FANTASTIC RELAXATIONS OF THE TSP AND HOW TO BOUND THEM: RELAXATIONS OF THE TRAVELING SALESMAN PROBLEM AND THEIR INTEGRALITY GAPS

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FANTASTIC RELAXATIONS OF THE TSP AND HOW TO BOUND THEM: RELAXATIONS OF THE TRAVELING SALESMAN PROBLEM AND THEIR INTEGRALITY GAPS

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The Traveling Salesman Problem (TSP) is a fundamental problem in combinatorial optimization, combinatorics, and theoretical computer science and is a canonical NP-hard problem. Given a set of n vertices and pairwise costs c_{ij} of traveling between vertices i and j, the TSP asks for a minimum-cost tour visiting each of the vertices exactly once (i.e., a minimum-cost Hamiltonian cycle). Despite the problem's ubiquity, the state-of-the-art TSP approximation algorithm dates back more than 40 years. Its performance guarantee can be derived using a linear program *relaxation* that is over 50 years old, but the best-known analysis of this linear program's *integrality gap* (which dictates its use in proving approximation guarantees) has not been improved in nearly 40 years of active research. This thesis contributes to two main avenues of TSP research towards breaking these bottlenecks, both of which involve analyzing TSP relaxations.

We first consider relaxations of the TSP that are based on *semidefinite programs* (SDPs). Recently, many such relaxations have been proposed as avenues towards better approximation algorithms. These SDPs exploit a breadth of mathematical structures and have shown considerable promise in small numerical experiments, but little has been known about their general performance. Our first main results fill this void: we provide the first theoretical analysis of the integrality gap of every major SDP relaxation of the TSP. Specifically, with standard costs that are symmetric and obey the triangle inequality, we show that every major SDP relaxation of the TSP has an unbounded integrality gap. To do so, we develop a systematic methodology that exploits symmetry. Our methodology allows us to analyze,

e.g., SDPs from [2, 18, 21, 23, 25, 43, 69, 82] (some of these SDPs are now known to find equivalent optimal values), and extends to SDP relaxations of the Quadratic Assignment Problem and the k-cycle cover problem. Our results contrast starkly with analysis of the 50-year-old linear program relaxation, whose integrality gap is at most $\frac{3}{2}$.

In the second part of this thesis, we turn to the prototypical linear program relaxation of the TSP, the subtour elimination LP. We analyze this relaxation on an important but non-metric set of instances: *circulant TSP* instances. Circulant TSP instances are particularly compelling because circulant instances impose enough structure to make some – but not all – NP-hard problems easy. De Klerk and Dobre [22] conjectured that, when instances are circulant, the subtour elimination LP is equivalent to a combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [77]. We resolve this conjecture in the affirmative, exploiting symmetry to find a readily-computable, analytic solution to the subtour elimination LP on circulant instances. Using this same symmetry, we show that the integrality gap of the subtour elimination LP on circulant instances is exactly 2; we show that this gap remains unchanged even when the crown, ladder, and chain inequalities are added (see [6, 61, 65]).

BIOGRAPHICAL SKETCH

Sam Gutekunst grew up in Connecticut. He earned his undergraduate degree in mathematics from Harvey Mudd College in 2014, working with Michael Orrison, Judith Grabiner, and Susan Martonosi. In 2015, he completed a MASt in mathematics at Cambridge University and then started his Ph.D. at Cornell ORIE.

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Work in this thesis is based off Gutekunst and Williamson [39] (Chapter 3), Gutekunst and Williamson [41] (Chapters 4-5), Gutekunst and Williamson [40] (Chapters 6-7), and Gutekunst and Williamson [42] (Chapter 8). Feedback from the referees on the original papers has improved the exposition of results in this thesis. Research in this thesis has been supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-1650441, and by National Science Foundation Grants No. CCF-1908517 and CCF-1552831.

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CHAPTER 1

INTRODUCTION: THE TRAVELING SALESMAN PROBLEM AND RELAXATIONS

The Traveling Salesman Problem (TSP) is a fundamental problem in combinatorial optimization, combinatorics, and theoretical computer science. An instance consists of a set $[n] := \{1, 2, 3, ..., n\}$ of n cities and, for each pair of distinct cities $i, j \in [n]$, an associated cost or distance $c_{ij} \geq 0$ reflecting the cost or distance of traveling from city i to city j. Throughout this thesis, we always implicitly assume that the edge costs c_{ij} are **symmetric** (so that $c_{ij} = c_{ji}$ for all $i, j \in [n]$). The TSP is then to find a minimum-cost tour visiting each city exactly once. Treating the cities as vertices of the complete, undirected graph K_n, and treating an edge $\{i, j\}$ of K_n as having cost c_{ij}, the TSP is equivalently to find a minimum-cost Hamiltonian cycle on K_n.

With just this set-up, the TSP is well known to be NP-hard. The TSP with general symmetric edge costs is, moreover, even hard to approximate within any constant factor α: An **α-approximation algorithm** for the TSP is an algorithm that finds a Hamiltonian cycle in polynomial time, and that cycle is guaranteed to cost no more than α times the (unknown) cost of the optimal solution. An α-approximation algorithm for the TSP for any constant α would imply P=NP (see, e.g., Theorem 2.9 in Williamson and Shmoys [79]).

Hence more restrictive assumptions are placed on the edge costs. If one assumes that edge costs are **metric** (i.e. $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct $i, j, k \in [n]$), it is known to be NP-hard to approximate TSP solutions in polynomial time to within any constant factor $\alpha < \frac{123}{122}$ (see Karpinski, Lampis, and Schmied [48]). Conversely, the polynomial-time Christofides-Serdyukov algorithm [14, 71] outputs a Hamiltonian cycle that is at most a factor of $\frac{3}{2}$ away from the optimal solution for any metric (and, as always, implicitly symmetric) instance:

on such instances, it is a $\frac{3}{2}$-approximation algorithm.

One powerful technique for analyzing TSP approximation algorithms – and one way to attain the $\frac{3}{2}$ performance guarantee on the Christofides-Serdyukov algorithm – is to *relax* the discrete set of Hamiltonian cycles. The prototypical example is the *subtour elimination linear program* (also referred to as the Dantzig-Fulkerson-Johnson relaxation [19] and the Held-Karp bound [44], and which we will refer to as the subtour LP; see Section 2.1 for the precise definition). The subtour LP is a *relaxation* of the TSP because 1) every Hamiltonian cycle has a corresponding feasible solution to the subtour LP, and 2) the value of the subtour LP for such a feasible solution equals the cost of the corresponding Hamiltonian cycle. As a result, the optimal value of the subtour LP is a lower bound on the optimal solution to the TSP. Wolsey [80], Cunningham [17], and Shmoys and Williamson [73] show that the Christofides-Serdyukov algorithm produces a (not-necessarily optimal) Hamiltonian cycle that is within a factor of $\frac{3}{2}$ of the optimal value of the subtour LP for any metric instance. Combining these two observations shows that the Christofides-Serdyukov algorithm satisfies the following chain of inequalities on any metric instance:

$$\begin{aligned}\text{Cost of optimal TSP solution} &\leq \text{Cost of cycle produced by Christofides-Serdyukov algorithm}\\ &\leq \frac{3}{2}\ \text{Optimal value of subtour LP}\\ &\leq \frac{3}{2}\ \text{Cost of optimal TSP solution.}\end{aligned}$$

This chain shows that the Christofides-Serdyukov algorithm is a $\frac{3}{2}$-approximation algorithm for the TSP on any metric instance: the algorithm efficiently finds a solution that is within a factor of $\frac{3}{2}$ of the optimal cost.

The chain of inequalities also bounds *integrality gap* of the subtour LP on metric instances, which measures the worst-case performance of a relaxation relative to the TSP. Specifically, the chain shows that the integrality gap is at most $\frac{3}{2}$: for any metric instance, the ratio of the optimal TSP solution's cost to the optimal value of the subtour LP cannot be more than $\frac{3}{2}$. Note that, if the subtour LP did not have a constant-factor integrality gap, it would not be possible to use the LP as above to show that a TSP algorithm was a constant-factor approximation algorithm: there would be instances for which the algorithm produced a cycle arbitrarily far in cost from the relaxation's optimal value. Goemans [32] conjectures that the integrality gap of the subtour LP is $\frac{4}{3}$, though the $\frac{3}{2}$ bound of Wolsey [80], Cunningham [17], and Shmoys and Williamson [73] remains state-of-the-art.

Relaxations of the TSP are of theoretical interest because they are a fundamental tool for proving performance guarantees on approximation algorithms, as used in the above chain. They are also, however, important from a practical perspective because of their potential to efficiently provide strong lower bounds on the cost of an optimal TSP solution. These bounds are used to improve branch-and-bound schemes that provide exact TSP solutions. While these schemes are not formally efficient, they can be used in-practice to solve structured TSP instances with tens of thousands of cities. Stronger lower bounds can help these schemes find an exact solution faster.

While relaxations are a ubiquitous tool in TSP research, the state-of-the-art relaxation remains the subtour LP (now well over 50 years old), and the state-of-the-art analysis remains the $\frac{3}{2}$ bound on its integrality gap (now roughly 40 years old). One prong of recent TSP research has been to search for potentially-stronger relaxations of the TSP that are based on *semidefinite programs (SDPs)* rather than linear programs. SDP-based relaxations have seen substantial recent interest because they are more powerful than linear programs but can still be solved efficiently. Hence, they may lead to improvements on the state-of-the-art subtour LP.

Over the past two decades, SDP relaxations for the TSP have been proposed based on a breadth of mathematical ideas: spectral graph theory [18], association schemes [23, 21], permutation matrices [82, 69], symmetry reduction [25], eigenvalue bounds [2, 43] and the matrix-tree theorem [23] (several of these SDPs are now known to be equivalent). These SDPs often have promising numerical performance, but very little has been known about their theoretical performance: some are known to find equivalent optimal values, and otherwise generally all that is known is their relationship to the subtour LP: Goemans and Rendl [34] showed that the one of the first SDP relaxations (due to Cvetković, Čangalović, and Kovačević-Vujčić [18], and based on spectral graph theory and algebraic connectivity) could never provide a better bound than the subtour LP: for any metric TSP instance,

$$\text{Optimal SDP value} \;\le\; \text{Optimal subtour LP value} \;\le\; \text{Optimal TSP solution.}$$

Most more recent SDPs are known to be incomparable to the subtour LP, but their integrality gaps – and even whether or not they are finite – had not been known prior to results from this thesis.

A second prong of recent TSP research has been to analyze relaxations when edge costs are more restrictive. Such settings may be easier to analyze than the general case of symmetric and metric edge costs, and techniques developed in these settings might translate to the more general TSP. Several settings have been studied. The $(1, 2)$-TSP restricts $c_{ij} \in \{1, 2\}$ for every edge $\{i, j\}$ (see, e.g., Papadimitriou and Yannakakis [66], Berman and Karpinski [4], Karpinski and Schmied [49]). On these instances, Qian, Schalekamp, Williamson, and van Zuylen show that the integrality gap of the subtour LP is strictly less than $\frac{4}{3}$. In graphic TSP, the input corresponds to a connected, undirected graph G on vertex set $[n]$, and for $i, j \in [n]$, the cost c_{ij} is the length of the shortest i-j path in G; approximation algorithms with stronger performance guarantees than the Christofides-Serdyukov algorithm are known in this case. Again the integrality gap of the subtour LP is known to be better than $\frac{3}{2}$ in this case (see, e.g., Oveis Gharan, Saberi, and Singh [64], Mömke and Svensson [58], Mucha [60], and Sebő and Vygen [70]).

One special case that is particularly intriguing, but where relatively little is known, is *circulant TSP*. Circulant TSP instances are those whose edge costs can be described by a *circulant matrix*, which imposes substantial symmetry: the cost of edge $\{i, j\}$ only depends on $(i - j) \bmod n$ (see Section 1.2 for the precise definition). Importantly, in circulant TSP we abandon the assumption that edge costs are metric.

Circulant TSP initially arose from questions of minimizing wallpaper waste in Garfinkel [27] and reconfigurable network design in Medova [56]. One of the reasons that circulant TSP has remained so compelling is that circulant instances seem to provide just enough structure to make an ambiguous set of instances: it is unclear whether or not a given combinatorial optimization problem should remain hard or become easy when restricted to circulant instances. Some classic combinatorial optimization problems become easy when restricted to circulant instances. In the late 80's, Burkard and Sandholzer [11] showed that the decidability question for whether or not a symmetric circulant graph is Hamiltonian can be solved in polynomial time and showed that bottleneck TSP is polynomial-time solvable on symmetric circulant graphs. Bach, Luby, and Goldwasser (cited in Gilmore, Lawler, and Shmoys [31]) showed that one could find minimum-cost Hamiltonian paths in (not-necessarily-symmetric) circulant graphs in polynomial time. In contrast, Codenotti, Gerace, and Vigna [15] showed that Max Clique and Graph Coloring remain NP-hard when restricted to circulant graphs and do not admit constant-factor approximation algorithms unless P=NP.

Because of this ambiguity, the complexity of circulant TSP has often been cited as an open problem (see, e.g., Burkhard [9], Burkhard, Deĭneko, Van Dal, Van der Veen, and Woeginger [10], and Lawler, Lenstra, Rinnooy Kan, and Shmoys [54]). Moreover, even the answers to fundamental TSP questions are not known when instances are circulant. It is not, for example, even known if the circulant TSP is solvable in polynomial-time or is NP-hard, even in highly restrictive cases (e.g. when only two types of edges have finite cost; See Greco and Gerace [37] and Gerace and Greco [30]).

1.1 Contributions and Outline

This thesis fits into both frameworks for TSP relaxation research: searching for stronger SDP relaxations using semidefinite programs, and better understanding the structure of the subtour LP. Our primary contributions provide a set of tools that allow us to analyze a broad variety of TSP relaxations, in both the metric and circulant cases.

The first part of the thesis, Chapters 3 to 5, focuses on semidefinite relaxations of the traveling salesman problem. We fill the void in the theoretical analysis of these SDP relaxations, analyzing the integrality gap of some of most promising recent SDP relaxations. We provide tools that allow us to analyze the integrality gap of all of the following relaxations on metric instances:

- An SDP TSP relaxation of Cvetković, Čangalović, and Kovačević-Vujčić [18] (based on algebraic connectivity); see Corollary 3.9.
- An SDP TSP relaxation of de Klerk, Pasechnik, and Sotirov [23] based on the theory of association schemes (see also de Klerk, de Oliveira Filho, and Pasechnik [21]); see Theorem 3.1.
- Any SDP attained by strengthening the association-scheme-based SDP of de Klerk, Pasechnik, and Sotirov [23] by adding arbitrary additional positive-semidefinite inequalities satisfied by linear combinations of the distance matrices of a Hamiltonian

cycle; see Proposition 3.16.

- An SDP TSP relaxation attained by adding a matrix-tree theorem-based constraint to any of the above SDPs due to de Klerk, Pasechnik, and Sotirov [23]; see Proposition 3.19.

- An SDP QAP[1] relaxation of Zhao, Karisch, Rendl, and Wolkowicz [82], when specialized to the TSP (based on permutation matrices, and shown by de Klerk et al. [23] to have an optimal value coinciding with the SDP of de Klerk et al. [23]); see Corollary 4.9. This also establishes the integrality gap of an SDP QAP relaxation of Povh and Rendl [69], when specialized to the TSP (based on permutation matrices, and shown by Povh and Rendl [69] to be equivalent to the SDP of Zhao et al. [82]).

- An SDP QAP relaxation of de Klerk and Sotirov [25], when specialized to the TSP (obtained by performing symmetry reduction on the SDP of de Klerk et al. [23]); see Theorem 7.1.

- An SDP QAP relaxation of Anstreicher [2], when specialized to the TSP (equivalent to the projected eigenvalue bound of Hadley, Rendl, and Wolkowicz [43]); see Corollary 5.22.

- Any of the above QAP SDPs with the triangle inequalities of de Klerk and Sotirov [24] added; see Section 5.4.

- An SDP relaxation of the k-cycle cover problem (which generalizes the TSP) of de Klerk, de Oliveira Filho, and Pasechnik [21]; see Theorem 3.13.

For all of these relaxations, we show that the integrality gap is unbounded. Our results follow by defining a family of highly symmetric *simplicial TSP instances*. This family of instances not only implies the unbounded integrality gaps, but it also shows that these SDP

[1]The *Quadratic Assignment Problem* (QAP) is a generalization of the TSP; see Chapter 2.

relaxations have a counter-intuitive non-monotonicity property: the lower bounds provided by these SDPs can be made arbitrarily weaker by adding more cities to visit (in a way that preserves metric and symmetric edge costs). Intuitively, adding more cities to visit should should increase the cost of the optimal solution (and indeed, that is the case for the TSP). Our results in this first section thus highlight fundamental issues with every major proposed SDP relaxation of the TSP; the above list includes, for example, every SDP relaxation of the TSP mentioned in the survey Sotirov [74]. Moreover, they apply to broader problems in combinatorial optimization: the QAP and k-cycle cover.

In the second part of this thesis, Chapters 6 to 8, we focus on the subtour LP. Our main results are as follows:

- We find a closed-form, directly computable optimal solution to the subtour LP on any circulant instance; see Theorem 7.1. In doing so, we resolve a conjecture of de Klerk and Dobre [22] relating the subtour LP to an entirely-combinatorial lower bound; see Conjecture 6.6.
- We show that the subtour LP has an integrality gap of exactly 2 on circulant instances; see Theorem 7.11.
- We show that the integrality gap of the subtour LP remains unchanged, even when the crown, ladder, and chain inequalities are added (see Naddef and Rinaldi [61], Boyd and Cunningham [6], and Padberg and Hong [65]); see Proposition 7.14.
- We show that subtour LP constraints imply the matrix-tree theorem-based constraint of de Klerk, Pasechnik, and Sotirov [23]; see Theorem 8.1.

While this thesis has two separate core sets of results, the techniques to derive them are unified by a single theme: understanding and exploiting symmetry. Our simplicial

TSP instances, for example, are highly symmetric instances that form an extremal test for connectivity. Our results in the first section lean heavily on their symmetry, which allows us to analytically find feasible solutions to involved SDP constraints. In the second section, we similarly exploit circulant symmetry: our main result will follow from using that symmetry to "spread out" a combinatorial lower bound among the appropriate subtour LP variables.

Outside of the technical contributions of this thesis to two TSP research prongs, we hope to provide a coherent treatment of both SDP relaxations of the TSP and circulant TSP. To that end, this thesis includes several survey components:

- We provide an updated survey on SDP relaxations of the TSP, including new, direct proofs of previous results; see Chapter 2.
- We survey and sketch the proofs of previous results on circulant TSP, many of which come from sources that are quite difficult to access; see Chapter 6.
- We make explicit new connections between number theory and combinatorial optimization via circulant TSP, and in particular the connection between circulant TSP and a conjecture from Marco Buratti in 2007 (see Buratti and Merola [7], Horak and Rosa [45], and Pasotti and Pellegrini [67]); see Section 6.4.
- We highlight a breadth of open problems; see Chapter 9.

Our hope is that results in this dissertation are self-contained and accessible to an early-career Ph.D. student with basic working knowledge of combinatorial optimization. To that end, Chapters 2, 4, and 6 are largely expository; they can be read for context and background results, without needing to work through the proofs.

Below, in Section 1.2, we introduce the notation used throughout this thesis. Chapter 2 introduces the relaxations we will focus on: the subtour LP and several SDP relaxations of the TSP. We pay particular attention to the intuition for each relaxation, and we describe the pertinent theoretical and empirical results about the strength of these SDPs.

In Chapter 3, we introduce the simplest case of simplicial TSP instances; they are sufficient to imply the unbounded integrality gap of several of the SDP relaxations listed above. Our techniques in that section are not, however, strong enough to imply the unbounded integrality gap of the SDP relaxation of de Klerk and Sotirov [25]. This SDP is notably strong: de Klerk and Sotirov run experiments showing that, on 24 instances on 8 vertices, it was outperformed by the subtour LP only once! In Chapter 4 we sketch how this SDP is robust to the techniques of Chapter 3. Doing so motivates the general simplicial instances we introduce in Chapter 5, which in turn allow us to attain the unbounded integrality gap of all the SDPs mentioned above.

In Chapter 6, we switch gears and focus on the subtour LP and circulant TSP. We survey pertinent results on lower bounds for circulant TSP, sketch the state-of-the-art 2-approximation algorithm for circulant TSP instances, and state de Klerk and Dobre [22]'s conjecture on circulant TSP. In Chapter 7 we then prove this conjecture and characterize the integrality gap of the subtour LP on circulant TSP instances.

In Chapter 8 we provide our final result on the subtour LP: we show that the subtour LP constraints imply a matrix-tree-theorem-based SDP constraint of de Klerk, Pasechnik, and Sotirov [23]; this result is not specialized to metric or circulant instances.

1.2 Notation and Preliminaries

Linear Algebra

We adopt standard notation from linear algebra. We use J_m and I_m to respectively denote the all-ones and identity matrix in $\mathbb{R}^{m\times m}$. We let $e_i^{(m)}$ denote the ith standard basis vector in $\mathbb{R}^m$ and let $e^{(m)} := e_1^{(m)} + \cdots + e_m^{(m)}$ denote the all-ones vector in $\mathbb{R}^m$. We let $E_{ij}^{(m)} := e_i^{(m)} \left(e_j^{(m)}\right)^T$ denote the $m \times m$ matrix with a one in the i, jth position and zeros elsewhere. When clear from context, we will suppress the dependence on the dimension and use, e.g., E_{ij} rather than $E_{ij}^{(m)}$.

We let $\mathcal{S}^{m\times m}$ denote the set of real, symmetric matrices in $\mathbb{R}^{m\times m}$ and let Π_m be the set of $m \times m$ permutation matrices. $Y \succeq 0$ denotes that Y is a positive semidefinite matrix; for $Y \in \mathcal{S}^{m\times m}$, $Y \succeq 0$ means that all eigenvalues of Y are nonnegative. $Y \geq 0$ denotes that Y is a nonnegative matrix entrywise.

We will also use several matrix operations. For a matrix $M \in \mathbb{R}^{m\times m}$ and $S_1, S_2 \subset [m]$, let $M[S_1, S_2]$ denote the submatrix of M with rows in S_1 and columns in S_2. When $S_1 = S_2$, we simplify notation and write $M[S_1] := M[S_1, S_1]$. For a vector $x \in \mathbb{R}^m$, let $Diag(x)$ be the $m \times m$ diagonal matrix whose i, i-th entry is x_i. For a matrix Y, let $\text{trace}(Y)$ denote the trace of Y, i.e., the sum of its diagonal entries. For $A, B \in \mathcal{S}^{m\times m}$, note that

$$\text{trace}(AB) = \sum_{i=1}^{m} \sum_{j=1}^{m} A_{ij} B_{ij} = \langle A, B \rangle,$$

the matrix inner product. For an $m \times m$ matrix Y, let $vec(Y)$ be the vector in $\mathbb{R}^{m^2}$ that stacks the columns of Y. Let A_{-i} denotes the matrix obtained by deleting the ith row and column of A.

Finally, for matrices $A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{p\times q}$ of arbitrary dimension, $A \otimes B$ denotes the **Kronecker product** of A and B, the $mp \times nq$ matrix

$$\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

The Kronecker product has particularly nice spectral properties.

Theorem 1.1. *Let $A \in \mathbb{R}^{a\times a}$ and $B \in \mathbb{R}^{b\times b}$ have respective eigenvalues $\lambda_i(A)$ and $\lambda_j(B)$ for $i = 1, ..., a$ and $j = 1, ..., b$. Then the ab eigenvalues of $A \otimes B$ are the ab products $\lambda_i(A)\lambda_j(B)$.*

See, e.g., Theorem 4.2.12 in Chapter 4 of Horn and Johnson [46]. Moreover, the Kronecker product is bilinear and associative:

Theorem 1.2. *If $A, B,$ and C are arbitrary matrices, and $\alpha \in \mathbb{R}$,*

$$\alpha(A \otimes B) = (\alpha A) \otimes B = A \otimes (\alpha B), \quad \textit{and} \quad (A \otimes B) \otimes C = A \otimes (B \otimes C).$$

If $A \in \mathbb{R}^{m\times n}$ and $B, C \in \mathbb{R}^{p\times q}$,

$$A \otimes (B + C) = A \otimes B + A \otimes C, \quad \textit{and} \quad (B + C) \otimes A = B \otimes A + C \otimes A.$$

See Theorems 4.2.3, 4.2.6, 4.2.7, and 4.2.8 in Chapter 4 of Horn and Johnson [46].

Circulant Matrices

We will regularly work with **circulant** matrices and use their structure when simplifying SDPs. A circulant matrix in $\mathbb{R}^{m\times m}$ has the form

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \cdots & c_{m-1} \\ c_{m-1} & c_0 & c_1 & c_2 & \cdots & c_{m-2} \\ c_{m-2} & c_{m-1} & c_0 & c_1 & \ddots & c_{m-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & c_4 & \cdots & c_0 \end{pmatrix} = \left(c_{(t-s) \bmod m}\right)_{s,t=1}^{m}.$$

Such a matrix is symmetric if $c_i = c_{m-i}$ for $i = 1, ..., m-1$. Let $d = \lfloor n/2 \rfloor$. We use a standard basis of symmetric circulant matrices in $\mathbb{R}^{m\times m}$ consisting of matrices $C_0^{(m)}, C_1^{(m)}, ..., C_d^{(m)}$ where, for $i = 1, ..., d-1$, $C_i^{(m)}$ is the symmetric circulant $m \times m$ matrix with $c_i = c_{m-i} = 1$ and $c_j = 0$ otherwise. We set $C_0^m = 2I$ and, when m is even, set $C_{m/2}^{(m)}$ to be the matrix where $c_{m/2} = 2$ and $c_j = 0$ otherwise. Note that, following these definitions, each $C_i^{(m)}$ has the property that all rows sum to 2. We use $\mathcal{A}(G)$ to denote the **adjacency matrix** of a graph G: the symmetric $n \times n$ matrix where $(\mathcal{A}(G))_{i,j} = 1$ if $\{i, j\}$ is an edge in G, and $(\mathcal{A}(G))_{i,j} = 0$ otherwise. We use $\mathcal{C}_m$ to denote a Hamiltonian cycle m vertices; when the vertices are in lexicographic order, $\mathcal{A}(\mathcal{C}_m) = C_1^{(m)}$.

Circulant matrices have well-studied structure (see, e.g., Davis [20] and Gray [36]), and in particular have easily-described eigenvalues.

Lemma 1.3 (Gray [36]). *The circulant matrix* $M = \left(m_{(s-t) \bmod n}\right)_{s,t=1}^{n}$ *has eigenvalues*

$$\lambda_t(M) = \begin{cases} \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi st\sqrt{-1}}{n}}, & \text{if } t = 1, ..., n-1 \\ \sum_{s=0}^{n-1} m_s, & \text{if } t = n. \end{cases}$$

To avoid ambiguity with index variables, we explicitly write $\sqrt{-1}$ and reserve i and j as index variables.

In a circulant matrix, the i, j-th entry only depends on $i - j \mod n$, and we use $\equiv_n$ to denote the mod-n equivalence relationship. When doing computations on the vertices of a graph, we assume all computations are done mod n. (e.g. writing "$v + 5$" corresponds to the vertex attained by starting at v and following an edge of length 5; we just write $v + 5$ instead of $(v + 5) \mod n$).

Combinatorial Optimization

We also use standard notation from combinatorial optimization for the TSP. Throughout, we will take n to be the number of cities/vertices of a TSP instance. We let $d = \lfloor \frac{n}{2} \rfloor$. We reserve C as the matrix of edge costs or distances (so that for $1 \leq i \leq n$ and $1 \leq j \leq n$, $C_{ii} = 0$ and $C_{ij} = C_{ij}$ is the cost of traveling between cities i and j).

Let $\mathrm{OPT}_{\mathrm{Relaxation}}(C)$ and $\mathrm{OPT}_{\mathrm{TSP}}(C)$ respectively denote the optimal value to a relaxation and the cost of an optimal TSP solution for a given matrix of costs C. Then because the relaxation is minimizing over a broader search space,

$$\mathrm{OPT}_{\mathrm{Relaxation}}(C) \leq \mathrm{OPT}_{\mathrm{TSP}}(C).$$

If $\mathcal{I}$ is a set of cost matrices (e.g. the set of all cost matrices corresponding to metric and symmetric edge costs), the **integrality gap** of the relaxation on instances in $\mathcal{I}$ is

$$\sup_{C \in \mathcal{I}} \frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{Relaxation}}(C)}.$$

The integrality gap measures the worst-case performance of the relaxation relative to the TSP; this ratio is bounded below by 1 by the above inequality. The ratio $\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{Relaxation}}(C)}$

for any TSP cost matrix $C \in \mathcal{I}$ provides a lower bound on the integrality gap. We will always assume matrices in $\mathcal{I}$ are symmetric; in Chapters 3 to 5, we will take $\mathcal{I}$ to be metric (but not necessarily circulant) instances, whereas in Chapters 6 and 7, $\mathcal{I}$ will generally be circulant (but not necessarily metric).

We take $x \in \mathbb{R}^{\binom{n}{2}}$ to mean that x is a vector whose entries are indexed by the edges of K_n. For a set of edges $F \subset E$, $x(F)$ denotes the total weight of edges in F: $x(F) = \sum_{e \in F} x_e$. For $S \subset V$, we denote the set of edges with exactly one endpoint in S by

$$\delta(S) := \{e = \{i, j\} : |\{i, j\} \cap S| = 1\}$$

and let $\delta(v) := \delta(\{v\})$. Similarly, we treat $E(S)$ as the set of edges with both endpoints in S, i.e. $E(S) := \{\{i, j\} : i, j \in S\}$. We use $\sqcup$ to denote a disjoint union (i.e. a partition): $A = B \sqcup C$ means $A = B \cup C$ and $B \cap C = \emptyset$. Finally, we use $\backslash$ for set-minus so that $A \backslash B = \{a \in A : a \notin B\}$.

CHAPTER 2

BACKGROUND: AN ABRIDGED HISTORY OF TSP RELAXATIONS

This chapter introduces the TSP relaxations discussed throughout this dissertation, most of which are based on semidefinite programs. Our main emphasis is on the intuition for each relaxation – why it is valid, and what types of solutions it is searching for – and on the previous work that provide context for our results. For a more detailed survey on the SDP relaxations of the TSP, see Section 2 of Sotirov [74]; for background on SDPs and combinatorial optimization, see Alizadeh [1], Goemans and Rendl [34], and Sotirov [74].

In Section 2.1 we begin with a fundamental relaxation: the subtour elimination linear program. We then discuss three of the most prominent SDP relaxations, each of which is progressively stronger: a relaxation based on algebraic connectivity in Section 2.2, a relaxation based on association schemes in Section 2.3, and a strengthening of this association scheme-based relaxation via symmetry reduction in Section 2.4. Finally, in Section 2.5, we end by briefly discussing relaxations based on properties of permutation matrices and the matrix-tree theorem.

2.1 The Subtour Elimination Linear Program

The prototypical TSP relaxation is the subtour elimination linear program (also referred to as the Dantzig-Fulkerson-Johnson relaxation [19] and the Held-Karp bound [44], and which we will refer to as the **subtour LP**). The subtour LP has a variable x_e associated to each edge:

$$\begin{array}{lll} \min & \sum_{e\in E} c_e x_e & \\ \text{subject to} & \sum_{e\in\delta(v)} x_e = 2, & v = 1,\dots,n \\ & \sum_{e\in\delta(S)} x_e \geq 2, & S\subset V : S\neq\emptyset, S\neq V \\ & 0\leq x_e\leq 1, & e = 1,\dots,n. \end{array} \tag{2.1}$$

Given a Hamiltonian cycle $\mathcal{C}$, there is a feasible solution to the subtour LP attained by setting $x_e = 1$ for each $e \in \mathcal{C}$ and $x_e = 0$ otherwise. Moreover, the objective value of this solution equals the cost of $\mathcal{C}$. Hence, it is a valid relaxation of the TSP. The constraints $\sum_{e\in\delta(v)} x_e = 2$ are known as the **degree constraints** and enforce that every node is incident to a net weight of two edges. The constraints $\sum_{e\in\delta(S)} x_e \geq 2$ are known as the **subtour elimination constraints**, and enforce that solutions are 2-edge-connected in a weighted sense: given any subset $S \subset V$, there is at least a net weight of 2 on the LP variables on edges connecting S to $V\backslash S$.

When edge costs are metric, Wolsey [80], Cunningham [17], and Shmoys and Williamson [73] show that solutions to this linear program are within a factor of $\frac{3}{2}$ of the optimal, integer solution to the TSP.

Theorem 2.1 (Wolsey [80], Cunningham [17], and Shmoys and Williamson [73])**.** *The integrality gap of the subtour LP on metric TSP instances is at most $\frac{3}{2}$. That is, for any input to the TSP with metric edge costs, the ratio*

$$\frac{OPT_{TSP}}{OPT_{LP}} \leq \frac{3}{2}.$$

It is conjectured that the integrality gap of the subtour LP on metric TSP instances is at most $\frac{4}{3}$ (Goemans [32]). The $\frac{3}{2}$ bound, however, remains state of the art. One recent approach to finding relaxations with better integrality gaps has been to consider relaxations that are instead based on more powerful semidefinite programs.

2.2 An Early Semidefinite Program-Based Relaxation and Algebraic Connectivity

We first discuss an early SDP relaxation of the TSP due to Cvetković, Čangalović, and Kovačević-Vujčić [18]. This relaxation can be viewed as a matrix analog of the subtour LP: In an integral solution to the subtour LP, variables x_e indicate whether or not an edge e is in a cycle. This SDP relaxation has a single matrix variable X. In an integral solution, this matrix variable will be the adjacency matrix of a Hamiltonian cycle. Like the subtour LP, the SDP constraints enforce local connectivity (using an equivalent matrix formulation of the degree constraints) and more global connectivity (using a notion of connectivity distinct from the subtour elimination constraints). Specifically, the relaxation is:

$$\begin{array}{lll} \min & \frac{1}{2}\mathrm{trace}\,(CX) & \\ \text{subject to} & Xe = 2e & \\ & X_{ii} = 0, & i = 1, ..., n \\ & 0 \leq X_{ij} \leq 1, & i, j = 1, ..., n \\ & 2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0 & \\ & X \in \mathcal{S}^n. & \end{array} \tag{2.2}$$

The constraint $Xe = 2e$ enforces that e is an eigenvector of X with corresponding eigenvalue 2, so that every row of X sums to 2. Since the ith row of an adjacency matrix captures information about all the edges incident to vertex i, this constraint is equivalent to the degree constraints of the subtour LP. The remaining semidefinite constraint is parallel to the subtour elimination constraints: it enforces that any solution X is as least as connected as a Hamiltonian cycle, but with respect to an algebraic notion of connectivity (instead of edge connectivity, as in the subtour LP).

Specifically, let G be a weighted, undirected graph on n vertices with weighted adjacency matrix A. Let D be the degree matrix of G (i.e., D is diagonal with $D_{ii} = \sum_{j=1}^{n} A_{ij}$). The **Laplacian** of G is defined as

$$L := D - A.$$

The Laplacian of a weighted graph is known to be positive semidefinite (see Spielman [75], which expresses the Laplacian as a quadratic form) and $Le = 0e$, so we can write the eigenvalues of L as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. The **algebraic connectivity** of G is the second smallest eigenvalue of the Laplacian, λ_2. It is, for example, well known to be zero if and only if the graph is disconnected. The algebraic connectivity of a Hamiltonian cycle[1] on n vertices is $h_n := 2 - 2\cos\left(\frac{2\pi}{n}\right)$. The positive semidefinite constraint of SDP (2.2) enforces that X – viewed as the weighted adjacency matrix of a weighted, undirected graph – is at least as 'algebraically connected' as a Hamiltonian cycle.

Claim 2.2 (See, e.g., Cvetković, Čangalović, and Kovačević-Vujčić [18])**.** *Let G be a weighted, undirected graph on n vertices and with weighted adjacency matrix X. Suppose that $Xe = 2e$ and that L has eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then $2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0$ if and only if $\lambda_2 \geq h_n$.*

Proof. Because $Xe = 2e$, we have that Laplacian L is $L = 2I - X$ and the positive semidefinite constraint can be rewritten

$$L + h_n(J - I) \succeq 0.$$

Since X is symmetric, we further assume that the eigenvalues of G correspond to an orthogonal basis of eigenvectors $v_1, ..., v_n$, where v_i corresponds to eigenvalue λ_i. Moreover, $Xe = 2e$ implies that

$$Le = (2I - X)e = 2e - Xe = 0e,$$

[1]One can directly compute the spectrum of the Laplacian matrix of a Hamiltonian cycle $\mathcal{C}$ using Lemma 1.3, since its Laplacian is circulant.

so that we can choose $v_1 = e$ and $\lambda_1 = 0$. Orthogonality of the eigenvectors implies that $Jv_i = ee^T v_i = 0$ if $i \neq 1$, while $Jv_1 = ee^T e = nv_1$.

Thus

$$(L + h_n(J - I))\, v_i = \lambda_i v_i + h_n J v_i - h_n v_i = \begin{cases} 0v_1 + nh_n v_1 - h_n v_1, & i = 1 \\ \lambda_i v_i + 0v_i - h_n v_i, & i \neq 1 \end{cases}.$$

That is, $v_1, ..., v_n$ remain eigenvectors of $L + h_n(J - I)$, but shifted so that their respective eigenvalues are

$$(n - 1)h_n, \lambda_2 - h_n, ..., \lambda_n - h_n.$$

Since $h_n \geq 0$, the positive semidefinite constraint is equivalent to

$$\lambda_i - h_n \geq 0, \quad i = 1, ..., n - 1,$$

and since $\lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, the positive semidefinite constraint holds if and only if

$$\lambda_2 \geq h_n.$$

□

Consequently, we will refer SDP (2.2) as the **algebraic connectivity SDP**.

Unfortunately, this relaxation is known to be weaker than the subtour LP: using algebraic connectivity instead of edge connectivity leads to a worse relaxation.

Theorem 2.3 (Goemans and Rendl [34])**.** *Any feasible solution to the subtour LP corresponds to a feasible solution to the algebraic connectivity SDP (2.2) of the same value, attained by setting*

$$X = \left(x_{\{i,j\}}\right)_{i,j=1}^{n}.$$

The algebraic connectivity SDP is therefore minimizing over a wider search space than the subtour LP. Consequently,

$$\mathrm{OPT}_{\mathrm{SDP}\ (2.2)} \leq \mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{TSP}},$$

and the algebraic connectivity can never provide a better bound (i.e., closer to the TSP optimal value) than the subtour LP.

2.3 A Semidefinite Program-Based Relaxation Incomparable to the Subtour LP

Goemans and Rendl [34] motivated a search for improved SDP relaxations, with the goal of finding relaxations that are not dominated by the subtour LP. De Klerk, Pasechnik, and Sotirov [23] introduced another, stronger SDP relaxation of the TSP that answered this challenge. Their SDP can be motivated and derived through a general framework for SDP relaxations based on the theory of association schemes (see de Klerk, de Oliveira Filho, and Pasechnik [21]). Moreover, de Klerk et al. [23] show computationally that this new SDP is incomparable to the subtour LP: there are cases for which their SDP provides a closer approximation to the TSP than the subtour LP and vice versa. Moreover, one of instances where their SDP does better than the subtour LP is particularly notable: it is part of a family of instances where the subtour LP achieves its worst-known integrality gap!

The SDP introduced by de Klerk et al. [23] uses d matrix variables $X^{(1)}, ..., X^{(d)} \in \mathbb{R}^{n \times n}$, with the cost of a solution depending only on $X^{(1)}$:

$$
\begin{array}{lll}
\min & \frac{1}{2}\text{trace}\left(CX^{(1)}\right) & \\
\text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\
& \sum_{j=1}^{d} X^{(j)} = J - I, & \\
& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\
& X^{(k)} \in \mathcal{S}^n, & k = 1, \ldots, d.
\end{array} \tag{2.3}
$$

Both de Klerk et al. [23] and de Klerk et al. [21] show that this is a valid relaxation of the TSP by showing that the following solution is feasible: for a simple, undirected graph G, let $A_k(G)$ be the *k-th distance matrix* (the matrix with i, j-th entry equal to 1 if and only if the shortest path between vertices i and j in G is of distance k, and equal to 0 otherwise). Let $\mathcal{C}_n$ be a cycle of length n (i.e., any Hamiltonian cycle on $[n]$). The solution where $X^{(k)} = A_k(\mathcal{C}_n)$ for $k = 1, ..., d$ is feasible for the SDP (see Proposition 2.4). Hence, the optimal integer solution to the TSP has a corresponding feasible solution to the SDP. That SDP solution has the same value as the optimal integer solution to the TSP, since $X^{(1)} = A_1(\mathcal{C}_n)$ is the adjacency matrix of a Hamiltonian cycle; the objective function of the SDP is scaled by $\frac{1}{2}$ since each edge $e = \{i, j\}$ is represented twice in $X^{(1)}$ (as both $X^{(1)}_{ij}$ and $X^{(1)}_{ji}$).

These solutions are shown to be feasible in de Klerk et al. [23] by noting that the $A_k(\mathcal{C}_n)$ form an association scheme and are therefore simultaneously diagonalizable. This allows for the positive semidefinite inequalities to be verified after computing the eigenvalues of each $A_k(\mathcal{C}_n)$. A more systematic approach is taken in de Klerk et al. [21], which introduces general results about association schemes. The constraints of the SDP then represent an application of these results to a specific association scheme: that of the distance matrices $A_k(\mathcal{C}_n)$. Below, we provide a direct proof using properties of circulant matrices.

Proposition 2.4 (de Klerk et al. [23]). *Let $\mathcal{C}_n$ be any Hamiltonian cycle on $[n]$. Setting*

$X^{(j)} = A_j(\mathcal{C}_n)$ for $j = 1, ..., d$ yields a feasible solution to the SDP (2.3).

Our proof begins with a trigonometric identity that we will use repeatedly in later proofs:

Lemma 2.5. *Let n be even and $0 < k < n$ be an integer. Then*

$$\sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) = \frac{-1 + (-1)^k}{2}.$$

Proof. This identity is a consequence of Lagrange's trigonometric identity (see, e.g., Identity 14 in Section 2.4.1.6 of Jeffrey and Dai [47]), which states that, for $0 < \theta < 2\pi$,

$$\sum_{j=1}^{m} \cos(j\theta) = -\frac{1}{2} + \frac{\sin\left(\left(m + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}.$$

Taking $\theta = \frac{2\pi k}{n}$ and using $n = 2d$, we obtain:

$$\begin{aligned}\sum_{j=1}^{d} \cos\left(\frac{2\pi k}{n} j\right) &= -\frac{1}{2} + \frac{\sin\left(\pi k + \frac{\pi k}{n}\right)}{2\sin\frac{\pi k}{n}} \\ &= -\frac{1}{2} + (-1)^k \frac{1}{2},\end{aligned}$$

where we recall that $\sin(\pi + \theta) = -\sin(\theta)$. □

Notice that when $k = 0$ or $k = n$, the sum is d.

Proof (of Proposition 2.4). We first remark that each $A_j(\mathcal{C}_n)$ is a nonnegative symmetric matrix. Moreover, $\sum_{j=1}^{d} A_j(\mathcal{C}_n) = J - I$. This follows because, in $\mathcal{C}_n$, the shortest path between any pair of distinct vertices $u, v \in [n]$ is a unique element of the set $[d]$. Hence, exactly one of the terms in the sum $\sum_{j=1}^{d} A_j(\mathcal{C}_n)$ has a one in its u, v entry, and all other terms have a zero.

Now for any fixed $k \in [d]$ we compute the eigenvalues of the matrix

$$M := I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) A_j(\mathcal{C}_n).$$

First, suppose the vertices are labeled so that the cycle $\mathcal{C}_n$ visits the vertices in lexicographic order. We will later show that this ordering is without loss of generality.

Then $M = \left(m_{(t-s) \mod m}\right)_{s,t=1}^{n}$ is symmetric circulant with, for $j = 1, ..., d$, entries m_j and m_{n-j} given exactly by the coefficient of the j-th term in the sum. Namely:

$$m_0 = 1, \quad m_d = \cos\left(\frac{2\pi kd}{n}\right), \quad m_j = m_{n-j} = \cos\left(\frac{2\pi jk}{n}\right), j = 1, ..., d-1.$$

We can directly compute the t-th eigenvalue of M using Lemma 1.3. Our later proofs will include similar computations, so we pay particular emphasis to the details. For $t = 1, ..., n-1$, the t-th eigenvalue of M is:

$$\begin{aligned}\lambda_t(M) &= \sum_{s=0}^{n-1} m_s e^{-\frac{2\pi st\sqrt{-1}}{n}} \\ &= 1 + \cos\left(\frac{2\pi kd}{n}\right) e^{-\frac{2\pi dt\sqrt{-1}}{n}} + \sum_{s=1}^{d-1} \cos\left(\frac{2\pi sk}{n}\right)\left(e^{-\frac{2\pi st\sqrt{-1}}{n}} + e^{-\frac{2\pi(n-s)t\sqrt{-1}}{n}}\right),\end{aligned}$$

where we have first written the terms when $s = 0$ and $s = d$. We rewrite terms so that our sum is to d and simplify exponentials:

$$\begin{aligned}&= 1 - \cos\left(\frac{2\pi kd}{n}\right) e^{\frac{2\pi dt\sqrt{-1}}{n}} + \sum_{s=1}^{d} \cos\left(\frac{2\pi sk}{n}\right)\left(e^{-\frac{2\pi st\sqrt{-1}}{n}} + e^{\frac{2\pi st\sqrt{-1}}{n}}\right) \\ &= 1 - (-1)^k(-1)^t + 2\sum_{s=1}^{d} \cos\left(\frac{2\pi sk}{n}\right)\cos\left(\frac{2\pi st}{n}\right).\end{aligned}$$

Recalling the product-to-sum identity for cosines (that $2\cos(\theta)\cos(\phi) = \cos(\theta+\phi)+\cos(\theta-\phi)$), we get

$$= 1 - (-1)^{k+t} + \sum_{s=1}^{d} \cos\left(\frac{2\pi s}{n}(k+t)\right) + \sum_{s=1}^{d} \cos\left(\frac{2\pi s}{n}(k-t)\right).$$

Using Lemma 2.5 and $(-1)^{k+t} = (-1)^{k-t}$:

$$= \begin{cases} 1-(-1)^{2d}+2d, & \text{if } k=t=d \\ -\frac{1}{2}+(-1)^{k+t}\frac{1}{2}+d, & \text{if } k \neq d, t \in \{k, n-k\} \\ 1-(-1)^{k+t}-\frac{1}{2}+(-1)^{k+t}\frac{1}{2}-\frac{1}{2}+(-1)^{k-t}\frac{1}{2}, & \text{else} \end{cases}$$

$$= \begin{cases} 2d, & \text{if } k=t=d \\ d, & \text{if } k \neq d, t \in \{k, n-k\} \\ 0, & \text{else.} \end{cases}$$

The eigenvalue λ_n is

$$\begin{aligned} \lambda_n(M) &= \sum_{s=0}^{n-1} m_s \\ &= 1 - \cos\left(\frac{2\pi kd}{n}\right) + 2\sum_{s=1}^{d} \cos\left(\frac{2\pi sk}{n}\right) \\ &= 1-(-1)^k - 1 + (-1)^k \\ &= 0. \end{aligned}$$

The matrix M thus has all nonnegative eigenvalues, so the positive semidefinite constraints hold for each $k \in \{1, ..., d\}$.

Finally, we note that our assumption that the cycle $\mathcal{C}_n$ is $1, 2, 3, ..., n-1, n, 1$ is without loss of generality: we can replace the $A_j(\mathcal{C}_n)$ with $P^T A_j(\mathcal{C}_n) P = P^{-1} A_j(\mathcal{C}_n) P$ for a permutation matrix P that permutes the labels of the vertices so that the cycle is $1, 2, 3, ..., n-1, n, 1$. Then M and $P^{-1}MP$ are similar matrices and share the same spectrum. Thus M is positive semidefinite if and only if $P^{-1}MP$ is positive semidefinite; $P^{-1}MP$ is the circulant matrix above, with

$$m_0 = 1, \quad m_d = \cos\left(\frac{2\pi kd}{n}\right), \quad m_j = m_{n-j} = \cos\left(\frac{2\pi jk}{n}\right), j = 1, ..., d-1,$$

and thus both $P^{-1}MP$ and M are positive semidefinite. □

We briefly remark that de Klerk et al. [23] also use the eigenvalue properties of circulant matrices in proving that this SDP is a relaxation of the TSP. They use the fact that each individual $A_k(\mathcal{C}_n)$ is circulant to compute the eigenvalues of each $A_k(\mathcal{C}_n)$, while we use the fact that the linear combinations of those matrices denoted above by M is circulant. We will refer to the SDP (2.3) as the **association scheme SDP**.

After showing that the association scheme SDP is a valid relaxation, de Klerk et al. [23] discuss the relationship between the association scheme SDP and other relaxations of the TSP. De Klerk et al. [23] solve both the association scheme SDP and subtour LP on the 24 classes of facet defining inequalities for the TSP on 8 vertices: in three instances the SDP provides a tighter bound, in nine instances they provide the same bound, and in the remaining 12 instances the subtour LP provides a tighter bound. Notably, the solution indicated in Figure 2.1 is feasible for the subtour LP, but not for the association scheme SDP; the solution indicated in Figure 2.1 is part of a family of solutions that show (as more intermediate nodes are added to each of the three horizontal, full-weight paths in Figure 2.1) the worst-known integrality gap for the subtour LP on metric, symmetric instances. While Wolsey [80], Cunningham [17], and Shmoys and Williamson [73] show that the integrality gap of the subtour LP is at most $\frac{3}{2}$, the best-known lower bound is $\frac{4}{3}$ and is achieved using solutions in this family. See Williamson [78] and Monma, Munson, and Pulleyblank [59].

Similarly, de Klerk et al. [23] show that their association scheme SDP is stronger than the algebraic connectivity SDP: any feasible solution for the association scheme SDP implies a feasible solution for the algebraic connectivity SDP of the same cost. Summarizing:

Theorem 2.6 (de Klerk et al. [23])**.** *The association scheme SDP (2.3) is a valid relaxation to the TSP. It is incomparable to the subtour LP and dominates the algebraic connectivity*

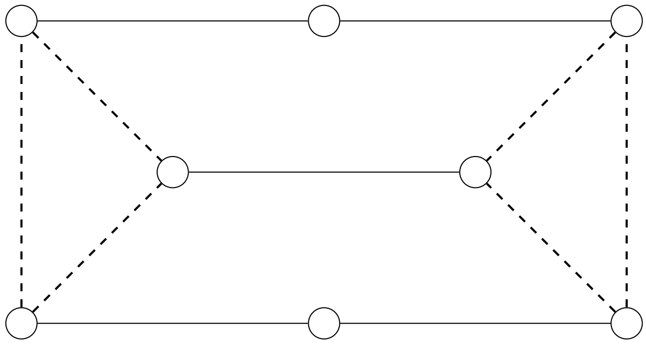

Figure 2.1: A solution feasible to the subtour LP, but not to the association scheme SDP (2.3). Dashed edges have weight $\frac{1}{2}$ while full edges have weight 1.

SDP (2.2).

Finally, we note that the association scheme SDP has many equivalent derivations and formulations. Zhao, Karisch, Rendl, and Wolkowicz [82] introduce a relaxation to the more general Quadratic Assignment Problem (QAP, discussed below), a special case of which is the TSP. Their relaxation is based on properties of permutation matrices; de Klerk et al. [23] show the optimal value of their SDP coincides with the optimal value of the SDP introduced by Zhao et al. [82] when specialized to the TSP. Sotirov [74] summarizes two interpretations of this latter SDP relaxation of the QAP: First, it is equivalent to a similar SDP relaxation of the QAP also based on permutation matrices from Povh and Rendl [69] (with equivalence shown in Povh and Rendl [69]). Second, it is equivalent to applying the N^+ lift-and-project operator of Lovász and Schrijver [55] to a QAP polytope; this equivalence is shown in Burer and Vandenbussche [8] and Povh and Rendl [69].

2.4 Strengthening SDP Relaxations and Symmetry Reduction

In de Klerk and Sotirov [25], symmetry reduction is applied to provide a stronger SDP than the association scheme SDP. While computationally involved, this new SDP performs extremely well on small instances: on the same 24 classes of facet-defining inequalities where the association scheme SDP is a strictly worse bound than the subtour LP on half the instances, this new SDP is a strictly worse bound on only one instance! Of the remaining 23 instances, the new SDP coincides with the subtour LP on three instances (where both coincide with the optimal value) and otherwise the new SDP outperforms the subtour LP.

To derive this new SDP, de Klerk and Sotirov [25] begin with an SDP for the Quadratic Assignment Problem (QAP), introduced in Koopmans and Beckmann [51]. In this problem there are n facilities to be assigned to n locations. Let $A = (a_{ij}) \in \mathbb{S}^{n\times n}$ denote the pairwise required commodity flows between facilities and let $B = (b_{ij}) \in \mathbb{S}^{n\times n}$ denote the pairwise costs of shipping a unit of flow between locations. Let $D = (d_{ij})$ be a matrix of placement costs where d_{ij} denotes the cost of placing facility i at location j. The QAP is to assign each facility to a distinct location so as to minimize total cost, where the cost depends quadratically on flows and distances and linearly on placement costs (e.g. assigning facility i to location $\sigma(i)$ incurs a linear placement cost $d_{i,\sigma(i)}$, while assigning facilities i and j to respective locations $\sigma(i)$ and $\sigma(j)$ incurs a quadratic cost $a_{i,j}b_{\sigma(i),\sigma(j)}$ to ship $a_{i,j}$ units of flow at $b_{\sigma(i),\sigma(j)}$ cost per unit). In full, the objective is

$$\min\{\text{trace}\left((AXB + D)X^T\right) : X \in \Pi_n\}.$$

The TSP for n cities is obtained in the special case where $B = C$, the matrix of TSP costs, where $D = 0$ (the all zeros matrix), and where $A = \frac{1}{2}\mathcal{A}(\mathcal{C}_n) = \frac{1}{2}C_1^{(n)}$ is a scaled adjacency matrix of a Hamiltonian cycle in lexicographic order. In this case, using the cyclic and linear properties of trace, the objective function becomes

$$\text{trace}\left(\frac{1}{2}C_1^{(n)}XCX^T\right) = \frac{1}{2}\langle X^T C_1^{(n)} X, C\rangle,$$

so that the permutation matrix X can interpreted as finding the optimal tour and relabeling the vertices according to the order of that tour; $X^T C_1^{(n)} X$ is then the adjacency matrix of the relabeled tour.

De Klerk and Sotirov [25] specifically consider a QAP relaxation of Povh and Rendl [69] which, when specialized to the TSP, has the same optimal value as the association scheme SDP (2.3). Their formulation is:

$$\begin{array}{lll}
\min & \frac{1}{2}\text{trace}\left(\left(D \otimes C_1^{(n)}\right)Y\right) & \\
\text{subject to} & \text{trace}((I_n \otimes E_{jj}^{(n)})Y) = 1, & j = 1, ..., n \\
& \text{trace}((E_{jj}^{(n)} \otimes I_n)Y) = 1, & j = 1, ..., n \\
& \text{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0 & \\
& \text{trace}(J_{n^2}Y) = n^2 & \\
& Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{n^2 \times n^2}. &
\end{array} \tag{2.4}$$

That this is a valid relaxation can be seen by setting $Y = vec(X)vec(X)^T$ for any permutation matrix $X \in \Pi_n$ (taking the convention that the ith row of X indicates the ith vertex visited). Then letting $X_{:i} = X[[n], \{i\}]$ denote the ith column of X,

$$vec(X) = \begin{pmatrix} X_{:1} \\ X_{:2} \\ \vdots \\ X_{:n} \end{pmatrix}$$

so that Y has the block structure

$$Y = \begin{pmatrix} Y^{(11)} & Y^{(12)} & \dots & Y^{(1n)} \\ Y^{(21)} & Y^{(22)} & \dots & Y^{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{(n1)} & Y^{(n2)} & \dots & Y^{(nn)} \end{pmatrix}$$

where $Y^{(ij)} = X_{:i}X_{:j}^T \in \mathbb{R}^{n \times n}$. If X is a permutation matrix, each $Y^{(ij)} = E_{st}$ for some s, t. Specifically, $Y^{(ij)} = E_{st}$ for the s, t such that $Xe_i = e_s$ and $Xe_j = e_t$. That is, $Y^{(ij)} = E_{st}$

means vertices i and j are respectively the sth and tth vertex visited. That the constraints hold then readily follows: Each $Y^{(ii)} = E_{ss}$ for some s (so that $\text{trace}((E_{ii} \otimes I_n)Y) = 1$) and because X is a permutation matrix, $Y^{(ii)} \neq Y^{(kk)}$ for $i \neq k$ (so that $\text{trace}((I_n \otimes E_{jj})Y) = 1$). Similarly, each $Y^{(ii)}$ is diagonal while each $Y^{(ij)}$ with $i \neq j$ has zero diagonal (so $\text{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0$) and since each of the n^2 blocks $Y^{(ij)}$ consists of a single 1 and zeros elsewhere, the sum of all entries in Y is n^2, i.e. $\text{trace}(J_{n^2}Y) = n^2$. The factored form $Y = vec(X)vec(X)^T$ implies that Y is a rank-1 positive semidefinite matrix and, since Y is 0-1, $Y \geq 0$. Finally, $(Y^{(ij)})^T = (X_{:i}X_{:j}^T)^T = X_{:j}X_{:i}^T = Y^{(ji)}$ so that Y is symmetric.

De Klerk and Sotirov [25] strengthen this SDP by appealing to symmetry. All that matters for the TSP is the order in which the vertices are visited in an optimal tour; there are $\frac{(n-1)!}{2}$ distinct tours, but $n!$ permutation matrices. One may thus assume an optimal solution $X \in \Pi_n$ is such that $X_{r,s} = 1$ (i.e., that the sth vertex visited is vertex r): an optimal tour includes vertex r and can be reindexed (without changing the cost of the tour) so that vertex r is the sth vertex visited. Making this assumption leaves the $n-1$ vertices $\alpha = [n]\backslash r$ to be visited at the $n-1$ positions $\beta = [n]\backslash s$, so one can effectively write a QAP for $X[\alpha, \beta] \in \Pi_{n-1}$ (the submatrix of X for which entries are not fixed by $X_{r,s} = 1$). Following through this process obtains a QAP on $(n-1)$ vertices; appropriately adjusting the objective function and writing the SDP relaxation of the QAP on $(n-1)$ vertices yields the following SDP relaxation (see de Klerk and Sotirov [25] for full details):

$$
\begin{array}{lll}
\min & \operatorname{trace}\left((C[\beta] \otimes \frac{1}{2}C_1^{(n)}[\alpha] + Diag(\bar{c}))Y\right) & \\
\text{subject to} & \operatorname{trace}\left(\left(I_{n-1} \otimes E_{jj}^{(n-1)}\right)Y\right) = 1, & j = 1, ..., n-1 \\
& \operatorname{trace}\left(\left(E_{jj}^{(n-1)} \otimes I_{n-1}\right)Y\right) = 1, & j = 1, ..., n-1 \\
& \operatorname{trace}\left((I_{n-1} \otimes (J_{n-1} - I_{n-1}) + (J_{n-1} - I_{n-1}) \otimes I_{n-1})\, Y\right) = 0 & \\
& \operatorname{trace}((J_{n-1} \otimes J_{n-1})Y) = (n-1)^2 & \\
& Y \geq 0, Y \succeq 0, Y \in \mathbb{S}^{(n-1)^2 \times (n-1)^2}, &
\end{array}
\tag{2.5}
$$

where $s, r \in [n]$, $\alpha = [n] \backslash r$ and $\beta = [n] \backslash s$, and $\bar{c} = vec(C_1[\alpha, \{r\}]C[\{s\}, \beta])$. We will refer to SDP (2.5) as the **symmetry reduction** SDP. Summarizing known results about the symmetry reduction SDP:

Theorem 2.7 (de Klerk and Sotirov [25])**.** *The symmetry reduction SDP (2.5) is a valid relaxation to the TSP. It is incomparable to the subtour LP and dominates the association scheme SDP (2.3).*

Figure 2.2 presents a diagram of the relationship between the feasible regions of the relaxations discussed in this chapter.

2.5 Other SDP Relaxations and Constraints

Most of this thesis will focus on the association scheme and symmetry reduction SDPs. In addition to these SDPs and their various equivalences, we will consider two other SDP relaxations of the TSP. First, Anstreicher [2] gives another SDP relaxation of the QAP. When specialized to the TSP, it is equivalent to the projected eigenvalue bound of Hadley, Rendl, and Wolkowicz [43]. This SDP is again in terms of an $n^2 \times n^2$ matrix, and we use

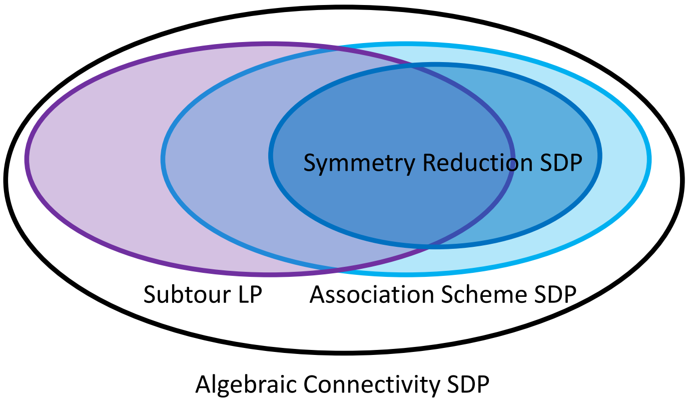

Figure 2.2: The relationship between the feasible regions of the relaxations discussed in this chapter.

the same block structure

$$Y = \begin{pmatrix} Y^{(11)} & Y^{(12)} & \dots & Y^{(1n)} \\ Y^{(21)} & Y^{(22)} & \dots & Y^{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{(n1)} & Y^{(n2)} & \dots & Y^{(nn)} \end{pmatrix}$$

with $Y^{(ij)} \in \mathbb{R}^{n\times n}$. The SDP is then:

$$\begin{array}{lll} \min & \frac{1}{2}\mathrm{trace}\left(\left(C \otimes C_1^{(n)}\right) Y\right) & \\ \text{subject to} & \sum_{i=1}^{n} Y^{(ii)} & = I_n \\ & \left(\mathrm{trace}(Y^{(ij)})\right)_{i,j=1}^{n} & = I_n \\ & \mathrm{trace}\left(YF^TF\right) & = 2n \\ & Y - \frac{1}{n^2}J_{n^2} & \succeq 0 \\ & Y \geq 0, Y \in \mathbb{S}^{n^2\times n^2}, & \end{array} \tag{2.6}$$

where

$$F = \begin{pmatrix} (e^{(n)})^T \otimes I_n \\ I_n \otimes (e^{(n)})^T \end{pmatrix}.$$

Let $Y' = vec(X)vec(X)^T$ for any $X \in \Pi_n$; that this is a valid relaxation can be seen by showing that

$$Y' - \frac{1}{n^2}\left(Y' J_{n^2} + J_{n^2} Y'\right) + \frac{2}{n^2} J_{n^2}$$

is feasible and has the same objective value as Y'. See Theorem 3.6 of Anstreicher [2] for more details.

Second, de Klerk et al. [23] give a constraint based on Kirchoff's matrix-tree theorem that can be added to SDPs. The matrix-tree theorem dates back to the mid-19th century (Kirchoff [50]) and connects the number of spanning trees of a graph to the Laplacian matrix of that graph. Consider a simple, undirected graph where each edge e has weight $x_e \geq 0$. Let X be the corresponding weighted adjacency matrix, so that X has zero diagonal and $X_{ij} = X_{ji} = x_{\{i,j\}}$. Suppose that $\mathcal{T}_G$ is the set of spanning trees of G. The **matrix-tree theorem** is a remarkable result about the Laplacian L of G. Specifically, the theorem states that any **principal minor** of L (i.e., the determinant of the matrix L_{-i} obtained by removing the ith row and column of L) equals $\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e$. In the case that $x_e = 1$ for every edge in G, the term $\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e$ exactly counts the number of spanning trees of G! See Theorem VI.29 in Tutte [76], e.g., for a proof of this general version of the matrix-tree theorem.

Any Hamiltonian cycle has n spanning trees (each attained by deleting a single edge). Here, if $X = A_1(\mathcal{C}_n)$ is the adjacency matrix of any Hamiltonian cycle G, then $|\mathcal{T}_G| = n$ and $L = 2I - X$. By the matrix-tree theorem,

$$\det\left((2I - X)_{-1}\right) = |\mathcal{T}_G| = n,$$

where A_{-i} denotes the matrix obtained by deleting the ith row and column of A. Since Laplacian matrices are positive semidefinite, their principal submatrices are as well, and $(2I - X)_{-1}$ is positive semidefinite.

De Klerk et al. [23] uses these observations to write an SDP constraint based on the matrix-tree theorem, which can be added to SDPs where a variable plays the role of a weighted adjacency matrix (e.g. X in the algebraic connectivity SDP, or $X^{(1)}$ in the association scheme SDP):

$$\det\left((2I - X)_{-1}\right) \geq n. \tag{2.7}$$

This constraint relies on the fact that $\{Z \succeq 0 : \det(Z) \geq n\}$ can be expressed as a linear matrix inequality (see, e.g., Section 3.2 of Nemirovskii [62]).

De Klerk et al. [23] note that this constraint strengthens the algebraic connectivity SDP. We refer to Equation (2.7) as the **matrix-tree theorem** constraint.

CHAPTER 3

TWO-GROUP SIMPLICIAL TSP INSTANCES AND THE UNBOUNDED INTEGRALITY GAP OF THE ASSOCIATION SCHEME AND ALGEBRAIC CONNECTIVITY SDPS

In this chapter, we introduce two-group simplicial TSP instances. These are highly structured instances, having several simple combinatorial interpretations, and are part of a class of instances on which the TSP can be solved easily. Despite their simplicity, they will be sufficient to imply the unbounded integrality gap of the association scheme SDP (2.3) and the algebraic connectivity SDP (2.2). This result holds even if the matrix-tree theorem constraint (2.7) is added. In contrast, the lower bound provided by the subtour LP is optimal on these instances. These instances also imply an unsettling non-monotonicity property of these SDPs, which contrasts with both the subtour LP and TSP.

We begin, in Section 3.1, by introducing the two-group simplicial TSP instances. In Section 3.2 we then use those instances to prove the main theorem of this chapter: that the association scheme SDP has an unbounded integrality gap. The crux of this result is constructing feasible solutions that match the symmetry of two-group simplicial TSP instances. Appealing to this symmetry allows us to rewrite the positive semidefinite constraints in the association scheme SDP as linear inequalities, in turn allowing us to find feasible solutions that imply the unbounded integrality gap.

In Section 3.3 we discuss three corollaries of this chapter's main result. First, we show the unsettling non-monotonicity property of the association scheme SDP (effectively, that you can take an instance for which the SDP provides a non-zero lower bound and, by adding extra cities to visit in a way that preserves metric and symmetric edge costs, get arbitrarily cheap SDP solutions). Second, the theorem implies the unbounded integrality gap of the algebraic connectivity SDP and provides a class of weighted graphs that 1) have the same algebraic connectivity as a Hamiltonian cycle while 2) having arbitrarily small edge connectivity. Third, our argument can be extended to show the unbounded integrality gap of an SDP relaxation for the k-cycle cover problem, a generalization of the TSP.

We conclude this chapter with a brief discussion of ways to strengthen this chapter's main result: there are several additional constraints that can be added to the association scheme SDP, but none strengthen the integrality gap. These constraints are based on hierarchies, association-scheme-based linear matrix inequalities, and the matrix-tree theorem constraint (2.7).

3.1 Two-Group Simplicial TSP Instances

Throughout this chapter we consider a family of instances with two groups of $d = n/2$ vertices and assume n is even. The costs associated to intergroup edges will be expensive (1), while the costs of intragroup edges, zero. Note that this family of instances corresponds to a **cut semimetric**: a subset $S \subset V$ such that $c_{ij} = 1$ if $\{i, j\} \in \delta(S)$ and $c_{ij} = 0$ otherwise; here we take a subset S such that $|S| = \frac{n}{2}$. Instances in this family are, moreover, **Euclidean TSP instances** in $\mathbb{R}^1$: instances where each city has a location in $\mathbb{R}^1$ and the cost c_{ij} is the Euclidean distance between the locations of city i and city j. In this case, $n/2$ of the cities are located at the point $(0) \in \mathbb{R}^1$, and the remaining $n/2$ cities are located at $(1) \in \mathbb{R}^1$. Note that a polynomial-time approximation scheme is known in the more general case of Euclidean TSP in $\mathbb{R}^2$ (Arora [3] and Mitchell [57]).

It is trivial to solve Euclidean TSP instances in $\mathbb{R}^1$: Suppose your cities are at coordinates $x_1, x_2, ..., x_n \in \mathbb{R}^1$ with $x_1 \leq x_2 \leq \cdots \leq x_n$. An optimal tour costs $2(x_n - x_1)$ and starts by visiting $x_1, x_2, ..., x_n$ and then returning to x_1. Hence showing an unbounded integrality gap on two-group simplicial TSP instances not only implies an unbounded integrality gap on metric, symmetric instances, but also in the highly restrictive setting of Euclidean TSP in $\mathbb{R}^1$.

Explicitly, we will use the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$

The edge costs embedded in this matrix are metric and symmetric.

Throughout this chapter, we reserve U and W to refer to the two groups of vertices, so that $|U| = |W| = d$ and $V = U \sqcup W$. In a Hamiltonian cycle $\delta(U) \geq 2$, so that any feasible solution to the TSP must use the expensive intergroup edges at least twice. We can achieve a tour costing 2 with a tour that starts in U, goes through all the vertices in U, crosses to W, goes through the vertices in W, and then returns to U. Hence $\mathrm{OPT}_{\mathrm{TSP}}(\hat{C}) = 2$.

Finally, we note that two-group simplicial TSP instances have two properties that make them particularly natural tests for SDP relaxations. First, they are extremely symmetric; searching for solutions that respect this symmetry – that can only distinguish between intra- and inter-group edges – makes analyzing SDP relaxations significantly more tractable. Second, they are an extremal test for connectivity. Finding optimal solutions to SDP relaxations on these instances is equivalent to finding solutions that place as little weight as possible on exactly those edges in the cut $\delta(U)$.

3.2 The Unbounded Integrality Gap of the Association Scheme SDP

We now use two-group simplicial TSP instances to prove our first main theorem. Throughout this section, OPT_{SDP} refers to the optimal value to the association scheme SDP (2.3).

Theorem 3.1. *For the association scheme SDP (2.3),*

$$OPT_{SDP}(\hat{C}) \leq \frac{\pi^2}{2n} OPT_{TSP}(\hat{C}).$$

Theorem 3.1 directly implies our first main result about unbounded integrality gaps.

Corollary 3.2. *The association scheme SDP (2.3) has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that*

$$\frac{OPT_{TSP}(C)}{OPT_{SDP}(C)} \leq \alpha$$

for all cost matrices C representing metric and symmetric edge costs.

To prove Theorem 3.1, it suffices to find a feasible solution to the association scheme SDP of cost $\frac{\pi^2}{n}$. We specifically consider solutions respecting the symmetry of $\hat{C}$: matrices $X^{(i)}$ that place a weight of a_i on each intragroup edge and a weight of b_i on each intergroup edge. Moreover, we choose[1] the b_i so as to enforce that the row sums of the $X^{(i)}$ match those of the distance matrices $A_i(\mathcal{C}_n)$ introduced earlier: $X^{(i)}e = A_i(\mathcal{C}_n)e = 2e$ for $i = 1, ..., d-1$ and $X^{(d)}e = A_d(\mathcal{C}_n)e = e$. Since every vertex is incident to $d-1$ edges in its group (with

[1]Note that de Klerk et al. [23] actually show that every feasible solution must satisfy $X^{(i)}e = 2e$ for $i = 1, ..., d-1$ and $X^{(i)}e = e$ for $i = d$ (when n is even). The fact that *every* feasible solution matches these row sums is not something we will need, though we implicitly use it to inform the solutions we search for.

weight a_i) and d edges in the other group (with weight b_i), we have

$$(d-1)a_i + db_i = \begin{cases} 2, & \text{if } i = 1, ..., d-1 \\ 1, & \text{if } i = d. \end{cases}$$

Rearranging for the b_i lets us express the i-th solution matrix of this form as

$$X^{(i)} = \left(\begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \otimes J_d \right) - a_i I_n, \quad b_i = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_i, & \text{if } i = 1, ..., d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_i, & \text{if } i = d, \end{cases} \tag{3.1}$$

where we subtract $a_i I_n$ so that the diagonal is zero. The cost of such a solution is entirely determined by the $(n/2)^2$ intergroup edges, each of cost b_1. Each edge is accounted for twice in trace$(\hat{C}X^{(1)})$, but the objective scales by $1/2$, so the cost of this solution is

$$\left(\frac{n}{2}\right)^2 b_1.$$

Theorem 3.1 then will follow from the claim below.

Claim 3.3. *Choosing the parameters*

$$a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right), \quad i = 1, ..., d,$$

so that

$$b_i = \begin{cases} \frac{2}{n}\left(1 - \cos\left(\frac{\pi i}{d}\right)\right), & \textit{if } i = 1, ..., d-1, \\ \frac{2}{n}, & \textit{if } i = d, \end{cases}$$

leads to a feasible solution for the association scheme SDP (2.3) with matrices $X^{(i)}$ as given in Equation (3.1).

In particular $b_1 = \frac{2}{n}\left(1 - \cos\left(\frac{\pi}{d}\right)\right)$. Basic facts from calculus will show that this is roughly $\frac{1}{n^3}$, so that the cost of our solution $(n/2)^2 b_1$ is roughly $\frac{1}{n}$, which gets arbitrarily small with n; because the optimal TSP solution always costs 2, we get the unbounded integrality gap.

The main work in proving this claim involves showing that the $X^{(i)}$ satisfy the semidefinite constraints. We first characterize the choices of the a_i that lead to feasible association scheme SDP solutions of the form in Equation (3.1). To do so we exploit the symmetry of matrices in the form of Equation (3.1) to rewrite the semidefinite constraints on the $X^{(i)}$ as linear inequality constraints on the a_i which ensure that all eigenvalues of the terms $I + \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(i)}$ are nonnegative.

Proposition 3.4. *For the association scheme SDP (2.3), solutions of the form*

$$X^{(i)} = \left(\begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \otimes J_d \right) - a_i I_n \quad \textit{where} \quad b_i = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_i, & \textit{if } i = 1, ..., d-1, \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_i, & \textit{if } i = d, \end{cases}$$

for $i = 1, ..., d$ are feasible if and only if

$$\begin{cases} -\frac{2}{n-2} \leq \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i \leq 1, & k = 1, ..., d \\ \sum_{i=1}^{d} a_i = 1 & \\ 0 \leq a_i \leq \frac{4}{n-2}, & i = 1, ..., d-1 \\ 0 \leq a_d \leq \frac{2}{n-2}. & \end{cases}$$

Proof. First, note that the final two constraints are equivalent to the constraint that the $X^{(i)}$ are nonnegative: The $X^{(i)}$ are nonnegative if and only if $a_i \geq 0, b_i \geq 0$, for $i = 1, ..., d$. The constraints $a_i \geq 0$ are explicit in the linear program, and $b_i \geq 0$ is equivalent to $a_i \leq \frac{4}{n-2}, i = 1, ..., d-1$ and $a_d \leq \frac{2}{n-2}$.

Similarly, the constraint that the $X^{(i)}$ sum to $J - I$ is equivalent to $\sum_{i=1}^{d} a_i = 1$ and

$\sum_{i=1}^{d} b_i = 1$. However, $\sum_{i=1}^{d} b_i = 1$ follows from requiring $\sum_{i=1}^{d} a_i = 1$:

$$\begin{aligned}\sum_{i=1}^{d} b_i &= \sum_{i=1}^{d-1} \left(\frac{4}{n} - \left(1 - \frac{2}{n}\right) a_i\right) + \left(\frac{2}{n} - \left(1 - \frac{2}{n}\right) a_i\right) \\ &= (d-1)\frac{4}{n} + \frac{2}{n} - \left(1 - \frac{2}{n}\right) \sum_{i=1}^{d} a_i \\ &= 2 - \frac{2}{n} - \left(1 - \frac{2}{n}\right) \\ &= 1.\end{aligned}$$

It remains to show that the k-th semidefinite constraint, for $1 \leq k \leq d$, is equivalent to

$$-\frac{2}{n-2} \leq \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i \leq 1.$$

The k-th semidefinite constraint is:

$$I + \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(i)} \succeq 0.$$

Using properties of the Kronecker product from Theorem 1.2 and the structure of our $X^{(j)}$, we simplify this:

$$\begin{aligned}I_n + \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(i)} &= I_n + \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) \left(\left(\left(\begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix} \otimes J_d\right) - a_i I_n\right)\right) \\ &= \left(1 - \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i\right) I_n + \left(\sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}\right) \otimes J_d \\ &= (1 - a^{(k)}) I_n + \begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d,\end{aligned}$$

where

$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \quad b^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) b_i$$

depend on the full sequences $a_1, ..., a_d, b_1, ..., b_d$ and on k.

To explicitly write the eigenvalues of the k-th semidefinite constraint, we use several helpful facts from linear algebra:

- The pq eigenvalues of $A \otimes B$ with $A \in \mathbb{R}^{p \times p}$ and $B \in \mathbb{R}^{q \times q}$ are $\lambda_i(A)\lambda_j(B)$ for $i = 1, ..., p$ and $j = 1, ..., q$. See Theorem 1.1.
- The rank one matrix $J_d = ee^T$, with e of dimension d, has one eigenvalue d corresponding to eigenvector e, and all other eigenvalues are zero. (Choose, e.g., any $d-1$ linearly independent vectors that are orthogonal to e.)
- $\lambda(A)$ is an eigenvalue of A with eigenvector v if and only if $\lambda(A) + c$ is an eigenvalue of $A + cI$ with eigenvector v. This follows by direct computation.
- The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are $a + b$ and $a - b$ with respective eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

From these facts, we obtain that the eigenvalues of $I + \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) X^{(i)}$ are:

$$1 - a^{(k)}, \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} + b^{(k)}\right), \quad \text{and } 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} - b^{(k)}\right).$$

For example, $1 - a^{(k)}$ has multiplicity $n - 2$. It corresponds to the $d - 1$ zero eigenvalues of J_d, each of which gives rise to 2 zero eigenvalues of $\begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d$.

Therefore, the k-th semidefinite constraint of the association scheme SDP (2.3) holds if and only if the following inequalities hold:

$$1 - a^{(k)} \geq 0, \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} + b^{(k)}\right) \geq 0, \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} - b^{(k)}\right) \geq 0. \tag{3.2}$$

We have thus derived a system of inequalities on the a_i, b_i that, if satisfied, imply a set of feasible solutions to the association scheme SDP. We can further simplify these by writing the b_i in terms of the a_i. As in Proposition 2.4, we begin by writing the sum $b^{(k)}$ so that we can use Lemma 2.5. We compute

$$\begin{aligned}
b^{(k)} &= \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) b_i \\
&= \left(\sum_{i=1}^{d-1} \cos\left(\frac{2\pi ik}{n}\right)\left(\frac{4}{n} - \left(1-\frac{2}{n}\right)a_i\right)\right) + \cos\left(\frac{2\pi dk}{n}\right)\left(\frac{2}{n} - \left(1-\frac{2}{n}\right)a_d\right) \\
&= \frac{4}{n}\left(\sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right)\right) - \left(1-\frac{2}{n}\right)\left(\sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right)a_i\right) - \cos(\pi k)\left(\frac{2}{n}\right)
\end{aligned}$$

Using Lemma 2.5:

$$\begin{aligned}
&= \frac{4}{n}\left(\frac{-1+(-1)^k}{2}\right) - \left(1-\frac{2}{n}\right)a^{(k)} - (-1)^k\left(\frac{2}{n}\right). \\
&= -\left(1-\frac{2}{n}\right)a^{(k)} - \frac{2}{n}.
\end{aligned}$$

We use this relationship to simplify the second and third inequalities in Equation (3.2) by writing them only in terms of $a^{(k)}$. We obtain

$$1 - a^{(k)} + \frac{n}{2}(a^{(k)} + b^{(k)}) = 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} - \left(1-\frac{2}{n}\right)a^{(k)} - \frac{2}{n}\right) = 0,$$

and

$$1 - a^{(k)} + \frac{n}{2}(a^{(k)} - b^{(k)}) = 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} + \left(1-\frac{2}{n}\right)a^{(k)} + \frac{2}{n}\right) = 2 + (n-2)a^{(k)}.$$

Hence, the three inequalities in Equation (3.2) become

$$-\frac{2}{n-2} \leq a^{(k)} \leq 1,$$

and these inequalities are equivalent to ensuring that the k-th semidefinite constraint of the association scheme SDP hold. □

We now show that our choice of the a_i leads to $X^{(i)}$ that are feasible for the association scheme SDP, which will prove Claim 3.3:

$$a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right), \qquad i = 1, ..., d.$$

As argued above, to show feasibility we need only verify that the constraints of Proposition 3.4 hold. Notice that $-1 \leq \cos(\pi i/d) \leq 1$ so that, for $i = 1, ..., d-1$, we have $0 \leq a_i \leq \frac{4}{n-2}$. Moreover, $a_d = 0$. Hence we need only show that $\sum_{i=1}^d a_i = 1$ and that the $a^{(k)}$ live in the appropriate range.

Claim 3.5. *For* $a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right)$,

$$\sum_{i=1}^{d} a_i = 1.$$

Proof. We directly compute $\sum_{i=1}^d a_i$ using Lemma 2.5 with $k = 1$. Then:

$$\sum_{i=1}^{d} a_i = \frac{2}{n-2}\sum_{i=1}^{d}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right) = \frac{2}{n-2}(-1 + d) = 1.$$

□

Claim 3.6. *For* $a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right)$,

$$a^{(k)} = \begin{cases} \frac{d-2}{n-2}, & \textit{if } k = 1, \\ -\frac{2}{n-2}, & \textit{otherwise.} \end{cases}$$

Proof. As in Proposition 2.4, we use the product-to-sum identity for cosines and then do casework using Lemma 2.5. We have:

$$\begin{aligned}
a^{(k)} &= \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i \\
&= \frac{2}{n-2}\sum_{i=1}^{d}\left(\cos\left(\frac{2\pi ik}{n}\right) + \cos\left(\frac{2\pi ik}{n}\right)\cos\left(\frac{\pi i}{d}\right)\right) \\
&= \frac{2}{n-2}\sum_{i=1}^{d}\left(\cos\left(\frac{2\pi ik}{n}\right) + \frac{1}{2}\cos\left(\frac{2\pi i(k+1)}{n}\right) + \frac{1}{2}\cos\left(\frac{2\pi i(k-1)}{n}\right)\right)
\end{aligned}$$

We cannot apply Lagrange's trigonometric identity from Lemma 2.5 only when $k = 1$, so that

$$= \begin{cases} \frac{2}{n-2}\left(\frac{-1+(-1)^k}{2} + \frac{-1+(-1)^{k+1}}{4} + \frac{-1+(-1)^{k-1}}{4}\right), & \text{if } k > 1 \\ \frac{2}{n-2}\left(-1 + 0 + \frac{1}{2}d\right), & \text{if } k = 1 \end{cases}$$

$$= \begin{cases} -\frac{2}{n-2}, & \text{if } k > 1 \\ \frac{d-2}{n-2}, & \text{if } k = 1. \end{cases}$$

□

That our solutions from Claim 3.3 are feasible now follows immediately.

Proof (of Claim 3.3). Claim 3.3 follows from Proposition 3.4 and Claims 3.5 and 3.6. □

We are now able to prove our main theorem, using solutions of the form in Equation (3.1) with

$$a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right), i = 1, ..., d.$$

Theorem 3.1 For the association scheme SDP (2.3), $OPT_{SDP}(\hat{C}) \leq \frac{\pi^2}{2n} OPT_{TSP}(\hat{C})$.

Proof. Earlier we saw that a feasible solution of the form in Equation (3.1) had cost $\frac{n^2}{4}b_1$, while $\mathrm{OPT}_{\mathrm{TSP}}(\hat{C}) = 2$. Hence, assuming a feasible solution, we can bound

$$\frac{\mathrm{OPT}_{\mathrm{SDP}}(\hat{C})}{\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})} \leq \frac{n^2 b_1}{8}.$$

We have found a feasible solution with parameter

$$a_1 = \frac{2}{n-2}\left(\cos\left(\frac{\pi}{d}\right) + 1\right)$$

so that

$$b_1 = \frac{4}{n} - \left(1 - \frac{2}{n}\right) \frac{2}{n-2} \left(\cos\left(\frac{\pi}{d}\right) + 1\right) = \frac{2}{n}\left(1 - \cos\left(\frac{\pi}{d}\right)\right).$$

Using Taylor series with remainder,

$$\cos\left(\frac{\pi}{d}\right) = 1 - \frac{\pi^2}{2d^2} + \frac{1}{4!}\frac{\pi^4}{d^4}\cos\left(\xi_{1/d}\right) \geq 1 - \frac{\pi^2}{2d^2},$$

where $\xi_{1/d} \in [0, \frac{1}{d}]$

Hence, we bound:

$$\begin{aligned} \frac{\text{OPT}_{\text{SDP}}(\hat{C})}{\text{OPT}_{\text{TSP}}(\hat{C})} &\leq \frac{n^2 b_1}{8} \\ &\leq \frac{n^2}{8}\frac{2}{n}\left(\frac{\pi^2}{2d^2}\right) \\ &= \frac{\pi^2}{2n}. \end{aligned}$$

□

We note the following:

Remark 3.7. *For the examples we have shown where the association scheme SDP gives an arbitrarily bad approximation, the subtour LP provides a perfect solution. Indeed, the subtour elimination constraints applied to $\delta(U)$ imply that the optimal solution to the subtour LP (with input costs $c_{ij} = \hat{C}_{ij}$) matches OPT_{TSP}. Figure 2.1, conversely, provides a worst-case instance for the subtour LP where the association scheme SDP outperforms the subtour LP. Hence we might hope that cases where the association scheme SDP does well may correspond to cases where the linear program does poorly, and vice versa. This suggests incorporating the subtour elimination constraints into the association scheme SDP or running both the association scheme SDP and subtour LP and taking the best result.*

3.3 Corollaries of Theorem 3.1

In this section, we discuss three corollaries of Theorem 3.1. First we show the counterintuitive result that adding vertices (in a way that retains costs being metric) can arbitrarily decrease the cost of some solutions to the association scheme SDP. We state this as a *non-monotonicity* property that contrasts with both TSP and subtour LP solutions. We then show that Theorem 3.1 implies the unbounded integrality gap of two other SDP relaxations: the algebraic connectivity SDP relaxation of the TSP, and a generalization of the association scheme SDP relaxation for a generalization of the TSP (the k-cycle cover problem).

Non-Monotonicity of the Association Scheme SDP (2.3)

Consider an optimization problem whose variables correspond to edges of the complete graph K_n and whose input consists of edge costs and a size n. Let $S \subset [n]$ be a subset of the vertices. Let OPT denote the cost of the optimal solution to the optimization problem on the full set of vertices, and let OPT$[S]$ denote the the cost of the optimal solution to the optimization problem *induced* on the set S. Formally, if C denotes the matrix of edge costs corresponding to the original input, then the induced problem on S uses the edge cost matrix $C[S]$, the principle submatrix of C obtained by deleting the rows and columns in $[n]\backslash S$. If OPT$[S] \leq$ OPT for all possible input costs, values of n, and subsets S, we say that the the optimization property has a *monotonicity* property.

The TSP (as usual, assuming metric and symmetric edge costs) is well-known to be monotonic (this can be seen as an application of *shortcutting.* See Section 2.4 of Williamson and Shmoys [79] for details of shortcutting.) Moreover, Shmoys and Williamson [73] show that the subtour LP is also monotonic. Our example shows that the association scheme SDP of de Klerk et al. [23], however, is not: the cost of our association scheme SDP solutions get arbitrarily small as n grows, and our instance on n' vertices can be viewed as induced by a larger instance on $n > n'$ vertices[2].

Corollary 3.8. *The association scheme SDP (2.3) is not monotonic.*

The Unbounded Integrality Gap of the Algebraic Connectivity SDP (2.2)

Theorem 2.4 summarized de Klerk et al. [23]'s result that any feasible solution for the association scheme SDP implies a feasible solution for the algebraic connectivity SDP of the same cost. Theorem 3.1 thus implies the following:

Corollary 3.9. *The algebraic connectivity SDP (2.2) has an unbounded integrality gap.*

To show this result directly, one would need only take our first feasible solution matrix $X^{(1)}$ above and show that it meets the constraints of the algebraic connectivity SDP:

[2]As a technical point, the lack of monotonicity requires showing that, for some n the given cost matrix $\hat{C}$, the association scheme SDP does not admit solutions of cost zero (this is to rule out the case where, for all n, the optimal solution to the association scheme SDP given cost matrix C is zero and hence does not arbitrarily decrease). One way to see that there cannot be solutions of cost zero is to use methods from spectral graph theory. As we will discuss below, any feasible association scheme SDP solution $X^{(1)}$ can be viewed as the weighted adjacency matrix of a graph that must have a sufficiently large algebraic connectivity. Hence, Cheeger's inequality can be used to get a strictly positive lower bound on the value of the edges in $\delta(U)$, the weight of edges of cost 1 in any feasible association scheme SDP solution. See Spielman [75].

$$\begin{array}{lll} \min & \frac{1}{2}\text{trace}\,(CX) & \\ \text{subject to} & Xe = 2e & \\ & X_{ii} = 0, & i = 1, ..., n \\ & 0 \le X_{ij} \le 1, & i, j = 1, ..., n \\ & 2I - X + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0 & \\ & X \in \mathcal{S}^n. & \end{array}$$

The constraint that $X^{(1)}e = 2e$ follows from enforcing that $(d-1)a_1 + db_1 = 2$; that $X^{(1)}_{ii} = 0, 0 \le X^{(1)}_{ij} \le 1$, and $X^{(1)} \in \mathcal{S}^n$ similarly follows from the assumed form of the $X^{(i)}$. By Claim 2.2, showing that

$$2I - X^{(1)} + \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right)(J - I) \succeq 0$$

is equivalent to showing that the algebraic connectivity of $X^{(1)}$ (when viewed the weighted adjacency matrix of a graph) is at least $h_n = 2 - 2\cos\left(\frac{2\pi}{n}\right)$ (the algebraic connectivity of a Hamiltonian cycle).

Proposition 3.10. *Taking*

$$X = X^{(1)} = \left(\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \otimes J_d\right) - a_1 I_n,$$

with $a_1 = \frac{2}{n-2}\left(\cos\left(\frac{\pi}{d}\right) + 1\right)$ *and* $b_1 = \frac{2}{n}\left(1 - \cos\left(\frac{\pi}{d}\right)\right)$ *yields a feasible solution for the algebraic connectivity SDP. Moreover, the algebraic connectivity of* X *is exactly that of an* n*-cycle.*

Proof. By construction, $X^{(1)}$ satisfies all conditions of (2.2) except possibly that

$$2I_n - X^{(1)} + h_n(J_n - I_n) \succeq 0.$$

The eigenvalues of

$$2I_n - X^{(1)} = (2 + a_1)I_n - \left(\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \otimes J_d \right)$$

are $2+a_1$, with multiplicity $n-2$, and $2+a_1-d(a_1 \pm b_1)$, each with multiplicity 1. Simplifying these later eigenvalues, we have the two eigenvalues

$$2 + a_1 - d(a_1 + b_1) = 0, \quad 2 + a_1 - d(a_1 - b_1) = h_n.$$

Hence, the second smallest eigenvalue of $I_n - X^{(1)}$ is indeed h_n. □

These calculations reveal the following surprising result: despite having extremely low edge connectivity, our solutions are "as connected as a Hamiltonian cycle" with respect to algebraic connectivity; they suggest that algebraic connectivity is a poor form of connectivity for TSP applications.

Corollary 3.11. *The algebraic connectivity of $X^{(1)}$, viewed as the weighted adjacency matrix of a complete graph on n vertices, is equal to the algebraic connectivity of Hamiltonian cycle on n vertices.*

The Unbounded Integrality Gap of a k-Cycle Cover SDP

In the TSP, we try to find a minimum-cost cycle that covers all vertices. This problem is generalized by the k-cycle cover problem which involves finding k equally sized cycles that cover all of the vertices (and assumes n is divisible by k). Just as in the TSP, the goal is to do so with minimum-cost. Goemans and Williamson [35] give a 4-approximation algorithm for this problem.

De Klerk et al. [21] notice that the association scheme SDP can be modified to become a relaxation of the k-cycle problem by changing only the objective function. They argue the following:

Proposition 3.12. *The following SDP is a relaxation of the minimum-cost k-cycle cover problem.*

$$
\begin{array}{lll}
\min & \frac{1}{2}\text{trace}\left(CX^{(k)}\right) & \\
\text{subject to} & X^{(j)} \geq 0, & j = 1, \ldots, d \\
& \sum_{j=1}^{d} X^{(j)} = J - I & \\
& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi ij}{n}\right) X^{(j)} \succeq 0, & i = 1, \ldots, d \\
& X^{(i)} \in S^n, & i = 1, \ldots, d.
\end{array}
\tag{3.3}
$$

We refer to SDP (3.3) as the k**-cycle cover SDP**.

Proof (from de Klerk et al. [21]). The proof uses exactly the same feasible solutions as Proposition 2.4. The key observation is that the k-th distance matrix $A_k(\mathcal{C}_n)$ represents a partition of the vertices into k equally sized cycles. In particular, if $\mathcal{C}_n$ is the cycle $v_1, v_2, ..., v_n, v_1$, then $A_k(\mathcal{C}_n)$ consists of the cycles $v_i, v_{i+k}, v_{i+2k}, ..., v_{i+(n-k)}, v_i$ for $i = 1, 2, ..., k$. See, for example, Figure 3.1. Any k-cycle cover of the vertices can similarly be represented as the k-th distance matrix of some Hamiltonian cycle. □

Since the TSP is a special case of the k-cycle cover problem when $k = 1$, our two-group simplicial TSP instances show that the k-cycle cover SDP also has an unbounded integrality gap. Further, our techniques can be modified to show that the integrality gap is unbounded for every possible k.

Let $\text{OPT}_{\text{SDP}}(C)$ and $\text{OPT}_{\text{k-Cycle}}(C)$ respectively denote the optimal solutions to the k-cycle cover SDP (3.3) and to the k-cycle cover problem for a given matrix of costs C and

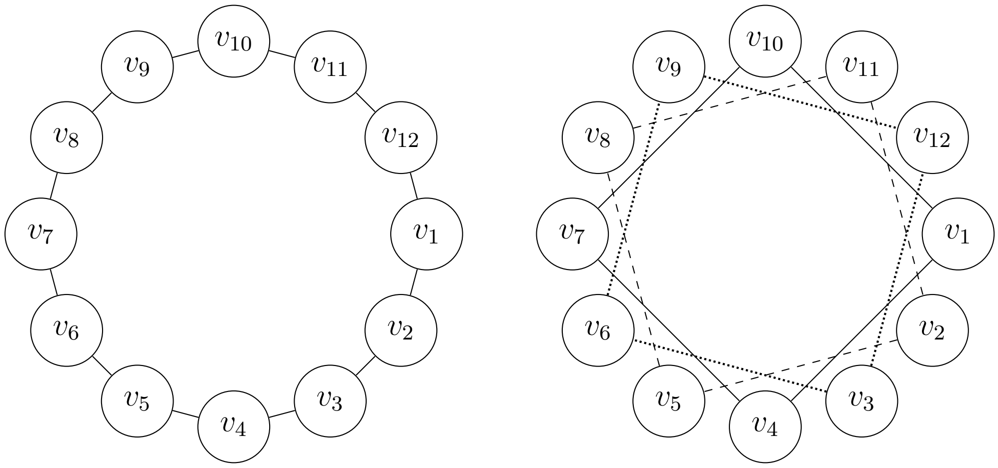

Figure 3.1: The graphs corresponding to $A_1(\mathcal{C}_n)$ and $A_3(\mathcal{C}_n)$ when $n = 12$. Notice that the right graph is a 3-cycle cover, and each cycle is drawn with a different edge style.

fixed $k \geq 2$. Our earlier result generalizes as follows:

Theorem 3.13.

$$OPT_{SDP}(\hat{C}) \leq \frac{\pi^2}{n} \frac{k}{k+1} OPT_{k\text{-}Cycle}(\hat{C}).$$

Corollary 3.14. *For every $k \geq 1$, the k-cycle cover SDP (3.3) has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that*

$$\frac{OPT_{k\text{-}Cycle}(C)}{OPT_{SDP}(C)} \leq \alpha$$

for all cost matrices C.

The full proof of Theorem 3.13 uses almost exactly the same ideas as in the proof of Theorem 3.1: For sufficiently structured solutions, the positive semidefinite constraints devolve to linear inequalities. We find feasible solutions using those linear inequalities and use Taylor series with remainder to obtain the stated bounds. Below we briefly sketch the main ways the proof of Theorem 3.1 is modified; the full proof can be found in the appendix of Gutekunst and Williamson [39].

Proof (Sketch). First, we modify our instances from the proof of Theorem 3.1: On the 1-cycle TSP version, we had two groups of vertices. For k cycles, we consider cost matrices reflecting $k+1$ equally sized groups of vertices. As before, the costs associated to intergroup edges are 1, while the costs of intragroup edges, 0.We consider instances where $n = ck(k+1)$ and will scale n by scaling $c \in \mathbb{N}$ (to reduce casework, we also take c to be even when k is even). Our cost matrix now becomes

$$\hat{C} := (J_{k+1} - I_{k+1}) \otimes J_{ck}.$$

Notice that this parameterization of n means that any integer solution to the k-cycle problem will use cycles of length $c(k+1)$, while each group is of size ck. Hence, any cycle in any integer solution will need to use at least two expensive edges; in analogy to $\mathrm{OPT}_{\mathrm{TSP}}$ in Theorem 3.1, $\mathrm{OPT}_{\text{k-Cycle}}(\hat{C}) = 2k$.

Our proof of Theorem 3.13 then proceeds as in the proof of Theorem 3.1. We find solutions whose structure respects the symmetry of $\hat{C}$: solutions that place a weight a_i on each intragroup edge, a weight b_i on each intergroup edge, and zeros on the diagonal. Now,

$$X^{(i)} = ((b_i J_{k+1} + (a_i - b_i) I_{k+1}) \otimes J_{ck}) - a_i I_n.$$

The analog of Proposition 3.4 is then as follows.

Proposition 3.15. *For the k-cycle cover SDP (3.3), solutions of the form*

$$X^{(i)} = ((b_i J_{k+1} + (a_i - b_i) I_{k+1}) \otimes J_{ck}) - a_i I_n$$

for $i = 1, ..., d$ with

$$b_i = \frac{1}{ck^2} \begin{cases} (2 - (ck-1)a_i)\,, & \text{if } i < d, \\ (1 - (ck-1)a_i)\,, & \text{if } i = d, \end{cases}$$

are feasible if and only if

$$\begin{cases} -\frac{1}{ck-1} \leq \sum_{i=1}^{d} \cos\left(\frac{2\pi ij}{n}\right) a_i \leq 1, & j = 1, ..., d \\ \sum_{i=1}^{d} a_i = 1 \\ 0 \leq a_i \leq \frac{2}{ck-1}, & i = 1, ..., d-1 \\ 0 \leq a_d \leq \frac{1}{ck-1}. \end{cases} \tag{3.4}$$

To prove Theorem 3.13, it then suffices to show that the following choice of a_i, for $i = 1, ..., d$, leads to a feasible SDP solution:

$$a_i = \begin{cases} 0, & \text{if } i \not\equiv_k k \\ \frac{2}{n-k-1}\left(\cos\left(\frac{\pi i}{d}\right) + k\right), & \text{if } i \equiv_k k, i \neq d \\ \frac{1}{n-k-1}\left(\cos\left(\frac{\pi i}{d}\right) + k\right), & \text{if } i = d. \end{cases}$$

Note that these imply that $b_k = \frac{\pi^2}{cd^2(k+1)}$, which is roughly $1/n^3$ since $c(k+1) = \frac{n}{k}$. □

3.4 Strengthening Theorem 3.1: Additional Constraints that Fail to Bound the Integrality Gap on Two-Group Simplicial Instances

In this section, we show that the two-group simplicial instances imply even stronger results. We consider three ways to strengthen the association scheme SDP; the feasible solutions we found in the proof of Theorem 3.1 remain feasible for all.

Adding Additional Semidefinite Constraints

One might wonder if other constraints of the form

$$c_0 I + \sum_{j=1}^{d} c_j X^{(j)} \succeq 0$$

can be added to the Association Scheme SDP to strengthen its integrality gap. For such solutions to be valid, they would need to be satisfied by the distance matrices of a Hamiltonian cycle, but not by the solutions in Claim 3.3. We show that no such constraints exist.

Proposition 3.16. *Let $X^{(1)}, ..., X^{(d)}$ be defined as in Equation 3.1 and let $A^{(j)} = A_j(\mathcal{C}_n)$ be the jth distance matrix of a Hamiltonian cycle. There do not exist any $c_0, c_1, ..., c_d$ such that*

$$c_0 I + \sum_{j=1}^{d} c_j A^{(j)} \succeq 0,$$

but

$$c_0 I + \sum_{j=1}^{d} c_j X^{(j)} \not\succeq 0.$$

This proof is similar to parts of the proof of Theorem 3.1 (and, indeed, is a more general way to show that the solutions in Claim 3.3 satisfy the positive semidefinite constraints of the association scheme SDP). As in Theorem 3.1, we will appeal to symmetry to compute the spectrum of both $\sum_{j=1}^{d} c_j X^{(j)}$ and $\sum_{j=1}^{d} c_j A^{(j)}$. We will show that, considering the smallest eigenvalues of each,

$$\lambda_{min}\left(\sum_{j=1}^{d} c_j A^{(j)}\right) \leq \lambda_{min}\left(\sum_{j=1}^{d} c_j X^{(j)}\right).$$

Doing so implies Proposition 3.16 holds.

Claim 3.17. *The spectrum (i.e. eigenvalue/eigenvector pairs λ_i/v_i) of $X^{(j)}$, as defined in Proposition 3.4, is:*

$$v_1 = e_n, \quad \lambda_1 = d(a_j + b_j) - a_j = \begin{cases} 2 & j < d, \\ 1, & j = d, \end{cases}$$

$$v_2 = (1, -1)^T \otimes e_d, \quad \lambda_2 = d(a_j - b_j) - a_j = \begin{cases} 2\cos(\frac{\pi j}{d}) & j < d \\ -1, & j = d, \end{cases}$$

$$\lambda_3, ..., \lambda_n = -a_j = -\frac{2}{n-2}\left(\cos(\frac{\pi j}{d}) + 1\right).$$

For $\lambda_3, ..., \lambda_n$ we can choose the corresponding eigenvectors to be any $n-2$ linearly independent real vectors orthogonal to v_1 and v_2 (and orthogonal to themselves).

Proof. The eigenvectors $v_1, ..., v_n$ follow by direct computation and the same properties of Kronecker products used in proving Proposition 3.4. The eigenvalue simplifications follow by arithmetic, and recalling that Proposition 3.4 enforces

$$(d-1)a_j + db_j = \begin{cases} 2, & j < d, \\ 1, & j = d. \end{cases}$$

Then:

$$d(a_j + b_j) - a_j = (d-1)a_j + db_j = \begin{cases} 2, & j < d, \\ 1, & j = d. \end{cases}$$

And

$$
\begin{aligned}
d(a_j - b_j) - a_j &= (d-1)a_j + db_j - nb_j \\
&= \begin{cases} 2 - nb_j, & j < d, \\ 1 - nb_d, & j = d. \end{cases} \\
&= \begin{cases} 2\cos(\frac{\pi j}{d}) & j < d, \\ -1, & j = d. \end{cases}
\end{aligned}
$$

□

To compute the spectra of the $A^{(j)}$, we use Lemma 1.3 which implies:

Claim 3.18. *The spectrum of each $A^{(j)}$ is:*

$$\mu_t = 2\cos\left(\frac{2\pi jt}{n}\right), \qquad t = 1, ...n, j = 1, ..., d-1,$$

and

$$\mu_t = \cos\left(\frac{2\pi jt}{n}\right), \qquad t = 1, ...n,$$

where the t-th eigenvalue corresponds to eigenvector

$$u_t = (1, w_t, w_t^2, ..., w_t^{n-1})^T, \qquad w_j = e^{2\pi t\sqrt{-1}/n}.$$

Proof (of Proposition 3.16). Recall that we want to show that

$$\lambda_{min}\left(\sum_{j=1}^{d} c_j X^{(j)}\right) \geq \lambda_{min}\left(\sum_{j=1}^{d} c_j A^{(j)}\right).$$

It will be convenient to rescale

$$A^{(d)} = 2A^{(d)}, \qquad X^{(d)} = 2X^{(d)}, \qquad c_d = c_d/2$$

(just for this proof), so that we don't have to consider a special case for the eigenvalue of the $j = d$ term in the summands.

Note that we have the same basis of eigenvectors for all matrices $X^{(1)}, ..., X^{(d)}$, so that they remain eigenvectors of $\sum_{j=1}^{d} c_j X^{(j)}$. The eigenvalues of this sum are thus:

$$\begin{cases} \lambda_1 = 2\sum_{j=1}^{d} c_j, & v_1 = e_n \\ \lambda_2 = 2\sum_{j=1}^{d} c_j \cos(\frac{\pi j}{d}), & v_2 = (1, -1)^T \otimes e_d \\ \lambda_3 = -\frac{2}{n-2}\sum_{j=1}^{d} c_j(\cos(\frac{\pi j}{d}) + 1) = \frac{-1}{n-2}(\lambda_1 + \lambda_2), & v_3, ..., v_n. \end{cases}$$

The eigenvalues of $\sum_{j=1}^{d} c_j A^{(j)}$ are similarly

$$\mu_t = 2\sum_{j=1}^{d} c_j \cos\left(\frac{2\pi jt}{n}\right), \qquad t = 1, ..., n,$$

where μ_t is associated with eigenvector $u_t = (1, w_t, w_t^2, ..., w_t^{n-1})^T$.

Note that $\mu_n = \lambda_1$ and $\mu_1 = \lambda_2$. Hence, for $\lambda_{min}\left(\sum_{j=1}^{d} c_j X^{(j)}\right) < \lambda_{min}\left(\sum_{j=1}^{d} c_j A^{(j)}\right)$, we need $\lambda_3 < \min\{\lambda_1, \lambda_2\}$. To finish proving the proposition, we need only show that, for any $m \in \mathbb{R}$,

$$\mu_1, ..., \mu_n \geq m \text{ implies } \lambda_3 \geq m.$$

To show this, we argue that λ_3 is a convex combination of $\mu_1, ..., \mu_n$. Specifically:

$$\begin{aligned} \frac{1}{n-2}\sum_{i=2}^{n-1} \mu_i &= \frac{1}{n-2}\sum_{i=1}^{n} \mu_i - \frac{1}{n-2}(\mu_1 + \mu_n) \\ &= \frac{1}{n-2}\left(\sum_{i=1}^{n} 2\sum_{j=1}^{d} c_j \cos\left(\frac{2\pi ji}{n}\right)\right) - \frac{1}{n-2}(\lambda_2 + \lambda_1) \\ &= \frac{2}{n-2}\left(\sum_{j=1}^{d} c_j \sum_{i=1}^{n} \cos\left(\frac{2\pi ji}{n}\right)\right) - \frac{1}{n-2}(\lambda_2 + \lambda_1). \end{aligned}$$

By Lagrange's trigonometric identity (see, e.g., the proof of Lemma 2.5 taking $m = n$ instead of $m = d$), $\sum_{i=1}^{n} \cos\left(\frac{2\pi ji}{n}\right) = 0$, so that

$$= -\frac{1}{n-2}\left(\lambda_2 + \lambda_1\right)$$
$$= \lambda_3,$$

□

Adding a Matrix-Tree Theorem Constraint

De Klerk et al. [23] also provide the matrix-tree theorem constraint (2.7):

$$\det\left((2I - X)_{-1}\right) \geq n.$$

In the notation of the association scheme SDP, this constraint is equivalent to

$$\sum_{T \in \mathcal{T}_G} \prod_{\{i,j\} \in T} (X^{(1)})_{ij} \geq n,$$

where $\mathcal{T}_G$ is the set of spanning trees of G. This constraint, however, does not reduce the integrality gap of the association scheme SDP.

Proposition 3.19. *The association scheme SDP (2.3) with the addition of the matrix-tree theorem constraint (2.7) still has an unbounded integrality gap.*

Proof. To prove this result, we show that the bad instances constructed in Theorem 3.1 do not violate the matrix-tree theorem constraint. In those instances,

$$X^{(1)} = \left(\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix} \otimes J_d\right) - a_1 I_n,$$

where

$$a_1 = \frac{2}{n-2}\left(\cos\left(\frac{\pi}{d}\right) + 1\right), \quad b_1 = \frac{2}{n}\left(1 - \cos\left(\frac{\pi}{d}\right)\right).$$

Let $\mathcal{T}'$ be the set of spanning trees with a single edge of weight b_1. We will show that the aggregate weight of just these trees is enough to imply the matrix-tree theorem constraint.

Claim 3.20. $\sum_{T\in\mathcal{T}'}\prod_{\{i,j\}\in T}(X^{(1)})_{ij} = \left(\frac{a_1 n}{2}\right)^{n-2} b_1$.

This claim follows from two facts. First, for any $T \in \mathcal{T}'$,

$$\prod_{\{i,j\}\in T}(X^{(1)})_{ij} = a_1^{n-2} b_1$$

so that

$$\sum_{T\in\mathcal{T}'}\prod_{\{i,j\}\in T}(X^{(1)})_{ij} = |\mathcal{T}'| a_1^{n-2} b_1.$$

Second, we compute $|\mathcal{T}'|$. To do so, we note that a tree $T \in \mathcal{T}'$ consists of any spanning tree on the vertices of the first group (of $n/2$ vertices), any spanning tree on the vertices of the second group (also of $n/2$ vertices), and any arbitrary intragroup edge (of which there are $n^2/4$ choices). It is well known that there are $\left(\frac{n}{2}\right)^{\frac{n}{2}-2}$ spanning trees on the complete graph on $n/2$ vertices (see, e.g., See Theorem VI.30 in Tutte [76]). Thus

$$|\mathcal{T}'| = \left(\frac{n}{2}\right)^{\frac{n}{2}-2}\left(\frac{n}{2}\right)^{\frac{n}{2}-2}\frac{n^2}{4} = \left(\frac{n}{2}\right)^{n-2}.$$

Claim 3.21. *For n sufficiently large, $\frac{a_1 n}{2} \geq \frac{3}{2}$.*

Note that

$$\frac{n}{2}a_1 = \frac{n}{n-2}\left(\cos\left(\frac{\pi}{d}\right) + 1\right) \geq \cos\left(\frac{\pi}{d}\right) + 1 \geq \frac{3}{2},$$

provided that n is sufficiently large for $\cos\left(\frac{\pi}{d}\right) \geq \frac{1}{2}$.

Claim 3.22. *There exists some constant $c > 0$ such that, for all n sufficiently large,*

$$b_1 \geq \frac{c}{n^3}.$$

Taylor series bounds show that

$$\cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Hence

$$\begin{aligned}
b_1 &= \frac{2}{n}\left(1 - \cos\left(\frac{\pi}{d}\right)\right) \\
&\geq \frac{2}{n}\left(1 - \left(1 - \frac{\pi^2}{2d^2} + \frac{\pi^4}{24d^4}\right)\right) \\
&= \frac{c'}{n^3} - \frac{c''}{n^5} \geq \frac{c}{n^3},
\end{aligned}$$

for constants $c, c', c'' > 0$.

Putting these three claims together,

$$\begin{aligned}
\frac{1}{n}\sum_{T \in \mathcal{T}_G} \prod_{\{i,j\} \in T} (X^{(1)})_{ij} &\geq \frac{1}{n}\sum_{T \in \mathcal{T}'} \prod_{\{i,j\} \in T} (X^{(1)})_{ij} \\
&= \frac{1}{n}\left(\frac{a_1 n}{2}\right)^{n-2} b_1 \\
&\geq \left(\frac{3}{2}\right)^{n-2} \frac{c}{n^4}.
\end{aligned}$$

This ratio grows arbitrarily large (as can be made precise, e.g., with L'Hôpitals rule), so that for n sufficiently large it exceeds 1 and

$$\sum_{T \in \mathcal{T}_G} \prod_{\{i,j\} \in T} (X^{(1)})_{ij} \geq n.$$

□

Strengthened Relaxations via Hierarchies

Several hierarchies exist that strengthen convex relaxations of combinatorial optimization problems, including those of Sherali and Adams [72], Lovász and Schrijver [55], and Lasserre

[53]. These hierarchies iteratively add constraints to the relaxation; after sufficiently many iterations, the surviving feasible solutions correspond exactly to convex combinations of integer solutions. See Chlamtac and Tulsiani [13] for a detailed survey.

Cheung [12], for example, applies hierarchies to show that certain feasible solutions for the subtour LP survive applying the Lovász and Schrijver hierarchy any constant number of times. In particular, those solutions violated certain constraints (specifically, *2-matching inequalities*) satisfied by Hamiltonian cycles. One might analogously wonder how long our solution survives iteratively adding constraints to an appropriate linear program. $X^{(1)}$ is not feasible for the subtour LP for sufficiently large n, so that it trivially doesn't survive any rounds of these hierarchies applied to the subtour LP. In contrast, it can be shown that the feasible $X^{(1)}$ we found is in the convex hull of cycle covers. Hence our solution would survive arbitrarily many rounds of any of these hierarchies applied to a linear program obtained by using only the degree constraints of the subtour LP.

It is also known that at some level of the Lasserre, Sherali-Adams, or Lovász-Schrijver hierarchies [53, 55, 72] the corresponding SDPs give the convex hull of tours. It would be of interest to know if a constant integrality gap can be obtained at some constant level of the hierarchy.

CHAPTER 4

TOWARDS GENERAL SIMPLICIAL TSP INSTANCES: SYMMETRY REDUCTION STRENGTHENS THE ASSOCIATION SCHEME SDP

In Chapter 3, we concluded by discussing several potential approaches for strengthening the association scheme SDP. All of these techniques failed to eliminate our bad instances, and hence failed to produce an SDP with bounded integrality gap. In this chapter, we discuss one approach that is promising because it is robust to the feasible solutions from Claim 3.3 of Chapter 3: the symmetry reduction SDP (2.5).

In Chapter 5, we will still show that the symmetry reduction SDP still has an unbounded integrality gap. To do so, however, we will need to generalize our two-group simplicial TSP instances. Moreover, the feasible solutions to the symmetry reduction SDP look substantially different than those of the association scheme SDP: In the latter, we had $n/2$ matrix variables, each $n \times n$ and designed to emulate the distance matrices of a Hamiltonian cycle. Now, we have a single matrix variable encoding the result of a two-step process: it begins with an $n^2 \times n^2$ matrix variable designed to emulate a block matrix where the i, j-th block has a single non-zero entry encoding the images of i and j under some permutation. Second, it applies symmetry reduction to attain a $(n-1)^2 \times (n-1)^2$ matrix variable.

To build intuition for our main results in Chapter 5, this chapter lays the groundwork for translating between the setting of the association scheme SDP and the symmetry reduction SDP. We specifically sketch this two-step process for two-group simplicial TSP instances. In Section 4.1 we will translate the feasible solutions from in Claim 3.3 to the setting of SDP (2.4) – the SDP with an $n^2 \times n^2$ matrix variable before symmetry reduction – to show its unbounded integrality gap. In Section 4.2, however, we will sketch how they fail to survive the second step of the translation – that they only show that the symmetry reduction SDP has an integrality gap of 2. Doing so motivates how we will use the general simplicial TSP instances in Chapter 5 to show the unbounded integrality gap of the symmetry reduction SDP. Finally, the proofs in Chapter 5 are considerably more computationally involved than their analogs for two-group simplicial TSP instances. The proofs sketched below illustrate the main ideas for those more involved proofs.

4.1 The Unbounded Integrality Gap of Povh and Rendl's SDP

De Klerk and Sotirov [25] provide promising numerical results for their symmetry reduction SDP, which strengthens the association scheme SDP. De Klerk and Sotirov [25] attain their strengthened SDP by performing symmetry reduction on an SDP relaxation of Povh and Rendl [69], which de Klerk, Pasechnik, and Sotirov [23] show coincides with the optimal value of the association scheme SDP. The SDP relaxation of Povh and Rendl [69] is:

$$\begin{array}{lll} \min & \frac{1}{2}\text{trace}\left(\left(C \otimes C_1^{(n)}\right) Y\right) & \\ \text{subject to} & \text{trace}((I_n \otimes E_{jj}^{(n)})Y) = 1, & j = 1, ..., n \\ & \text{trace}((E_{jj}^{(n)} \otimes I_n)Y) = 1, & j = 1, ..., n \\ & \text{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0 & \\ & \text{trace}(J_{n^2}Y) = n^2 & \\ & Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{n^2 \times n^2}. & \end{array} \tag{2.4}$$

Theorem 3 of de Klerk et al. [23] specifically shows that the optimal value of this SDP coincides with the optimal value of the association scheme SDP. In their proof, de Klerk et al. [23] show how to take a solution to the association scheme SDP and convert it to a solution to SDP (2.4). Applying that translation shows directly that SDP (2.4) has an unbounded integrality gap. Those translated solutions, however, do not suffice to show that the stronger, symmetry reduction SDP has an unbounded integrality gap.

We first state the analog of our feasible solutions to the association scheme SDP, defined using the basis of symmetric, circulant matrices.

Theorem 4.1. *Let $d = \frac{n}{2}$ where n is even, and define*

$$a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right), \qquad i = 1, ..., d,$$

and

$$b_i = \begin{cases} \frac{2}{n}\left(1 - \cos\left(\frac{\pi i}{d}\right)\right), & \text{if } i = 1, ..., d-1 \\ \frac{2}{n}, & \text{if } i = d, \end{cases}$$

as in Claim 3.3. Let $A = \sum_{i=1}^{d} a_i C_i$ and $B = \sum_{i=1}^{d} b_i C_i$. Then

$$Y = \frac{1}{2n}\left((I_2 \otimes J_d - I_n) \otimes A + (J_2 - I_2) \otimes J_d \otimes B + 2I_n \otimes I_n\right)$$

is feasible for SDP (2.4).

Example 4.2. Note that Y is an $n^2 \times n^2$ symmetric matrix that can be partitioned into blocks of size $n \times n$. The n blocks on the diagonal are scaled copies of the identity matrix. The other blocks are all scaled copies of A or B. For example, when $n = 6$ we have

$$Y = \frac{1}{2n}\begin{pmatrix} 2I & A & A & B & B & B \\ A & 2I & A & B & B & B \\ A & A & 2I & B & B & B \\ B & B & B & 2I & A & A \\ B & B & B & A & 2I & A \\ B & B & B & A & A & 2I \end{pmatrix}, \quad A = \begin{pmatrix} 0 & a_1 & a_2 & 2a_3 & a_2 & a_1 \\ a_1 & 0 & a_1 & a_2 & 2a_3 & a_2 \\ a_2 & a_1 & 0 & a_1 & a_2 & 2a_3 \\ 2a_3 & a_2 & a_1 & 0 & a_1 & a_2 \\ a_2 & 2a_3 & a_2 & a_1 & 0 & a_1 \\ a_1 & a_2 & 2a_3 & a_2 & a_1 & 0 \end{pmatrix},$$

with B defined analogously to A.

To prove Theorem 4.1, we will reuse several facts from Chapter 3. We extend the definition of the $a^{(k)}$ and $b^{(k)}$ to be for $k = 0, ..., n-1$:

$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \quad b^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) b_i, \quad k = 0, 1, ..., n-1.$$

Note that

$$\cos\left(\frac{2\pi i(n-k)}{n}\right) = \cos\left(2\pi i - \frac{2\pi ik}{n}\right) = \cos\left(\frac{2\pi ik}{n}\right),$$

so that $a^{(k)} = a^{(n-k)}$ and $b^{(k)} = b^{(n-k)}$. We will use the following facts proved in Chapter 3.

Claim 4.3.

1. $\sum_{i=1}^{d} a_i = \sum_{i=1}^{d} b_i = 1$. *Equivalently,* $a^{(0)} = b^{(0)} = 1$ *(see Claim 3.5 and the proof of Proposition 3.4).*
2. $b^{(k)} = -\left(1 - \frac{2}{n}\right) a^{(k)} - \frac{2}{n}$ *(see the proof of Proposition 3.4).*
3. *For* $k = 1, ..., d$,
$$a^{(k)} = \begin{cases} \frac{d-2}{n-2}, & \text{if } k = 1 \\ -\frac{2}{n-2}, & \text{otherwise} \end{cases}$$
(see Claim 3.6).
4. $b_1 \le \frac{4\pi^2}{n^3}$ *(see the proof of Theorem 3.1).*

We first show that Y satisfies each of the constraints of SDP (2.4).

Claim 4.4. $\mathit{trace}((I_n \otimes E_{jj})Y) = 1$ *and* $\mathit{trace}((E_{jj} \otimes I_n)Y) = 1$ *for* $j = 1, ..., n$.

Proof. Each of the n^2 diagonal entries of Y is $\frac{1}{n}$. Both $I_n \otimes E_{jj}$ and $E_{jj} \otimes I_n$ are diagonal matrices with exactly n nonzero entries, all of which are equal to 1. □

Claim 4.5. $\mathit{trace}((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0$.

Proof. The $n \times n$ blocks of Y have sparsity patterns that imply this constraint: I is a diagonal matrix, while A and B have zero diagonal (there is no coefficient of C_0 in the sums defining A and B). □

Claim 4.6. *trace*$(J_{n^2}Y) = n^2$.

Proof. To show this constraint holds, we note that Y is expressed in terms of n^2 blocks, each of size $n \times n$ and each of which is either $\frac{1}{2n}A$, $\frac{1}{2n}B$, or $\frac{1}{n}I$. In the first row of A, we have that $A_{1,i} = a_i = a_{n-i}$ for $i = 1, ..., d-1$, while $A_{1,d} = 2a_d$. Since A is circulant, each of the n rows of A then sums to $2\sum_{i=1}^{d} a_i$. Using the first result of Claim 4.3, the entries in A thus sum to $2n$ so that trace$(J_n \frac{1}{2n}A) = 1$. Analogously, trace$(J_n \frac{1}{2n}B) = $ trace$(J_n \frac{1}{n}I_n) = 1$. That is, each of the n^2 blocks defining Y sums to 1 so that, when we sum all the entries in Y,

$$\text{trace}(J_{n^2}Y) = n^2.$$

□

Claim 4.7. $Y \geq 0$.

Proof. The penultimate constraint follows because $a_i, b_i \geq 0$. □

To show feasibility, we thus must finally show

Claim 4.8. $Y \succeq 0$.

Proof. From Lemma 1.3, we have that the eigenvectors of a circulant matrix with first row $(m_0, m_1, ..., m_{n-1})$ are of the form $v_j = (1, w_j, w_j^2, ..., w_j^{n-1})$ for $j = 0, 1, ..., n-1$ with $w_j = e^{-\frac{2\pi j\sqrt{-1}}{n}}$. The eigenvalue corresponding to v_j is

$$\lambda_j = m_0 + m_1 w_j + m_2 w_j^2 + \cdots m_{n-1} w_j^{n-1}.$$

Hence, v_j is a simultaneous eigenvector of A, B, and I_n. Let λ_j^A and λ_j^B respectively indicate the eigenvalues of A and B corresponding to v_j. Note that

$$w_j^i + w_j^{n-i} = e^{-\frac{2\pi j i\sqrt{-1}}{n}} + e^{-\frac{2\pi j(n-i)\sqrt{-1}}{n}} = 2\cos\left(\frac{2\pi ij}{n}\right).$$

Then since $A_{1,i} = A_{1,n-i} = a_i$ for $i = 1, ..., d-1$ and $A_{1,d} = 2a_d$,

$$\lambda_j^A = \left(\sum_{i=1}^{d-1} a_i(w_j^i + w_j^{n-i})\right) + 2a_d w_j^d = 2\sum_{i=1}^{d} a_i \cos\left(\frac{2\pi ij}{n}\right) = 2a^{(j)}.$$

Similarly, $\lambda_j^B = 2b^{(j)}$.

Recall that

$$Y = \frac{1}{2n}\left(((I_2 \otimes J_d) - I_n) \otimes A + (J_2 - I_2) \otimes J_d \otimes B + 2I_n \otimes I_n\right).$$

By finding a shared set of eigenvectors of $(I_2 \otimes J_d) - I_n$, $(J_2 - I_2) \otimes J_d$ and $2I_n$, we can use properties of the Kronecker product from Theorems 1.1 and 1.2 to explicitly compute the eigenvalues of Y as a function of the $a^{(j)}$ and $b^{(j)}$; results from Claim 4.3 will suffice to show that they are all nonnegative. We will use the following as our shared set of eigenvectors[1]. We first have $u_1 = e^{(n)}$ and $u_2 = (e_1^{(2)} - e_2^{(2)}) \otimes e^{(d)}$. The remaining $u_3, ..., u_n$ are the $n-2$ vectors of the form $e^{(2)} \otimes (e_1^{(d)} - e_i^{(d)})$ and $(e_1^{(2)} - e_2^{(2)}) \otimes (e_1^{(d)} - e_i^{(d)})$ for $i = 2, ..., d$ (in any order). Denote by μ_j^A and μ_j^B the respective eigenvalues of $(I_2 \otimes J_d) - I_n$ and $(J_2 - I_2) \otimes J_d$ associated with u_j. Then

$$\mu_1^A = d-1, \quad \mu_2^A = d-1, \quad \mu_j^A = -1 \text{ otherwise}$$

[1] To find this shared set of eigenvectors, recall that $J_m = e^{(m)}(e^{(m)})^T$ is a rank-1 matrix and that

$$J_m e^{(m)} = e^{(m)}(e^{(m)})^T e^{(m)} = me^{(m)}.$$

The only nonzero eigenvector of J_m is thus $e^{(m)}$ with corresponding eigenvalue m. All other eigenvectors have corresponding eigenvalue zero, and a convenient basis for them is $e_1^{(m)} - e_i^{(m)}$ for $i = 2, ..., m$. Then

$$J_m\left(e_1^{(m)} - e_i^{(m)}\right) = e^{(m)} - e^{(m)} = 0\left(e_1^{(m)} - e_i^{(m)}\right).$$

The vectors $e^{(m)}, e_1^{(m)} - e_2^{(m)}, ..., e_1^{(m)} - e_m^{(m)}$ are linearly independent and so form an eigenbasis for J_m. To extend these to find eigenvectors of $(I_2 \otimes J_d) - I_n$, $(J_2 - I_2) \otimes J_d$ and $2I_n$, we use the eigenvalue facts from the proof of Proposition 3.4 .

and

$$\mu_1^B = d, \quad \mu_2^B = -d, \quad \mu_j^B = 0 \text{ otherwise.}$$

Now note that

$$(((I_2 \otimes J_d) - I_n) \otimes A + (J_2 - I_2) \otimes J_d \otimes B + 2I_n \otimes I_n)\,(u_i \otimes v_j) = (\mu_i^A \lambda_j^A + \mu_i^B \lambda_j^B + 2)(u_i \otimes v_j),$$

so that the eigenvalues of $2nY$ must be the values of $(\mu_i^A \lambda_j^A + \mu_i^B \lambda_j^B + 2)$ over $i = 1, ..., n$ and $j = 0, ..., n-1$. That is,

$$2(\mu_i^A a^{(j)} + \mu_i^B b^{(j)} + 1), \quad i = 1, ..., n; j = 0, ..., n-1.$$

To show $Y \succeq 0$, it suffices to show that these are all nonnegative. For $j = 0$, we have that $a^{(0)} = b^{(0)} = 1$ and thus that

$$\mu_i^A a^{(0)} + \mu_i^B b^{(0)} + 1 = \begin{cases} d - 1 + d + 1 = 2d \geq 0, & i = 1 \\ d - 1 - d + 1 = 0 \geq 0, & i = 2 \\ -1 + 0 + 1 = 0 \geq 0, & i \geq 3. \end{cases}$$

Otherwise, for $j \neq 0$, we have

$$\mu_i^A a^{(j)} + \mu_i^B b^{(j)} + 1 = \begin{cases} (d-1)a^{(j)} + db^{(j)} + 1, & i = 1 \\ (d-1)a^{(j)} - db^{(j)} + 1, & i = 2 \\ -a^{(j)} + 1, & i \geq 3. \end{cases}$$

Using $b^{(k)} = -\left(1 - \frac{2}{n}\right) a^{(k)} - \frac{2}{n}$ from Claim 4.3:

$$= \begin{cases} 0, & i = 1 \\ (n-2)a^{(j)} + 2 & i = 2 \\ -a^{(j)} + 1, & i \geq 3. \end{cases}$$

By the final case of Claim 4.3, for $j = 1, ..., d$,

$$a^{(j)} = \begin{cases} \frac{d-2}{n-2}, & \text{if } j = 1 \\ \\ -\frac{2}{n-2}, & \text{otherwise.} \end{cases}$$

Hence, the eigenvalues are all nonnegative and $Y \succeq 0$. □

Proof (of Theorem 4.1). Claims 4.4 to 4.8 imply Y is feasible for SDP (2.4). □

Corollary 4.9. *The integrality gap of SDP (2.4) is unbounded.*

To show that the integrality gap is unbounded, we recall the cost matrix C defined for two-group simplicial TSP instances,

$$C = \begin{pmatrix} 0 & 1 \\ \\ 1 & 0 \end{pmatrix} \otimes J_d.$$

Proof. The integrality gap of SDP (2.4) is again at least

$$\frac{\mathrm{OPT}_{\mathrm{TSP}}(C)}{\mathrm{OPT}_{\mathrm{SDP\ (2.4)}}(C)} = \frac{2}{\mathrm{OPT}_{\mathrm{SDP\ (2.4)}}(C)}.$$

When computing the cost of the SDP (2.4), we evaluate $\mathrm{trace}\left(\left(C \otimes C_1\right) Y\right)$. The $n^2 \times n^2$ matrix $C \otimes C_1$ consists of $n \times n$ blocks, either of which is an $n \times n$ block of zeros (exactly where Y has an $\frac{1}{2n}A$ block or a $\frac{1}{n}I$ block) or a C_1 (exactly in the $2d^2$ places where Y has a $\frac{1}{2n}B$ block). Hence:

$$\begin{aligned} \mathrm{OPT}_{\mathrm{SDP\ (2.4)}}(C) &\leq \frac{1}{2}\mathrm{trace}\left((C \otimes C_1)Y\right) \\ &= \frac{1}{2}2d^2\frac{1}{2n}\mathrm{trace}(C_1 B) \\ &= \frac{d^2}{2n}2nb_1 \\ &= d^2 b_1 \\ &\leq 4\pi^2\frac{d^2}{n^3}, \end{aligned}$$

using the final result of Claim 4.3. Thus $\text{OPT}_{\text{SDP (2.4)}}(C) \leq c\frac{1}{n}$ for some constant c. Hence the integrality gap is at least

$$\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP (2.4)}}(C)} \geq \frac{2}{c\frac{1}{n}} = \frac{2}{c}n,$$

which grows without bound. □

4.2 Our Solutions Fail to Show an Unbounded Integrality Gap for the Symmetry Reduction SDP

This instance and feasible solution Y do not, however, show that the integrality gap of the symmetry reduction SDP is unbounded. We recall that this SDP has the form

$$\begin{array}{lll}
\min & \text{trace}\left((C[\beta] \otimes \frac{1}{2}C_1^{(n)}[\alpha] + Diag(\bar{c}))Y\right) & \\
\text{subject to} & \text{trace}\left(\left(I_{n-1} \otimes E_{jj}^{(n-1)}\right)Y\right) = 1, & j = 1, ..., n-1 \\
& \text{trace}\left(\left(E_{jj}^{(n-1)} \otimes I_{n-1}\right)Y\right) = 1, & j = 1, ..., n-1 \\
& \text{trace}\left((I_{n-1} \otimes (J_{n-1} - I_{n-1}) + (J_{n-1} - I_{n-1}) \otimes I_{n-1})Y\right) = 0 & \\
& \text{trace}((J_{n-1} \otimes J_{n-1})Y) = (n-1)^2 & \\
& Y \geq 0, Y \succeq 0, Y \in \mathbb{S}^{(n-1)^2 \times (n-1)^2}, &
\end{array} \tag{4.1}$$

where $s, r \in [n]$, $\alpha = [n]\backslash r$ and $\beta = [n]\backslash s$, and $\bar{c} = vec(C_1[\alpha, \{r\}]C[\{s\}, \beta])$. The constraints mirror those of SDP (2.4), and our previous solution is feasible for any instance of the appropriate dimension. Namely

$$Y = \frac{1}{2n}\left((I_2 \otimes J_d - I_n) \otimes A + (J_2 - I_2) \otimes J_d \otimes B + 2I_n \otimes I_n\right)$$

with

$$a_i = \frac{2}{n-2}\left(\cos\left(\frac{\pi i}{d}\right) + 1\right), \quad i = 1, ..., d,$$

$$b_i = \begin{cases} \frac{2}{n}\left(1 - \cos\left(\frac{\pi i}{d}\right)\right), & \text{if } i = 1, ..., d-1 \\ \frac{2}{n}, & \text{if } i = d, \end{cases}$$

$A = \sum_{i=1}^{d} a_i C_i$ and $B = \sum_{i=1}^{d} b_i C_i$ is feasible for any instance on $n+1$ vertices. However, the modified objective function means that it no longer implies an unbounded integrality gap. Instead, we have the following result.

Corollary 4.10. *The integrality gap of the symmetry reduction SDP (2.5) is at least 2.*

Proof. We consider an instance on $n+1$ vertices with two groups of vertices, $\{1, ..., d, d+1\}$ and $\{d+2, ..., n+1\}$. As before, we define the cost of traveling between vertices in the same group to be zero and the cost of traveling between vertices in distinct groups to be 1. By taking $r = s = 1$ we have that

$$C[\beta] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$$

and $C_1^{(n+1)}[\alpha] \le C_1^{(n)}$ entrywise. As in Corollary 4.9, the integrality gap is at least

$$\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP (2.5)}}(C)},$$

and again $\text{OPT}_{\text{TSP}}(C) = 2$. To upper bound the denominator, we note that feasibility of Y implies

$$\begin{aligned} \text{OPT}_{\text{SDP (2.5)}}(C) &\le \text{trace}\left(\left(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\bar{c})\right) Y\right) \\ &= \text{trace}\left(\left(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha]\right) Y\right) + \text{trace}(Diag(\bar{c})Y). \end{aligned}$$

We can bound the first term by Corollary 4.9, since

$$\text{trace}\left(\left(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha]\right) Y\right) \le \text{trace}\left(\left(\left(\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d\right) \otimes \frac{1}{2}C_1^{(n)}\right) Y\right) \le c\frac{1}{n}.$$

We can compute the second term. Note that $C_1[\alpha, \{r\}] = e_1^{(n)} + e_n^{(n)}$ and $C[\{s\}, \beta] = (e_{d+1}^{(n)} + e_{d+2}^{(n)} + ... + e_n^{(n)})^T$. Thus $C_1[\alpha, \{r\}]C[\{s\}, \beta]$ is an $n \times n$ matrix with exactly n ones and all other entries zero. Hence

$$Diag(\bar{c}) = Diag(vec(C_1[\alpha, \{r\}]C[\{s\}, \beta]))$$

is a diagonal matrix with exactly n ones on the diagonal. Since each diagonal entry of Y is $\frac{1}{n}$, we have

$$\text{trace}(Diag(\bar{c})Y) = 1.$$

Putting everything together, we get that

$$\text{OPT}_{\text{SDP (2.5)}}(C) \leq 1 + \frac{c}{n},$$

so that the integrality gap is at least

$$\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP (2.5)}}(C)} \geq \frac{2}{1 + \frac{c}{n}} = \frac{2n}{n + c}$$

for some constant c, which gets arbitrarily close to 2 as n grows. □

Note also that the solution Y is not necessarily optimal for the symmetry reduction SDP and hence this family of instances may in fact still imply an integrality gap larger then 2. Numerical experiments on this family suggest that the optimal solutions to the symmetry reduction SDP have value strictly less than 1 as n grows sufficiently large, but are far less symmetric than the feasible Y we used above.

We highlight a few important pieces of the analysis above. We decomposed the SDP objective function into two terms

$$\text{trace}\left(\left(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha]\right)Y\right) + \text{trace}(Diag(\bar{c})Y).$$

We also took $r = s = 1$ and $\alpha = \beta = [n]\backslash 1$. In this case, the first term is in the objective function is then a modified QAP objective, corresponding to the cost of visiting vertices $2, 3, ..., n$ at positions $2, 3, ..., n$. The second term accounts for the reinsertion cost: if vertices i and j are visited at positions 2 and n respectively, we also incur the costs d_{1i} and d_{j1} when we "reinsert" vertex 1 in at position 1. That we can find solutions where the first term decays like $\frac{1}{n}$ is not surprising; this result is consistent with our previous unbounded integrality gap results where we found a feasible solution spreading low weight across the expensive edges. The second term, which evaluates to 1 on our feasible solutions, is what leads to a bounded integrality gap of 2 and is the bottleneck in our analysis: it costs $\frac{1}{2}\mathrm{OPT}_{\mathrm{TSP}}(C)$ regardless of n.

To show that the integrality gap of the symmetry reduction SDP is unbounded, we instead modify the family of instances considered. We will continue to work with instances that have a finite reinsertion cost. Specifically, we will continue to consider instances where all edge costs are in $\{0, 1\}$ and hence the reinsertion cost can be bounded by 2. We will also continue to consider instances of a similar spirit to the two-group simplicial TSP instances; by considering similar instances, we can expect feasible solutions where the modified QAP term in the objective function decays like $\frac{1}{n}$.

Both of these two ingredients will hold, so that for this new family of instances, we will be able to bound $\mathrm{OPT}_{\mathrm{SDP}\ (2.5)}(C) \leq 2 + \frac{b}{n}$ for some constant b. The key difference, however, is that we will design this new family of instances so that $\mathrm{OPT}_{\mathrm{TSP}}(C)$ grows arbitrarily large. Together these two features will imply the unbounded integrality gap. Finding feasible solutions to these instances will otherwise proceed similarly: we will again find feasible (but not necessarily optimal) solutions that have a simple block structure that respects that of the cost matrix; that can be decomposed into terms, each of which is the Kronecker product of a matrix constructed using J and I and a circulant matrix; and for which we will thus be able to explicitly write down the spectrum.

CHAPTER 5

SIMPLICIAL TSP INSTANCES AND THE SYMMETRY REDUCTION SDP'S INTEGRALITY GAP

In this chapter, we generalize the results of Chapter 3 to a more involved set of instances: simplicial TSP instances. Doing so will allow us to prove our main theorem:

Theorem 5.1. *Let $z \in \mathbb{N}$. Then the integrality gap of the symmetry reduction SDP (2.5) is at least z.*

An immediate corollary is:

Corollary 5.2. *The integrality gap of the symmetry reduction SDP (2.5) is unbounded.*

Section 5.1 describes how we generalize the two-group simplicial TSP instances of Chapter 3, and then we follow the flow of Chapter 4: First, in Section 5.2, we find feasible solutions to the SDP (2.4) which are, in turn, feasible to the symmetry reduction SDP. In Section 5.3 we use these solutions to show that the integrality gap of the symmetry reduction SDP is again unbounded. Finally, in Section 5.4, we show that our feasible solutions/simplicial TSP instances are robust to the triangle inequalities of De Klerk and Sotirov [24] and imply the unbounded integrality gap of SDP (2.6).

5.1 General Simplicial TSP Instances

Following Chapter 4, we begin by generalizing our two-group simplicial TSP instances. We consider an instance with g equally sized groups of $\frac{n}{g}$ vertices. If u, v are two vertices in the same group, then the cost of traveling between u and v is zero; otherwise the cost is 1. Labeling the vertices so that the ith group consists of vertices $\{(i-1)\frac{n}{g}+1, ..., i\frac{n}{g}\}$, the cost matrix is

$$C = (J_g - I_g) \otimes J_{n/g}.$$

These instances are metric and can be viewed as Euclidean TSP in $\mathbb{R}^{g-1}$; we refer to this family of instances as **simplicial TSP instances**: In a regular $g-1$ simplex, there are g extreme points, each pair of which is a distance 1 apart. One way to interpret an instance with g groups is as embedded into a regular $g-1$ simplex in $\mathbb{R}^{g-1}$ where each group of $\frac{n}{g}$ vertices is placed at an extreme point of the simplex. Note also that the instances in Chapter 3 are the special case when $g = 2$.

For an instance in this family, $\text{OPT}_{\text{TSP}}(C) = g$: any optimal solution must visit each group at least once before returning to the start, and such a tour is readily achieved by visiting vertices lexicographically. To prove Theorem 7.1, we will take $g = 2z$. To simplify the our proofs, we thus assume that g is even throughout.

5.2 Feasible Solutions to the Symmetry Reduction SDP

Throughout this chapter, it will be helpful to view the symmetry reduction SDP as an SDP for an $n+1$ vertex instances (with n still even). Doing so allows the symmetry reduction

SDP to be with respect to an $n^2 \times n^2$ matrix variable and the SDP becomes:

$$\begin{array}{lll} \min & \operatorname{trace}\left((C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\bar{c}))Y\right) & \\ \text{subject to} & \operatorname{trace}((I_n \otimes E_{jj}^{(n)})Y) = 1 & j = 1, ..., n \\ & \operatorname{trace}\left(\left(E_{jj}^{(n)} \otimes I_n\right) Y\right) = 1 & j = 1, ..., n \\ & \operatorname{trace}\left((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n) Y\right) = 0 & \\ & \operatorname{trace}((J_n \otimes J_n)Y) = n^2 & \\ & Y \geq 0, Y \succeq 0, Y \in \mathcal{S}^{n^2 \times n^2}. & \end{array} \tag{2.5}$$

We use solutions of the form

$$Y = \frac{1}{2n}\left[(J_g - I_g) \otimes J_{n/g} \otimes B + I_g \otimes J_{n/g} \otimes A + I_g \otimes I_{n/g} \otimes (2I_n - A)\right], \tag{5.1}$$

where

$$A = \sum_{i=1}^{d} a_i C_i, \quad B = \sum_{i=1}^{d} b_i C_i$$

are symmetric circulant matrices defined in terms of parameters $a_1, ..., a_d$ and $b_1, ..., b_d$. We set

$$a_i = \begin{cases} \frac{1}{n-g}\left[2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi i j}{d}\right)\right], & i < d \\ \frac{1}{n-g}\left[1 + \frac{2}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi i j}{d}\right)\right], & i = d. \end{cases}$$

We also set[1]

$$b_i = \begin{cases} \frac{2g-(n-g)a_i}{n(g-1)}, & i < d \\ \frac{g-(n-g)a_i}{n(g-1)}, & i = d. \end{cases}$$

We will often take sums of the a_i or b_i. It will be helpful to note that

$$a_d = \frac{1}{n-g}\left[2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi d j}{d}\right)\right] - \frac{1}{n-g}\left[1 + \frac{2}{g}\sum_{j=1}^{g-1}(g-j)\cos(\pi j)\right]$$

[1]These values come from assuming

$$\left(\frac{n}{g} - 1\right) a_i + \frac{g-1}{g} n b_i = \begin{cases} 2, & i < d \\ 1, & i = d. \end{cases}$$

These are analogous to the degree constraints assumed in Chapter 3.

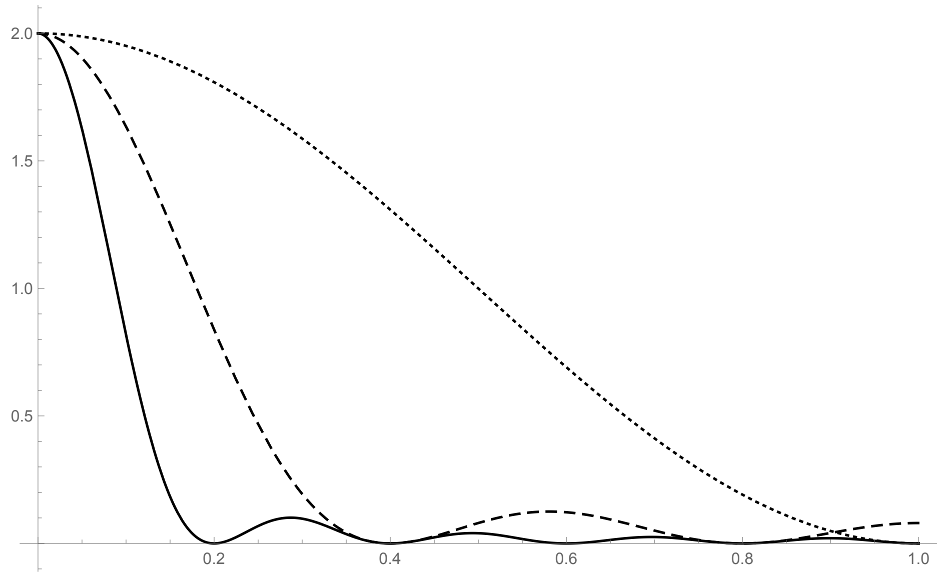

Figure 5.1: $\frac{n-g}{g} a_i$ for $g = 2, 5$, and 10. For each curve (and any value of n), the values $a_1, ..., a_{d-1}$ are taken by sampling the curve at $x = \frac{1}{d}, \frac{2}{d}, ..., \frac{d-1}{d}$; the value of a_d is half the value at $x = 1$. The dotted curve shows $g = 2$, the dashed curve shows $g = 5$, and the remaining curve shows $g = 10$.

and

$$b_d = \frac{2g - (n-g)a_d}{n(g-1)} - \frac{g}{n(g-1)}.$$

Figure 5.1 provides intuition for how the a_i depend on g. The sequence $a_1, a_2, ..., a_d$ can be viewed as a uniformly-spaced sample from a sum of cosines that places a larger weight on smaller indices. As g increases, the proportion of the a_i that are close to zero grows. As in Chapter 4, the only parameter that the symmetry reduction SDP cost depends on will be b_1, and large a_1 implies small b_1.

Note that Y is a large block matrix that respects symmetry of our cost matrix C in the exact same way as in Chapter 4: each diagonal block is $\frac{1}{n} I_n$; everywhere that C has a 0, Y places a block $\frac{1}{2n} A$; everywhere C has a 1, Y has a block $\frac{1}{2n} B$. In the proofs below, it will help to refer to multiple types of blocks of Y. Y can be partitioned into larger blocks of size $\frac{n^2}{g} \times \frac{n^2}{g}$, each of which is either $J_{n/g} \otimes \frac{1}{2n} B$ or $\frac{1}{2n} \left((J_{n/g} \otimes A) + I_{n/g} \otimes (2I_n - A) \right)$; we will refer to these blocks as *major blocks.* The former are off-diagonal, so we will refer to them as *major off-diagonal blocks* while the latter are on the diagonal of Y, so we will refer to them as *major diagonal blocks.* Each of these major blocks consists of $(n/g)^2$ smaller, $n \times n$ blocks, each of which is a $\frac{1}{2n}A$, $\frac{1}{2n}B$, or $\frac{1}{n}I_n$. We will refer to each as a *minor block.* We refer to each of the n blocks of $\frac{1}{n}I_n$ as a *minor diagonal block*, and the remaining $n \times n$ blocks (each of which is a single $n \times n$ block equal to $\frac{1}{2n}A$ or $\frac{1}{2n}B$) as a *minor off-diagonal block.*

Example 5.3. Suppose $g = 3$ and $n = 12$. Pictorially, the minor blocks are those blocks proportional to I_n, A, and B; the major blocks are those delineated below that each consist of 16 minor blocks; those with just B's are the major off-diagonal blocks, while those with I_n's and A's are the major diagonal blocks.

$$Y = \frac{1}{2n}\left(\begin{array}{cccc|cccc|cccc}
2I_n & A & A & A & B & B & B & B & B & B & B & B \\
A & 2I_n & A & A & B & B & B & B & B & B & B & B \\
A & A & 2I_n & A & B & B & B & B & B & B & B & B \\
A & A & A & 2I_n & B & B & B & B & B & B & B & B \\
\hline
B & B & B & B & 2I_n & A & A & A & B & B & B & B \\
B & B & B & B & A & 2I_n & A & A & B & B & B & B \\
B & B & B & B & A & A & 2I_n & A & B & B & B & B \\
B & B & B & B & A & A & A & 2I_n & B & B & B & B \\
\hline
B & B & B & B & B & B & B & B & 2I_n & A & A & A \\
B & B & B & B & B & B & B & B & A & 2I_n & A & A \\
B & B & B & B & B & B & B & B & A & A & 2I_n & A \\
B & B & B & B & B & B & B & B & A & A & A & 2I_n
\end{array}\right).$$

We now show that this solution meets each constraint, mirroring the flow of Chapter 4.

Proposition 5.4. *Y, as given by Equation (5.1), is feasible for SDP (2.4).*

Claim 5.5. *For each $j = 1, ..., n$, we have*

$$trace((I_n \otimes E_{jj})Y) = 1 \quad and \quad trace((E_{jj} \otimes I_n)Y) = 1.$$

Proof. Note that these constraints only impact the diagonal entries of Y, each of which is equal to $\frac{1}{n}$. Each constraints expands to a sum consisting of n terms, each of which is equal to $\frac{1}{n}$, so both hold. □

Claim 5.6. $trace((I_n \otimes (J_n - I_n) + (J_n - I_n) \otimes I_n)Y) = 0.$

Proof. This constraint holds because of Y's sparsity pattern: First note that

$$\text{trace}((I_n \otimes (J_n - I_n))Y) = 0,$$

as each $n \times n$ minor diagonal block of Y is $\frac{1}{n}I_n$, which is diagonal. Second

$$\text{trace}(((J_n - I_n) \otimes I_n)Y) = 0,$$

as every minor off-diagonal block is either $\frac{1}{2n}A$ or $\frac{1}{2n}B$; the matrices A and B are a linear combination of $C_1, ..., C_d$ all of which have every diagonal entry zero. □

Claim 5.7. $trace(J_{n^2}Y) = n^2.$

This proof involves some involved bookkeeping and uses a handful of lemmas. We use $\mathbb{1}_{\{\circ\}}$ to denote the indicator function that is 1 if event $\circ$ happens and zero otherwise.

Lemma 5.8. *Let g be even. Then*

$$\sum_{j=1}^{g-1}(g-j)\mathbb{1}_{\{j\ odd\}} = \frac{g^2}{4} \quad and \quad \sum_{j=1}^{g-1}(g-j)(-1)^j = -\frac{g}{2}.$$

Proof. The first claim of this lemma readily follows from the fact that the sum of the first m positive odd integers is m^2.

$$\sum_{j=1}^{g-1}(g-j)\mathbb{1}_{\{j \text{ odd}\}} = (g-1)+(g-3)+...+1 = \left(\frac{g}{2}\right)^2,$$

where we note that we added $\frac{g}{2}$ odd numbers. The second claim follows since:

$$\begin{aligned}\sum_{j=1}^{g-1}(g-j)(-1)^j &= [-(g-1)+(g-2)]+[-(g-3)+(g-4)]+\cdots+[-3+2]-1,\\ &= -1\frac{g-2}{2}-1\\ &= -\frac{g}{2}.\end{aligned}$$

□

Lemma 5.9. $\sum_{i=1}^{d} a_i = 1.$

Proof. This lemma follows by direct computation using the preceding identities.

$$\begin{aligned}\sum_{i=1}^{d} a_i &= \left(\sum_{i=1}^{d}\frac{1}{n-g}\left[2+\frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right)\right]\right)-\frac{1}{n-g}\left[1+\frac{2}{g}\sum_{j=1}^{g-1}(g-j)\cos(\pi j)\right]\\ &= \frac{1}{n-g}\left(2d+\frac{4}{g}\left[\sum_{i=1}^{d}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right)\right]-1-\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right)\\ &= \frac{1}{n-g}\left(2d+\frac{4}{g}\left[\sum_{j=1}^{g-1}(g-j)\sum_{i=1}^{d}\cos\left(\frac{\pi ij}{d}\right)\right]-1-\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right).\end{aligned}$$

By Lemma 2.5:

$$\begin{aligned}&= \frac{1}{n-g}\left(2d+\frac{4}{g}\left[\sum_{j=1}^{g-1}(g-j)\frac{(-1)+(-1)^j}{2}\right]-1-\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right)\\ &= \frac{1}{n-g}\left(2d-\frac{4}{g}\left[\sum_{j=1}^{g-1}(g-j)\mathbb{1}_{\{j \text{ odd}\}}\right]-1-\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right).\end{aligned}$$

By Lemma 5.8, and using that g is even:

$$
\begin{aligned}
&= \frac{1}{n-g}\left(2d - \frac{4}{g}\left[\frac{g^2}{4}\right] - 1 + \frac{2}{g}\frac{g}{2}\right) \\
&= \frac{1}{n-g}(2d-g) \\
&= 1,
\end{aligned}
$$

since $n = 2d$. □

Lemma 5.10. $\sum_{i=1}^{d} b_i = 1$.

Proof. This lemma readily follows from the definition of the b_i in terms of the a_i.

$$
\begin{aligned}
\sum_{i=1}^{d} b_i &= \left(\sum_{i=1}^{d} \frac{2g-(n-g)a_i}{n(g-1)}\right) - \frac{g}{n(g-1)} \\
&= \frac{2gd}{n(g-1)} - \frac{n-g}{n(g-1)}\sum_{i=1}^{d} a_i - \frac{g}{n(g-1)}.
\end{aligned}
$$

By Lemma 5.9:

$$
\begin{aligned}
&= \frac{2gd}{n(g-1)} - \frac{n-g}{n(g-1)} - \frac{g}{n(g-1)} \\
&= \frac{1}{n(g-1)}\,(ng-n+g-g) \\
&= \frac{1}{n(g-1)}\,(n(g-1)) \\
&= 1.
\end{aligned}
$$

□

Proof (of Claim 5.7). To show that $\operatorname{trace}(J_{n^2}Y) = n^2$, we want to sum the entries of Y. We mirror the proof of Claim 4.6 and first compute the sum of the entries in each minor block, which is either a $\frac{1}{n}I_n$, $\frac{1}{2n}A$, or $\frac{1}{2n}B$. As in Claim 4.6, Lemma 5.9 implies that $\operatorname{trace}(J_n\frac{1}{2n}A) =$

$\frac{1}{2n}2n\sum_{i=1}^{d} a_i = \frac{1}{2n}2n = 1$, and analogously Lemma 5.10 implies that $\text{trace}(J_n \frac{1}{2n} B) = 1$. Moreover, $\text{trace}(J_n \frac{1}{n} I_n) = 1$. Hence, each of the n^2 minor blocks of Y sums to 1, so that the total sum of entries in Y is

$$\text{trace}(J_{n^2} Y) = n^2.$$

□

Claim 5.11. $Y \geq 0$.

To show that $Y \geq 0$, we show that the a_i and b_i are nonnegative. We will use the following trigonometric identity.

Lemma 5.12.

$$(2\cos(\theta) - 2)\sum_{j=1}^{g-1}(g-j)\cos(j\theta) = \cos(g\theta) - g\cos(\theta) + (g-1).$$

Proof.

$$\begin{aligned}
&(2\cos(\theta) - 2)\sum_{j=1}^{g-1}(g-j)\cos(j\theta) \\
&\quad = 2\sum_{j=1}^{g-1}(g-j)\cos(j\theta)\cos(\theta) - 2\sum_{j=1}^{g-1}(g-j)\cos(j\theta).
\end{aligned}$$

Applying the product-to-sum identity for cosine:

$$= \sum_{j=1}^{g-1}(g-j)\cos((j+1)\theta) + \sum_{j=1}^{g-1}(g-j)\cos((j-1)\theta) - 2\sum_{j=1}^{g-1}(g-j)\cos(j\theta).$$

Reindexing to combine terms:

$$\begin{aligned}
&= \sum_{j=2}^{g}(g-j+1)\cos(j\theta) + \sum_{j=0}^{g-2}(g-j-1)\cos(j\theta) - 2\sum_{j=1}^{g-1}(g-j)\cos(j\theta) \\
&= \left(\sum_{j=1}^{g-1}[(g-j+1) + (g-j-1) - 2(g-j)]\cos(j\theta)\right) + \cos(g\theta) - g\cos(\theta) + (g-1)\cos(0) - 0 \\
&= \cos(g\theta) - g\cos(\theta) + (g-1).
\end{aligned}$$

$\square$

Proof (of Claim 5.11). We first show that the a_i are nonnegative. Recall that

$$a_i \propto 2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right)$$

(where the constant of proportionality is different for $a_1, ..., a_{d-1}$ and for a_d, but in both cases is positive). To show that the a_i are nonnegative, we thus want to show that, for $i = 1, ..., d$,

$$\frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right) \geq -2,$$

or equivalently

$$\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right) \geq -\frac{g}{2}.$$

We appeal to Lemma 5.12 with $\theta = \frac{\pi i}{d}$. For $i = 1, ..., d$, $\cos(\theta) \neq 1$, so we have that:

$$\begin{aligned}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right) &= \frac{\cos(g\theta) - g\cos(\theta) + g - 1}{2\cos(\theta) - 2}\\ &= \frac{g(1-\cos(\theta))}{2(\cos(\theta)-1)} + \frac{\cos(g\theta) - 1}{2\cos(\theta) - 2}\\ &= -\frac{g}{2} + \frac{1-\cos(g\theta)}{2-2\cos(\theta)}\\ &\geq -\frac{g}{2},\end{aligned}$$

since $1 - \cos(g\theta) \geq 0$ and $2 - 2\cos(\theta) \geq 0$.

We now need only show that the $b_i \geq 0$. Recall that

$$b_i = \begin{cases} \frac{2g-(n-g)a_i}{n(g-1)}, & i < d\\ \frac{g-(n-g)a_i}{n(g-1)}, & i = d. \end{cases}$$

For $i = 1, ..., d-1$ it suffices to show that $2g \geq (n-g)a_i$. In these cases, we have

$$(n-g)a_i = 2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right).$$

Using $\cos(\theta) \leq 1$:

$$\begin{aligned}
&\leq 2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j) \\
&= 2 + \frac{4}{g}(1 + 2 + ... + (g-1)) \\
&= 2 + \frac{4}{g}\frac{(g-1)g}{2} \\
&= 2 + 2(g-1) \\
&= 2g,
\end{aligned}$$

as desired.

For $i = d$ the situation is analogous. We want $(n-g)a_d \leq g$ which follows by the exact computations as above.

$$\begin{aligned}
(n-g)a_d &= \frac{1}{2}\left[2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right)\right] \\
&\leq \frac{1}{2}2g \\
&= g.
\end{aligned}$$

□

Proposition 5.13. $Y \succeq 0$.

As before, define $a^{(k)} = \sum_{i=1}^{d} a_i \cos\left(\frac{2\pi ik}{n}\right)$ and $b^{(k)} = \sum_{i=1}^{d} b_i \cos\left(\frac{2\pi ik}{n}\right)$. Recall, as in Claim 4.8, that the eigenvectors of a general circulant matrix are of the form $v_j = (1, w_j, w_j^2, ..., w_j^{n-1})$ for $j = 0, 1, ..., n-1$. As in Claim 4.8, we find the eigenvalues of Y using simultaneous eigenvectors and properties of the Kronecker product.

Claim 5.14. *A and B are simultaneously diagonalizable. The eigenvalues of A are*

$$\lambda_k(A) = 2a^{(k)}$$

for $k = 0, ..., n-1$. *The eigenvalues of* B *are*

$$\lambda_k(B) = 2b^{(k)}$$

for $k = 0, ..., n-1$, *where* $\lambda_k(A)$ *and* $\lambda_k(B)$ *correspond to the same eigenvector* v_k.

Proof. This is exactly as in Claim 4.8, since A and B are constructed using the same basis of symmetric circulant matrices. □

Claim 5.15. *The distinct eigenvalues of* $2nY$ *are*

$$\begin{cases} 2(g-1)\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\ -2\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\ 2 - 2a^{(k)}, \end{cases}$$

over $k = 0, ..., n-1$.

Proof. Note that

$$2nY = (J_g - I_g) \otimes J_{n/g} \otimes B + I_g \otimes J_{n/g} \otimes A + I_g \otimes I_{n/g} \otimes (2I_n - A).$$

Claim 5.14 gives a set of simultaneous eigenvectors/eigenvalues for B and A (and thus also $2I_n - A$) which we denote by v_k, for $k = 1, ..., n$. We can similarly obtain a simultaneous set of eigenvectors/eigenvalues of $(J_g - I_g) \otimes J_{n/g}, I_g \otimes J_{n/g}$ and $I_g \otimes I_{n/g}$, so that we will again use properties of the Kronecker product to explicitly compute the eigenvalues of Y as a function of the $a^{(k)}$ and $b^{(k)}$. Note $(J_g - I_g) \otimes J_{n/g}$ has three distinct eigenvalues: $J_g - I_g$ has two distinct eigenvalues ($g-1$ with associated eigenvector $e^{(g)}$ and -1 with associated eigenvectors $e_1^{(g)} - e_i^{(g)}$, for $i = 2, ..., g$) and $J_{n/g}$ has two distinct eigenvalues (n/g with associated eigenvector $e^{(n/g)}$ and 0 with associated eigenvectors $e_1^{(n/g)} - e_i^{(n/g)}$

for $i = 2, ..., \frac{n}{g}$). Hence spectral products of Kronecker products imply that the distinct eigenvalues of $(J_g - I_g) \otimes J_{n/g}$ are

$$\mu_i^B := \begin{cases} (g-1) \times \frac{n}{g}, & i = 1 \text{ using } e^{(g)} \otimes e^{(n/g)} \\ -1 \times \frac{n}{g}, & i = 2 \text{ using } \left(e_1^{(g)} - e_i^{(g)}\right) \otimes e^{(n/g)} \\ (g-1) \times 0 = -1 \times 0, & i = 3 \text{ using } e^{(g)} \otimes \left(e_1^{(n/g)} - e_i^{(n/g)}\right) \text{ or } \left(e_1^{(g)} - e_i^{(g)}\right) \otimes \left(e_1^{(n/g)} - e_i^{(n/g)}\right). \end{cases}$$

In exactly the same way, the distinct eigenvalues of $I_g \otimes J_{n/g}$ are

$$\mu_i^A := \begin{cases} 1 \times \frac{n}{g}, & i = 1 \text{ using } e^{(g)} \otimes e^{(n/g)} \\ 1 \times \frac{n}{g}, & i = 2 \text{ using } \left(e_1^{(g)} - e_i^{(g)}\right) \otimes e^{(n/g)} \\ 1 \times 0, & i = 3 \text{ using } e^{(g)} \otimes \left(e_1^{(n/g)} - e_i^{(n/g)}\right) \text{ or } \left(e_1^{(g)} - e_i^{(g)}\right) \otimes \left(e_1^{(n/g)} - e_i^{(n/g)}\right). \end{cases}$$

For $1 \leq i \leq 3$, let u_i be a shared eigenvector of $(J_g - I_g) \otimes J_{n/g}$ and $I_g \otimes J_{n/g} \otimes A$ with respective associated eigenvalues μ_i^B and μ_i^A. Then:

$$\begin{aligned} &\left((J_g - I_g) \otimes J_{n/g} \otimes B + I_g \otimes J_{n/g} \otimes A + I_g \otimes I_{n/g} \otimes (2I_n - A)\right)(u_i \otimes v_k) \\ &\quad = (\mu_i^B \lambda_k^B + \mu_i^A \lambda_k^A + (2 - \lambda_k^A))(u_i \otimes v_k). \end{aligned}$$

Plugging in for the three cases of μ_i^A and μ_i^B, we get that the distinct eigenvalues of $2nY$ are

$$\begin{cases} 2(g-1)\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\ -2\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\ 2 - 2a^{(k)}, \end{cases} \tag{5.2}$$

over $k = 0, ..., n-1$ as claimed. □

Claim 5.16. *For $k = 1, ..., n-1$,*

$$b^{(k)} = -\frac{g}{n(g-1)} - \frac{n-g}{n(g-1)}a^{(k)}.$$

Proof. We have that

$$\begin{aligned}
b^{(k)} &= \sum_{i=1}^{d} b_i \cos\left(\frac{2\pi ik}{n}\right) \\
&= \left[\sum_{i=1}^{d} \frac{2g-(n-g)a_i}{n(g-1)} \cos\left(\frac{2\pi ik}{n}\right)\right] - \frac{g}{n(g-1)} \cos(\pi k).
\end{aligned}$$

Using Lemma 2.5 and the definition of $a^{(k)}$:

$$\begin{aligned}
&= \frac{2g}{n(g-1)} \left[\sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right)\right] - \frac{n-g}{n(g-1)} \left[\sum_{i=1}^{d} a_i \cos\left(\frac{2\pi ik}{n}\right)\right] - \frac{g}{n(g-1)}(-1)^k \\
&= \frac{g}{n(g-1)} \left((-1)+(-1)^k\right) - \frac{n-g}{n(g-1)} a^{(k)} - \frac{g}{n(g-1)}(-1)^k \\
&= -\frac{g}{n(g-1)} - \frac{n-g}{n(g-1)} a^{(k)}.
\end{aligned}$$

□

Plugging in for the $b^{(k)}$ we can simplify the eigenvalues of $2nY$.

Claim 5.17. *The eigenvalues of $2nY$ are*

$$\begin{cases} 0 \\ 2\left(\frac{g}{g-1}\right) + 2\left(\frac{n-g}{g-1}\right) a^{(k)} \\ 2 - 2a^{(k)}, \end{cases}$$

over $k = 1, ..., n-1$ and

$$\begin{cases} 2n \\ 0, \end{cases}$$

corresponding to $k = 0$.

Proof. The $k = 0$ follows by simplifying Equation (5.2) using $a^{(0)} = b^{(0)} = 1$ from Lemmas 5.9 and 5.10. Otherwise, notice that

$$\begin{aligned}
2(g-1)\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) &= -2(g-1)\frac{n}{g}\left(\frac{g}{n(g-1)} + \frac{n-g}{n(g-1)}a^{(k)}\right) + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\
&= 2\left[-1 - \frac{n-g}{g}a^{(k)} + \frac{n}{g}a^{(k)} + 1 - \frac{g}{g}a^{(k)}\right] \\
&= 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
-2\frac{n}{g}b^{(k)} + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) &= 2\frac{n}{g}\left(\frac{g}{n(g-1)} + \frac{n-g}{n(g-1)}a^{(k)}\right) + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\
&= 2\left(\frac{1}{g-1} + \frac{n-g}{g(g-1)}a^{(k)}\right) + 2\frac{n}{g}a^{(k)} + (2 - 2a^{(k)}) \\
&= 2\left(1 + \frac{1}{g-1}\right) + 2\left(\frac{n-g}{g(g-1)} + \frac{n(g-1)}{g(g-1)} - \frac{g(g-1)}{g(g-1)}\right)a^{(k)} \\
&= 2\left(1 + \frac{1}{g-1}\right) + 2\left(\frac{ng-g^2}{g(g-1)}\right)a^{(k)} \\
&= 2\left(\frac{g}{g-1}\right) + 2\left(\frac{n-g}{g-1}\right)a^{(k)}.
\end{aligned}$$

□

To complete the proof of Proposition 5.13, we will show that the eigenvalues for the cases $k = 1, ..., n-1$ are nonnegative. We will use two lemmas.

Lemma 5.18.

$$\sum_{i=1}^{d} \cos\left(\frac{\pi i j}{d}\right)\cos\left(\frac{\pi i k}{d}\right) \geq -\mathbb{1}_{j-k\ odd}.$$

Proof. By the product-to-sum identity,

$$\sum_{i=1}^{d} \cos\left(\frac{\pi ij}{d}\right)\cos\left(\frac{\pi ik}{d}\right) = \frac{1}{2}\left(\sum_{i=1}^{d} \cos\left(\frac{\pi i(j+k)}{d}\right) + \cos\left(\frac{\pi i(j-k)}{d}\right)\right).$$

Applying Lemma 2.5 and considering separately the cases where we cannot apply it (those that devolve down to summing $\cos(\theta i)$ over i when θ is an integer multiple of 2π):

$$= \begin{cases} \frac{1}{4}\left[-1 + (-1)^{j+k} + -1 + (-1)^{j-k}\right], & j-k, j+k \notin \{0, n\} \\ \frac{1}{4}\left[-1 + (-1)^{j+k} + 2d\right], & j-k \in \{0, n\}, j+k \notin \{0, n\} \\ \frac{1}{4}\left[2d - 1 + (-1)^{j-k}\right], & j-k \notin \{0, n\}, j+k \in \{0, n\} \\ \frac{1}{4}\left[2d + 2d\right], & j-k, j+k \in \{0, n\}. \end{cases}$$

Noting that $(-1)^{j+k} = (-1)^{j-k}$:

$$\begin{aligned} &\geq \frac{1}{2}\left(-1 + (-1)^{j-k}\right) \\ &= -\mathbb{1}_{j-k \text{ odd}}. \end{aligned}$$

Note that $j-k, j+k \in \{0, n\}$ requires $j-k = 0$ and $j+k = n$, i.e. $j = k = d$. Since j ranges from 1 to $g-1$, our final case is irrelevant if each group contains at least 2 vertices.

□

Lemma 5.19. *For g even,*

$$\sum_{j=1}^{g-1} (g-j)\mathbb{1}_{j-k \text{ odd}} = \frac{1}{4}\left(g(g-1) + g(-1)^k\right).$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{g-1}(g-j)\mathbb{1}_{j-k \text{ odd}} &= -\frac{1}{2}\sum_{j=1}^{g-1}\left(-1+(-1)^{j-k}\right)(g-j) \\
&= \frac{1}{2}\sum_{j=1}^{g-1}(g-j) - \frac{1}{2}\sum_{j=1}^{g-1}(-1)^{j-k}(g-j) \\
&= \frac{1}{2}(1+2+...+(g-1)) - \frac{1}{2}(-1)^{-k}\sum_{j=1}^{g-1}(-1)^{j}(g-j).
\end{aligned}$$

Using Lemma 5.8:

$$\begin{aligned}
&= \frac{1}{2}\frac{g(g-1)}{2} + \frac{1}{2}(-1)^k\frac{g}{2} \\
&= \frac{1}{4}\left(g(g-1)+g(-1)^k\right).
\end{aligned}$$

□

Proof (of Proposition 5.13). To complete the proof of Proposition 5.13 and show that Y is positive semidefinite, we need only show that the eigenvalues listed in Claim 5.17 are nonnegative. We thus need to show that

$$2\left(\frac{g}{g-1}\right)+2\left(\frac{n-g}{g-1}\right)a^{(k)} \geq 0 \qquad \text{and} \qquad 2-2a^{(k)} \geq 0$$

for $k = 1, ..., n-1$. The latter is a direct consequence of Claim 5.11, since $a_i \geq 0$ implies

$$a^{(k)} = \sum_{i=1}^{d} a_i \cos\left(\frac{2\pi i k}{n}\right) \leq \sum_{i=1}^{d} a_i = 1.$$

Hence we need only show that $2\left(\frac{g}{g-1}\right)+2\left(\frac{n-g}{g-1}\right)a^{(k)} \geq 0$. Equivalently, we need to show that

$$a^{(k)} \geq -\frac{g}{n-g}.$$

This result holds since:

$$
\begin{aligned}
a^{(k)} &= \sum_{i=1}^{d} a_i \cos\left(\frac{2\pi ik}{n}\right) \\
&= \frac{1}{n-g}\left[\sum_{i=1}^{d}\left(2+\frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi ij}{d}\right)\right)\cos\left(\frac{2\pi ik}{n}\right)\right] \\
&\qquad - \frac{1}{n-g}\left[1+\frac{2}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi dj}{d}\right)\right]\cos\left(\frac{2\pi dk}{n}\right).
\end{aligned}
$$

By Lemma 2.5:

$$
\begin{aligned}
&= \frac{1}{n-g}\left[(-1)+(-1)^k+\frac{4}{g}\sum_{j=1}^{g-1}(g-j)\sum_{i=1}^{d}\cos\left(\frac{\pi ij}{d}\right)\cos\left(\frac{2\pi ik}{n}\right)\right] \\
&\qquad - \frac{1}{n-g}\left[1+\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right](-1)^k.
\end{aligned}
$$

By Lemma 5.18:

$$
\geq \frac{1}{n-g}\left[\left[(-1)+(-1)^k-\frac{4}{g}\sum_{j=1}^{g-1}(g-j)\mathbb{1}_{j-k \text{ odd}}\right]-\left[1+\frac{2}{g}\sum_{j=1}^{g-1}(g-j)(-1)^j\right](-1)^k\right].
$$

By Lemmas 5.8 and 5.19:

$$
\begin{aligned}
&= \frac{1}{n-g}\left[\left[(-1)+(-1)^k-\frac{1}{g}\left(g(g-1)+g(-1)^k\right)\right]-\left[1-\frac{2}{g}\frac{g}{2}\right](-1)^k\right] \\
&= \frac{1}{n-g}\left[(-1)+(-1)^k-(g-1)-(-1)^k\right] \\
&= -\frac{g}{n-g}.
\end{aligned}
$$

□

Proof (of Proposition 5.4). Feasibility of Y follows directly from Claims 5.4 to 5.7, Claim 5.11, and Proposition 5.13. □

5.3 The Unbounded Integrality Gap of the Symmetry Reduction SDP

We first compute the objective function value of Y.

Theorem 5.20. *For Y as above, there exists a constant $\tilde{c}_g$ (depending on g but not n) such that*

$$\frac{1}{2} trace((C \otimes C_1)Y) \leq \frac{\tilde{c}_g}{n}.$$

Proof. Recalling that $C = (J_g - I_g) \otimes J_{n/g}$, we see that $C \otimes C_1$ has blocks of zeros in each of the g major $\frac{n^2}{g} \times \frac{n^2}{g}$ diagonal blocks of Y. Hence the only places where $C \otimes C_1$ places a nonzero entry are exactly those where Y has a B block; on each such block, $C \otimes C_1$ has a block C_1. There are $g(g-1)$ blocks of B matrices, each containing $\frac{n^2}{g^2}$ copies of B. Accounting for the fact that Y is scaled by $\frac{1}{2n}$, the value of the objective function is thus

$$\frac{1}{2}\text{trace}((C \otimes C_1)Y) = \frac{1}{2} g(g-1) \frac{n^2}{g^2} \frac{1}{2n} \text{trace}(C_1 B).$$

Since $\text{trace}(C_1 B) = 2nb_1$:

$$\begin{aligned} &= \frac{1}{2} g(g-1) \frac{n^2}{g^2} b_1 \\ &= \frac{1}{2} \frac{g-1}{g} n^2 b_1. \end{aligned}$$

Recall that

$$\cos(x) \geq 1 - \frac{1}{2}x^2.$$

Hence

$$\begin{aligned}
b_1 &= \frac{2g - (n-g)a_1}{n(g-1)} \\
&= \frac{2g - \left[2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\cos\left(\frac{\pi j}{d}\right)\right]}{n(g-1)} \\
&\leq \frac{2g - \left[2 + \frac{4}{g}\sum_{j=1}^{g-1}(g-j)\left(1 - \frac{1}{2}\frac{\pi^2 j^2}{d^2}\right)\right]}{n(g-1)} \\
&= \frac{2(g-1) - \frac{4}{g}\sum_{j=1}^{g-1}(g-j) + \frac{2}{g}\frac{\pi^2}{d^2}\sum_{j=1}^{g-1}(g-j)j^2}{n(g-1)}.
\end{aligned}$$

Define $c_g = \frac{2}{g}\pi^2 \sum_{j=1}^{g-1}(g-j)j^2$, a constant depending on g but not n.

$$\begin{aligned}
&= \frac{2(g-1) - \frac{4}{g}\frac{(g-1)g}{2} + \frac{c_g}{d^2}}{n(g-1)} \\
&= \frac{c_g}{d^2 n(g-1).}
\end{aligned}$$

Setting $\hat{c}_g = \frac{4}{g-1}c_g$, a constant depending on g but not n:

$$= \frac{\hat{c}_g}{n^3}.$$

Putting everything together,

$$\frac{1}{2}\mathrm{trace}((C \otimes C_1)Y) = \frac{1}{2}\frac{g-1}{g}n^2 b_1 \leq \frac{1}{2}\frac{g-1}{g}n^2\frac{\hat{c}_g}{n^3},$$

from which the result follows. □

We can now prove our main theorem, which we restate below.

Theorem (Theorem 7.1)**.** *Let $z \in \mathbb{N}$. Then the integrality gap of the symmetry reduction SDP (2.5) is at least z.*

Proof. We again consider the symmetry reduction SDP corresponding to an instance on $n+1$ vertices. Let $s = r = 1$ and consider an instance of the TSP on $n+1$ vertices with $g = 2z$ groups of vertices. Specifically, let groups $2, ..., g$ be equally sized, each of size $\frac{n}{g} \in \mathbb{N}$, and let group 1 have one extra vertex, so that group one is of size $\frac{n}{g} + 1$. Note also that

$$\mathrm{OPT}_{\mathrm{TSP}} = g = 2z$$

since each group of vertices must be visited at least once. Set

$$Y = \frac{1}{2n}\left[(J_g - I_g) \otimes J_{n/g} \otimes B + I_g \otimes J_{n/g} \otimes A + I_g \otimes I_{n/g} \otimes (2I_n - A)\right],$$

which is feasible for the symmetry reduction SDP by our earlier computations. Then the integrality gap is bounded below by

$$\frac{\mathrm{OPT}_{\mathrm{TSP}}}{\mathrm{OPT}_{\mathrm{SDP}\ (2.5)}} \geq \frac{2z}{\mathrm{trace}((C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\overline{c}))Y)}.$$

To bound the right-hand side, we note that linearity of the trace operator implies

$$\mathrm{trace}(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\overline{c}))Y = \mathrm{trace}((C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha])Y) + \mathrm{trace}(Diag(\overline{c})Y). \tag{5.3}$$

We upper bound each term, which respectively account for the modified QAP objective and the reinsertion costs described at the end of Chapter 4. First note that

$$C[\beta] = C[\{2, ..., n+1\}] = (J_g - I_g) \otimes J_{(n/g)}.$$

Similarly,

$$C_1^{(n+1)}[\alpha] = C_1^{(n)} - e_1 e_n^T - e_n e_1^T \leq C_1^{(n)},$$

where $\leq$ is taken entry-wise. By non-negativity,

$$\mathrm{trace}((C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha])Y) \leq \frac{1}{2}\mathrm{trace}((((J_g - I_g) \otimes J_{(n/g)}) \otimes (C_1^{(n)}))Y) \leq \frac{\tilde{c}_g}{n}, \tag{5.4}$$

by Theorem 5.20. As in Theorem 5.20, $\tilde{c}_g$ remains independent of n.

Second, consider trace$(Diag(\bar{c})Y)$. We compute that

$$\bar{c} = vec(C_1^{(n)}[\alpha, \{1\}]C[\{1\}, \beta]) = vec((e_1^{(n)} + e_n^{(n)})C[\{1\}, \beta]) \leq vec(e_1^{(n)} + e_n^{(n)})(e^{(n)})^T.$$

We note that $(e_1^{(n)} + e_n^{(n)})(e^{(n)})^T$ is an $n \times n$ matrix with $2n$ ones and the rest of the entries zero. The vec operator stacks the columns of this matrix, creating a vector in $\mathbb{R}^{n^2}$ with $2n$ ones and the remaining entries zero. Finally, $Diag(\bar{c})$ creates a diagonal matrix with $2n$ ones on the diagonal and the remaining entries zero. Since all diagonal entries of Y are equal to $\frac{1}{n}$, we have that

$$\text{trace}(Diag(\bar{c})Y) \leq 2n * \frac{1}{n} = 2. \tag{5.5}$$

Plugging Equations (5.4) and (5.5) into Equation (5.3) we obtain

$$\text{trace}\left(\left(C[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\bar{c})\right)Y\right) \leq \frac{\tilde{c}_g}{n} + 2.$$

Hence the integrality gap is at least:

$$\begin{aligned}
\frac{\text{OPT}_{\text{TSP}}}{\text{OPT}_{\text{SDP (2.5)}}} &\geq \frac{2z}{\text{trace}((D[\beta] \otimes \frac{1}{2}C_1^{(n+1)}[\alpha] + Diag(\bar{c}))Y)} \\
&\geq \frac{2z}{2 + \frac{\tilde{c}_g}{n}} \\
&= \frac{2zn}{2n + \tilde{c}_g} \\
&\to z,
\end{aligned}$$

as $n \to \infty$. □

5.4 Consequences of Theorem 7.1

Non-Montonicity of the Symmetry Reduction SDP (2.5)

The results of Section 3.3 implied that the algebraic connectivity and association scheme SDPs have a counterintuitive non-monotonicity property: adding vertices (in a way that retains costs being metric) can arbitrarily decease the cost of some solutions to the corresponding SDPs. Corollary 4.10 shows that in the $g = 2$ case the cost of the symmetry reduction SDP,

$$\mathrm{OPT}_{\mathrm{SDP}\ (2.5)}(D) \leq 1 + \frac{c}{n},$$

decays arbitrarily close to 1 as the number of vertices in each group grew. Any $g = 2$ instance with cost strictly greater than one thus shows that the symmetry reduction SDP is non-monotonic. Moreover, such an instance implies the non-montonicity property in $\mathbb{R}^1$: the symmetry reduction SDP can find a smaller optimal value by only adding more points to visit on the real line.

Corollary 5.21. *The Symmetry Reduction SDP (2.5) is non-monotonic.*

Proof (sketch). It suffices to show a single two group instance with cost strictly greater than 1. Consider such an instance on $n + 1 = 3$ vertices, where the first group has two vertices and the second has one. Explicitly writing down the constraints shows that any feasible solution to the symmetry reduction SDP has cost 2. $\square$

The Unbounded Integrality Gap of Another SDP

Our simplicial TSP instances and associated feasible solutions show the the SDP relaxation (2.6), due to Anstreicher [2] and equivalent to the projected eigenvalue bound of Hadley, Rendl, and Wolkowicz [43], has an unbounded integrality gap. Recall that this SDP is again in terms of an $n^2 \times n^2$ matrix, and we use the same block structure

$$Y = \begin{pmatrix} Y^{(11)} & Y^{(12)} & \dots & Y^{(1n)} \\ Y^{(21)} & Y^{(22)} & \dots & Y^{(2n)} \\ \vdots & \vdots & \ddots & \vdots \\ Y^{(n1)} & Y^{(n2)} & \dots & Y^{(nn)} \end{pmatrix}$$

with $Y^{(ij)} \in \mathbb{R}^{n\times n}$. The SDP is then:

$$\begin{array}{lll} \min & \frac{1}{2}\text{trace}\left(\left(C \otimes C_1^{(n)}\right) Y\right) & \\ \text{subject to} & \sum_{i=1}^{n} Y^{(ii)} & = I_n \\ & \left(\text{trace}(Y^{(ij)})\right)_{i,j=1}^{n} & = I_n \\ & \text{trace}\left(YF^TF\right) & = 2n \\ & Y - \frac{1}{n^2}J_{n^2} & \succeq 0 \\ & Y \geq 0, Y \in \mathcal{S}^{n^2\times n^2}, & \end{array} \tag{2.6}$$

where

$$F = \begin{pmatrix} (e^{(n)})^T \otimes I_n \\ I_n \otimes (e^{(n)})^T \end{pmatrix}.$$

Corollary 5.22. *SDP (2.6) has an unbounded integrality gap.*

Proof. We show that Y, as defined in Theorem 4.1, remains feasible. The objective function remains unchanged from SDP (2.4), so the analysis in Corollary 4.9 then implies that SDP (2.6) has an unbounded integrality gap.

By definition of Y, $Y^{(ii)} = \frac{1}{n}I_n$ for $i = 1, ..., n$ so that $\sum_{i=1}^n Y^{(ii)} = I_n$. Moreover $\text{trace}(Y^{(ii)}) = \frac{1}{n}\text{trace}(I_n) = 1$ for $i = 1, ..., n$, while A and B have zero diagonal so that the trace of any minor off-diagonal block is zero. Hence $\left(\text{trace}(Y^{(ij)})\right)_{i,j=1}^n = I_n$.

Next, note that $F^T F = J_n \otimes I_n + I_n \otimes J_n$. Thus $F^T F$ has a $J_n + I_n$ on each minor diagonal block and an I_n on each minor off-diagonal block. Since A and B have zero diagonal, $\text{trace}(AI_n) = \text{trace}(BI_n) = 0$ and the minor off-diagonal blocks make no contribution to $\text{trace}\left(YF^T F\right)$. Hence

$$\text{trace}\left(YF^T F\right) = n\text{trace}\left(\frac{1}{n}I_n(J_n + I_n)\right) = \text{trace}\left(I_n(J_n + I_n)\right) = 2n.$$

By Claim 4.7 and the definition of Y, Y is nonnegative and symmetric. Hence it remains to show that $Y - \frac{1}{n^2}J_{n^2} \succeq 0$. We note that $e^{(n^2)}$ is an eigenvector of Y. In the notation of Claim 4.8, it is the eigenvector when $j = 0$ and $i = 1$. In Claim 4.8, we showed that the corresponding eigenvalue of nY was $2d = n$, so that the corresponding eigenvalue of Y is 1. Then

$$\left(Y - \frac{1}{n^2}J_{n^2}\right)e^{(n^2)} = Ye^{(n^2)} - \frac{1}{n^2}e^{(n^2)}\left(e^{(n^2)}\right)^T e^{(n^2)} = e^{(n^2)} - \frac{1}{n^2}e^{(n^2)}n^2 = 0e^{(n^2)}.$$

Any other eigenvector v of Y is orthogonal to $e^{(n^2)}$. Letting λ denote the corresponding eigenvalue,

$$\left(Y - \frac{1}{n^2}J_{n^2}\right)v = Yv - \frac{1}{n^2}e^{(n^2)}\left(e^{(n^2)}\right)^T v = \lambda v - 0v = \lambda v.$$

Thus $Y - \frac{1}{n^2}J_{n^2}$ has the same spectrum as Y except that one eigenvalue (the eigenvalue 1 corresponding to eigenvector $e^{(n^2)}$) is shifted down by 1 (to eigenvalue 0). Consequently all eigenvalues of $Y - \frac{1}{n^2}J_{n^2}$ are nonnegative, and $Y - \frac{1}{n^2}J_{n^2} \succeq 0$. □

The Triangle Inequalities of De Klerk and Sotirov [24]

De Klerk and Sotirov [24] provide a set of linear inequalities that can be added to SDP relaxations where $Y = vec(X)vec(X)^T$ is feasible for any permutation matrix $X \in \Pi_n$, the *triangle inequalities*. For all distinct r, s, t these are that:

$$0 \leq Y_{rs} \leq Y_{rr}, \quad Y_{rr}+Y_{ss}-Y_{rs} \leq 1, \quad -Y_{tt}-Y_{rs}+Y_{rt}+Y_{st} \leq 0, \quad Y_{tt}+Y_{rr}+Y_{ss}-Y_{rs}-Y_{rt}-Y_{st} \leq 1.$$

We note that these inequalities do not eliminate the feasible solutions defined in Equation (5.1):

$$Y_{ss} = Y_{rr} = Y_{tt} = \frac{1}{n}$$

while

$$0 \leq Y_{rs}, Y_{rt}, Y_{st} \leq \frac{1}{2n}.$$

The upper bound on off-diagonal entries can be seen directly, by noting that $Y_{rs}, Y_{rt}, Y_{st} \in \{0, \frac{a_1}{2n}, ..., \frac{a_{d-1}}{2n}, \frac{a_d}{n}, \frac{b_1}{2n}, ..., \frac{b_{d-1}}{2n}, \frac{b_d}{n}\}$; by Lemmas 5.9 and 5.10, $0 \leq a_i, b_i \leq 1$, and it is straightforward to directly bound $a_d, b_d \leq \frac{1}{2}$ provided $n/g \geq 3$.

CHAPTER 6

BACKGROUND ON THE CIRCULANT TRAVELING SALESMAN PROBLEM AND RELAXATIONS

The circulant traveling salesman problem – when the input edge costs form a circulant matrix – is a highly symmetric (and not necessarily metric) special case of the traveling salesman problem. There are some indications that circulant edge costs should make the TSP significantly easier. For example, by the late 80's it was known that bottleneck TSP is polynomial-time solvable on symmetric circulant graphs and that one could efficiently decide if a symmetric circulant graph was Hamiltonian (Burkard and Sandholzer [11]), as well as that one could find minimum-cost Hamiltonian paths in (not-necessarily-symmetric) circulant graphs in polynomial time (Bach, Luby, and Goldwasser cited in Gilmore, Lawler, and Shmoys [31]). In contrast, the circulant TSP is intimately connected to long-standing open questions in number theory, and the complexity of circulant TSP is unknown even in highly restrictive settings. As a result, the complexity of circulant TSP is often cited as a compelling open problem (see, e.g., Burkhard [9], Burkhard, Deĭneko, Van Dal, Van der Veen, and Woeginger [10], and Lawler, Lenstra, Rinnooy Kan, and Shmoys [54]).

In Section 6.1, we define the circulant traveling salesman problem and briefly sketch its history. In Section 6.2, we more formally describe previous results on circulant TSP that provide context for Chapter 7: we define classic tools used to analyze circulant TSP instances, describe the Van der Veen lower bound, and describe De Klerk and Dobre [22]'s conjecture and partial progress proving that it is equivalent to the subtour LP (2.1). In 6.3, we then sketch a 2-approximation algorithm due to Gerace and Greco [29]. Finally, Section 6.2 describes connections between circulant TSP and a number-theoretic conjecture of Buratti.

6.1 Circulant TSP, Formally

Circulant TSP instances are those instances whose corresponding cost matrices $C = (c_{i,j})_{i,j=1}^n$ are symmetric and circulant, which implies that the cost of edge $\{i, j\}$ only depends on $i - j \bmod n$. We can thus write our cost matrix in terms of $\lfloor \frac{n}{2} \rfloor$ parameters:

$$C = (c_{(j-i) \bmod n})_{i,j=1}^n = \begin{pmatrix} 0 & c_1 & c_2 & c_3 & \cdots & c_1 \\ c_1 & 0 & c_1 & c_2 & \cdots & c_2 \\ c_2 & c_1 & 0 & c_1 & \ddots & c_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & c_4 & \cdots & 0 \end{pmatrix}, \tag{6.1}$$

with $c_0 = 0$ and $c_i = c_{n-i}$ for $i = 1, ..., \lfloor \frac{n}{2} \rfloor$. Importantly, in circulant TSP we do not implicitly assume that the edge costs are also metric. A **circulant graph** is a graph whose weighted adjacency matrix is circulant.

Circulant matrices have well-studied structure (see, e.g., Davis [20] and Gray [36]), and form an intriguing class of instances for combinatorial optimization problems. They seem to provide just enough structure to make a compelling, ambiguous set of instances; it is unclear whether or not a given combinatorial optimization problem should remain hard or become easy when restricted to circulant instances: As mentioned above, circulant structure is enough to make it easy to determine whether or not a graph is Hamiltonian, to find a minimum-cost Hamiltonian path, and to solve bottleneck TSP. In contrast, Codenotti, Gerace, and Vigna [15] show that Max Clique and Graph Coloring remain NP-hard when restricted to circulant graphs and do not admit constant-factor approximation algorithms unless P=NP.

It is not known if the circulant TSP is solvable in polynomial-time or is NP-hard, even when restricted to instances where only two of the edge costs $c_1, ..., c_{\lfloor \frac{n}{2} \rfloor}$ are finite: the *two-stripe circulant TSP*. See Greco and Gerace [37] and Gerace and Greco [30]. Yang, Burkard, Çela, and Woeginger [81] provide a polynomial-time algorithm for asymmetric TSP in circulant graphs with only two stripes having finite edge costs. The symmetric two-stripe circulant TSP is not, however, a special case of the asymmetric two-stripe version (In the symmetric case, edges $\{v, v+i\}$ and $\{v, v-i\}$ of cost c_i connect v to both $v+i$ and $v-i$; in the asymmetric case, there are edge costs $c_1, ..., c_{n-1}$ and an edge $(v, v+i)$ of cost c_i only connects from v to $v+i$. To encode two general symmetric circulant edges would require four asymmetric circulant edges.) In addition to questions of minimizing wallpaper waste, circulant TSP has applications in reconfigurable network design (see Medova [56]).

Motivated by positive results on Hamiltonicity and minimum-cost Hamiltonian paths, Van der Veen, Van Dal, and Sierksma [77] developed two heuristic algorithms for circulant TSP. In the case where all costs $c_1, ..., c_{\lfloor \frac{n}{2} \rfloor}$ are distinct, one heuristic provides tours within a factor of two of the optimal solution. In addition, Van der Veen, Van Dal, and Sierksma [77] give an explicit combinatorial formula as a lower bound for circulant TSP. Gerace and Greco [29] give a 2-approximation algorithm for the general case of circulant TSP when costs may not be distinct. Gerace and Irving [28] give a $\frac{4}{3}$-approximation algorithm for circulant TSP when edge costs are also metric. See also Greco and Gerace [38].

De Klerk and Dobre [22] consider several lower bounds for the circulant TSP, including the subtour LP. They show that, in the context of circulant TSP, the subtour LP is at least as strong as the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [77]. They also conjecture that, on any instance of circulant TSP, the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [77] exactly equals the optimal solution to the subtour LP. In Chapter 7 we will prove this conjecture and use it to exactly characterize

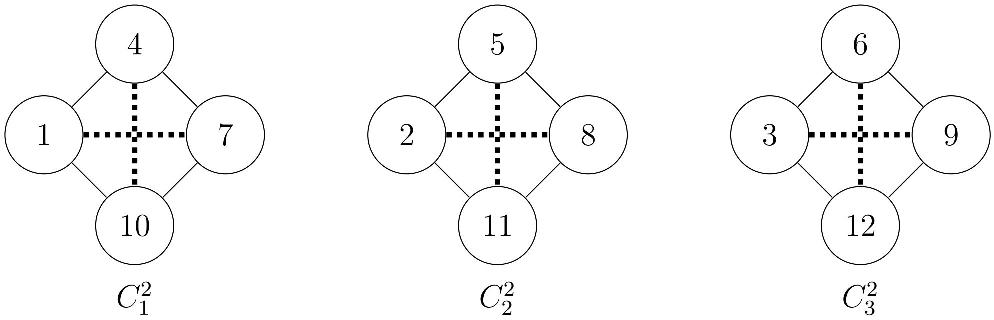

Figure 6.1: The graph $C\langle\{6,3\}\rangle$ for $n = 12$. If $\{\phi(1), \phi(2)\} = \{3, 6\}$, the three components are C_1^2, C_2^2, and C_3^2. Dashed edges are of length 6. In this example, $g_2 = 3$.

the integrality gap of the subtour LP on circulant instances.

6.2 Previous Results on the Subtour LP and Circulant TSP

In circulant TSP, all edges $\{i, j\}$ such that $i - j \equiv_n k$ or $i - j \equiv_n (n - k)$ have the same cost c_k. We refer to such edges as being in the k**-th stripe,** and we describe k as the **length** of the stripe. Classic algorithms and bounds for circulant TSP depend only on the ordering of the stripes with respect to their costs.

Definition 6.1. *Let $S \subset \{1, ..., d\}$. The* **circulant graph** $C\langle S\rangle$ *is the (simple, undirected, unweighted) graph including exactly the edges associated with the stripes S. I.e., the graph with adjacency matrix*

$$A = (a_{ij})_{i,j=1}^n, \quad a_{ij} = \begin{cases} 1, & (i-j) \bmod n \in S \text{ or } (j-i) \bmod n \in S \\ 0, & \text{else.} \end{cases}$$

For a set of stripes S, the graph $C\langle S\rangle$ includes exactly the edges associated with those stripes. See, for example, Figure 6.1 for a circulant graph with two stripes.

Given such an input to circulant TSP, we associate a permutation $\phi : [d] \to [d]$ that sorts the stripes in order of nondecreasing cost as well as a sequence that encodes the connectivity of $C\langle\{\phi(1), ..., \phi(k)\}\rangle$ for $1 \leq k \leq d$.

Definition 6.2 (Van der Veen, Van Dal, and Sierksma [77]). *Consider an instance of circulant TSP with edge costs* $c_1, ..., c_d$. *A* **stripe permutation** $\phi : [d] \to [d]$ *is a permutation such that* $c_{\phi(1)} \leq c_{\phi(2)} \leq \cdots \leq c_{\phi(d)}$. *The* g**-sequence associated to** ϕ *is* $g^\phi = (g_0^\phi, g_1^\phi, ..., g_d^\phi)$, *recursively defined by*

$$g_i^\phi = \begin{cases} n, & i = 0 \\ \gcd\left(\phi(i), g_{i-1}^\phi\right), & \textit{else.} \end{cases}$$

Proposition 6.3 will allow us to interpret g_i^ϕ as the number of components of $C\langle\{\phi(1), ..., \phi(i)\}\rangle$, the graph of all edges from the cheapest i stripes. See, e.g., Figure 6.1.

Note that, if edge costs are not distinct for a given instance of circulant TSP, there may be multiple associated stripe permutations. In this case, we will take ϕ to be an arbitrary stripe permutation sorting the costs. In Van der Veen, Van Dal, and Sierksma [77], the g-sequence is denoted as $(\mathcal{GCD}(\phi(0)), ..., \mathcal{GCD}(\phi(d)))$ with $\phi(0) := n$. In Greco and Gerace [37], ϕ is referred to as a presentation.

An early result from Burkard and Sandholzer [11] characterizes when Hamiltonian cycles exist in (symmetric) circulant graphs: Hamiltonian cycles exist whenever the (undirected) circulant graph is connected.

Proposition 6.3 (Burkard and Sandholzer [11]). *Let* $\{a_1, ..., a_t\} \subset [d]$ *and let* $\mathcal{G} = \gcd(n, a_1, ..., a_t)$. *The circulant graph* $C\langle\{a_1, ..., a_t\}\rangle$ *has* $\mathcal{G}$ *components. The* i*th component, for* $0 \leq i \leq \mathcal{G} - 1$, *consists of* $n/\mathcal{G}$ *nodes*

$$\{i + \lambda\mathcal{G} \bmod n : 0 \leq \lambda \leq \frac{n}{\mathcal{G}} - 1\}.$$

$C\langle\{a_1, ..., a_t\}\rangle$ *is Hamiltonian if and only if* $\mathcal{G} = 1$.

Set

$$\ell := \min\{i : 1 \leq i \leq d, g_i^\phi = 1\}.$$

By Proposition 6.3, the graph $C\langle\{\phi(1), ..., \phi(\ell - 1)\}\rangle$ is not Hamiltonian, while $C\langle\{\phi(1), ..., \phi(\ell)\}\rangle$ is. Hence any Hamiltonian tour uses an edge of cost at least $c_{\phi(\ell)}$, and tours can be constructed where $c_{\phi(\ell)}$ is the most expensive edge. Thus this proposition not only resolves Hamiltonicity in circulant graphs, but it also resolves bottleneck TSP in circulant graphs. In bottleneck TSP, the objective is to find a Hamiltonian tour for which the cost of the most expensive edge is minimized. Burkard and Sandholzer [11] use Proposition 6.3 to give a constructive algorithm for bottleneck TSP on circulant instances. We will use Proposition 6.3 to partition the vertices of circulant graphs.

Moreover, Proposition 6.3 immediately gives rise to an easily solvable case of circulant TSP: if there exists a stripe permutation ϕ such that $g_1^\phi = 1$, or equivalently, the length $\phi(1)$ of a cheapest stripe is relatively prime to n. For example, if n is prime, circulant TSP is easily solvable: you obtain a Hamiltonian tour by following edges of the cheapest stripe; after n edges you will have visited every node and returned to the start. These observations were first made in Garfinkel [27].

Proposition 6.3 can be used to solve the minimum-cost Hamiltonian path problem on circulant instances.

Proposition 6.4 (Bach, Luby, and Goldwasser, cited in Gilmore, Lawler, and Shmoys [31])**.** *Let* $c_1, ..., c_d$ *be the edge costs of a circulant instance and let* ϕ *be an associated stripe permutation. The minimum-cost Hamiltonian path has cost*

$$\sum_{i=1}^{\ell} (g_{i-1}^\phi - g_i^\phi) c_{\phi(i)}.$$

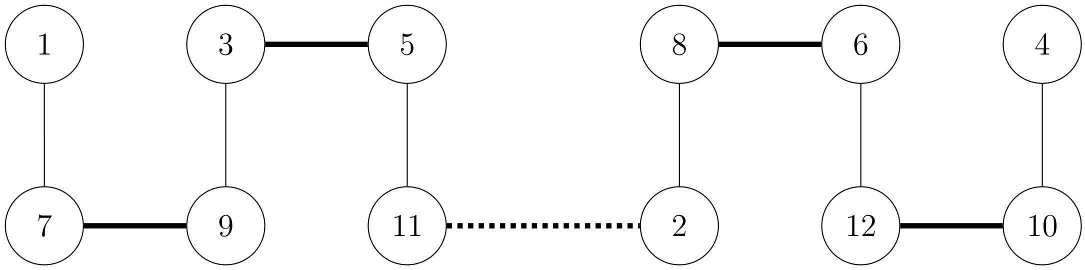

Figure 6.2: Constructing a minimum-cost Hamiltonian Path via the nearest neighbor heuristic. In this case, $n = 12$, $\phi(1) = 6$ (thin edges), $\phi(2) = 2$ (thick edges), and $\phi(3) = 3$ (dotted edges). This process fully connects a component of $C\langle\{\phi(1), ..., \phi(i)\}\rangle$, uses an edge of length $\phi(i+1)$ to move to the new component of $C\langle\{\phi(1), ..., \phi(i)\}\rangle$, and recursively fully connects that component. When all possible edges of length $\phi(1), ..., \phi(i), \phi(i+1)$ have been added, the path connects a component of $C\langle\{\phi(1), ..., \phi(i+1)\}\rangle$ and the process repeats using edges of length $\phi(i+2)$.

Sketch. Van der Veen, Van Dal, and Sierksma [77] argue that the nearest neighbor heuristic[1] constructs a Hamiltonian path using exactly $g^{\phi}_{i-1} - g^{\phi}_{i}$ edges from the ith cheapest stripe (see Figure 6.2). This path thus has cost

$$\sum_{i=1}^{\ell} (g^{\phi}_{i-1} - g^{\phi}_{i}) c_{\phi(i)}.$$

The optimality of such a path can be seen by applying Kruskal's algorithm [52] for minimum-cost spanning trees: For $1 \le i \le \ell$, Proposition 6.3 indicates that the graph $C\langle\{\phi(1), \phi(2), ..., \phi(i)\}\rangle$ has g^{ϕ}_{i} components. Hence, at most $n - g^{\phi}_{i}$ edges can be used from the cheapest i stripes without creating a cycle. Kruskal's algorithm will find a minimum-cost spanning tree using $n - g^{\phi}_{1} = g^{\phi}_{0} - g^{\phi}_{1}$ edges from the cheapest stripe, $g^{\phi}_{1} - g^{\phi}_{2}$ edges from the second cheapest stripe, and in general $g^{\phi}_{i-1} - g^{\phi}_{i}$ edges from the ith cheapest stripe. This spanning tree thus also costs $\sum_{i=1}^{\ell} (g^{\phi}_{i-1} - g^{\phi}_{i}) c_{\phi(i)}$. Since any Hamiltonian path is itself a spanning tree, any Hamiltonian path must cost at least this much; the constructed Hamiltonian path achieves this lower bound and is therefore optimal. □

[1]Start at some vertex and and follow a cheapest edge from that vertex. Then, recursively grow a Hamiltonian path by adding a cheapest edge from the most recently added vertex to a vertex that has not yet been visited.

Proposition 6.4 yields a natural lower bound on the optimal solution to circulant TSP instances: delete the most expensive edge of a Hamiltonian tour (of cost at least $c_{\phi(\ell)}$), and compare the resultant Hamiltonian path to a minimum-cost Hamiltonian path.

Proposition 6.5 (Van der Veen, Van Dal, and Sierksma [77])**.** *Let $c_1, ..., c_d$ be the edge costs of a circulant instance and let ϕ be an associated stripe permutation. Any Hamiltonian tour costs at least*

$$VDV := \left(\sum_{i=1}^{\ell} (g_{i-1}^{\phi} - g_i^{\phi}) c_{\phi(i)} \right) + c_{\phi(\ell)}.$$

VDV is the aforementioned explicit combinatorial formula for circulant TSP provided by Van der Veen, Van Dal, and Sierksma [77].

If there are multiple stripe permutations associated with an instance (i.e., the c_i are not all distinct), the lower bound is independent of the stripe permutation chosen. The lower bound is, moreover, tight as can be shown by considering any instance where the cheapest stripe has length relatively prime to n. For example the lower bound is tight for any instance where $\phi(1) = 1$.

De Klerk and Dobre [22] compare the VDV lower bound to several other well-known TSP bounds. In a series of numerical experiments, they provide evidence to conjecture that the VDV lower bound is exactly equal to the value of the optimal solution to the subtour LP.

Conjecture 6.6 (De Klerk and Dobre [22])**.** *Let $c_1, ..., c_d$ be the edge costs of a circulant instance and let ϕ be an associated stripe permutation. Let OPT_{LP} denote the optimal value of the subtour LP and VDV denote the value of the lower bound in Proposition 6.5. Then*

$$VDV = OPT_{LP}.$$

Our first main result of Chapter 7 will be the proof of this conjecture.

De Klerk and Dobre [22] provide further evidence for this conjecture by showing the following.

Theorem 6.7 (De Klerk and Dobre [22])**.** *Let* $c_1, ..., c_d$ *be the edge costs of a circulant instance and let* ϕ *be an associated stripe permutation. Let* OPT_{LP} *denote the optimal value of the subtour LP and VDV denote the value of the lower bound in Proposition 6.5. Then:*

$$VDV \leq OPT_{LP}.$$

To prove this result, de Klerk and Dobre [22] relax the subtour LP by dropping the degree constraints. Denote by $\mathrm{OPT}_{\mathrm{Relaxed}}$ the value of an optimal solution to this LP, so that:

$$\begin{aligned} \mathrm{OPT}_{\mathrm{Relaxed}} = \quad & \min && \textstyle\sum_{e \in E} c_e x_e \\ & \text{subject to} && \textstyle\sum_{e \in \delta(S)} x_e \geq 2, \quad S \subset V : S \neq \emptyset, S \neq V \\ & && 0 \leq x_e \leq 1, \qquad e \in E, \end{aligned}$$

and

$$\mathrm{OPT}_{\mathrm{Relaxed}} \leq \mathrm{OPT}_{\mathrm{LP}}.$$

Any feasible solution to the dual of this relaxed LP thus also provides a lower bound on $\mathrm{OPT}_{\mathrm{LP}}$. De Klerk and Dobre [22] provide a feasible solution to this dual of value equal to VDV, thus showing

$$\mathrm{VDV} \leq \mathrm{OPT}_{\mathrm{Relaxed}} \leq \mathrm{OPT}_{\mathrm{LP}}.$$

Theorem 6.7 can also be used to show that, on circulant (but not necessarily metric) instances, the subtour LP also has a bounded integrality gap.

Theorem 6.8. *The integrality gap of the subtour LP restricted to circulant TSP instances is at most 2. That is,*

$$\sup_{(c_1,\ldots,c_d)\in\mathbb{R}^d_{\geq 0}} \frac{OPT_{TSP}(c_1,\ldots,c_d)}{OPT_{LP}(c_1,\ldots,c_d)} \leq 2,$$

Proof. Consider any circulant instance. Let $\mathrm{OPT_{TSP}}$ denote the value of the optimal solution to the TSP on this instance, $\mathrm{OPT_{LP}}$ denote the value of the optimal solution to the subtour LP on this instance, and let VDV denote the value of the Van der Veen, Van Dal, and Sierksma [77] lower bound on this instance. By Theorem 6.7,

$$\frac{\mathrm{OPT_{TSP}}}{\mathrm{OPT_{LP}}} \leq \frac{\mathrm{OPT_{TSP}}}{\mathrm{VDV}}.$$

Theorem 6.3 in Gerace and Greco [29] argues that $\frac{\mathrm{OPT_{TSP}}}{\mathrm{VDV}} \leq 2$, by constructing Hamiltonian tours of cost at most $2 \cdot \mathrm{VDV}$. This construction is shown below in Section 6.3. □

6.3 Constructing Approximate Hamiltonian Tours

In this section, we describe the 2-approximation algorithm of Gerace and Greco [29]. Its exact details will not be relevant for our results in Chapter 7, but we sketch them for completeness. The algorithm is motivated by a heuristic Van der Veen, Van Dal, and Sierksma [77] developed for the case where every stripe has distinct cost. The algorithm only adds edges of length $\phi(i)$ if $g_i^\phi < g_{i-1}^\phi$. For simplicity of exposition, we'll suppress the dependence on ϕ and assume that

$$n = g_0 > g_1 > g_2 > \cdots > g_\ell = 1.$$

Doing so makes the details cleaner, and is without loss of generality: If $g_i^\phi = g_{i-1}^\phi$ for $i < \ell$, then zero weight will be placed on any edge of length $\phi(i)$, and the algorithm below treats it as equivalent to an instance where $c_{\phi(i)}$ is increased beyond $c_{\phi(\ell)}$. By applying this argument iteratively, we can obtain an instance of circulant TSP for which the g-sequence is strictly decreasing until it reaches 1.

Case 1: $g_{\ell-1}$ is Even

This algorithm is most straightforward when $g_{\ell-1}$ is even: First, it builds Hamiltonian paths on each component of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$. It then deletes one edge from $g_{\ell-1}-1$ of these paths. Finally, it adds $2(g_{\ell-1}-1)$ edges of length $\phi(\ell)$.

More specifically, the algorithm constructs a Hamiltonian path on the vertices in the component of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$ containing vertex 1 using the nearest neighbor rule starting at vertex 1. Call this path P_1 and let z be the other endpoint of P_1. Let $C_i^{\ell-1}$ be the component of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$ containing vertex $1+(i-1)\phi(\ell)$ (as usual, here and throughout we implicitly consider all vertices mod n). The algorithm translates P_1 to a Hamiltonian path P_i on the vertices in $C_i^{\ell-1}$: by adding $(i-1)\phi(\ell)$ to the label of every vertex in P_1. See Figure 6.3.

If $g_{\ell-1}$ is even, the algorithm deletes $g_{\ell-1}-2$ edges. It picks some edge $\{u, v\}$ in P_1 and deletes the corresponding edge in each $P_2, P_3, ..., P_{g_{\ell-1}-1}$ by deleting the edge $\{u+(i-1)\phi(\ell), v+(i-1)\phi(\ell)\}$ from P_i. It forms a Hamiltonian cycle on the entire vertex set by adding $2(g_{\ell-1}-1)$ edges of length $\phi(\ell)$ as in Figure 6.4[2].

[2]Specifically, it adds the following edges:

- Add the edges $\{1, 1+\phi(\ell)\}, \{1+2\phi(\ell), 1+3\phi(\ell)\}, ..., \{1+(g_{\ell-1}-2)\phi(\ell), 1+(g_{\ell-1}-1)\phi(\ell)\}$. Also add the edges $\{z, z+\phi(\ell)\}, \{z+2\phi(\ell), z+3\phi(\ell)\}, ..., \{z+(g_{\ell-1}-2)\phi(\ell), z+(g_{\ell-1}-1)\phi(\ell)\}$. This adds $g_{\ell-1}$ edges of length $\phi(\ell)$.
- Add the edges $\{u+\phi(\ell), u+2\phi(\ell)\}, \{u+3\phi(\ell), u+4\phi(\ell)\}, ..., \{u+(g_{\ell-1}-3)\phi(\ell), u+(g_{\ell-1}-2)\phi(\ell)\}$. Also add the edges$\{v+\phi(\ell), v+2\phi(\ell)\}, \{v+3\phi(\ell), v+4\phi(\ell)\}, ..., \{v+(g_{\ell-1}-3)\phi(\ell), v+(g_{\ell-1}-2)\phi(\ell)\}$. This adds $g_{\ell-1}-2$ edges of length $\phi(\ell)$.

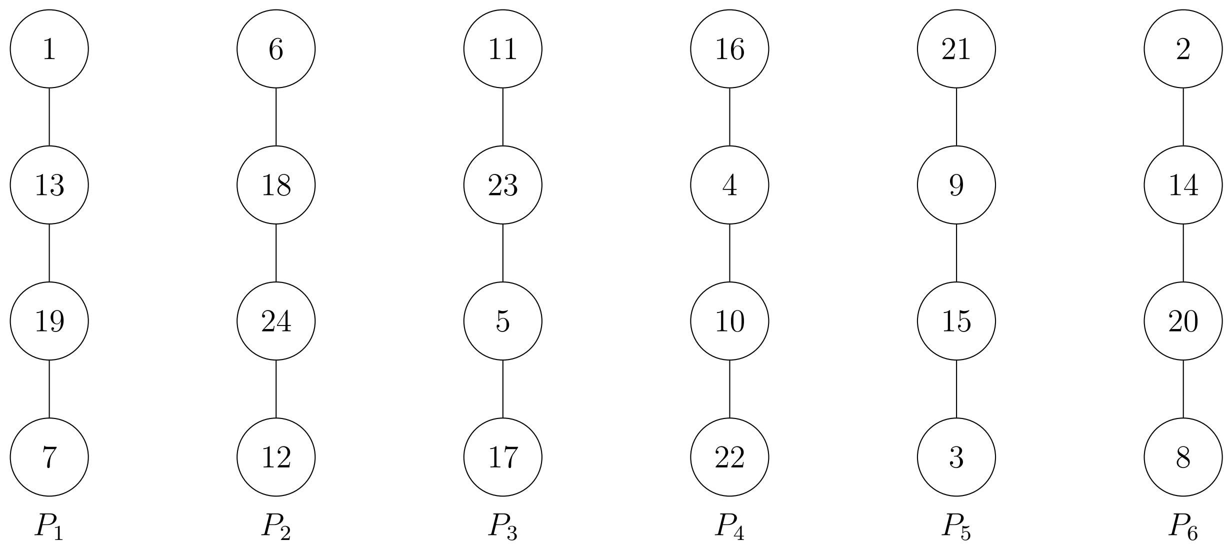

Figure 6.3: Translations of a Hamiltonian path P_1 to other components of $C\langle\{12, 6\}\rangle$ for a graph where $n = 24, \phi(1) = 12, \phi(2) = 6$, and $\phi(3) = 5$. In this example, $z = 7$.

Proposition 6.9 (Gerace and Greco [29]). *Consider any circulant instance where $g_{\ell-1}$ is even. Let OPT_{TSP} denote the optimal cost of a Hamiltonian tour on the circulant instance. Then the above algorithm produces a Hamiltonian tour of cost at most $2OPT_{TSP}$.*

Proof (Sketch). By construction, the above algorithm produces a Hamiltonian tour. We can analyze its cost in 3 steps:

1. When we start with $g_{\ell-1}$ paths (each Hamiltonian on a component of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$), we have used all of the edges in a minimum-cost Hamiltonian path on $[n]$ except those of length $\phi(\ell)$. In total, these edges cost
$$\sum_{i=1}^{\ell-1}(g^{\phi}_{i-1} - g^{\phi}_{i})c_{\phi(i)}.$$
2. We then delete some edges (translates of $\{u, v\}$), which cannot increase the cost.
3. Finally, we add $2(g_{\ell-1} - 1) = 2(g_{\ell-1} - g_\ell)$ edges of cost $\phi(\ell)$.

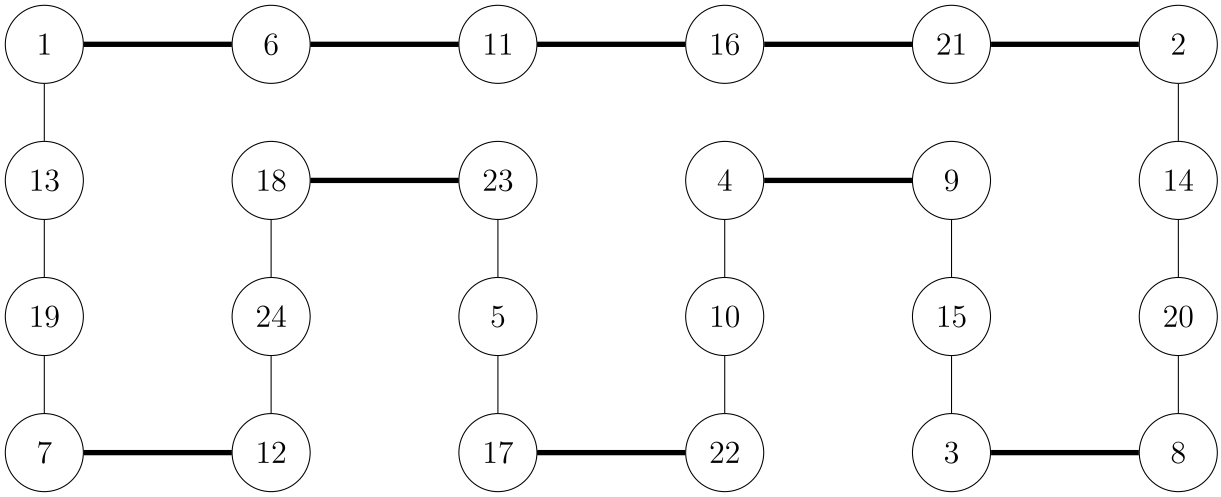

Figure 6.4: Constructing a Hamiltonian cycle when $g_{\ell-1}$ is even. In this case, $n = 24, \phi(1) = 12, \phi(2) = 6$ and $\phi(3) = 5$. We pick $\{u, v\} = \{1, 13\}$.

Hence, we end with a tour costing at most

$$\sum_{i=1}^{\ell-1}(g_{i-1}^{\phi} - g_i^{\phi})c_{\phi(i)} + 2(g_{\ell-1} - g_\ell)c_{\phi(\ell)} \leq 2\sum_{i=1}^{\ell}(g_{i-1}^{\phi} - g_i^{\phi})c_{\phi(i)} \leq 2\mathrm{OPT}_{\mathrm{TSP}}.$$

The second inequality follows because $\sum_{i=1}^{\ell}(g_{i-1}^{\phi} - g_i^{\phi})c_{\phi(i)}$ is the cost of a minimum-cost Hamiltonian path, which lower-bounds the cost of a Hamiltonian tour. □

Case 2: $g_{\ell-1}$ is Odd

If $g_{\ell-1}$ is odd, the algorithm of Gerace and Greco [29] proceeds similarly, but the analysis is more involved because the paths $P_1, ..., P_{g_{\ell-1}}$ cannot be connected into a Hamiltonian cycle as before. Instead, the algorithm recursively calls itself to produce a Hamiltonian cycle H in component $C_1^{\ell-1}$, as explained below. As before, we take $P_1, ..., P_{g_{\ell-1}}$ to be Hamiltonian paths on the components of $C\langle\{\phi(1), ..., \phi(\ell-1)\}\rangle$, where the endpoints of P_1 are vertex 1 and vertex z, and each other P_i is a translate of P_1. We arbitrarily select edge $\{u, v\}$ of length $\phi(\ell-1)$ in path P_1. Without loss of generality, we can assume H contains edge

$\{u, v\}$: H contains some edge of length $\phi(\ell - 1)$, and we can shift all the vertices in H (adding some multiple of $g_{\ell-1}$ to each vertex) until that edge is $\{u, v\}$.

We then delete edge $\{u, v\}$ and its translates from $H, P_2, P_3, ..., P_{g_{\ell-1}-1}$ and add $2(g_{\ell-1} - 1)$ edges of length $\phi(\ell)$ as in Figure 6.5[3].

This recursive process will eventually reach one of two halting conditions:

1. It is called to find a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ where $\frac{g_{t-1}}{g_t}$ is even, in which case it proceeds as in Case 1. This cycle is then recursively used to create a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t + 1)\}\rangle$, and then on a component of $C\langle\{\phi(1), ..., \phi(t + 2)\}\rangle$, and so on until it creates a Hamiltonian cycle on $C\langle\{\phi(1), ..., \phi(\ell - 1)\}\rangle$ (following the process described above). Note that $\frac{g_{t-1}}{g_t}$ counts the number of components of $C\langle\{\phi(1), ..., \phi(t - 1)\}\rangle$ that get merged into a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$.
2. Otherwise, we recursively call the algorithm until it attempts to produce a Hamiltonian cycle on a component of $C\langle\{\phi(1)\}\rangle$, in which case the Hamiltonian cycle on $C\langle\{\phi(1)\}\rangle$ can be found by following edges of length $\phi(1)$ until a cycle is created. In the case where $\phi(1) = n/2$, we treat $\{1, 1 + n/2\}$ as a cycle on $C\langle\{\phi(1)\}\rangle$ consisting of two length d edges.

Proposition 6.10 (Gerace and Greco [29]). *Consider any circulant instance where $g_{\ell-1}$ is*

[3]Specifically:

- Add the edges $\{1+\phi(\ell), 1+2\phi(\ell)\}, \{1+3\phi(\ell), 1+4\phi(\ell)\}, ..., \{1+(g_{\ell-1}-2)\phi(\ell), 1+(g_{\ell-1}-1)\phi(\ell)\}$. Also add the edges $\{z+\phi(\ell), z+2\phi(\ell)\}, \{z+3\phi(\ell), z+4\phi(\ell)\}, ..., \{z+(g_{\ell-1}-2)\phi(\ell), z+(g_{\ell-1}-1)\phi(\ell)\}$. This adds $g_{\ell-1} - 1$ edges of length $\phi(\ell)$.
- Add the edges $\{u+\phi(\ell), u+2\phi(\ell)\}, \{u+3\phi(\ell), u+4\phi(\ell)\}, ..., \{u+(g_{\ell-1}-3)\phi(\ell), u+(g_{\ell-1}-2)\phi(\ell)\}$. Also add the edges$\{v+\phi(\ell), v+2\phi(\ell)\}, \{v+3\phi(\ell), v+4\phi(\ell)\}, ..., \{v+(g_{\ell-1}-3)\phi(\ell), v+(g_{\ell-1}-2)\phi(\ell)\}$. This adds $g_{\ell-1} - 1$ edges of length $\phi(\ell)$.

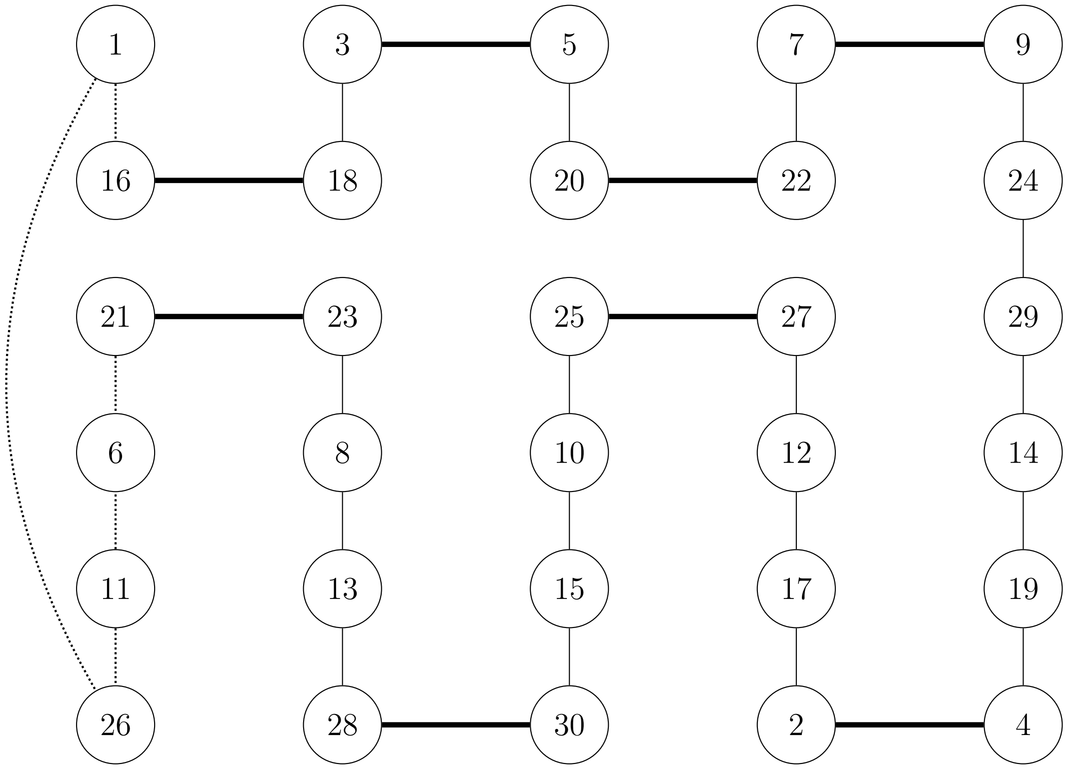

Figure 6.5: The 2-approximation algorithm for circulant TSP when $g_{\ell-1}$ is odd. In this case, $n = 30, \phi(1) = 15, \phi(2) = 5$ and $\phi(3) = 2$. We find the Hamiltonian path $P_1 = \{1,16\},\{16,21\},\{21,6\},\{6,11\},\{11,26\}$ so that, e.g., $P_2 = \{3,18\},\{18,23\},\{23,8\},\{8,13\},\{13,28\}$ is the path translated by $1 \times \phi(2) = 2$. We pick $\{u,v\} = \{16,21\}$, an edge of length $\phi(2) = 5$. Since $g_{\ell-1} = g_2 = 5$ is odd, we apply the recursive algorithm to find a Hamiltonian cycle on the vertices in P_1 (i.e., C_1^2). This yields the cycle $\{1,26\},\{26,11\}\{11,6\},\{6,21\},\{21,16\},\{16,1\}$, including the edge $\{u,v\}$, so we don't need to shift it. We then delete the edge $\{u,v\}$ and its translates from H, P_2, P_3, and P_4 and reconnect using the thick edges (of length $\phi(3) = 2$). Bold edges are of length $\phi(\ell)$, while the dotted edges correspond to the edges from H (after $\{u,v\}$ is removed).

odd. Let OPT_{TSP} denote the optimal cost of a Hamiltonian tour on the circulant instance. Then the above algorithm produces a Hamiltonian tour of cost at most $2OPT_{TSP}$.

Proof (Sketch). By construction, the above algorithm produces a Hamiltonian tour. We analyze its cost inductively at each stage of the recursion.

Suppose the algorithm recurses until it finds a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ (where possibly $t = 1$). We claim that the cost of the Hamiltonian cycle produced on this component is at most

$$\frac{2}{g_t}\sum_{i=1}^{t}(g_{i-1}-g_i)c_{\phi(i)}.$$

Indeed, if the algorithm halts because $t = 1$, it produces a Hamiltonian cycle consisting of $\frac{n}{g_1}$ edges of cost $c_{\phi(1)}$ and

$$\frac{n}{g_1}c_{\phi(1)} \leq 2\left(\frac{n}{g_1}-1\right)c_{\phi(1)} = \frac{2}{g_1}\sum_{i=1}^{1}(g_{i-1}-g_i)c_{\phi(i)}.$$

If instead $t > 1$, we view the component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ as the graph $C\langle\{\frac{\phi(1)}{g_t}, \frac{\phi(2)}{g_t}..., \frac{\phi(t)}{g_t}\}\rangle$ with $\frac{n}{g_t}$ vertices where edges of length $\frac{\phi(i)}{g_t}$ have cost $c_{\phi(i)}$; since $g_t = \gcd(n, \phi(1), ..., \phi(t))$, this is a well-defined circulant graph[4]. Moreover, the algorithm reaching a base case of the recursion and $t > 1$ implies that $\frac{g_{t-1}}{g_t}$ is even, so that the graph $C\langle\{\frac{\phi(1)}{g_t}, \frac{\phi(2)}{g_t}..., \frac{\phi(t-1)}{g_t}\}\rangle$ with $\frac{n}{g_t}$ vertices has an even number of components. Thus we can appeal to the analysis of the algorithm introduced Proposition 6.9 and, at the base case of recursion, the algorithm will produce a Hamiltonian tour on a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ of cost at most

$$2\sum_{i=1}^{t}\frac{g_{i-1}-g_i}{g_t}c_{\phi(i)} = \frac{2}{g_t}\sum_{i=1}^{t}(g_{i-1}-g_i)c_{\phi(i)}.$$

We now analyze the algorithm inductively, claiming that at each subsequent iteration of the algorithm, it extends a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(k)\}\rangle$ of cost at most $\frac{2}{g_k}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}$ to a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(k+1)\}\rangle$ of cost at most $\frac{2}{g_{k+1}}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}$. We do so in the following steps:

[4]Consider a component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ whose smallest vertex is labeled i. Any vertex in this component with label v can be relabeled with $\frac{v-i}{g_t}$, which is an integer: v, i in the same component of $C\langle\{\phi(1), ..., \phi(t)\}\rangle$ implies $v \equiv_{g_t} i$. Any edge in this component is of length $\phi(i)$ for $1 \leq i \leq t$, and

$$u - v = \phi(t) \text{ if and only if } \frac{u-i}{g_t} - \frac{v-i}{g_t} = \frac{\phi(t)}{g_t}.$$

1. By assumption, the Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(k)\}\rangle$ costs at most
$$\frac{2}{g_k}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}.$$

2. There are $\frac{g_k}{g_{k+1}}$ components of $C\langle\{\phi(1), ..., \phi(k)\}\rangle$ that get joined into a component of $C\langle\{\phi(1), ..., \phi(k+1)\}\rangle$. The algorithm produces a minimum Hamiltonian path on the other $\frac{g_k}{g_{k+1}}-1$ components of $C\langle\{\phi(1), ..., \phi(k)\}\rangle$ that merge into $C\langle\{\phi(1), ..., \phi(k+1)\}\rangle$. As in bounding the cost of the base case of the recursion, each of these components is equivalent to the circulant graph $C\langle\{\frac{\phi(1)}{g_k}, \frac{\phi(2)}{g_k} ..., \frac{\phi(k)}{g_k}\}\rangle$ on $\frac{n}{g_k}$ vertices so that the Hamiltonian path on each of these components will cost
$$\sum_{i=1}^{k}\frac{g_{i-1}-g_i}{g_k}c_{\phi(i)}.$$
These paths, with our Hamiltonian cycle, together cost at most
$$\begin{aligned}\frac{2}{g_k}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}+\left(\frac{g_k}{g_{k+1}}-1\right)\sum_{i=1}^{k}\frac{g_{i-1}-g_i}{g_k}c_{\phi(i)} &= \left(\frac{2}{g_k}+\frac{1}{g_{k+1}}-\frac{1}{g_k}\right)\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}\\ &= \frac{1}{g_{k+1}}\left(\frac{g_{k+1}}{g_k}+1\right)\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)}\\ &\le \frac{2}{g_{k+1}}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)},\end{aligned}$$
since $g_{k+1}\le g_k$.

3. We then delete some edges, which cannot increase the cost.

4. Finally, we add $2\left(\frac{g_k}{g_{k+1}}-1\right)$ edges of length $\phi(k+1)$ to form the Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(k+1)\}\rangle$. In total, these edges cost
$$2\left(\frac{g_k}{g_{k+1}}-1\right)c_{\phi(k+1)}=\frac{2}{g_{k+1}}(g_k-g_{k+1})c_{\phi(k+1)}.$$

Hence, we end with a Hamiltonian cycle on a component of $C\langle\{\phi(1), ..., \phi(k+1)\}\rangle$ costing at most

$$\frac{2}{g_{k+1}}\sum_{i=1}^{k}(g_{i-1}-g_i)c_{\phi(i)} + \frac{2}{g_{k+1}}(g_k - g_{k+1})c_{\phi(k+1)} = \frac{2}{g_{k+1}}\sum_{i=1}^{k+1}(g_{i-1}-g_i)c_{\phi(i)},$$

completing an inductive step.

Applying iteratively until we have a Hamiltonian cycle on the full instance, the total cost of this is at most

$$2\sum_{i=1}^{\ell}(g_{i-1}^{\phi} - g_i^{\phi})c_{\phi(i)} \leq 2\mathrm{OPT}_{\mathrm{TSP}}.$$

The inequality again follows because $\sum_{i=1}^{\ell}(g_{i-1}^{\phi} - g_i^{\phi})c_{\phi(i)}$ is the cost of a minimum-cost Hamiltonian path, which lower-bounds the cost of a Hamiltonian tour. □

6.4 Connections to Number Theory

The circulant traveling salesman problem is closely connected to several results and conjectures in number theory. In particular, the following conjecture dates to Marco Buratti in 2007 and conjectures conditions for a Hamiltonian path using prescribed edge lengths (see Buratti and Merola [7] for an initial statement, Horak and Rosa [45] for a generalization, and Pasotti and Pellegrini [67] for a rephrasal).

Conjecture 6.11 (Buratti)**.** *Let L be a multiset of size $n-1$ consisting of edge lengths in $1, 2, ..., \lfloor \frac{n}{2} \rfloor$. There exists a Hamiltonian path in K_n using edge lengths L if and only if: for every q that divides n,*

$$\#\{e \in L : q|e\} \leq n - q.$$

Here $\{e \in L : q|e\}$ is taken as a multiset consisting of all edge lengths in L that are a multiple of q. In the case where $n = 8$, for example, this condition says that L must contain at most $8 - 2 = 6$ edges of even length.

Buratti's conjecture remains open, with limited partial progress hinting at its difficulty. Pasotti and Pellegrini [67] show that the conjecture is true when the possible edge lengths are in $\{1, 2, 3, 5\}$, Dinitz and Janiszewski [26] show that it is true when n is prime and L has at most two distinct elements, and Pasotti and Pellegrini [68] show that it is true when L consists of elements 1, 2, and t where $t \geq 4$ and there are at least $t-1$ edges of length a and b. Costa, Morini, Pasotti, and Pellegrini [16] leave as open finding necessary and sufficient conditions for a graph to have a Hamiltonian cycle using prescribed edge lengths.

We note that Buratti's conjecture would lead to an alternative proof of Proposition 6.4: Buratti's conjecture implies the existence of a Hamiltonian path with the maximal number of edges from the cheapest stripe (i.e. $n - g_1$ such edges), from the two cheapest stripes (i.e. $n - g_2$ such edges), and so on all the way until reaching the ℓ-th cheapest stripe. This is exactly the path found in Proposition 6.4.

An analogue of Buratti's conjecture for Hamiltonian cycles would also resolve the 2-stripe circulant TSP (and, indeed, the $k-$stripe TSP for any constant k). In the 2-stripe case, for example, suppose the possible edges had length a and b with $c_a \leq c_b$. One would need only conditions to check if it were possible to have a Hamiltonian cycle with $n - i$ edges of length a and i edges of length b for $i = 0, ..., n$; the smallest possible i for which a Hamiltonian cycle were possible would give the optimal solution to the 2-stripe circulant TSP instance.

CHAPTER 7

THE SUBTOUR LP AND CIRCULANT TSP: A COMBINATORIAL INTERPRETATION AND THE INTEGRALITY GAP

In Chapter 6, we saw two lower bounds for circulant TSP: the bound stemming from the subtour LP relaxation and a combinatorial lower bound due to Van der Veen, Van Dal, and Sierksma [77] (denoted VDV; see Proposition 6.5). De Klerk and Dobre [22] conjectured that these two bounds were equal and showed that

$$\mathrm{VDV} \leq \mathrm{OPT}_{\mathrm{LP}}.$$

The main results of this chapter are twofold. First, in Section 7.1, we prove that

$$\mathrm{VDV} \geq \mathrm{OPT}_{\mathrm{LP}},$$

resolving De Klerk and Dobre's conjecture in the affirmative. To do so, we construct a combinatorial solution to the subtour LP of cost VDV that uses symmetry to "spread out" the Van der Veen lower bound. Second, in Section 7.2, we apply this theorem to show that the integrality gap of the subtour LP is exactly two: Theorem 6.8 implies that it is at most two, and here we construct a family of instances showing that the gap is two. To argue that the integrality gap is two, we use the optimal combinatorial solution found in Section 7.1. Finally, Section 7.3 discusses potential avenues to strengthening the subtour LP on circulant instances.

7.1 A Combinatorial Interpretation of the Subtour LP

We begin by resolving Conjecture 6.6. Recall that

$$\ell = \min\{i : 1 \leq i \leq d, g_i^{\phi} = 1\}.$$

Theorem 7.1. *Let $c_1, ..., c_d$ be the edge costs of a circulant instance and let ϕ be an associated stripe permutation. Let OPT_{LP} denote the optimal value of the subtour LP and let VDV denote the value of the lower bound in Proposition 6.5. Then:*

$$VDV = OPT_{LP}.$$

Moreover, an optimal solution to the subtour LP is achieved by setting, for $1 \leq i \leq d$, the weight on every edge e of length $\phi(i)$ to be

$$x_e = \begin{cases} \frac{g^{\phi}_{i-1} - g^{\phi}_i}{n}, & i \neq \ell, \phi(i) \neq \frac{n}{2} \\ 2\frac{g^{\phi}_{i-1} - g^{\phi}_i}{n}, & i \neq \ell, \phi(i) = \frac{n}{2} \\ \frac{g^{\phi}_{i-1}}{n}, & i = \ell, \phi(i) \neq \frac{n}{2} \\ 2\frac{g^{\phi}_{i-1}}{n}, & i = \ell, \phi(i) = \frac{n}{2}. \end{cases}$$

The explicit x_e values given in Theorem 7.1 spread out the weight placed by the Van der Veen, Van Dal, and Sierksma [77] bound,

$$\text{VDV} = \left(\sum_{i=1}^{\ell} (g^{\phi}_{i-1} - g^{\phi}_i) c_{\phi(i)} \right) + c_{\phi(\ell)}.$$

The coefficient of $c_{\phi(i)}$ is spread uniformly over all edges of length $\phi(i)$. For n even and $\phi(i) = d = n/2$, there are only $\frac{n}{2}$ such edges; otherwise there are n edges. As a result, we remark the following.

Remark 7.2. *Let x be defined as in Theorem 7.1. Then*

$$\sum_{e \in E} c_e x_e = VDV.$$

Note also that the solution places zero weight on edges of length $\phi(\ell + 1), ..., \phi(d)$ as well as zero weight on edges of any length $\phi(i)$ such that $g^{\phi}_i = g^{\phi}_{i-1}$. The optimal solution

x, therefore, only depends on the relative ordering of edge costs ϕ, and specifically, those stripes $\phi(i)$ for which $C\langle\{\phi(1), ..., \phi(i)\}\rangle$ has fewer components than $C\langle\{\phi(1), ..., \phi(i-1)\}\rangle$

To simplify our work that follows, and as in Section 6.3, we assume that the edges are ordered so that

$$g_0^\phi > g_1^\phi > \cdots > g_\ell^\phi = 1. \quad (7.1)$$

We can again make this assumption without loss of generality: If $g_i^\phi = g_{i-1}^\phi$ for $i < \ell$, then zero weight is placed on any edge of length $\phi(i)$ by both the Van der Veen, Van Dal, and Sierksma [77] bound and in the edge weights in Theorem 7.1. Both the Van der Veen, Van Dal, and Sierksma [77] bound and the subtour LP solution we find in Theorem 7.1 thus remain the same on an instance where $c_{\phi(i)}$ is increased beyond $c_{\phi(\ell)}$. By applying this argument iteratively, we can obtain an instance of circulant TSP for which the g-sequence is strictly decreasing until it reaches 1, and which the Van der Veen, Van Dal, and Sierksma [77] bound and the subtour LP treat equivalently.

For $0 \le i \le \ell - 1$ and $1 \le k \le g_i$, we use C_k^i to denote the vertex set of the kth connected component of the graph $C\langle\{\phi(1), ..., \phi(i)\}\rangle$. Note that C_k^i and $C_{k'}^i$ are isomorphic. See Figure 6.1. We let C^i denote an arbitrary representative of $C_1^i, ..., C_{g_i}^i$.

Our proof of Theorem 7.1 involves several steps. In Lemma 7.3, we show that the solution x posited satisfies the degree constraints. We then characterize the components C^i for $1 \le i \le \ell - 1$ as maximally dense: in Lemma 7.5 we show they satisfy the subtour elimination constraints with equality. To complete the proof, we look at arbitrary subsets $S \subset V$ in Proposition 7.8.

Throughout the proof, we suppress the dependence of g^ϕ on ϕ to simplify notation. It will also be helpful to treat our graph as a directed graph. Each edge from the ith stripe,

$i \neq n/2$, is directed $(v, v+i)$ (with the convention that $v+i$ is taken mod n). If n is even, we treat each edge of length $n/2$ incident to v as two directed edges, $(v, v+(n/2))$ and $(v+(n/2), v)$, each of which is assigned half the weight of an edge with length $n/2$. Thinking of our graph in this way means that every vertex v is incident to exactly two edges from each stripe $i = 1, ..., d$, with one edge directed into v and one edge directed out of v. That is, the edges of stripe $\phi(i)$ form a cycle cover on V. Moreover, this simplifies the number of cases for x_e since, if n is even and $\phi(i) = n/2$, we still spread the weight over n edges; the weight on every directed edge e of length $\phi(i)$ is then:

$$x_e = \begin{cases} \frac{g_{i-1}-g_i}{n}, & i \neq \ell \\ \frac{g_{i-1}}{n}, & i = \ell. \end{cases}$$

We fix $x \in \mathbb{R}^E$ to be the edge-weight vector with these weights.

With directed edges, we still treat $\delta(S)$ as the set of all edges with exactly one endpoint in S, whether that edge is directed into or out of S. Similarly, we treat $E(S)$ as the set of edges with both endpoints in S, i.e. $E(S) := \{(i,j) : i,j \in S\}$. For $A, B \subset V$, let $\delta^+(A,B) := \{e = (u,v) : u \in A, v \in B\}$ denote the set of edges starting in A and ending in B.

Lemma 7.3. *For any vertex $v \in V$, $x(\delta(v)) = 2$.*

Proof. Let $i < \ell$ and consider edges of length $\phi(i)$ incident to v. There are two edges of weight $\frac{g_{i-1}-g_i}{n}$: $(v, v+\phi(i))$ and $(v, v-\phi(i))$, so the total weight of edges of length $\phi(i)$ incident to v is $2\frac{g_{i-1}-g_i}{n}$. Analogously, the weight of edges of length $\phi(\ell)$ incident to v is $\frac{2g_{\ell-1}}{n}$. Thus

$$x(\delta(v)) = \sum_{i=1}^{\ell} \sum_{\substack{e \in \delta(v): \\ \text{length}(e)=\phi(i)}} x_e = \frac{2}{n}\left(\left(\sum_{i=1}^{\ell-1}(g_{i-1}-g_i)\right) + g_{\ell-1}\right) = \frac{2}{n}g_0 = 2,$$

since $g_0 = n$. □

We next argue that, for a set of vertices $S = C_k^i$, the only edges within $E(S)$ that have nonzero weight are those of length $\phi(1), ..., \phi(i)$.

Lemma 7.4. *Let $S = C_k^i$ where $0 \leq i \leq \ell - 1$ and $1 \leq k \leq g_i$. Let $e \in E(S)$. Then $x_e > 0$ implies e is an edge in stripes $\phi(1), ..., \phi(i)$.*

Proof. By Proposition 6.3, $S = \{v : v \equiv_{g_i} j\}$ for some $0 \leq j \leq g_i - 1$. Consider an edge of $e = (v, v + \phi(t)) \in E(S)$ of length $\phi(t)$ with $t > i$. Then, since e has both endpoints in C_k^i, $\phi(t) = c \cdot g_i$ for some $c \in \mathbb{N}$. Hence $g_t = \gcd(g_{t-1}, \phi(t)) = \gcd(g_{t-1}, c \cdot g_i) = g_{t-1}$, since g_{t-1} divides g_i, and so $x_e = 0$. □

Lemma 7.4 lets us now show that the C_k^i are maximally dense.

Lemma 7.5. *Let $S = C_k^i$ for $0 \leq i \leq \ell - 1$ and $1 \leq k \leq g_i$. Then $x(\delta(S)) = 2$.*

Proof. By Lemma 7.4, we can compute $x(E(S))$ by only summing up the weights of edges in the cheapest i stripes. Consider any fixed j with $j \leq i < \ell$. There are n total edges of length $\phi(j)$ and, since $j \leq i$, none of these edges are in any $\delta(C^i)$ (each C^i is a connected component using all edges of stripes $\phi(1), \phi(2)..., \phi(j), ..., \phi(i)$). Thus each isomorphic component $C^i \in \{C_1^i, ..., C_{g_i}^i\}$ has $\frac{n}{g_i}$ edges of length $\phi(j)$ in $E(C^i)$, and each edge has weight $\frac{g_{j-1}-g_j}{n}$. Hence

$$\sum_{\substack{e \in E(S): \\ \text{length}(e) = \phi(j)}} x_e = \frac{n}{g_i} \frac{g_{j-1} - g_j}{n} = \frac{g_{j-1} - g_j}{g_i}.$$

We can now compute:

$$\begin{aligned}
x(E(S)) &= \sum_{j=1}^{i} \sum_{e \in E(S):\text{length}(e)=\phi(j)} x_e \\
&= \frac{1}{g_i} \sum_{j=1}^{i} (g_{j-1} - g_j) \\
&= \frac{g_0 - g_i}{g_i} \\
&= \frac{n}{g_i} - 1 \\
&= |C_k^i| - 1.
\end{aligned}$$

The lemma then follows from the degree constraints: every vertex in $v \in S$ satisfies $x(\delta(v)) = 2$, so

$$2|S| = \sum_{v \in S} x(\delta(v)) = x(\delta(S)) + 2x(E(S)). \tag{7.2}$$

Substituting in $x(E(S)) = |C_k^i| - 1$ into $2|S| = x(\delta(S)) + 2x(E(S))$ completes the proof. □

We now want to extend Lemma 7.5 to show that $x(\delta(S)) \geq 2$ for any $S \subset V$, not just those corresponding to components connected by a set of cheapest stripes. We will consider any set S^* and partition it into its intersections with C^j (for an appropriate choice of j). We label the C_i^j such that $C_1^j, C_2^j, ..., C_s^j$ each intersect nontrivially with S^*, and $S^* \subset \bigcup_{i=1}^{s} C_i^j$. Expanding

$$x(E(S^*)) = \sum_{i=1}^{s} x(E(S^* \cap C_i^j)) + \sum_{\substack{1 \leq i_1, i_2 \leq s \\ i_1 \neq i_2}} x(\delta^+(S^* \cap C_{i_1}^j, S^* \cap C_{i_2}^j)),$$

we will bound each term of the sum. To do so, we will bound $x(\delta^+(S^* \cap C_{i_1}^j, S^* \cap C_{i_2}^j))$ by $x(\delta^+(C_{i_1}^j, C_{i_2}^j))$. Our first step is thus to understand the edges between distinct C^j.

Proposition 6.3 implies that the vertices in a component C^j are defined as $\{v : v \bmod g_j = i\}$ for some fixed $1 \leq i \leq \frac{n}{g_j}$. Because g_{j+1} divides g_j, $u \equiv_{g_j} v$ means that $u \equiv_{g_{j+1}} v$:

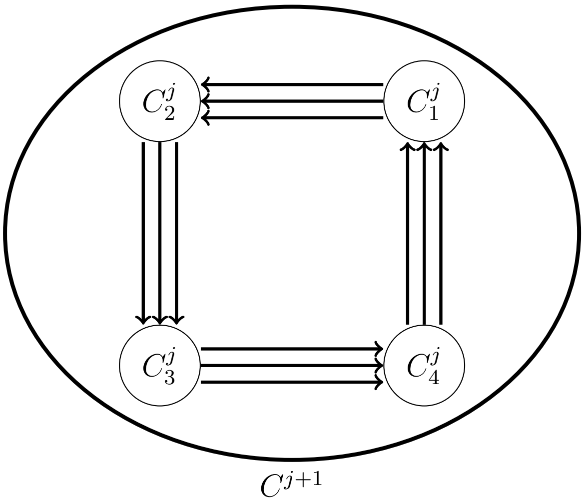

Figure 7.1: The structure of edges from stripe $\phi(j+1)$ (marked by arrows) from Lemma 7.6. Here $\frac{g_j}{g_{j+1}} = 4$.

if u, v are in the same C^j then u, v are in the same C^{j+1}. Consequently, the edges of stripe $\phi(j+1)$ merge the C^j components into a smaller number of C^{j+1} components. The facts that the C^i are all isomorphic and that C^i has g_i components implies that $\frac{g_j}{g_{j+1}}$ components C^j get merged into each C^{j+1}. See, for example, Figure 7.1. Our next lemma describes the role of the edges from stripe $\phi(j+1)$ in this merging process. It says that the subgraph of C^{j+1} obtained by contracting each $C^j \subset C^{j+1}$ into a single vertex is a cycle.

Lemma 7.6. *Suppose that* $C^{j+1} = C_1^j \sqcup \cdots \sqcup C^j_{\frac{g_j}{g_{j+1}}}$. *Consider the directed graph* G' *on* $V' = \left[\frac{g_j}{g_{j+1}}\right]$ *where* $(u, v) \in E'$ *if and only if there is an edge of stripe* $\phi(j+1)$ *in* $\delta^+(C_u^j, C_v^j)$. *Then* G' *is a directed cycle.*

Proof. First, suppose that $u \in V$ is such that $u \in C_i^j$ (so that $i \in V'$). For any other $v \in V$ with $v \in C_i^j$, we have $u \equiv_{g_j} v$ and so $u + \phi(j+1) \equiv_{g_j} v + \phi(j+1)$. Hence, the vertex $i \in V'$ has a single outgoing edge. Analogously $u - \phi(j+1) \equiv_{g_j} v - \phi(j+1)$ so that the vertex $i \in V'$ has a single incoming edge. These facts establish that every vertex of G' has a single outgoing edge and a single incoming edge and G' is a directed cycle cover. However, G'

must also be connected: C^{j+1} is a connected component of the graph $C\langle\{\phi(1),...,\phi(j+1)\}\rangle$, and edges of $\phi(1),...,\phi(j)$ are internal to the C^j. The only connected, directed cycle cover is a directed cycle. □

Lemma 7.6 allows us to bound the total weight of edges of stripe $\phi(j+1)$ going between some C^j in a C^{j+1}.

Lemma 7.7. *Suppose that $C^j_1,...,C^j_s \subset C^{j+1}$ with $1 < s \le \frac{g_j}{g_{j+1}}$ and $j < \ell$. Provided $j < \ell-1$ or $s < \frac{g_j}{g_{j+1}}$,*

$$\sum_{\substack{1\le i_1,i_2\le s\\ i_1\ne i_2}} x(\delta^+(C^j_{i_1}, C^j_{i_2})) \le s-1.$$

Proof. By Lemma 7.4, the only edges with both endpoints in C^{j+1} with nonzero weight are those in stripes $\phi(1),...,\phi(j+1)$. Moreover, any edge of stripe $\phi(i)$ with $i < j+1$ has both endpoints in the same C^j: $i \le j$ implies $\phi(i)$ divides g_j, so $u+\phi(i) \equiv_{g_j} u$; Proposition 6.3 implies that $u+\phi(i)$ and u are in the same component of $C\langle\{\phi(1),...,\phi(j)\}\rangle$. Hence the only edges contributing to the sum $\sum_{1\le i_1<i_2\le s} x(\delta^+(C^j_{i_1}, C^j_{i_2}))$ are those from stripe $\phi(j+1)$.

Consider the graph G' from Lemma 7.6, a cycle with vertices corresponding to $C^j_1,...,C^j_{\frac{g_j}{g_{j+1}}}$. A subset of s vertices of a cycle on $\frac{g_j}{g_{j+1}}$ vertices contains at most $s-1+\mathbb{1}_{\{s=\frac{g_j}{g_{j+1}}\}}$ edges, where $\mathbb{1}_{\{\circ\}}$ denotes the indicator function that is 1 if $\circ$ is true and 0 otherwise.

Hence at most $s-1+\mathbb{1}_{\{s=\frac{g_j}{g_{j+1}}\}}$ terms in the sum $\sum_{\substack{1\le i_1,i_2\le s\\ i_1\ne i_2}} x(\delta^+(C^j_{i_1}, C^j_{i_2}))$ are nonzero. Now consider any nonzero term $x(\delta^+(C^j_{i_1}, C^j_{i_2})) \ne 0$. Since the only edges contributing to this term are from stripe $\phi(j+1)$, we need only count the number of edges of stripe $\phi(j+1)$ starting in $C^j_{i_1}$ and ending in $C^j_{i_2}$. There are $\frac{n}{g_j}$ vertices in $C^j_{i_1}$, each of which has one outgoing edge of length $\phi(j+1)$ ending in $C^j_{i_2}$. If $j < \ell-1$, each of these has weight

$\frac{g_j - g_{j+1}}{n}$. Thus

$$\sum_{\substack{1 \le i_1, i_2 \le s \\ i_1 \ne i_2}} x(\delta^+(C^j_{i_1}, C^j_{i_2})) \le \left(s - 1 + \mathbb{1}_{\{s = \frac{g_j}{g_{j+1}}\}}\right) \frac{n}{g_j} \frac{g_j - g_{j+1}}{n}$$

$$= \left(s - 1 + \mathbb{1}_{\{s = \frac{g_j}{g_{j+1}}\}}\right) \left(1 - \frac{g_{j+1}}{g_j}\right).$$

If $s \ne \frac{g_j}{g_{j+1}}$, then the result follows because $\left(1 - \frac{g_{j+1}}{g_j}\right) \le 1$. Otherwise, when $s = \frac{g_j}{g_{j+1}}$, the right hand side is

$$\frac{g_j}{g_{j+1}} \left(1 - \frac{g_{j+1}}{g_j}\right) = \frac{g_j}{g_{j+1}} - 1 = s - 1.$$

The final case we must consider is when $j = \ell - 1$ but $s < \frac{g_j}{g_{j+1}}$. In this case every edge of length $\phi(j+1)$ has weight $\frac{g_{\ell-1}}{n}$ so that

$$\sum_{1 \le i_1 < i_2 \le s} x(\delta^+(C^j_{i_1}, C^j_{i_2})) \le \left(s - 1 + \mathbb{1}_{\{s = \frac{g_j}{g_{j+1}}\}}\right) \frac{n}{g_{\ell-1}} \frac{g_{\ell-1}}{n}$$

$$= (s-1) \frac{n}{g_{\ell-1}} \frac{g_{\ell-1}}{n}$$

$$= s - 1.$$

□

Proposition 7.8. *Let $S \subset V$ $(2 \le |S| \le n-2)$. Then $x(\delta(S)) \ge 2$.*

Proof. From Equation (7.2), $x(\delta(S)) + 2x(E(S)) = 2|S|$, so it suffices to show that $x(E(S)) \le |S| - 1$ for all S (with $2 \le |S| \le n-2$). Suppose towards a contradiction that there is some S^* with $x(E(S^*)) > |S^*| - 1$. We consider three cases.

Case 1: Suppose there exists an S^* that is disjoint from at least one $C^{\ell-1}$. Then consider any such S^* that is minimal by inclusion. By Lemma 7.5, $S^* \ne C^i_k$ for any $0 \le i \le \ell - 1$ and $1 \le k \le g_i$. Since $S^* \subset C^\ell = V$ and the C^i nest within the C^{i+1}, there are some j

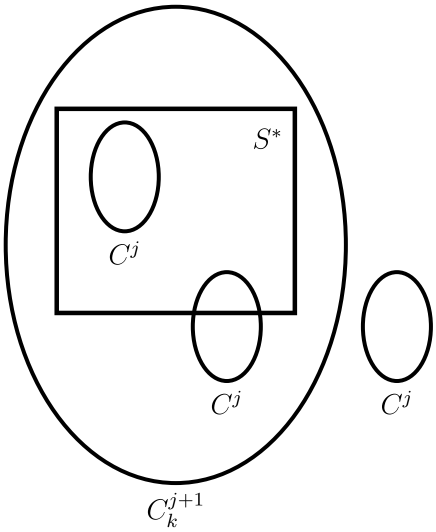

Figure 7.2: S^* and choice of j in Proposition 7.8. In this example, $s = 2$ of the C^j have nonempty intersections with S^*.

and k such that $S^* \subset C_k^{j+1}$ but S^* is not contained in any single $C_1^j, ..., C_{g_j}^j$ (i.e. $j+1$ is the smallest value such that S^* is properly contained in a C^{j+1} which we denote C_k^{j+1}). See Figure 7.2.

Without loss of generality, suppose that the C_i^j are labeled so that $C_1^j, ..., C_s^j$ have nonempty intersections with S^* while $C_{s+1}^j, ..., C_{g_j}^j$ have empty intersections with S^*. Note that, by choice of j, $C_1^j, ..., C_s^j \subset C_k^{j+1}$ so that $s \leq \frac{g_j}{g_{j+1}}$. We partition $S^* = \left(S^* \cap C_1^j\right) \bigsqcup \cdots \bigsqcup \left(S^* \cap C_s^j\right)$, so that

$$x(E(S^*)) = \sum_{i=1}^{s} x(E(S^* \cap C_i^j)) + \sum_{\substack{1 \leq i_1, i_2 \leq s \\ i_1 \neq i_2}} x(\delta^+(S^* \cap C_{i_1}^j, S^* \cap C_{i_2}^j)).$$

By minimality of S^*

$$x(E(S^*)) \leq \sum_{i=1}^{s} \left(|S^* \cap C_i^j| - 1\right) + \sum_{\substack{1 \leq i_1, i_2 \leq s \\ i_1 \neq i_2}} x(\delta^+(S^* \cap C_{i_1}^j, S^* \cap C_{i_2}^j))$$

$$= |S^*| - s + \sum_{\substack{1 \leq i_1, i_2 \leq s \\ i_1 \neq i_2}} x(\delta^+(S^* \cap C_{i_1}^j, S^* \cap C_{i_2}^j)).$$

By expanding sets in the rightmost term:

$$x(E(S^*)) \leq |S^*| - s + \sum_{\substack{1 \leq i_1, i_2 \leq s \\ i_1 \neq i_2}} x(\delta^+(C^j_{i_1}, C^j_{i_2})).$$

Note that our assumption that S^* doesn't intersect with every single $C^{\ell-1}$ means that Lemma 7.7 applies, so that

$$\begin{aligned} x(E(S^*)) &\leq |S^*| - s + (s-1) \\ &= |S^*| - 1. \end{aligned}$$

This contradicts our choice of S^* as a counterexample.

The cases that remain are those where every single S^* with $x(E(S^*)) > |S^*| - 1$ has $j = \ell - 1$ (so that the smallest C^{j+1} fully containing S^* is $C^\ell = V$) and $s = \frac{g_{\ell-1}}{g_\ell}$. This case means that S^* has a nonempty intersection with every $C^{\ell-1}$.

Case 2: Suppose that there is some $C^{\ell-1}$ fully contained in S^*. Then $2x(E(S^*)) + x(\delta(S^*)) = 2|S^*|$, so $x(E(S^*)) > |S^*| - 1$ implies $x(\delta(S^*)) < 2$. Applying the same argument to $S^{*^c} := V \backslash S^*$, we get $x(E(S^{*^c})) > |S^{*^c}| - 1$. But S^{*^c} is entirely disjoint from at least one $C^{\ell-1}$, contradicting the assumption that case 1 does not apply.

Case 3: The only remaining case is that $1 \leq |S^* \cap C^{\ell-1}| < |C^{\ell-1}|$ for every $C^{\ell-1}$. For this case we contradict that $x(E(S^*)) > |S^*| - 1$ by showing that $x(\delta(S^*)) \geq 2$. Since $g_{\ell-1} > g_\ell = 1$, there are at least two $C^{\ell-1}$ and they are disjoint. We will use the following claim to argue that each of them contributes at least 1 to $x(\delta(S^*))$.

Claim 7.9. *Let C be a set such that $x(\delta(C)) = 2$. Suppose that $C = A \sqcup B$ where $x(\delta(A)) \geq 2$ and $x(\delta(B)) \geq 2$. Then $x(\delta^+(A, B)) + x(\delta^+(B, A)) \geq 1$.*

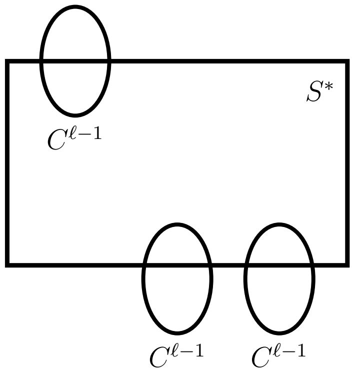

Figure 7.3: Case 3 in the proof of Proposition 7.8. S^* intersects with every $C^{\ell-1}$ but does not fully contain any of the $C^{\ell-1}$.

This claim follows by expanding $\delta(A)$ and $\delta(B)$ and rearranging.

$$
\begin{aligned}
4 &\leq x(\delta(A)) + x(\delta(B)) \\
&= \big(x(\delta^+(A, V\backslash C)) + x(\delta^+(V\backslash C, A)) + x(\delta^+(A, B)) + x(\delta^+(B, A))\big) \\
&\qquad + \big(x(\delta^+(B, V\backslash C)) + x(\delta^+(V\backslash C, B)) + x(\delta^+(A, B)) + x(\delta^+(B, A))\big) \\
&= x(\delta(C)) + 2\left(x(\delta^+(A, B)) + x(\delta^+(B, A))\right) \\
&= 2 + 2\left(x(\delta^+(A, B)) + x(\delta^+(B, A))\right),
\end{aligned}
$$

from which the claim follows.

We now apply Claim 7.9. Let $C_i^{\ell-1}$ take the role of C, since by Lemma 7.5 $x(\delta(C_i^{\ell-1})) = 2$. We partition $C_i^{\ell-1} = A \sqcup B$ where $A = S^* \cap C_i^{\ell-1}$ and $B = C_i^{\ell-1}\backslash A$. Then $A, B \subset C_i^{\ell-1}$ and the fact that we are not in case 1 implies that $x(\delta(A)) \geq 2$ and $x(\delta(B)) \geq 2$ and the claim yields

$$
x(\delta^+(S^* \cap C_i^{\ell-1}, C_i^{\ell-1}\backslash A)) + x(\delta^+(C_i^{\ell-1}\backslash A, S^* \cap C_i^{\ell-1}) \geq 1.
$$

All together,

$$\begin{aligned} x(\delta(S^*)) &\geq \sum_{i=1}^{g_{\ell-1}} x(\delta^+(S^* \cap C_i^{\ell-1}, C_i^{\ell-1} \backslash A)) + x(\delta^+(C_i^{\ell-1} \backslash A, S^* \cap C_i^{\ell-1}) \\ &\geq \sum_{i=1}^{g_{\ell-1}} 1 \\ &= g_{\ell-1} \geq 2. \end{aligned}$$

Hence we contradict that $x(E(S^*)) > |S^*| - 1$ and we have handled all cases.

□

Proof (Theorem 7.1). This proof follows immediately from Lemma 7.3 and Proposition 7.8.
□

We note that Theorem 7.1, together with the proof of Theorem 6.7 in De Klerk and Dobre [22], indicate the following result.

Corollary 7.10. *The degree constraints do not strengthen the subtour elimination LP for circulant TSP. That is, letting $OPT_{Relaxed}$ denote the value of an optimal solution to the subtour LP relaxation obtained by dropping the degree constraints,*

$$OPT_{Relaxed} = OPT_{LP} = VDV.$$

Proof. Our proof of Theorem 7.1 shows that $\mathrm{OPT}_{\mathrm{Relaxed}} \leq \mathrm{VDV}$, while the proof of Theorem 6.7 shows that $\mathrm{VDV} \leq \mathrm{OPT}_{\mathrm{Relaxed}}$. □

Note that this property mirrors the *parsimonious property* of metric TSP instances shown in Goemans and Bertsimas [33]: for metric edge costs, Goemans and Bertsimas show also that $\mathrm{OPT}_{\mathrm{Relaxed}} = \mathrm{OPT}_{\mathrm{LP}}$

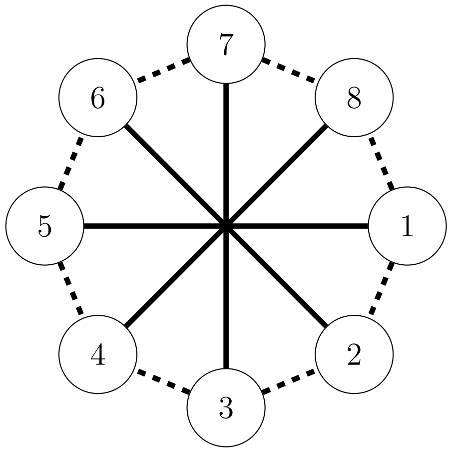

Figure 7.4: An example of a class of instances showing that the integrality gap of the subtour LP restricted to circulant instances is at least 2. The dashed edges have weight 1/2 and cost 1, while the full edges have weight 1 and cost 0.

7.2 The Integrality Gap of the Subtour LP

Theorem 7.1 allows us to exactly characterize the integrality gap of the subtour LP on circulant instances by considering the Van der Veen, Van Dal, and Sierksma [77] bound. In this section we provide an example showing that this bound can be off by a factor of 2 asymptotically. This, together with Theorem 6.8, will imply our second main theorem of this chapter.

Theorem 7.11. *The integrality gap of the subtour LP restricted to circulant instances is exactly 2.*

The example we use to prove this theorem is intimately related to the crown inequalities for the TSP, as we discuss in Section 7.3.

Proof. Theorem 6.8 implies that the integrality gap is at most 2. To prove the theorem it thus suffices to demonstrate an example where the Van der Veen, Van Dal, and Sierksma [77] bound is a factor of two away from the optimal TSP solution. For such an example, we take $n = 2^{k+1}$ so that $d = n/2 = 2^k$. Suppose that $c_1 = 1, c_d = 0$, and $c_i > 2^{k+1}$ otherwise. Then $\phi(1) = d$ and $\phi(2) = 1$, so that $g_1^\phi = d$, $g_i^\phi = 1$ for $i \geq 2$, and $\ell = 2$. By Theorem 7.1, the optimal solution to the subtour LP has cost

$$\text{VDV} = \left(\sum_{i=1}^{\ell}(g_{i-1}^\phi - g_i^\phi)c_{\phi(i)}\right) + c_{\phi(\ell)} = d \cdot 0 + d \cdot 1 = d = 2^k.$$

See Figure 7.4 for a picture of the corresponding subtour LP solution.

Now consider an optimal solution to the TSP. It cannot use any edges other than those of lengths 1 and d: we can find a tour of cost 2^{k+1} by just taking edges of length 1 (i.e. $\{1, 2\}, \{2, 3\}, ..., \{n-1, n\}, \{n, 1\}$), while edges of any length other than 1 or d cost strictly greater than 2^{k+1}. Now consider any Hamiltonian cycle using only edges from these cheapest two stripes, and consider it as a directed cycle as in Theorem 7.1. Suppose that it uses s_1 edges of length 1 (where we interpret a directed edge $(u, u+1)$ as having length 1), s_{-1} edges of length -1 (where we interpret a directed edge $(u, u-1)$ as having length -1), and $n - s_1 - s_{-1}$ edges of length $d = 2^k$. Because n is even, there is no difference between an edge of length d and $-d$: $v + d \equiv_n v - d$.

Claim 7.12. *Any Hamiltonian cycle satisfies*

$$s_1 - s_{-1} \equiv_n 0.$$

With the notation above, a Hamiltonian tour uses $n - s_1 - s_{-1}$ edges of length $d = 2^k$. Since it starts and ends at the same vertex,

$$2^k(n - s_1 - s_{-1}) + s_1 - s_{-1} \equiv_n 0 \implies s_1 - s_{-1} \equiv_n 2^k(s_1 + s_{-1}). \tag{7.3}$$

Since n and 2^k are even, $(2^k(s_1 + s_{-1})) \bmod n$ is even; for the left and right sides to have the same parity, $s_1 - s_{-1}$ must therefore also be even. Moreover

$$s_1 + s_{-1} = s_1 - s_{-1} + 2s_{-1}$$

so that $s_1 + s_{-1}$ is the sum of two even numbers and is therefore even. Consider again Equation (7.3). Since $s_1 + s_{-1}$ is even and $2^k = \frac{n}{2}$,

$$2^k(s_1 + s_{-1}) \equiv_n 0.$$

Thus Equation (7.3) implies

$$s_1 - s_{-1} \equiv_n 0,$$

and Claim 7.12 follows.

Since $s_1, s_{-1} \in [n]$, we have that

$$s_1 - s_{-1} \in \{-n, 0, n\}.$$

The cases where $|s_1 - s_{-1}| = n$ imply a tour only using edges of length 1; i.e., a tour of cost $n = 2^{k+1}$. Thus we need only consider the case where $s_1 = s_{-1}$. Here we analogize an argument from Theorem 5.2 in Greco and Gerace [37].

Claim 7.13. *A tour using just edges of lengths* $1, -1$ *and* d *visits* $\max\{s_1, s_{-1}\} + 1$ *components in of* $C\langle\{2^k\}\rangle$*. Hence, a Hamiltonian tour requires* $\max\{s_1, s_{-1}\} + 1 \geq \frac{n}{2} = 2^k$.

We note that the graph $C\langle\{2^k\}\rangle$ using just edges of length 2^k has 2^k connected components $C_1^1, C_2^1, ..., C_{2^k}^1$. We identify C_i^1 as consisting of the two vertices $\{i, 2^k + i\}$ connected by a single edge of length 2^k.

Let $L = (e_1, ..., e_n)$ be a list of edges in any Hamiltonian tour using just edges of lengths $1, -1$ and d, so that $e_i \in \{-1, 1, d\}$ for $i = 1, ..., n$. From this list, we can bound the number of components of $C\langle\{2^k\}\rangle$ visited: first, we can delete any edges of length d: they do not cause us to change components of $C\langle\{2^k\}\rangle$; any length 1 edge connects C_i^1 to C_{i+1}^1, while any length -1 edge connects C_i^1 to C_{i-1}^1 (regardless of whether or not any length d edges are used). Hence we need only consider the subsequence L' of L just consisting of edges of lengths 1 and -1 obtained by deleting the edges of length d. Formally,

$$L' = (e_{i_1}, ..., e_{i_k}) : i_1 < i_2 < \cdots i_k, e_{i_j} \in \{\pm 1\}.$$

We upper bound the number of components of $C\langle\{2^k\}\rangle$ visited directly from L' as follows: Set $U = 1$, corresponding to starting at some component. Until L' is either all 1s or all -1s, find an occurrence of a 1 followed by a -1 in L' (or a -1 followed by a 1); delete these two elements and increment U by 1. Once this process terminates, increment U by $|L'|$ (the number of 1s or -1s remaining when L' is either all 1s or all -1s). Note that, at the end, $U = \max\{s_1, s_{-1}\} + 1$. U provides an upper bound on the number of components of $C\langle\{2^k\}\rangle$ visited: Any time a 1 is followed by a -1 in L, the effect is to move from C_i^1 to C_{i+1}^1, then back to C_i^1. Hence we visit at most one new component, C_{i+1}^1. It is analogous any time a -1 is followed by a 1. Thus Claim 7.13 holds.

Since any Hamiltonian cycle must visit every component of $C\langle\{2^k\}\rangle$, we need

$$\max\{s_1, s_{-1}\} + 1 \geq \frac{n}{2} = 2^k.$$

That is, we need at least $2^k - 1$ length 1 edges, or $2^k - 1$ length -1 edges, to connect all components.

Putting Claims 7.12 and 7.13 together, we find that we need

$$s_1, s_{-1} \geq 2^k - 1,$$

so that $\mathrm{OPT_{TSP}} \geq 2^{k+1} - 2$. We can find such a tour to establish equality:

$$\{1,2\}, \{2,3\}, ..., \{2^k - 1, 2^k\}, \{2^k, n\}, \{n, n-1\}, ..., \{2^k + 2, 2^k + 1\}, \{2^k + 1, 1\}.$$

See, for example, Figure 7.5. Thus

$$\frac{\mathrm{OPT_{TSP}}}{\mathrm{VDV}} = \frac{2^{k+1} - 2}{2^k} \to 2.$$

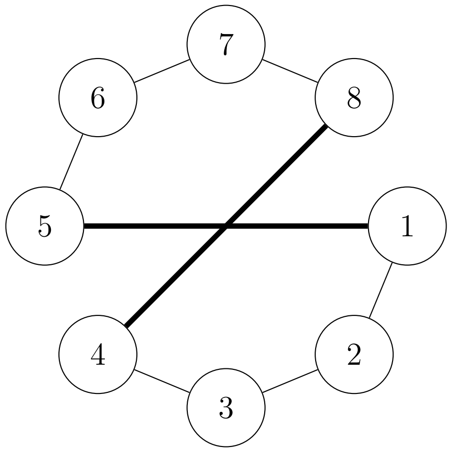

Figure 7.5: An optimal TSP solution for the instance in Theorem 7.11. Thick edges have cost 0 while thin edges have cost 1 so that this solution has cost $2^3 - 2 = 6$.

□

7.3 Avenues to Improving the Integrality Gap

Having exactly found the integrality gap of the subtour LP on circulant TSP instances, the natural question is how to strengthen it. Several inequalities have been investigated that might strengthen the subtour LP on metric instances (see, e.g., Goemans [32]). A first step is to see if these (or any other inequalities) might remove our bad instances (see Figure 7.4). We note that our instance achieving the worst-case integrality gap also appears in Naddef and Rinaldi [61] where they construct explicit facet-defining inequalities that remove it from the subtour LP in a non-circulant setting. These inequalities are the **crown inequalities** and take the form

$$\alpha^T x \geq \alpha_0 := 12s(s-1) - 2, \quad n = 4s,$$

where the weight α_e that α places on edge e is based only on the length of edge e:

$$\alpha(v, v+j) = \begin{cases} 4s - 6 + j, & j < d \\ 2(s-1), & j = d. \end{cases}$$

Here, for example, the crown inequalities place a weight of $2(s-1)$ on each of the d edges from the dth stripe, and a weight of $4s-5$ on each edge in the first stripe. The subtour LP solution places a weight of 1 on each of the d edges of length d, and 1/2 on each of the n length 1 edges. Since $d = 2s$:

$$\alpha^T x = 2s(2s-2) + \frac{1}{2}4s(4s-5) = 2s(6s-7) = 12s^2 - 14s < \alpha_0 = 12s^2 - 12s - 2$$

so that they are violated for any example where $n = 4s$ and $s > 1$.

Unfortunately, adding these constraints does not reduce the integrality gap from 2. We can instead consider solutions to the subtour LP that place marginally less weight on the d-edges and marginally more weight on the 1-edges. If we let λ be the weight on the n edges of length 1 (on which α places weight $4s - 5 = n - 5$), then $2 - 2\lambda$ is the weight on each of the $d = \frac{n}{2}$ edges of length d (on which α places a weight of $2s - 2 = \frac{n}{2} - 2$). The right hand side of the crown inequalities is $12\frac{n}{4}\left(\frac{n}{4} - 1\right) - 2 = \frac{3}{4}n^2 - 3n - 2$, so we can solve for

$$\lambda n(n-5) + (2-2\lambda)\frac{n}{2}\left(\frac{n}{2} - 2\right) \geq \frac{3}{4}n^2 - 3n - 2 \rightarrow \lambda \geq \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}$$

(assuming that $n > 6$). Hence, setting

$$\lambda = \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}$$

suffices to find a solution that satisfies the subtour elimination constraints and the crown inequalities, but does not reduce the integrality gap.

Proposition 7.14. *Adding the crown inequalities does not change the integrality gap of the subtour LP when restricted to circulant instances.*

Proof. We take our solution above, setting

$$\lambda = \frac{n^2 - 4n - 8}{2n^2 - 12n} = \frac{1}{2} + \frac{2}{3n} + \frac{1}{3(n-6)}$$

and placing a weight λ on the 1-edges (the dashed edges in Figure 7.4) and $2 - 2\lambda$ on the edges of length d (the full edges in Figure 7.4). Note that this solution is still feasible for the subtour LP: we are taking a convex combination of the instance in Theorem 7.11 and the Hamiltonian cycle using just 1-edges. This thus lower bounds the integrality gap as:

$$\frac{\text{OPT}_{\text{TSP}}}{\text{OPT}_{\text{LP}}} = \frac{n-2}{n\lambda} \to 2$$

as $n \to \infty$, where $n = 2^{k+1}$. □

We note that the ladder and chain inequalities (see Boyd and Cunningham [6], Padberg and Hong [65]) can similarly be added to remove the solutions constructed in Theorem 7.11 but do not reduce the integrality gap from 2.

We conjecture that the following inequalities are valid.

Conjecture 7.15. *The following inequality, if valid, would strengthen the subtour LP in the symmetric circulant case. If* $4|n$, *then*

$$\sum_{i=1}^{d} \alpha_i \left(\sum_{\substack{e \in E: \\ \mathit{length}(e)=i}} x_e \right) \geq n - 2, \quad \alpha_i = \begin{cases} i, & \text{if } i \text{ odd} \\ d - i, & \text{if } i \text{ even.} \end{cases}$$

Finally, as noted earlier, it is a major open question whether or not circulant TSP is polynomial-time solvable. The answer is not known even in the case where only two stripes have finite cost. It would be interesting to see if some of the tools developed recently for the metric TSP might be able to resolve this decades-long open question.

CHAPTER 8

THE SUBTOUR LP CONSTRAINTS AND THE MATRIX-TREE THEOREM CONSTRAINT

In Chapter 2, we discussed an SDP constraint for the TSP based on Kirchoff [50]'s matrix-tree theorem and due to de Klerk, Pasechnik, and Sotirov [23]. This constraint says that "the aggregate weight of spanning trees in a TSP solution must be at least n" and has the form

$$\det\left((2I - X)_{-1}\right) \geq n. \tag{2.7}$$

Equivalently,

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

De Klerk et al. note that this constraint strengthens the algebraic connectivity SDP (2.2), but in Chapter 3 we showed that it does not strengthen in the integrality gap of the association scheme SDP (2.3).

The main result of this short chapter is that the matrix-tree theorem constraint of de Klerk et al. [23] is also implied by the subtour elimination linear program constraints. This result is akin to Goemans and Rendl [34], which shows that the constraints used in the SDP relaxation of Cvetković et al. [18] are implied by the subtour LP.

8.1 The Subtour LP Constraints Imply the Matrix-Tree Theorem Constraint

The main result of this chapter is:

Theorem 8.1. *Let $x \in \mathbb{R}^E$ be a feasible solution to the subtour LP and let G be the complete* *graph. Let X be the symmetric matrix where $X_{ij} = X_{ji} = x_{\{i,j\}}$ and $X_{ii} = 0$ for all i. Then X satisfies the matrix-tree theorem constraint (2.7):*

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

To prove our main result, we will use the following result from Ok and Thomassen [63] which relates edge-connectivity to spanning trees. An unweighted, undirected, loopless multigraph $G = (V, E)$ is k-**edge-connected** if G is still connected after the removal of any $k - 1$ edges.

Theorem 8.2 (Theorem 1 in Ok and Thomassen [63])**.** *Let G be a weighted, loopless, undirected multigraph that is k-edge-connected. Then G has at least $n\left(\frac{k}{2}\right)^{n-1}$ spanning trees.*

We first use it to prove the following:

Proposition 8.3. *Let $G = (V, E)$ be a weighted simple graph with rational edge weights given by $x \in \mathbb{R}^E$. If x is an extreme point of the subtour LP, then*

$$\sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

Theorem 8.1 will then follow as an immediate consequence.

To prove Proposition 8.3, we start with a symmetric, simple weighted graph $G = (V, E)$ with edge weights given by $x \in \mathbb{R}^E$, an extreme point of the subtour LP. Because x is an extreme point, x is rational[1], and we will be able to scale x so that $Rx \in \mathbb{Z}^E$. Then we let $G' = (V, E')$ be an undirected, loopless, unweighted multigraph with Rx_e copies of

[1] Extreme points occur where a certain number of constraints hold with equality. Cramer's rule, e.g., shows that if the constraints of a linear program have rational coefficients, then every extreme point is rational. This is the case for the subtour LP.

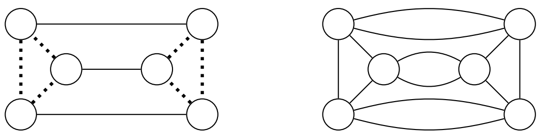

Figure 8.1: The left shows a simple, weighted graph G where dashed edges have weight 1/2 and full edges have weigh 1. In this case $R = 2$ and the right shows the corresponding unweighted multigraph G'.

edge e. See Figure 8.1. Moreover, if $x(\delta(S)) \geq 2$ then $Rx(\delta(S)) \geq 2R$ so that G' will be $2R$-edge-connected. We can then appeal to Theorem 8.2, find a large number of spanning trees, and find corresponding spanning trees in G.

We first verify that the aggregate weight of spanning trees in G' (as an unweighted multigraph with Rx_e copies of edge e) matches the aggregate weight of spanning trees in G (as a weighted simple graph where edge e has weight Rx_e). To do so, we apply the following lemma iteratively.

Lemma 8.4. *Let G be a weighted loopless multigraph. Let $e = \{u, v\} \in G$ and let G' be obtained from G by splitting e into two copies $e_1 = e_2 = \{u, v\}$ and assigning nonnegative weights x' to the edges in G' so that $x_e = x'_{e_1} + x'_{e_2}$ (and $x_f = x'_f$ for all other edges f in G). Then*

$$\sum_{T \in \mathcal{T}_G} \prod_{f \in T} x_f = \sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f.$$

In the proof, we use $\sqcup$ to denote a partition: $S = A \sqcup B$ means $S = A \cup B$ and $A \cap B = \emptyset$. We also use $\setminus$ for set-minus, so that $S \setminus A = \{x \in S : x \notin A\}$.

Proof. This result follows by partitioning $\mathcal{T}_{G'}$. No $T \in \mathcal{T}_{G'}$ can contain both e_1 and e_2 so we write

$$\mathcal{T}_{G'} = \mathcal{T}^0_{G'} \sqcup \mathcal{T}^1_{G'} \sqcup \mathcal{T}^2_{G'}$$

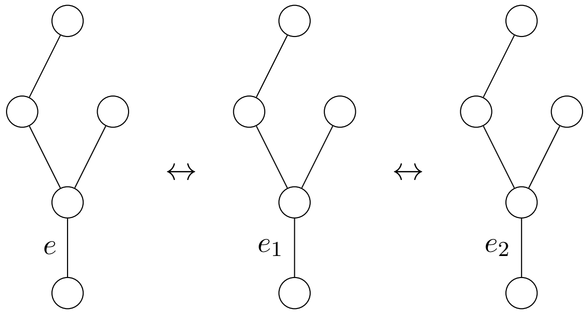

Figure 8.2: A sample tree instantiation in $\mathcal{T}_G^e, \mathcal{T}_{G'}^1$, and $\mathcal{T}_{G'}^2$.

where $\mathcal{T}_{G'}^i$ consists of those spanning trees including edge e_i for $i = 1, 2$ and $\mathcal{T}_{G'}^0$ consists of those trees including neither e_1 nor e_2. We analogously partition

$$\mathcal{T}_G = \mathcal{T}_G^0 \sqcup \mathcal{T}_G^e,$$

where $\mathcal{T}_G^0$ consists of spanning trees not using e and $\mathcal{T}_G^e$ consists of spanning trees using e.

The trees in $\mathcal{T}_{G'}^1$, $\mathcal{T}_{G'}^2$, and $\mathcal{T}_G^e$ all use exactly one edge $\{u, v\}$ (and other than using exactly one of e_1, e_2, or e as $\{u, v\}$, draw from exactly the same set of edges). Hence if $T \in \mathcal{T}_{G'}^1$, then $(T \backslash e_1) \cup e_2 \in \mathcal{T}_{G'}^2$ and $(T \backslash e_1) \cup e \in \mathcal{T}_G^e$. This process gives a one-to-one correspondence between trees in $\mathcal{T}_{G'}^1$, $\mathcal{T}_{G'}^2$, and $\mathcal{T}_G^e$; see Figure 8.2. Analogously, $\mathcal{T}_{G'}^0 = \mathcal{T}_G^0$. Hence:

$$\begin{aligned}
\sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f &= \sum_{T \in \mathcal{T}_{G'}^0} \prod_{f \in T} x'_f + \sum_{T \in \mathcal{T}_{G'}^1} \prod_{f \in T} x'_f + \sum_{T \in \mathcal{T}_{G'}^2} \prod_{f \in T} x'_f \\
&= \sum_{T \in \mathcal{T}_G^0} \prod_{f \in T} x_f + (x'_{e_1} + x'_{e_2}) \sum_{T \in \mathcal{T}_{G'}^1} \prod_{f \in T, f \neq e_1} x'_f \\
&= \sum_{T \in \mathcal{T}_G^0} \prod_{f \in T} x_f + x_e \sum_{T \in \mathcal{T}_G^e} \prod_{f \in T, f \neq e} x_f \\
&= \sum_{T \in \mathcal{T}_G^0} \prod_{f \in T} x_f + \sum_{T \in \mathcal{T}_G^e} \prod_{f \in T} x_f \\
&= \sum_{T \in \mathcal{T}_G} \prod_{f \in T} x_f.
\end{aligned}$$

□

We now prove our main theorem in the special case of subtour LP extreme points.

Proof (of Proposition 8.3). Let x be a feasible extreme point of the subtour LP. Then $x(\delta(S)) \geq 2$ for all S with $1 \leq |S| \leq |V| - 1$ and, moreover, it is well-known that x is rational (as mentioned above, this can be shown via Cramer's rule).

We first convert G into a loopless, unweighted multigraph. To do so, suppose that $x_{e_i} = \frac{s_i}{r_i}$ in lowest terms. Let

$$R = \mathcal{LCM}(r_1, ..., r_m).$$

Let G' denote the graph G with weights $x'_e = Rx_e$ for all $e \in G$ and let $X' = RX$. Then we make two observations: $x'_e \in \mathbb{Z}$ for all e, and properties of deteriminants imply

$$\det\left((L(X')_{-1})\right) = R^{n-1} \det\left((L(X)_{-1})\right). \tag{8.1}$$

To compute $\det\left((L(X')_{-1})\right)$ we appeal to the matrix-tree theorem; by Lemma 8.4 it is equivalent (in terms of the aggregate weight of spanning trees) to view G' as a loopless unweighted multigraph where there are x'_e copies of edge e (so that there are $s_i \frac{R}{r_i} \in \mathbb{Z}$ copies of edge e_i, each of weight 1). See Figure 8.1.

Note that $x(\delta(S)) \geq 2$ implies that $x'(\delta(S)) \geq 2R$. Thus G' is $2R$-edge-connected and by Theorem 8.2, G' has at least nR^{n-1} spanning trees; since every edge of G' has weight 1, the matrix-tree theorem states

$$\det\left((L(X')_{-1})\right) = \sum_{T \in \mathcal{T}_{G'}} \prod_{f \in T} x'_f \geq nR^{n-1}.$$

Combining with Equation 8.1 we have:

$$R^{n-1}\det\left((L(X)_{-1})\right) = \det\left((L(X')_{-1})\right)$$
$$\geq nR^{n-1}.$$

That is,

$$\det\left((L(X)_{-1})\right) \geq n$$

and the matrix-tree theorem implies

$$\sum_{T\in\mathcal{T}_{\mathcal{G}}} \prod_{e\in T} x_e \geq n.$$

□

We can now show that the matrix-tree theorem constraint (2.7) holds for any feasible point of the subtour LP. We restate our main theorem in slightly more detail.

Theorem (Theorem 8.1, restated)**.** *Let $x \in \mathbb{R}^E$ be a feasible solution to the subtour LP and let G be the complete graph. Let X be the symmetric matrix where $X_{ij} = X_{ji} = x_{\{i,j\}}$ and $X_{ii} = 0$ for all i. Then X satisfies the matrix-tree theorem constraint:*

$$det\left((2I - X)_{-1}\right) \geq n.$$

Equivalently,

$$\sum_{T\in\mathcal{T}_G} \prod_{e\in T} x_e \geq n.$$

Proof. The subtour LP is bounded, so that every feasible x for the subtour LP can be written as a convex combination of extreme points to the subtour LP. For any extreme point of the subtour LP y, let Y be the matrix where $Y_{ij} = Y_{ji} = y_{\{i,j\}}$ and $Y_{ii} = 0$. Feasibility of the subtour LP means $y(\delta(i)) = 2$ for all $i \in V$, so the associated Laplacian is $2I - Y$. By Proposition 8.3 and the matrix-tree theorem, $\det\left((2I - Y)_{-1}\right) \geq n$.

We now show that any convex combination of two extreme points of the subtour LP also satisfies the matrix-tree theorem constraint; extending to general convex combinations is left as an exercise. Note that the determinant is well-known to be log concave on symmetric positive definite matrixes (see, e.g., section 3.1 of Boyd and Vandenberghe [5]), so that $\det(tA + (1-t)B) \geq \det(A)^t \det(B)^{1-t}$ for $0 \leq t \leq 1$ if $A, B \succ 0$.

Consider two extreme points of the subtour LP, with weighted adjacency matrices A and B. Denote their graph Laplacians as $L(A) = 2I - A$ and $L(B) = 2I - B$ respectively. For a graph with weighted adjacency matrix X, all principal subminors of $L(X)$ are nonnegative so that all principal subminors of $L(X)_{-1}$ are as well: these are just the principal subminors of $L(X)$ that include row/column 1 being removed. This implies that $L(X)_{-1} \succeq 0$. By Proposition 8.3, $\det\left((L(A)_{-1})\right), \det\left((L(B)_{-1})\right) \geq n$ so that zero cannot be an eigenvalue of $(L(A)_{-1})$ or $(L(N)_{-1})$ and so both are positive definite. By log-concavity,

$$\begin{aligned}
&\det(t\,(L(A)_{-1}) + (1-t)\,(L(B)_{-1})) \\
&\quad \geq \left(\det\left(L(A)_{-1}\right)\right)^t \left(\det\left(L(B)_{-1}\right)\right)^{1-t} \\
&\quad \geq n^t n^{1-t} \\
&\quad = n.
\end{aligned}$$

Hence, $tA + (1-t)B$ satisfies the matrix-tree-theorem constraint (2.7). □

Remark 8.5. *Note that the proof of Theorem 8.1 holds for any x such that $x(\delta(S)) \geq 2$ for each $S \subset V$ with $1 \leq |S| \leq |V| - 1$. Hence, any x with $x(\delta(S)) \geq 2$ for all such S and corresponding weighted adjacency matrix X satisfies*

$$det\left(L(X)_{-1}\right) = \sum_{T \in \mathcal{T}_G} \prod_{e \in T} x_e \geq n.$$

However, it need not be the case that that rows of X sum to 2, so possibly $L(X) \neq 2I - X$; the result holds for $det\left(L(X)_{-1}\right)$ rather than $det\left(\left(2I - X\right)_{-1}\right)$.

Theorem 8.1 has several implications. Goemans and Rendl [34] show that the subtour LP is stronger than a TSP SDP relaxation of Cvetković et al. [18] in the following sense: Any feasible solution for the subtour LP corresponds to a feasible solution of the same cost for the SDP. Hence, on any given instance, the optimal value of the subtour LP is at least as close to the cost of a TSP solution as the optimal value of the SDP. Theorem 8.1 gives a comparable weakness result for the matrix-tree theorem constraint (2.7). Moreover, it implies that Goemans and Rendl [34]'s result extends to the case where the matrix-tree theorem constraint (2.7) is added to the SDP of Cvetković et al. [18]. More generally, our results show that matrix semidefinite inequalities used to impose the matrix-tree theorem are implied by a set of linear inequalities.

CHAPTER 9

CONCLUSION

This thesis contributes to two prongs of TSP research. First, a variety of SDP relaxations of the TSP have recently been proposed. These relaxations have been motivated by a breadth of mathematical ideas, and often perform very strongly in small numerical experiments. The symmetry reduction SDP (2.5), in particular, provided a same or better lower bound than the state-of-the-art subtour LP on 23 of 24 instances tested in de Klerk and Sotirov [25].

However, there has been no substantial theoretical analysis of these SDP's since Goemans and Rendl [34] showed that that the algebraic connectivity SDP of Cvetković, Čangalović, and Kovačević-Vujčić [18] was never better than the subtour LP in 2000. The first main results of this thesis fill the void through a unified set of techniques: by using highly symmetric simplicial TSP instances, we showed that every major SDP relaxation of the TSP has an unbounded integrality gap. For these SDPs, our instances reveal an unsettling lack of monotonicity.

Second, we turned to the subtour LP. We showed that its constraints imply a matrix-tree theorem-based SDP constraint of de Klerk, Pasechnik, and Sotirov [23]. We then studied it on circulant TSP instances, a setting where many fundamental TSP questions remain open. In this setting, de Klerk and Dobre [22] conjectured that the subtour LP was equivalent to a closed-form combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [77]. We prove that this conjecture is true, allowing us to exactly compute the integrality gap of the subtour LP on circulant instances.

In both cases, the main results of this thesis followed by exploiting symmetry. When analyzing SDP relaxations, we found solutions that respect the symmetry of simplicial TSP instances; restricting ourselves to such a structured class allowed us to simplify complex SDP constraints. When analyzing circulant TSP instances, we "spread out" the combinatorial lower bound of Van der Veen, Van Dal, and Sierksma [77] along all edges of the same stripe.

Our hope is that this thesis invigorates interest in SDP relaxations of the TSP and in circulant TSP, and both settings have many open questions. Our simplicial TSP instances showed that every major SDP relaxation of the TSP had an unbounded integrality gap, and the natural subsequent question is to find an SDP relaxation robust to those specific instances. The subtour LP lower bound exactly matches the TSP optimal value on simplicial TSP instances, and a related question is to find the integrality gap of the SDP relaxations discussed in this thesis with the subtour elimination constraints added (or of a 'best of SDP and subtour LP' relaxation that independently runs an SDP and the subtour LP on the same instance, and then takes the best of both bounds). Using any of these approaches to find an integrality gap of $1.5 - \epsilon$ for some absolute constant $\epsilon > 0$ would be a major TSP development.

We have also showed that these SDP relaxations have an unbounded integrality gap on general metric instances as well as specialized to Euclidean TSP instances. Gutekunst and Williamson [39] show that the association scheme SDP (2.3)'s integrality gap on *graphic TSP instances* is at most 2. The same result holds for any SDP imposing the degree constraints: the minimum-cost Hamiltonian cycle is at most twice the cost of a MST (see Section 2.4 of Williamson and Shmoys [79], for example), and the cost of an MST in a connected graph with unit edge weights is $n - 1$. Hence the maximum cost of any graphic TSP instance is $2n - 2$. Conversely, in graphic TSP the minimum cost of an edge is 1, so that any relaxation imposing the degree constraints (or even just that the aggregate weight of edges used is n) will output a lower bound of at least n. An open question is to exactly compute the integrality gap of the SDPs in this thesis on graphic TSP instances; we conjecture that the integrality gap of the association scheme SDP, for example, is at least 1.5 and is asymptotically achieved when G is a path, and we conjecture that the symmetry reduction SDP does not strengthen the association scheme SDP's integrality gap.

The major question in circulant TSP is whether or not circulant TSP is NP-hard. A starting point is circulant TSP with a constant number of stripes of finite cost: even in the case where just two stripes have finite cost, it is unknown if there exists a polynomial-time algorithm to find the optimal tour!

Our work in this thesis exactly characterizes the performance of the subtour LP relaxation on circulant TSP instances. In searching for stronger relaxations, one might first ask if any inequalities can be added to the subtour LP to strengthen its integrality gap on circulant instances? For example, are the inequalities conjectured at the end of Chapter 7 valid and do they strengthen the integrality gap? Similarly, the integrality gap of the SDPs described in this thesis on circulant instances remains open.

Finally, in circulant TSP, the cost is solely determined by the total number of edges used from each stripe. Denoting by t_i the total number of edges used from the ith stripe,

$$t_i = \sum_{\substack{e \in E: \\ \text{length}(e) = i}} x_e,$$

one might wonder if there is an easily-describable relaxation in terms of the variables $t_1, t_2, ..., t_d$.

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