GENERAL BINARY BALANCED BLOCK DESIGN

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BU-599-M* January 1977

SUMMARY

The concept of binary balanced incomplete block designs whose incidence matrix $\underline{n}$ takes only two values 0 or 1, is extended to general binary balanced block designs whose incidence matrix $\underline{n}^*$ takes two values m_0 and m_1, where $0 \leq m_0 < m_1$. Parameters and necessary conditions for $\underline{n}^*$ are evaluated. Given a fixed number of units N^* (say) and a fixed number of treatments v (say), more than one general binary balanced block designs for different values of m_0 and m_1 are possible. A criterion to pick an optimal design from its class is derived.

Some key words: Basic binary balanced block design (BBBIBD). General binary balanced block design (GBBBD). Incidence matrix, orthogonal, variance optimality.

1. INTRODUCTION

Given a set $V = \{(a_i), i=1,2,\cdots,v\}$ of v elements, a collection $\mathcal{B} = \{(B_j); |B_j| = k_j \in N^+, j=1,2,\cdots,b\}$ is called a block design if:

(a) k_j elements (distinct or otherwise) of the set V are assigned to B_j and

(b) a_i is assigned to r_i elements of the collection $\mathcal{B}$ such that $\Sigma_{i=1}^{v} a_i = \Sigma_{j=1}^{b} k_j = N$.

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* Paper No. BU-599-M in Mimeo Series of the Biometrics Unit, Cornell University.

The elements of V and $\mathcal{B}$ are called treatments and blocks respectively; N^+ is the set of block sizes k_j . The combinatorial properties of the above setting are invariant to any ordering of the blocks or of the treatments within blocks; hence, when the treatments and blocks are randomly permuted in an experiment, the combinatorial properties are preserved. A block design may be presented by a v × b matrix, the incidence matrix and denoted by $\underline{n} = (n_{ij})$. The $(i,j)^{th}$ entry of $\underline{n}$ represents the frequency n_{ij} of the i^{th} treatment in the j^{th} block. A block design is said to be <u>general binary</u> if for some non-negative integers, m_0 and m_1, $n^*_{ij} = m_0$ or m_1, where $0 \leq m_0 < m_1$ for all i and j. If $n_{ij} = 0$ or 1, the design is said to be <u>basic binary</u> or more simply <u>binary</u>.

Writers of statistical and mathematical papers have confined their attention predominantly to the case where $n_{ij} = 0$ or 1, that is, the basic binary. Consequently, experimenters are left with the impression that no other design exists. In several situations, the block sizes k_j may be larger than the number of treatments v. For example, the litter sizes or the family sizes may be greater than the number of nutritional treatments under consideration. One procedure in current usage is to discard material randomly until the number of treatments equals the block size. Discarding experimental material to achieve equality of treatment numbers and block sizes is an unjustifiable procedure in terms of cost of experimental material. If one would consider generalized block designs, all the experimental material could be utilized.

Another situation wherein $n_{ij} = 0$ or 1, $k_j < v$, and a binary block design may not be appropriate, is when an estimate of the variance for individual treatments or for pooled variance within blocks is desired to test for block by treatment interaction. Here one could use an incomplete block design with $k_j \leq v$ and use $n_{ij} = 0$ or $m > 1$, or one could use a binary block design with

$n_{ij} = m_0$ or m_1 for $m_1 > m_0 > 0$, for example. Of course, there is no reason to confine ones self to binary designs.

Since balanced incomplete block designs have equal variance (variance balance) for all differences among treatment effects (under homoscedasticity assumptions), such designs may be preferred if they exist. If not, one may use a partially balanced block design which is as near variance balance as possible. It should be noted, however, that variance balanced block designs may not always be variance optimal designs. For example, consider the following two binary block designs:

Design (i)

Balanced block design

$m_0 = 1$, $m_1 = 3$

$v = b = 4$, $k = r = 6$, $\lambda = 8$

Blocks

1	2	3	4
A	A	A	A
B	B	B	B
C	C	C	C
D	D	D	D
A	B	C	D
A	B	C	D

average intrablock variance = $.40\sigma^2$

Design (ii)

Partially balanced block design

$m_0 = 1$, $m_1 = 2$

$v = b = 4$, $k = r = 6$, $\lambda_1 = 8$, $\lambda_2 = 9$

Blocks

1	2	3	4
A	A	A	A
B	B	B	B
C	C	C	C
D	D	D	D
A	C	A	B
B	D	C	D

average intrablock variance

$= [4(23) + 2(24)]/6(66)$

$= .35\sigma^2$

Likewise, if one extends the class of designs to all n-ary designs (Tocher, 1952), then it is possible to have some partially balanced incomplete block designs which have smaller average variance than a balanced block design. The following examples will illustrate this.

Design 1

Balanced Incomplete Block Design $n_{ij} = 0$ or 2

Blocks

1	2	3	4	5	6
A	A	A	B	B	C
A	A	A	B	B	C
B	C	D	C	D	D
B	C	D	C	D	D

average variance = $\sigma^2/2$

Design 2

Partially Balanced Incomplete Block Design $n_{ij} = 0, 1$ or 2

Blocks

1	2	3	4	5	6
A	A	A	A	A	C
B	B	B	B	A	C
D	D	D	D	B	D
D	D	D	D	B	D

average variance = $7\sigma^2/18$

In this paper we confine our attention to general binary balanced block design (GBBBD) as a selected subset of all n-ary designs, and to equal block size; and in the next section the parameters of a GBBBD are constructed and some definitions are presented. The relationship of the parameters to the binary balanced incomplete block design, as a special case, is demonstrated. Some results on existence of GBBBD and on their variance optimality are presented in section three. Two examples, illustrating the results, where a most optimal GBBBD exists and where it doesn't exist, are given in the last section.

2. PARAMETERS OF GBBBD AND SOME DEFINITIONS

Let $k_j = k$ for all j, let $r_i = r$ for all i and let $\underline{n}$ be the incidence matrix of a balanced incomplete block design with parameters $(v,b,r,k,\lambda;$ $n_{ij} = 0,1)$; then we may define the incidence matrix of a general binary balanced block design to be

$$\underline{n}^* = \underline{n}(m_1 - m_0) + \underline{J}m_0 , \tag{2.1}$$

where $\underline{J}$ is $v \times b$ matrix whose elements are all one and where $0 \leq m_0 < m_1$.
The parameters of the GBBBD are $(v, b, r^*, k^*, \lambda^*; n^*_{ij} = m_0 \text{ or } m_1)$ where

$$r^* = rm_1 + (b-r)m_0 \tag{2.2}$$

$$k^* = km_1 + (v-k)m_0 \tag{2.3}$$

$$\lambda^* = \frac{r^*(k^* - m_1 - m_0) + bm_1m_0}{v - 1} \tag{2.4}$$

where $\lambda^* = \Sigma_{j=1}^{b} n^*_{ij} n^*_{i'j}$ for all $i \neq i'$ $(1, 2, \cdots, v)$

$$vr^* = bk^* = N^* \tag{2.5}$$

$$v \leq b \tag{2.6}$$

In order to be precise, some formal definitions are needed. These are presented below.

Definition 2.1. A GBBBD is said to be incomplete if $m_0 = 0$; otherwise it is said to be complete.

Design 1 given above is an incomplete block binary design and design 2 is an incomplete ternary design.

An example of a complete GBBBD is

Blocks

1	2	3
A	A	A
B	B	B
C	C	C
A	B	C

where $v = b = 3$, $r = k = 1$, $r^* = k^* = 4$; $\lambda^* = 5$, $m_0 = 1$, $m_1 = 2$.

Definition 2.2. A complete GBBBD is said to be orthogonal if $n^*_{ij} = \frac{r^*_i k^*_j}{N^*}$. Otherwise it is said to be non-orthogonal.

A design may be complete and non-orthogonal; for example, the design given under definition 2.1 is of this type, since $(r^*_i k^*_j)/N^* = 16/12 \neq n^*_{ij}$. Consider the following designs:

Blocks

1	2	3
A	A	A
A	A	A
B	B	B
C	C	C

Blocks

1	2	3
A	A	A
B	B	B
C	C	C
A	B	C

Both designs are binary and complete. The first one is orthogonal but not balanced, whereas the second is balanced but not orthogonal.

Definition 2.3. A complete GBBBD is variance balance if its coefficient matrix $\underline{c}^* = c^*_1 \underline{I} + c^*_2 \underline{J}$ where c^*_1 and c^*_2 are scalars and c^*_1 is the non-zero eigenvalue of $\underline{c}^*$ and $c^*_2 = c^*_1/v$. $\underline{I}$ is the identity matrix and $\underline{J}$ is a matrix whose all elements are ones.

Note a design could be complete but not balanced, for example, the first two blocks of the design under definition 2.1.

3. EXISTENCE AND VARIANCE OPTIMALITY

Theorem 3.1. Existence of a balanced incomplete block design with parameters $(v,b,r,k,\lambda;\ n_{ij} = 0,1)$ implies the existence of GBBBD with parameters $(v,b,r^*,k^*,\lambda^*;\ n^*_{ij} = m_0 \text{ or } m_1)$.

Proof: From the definition of a GBBBD note that $\underline{n}^* = \underline{n}(m_1-m_0) + \underline{J}m_0$. The $(i,j)^{th}$ entry of n^* is

$$n^*_{ij} = n_{ij}(m_1-m_0) + m_0 = m_0 \quad \text{if} \quad n_{ij} = 0$$

$$= m_1 \quad \text{if} \quad n_{ij} = 1$$

To construct a GBBBD, start with a balanced incomplete block design with incidence matrix $\underline{n}$, replace the ones by m_1 and the zero's by m_0 to obtain the incidence matrix for a GBBBD with parameters $(v, b, r^*, k^*, \lambda^*; m_0, m_1)$ where r^*, k^* and λ^* satisfy equations (2.2) to (2.5). From this construction of $\underline{n}^*$, it is clear that m_1 and m_0 appear r and (b-r) times in each row and k and v - k times in each column respectively. Hence, row and column totals are

$$r^* = rm_1 + (b-r)m_0$$

and

$$k^* = km_1 + (v-k)m_0$$

or more formally, these can be obtained by post-multiplication of $\underline{n}^*$ and $\underline{n}^{*\prime}$ by $\underline{1}_{b\times 1}$ and $\underline{1}_{v\times 1}$, column vectors of ones, respectively. Thus

$$r^*\underline{1}_{v\times 1} = \underline{n}^*\underline{1}_{b\times 1} = [\underline{n}(m_1-m_0) + \underline{J}m_0]\underline{1}_{b\times 1}$$

$$= \underline{n}\underline{1}_{b\times 1}(m_1-m_0) + \underline{J}\underline{1}_{b\times 1}m_0$$

$$= r\underline{1}_{v\times 1}(m_1-m_0) + b\underline{1}_{v\times 1}m_0$$

$$= [rm_1 + (b-r)m_0]\underline{1}_{v\times 1} \qquad (3.1)$$

$$
\begin{aligned}
k^{*}\underline{1}_{b\times 1} = n^{*\prime}\underline{1}_{v\times 1} &= [\underline{n}'(m_1-m_0) + \underline{J}'m_0]\underline{1}_{v\times 1} \\
&= \underline{n}'\underline{1}_{v\times 1}(m_1-m_0) + \underline{J}'\underline{1}_{v\times 1}m_0 \\
&= k\underline{1}_{b\times 1}(m_1-m_0) + b\underline{1}_{b\times 1}m_0 \\
&= [km_1 + (v-k)m_0]\underline{1}_{b\times 1} \qquad (3.2)
\end{aligned}
$$

In any two rows of $\underline{n}^{*}$, pairs $\begin{pmatrix} m_1 \\ m_1 \end{pmatrix}$, $\begin{pmatrix} m_1 \\ m_0 \end{pmatrix}$, $\begin{pmatrix} m_0 \\ m_1 \end{pmatrix}$ and $\begin{pmatrix} m_0 \\ m_0 \end{pmatrix}$ appear λ, $r - \lambda$, $r - \lambda$ and $(b-2r+\lambda)$ times respectively. The inner product of any two rows of $\underline{n}^{*}$, denoted by λ^{*}, is

$$
\begin{aligned}
\lambda^{*} &= \lambda m_1^2 + 2(r-\lambda)m_1 m_0 + (b-2r+\lambda)m_0^2 \\
&= \lambda(m_1-m_0)^2 + 2r(m_1-m_0)m_0 + bm_0^2 \qquad (3.3)
\end{aligned}
$$

since $\underline{n}$ is BIBD $\therefore \lambda = r(k-1)/(v-1)$

$$
\begin{aligned}
\lambda(m_1-m_0)^2 &= [rk(m_1-m_0)^2 - r(m_1-m_0)^2]/(v-1) \\
&= [(r^{*}-bm_0)(k^{*}-vm_0) - (r^{*}-bm_0)(m_1-m_0)]/(v-1) \\
&= [(r^{*}-bm_0)(k^{*}-vm_0-m_1+m_0)]/(v-1)
\end{aligned}
$$

$$
2r(m_1-m_0)m_0 = 2(r^{*}-bm_0)m_0 ,
$$

hence

$$
\lambda^{*} = [(r^{*}-bm_0)\{k^{*}+vm_0-m_1-m_0\} + vbm_0^2 - bm_0^2]/(v-1)
$$

or

$$
\lambda^{*} = [r^{*}(k^{*}-m_1-m_0) + bm_1m_0]/(v-1) \qquad (3.4)
$$

This result may be obtained directly from $\underline{n}^*\underline{n}^{*'}$ is

$$(\underline{n}^*\underline{n}^{*'})_{ij} = r^*(m_1+m_0) - bm_1m_0 = r^*k^* - (v-1)\lambda^* \quad \text{if} \quad i = j$$

$$= \lambda^* \qquad\qquad \text{if} \quad i \neq j \quad .$$

When $m_0 = 0$ and $m_1 = 1$, equation (2.6) has been called Fisher's inequality. Note that in our formulation for arbitrary m_0 and m_1, such that $m_0 < m_1$, equation (2.6) represents a generalization of Fisher's inequality. This inequality has been proven by evaluating the value of the determinant of $v \times v$ matrix $\underline{n}^*\underline{n}^{*'}$. Thus

$$|\underline{n}^*\underline{n}^{*'}| = [r^*k^* - (v-1)\lambda^* + (v-1)\lambda^*][r^*k^* - (v-1)\lambda^* - \lambda^*]^{v-1} \quad .$$

Also,

$$r^*k^* = rk(m_1-m_0)^2 + 2bk(m_1-m_0)m_0 + bvm_0^2 \quad ,$$

$$\lambda^*v = \lambda v(m_1-m_0)^2 + 2bk(m_1-m_0)m_0 + bvm_0^2 \quad ,$$

and

$$r^*k^* - \lambda^*v = (rk-\lambda v)(m_1-m_0)^2 = (r-\lambda)(m_1-m_0)^2 \quad .$$

Hence,

$$|\underline{n}^*\underline{n}^{*'}| = r^*k^*[(r-\lambda)(m_1-m_0)^2]^{(v-1)} \quad , \tag{3.6}$$

as $\lambda < r$. Therefore in balanced designs, where $m_0 < m_1$, the value of $|\underline{n}^*\underline{n}^{*'}| > 0$. Since $\underline{n}^*$ is a $v \times b$ matrix and $\underline{n}^*\underline{n}^{*'}$ is $v \times v$, therefore $v \leq b$.

The form of the coefficient matrix $\underline{c}^*$ is

$$\underline{c}^* = r^*\underline{I} - \frac{\underline{n}^*\underline{n}^{*'}}{k^*} = \frac{\lambda^*(v\underline{I}-\underline{J})}{k^*} \tag{3.7}$$

$$= \frac{v\lambda^*}{k^*}\,\underline{I} - \frac{\lambda^*}{k^*}\,\underline{J} = c_1^*\underline{I} + c_2^*\underline{J} \quad \text{(def. 2.3)} \quad .$$

This form is identical for BIBD if * is dropped. The rank of $\underline{c}^*$ is v-1 and the covariance matrix (intrablock) of treatment effects if $\frac{k^*}{v\lambda^*}\sigma^2\underline{I}$ under the usual constraints that the sum of the estimated treatments effects $\hat{t}_i$, (i-1,2,$\cdots$,v) equals zero.

Some particular members of the family of GBBBD are given below:

i) Basic binary balanced block design. Parameters (v, b, r, k, λ; 0, 1).

Here $\underline{n}^* = n, \quad r^* = r, \quad k^* = k$

$$\lambda^* = \lambda = \frac{r(k-1)}{v-1}, \quad m_0 = 0, \; m_1 = 1 .$$

$$\underline{c}^* = \frac{\lambda}{k}(v\underline{I}-\underline{J}), \; \mathrm{Cov}(\hat{\underline{t}}) = \frac{k}{v\lambda}\sigma^2\underline{I}$$

$$\text{Efficiency factor} = r\,\frac{(1-\frac{1}{k})}{(1-\frac{1}{v})}$$

ii) Randomized block design. (Note this is not a binary design, but could be obtained from equation (2.1) by setting $m_0 = m_1 = 1$.) Thus the parameters are (v = k, b = r = λ, 1, 1).

Here $\underline{n}^* = \underline{J}, \; r^* = r = \lambda^*, \; k^* = v$

$$m_0 = m_1 = 1$$

$$\underline{c}^* = \frac{r}{v}(v\underline{I}-\underline{J}) \quad \mathrm{Cov}(\hat{\underline{t}}) = \frac{\sigma^2}{r}\underline{I}$$

Efficiency factor = r

iii) $m_0 = m, \; m_1 = m$. Then the parameters are (v, b, rm, km, λm^2; 0, m).

$$\underline{n}^* = \underline{n}m, \quad r^* = rm, \quad k^* = km ,$$

$$\lambda^* = \frac{r(k-1)}{v-1}\,m^2, \quad m_0 = 0, \; m_1 = m,$$

$$\underline{c}^* = \frac{\lambda}{k}(v\underline{I}-\underline{J})m, \quad \mathrm{Cov}(\hat{\underline{t}}) = \frac{k\sigma^2}{v\lambda m}\underline{I},$$

$$\text{Efficiency factor} = r\,\frac{(1-\frac{1}{k})}{(1-\frac{1}{v})m} .$$

In the class of all equireplicated and equisized block designs which are GBBBD, the question arises as to which of these balanced block designs has the smallest variance. As may be noted from the definition of a GBBBD, there are many possible values of m_0 and m_1 and BBBIBD $(v, b_d, r_d, k_d, \lambda_d; 0, 1)$ leading to GBBBD. In searching for an optimal design, we note that maximizing $\frac{\lambda^* v}{k^*}$ will minimize the variance. Since v is the same, we need confine our attention only to $\frac{\lambda^*}{k^*}$. The following theorem is in this spirit.

<u>Theroem 3.2</u>. <u>Among all equireplicated and equiblock sized</u> GBBBD <u>with parameters</u> $(v, b_d, r^*, k_d^*, \lambda_d^*; m_{0d}, m_{1d})$ <u>the design(s) having the minimal value of</u> $r_d(b_d-r_d)(m_{1d}-m_{0d})^2$ <u>is(are) optimal in the sense of</u> A-, D-, E-<u>optimality</u>.

<u>Proof</u>. The three criteria of optimality, A-, D-, E-optimality involve functions of non-zero eigen-values of the coefficient matrix $\underline{c}^*$. Let $(\gamma_i, i=1,2,\cdots,v-1)$ be the set of non-zero eigen-values of $\underline{c}^*$; then,

i) A-optimality: $$f_A(\underline{c}^*) = \sum_{i=1}^{v-1} \gamma_i^{-1}$$

ii) D-optimality: $$f_D(\underline{c}^*) = \prod_{i=1}^{v-1} \gamma_i^{-1}$$

iii) E-optimality: $$f_E(\underline{c}^*) = \max_{1\leq i\leq v-1} \gamma_i^{-1}$$

(For a discussion of these criteria, see Kiefer [1958],[1959].) Since, in our case, the v-1 non-zero eigen-values of $\underline{c}^*$ are all equal to $\frac{v\lambda_d^*}{k_d^*} = \gamma$ for each d; by minimizing $\left(\frac{\lambda_d^*}{k_d^*}\right)^{-1}$, we shall achieve all the three optimality criteria. Thus,

$$\max_d \left(\frac{\lambda^*_d}{k^*_d} \right) \equiv \max_d \left[r^* - \frac{r^*(m_{1d}+m_{Od}) - b_d m_{1d} m_{Od}}{k^*_d} \right]$$

$$\equiv \min_d \left[\frac{r^*(m_{1d}+m_{Od}) - b\, m_{1d} m_{Od}}{k^*_d} \right] \text{ as } r^* \text{ is constant,}$$

$$\equiv \min_d \left[r^*(m_{1d}+m_{Od}) b_d - b_d^2 m_{1d} m_{Od} \right]$$

$$\equiv \min_d \left[r^{*2} - (r^*-b_d m_{1d})(r^*-b_d m_{Od}) \right]$$

$$\equiv \min_d \left[r^{*2} + r_d(b_d-r_d)(m_{1d}-m_{Od})^2 \right]$$

$$\equiv \min_d \left[r_d(b_d-r_d)(m_{1d}-m_{Od})^2 \right] .$$

<u>Corollary 3.1</u>. <u>In a subclass of</u> GBBBD <u>with parameters</u> (v, b_d, r^*, k^*_d, λ^*_d; m_{Od}, m_{1d}), <u>constructed from the</u> BIBD (v, b_d, r_d, k_d, λ_d; 0, 1) <u>and for which the difference</u> $(m_{1d}-m_{Od}) > 0$ <u>is constant</u>, <u>the design(s)</u> <u>having minimal value of</u> $r_d(b_d-r_d)$ <u>is(are) optimal</u>.

The proof follows from Theorem (3.2).

<u>Corollary 3.2</u>. <u>In a subclass of</u> GBBBD <u>with parameters</u> (v, b_d, r^*, k^*_d, λ^*_d; m_{Od}, m_{1d}) <u>constructed from the</u> BIBD (v, b_d, r_d, k_d, λ_d; 0, 1) <u>for which the value</u> $r_d(b_d-r_d)$ <u>is constant</u>, <u>the design(s) having minimal value of</u> $(m_{1d}-m_{Od})$ <u>is(are) optimal</u>.

The proof follows from Theorem (3.2).

Corollary 3.3. In a subclass of GBBBD $(v, b_d, r^*, k_d^*, \lambda_d^*; m_{0d}, m_{1d},$ constructed from BIBD $(v, b_d, r, k_d, \lambda; 0, 1)$, which are either symmetrical or $b_d = b$ for all d, the design(s) having the minimal value of $k_d(v-k_d)(m_{1d}-m_{0d})^2$ is(are) optimal.

In a symmetrical BIBD, $v = b_d$, $r_d = k_d$ the result then follows by substituting these values in Theorem (3.2). In designs where $b_d = b$ for all d,

$$r_d(b_d-r_d) = \frac{bk_d}{v}\left(b - \frac{bk_d}{v}\right)$$

$$= \left(\frac{b^2}{v}\right) k_d(v-k_d) ,$$

as b and v are fixed. This result follows from Theorem 3.2.

4. EXAMPLES

Two examples are presented below to illustrate some consequences of the theorems in corollaries given in the previous section. In the first example, members of a class of balanced block designs are presented to illustrate the need to consider more than one member of the class in searching for a variance-optimal design and to indicate that a most optimal design may not exist in a given class. A most optimal design with respect to A-, D-, and E-optimality is defined here to be one which has $m_{1d} - m_{0d} = 1$ and has $r_d\ (b_d-r_d)$ a minimum.

Examples 4.1. The following BBBIBD's are used to construct GBBBD with $v = 4$ and $r^* = 12$:

BBBIBD - 1

treatments	blocks 1	2	3	4
A	1	1	1	0
B	1	1	0	1
C	1	0	1	1
D	0	1	1	1

BBBIBD - 2

treatments	blocks 1	2	3	4
A	1	0	0	0
B	0	1	0	0
C	0	0	1	0
D	0	0	0	1

BBBIBD - 3

treatments	blocks 1	2	3	4	5	6
A	1	1	1	0	0	0
B	1	0	0	1	1	0
C	0	1	0	1	0	1
D	0	0	1	0	1	1

Table 4.1.

BBBIBD		Parameters of BBBIBD				Parameters of GBBBD				Optimality Criteria		
	d	b_d	r_d	k_d	λ_d	k^*_d	λ^*_d	m_{od}	m_{1d}	$m_{1d}-m_{Od}$	I^*_d	II^*_d
1	1	4	3	3	2	12	36	3	3	0	3	0
1	2	4	3	3	2	12	32	0	4	4	3	4
2	3	4	1	1	0	12	36	3	3	0	3	0
2	4	4	1	1	0	12	32	2	6	4	3	4
2	5	4	1	1	0	12	20	1	9	8	3	16
2	6	4	1	1	0	12	0	0	12	12	3	36
3	7	6	3	2	1	8	24	2	2	0	9	0
3	8	6	3	2	1	8	22	1	3	2	9	3
3	9	6	3	2	1	8	16	0	4	4	9	12

$I^*_d = r_d(b_d-r_d)$ $\qquad II^*_d = r_d(b_d-r_d)(m_{1d}-m_{Od})^2/r^*$

In designs 1, 3, and 7, $m_0 = m_1$ and are orthogonal. A most optimal design, where $m_{1d} - m_{0d} = 1$ and $r_d(b_d-r_d)$ is minimal, does not exist here. Of the other designs, design 8 is the best one in the sense of our optimality criterion. Design 1 to 6 fall in the subclass of designs in which I^*_d is constant and is equal to 3. Therefore, the difference $(m_{1d}-m_{0d})$ serves as an optimality criterion for this class. Similarly, designs 7 to 9 also belong to the class, as $r_d(b_d-r_d)$ is constant and is equal to 9. Difference criterion $(m_{1d}-m_{0d})$ does not serve as an optimality criterion to make comparisons between members of these two classes. In a class of designs having the same value of the difference $m_{1d}-m_{0d}$, there is(are) design(s) which is(are) optimal. Designs 2, 4, and 9 belong to this class. Of the three designs, designs 2 and 4 have smaller values of $r_d(b_d-r_d)$, hence are optimal in this class.

<u>Example 4.2</u>. Example 1 deals with a class of designs in which most optimal designs do not exist. This example deals with a class of designs in which a most optimal design exists. BBBIBD's 1 to 3 are used to construct GBBBD with $v = 4$ and $r^* = 15$.

Table 4.2.

BBBIBD	Parameters of BBBIBD					Parameters of GBBBD				Optimality Criteria		
	d	b_d	r_d	k_d	λ_d	k^*_d	λ^*_d	m_{0d}	m_{1d}	$m_{1d}-m_{0d}$	I^*_d	II^*_d
1	1	4	3	3	2	15	56	3	4	1	3	.2
1	2	4	3	3	2	15	50	0	5	5	3	5.0
2	3	4	1	1	0	15	54	3	6	3	3	1.8
2	4	4	1	1	0	15	44	2	9	7	3	9.8
2	5	4	1	1	0	15	26	1	12	11	3	24.2
2	6	4	1	1	0	15	0	0	15	15	3	45.0
3	7	6	3	2	1	10	37	2	3	1	9	.6
3	8	6	3	2	1	10	33	1	4	3	9	5.4
3	9	6	3	2	1	10	25	0	5	5	9	15.0

$I^*_d = r_d(b_d-r_d)$, $\qquad II^*_d = r_d(b_d-r_d)(m_{1d}-m_{0d})^2/r^*$

Design 1 and 7 both have $m_{1d} - m_{0d} = 1$, but design 1 is most optimal since $I^*_1 < I^*_7$.

REFERENCES

Kiefer, J. (1958). On the nonrandomized optimality and randomized non-optimality of symmetrical designs. _Annals_ _Math_. _Statist_. _29_, 675-699.

Kiefer, J. (1959). Optimum experimental designs. _J.R.S.S_. _Ser_. _B_, _21_, 272-319.

Tocher, K. D. (1952). The design and analysis of block experiments 319. _J.R.R.S_. _Ser_. _B_, _14_, 45-100.

Hedayat, A. & Federer, W. T. (1974). Pairwise and variance balanced incomplete block designs. _Annals_ _Inst_. _Statist_. _Math_., _26_, 331-338.