+1}{2} - k)$, since the representations given by ν and $-\nu$ are dual to each other).

Future work on this problem includes providing the precise conditions under which the image of the intertwiner is non-zero. A shift functor similar to the shift functor μ defined in [4] can then be used obtain the composition series of the induced modules.

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