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ON THE EXPECTED SAMPLE SIZE FOR THE BECHHOFER-KULKARNI BERNOULLI SELECTION PROCEDURE

by

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Abstract

Bechhofer and Kulkarni (1982a) proposed a sequential procedure for selecting the best of $k \geq 2$ Bernoulli populations, and in a subsequent paper (1982b) gave an upper bound for the expected number of observations taken from each population by this procedure. In this note we present an asymptotically correct approximation to the expected sample size taken from each population and a slightly improved upper bound on these expected sample sizes.

Key words and phrases: Bernoulli selection problem, sequential selection problem, expected sample size.

1. Introduction

Let Π_i $(1 \leq i \leq k)$ denote $k \geq 2$ Bernoulli populations with corresponding single-trial "success" probabilities p_i. Denote the ordered values of the p_i by $p_{[1]} \leq \cdots \leq p_{[k]}$; the values of the $p_{[j]}$ are assumed to be unknown and the pairing of the Π_i with the $p_{[j]}$ $(1 \leq i,j \leq k)$ is assumed to be completely unknown. Let $\Pi_{(i)}$ denote the population associated with $p_{[i]}$ $(1 \leq i \leq k)$. The goal of the experimenter is to select as "best" $\Pi_{(k)}$, the population associated with $p_{[k]}$.

Bechhofer and Kulkarni (1982a) proposed a procedure P^* for this goal, consisting of a sequential sampling rule R^* which takes no more than a prespecified number $n \geq 1$ of observations from any one of the k populations, a stopping rule S^*, and a terminal decision rule T^*. When a total of m observations have been taken from all populations, let $n_{i,m}$ denote the number of observations taken from Π_i and $z_{i,m}$ the number of successes yielded by Π_i $(1 \leq i \leq k,\ 1 \leq m \leq kn-1)$. The sampling rule R^* takes the next observations from the population with the smallest number of failures, breaking ties according to the largest number of successes, and randomizing if there is a further tie. The stopping rule S^* stops sampling at the first stage m at which there exists at least one population Π_i satisfying

$$z_{i,m} \geq z_{j,m} + n - n_{j,m} \quad \text{for all} \quad j \neq i \ (1 \leq j \leq k). \qquad (1.1)$$

The terminal decision rule T^* selects the population Π_i satisfying (1.1), selecting one such population at random if there is more than one. We note that P^* always selects a population that would have had most successes if exactly n observations had been taken from each of the k populations. The sampling rule R^* can be thought of as operating in cycles; in a

complete cycle each population is sampled until it yields one failure and within a cycle, populations are sampled in order according to the number of successes yielded thus far, populations with the most successes being sampled first.

Let $N_{(i)}$ denote the total number of observations taken from $\Pi_{(i)}$ at the termination of sampling ($1 \leq i \leq k$). In Section 2 we derive, for $1 \leq i \leq k$, $p_{[k]} \neq 1$ and "large" n, the approximation

$$E\{N_{(i)}\} \simeq \frac{n(1-p_{[k]})}{1-p_{[i]}} . \tag{1.2}$$

Bechhofer and Kulkarni (1982b, Appendix D) obtained, for $1 \leq i < k$ and $p_{[i]} \neq 1$, the upper bound

$$E\{N_{(i)}\} \leq \frac{n(1-p_{[k]})}{1-p_{[i]}} + \frac{1}{1-p_{[i]}} ; \tag{1.3}$$

in Section 3 we improve this upper bound to

$$E\{N_{(i)}\} \leq \frac{n(1-p_{[k]})}{1-p_{[i]}} + \frac{1}{2(1-p_{[i]})} . \tag{1.4}$$

In this paper we compare (1.2) with exact values of $E\{N_{(i)}\}$ calculated by Bechhofer and Kulkarni (1982b). The approximation (1.2) also yields results which agree closely with precise Monte Carlo estimates of $E\{N_{(i)}\}$ ($1 \leq i \leq k$) for k = 4 and 5, obtained by Bechhofer and Frisardi (1983).

2. An approximation for $E\{N_{(i)}\}$

We suppose conceptually that a realization of an experiment gives rise to an infinite sequence of observations from each population; these observations are revealed in the order determined by the sampling rule until the experiment terminates and observations which are not 'seen' correspond to those which would have been obtained under further sampling. Let X_i be the number of failures in the first n observations on Π_i and let

$$W_i = \min_{j \neq i} (X_j).$$

Let A_i be the set of indices $j \neq i$ for which $X_j = W_i$. Finally, let M_i be the number of observations on Π_i required to obtain W_i failures.

N_i is defined to be the number of observations taken from Π_i. It follows from the definition of R^* and S^* that $N_i = M_i$, except in the following cases:

A1 $X_i > W_i$ and Π_i is sampled in the last cycle before Π_j for all $j \in A_i$; in this case $N_i =$ {the number of observations on Π_i required to obtain (W_i+1) failures} $\wedge\ n > M_i$, where $x \wedge y$ denotes the minimum of x and y.

A2 $X_i = W_i$, Π_i is selected as best and the n'th observation on Π_i is not a failure; in this case $N_i = n > M_i$.

B1 $X_i = W_i$, Π_i is selected as best, Π_i is sampled last in the last cycle and the n'th observation on Π_i is a failure; in this case $N_i = n-1 < M_i$.

B2 $X_i < W_i$ (and hence Π_i is selected); in this case $N_i = n$ or $(n-1) < M_i$.

If $p_i < p_{[k]}$, it is seen by considering the tail probabilities of the binomial distribution that $P\{X_i \leq W_i\} = O(e^{-\alpha n})$ as $n \to \infty$, for some $\alpha > 0$; hence the probabilities of A2, B1 and B2 are $O(e^{-\alpha n})$. Also if $p_i < p_{[k]} < 1$, the probability that Π_i is sampled in the last cycle before Π_j for all $j \in A_i = O(e^{-\beta n})$ as $n \to \infty$, for some $\beta > 0$ and hence $P\{A1\} = O(e^{-\beta n})$. If, however, $p_{[k]} = 1$, then $W_i = 0$ and there is exactly one cycle with probability one; hence $P\{A1\} = 1/(r+1) + O(e^{-\gamma n})$ as $n \to \infty$, where $\gamma > 0$ and r is the number of populations with success probability unity.

Now $E\{M_i \mid X_1,\ldots,X_k\} \leq n + n/(1-p_i)$, and hence $E\{|N_i-M_i| \mid X_1,\ldots,X_k\} = O(n)$. Also, $E\{M_i\} = E\{W_i\}/(1-p_{[i]})$. Combining all of the above results for the case $p_i < p_{[k]} < 1$, we have

$$E\{N_i\} = E\{M_i\} + E\{(N_i-M_i)\delta(N_i \neq M_i)\}$$

$$= \frac{E\{W_i\}}{1-p_i} + \underset{\sim}{o}(n) \qquad (2.1)$$

(where $\delta(\cdot)$ is the indicator random variable taking the value 1 if the event occurs and 0 otherwise), and for $p_i < p_{[k]} = 1$, when r populations have success probability unity,

$$E\{N_i\} = \frac{1}{(r+1)(1-p_i)} + \underset{\sim}{o}(n). \qquad (2.2)$$

If $p_i \neq p_{[k]}$ then $W_i \leq X_{(k)}$ and therefore $E\{W_i\} \leq n(1-p_{[k]})$. In fact, if $p_{[k]} > p_{[k-1]}$ we have

$$E\{W_i\} = n(1-p_{[k]}) + \underset{\sim}{o}(n). \tag{2.3}$$

Suppose that there are r populations with success probability $p_{[k]}$; for simplicity call these $\Pi_1,\ldots,\Pi_r$. Let $Z = \min_{1\le j\le r} (X_j)$. Then clearly $E\{Z\} \le n(1-p_{[k]})$. But also

$$Z \ge \frac{1}{r}\sum_{j=1}^{r} X_j - \max_{1\le \ell,m\le r} |X_\ell - X_m|$$

$$\ge \frac{1}{r}\sum_{j=1}^{r} X_j - \max_{1\le \ell,m\le r} |X_\ell - X_m| \ .$$

Since $E\{|X_\ell - X_m|\} \le (E(X_\ell - X_m)^2)^{1/2}$ we have $E\{Z\} \ge n(1-p_{[k]}) + O(\sqrt{n})$ as $n \to \infty$. It follows that in this case

$$E\{W_i\} = n(1-p_{[k]}) + O(\sqrt{n}). \tag{2.4}$$

Combining (2.1) to (2.4), for $p_{[i]} < p_{[k]}$ we have

$$E\{N_{(i)}\} = \begin{cases} \dfrac{n(1-p_{[k]})}{1-p_{[i]}} + \underset{\sim}{o}(n) & \text{(2.5a)} \\ \text{if } p_{[k]} < 1 \text{ is unique,} & \\ \dfrac{n(1-p_{[k]})}{1-p_{[i]}} + O(\sqrt{n}) & \text{(2.5b)} \\ \text{if } p_{[k]} < 1 \text{ is not unique,} & \\ \dfrac{1}{(r+1)(1-p_{[i]})} - \underset{\sim}{o}(n) & \text{(2.5c)} \\ \text{if } p_{[k]} = 1 \text{ and there are } r \text{ such populations.} & \end{cases}$$

The negative sign in (2.5c) indicates that $E\{N_{(i)}\} \leq 1/(r+1)(1-p_{[i]})$. Similar arguments show that for $p_{[i]} = p_{[k]}$

$$E\{N_{(i)}\} = \begin{cases} n - \underset{\sim}{o}(n) & (2.6a) \\ \text{if } p_{[k]} < 1 \text{ is unique,} & \\ n - O(\sqrt{n}) & (2.6b) \\ \text{if } p_{[k]} < 1 \text{ is not unique,} & \\ \frac{n}{r} - \underset{\sim}{o}(n) & (2.6c) \\ \text{if } p_{[k]} = 1 \text{ and there are } r \text{ such populations.} & \end{cases}$$

We note that for $p_{[k]} \neq 1$ the expressions for $E\{N_{(i)}\}$ are those obtained by a mean path approximation in which the proportion of successes from Π_i is exactly p_i. However, for unique $p_{[k]}$, the error term is of a smaller order than that usually associated with the mean path approximation.

We thus propose the following approximations:

For $p_{[k]} < 1$:

$$E\{N_{(i)}\} \simeq \frac{n(1-p_{[k]})}{1-p_{[i]}}, \quad \text{for } p_{[i]} < p_{[k]} \tag{2.7}$$

$$E\{N_{(i)}\} \simeq n, \quad \text{for } p_{[i]} = p_{[k]}. \tag{2.8}$$

For $p_{[k]} = 1$, and if there are r such populations:

$$E\{N_{(i)}\} \simeq \frac{1}{(r+1)(1-p_{[i]})}, \quad \text{for } p_{[i]} < p_{[k]} \tag{2.9}$$

$$E\{N_{(i)}\} \simeq \frac{n}{r}, \qquad \text{for } p_{[i]} = p_{[k]}. \tag{2.10}$$

The right-hand sides of (2.8), (2.9) and (2.10) are upper bounds for the cases considered; the right-hand side of (2.7) may be either too high or too low – events A1 and A2 lead to a low approximation, while events B1 and B2 and the fact that $W_{(i)}$ may be less than $X_{(k)}$ cause the approximation to be too high.

Tables 2.1 to 2.4 compare these approximations with exact values of $E\{N_{(i)}\}$ calculated by Bechhofer and Kulkarni (1982b, Tables 4.6 and 4.14). The results are representative of the comparisons for other values of $(p_{[1]},p_{[2]})$ and $(p_{[1]},p_{[2]},p_{[3]})$ considered by Bechhofer and Kulkarni. We note that the approximations are quite good, even for values of n as small as 5 or 10. In general the approximations tend to be higher than $E\{N_{(i)}\}$; exceptions occur when $p_{[k]}$ is close to 1 and in that case it appears that the event A1 occurs with a high enough probability to cause the approximation to underestimate $E\{N_{(i)}\}$. Bechhofer and Frisardi (1983) present Monte Carlo estimates of $E\{N_{(i)}\}$ for k = 3, 4 and 5 and n = 10, 20, 30, 40 and 50; our approximations also perform well for their vectors of success probabilities.

3. An improved upper bound for $E\{N_{(i)}\}$

We now suppose conceptually that a realization of an experiment gives rise to n observations from every population, observations being revealed in the order determined by the sampling rule until the experiment terminates – thus, not all of the n observations will be 'seen' for some populations.

Let X_i denote the number of failures in n observations on Π_i and let Y_i denote the number of failures actually observed on Π_i. It follows from the definition of the sampling rule R^* and the stopping rule S^* that

$$Y_i \leq \min_{1\leq j\leq k} (X_j) + 1. \tag{3.1}$$

Let $Z = \min_{1\leq j\leq k} (X_j)$. We now show that

$$P\{Y_i = Z + 1\} \leq \frac{1}{2}. \tag{3.2}$$

We argue conditionally on $\underset{\sim}{X} = (X_1,\ldots,X_n)$. If $X_i = Z$, then trivially

$$P\{Y_i = Z + 1 \mid \underset{\sim}{X}\} = 0. \tag{3.3}$$

If $X_i < Z$ and in particular $X_i < X_j$, then $Y_i = Z + 1$ only if the last cycle of sampling starts with Z failures on each population and in this cycle Π_i is sampled before Π_j. Since $X_i < X_j$ the conditional probability of Π_i being sampled before Π_j in this last cycle is at most 1/2 and therefore

$$P\{Y_i = Z + 1 \mid \underset{\sim}{X}\} \leq \frac{1}{2}. \tag{3.4}$$

From (3.3) and (3.4), $P\{Y_i = Z + 1 \mid \underset{\sim}{X}\} \leq \frac{1}{2}$ for all $\underset{\sim}{X}$ and (3.2) follows.

From (3.1)

$$E(Y_i) \leq E\{\min_{1\leq j\leq k} (X_j) + \delta(Y_i = Z + 1)\}$$

$$\leq E\{X_{(k)}\} + P\{Y_i = Z + 1\}$$

$$\leq n(1-p_{[k]}) + \frac{1}{2} . \tag{3.5}$$

Now $E\{Y_i\} = (1-p_i)E\{N_i\}$ and substituting into (3.5) gives, for $1 \leq i < k$ and $p_{[i]} \neq 1$,

$$E\{N_{(i)}\} \leq \frac{n(1-p_{[k]})}{1-p_{[i]}} + \frac{1}{2(1-p_{[i]})} . \tag{3.6}$$

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References

Bechhofer, R. E. and Frisardi, T. (1983). A Monte Carlo study of the performance of a closed adaptive sequential procedure for selecting the best Bernoulli population. Accepted for publication in Journal of Statist. Comp. & Simulation.

Bechhofer, R. E. and Kulkarni, R. V. (1982a). Closed adaptive sequential procedures for selecting the best of $k \geq 2$ Bernoulli populations. Proceedings of the Third Purdue Symposium on Statistical Decision Theory and Related Topics (ed. by S. S. Gupta and J. Berger), New York, Academic Press, I, 61-108.

Bechhofer, R. E. and Kulkarni, R. V. (1982b). On the performance characteristics of a closed adaptive sequential procedure for selecting the best Bernoulli population. Comm. Statist. Sequential Analysis, 1, No. 4, 315-354.

Table 2.1. Comparison of $E\{N_{(i)}\}$, (i = 1,2) with the approximations (2.7), (2.8) for $p_{[1]} = 0.5$, $p_{[2]} = 0.7$ and n = 5, 10, 20, 50 and 100.

n	$E\{N_{(1)}\}$	$\frac{n(1-p_{[2]})}{1-p_{[1]}}$	$E\{N_{(2)}\}$	n
5	2.82	3.0	4.15	5.0
10	5.76	6.0	9.26	10.0
20	11.80	12.0	19.52	20.0
50	29.94	30.0	49.88	50.0
100	59.99	60.0	99.99	100.0

Table 2.2. Comparison of $E\{N_{(i)}\}$, (i = 1,2) with the approximations (2.7), (2.8) for $p_{[1]} = 0.7$, $p_{[2]} = 0.9$ and n = 5, 10, 20, 50 and 100.

n	$E\{N_{(1)}\}$	$\frac{n(1-p_{[2]})}{1-p_{[1]}}$	$E\{N_{(2)}\}$	n
5	2.17	1.67	3.92	5.0
10	3.88	3.33	8.99	10.0
20	7.02	6.67	19.42	20.0
50	16.71	16.67	49.93	50.0
100	33.33	33.33	100.00	100.0

Table 2.3. Comparison of $E\{N_{(i)}\}$, (i = 1,2,3) with the approximations (2.7), (2.8) for $p_{[1]} = 0.4$, $p_{[2]} = 0.6$, $p_{[3]} = 0.8$ and n = 4, 6, 10, 20 and 40.

n	$E\{N_{(1)}\}$	$\frac{n(1-p_{[3]})}{1-p_{[1]}}$	$E\{N_{(2)}\}$	$\frac{n(1-p_{[3]})}{1-p_{[2]}}$	$E\{N_{(3)}\}$	n
4	1.44	1.33	1.99	2.0	3.05	4.0
6	2.04	2.00	2.99	3.0	5.04	6.0
10	3.26	3.33	4.96	5.0	9.18	10.0
20	6.54	6.67	9.92	10.0	19.53	20.0
40	13.28	13.33	19.95	20.0	39.84	40.0

Table 2.4. Comparison of $E\{N_{(i)}\}$, (i = 1,2,3) with the approximations (2.7), (2.8) for $p_{[1]} = 0.3$, $p_{[2]} = 0.6$, $p_{[3]} = 0.9$ and n = 4, 6, 10, 20 and 40.

n	$E\{N_{(1)}\}$	$\frac{n(1-p_{[3]})}{1-p_{[1]}}$	$E\{N_{(2)}\}$	$\frac{n(1-p_{[3]})}{1-p_{[2]}}$	$E\{N_{(3)}\}$	n
4	0.99	0.57	1.59	1.0	3.28	4.0
6	1.24	0.86	2.14	1.5	5.37	6.0
10	1.70	1.43	3.09	2.5	9.57	10.0
20	2.96	2.86	5.29	5.0	19.87	20.0
40	5.73	5.71	10.04	10.0	39.99	40.0