COMBINATORICS OF SCHUBERT POLYNOMIALS

A Dissertation
Presented to the Faculty of the Graduate School
of Cornell University
in Partial Fulfillment of the Requirements for the Degree of
Doctor of Philosophy

by
Avery J St. Dizier
August 2020

COMBINATORICS OF SCHUBERT POLYNOMIALS

Avery J St. Dizier, Ph.D.

Cornell University 2020

In this thesis, we study several aspects of the combinatorics of various important families of polynomials, particularly focusing on Schubert polynomials. Schubert polynomials arise as distinguished representatives of cohomology classes in the cohomology ring of the flag variety. As polynomials, they enjoy a rich and well-studied combinatorics. Through joint works with Fink and Mészáros, we connect the supports of Schubert polynomials to a class of polytopes called generalized permutahedra. Through a realization of Schubert polynomials as characters of flagged Weyl modules, we show that the exponents of a Schubert polynomial are exactly the integer points in a generalized permutahedron. We also prove a combinatorial description of this permutahedron. We then study characters of flagged Weyl modules more generally and give an interesting inequality on their coefficients.

We next shift our focus onto the coefficients of Schubert polynomials. We describe a construction due to Magyar called orthodontia. We use orthodontia together with the previous inequality for characters to give a complete description of the Schubert polynomials that have only zero and one as coefficients. Through joint work with Huh, Matherne, and Mészáros, we next show a discrete log-concavity property of the coefficients of Schubert polynomials. The main tool for this purpose is the Lorentzian property introduced by Brändén and Huh. We prove that something similar to Schubert polynomials is Lorentzian. We extract from this the discrete log-concavity of Schubert polynomials and the Lorentzian property of

Schur polynomials. We finish with various conjectures and partial results regarding other families of polynomials.

BIOGRAPHICAL SKETCH

Avery St. Dizier was born in Baton Rouge, Louisiana on April 29th, 1993. His family consists of his parents Amy and Paul, and older sibling Dustin. He graduated from Louisiana State University with a B.S. in Mathematics. He then completed a M.S. in Mathematics at Cornell University. He completed his dissertation under the supervision of Professor Karola Mészáros.

This document is dedicated to my partner Sam for her endless support.

ACKNOWLEDGEMENTS

I would first like to thank my advisor, Karola Mészáros, for her infinite patience, thoughtful guidance, and motivational emailing. She has invested so much time and effort in teaching me about many aspects of doing mathematics. She has also taken great care to support my professional career and connect me with many people. I am so grateful for the many exciting projects she has shared and worked on with me. A special thanks to Farbod Shokrieh as well, for helping me get through the door at Cornell and connecting me with Karola. Thanks also to all my coauthors for our collaborations, and to Alex Yong for taking on the role advisor for the next phase of my career.

Thanks also to the professors and staff of Cornell for showing me the breadth and beauty of mathematics, and for making graduate school the amazing experience it has been. A particular shout-out to Melissa Totman for putting out every fire and always being willing to chat about life. I would also like to thank Frank Neubrander and Karl Mahlburg at LSU for their support and guidance at a crucial stage of my undergraduate career. I would not be at Cornell today if not for their help. Going back further, I would like to thank my formative math teachers, Edie Boudreaux and Emily Coco, for fostering my interest in mathematics.

Finally, I would like acknowledge the many friends and family who constantly support me. In particular, I thank Ayah Almousa and Abby Turner for being the best office-mates I could ask for. I am grateful to Brian Hwang for help with algebraic geometry. I am thankful to my relatives Suzanne and Neil Granstrand, and parents Paul and Amy St. Dizier, for supporting me throughout my Ph.D. And lastly, I am grateful to Sam Castonguay for always believing in me and being such an amazing partner.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

GENERAL BACKGROUND

1.1 ℕ-otational Note

Throughout this document, we will take $\mathbb{N} = \{0, 1, 2, \ldots\}$. This minimizes notational complexity in various places. For example, consider $\mathbb{N}^n$ versus $\mathbb{Z}^n_{\geq 0}$. We will frequently use the shorthand $[n]$ for the set $\{1, 2, \ldots, n\}$.

1.2 Multi-index Disclaimer

Throughout, we will use undecorated math symbols to refer to lists, and subscripts to refer to their parts. For instance, $\alpha \in \mathbb{N}^n$ should be interpreted as $\alpha = (\alpha_1, \ldots, \alpha_n)$. This is particularly prevalent with regard to polynomials. An arbitrary monomial of a polynomial f in variables $x_1, \ldots, x_n$ may be denoted x^α, with the understanding that this means $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. There very few one-variable polynomials used, so little conflict should occur with this usage. When an object is one-dimensional, this will be made explicit in context. When the number of variables of a polynomial f, is clear, we will sometimes write $f(x)$ or simply f. This allows for compact expressions such as $f(1-x)$ in lieu of $f(1-x_1, \ldots, 1-x_n)$.

1.3 Note on Source Material

Many of the mathematical results presented in this thesis that are part of joint works [14, 15] (with Alex Fink and Karola Mészáros), and [22] (with June Huh, Jacob Matherne, and Karola Mészáros). Specific attributions are included at the beginning of each chapter and section where appropriate.

1.4 Polynomials and their Indexing Objects

In this section, we give the definitions of various families of polynomials from algebra and geometry indexed by combinatorial objects. We will refer to these polynomials throughout.

1.4.1 Partitions and Schur Polynomials

A *partition* is a weakly decreasing finite sequence of nonnegative integers. We will usually denote partitions by λ and μ. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$, the *Young diagram* of λ is the array of left aligned boxes that has λ_i boxes in row i for each $i \in [m]$. A *semistandard Young tableau* (SSYT) of shape λ is a filling of the Young diagram of λ with numbers from $[m]$ such that entries weakly increase left-to-right along rows, and entries strictly increase top-to-bottom along columns.

Schur polynomials are a generating function over SSYT of a given shape.

Definition 1.4.1. The *Schur polynomial* of a partition $\lambda \in \mathbb{N}^m$ is defined by

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x^{\mathrm{wt}(T)},$$

where the sum is over all SSYT T of shape λ, and $\mathrm{wt}(T)$ is the *weight* of T, the vector whose ith component is the number of i's in T.

Collecting SSYT of the same weight together, we get

$$s_\lambda(x_1, \ldots, x_m) = \sum_\alpha K_{\lambda\alpha} x^\alpha,$$

where $K_{\lambda\alpha}$ is the *Kostka number*, the number of SSYT of shape λ and weight α.

Example 1.4.2. The SSYT of shape $\lambda = (2, 1, 0)$ are

$$\mathrm{SSYT}(2,1,0): \quad \begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array}$$

so the corresponding Schur polynomial is

$$s_{(2,1,0)}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2.$$

Schur polynomials are symmetric polynomials, and form a basis for the space of symmetric polynomials. Schur polynomials can also be viewed as generating functions of integer points in a polytope.

Definition 1.4.3. Given a partition $\lambda \in \mathbb{N}^m$, the *Gelfand–Tsetlin polytope* $\mathrm{GT}(\lambda)$ is the set of all nonnegative triangular arrays

$$\begin{array}{ccccccc}
x_{11} & & x_{12} & & \cdots & & x_{1m} \\
 & x_{22} & & x_{23} & \cdots & x_{2m} & \\
 & & \cdots & & \cdots & & \\
 & & x_{m-1,m-1} & & x_{m-1,m} & & \\
 & & & x_{mm} & & &
\end{array}$$

such that

$$x_{1i} = \lambda_i \text{ for all } 1 \le i \le m,$$

$$x_{i-1,j-1} \ge x_{ij} \ge x_{i-1,j} \text{ for all } 1 \le i \le j \le m.$$

The *weight* $\mathrm{wt}(P)$ of an integer point $P \in \mathrm{GT}(\lambda)$ is the vector in $\mathbb{R}^m$ with components

$$\mathrm{wt}(P)_{m-i+1} = (x_{ii} + x_{i,i+1} + \cdots + x_{im}) - (x_{i+1,i+1} + x_{i+1,i+2} + \cdots + x_{i+1,i+1}).$$

Example 1.4.4. If $\lambda = (2, 1, 0)$, then $\mathrm{GT}(\lambda)$ is the polytope

$$\mathrm{GT}(2,1,0) = \left\{ \begin{matrix} 2 & & 1 & & 0 \\ & x_{11} & & x_{12} & \\ & & x_{22} & & \end{matrix} \;\middle|\; \begin{matrix} 2 \geq x_{11} \geq 1 \\ 1 \geq x_{12} \geq 0 \\ x_{11} \geq x_{22} \geq x_{12} \end{matrix} \right\}.$$

There is a classical weight-preserving bijection between the integer points of $\mathrm{GT}(\lambda)$ and $\mathrm{SSYT}(\lambda)$. Given an SSYT T, set x_{mm} to be the number of 1's in T. Iteratively, set $(x_{ii}, \ldots, x_{im})$ to be the shape of the tableau obtained by restricting T to boxes with entries at most $m - i + 1$. An example of this bijection is given for $\lambda = (2, 1, 0)$ in Figure 1.1.

$$\begin{array}{cccccccc}
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} &
\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \\[2em]
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 2 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 & \\ & & 1 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{smallmatrix} &
\begin{smallmatrix} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{smallmatrix}
\end{array}$$

Figure 1.1: Bjecting $\mathrm{SSYT}(2,1,0)$ to integer points of $\mathrm{GT}(2,1,0)$

For further exposition on Schur polynomials, see [19].

1.4.2 Permutations and Schubert and Grothendieck Polynomials

A *permutation* of $[n] = \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to itself. We will write permutations in *one-line notation*, listing the images of each element of $[n]$ in increasing order. For example 1342 is the permutation that sends 1 to 1, 2 to 3, 3 to 4, and 4 to 2. The *symmetric group* on $[n]$ will be denoted S_n, with the natural inclusions $S_n \subseteq S_{n+1}$. For each i, the adjacent transposition swapping i and $i+1$ will be denoted s_i. To each permutation w is associated a special polynomial called the Schubert polynomial $\mathfrak{S}_w$.

Schubert polynomials, introduced by Lascoux and Schützenberger in [28], represent cohomology classes of Schubert cycles in the flag variety. Knutson and Miller also showed them to be multidegrees of matrix Schubert varieties [25]. There are a plethora of combinatorial formulas for the Schubert polynomials [4, 6, 16, 17, 18, 27, 32, 49]. We give here the classical formula of Lascoux and Schützenberger [28].

First define the ith *divided difference operator* ∂_i on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\partial_i(f) := \frac{f - s_i f}{x_i - x_{i+1}} \text{ and } s_i = (i, i+1),$$

where s_i acts on f by swapping the variables x_i and x_{i+1}.

Definition 1.4.5. The Schubert polynomial of the longest permutation $w_0 = n\, n{-}1\, \cdots\, 2\, 1 \in S_n$ is given by

$$\mathfrak{S}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \in S_n$, $w \neq w_0$, there exists $i \in [n-1]$ such that $w_i < w_{i+1}$. For any such i, the *Schubert polynomial* $\mathfrak{S}_w$ is defined as

$$\mathfrak{S}_w(x_1,\ldots,x_n) := \partial_i \mathfrak{S}_{ws_i}(x_1,\ldots,x_n).$$

Since the ∂_i satisfy the braid relations, the Schubert polynomials $\mathfrak{S}_w$ are well-defined.

Example 1.4.6. For the permutation $w = 31542$, we have

$$\begin{aligned}
\mathfrak{S}_w &= \partial_2(\mathfrak{S}_{35142}) = \partial_2\partial_3(\mathfrak{S}_{35412}) = \partial_2\partial_3\partial_4(\mathfrak{S}_{35421}) = \partial_2\partial_3\partial_4\partial_1(\mathfrak{S}_{53421}) \\
&= \partial_2\partial_3\partial_4\partial_1\partial_2(\mathfrak{S}_{54321}) \\
&= \partial_2\partial_3\partial_4\partial_1\partial_2(x_1^4x_2^3x_3^2x_4) \\
&= x_2x_3x_1^3 + x_2x_4x_1^3 + x_3x_4x_1^3 + x_2x_3^2x_1^2 + x_2^2x_3x_1^2 + x_2^2x_4x_1^2 + x_3^2x_4x_1^2 + x_2x_3x_4x_1^2
\end{aligned}$$

The Schubert polynomials are homogeneous, and the set of all Schubert polynomials of permutations forms a basis of the infinite polynomial ring $\mathbb{Z}[x_1,x_2,\ldots]$. It is a classical result due to the transition rule of Lascoux and Schützenberger that the Schubert polynomials have nonnegative integer coefficients.

We now recall the connection between the Schubert and Schur polynomials. A permutation w is called *Grassmannian* if it has a unique *descent*; that is, there is a unique index i such that $w_i > w_{i+1}$. Grassmannian permutations are in bijection with partitions. If the unique descent of w occurs at position r, then the partition associated to w is $(w_r - r, w_{r-1} - (r-1), \ldots, w_1 - 1)$. To construct the Grassmannian permutation w associated to $\lambda = (\lambda_1, \ldots, \lambda_r)$, set $n = \lambda_1 + r$ and

$$w_i = \begin{cases} \lambda_{r-i+1} + r & \text{if } 1 \le i \le r, \\ \min\left([n]\backslash\{w_1,\ldots,w_{i-1}\}\right) & \text{if } r < i \le n. \end{cases}$$

Note the dependence on n is artificial due to the natural inclusions $S_n \subseteq S_{n+1}$. See for instance [35, Proposition 2.6.8] or any other standard reference for a proof of the following result.

Theorem 1.4.7. *If* $\lambda = (\lambda_1, \ldots, \lambda_r)$ *corresponds to the Grassmannian permutation* $w \in S_n$*, then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = s_\lambda(x_1, \ldots, x_r).$$

For further exposition on Schubert polynomials, refer to [35].

We now define Grothendieck polynomials, an inhomogeneous analogue of Schubert polynomials. Grothendieck polynomials occur as distinguished representatives of K-theory classes of Schubert cycles in the flag variety. Let $\widetilde{\pi}_i$ denote the ith *isobaric divided difference operator* defined by

$$\widetilde{\pi}_i f = \partial_i (1 - x_{i+1}) f.$$

Definition 1.4.8. The Grothendieck polynomial of the longest permutation $w_0 = n\, n{-}1 \, \cdots \, 2\, 1 \in S_n$ is given by

$$\mathfrak{G}_{w_0} := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \in S_n$, $w \neq w_0$, there exists $i \in [n-1]$ such that $w(i) < w(i+1)$. For any such i, the *Grothendieck polynomial* $\mathfrak{G}_w$ is defined as

$$\mathfrak{G}_w(x_1, \ldots, x_n) := \widetilde{\pi}_i \mathfrak{G}_{ws_i}(x_1, \ldots, x_n).$$

It is well-known that the Schubert polynomial $\mathfrak{S}_w$ is the lowest homogeneous component of the corresponding Grothendieck polynomial $\mathfrak{G}_w$ (see for instance [25, Lemma 1.1.4]).

1.4.3 Compositions and Key Polynomials

By a *composition*, we mean simply a tuple of nonnegative integers. Key polynomials, also known as Demazure characters, are a family of polynomials associated to compositions. Key polynomials were first introduced by Demazure for Weyl groups [10], and studied in the context of the symmetric group by Lascoux and Schützenberger in [29, 30]. Their definition is similar to that of Schubert and Grothendieck polynomials. Let π_i denote the ith *Demazure operator* defined by

Definition 1.4.9. If $\alpha \in \mathbb{N}^n$ is weakly decreasing, define the *key polynomial* κ_α to be

$$\kappa_\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Otherwise, find i with $\alpha_i < \alpha_{i+1}$ and set

$$\kappa_\alpha = \pi_i \kappa_{\hat{\alpha}} \text{ where } \hat{\alpha} = (\alpha_1, \ldots, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n).$$

The set of key polynomials κ_α where α ranges over all compositions of length n, is a basis for $\mathbb{Z}[x_1, \ldots, x_n]$. Every Schubert polynomial can be written uniquely as a sum of key polynomials. The key polynomials generalize the Schur polynomials, as s_λ is the key polynomial of the composition obtained by reversing the entries of λ. We refer to [11] for more about key polynomials.

1.4.4 Diagrams

A *diagram* on $[n]$ is a sequence $D = (C_1, C_2, \ldots, C_m)$ of finite subsets of $[n]$. The subsets C_i are called the *columns* of D. Equivalently, a diagram is a subset of $[n]^2$. We think of D as a collection of boxes (i, j) in a grid, viewing an element

$i \in C_j$ as a box in row i and column j of the grid. When drawing diagrams, we read the indices in a matrix notation: i increases top-to-bottom and j increases left-to-right.

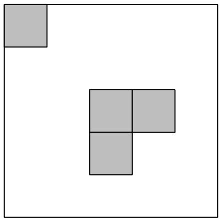
There are natural diagrams associated to any permutation or composition.

Definition 1.4.10. The *Rothe diagram* $D(w)$ of a permutation $w \in S_n$ is the diagram

$$D(w) = \{(i, j) \in [n]^2 \mid i < w^{-1}(j) \text{ and } j < w(i)\}.$$

The Rothe diagram of w can equivalently be obtained by starting with $[n]^2$ and then removing the boxes (i, w_i) for $i \in [n]$ together with any box below or to the right of an (i, w_i).

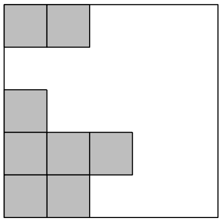
Example 1.4.11. The permutation $w = 21543$ has Rothe diagram

$D(w) =$  .

Definition 1.4.12. The *skyline diagram* $D(\alpha)$ of a composition $\alpha \in \mathbb{N}^n$ is the diagram

$$D(\alpha) = \{(i, j) \in [n]^2 \mid j \le \alpha_i\}.$$

Example 1.4.13. The composition $\alpha = (2, 0, 1, 3, 2)$ has skyline diagram

$D(\alpha) =$  .

Definition 1.4.14. A diagram D is said to be *northwest* or to have the *northwest property* if $(i, j'), (i', j) \in D$ with $i < i'$ and $j < j'$ implies that $(i, j) \in D$.

Note that both Rothe diagrams and skyline diagrams have the northwest property.

1.5 Polytopes

Recall that a subset S of $\mathbb{R}^n$ is *convex* if for all $x, y \in S$,

$$\{tx + (1-t)y \mid 0 \leq t \leq 1\} \subseteq S.$$

Given a finite set of points $V = \{v_1, \ldots, v_m\} \subseteq \mathbb{R}^n$, denote by $\mathrm{Conv}(V)$ the *convex hull* of V, the smallest convex set containing V.

A *polytope* in $\mathbb{R}^n$ is a set of the form $\mathrm{Conv}(V)$ for a finite set $V \subseteq \mathbb{R}^n$. Equivalently (though not obviously so [50]) a polytope is a bounded intersection of finitely many halfspaces. Consequently, we will interchangeably describe polytopes by a list of linear inequalities or a list of vectors. A polytope is called *integral* if it is the convex hull of a subset of $\mathbb{Z}^n$.

1.5.1 Newton Polytopes

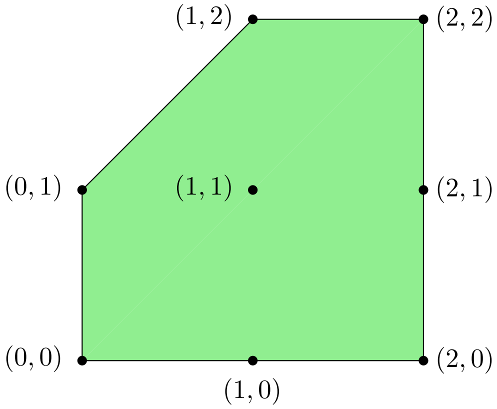
Associated to any polynomial is an integral polytope called its Newton polytope. If $f \in \mathbb{C}[x_1, \ldots, x_n]$ is given by

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha,$$

then the *Newton polytope* of f is the convex hull

$$\mathrm{Newton}(f) = \mathrm{Conv}\left\{\alpha \mid c_\alpha \neq 0\right\}.$$

Example 1.5.1. If $f = 1 + y + xy + xy^2 + x^2 + x^2y^2$, then $\mathrm{Newton}(f)$ is the polytope shown below.



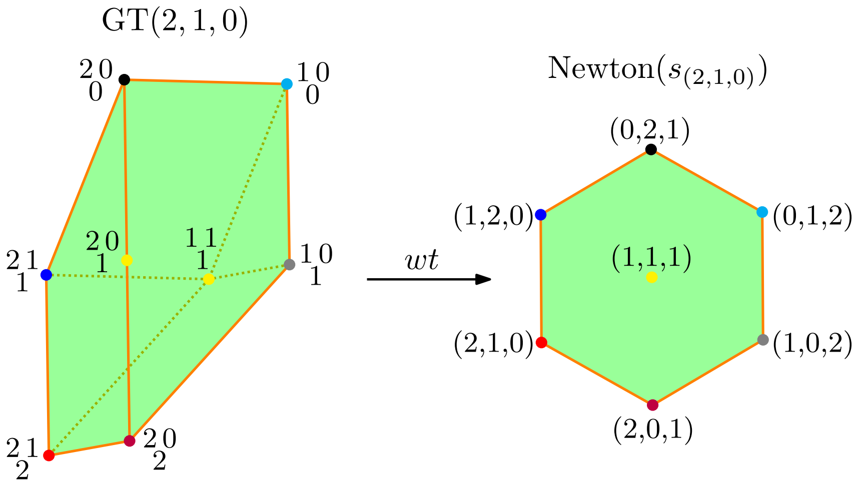
Note that integer points in $\mathrm{Newton}(f)$ need not correspond to monomials in f, for instance the point $(1, 0)$ or $(2, 1)$.

Example 1.5.2. Building on Example 1.4.4, we can view the map *wt* on Gelfand–Tsetlin patterns (Definition 1.4.3) as a map between the polytopes $\mathrm{GT}(\lambda)$ and $\mathrm{Newton}(s_\lambda)$. This illustrated for $\lambda = (2, 1, 0)$ in Figure 1.2.

1.5.2 Generalized Permutahedra

The *standard permutahedron* in $\mathbb{R}^n$ is the polytope whose vertices consist of all the permutations of $[n]$, that is all rearrangements of the vector $(1, \ldots, n)$. It is a theorem of Rado [44] that the standard permutahedron is characterized by the

$$\mathrm{GT}(2,1,0) = \left\{ \begin{array}{ccccc|c} 2 & & 1 & & 0 & 2 \geq x_{11} \geq 1 \\ & x_{11} & & x_{12} & & 1 \geq x_{12} \geq 0 \\ & & x_{22} & & & x_{11} \geq x_{22} \geq x_{12} \end{array} \right\}.$$


$\mathrm{GT}(2,1,0)$

$\mathrm{Newton}(s_{(2,1,0)})$

Figure 1.2: Viewing wt as a map $\mathrm{GT}(\lambda) \to \mathrm{Newton}(s_\lambda)$

inequalities

$$\sum_{i \in I} x_i \leq n + (n-1) + \cdots + (n - \#I + 1) \quad \text{for } I \subset [n]$$

$$\sum_{i \in [n]} x_i = n + (n-1) + \cdots + 1$$

Definition 1.5.3. A *generalized permutahedron* is a deformation of the standard permutahedron obtained by translating the vertices in such a way that all edge directions are preserved (edges are allowed to degenerate to points).

The study of generalized permutahedra originates in combinatorial optimization, where they are known as base polytopes of polymatroids. Their combinatorics have been studied extensively, see for example [1, 41, 42].

Analogous to the inequalities for standard permutahedra, generalized permutahedra are parametrized by certain tuples of real numbers $(z_I)_{I \subseteq [n]}$ indexed by

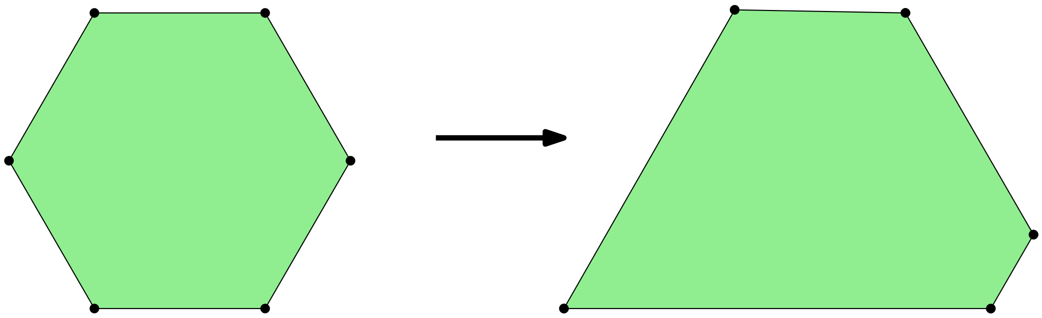


Figure 1.3: Deforming a standard permutahedron to a generalized permutahedron

nonempty subsets $I \subseteq [n]$. Such a tuple $(z_I)_I$ is called *submodular* if $z_I + z_J \geq z_{I\cup J} + z_{I\cap J}$ for all $I, J \subseteq [n]$.

Given a submodular $(z_I)_I$, the associated generalized permutahedron is

$$\left\{\mathbf{t} \in \mathbb{R}^n : \sum_{i\in I} t_i \leq z_I \text{ for } I \neq [n], \text{ and } \sum_{i=1}^{n} t_i = z_{[n]}\right\}.$$

Recall the *Minkowski sum* of polytopes P and Q in $\mathbb{R}^n$ is

$$P + Q = \{p + q \mid p \in P \text{ and } q \in Q\}.$$

An important fact is that the class of generalized permutahedra is closed under Minkowski sums. This follows from [2, Lemma 2.2]: If P and Q are parameterized by $(z_I)_{I\subseteq[n]}$ and $(z'_I)_{I\subseteq[n]}$ respectively, then $P+Q$ is parameterized by $(z_I+z'_I)_{I\subseteq[n]}$.

1.5.3 Matroid Polytopes

Matroids are combinatorial structures that generalize the notion of independence in linear algebra. A *matroid* M is a pair $(E, \mathcal{B})$ consisting of a finite set E and a nonempty collection of subsets $\mathcal{B}$ of E, called the *bases* of M. The set $\mathcal{B}$ is required to satisfy the *basis exchange axiom*: If $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 - B_2$, then there exists $b_2 \in B_2 - B_1$ such that $B_1 - b_1 \cup b_2 \in \mathcal{B}$. By choosing a labeling of the elements of E, we can always assume $E = [n]$ for some n.

Given a matroid $M = (E, \mathcal{B})$ with $E = [n]$ and a basis $B \in \mathcal{B}$, let ζ^B be the indicator vector of B. That is, let $\zeta^B = (\zeta_1^B, \ldots, \zeta_n^B) \in \mathbb{R}^n$ with $\zeta_i^B = 1$ if $i \in B$ and $\zeta_i^B = 0$ if $i \notin B$ for each i.

Definition 1.5.4. The *matroid polytope* of a matroid M is the polytope

$$P(M) = \mathrm{Conv}\{\zeta^B : B \in \mathcal{B}\}.$$

The *rank function* of M is the function

$$r_M : 2^E \to \mathbb{N}$$

defined by $r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}$. The sets $S \cap B$ where $S \subseteq [n]$ and $B \in \mathcal{B}$ are called the *independent sets* of M.

Matroid polytopes are actually a subclass of generalized permutahedra and admit the parametrization $(r_M(I))_{I \subseteq E}$:

$$P(M) = \left\{ \mathbf{t} \in \mathbb{R}^n : \sum_{i \in I} t_i \leq r_M(I) \text{ for } I \neq E, \text{ and } \sum_{i \in E} t_i = r_M(E) \right\}.$$

See [47, Corollary 40.2d] for a proof.

The following two lemmas about matroid polytopes are crucial to the proof of Theorem 2.2.3.

Lemma 1.5.5. *Let $M_1, M_2, \ldots, M_n$ be matroids and let Q be the Minkowski sum $Q = P(M_1) + \cdots + P(M_n)$. Then, every integer point $q \in Q$ can be written as $q = p_1 + \cdots + p_n$, where p_i is an integer point of $P(M_i)$ for each i.*

Lemma 1.5.5 is an easy consequence of the analogous result for integral polymatroids, see for instance [47, Corollary 46.2c]. Lemma 1.5.6 below is well-known, but we include a proof for completeness.

Lemma 1.5.6. *For any matroid M on E, the only integer points of the matroid polytope $P(M)$ are its vertices.*

Proof. Let $P(M)$ have vertices $v_1, \ldots, v_m \in \mathbb{R}^n$ corresponding to bases $B_1, \ldots, B_m$, and assume $q = (q_1, \ldots, q_n) \in P(M)$ is an integer point that is not a vertex. Then, q can be written $q = \sum_{i=1}^m \lambda_i v_i$ with $0 \le \lambda_i \le 1$ for all i and $\sum_{i=1}^m \lambda_i = 1$. It follows from a characterization of matroid polytopes [8, Theorem 1.11.1] that every face of a matroid polytope is also a matroid polytope, so there is no loss of generality in assuming that q lies in the interior of $P(M)$. In particular, this implies that all λ_i are positive. Note that since q is integral, if $q_j \neq 0$ for some j then $q_j = 1$. It follows that for any j, $q_j > 0$ if and only if $j \in B_i$ for all i. This implies that

$$\sum_{i=1}^n q_i = \#(B_1 \cap \cdots \cap B_m) < \#B_1 = r_M(E),$$

contradicting that $q \in P(M)$. $\square$

The main class of matroids whose associated polytopes we will need to consider is the Schubert matroids.

Definition 1.5.7 ([7], Section 2.4)**.** Fix positive integers $1 \le s_1 < \cdots < s_r \le n$. The sets $\{a_1, \ldots, a_r\}$ of positive integers with $a_1 < \cdots < a_r$ such that $a_1 \le s_1, \ldots, a_r \le s_r$ are the bases of a matroid, called the *Schubert matroid* $\mathrm{SM}_n(s_1, \ldots, s_r)$.

Example 1.5.8. The Schubert matroid $\mathrm{SM}_4(3,4)$ has as bases all 2-subsets of $[4]$. Its matroid polytope is the octahedron shown in Figure 1.4.

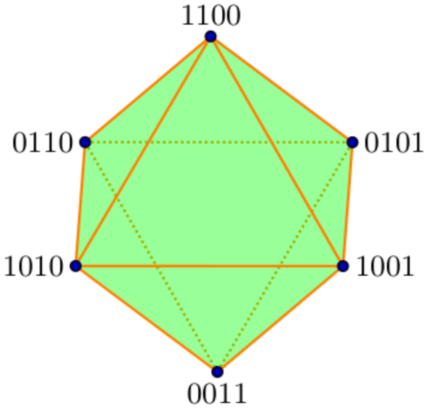


Figure 1.4: The matroid polytope $P(\mathrm{SM}_4(3,4))$

CHAPTER 2

FLAGGED WEYL MODULES

In this chapter, we describe modules of the general linear group whose characters include Schubert and key polynomials. We prove various properties of Schubert and key polynomials simultaneously by studying the combinatorics of these modules. The results of this chapter are from the papers [14, 15] joint with Alex Fink and Karola Mészáros.

2.1 Construction and Dual Characters

Let $G = \mathrm{GL}(n,\mathbb{C})$ be the group of $n \times n$ invertible matrices over $\mathbb{C}$ and B be the subgroup of G consisting of the $n \times n$ upper-triangular matrices. The flagged Weyl module is a representation $\mathcal{M}_D$ of B associated to a diagram D. The dual character of $\mathcal{M}_D$ has been shown in certain cases to be a Schubert polynomial [26] or a key polynomial [45]. We will explain the construction of $\mathcal{M}_D$ in terms of determinants given in [34].

Denote by Y the $n \times n$ matrix with indeterminants y_{ij} in the upper-triangular positions $i \leq j$ and zeros elsewhere. Let $\mathbb{C}[Y]$ be the polynomial ring in the indeterminants $\{y_{ij}\}_{i\leq j}$. Note that B acts on $\mathbb{C}[Y]$ on the right via left translation: if $f(Y) \in \mathbb{C}[Y]$, then a matrix $b \in B$ acts on f by $f(Y) \cdot b = f(b^{-1}Y)$. For any $R, S \subseteq [n]$, let Y_S^R be the submatrix of Y obtained by restricting to rows R and columns S.

For $R, S \subseteq [n]$, we say $R \leq S$ if $\#R = \#S$ and the kth least element of R does not exceed the kth least element of S for each k. For any diagrams $C =$

$(C_1, \ldots, C_n)$ and $D = (D_1, \ldots, D_n)$, we say $C \leq D$ if $C_j \leq D_j$ for all $j \in [n]$.

Definition 2.1.1. For a diagram $D = (D_1, \ldots, D_n)$, the *flagged Weyl module* $\mathcal{M}_D$ is defined by

$$\mathcal{M}_D = \operatorname{Span}_{\mathbb{C}} \left\{ \prod_{j=1}^{n} \det \left(Y_{D_j}^{C_j} \right) \,\middle|\, C \leq D \right\}.$$

$\mathcal{M}_D$ is a B-module with the action inherited from the action of B on $\mathbb{C}[Y]$.

Note that since Y is upper-triangular, the condition $C \leq D$ is technically unnecessary since $\det \left(Y_{D_j}^{C_j} \right) = 0$ unless $C_j \leq D_j$. Conversely, if $C_j \leq D_j$, then $\det \left(Y_{D_j}^{C_j} \right) \neq 0$.

For any B-module N, the character of N is defined by $\operatorname{char}(N)(x_1, \ldots, x_n) = \operatorname{tr}(X : N \to N)$ where X is the diagonal matrix $\operatorname{diag}(x_1, x_2, \ldots, x_n)$ with diagonal entries $x_1, \ldots, x_n$, and X is viewed as a linear map from N to N via the B-action. Define the *dual character* of N to be the character of the dual module N^*:

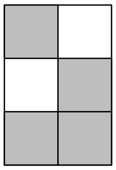
$$\begin{aligned} \operatorname{char}^*(N)(x_1, \ldots, x_n) &= \operatorname{tr}(X : N^* \to N^*) \\ &= \operatorname{char}(N)(x_1^{-1}, \ldots, x_n^{-1}). \end{aligned}$$

Definition 2.1.2. For a diagram $D \subseteq [n]^2$, let $\chi_D = \chi_D(x_1, \ldots, x_n)$ be the dual character

$$\chi_D = \operatorname{char}^* \mathcal{M}_D.$$

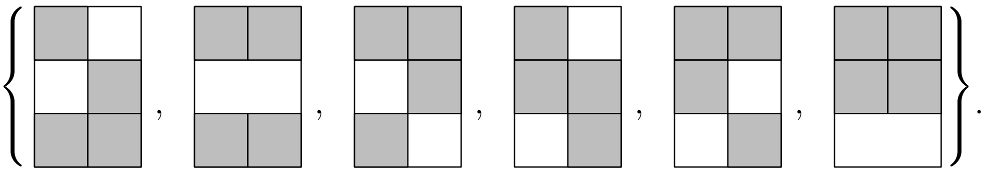
Example 2.1.3. Let D be the diagram

$$D = (\{1, 3\}, \{2, 3\}) =$$



.

Then the diagrams C with $C \leq D$ are



The polynomials corresponding to these diagrams are all linearly independent, so

$$\chi_D(x_1, x_2, x_3) = x_1x_2x_3^2 + x_1^2x_3^2 + 2x_1^2x_2x_3 + x_1x_2^2x_3 + x_1^2x_2^2.$$

2.1.1 Relation to Schubert and Key Polynomials

The following two theorems show that the set of dual characters of flagged Weyl modules of diagrams includes all Schubert and key polynomials. Recall the Rothe diagram of a permutation (Definition 1.4.10) and the skyline diagram of a composition (Definition 1.4.12).

Theorem 2.1.4 ([26])**.** *If w is a permutation with Rothe diagram $D(w)$, then*

$$\mathfrak{S}_w = \chi_{D(w)}.$$

Theorem 2.1.5 ([11])**.** *If α is a composition, with skyline diagram $D(\alpha)$, then*

$$\kappa_\alpha = \chi_{D(\alpha)}.$$

2.2 Saturation of Dual Characters

In this section, we prove the saturated Newton polytope property for Schubert and key polynomials. We accomplish these goals simultaneously by considering the more general family of dual characters of flagged Weyl modules. The results of this section appear in the joint work [14] with Alex Fink and Karola Mészáros.

Recall that for a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$, the support of f is the set $\{\alpha \mid c_\alpha \neq 0\}$, and the Newton polytope Newton(f) is the convex hull of the support of f.

Definition 2.2.1 ([38])**.** A polynomial f has *saturated Newton polytope*, abbreviated SNP, if every integer point in Newton(f) is a vector in the support of f.

Lemma 2.2.2. *For any diagram $D \subseteq [n]^2$, the monomials appearing in χ_D are exactly*

$$\left\{ \prod_{j=1}^{n} \prod_{i \in C_j} x_i \,\middle|\, C \leq D \right\}.$$

Proof. Let $X \in B$ be a diagonal matrix with diagonal entries $x_1, \dots, x_n \in \mathbb{C}$. First, note that by matrix multiplication, y_{ij} is an eigenvector of X with eigenvalue x_i^{-1}. Take a diagram $C = (C_1, \dots, C_n)$ with $C \leq D$. Then, the element $\prod_{j=1}^{n} \det\left(Y_{D_j}^{C_j}\right)$ is an eigenvector of X with eigenvalue

$$\prod_{j=1}^{n} \prod_{i \in C_j} x_i^{-1}.$$

Since M_D is spanned by elements $\prod_{j=1}^{n} \det\left(Y_{D_j}^{C_j}\right)$ and each is an eigenvector of D, the monomials appearing in the dual character χ_D with nonzero coefficient are exactly

$$\left\{ \prod_{j=1}^{n} \prod_{i \in C_j} x_i \mid C \leq D \right\}.$$

$\square$

Before proving SNP of dual characters, recall the Schubert matroid $\mathrm{SM}_n(s_1, \dots, s_r)$ (Definition 1.5.7) and its matroid polytope (Definition 1.5.4).

Theorem 2.2.3. *Let* $D = (D_1, \ldots, D_n)$ *be a diagram. Then* χ_D *has SNP, and the Newton polytope of* χ_D *equals the Minkowski sum of matroid polytopes*

$$\mathrm{Newton}(\chi_D) = \sum_{j=1}^{n} P(\mathrm{SM}_n(D_j)).$$

In particular, $\mathrm{Newton}(\chi_D)$ *is a generalized permutahedron.*

Proof. For a diagram $C = (C_1, \ldots, C_n)$, define a vector $\xi^C = (\xi^C_1, \ldots, \xi^C_n)$ by setting $\xi^C_i = \#\{j \mid i \in C_j\}$ for each i. The exponent vector of $\prod_{j=1}^{n} \prod_{i \in C_j} x_i$ is exactly ξ^C, so by Lemma 2.2.2, the support of χ_D is precisely the set $\left\{\xi^C \mid C \le D\right\}$.

However, for each $j \in [n]$, the sets $S \subseteq [n]$ with $S \le D_j$ are exactly the bases of the Schubert matroid $\mathrm{SM}_n(D_j)$. In particular, choosing a diagram $C \le D$ is equivalent to picking a basis C_j of $\mathrm{SM}_n(D_j)$ for each $j \in [n]$. If ζ^{C_j} is the indicator vector of C_j, then comparing components shows

$$\xi^C = \sum_{j=1}^{n} \zeta^{C_j}.$$

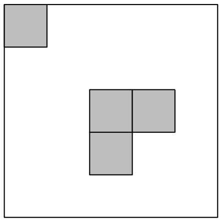
This shows that each vector ξ^C is a sum consisting of a vertex from each matroid polytope $P(\mathrm{SM}_n(D_j))$ for $j \in [n]$. Conversely, given any sum $\sum_{j=1}^{n} \zeta^{B_j}$ of a vertex ζ^{B_j} from each $P(\mathrm{SM}_n(D_j))$, let $C = (B_1, \ldots, B_n)$. Since each B_j is a basis of $\mathrm{SM}_n(D_j)$, $C \le D$. Thus, $\xi^C = \sum_{j=1}^{n} \zeta^{C_j}$ is in the support of χ_D. Consequently,

$$\mathrm{Newton}(\chi_D) = \sum_{j=1}^{n} P(\mathrm{SM}_n(D_j)). \tag{2.1}$$

To prove that χ_D has SNP, it remains to show that every integer point q in $\mathrm{Newton}(\chi_D)$ is in the support of χ_D. By (2.1), q is an integer point of a Minkowski sum of matroid polytopes, so by Lemmas 1.5.5 and 1.5.6, q can be written as a sum consisting of one vertex from each $P(\mathrm{SM}_n(D_j))$. As shown above, this is precisely what it means for q to be in the support of χ_D. $\square$

Corollary 2.2.4. *The support of any Schubert or key polynomial equals the set of integer points of a generalized permutahedron.*

Example 2.2.5. Consider $w = 21543$. The Rothe diagram of w is

$D(w) =$

.

The Minkowski sum decomposition of Newton($\mathfrak{S}_w$) is shown in Figure 2.1.

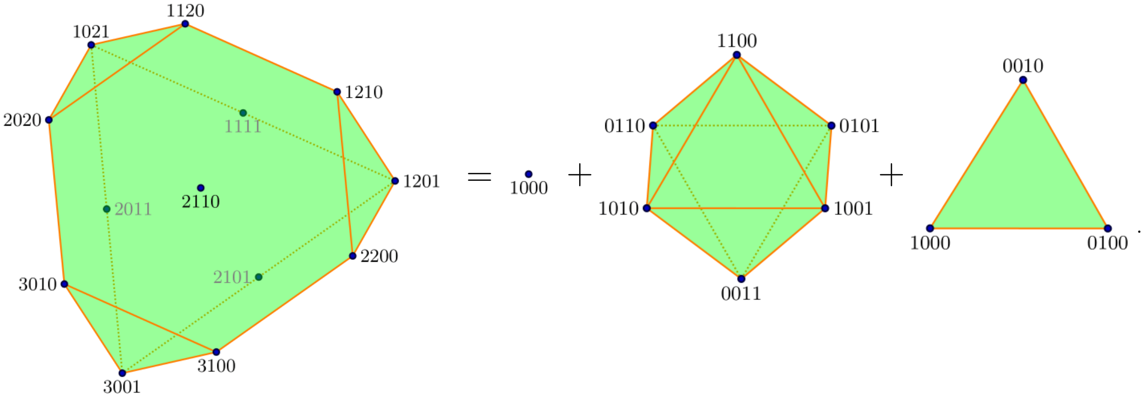


Figure 2.1: The Minkowski sum decomposition of Newton($\mathfrak{S}_{21543}$)

2.2.1 Newton Polytopes of Dual Characters

We now prove an inequality description of the Newton polytope of any dual character. The Schubert and key cases of this description were conjectured in [38]. We state this description and match it to the Minkowski sum decomposition proven in Theorem 2.2.3.

Let $D \subseteq [n]^2$ be any diagram with columns $D_j = \{i \mid (i,j) \in D\}$ for $j \in [n]$. Let $I \subseteq [n]$ be a set of row indices and $j \in [n]$ a column index. Construct a string $\mathrm{word}_{j,I}(D)$ by reading column j of the n by n grid from top to bottom and recording

- (if $(i,j) \notin D$ and $i \in I$;
-) if $(i,j) \in D$ and $i \notin I$;
- $\star$ if $(i,j) \in D$ and $i \in I$.

Let $\theta_D^j(I) = \#\text{paired ()'s in } \mathrm{word}_{j,I}(D) + \# \star\text{'s in } \mathrm{word}_{j,I}(D)$, and set

$$\theta_D(I) = \sum_{j=1}^{n} \theta_D^j(I).$$

Definition 2.2.6 ([38])**.** For any diagram $D \subseteq [n]^2$, define the *Schubitope* $\mathcal{S}_D$ by

$$\mathcal{S}_D = \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^{n} \alpha_i = \#D \text{ and } \sum_{i \in I} \alpha_i \leq \theta_D(I) \text{ for all } I \subseteq [n] \right\}.$$

Theorem 2.2.7. *For any diagram $D \subseteq [n]^2$,*

$$\mathcal{S}_D = \sum_{j=1}^{n} P\left(\mathrm{SM}_n\left(D_j\right)\right).$$

Proof. Let r_j be the rank function of the matroid $\mathrm{SM}_n(D_j)$. By [2, Lemma 2.2], the Minkowski sum $\sum_{j=1}^{n} P\left(\mathrm{SM}_n\left(D_j\right)\right)$ equals

$$\left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \sum_{i \in [n]} \alpha_i = \sum_{j=1}^{n} r_j([n]) \text{ and } \sum_{i \in I} \alpha_i \leq \sum_{j=1}^{n} r_j(I) \text{ for all } I \subseteq [n] \right\}.$$

Thus, it is sufficient to prove that $\theta_D^j(I) = r_j(I)$ for each $j \in [n]$ and $I \subseteq [n]$. Fix I and j, and let $\mathrm{word}_{j,I}(D)$ have p paired ()'s and q $\star$'s.

First, note that D_j is a basis of $\mathrm{SM}_n(D_j)$. Let B be any basis of $\mathrm{SM}_n(D_j)$ and pick elements r_1 and r_2 with $r_1 \notin B$, $r_2 \in B$, and $r_1 < r_2$. Consider the set

$B' = B \backslash \{r_2\} \cup \{r_1\}$. Then $B' \leq B \leq D_j$, so B' is also a basis of $\mathrm{SM}_n(D_j)$. Using this observation, we build a decreasing sequence of bases $D_j \geq B_1 \geq \cdots \geq B_p$.

Order the set of paired ()'s in $\mathrm{word}_{j,I}(D)$ from 1 to p. For the first pair, we get two grid squares (r_1, j) and (r_2, j) with $r_1 < r_2$, $r_1 \in I \backslash D_j$, and $r_2 \in D_j \backslash I$. Define B_1 to be the basis $D_j \backslash \{r_2\} \cup \{r_1\}$.

Inductively, the ith set of paired ()'s in $\mathrm{word}_{j,I}(D)$ gives two grid squares (r_1, j) and (r_2, j) with $r_1 < r_2$, $r_1 \in I \backslash B_{i-1}$, and $r_2 \in (B_{i-1} \cap D_j) \backslash I$. Define B_i to be the basis $B_{i-1} \backslash \{r_2\} \cup \{r_1\}$.

By construction, $\#(I \cap B_p) = p + \#(I \cap D_j) = p + q$. The proof will be complete if we can show $I \cap B_p$ is a maximal independent subset of I. If not, there is some $k \in I \backslash B_p$ and $l \in (B_p \cap D_j) \backslash I$ such that $B_p \backslash \{l\} \cup \{k\}$ is a basis. If $k < l$, then k and l correspond to a () in $\mathrm{word}_{j,S}(D)$, so $k \in B_p$ already, a contradiction. If $k > l$, then in $\mathrm{word}_{j,S}(D)$, k and l correspond to a subword)(where neither parenthesis was paired. Then, the position of l in B_p is the same as the original position of l in D_j, since it cannot have been changed by any of the swaps. In this case, $k > l$ implies $B \backslash \{l\} \cup \{k\}$ is not a basis. $\square$

2.3 An Inequality on Coefficients of Dual Characters

In this section, we prove an inequality relating the largest coefficients of dual characters of different diagrams. We start by explicitly describing the coefficients of dual characters. The results of this section are from the joint work [15] with Alex Fink and Karola Mészáros.

2.3.1 Coefficients of Dual Characters

Lemma 2.3.1. *Let* $D \subseteq [n]^2$ *be a diagram. Fix any diagram* $C^{(1)} \leq D$*, and set*

$$\boldsymbol{m} = \prod_{j=1}^{n} \prod_{i \in C_j^{(1)}} x_i.$$

Let $C^{(1)}, \ldots, C^{(r)}$ *be all the diagrams* C *such that* $C \leq D$ *and* $\prod_{j=1}^{n} \prod_{i \in C_j} x_i = \boldsymbol{m}$*. Then, the coefficient of* $\boldsymbol{m}$ *in* χ_D *is equal to*

$$\dim \left(\operatorname{Span}_{\mathbb{C}} \left\{ \prod_{j=1}^{n} \det \left(Y_{D_j}^{C_j^{(i)}} \right) \;\middle|\; i \in [r] \right\} \right).$$

Proof. The coefficient of $\boldsymbol{m}$ in χ_D equals the dimension of the eigenspace of $\boldsymbol{m}^{-1}$ in $\mathcal{M}_D$ ($\boldsymbol{m}^{-1}$ occurs here instead of $\boldsymbol{m}$ since χ_D is the dual character). This eigenspace equals

$$\operatorname{Span}_{\mathbb{C}} \left\{ \prod_{j=1}^{n} \det \left(Y_{D_j}^{C_j^{(i)}} \right) \;\middle|\; i \in [r] \right\},$$

so the result follows. □

2.3.2 The Inequality on the Coefficients

Given diagrams $C, D \subseteq [n]^2$ and $k, l \in [n]$, let $\widehat{C}$ and $\widehat{D}$ denote the diagrams obtained from C and D by removing any boxes in row k or column l. Fix a diagram D. For each diagram $\widehat{C}$, let

$$\widehat{C}_{\text{aug}} = \widehat{C} \cup \{(k, i) \mid (k, i) \in D\} \cup \{(i, l) \mid (i, l) \in D\} \subseteq [n]^2.$$

The following lemma is immediate.

Lemma 2.3.2. *Let* $C, D \subseteq [n]^2$ *be diagrams and* $k, l \in [n]$*. If* $\widehat{C} \leq \widehat{D}$*, then* $\widehat{C}_{\text{aug}} \leq D$*. In particular, every diagram* $C' \leq \widehat{D}$ *with no boxes in row* k *can be* *obtained from some diagram $C \leq D$ by removing any boxes in row k or column l from C.*

The following result is key. For a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and a monomial $\boldsymbol{m}$, let $[\boldsymbol{m}]f$ denote the coefficient of $\boldsymbol{m}$ in f.

Lemma 2.3.3. *Fix a diagram D and $k, l \in [n]$. Let $\{\widehat{C}^{(i)}\}_{i\in[m]}$ be a set of diagrams with $\widehat{C}^{(i)} \leq \widehat{D}$ for each i, and denote $\widehat{C}^{(i)}_{\mathrm{aug}}$ by $C^{(i)}$ for $i \in [m]$. If the polynomials $\left\{\prod_{j\in[n]} \det\left(Y_{D_j}^{C_j^{(i)}}\right)\right\}_{i\in[m]}$ are linearly dependent, then so are the polynomials $\left\{\prod_{j\in[n]\setminus\{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right)\right\}_{i\in[m]}$.*

Proof. We are given that

$$\sum_{i\in[m]} c_i \prod_{j\in[n]} \det\left(Y_{D_j}^{C_j^{(i)}}\right) = 0 \tag{2.2}$$

for some constants $(c_i)_{i\in[m]} \in \mathbb{C}^m$ not all zero. Since $C^{(i)} = \widehat{C}^{(i)}_{\mathrm{aug}}$ for $\widehat{C}^{(i)} \leq \widehat{D}$ we have that $C_l^{(i)} = D_l$ for every $i \in [m]$. Thus, (2.2) can be rewritten as

$$\det\left(Y_{D_l}^{D_l}\right)\left(\sum_{i\in[m]} c_i \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right)\right) = 0. \tag{2.3}$$

However, since $\det\left(Y_{D_l}^{D_l}\right) \neq 0$, we conclude that

$$\sum_{i\in[m]} c_i \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right) = 0. \tag{2.4}$$

First consider the case that the only boxes of D in row k or column l are those in D_l. If this is the case then

$$\prod_{j\in[n]\setminus\{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right) = \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right) \tag{2.5}$$

for each $i \in [m]$. Therefore,

$$\sum_{i\in[m]} c_i \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right) = \sum_{i\in[m]} c_i \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right). \tag{2.6}$$

Combining (2.4) and (2.6) we obtain that the polynomials $\left\{\prod_{j\in[n]\setminus\{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right)\right\}_{i\in[m]}$ are linearly dependent, as desired.

Now, suppose that there are boxes of D in row k that are not in D_l. Let $j_1 < \ldots < j_p$ be all indices $j \neq l$ such that $D_j = \widehat{D}_j \cup \{k\}$. Then also $C_{j_q}^{(i)} = \widehat{C}_{j_q}^{(i)} \cup \{k\}$ for each $i \in [m]$ and $q \in [p]$. Let us consider the left-hand side of (2.4) as a polynomial in y_{kk}. Then, (2.4) implies that the coefficient of y_{kk}^p is 0:

$$[y_{kk}^p] \sum_{i\in[m]} c_i \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right) = 0. \tag{2.7}$$

However,

$$[y_{kk}^p] \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right) = \prod_{j\in[n]\setminus\{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right), \tag{2.8}$$

as is seen by Laplace expansion on the kth row, and therefore

$$[y_{kk}^p] \sum_{i \in [m]} c_i \prod_{j \in [n] \setminus \{l\}} \det\left(Y_{D_j}^{C_j^{(i)}}\right) = \sum_{i \in [m]} c_i \prod_{j \in [n] \setminus \{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right). \tag{2.9}$$

Thus, (2.7) and (2.9) imply that

$$\sum_{i \in [m]} c_i \prod_{j \in [n] \setminus \{l\}} \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right) = 0, \tag{2.10}$$

as desired. □

Theorem 2.3.4. *Fix a diagram $D \subseteq [n]^2$, and let $\widehat{D}$ be the diagram obtained from D by removing any boxes in row k or column l. Then*

$$\chi_D(x_1, \ldots, x_n) = M(x_1, \ldots, x_n)\chi_{\widehat{D}}(x_1, \ldots, x_{k-1}, 0, x_{k+1}, \ldots, x_n) + F(x_1, \ldots, x_n),$$

where $F(x_1, \ldots, x_n) \in \mathbb{N}[x_1, \ldots, x_n]$ and

$$M(x_1, \ldots, x_n) = \left(\prod_{(k,i) \in D} x_k\right)\left(\prod_{(i,l) \in D} x_i\right).$$

Proof. Let $M = M(x_1, \ldots, x_n)$. We must show that $[M\boldsymbol{m}]\chi_D \geq [\boldsymbol{m}]\chi_{\widehat{D}}$ for each monomial $\boldsymbol{m}$ of $\chi_{\widehat{D}}$ not divisible by x_k. Let $C^{(1)}, \ldots, C^{(r)}$ be all the diagrams C such that $C \leq D$ and $\prod_{j=1}^n \prod_{i \in C_j} x_i = M\boldsymbol{m}$. By Lemma 2.3.1,

$$[M\boldsymbol{m}]\chi_D = \dim\left(\operatorname{Span}_{\mathbb{C}}\left\{\prod_{j=1}^n \det\left(Y_{D_j}^{C_j^{(i)}}\right) \;\middle|\; i \in [r]\right\}\right).$$

Let $1, 2, \ldots, q$ be the indices of the distinct diagrams among $\widehat{C}^{(1)}, \ldots, \widehat{C}^{(r)}$. By Lemma 2.3.2, $\widehat{C}^{(1)}, \ldots, \widehat{C}^{(q)}$ are all the diagrams C such that $C \leq \widehat{D}$ and $\prod_{j=1}^n \prod_{i \in C_j} x_i = \boldsymbol{m}$, as no diagram with this dual eigenvalue can have a box in row k. So Lemma 2.3.1 implies that

$$[\boldsymbol{m}]\chi_{\widehat{D}} = \dim\left(\operatorname{Span}_{\mathbb{C}}\left\{\prod_{j=1}^n \det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right) \;\middle|\; i \in [q]\right\}\right).$$

Finally, Lemma 2.3.3 implies that

$$\dim\left(\mathrm{Span}_{\mathbb{C}}\left\{\prod_{j=1}^{n}\det\left(Y_{D_j}^{C_j^{(i)}}\right)\ \middle|\ i\in[r]\right\}\right)\geq\dim\left(\mathrm{Span}_{\mathbb{C}}\left\{\prod_{j=1}^{n}\det\left(Y_{\widehat{D}_j}^{\widehat{C}_j^{(i)}}\right)\ \middle|\ i\in[q]\right\}\right),$$

so $[M\boldsymbol{m}]\chi_D\geq[\boldsymbol{m}]\chi_{\widehat{D}}$ for each monomial $\boldsymbol{m}$ of $\chi_{\widehat{D}}$ not divisible by x_k; that is

$$\chi_D(x_1,\ldots,x_n)-M\chi_{\widehat{D}}(x_1,\ldots,x_{k-1},0,x_{k+1},\ldots,x_n)\in\mathbb{N}[x_1,\ldots,x_n].$$

□

Corollary 2.3.5. *Fix a diagram* $D\subseteq[n]^2$*, and let* $\widehat{D}$ *be the diagram obtained from* D *by removing any boxes in row* k *or column* l*. If* c *is the coefficient of some monomial in* $\chi_{\widehat{D}}(x_1,\ldots,x_{k-1},0,x_{k+1},\ldots,x_n)$*, then* $\chi_D(x_1,\ldots,x_n)$ *has a monomial with coefficient at least* c*.*

CHAPTER 3

ZERO-ONE SCHUBERT POLYNOMIALS

In this chapter, we give a full characterization of the set of permutations whose Schubert polynomial is zero-one, that is has all its nonzero coefficients equal to one. These are exactly the permutations whose Schubert polynomial equals the integer point transform of its Newton polytope. We begin with a construction due to Magyar which gives two formulas for Schubert polynomials. We use the combinatorics of Magyar's construction to provide several equivalent conditions on a Schubert polynomial being zero-one. The results and presentation of this chapter are from the paper [15] joint with Alex Fink and Karola Mészáros.

3.1 Orthodontia for Rothe Diagrams

In this section, we describe a construction called orthodontia, due to Magyar [34]. We focus on the special case of Rothe diagrams.

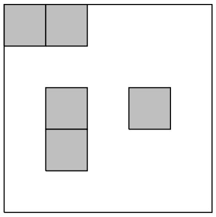
Recall that by *diagram*, we mean a sequence $D = (C_1, C_2, \ldots, C_n)$ of finite subsets of $[n]$, called the *columns* of D. We interchangeably think of D as a collection of boxes (i, j) in a grid, viewing an element $i \in C_j$ as a box in row i and column j of the grid. When we draw diagrams, we read the indices as in a matrix: i increases top-to-bottom and j increases left-to-right. Two diagrams D and D' are called *column-equivalent* if one is obtained from the other by reordering nonempty columns and adding or removing any number of empty columns. For a column $C \subseteq [n]$, let the *multiplicity* $\mathrm{mult}_D(C)$ be the number of columns of D which are equal to C. The sum of diagrams, denoted $D \oplus D'$, is constructed by concatenating the lists of columns; graphically this means placing D' to the right of D.

Recall that the Rothe diagram $D(w)$ of a permutation $w \in S_n$ is the diagram

$$D(w) = \{(i,j) \in [n]^2 \mid i < w^{-1}(j) \text{ and } j < w(i)\}.$$

Note that Rothe diagrams have the *northwest property*: If $(r,c'), (r',c) \in D(w)$ with $r < r'$ and $c < c'$, then $(r,c) \in D(w)$.

Example 3.1.1. If $w = 31542$, then

$D(w) =$  $= (\{1\}, \{1,3,4\}, \emptyset, \{3\}, \emptyset).$

We next recall Magyar's orthodontia. Let D be the Rothe diagram of a permutation $w \in S_n$ with columns $C_1, C_2, \ldots, C_n$. We describe an algorithm for constructing a reduced word $\boldsymbol{i} = (i_1, \ldots, i_l)$ and a multiplicity list $\boldsymbol{m} = (k_1, \ldots, k_n;\, m_1, \ldots, m_l)$ such that the diagram $D_{\boldsymbol{i},\boldsymbol{m}}$ defined by

$$D_{\boldsymbol{i},\boldsymbol{m}} = \bigoplus_{j=1}^{n} k_j \cdot [j] \quad \oplus \quad \bigoplus_{j=1}^{l} m_j \cdot (s_{i_1} s_{i_2} \cdots s_{i_j}[i_j]),$$

is column-equivalent to D. In the above, $m \cdot C$ denotes $C \oplus \cdots \oplus C$ with m copies of C; in particular $0 \cdot C$ should be interpreted as a diagram with no columns, not the empty column.

The algorithm to produce $\boldsymbol{i}$ and $\boldsymbol{m}$ from D is as follows. To begin the first step, for each $j \in [n]$ let $k_j = \text{mult}_D([j])$, the number of columns of D of the form $[j]$. Replace all such columns by empty columns for each j to get a new diagram D_-.

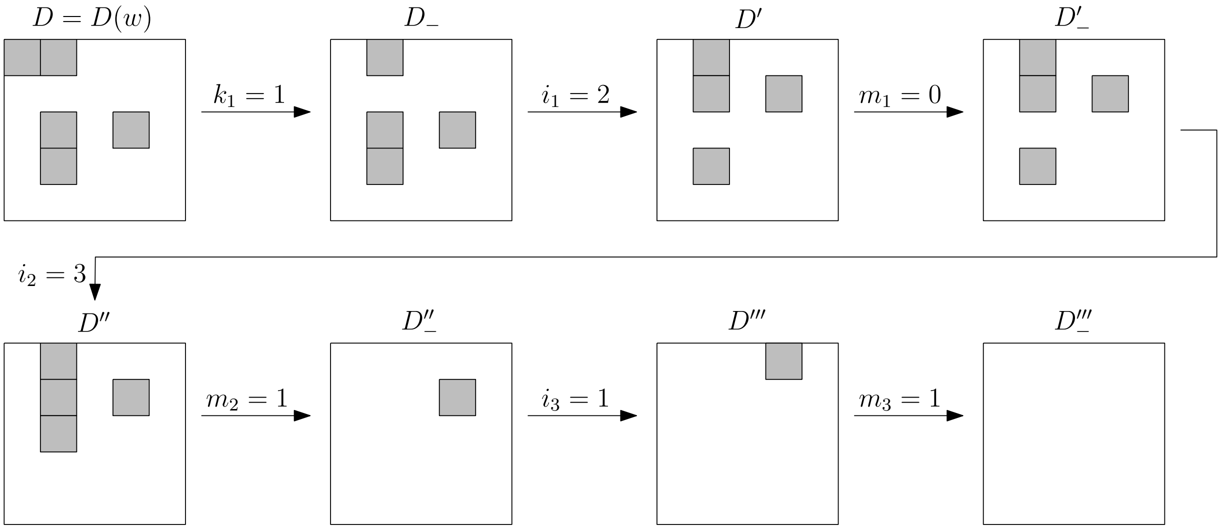
Given a column $C \subseteq [n]$, a *missing tooth* of C is a positive integer i such that $i \notin C$, but $i+1 \in C$. The only columns without missing teeth are the empty column and the intervals $[i]$. Hence the first nonempty column of D_- (if there is any) contains a smallest missing tooth i_1. Switch rows i_1 and i_1+1 of D_- to get a new diagram D'.

In the second step, repeat the above with D' in place of D. That is, let $m_1 = \mathrm{mult}_{D'}([i_1])$ and replace all columns of the form $[i_1]$ in D' by empty columns to get a new diagram D'_-. Find the smallest missing tooth i_2 of the first nonempty column of D'_-, and switch rows i_2 and i_2+1 of D'_- to get a new diagram D''.

Continue in this fashion until no nonempty columns remain. It is easily seen that the sequences $\boldsymbol{i} = (i_1, \ldots, i_l)$ and $\boldsymbol{m} = (k_1, \ldots, k_n; m_1, \ldots, m_l)$ just constructed have the desired properties.

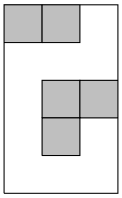
Definition 3.1.2. The pair $(\boldsymbol{i}, \boldsymbol{m})$ constructed from the preceding algorithm is called the *orthodontic sequence* of w.

Example 3.1.3. If $w = 31542$, then the orthodontic sequence algorithm produces the diagrams



The sequence of missing teeth gives $\boldsymbol{i} = (2, 3, 1)$ and $\boldsymbol{m} = (1, 0, 0, 0, 0;\ 0, 1, 1)$, so

$$D_{\boldsymbol{i},\boldsymbol{m}} =$$



.

3.2 Two Formulas for Schubert Polynomials

We recount two related "ascending formulas" for Schubert polynomials, due to Magyar [34]. These formulas stand in contrast to the classical descending formula due to Lascoux and Schützenberger [28].

Theorem 3.2.1 ([34, Proposition 15])**.** *Let $w \in S_n$ have orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$. If $\pi_j = \partial_j x_j$ denotes the jth Demazure operator and $\omega_j = x_1 x_2 \cdots x_j$, then*

$$\mathfrak{S}_w = \omega_1^{k_1} \cdots \omega_n^{k_n} \pi_{i_1}(\omega_{i_1}^{m_1} \pi_{i_2}(\omega_{i_2}^{m_2} \cdots \pi_{i_l}(\omega_{i_l}^{m_l}) \cdots)).$$

Example 3.2.2. For $w = 31542$, it is easily checked that (c.f. Example 1.4.6)

$$\mathfrak{S}_w = x_1 \pi_2 \pi_3 (x_1 x_2 x_3 \pi_1(x_1)).$$

Theorem 3.2.1 can also be realized on the level of tableaux, analogous to semistandard Young tableaux in the case of Schur polynomials. A **filling** (with entries in $\{1, ..., n\}$) of a diagram D is a map T assigning to each box in D an integer in $[n]$. A filling T is called *column-strict* if T is strictly increasing down each column of D. The *weight* of a filling T is the vector $\mathrm{wt}(T)$ whose ith component $\mathrm{wt}(T)_i$ is the number of times i occurs in T.

Given a permutation $w \in S_n$ with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, we will define a set $\mathcal{T}_w$ of fillings of the diagram $D_{\boldsymbol{i},\boldsymbol{m}}$ which satisfy

$$\mathfrak{S}_w = \sum_{T \in \mathcal{T}_w} x_1^{\mathrm{wt}(T)_1} x_2^{\mathrm{wt}(T)_2} \cdots x_n^{\mathrm{wt}(T)_n}.$$

We start by recalling the *root operators*, first defined in [30]. These are operators f_i which either take a filling T of a diagram D to another filling of D, or are undefined on T. To define root operators, we first encode a filling T in terms of its reading word. The *reading word* of a filling T of a diagram $D = D_{\boldsymbol{i},\boldsymbol{m}}$ is the sequence of the entries of T read in order, down each column going left-to-right along columns; that is the sequence

$$T(1,1), T(2,1), \ldots, T(n,1), T(1,2), T(2,2), \ldots, T(n,2), \ldots, T(n,n)$$

ignoring any boxes $(i, j) \notin D$.

If it is defined, the operator f_i changes an entry of i in T to an entry of $i+1$ according to the following rule. First, ignore all the entries in T except those which equal i or $i+1$. Now "match parentheses": if, in the list of entries not yet ignored, an i is followed by an $i+1$, then henceforth ignore that pair of entries as well; look again for an i followed (up to ignored entries) by an $i+1$, and henceforth ignore this pair; continue doing this until all no such pairs remain unignored. The remaining entries of T will be a subword of the form $i+1, i+1, \ldots, i+1, i, i, \ldots, i$.

If i does not appear in this word, then $f_i(T)$ is undefined. Otherwise, f_i changes the leftmost i to an $i+1$. Reading the image word back into D produces a new filling. We can iteratively apply f_i to a filling T.

Example 3.2.3. If $T = 3\,1\,2\,2\,2\,1\,3\,1\,2\,4\,3\,2\,4\,1\,3\,1$, applying f_1 iteratively to T yields:

$$\begin{array}{rcccccccccccccccc}
T = & 3&1&2&2&2&1&3&1&2&4&3&2&4&1&3&1\\
& \cdot&1&2&2&2&1&\cdot&1&2&\cdot&\cdot&2&\cdot&1&\cdot&1\\
& \cdot&\cdot&\cdot&2&2&1&\cdot&\cdot&\cdot&\cdot&\cdot&2&\cdot&1&\cdot&1\\
& \cdot&\cdot&\cdot&2&2&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\
f_1(T) = & 3&1&2&2&2&1&3&1&2&4&3&2&4&\mathbf{2}&3&1\\
f_1^2(T) = & 3&1&2&2&2&1&3&1&2&4&3&2&4&2&3&\mathbf{2}
\end{array}$$

$f_1^3(T)$ is undefined

Define the set-valued *quantized Demazure operator* $\widetilde{\pi}_i$ by $\widetilde{\pi}_i(T) = \{T, f_i(t), f_i^2(T), \ldots\}$. For a set $\mathcal{T}$ of tableaux, let

$$\widetilde{\pi}_i(\mathcal{T}) = \bigcup_{T \in \mathcal{T}} \widetilde{\pi}_i(T).$$

Next, consider the column $[j]$ and its minimal column-strict filling $\widetilde{\omega}_j$ (jth row maps to j). For a filling T of a diagram D with columns $(C_1, C_2, \ldots, C_n)$, define in the obvious way the composite filling of $[j] \oplus D$, corresponding to concatenating the reading words of $[j]$ and D. Define $[j]^r \oplus D$ analogously by adding r columns $[j]$ to D, each with filling $\widetilde{\omega}_j$.

Definition 3.2.4. Let $w \in S_n$ be a permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$. Define the set $\mathcal{T}_w$ of tableaux by

$$\mathcal{T}_w = \widetilde{\omega}_1^{k_1} \oplus \cdots \oplus \widetilde{\omega}_n^{k_n} \oplus \widetilde{\pi}_{i_1}(\widetilde{\omega}_{i_1}^{m_1} \oplus \widetilde{\pi}_{i_2}(\widetilde{\omega}_{i_2}^{m_2} \oplus \cdots \oplus \widetilde{\pi}_{i_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots)).$$

Theorem 3.2.5 ([34, Proposition 16])**.** *Let $w \in S_n$ be a permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$. Then,*

$$\mathfrak{S}_w = \sum_{T \in \mathcal{T}_w} x_1^{\mathrm{wt}(T)_1} x_2^{\mathrm{wt}(T)_2} \cdots x_n^{\mathrm{wt}(T)_n}.$$

Example 3.2.6. Consider again $w = 31542$, so the orthodontic sequence of w is $\boldsymbol{i} = (2, 3, 1)$ and $\boldsymbol{m} = (1, 0, 0, 0, 0; 0, 1, 1)$. The set $\mathcal{T}_w$ is built up as follows:

$$\begin{aligned}
\{\} &\xrightarrow{\widetilde{\omega}_1} \{1\} \xrightarrow{\widetilde{\pi}_1} \{1, 2\} \xrightarrow{\widetilde{\omega}_3} \{1231, 1232\} \xrightarrow{\widetilde{\pi}_3} \{1231, 1241, 1232, 1242\} \\
&\xrightarrow{\widetilde{\pi}_2} \{1231, 1241, 1341, 1232, 1233, 1242, 1342, 1343\} \\
&\xrightarrow{\widetilde{\omega}_1} \{11231, 11241, 11341, 11232, 11233, 11242, 11342, 11343\}
\end{aligned}$$

which agrees with (c.f. Example 1.4.6)

$$\mathfrak{S}_w = x_1^3x_2x_3 + x_1^3x_2x_4 + x_1^3x_3x_4 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^2x_2x_3x_4 + x_1^2x_3^2x_4.$$

We now describe a way to view each step of the construction of $\mathcal{T}_w$ as producing a set of fillings of a diagram.

Definition 3.2.7. Let w be a permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. For each $r \in [l]$, define

$$\mathcal{T}_w(r) = \widetilde{\omega}_{i_r}^{m_r} \oplus \widetilde{\pi}_{i_{r+1}}(\widetilde{\omega}_{i_{r+1}}^{m_{r+1}} \oplus \cdots \oplus \widetilde{\pi}_{i_l}(\widetilde{\omega}_{i_l}) \cdots).$$

Set $\mathcal{T}_w(0) = \mathcal{T}_w$.

Definition 3.2.8. Let w be a permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. For any $r \in [l]$, let $O(w, r)$ be the diagram obtained from $D(w)$ in the construction of $(\boldsymbol{i}, \boldsymbol{m})$ at the time when the row swaps of the missing teeth $i_1, \ldots, i_r$ have all been executed on $D(w)$, but after executing the row swap of the missing tooth i_r, columns without missing teeth have not yet been removed (m_r has not yet been recorded). Set $O(w, 0) = D(w)$. For each r, give $O(w, r)$ the same column indexing as $D(w)$, so any columns replaced by empty columns in the execution of the missing teeth $i_1, \ldots, i_{r-1}$ retain their original index in $D(w)$.

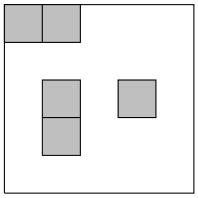
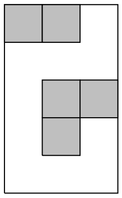
The motivation behind Definition 3.2.7 and Definition 3.2.8 is that the elements of $\mathcal{T}_w(r)$ can be viewed as column-strict fillings of $O(w, r)$ for each r. To do this, the choice of filling order for $O(w, r)$ is crucial. Let $w \in S_n$ and consider $D = D(w)$ and $D_{\boldsymbol{i},\boldsymbol{m}}$. Suppose D has z nonempty columns. There is a unique permutation τ of $[n]$ taking the column indices of D to the column indices of $D_{\boldsymbol{i},\boldsymbol{m}} \oplus \emptyset^{n-z}$ with the following properties:

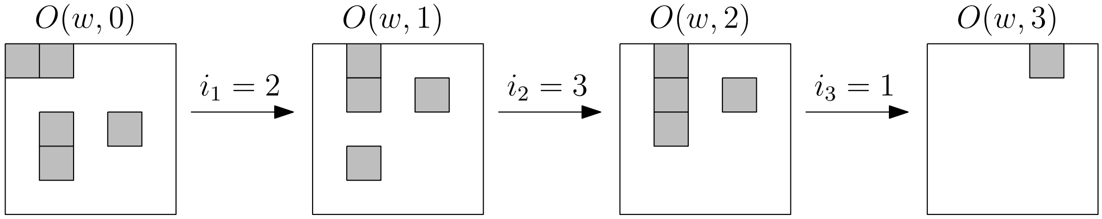
- Column c of D is the same as column $\tau(c)$ of $D_{\boldsymbol{i},\boldsymbol{m}}$.
- If column c and column c' of D are equal with $c < c'$, then $\tau(c) < \tau(c')$.

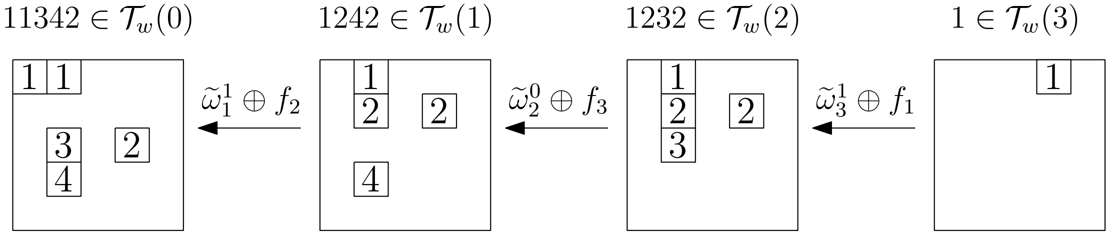
Recall that the columns of $O(w, r)$ have the same column labels as D. To read an element $T \in \mathcal{T}_w(r)$ into $O(w, r)$, read T left-to-right and fill in top-to-bottom columns $\tau^{-1}(n)$, $\tau^{-1}(n-1), \ldots, \tau^{-1}(1)$ (ignoring any column indices referring to empty columns).

Lemma 3.2.9. *Let $w \in S_n$ have orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. In the filling order specified above, the elements of $\mathcal{T}_w(r)$ are column-strict fillings of $O(w, r)$ for each $0 \le r \le l$.*

Example 3.2.10. Take again $w = 31542$ with orthodontic sequence $\boldsymbol{i} = (2, 3, 1)$ and $\boldsymbol{m} = (1, 0, 0, 0, 0; 0, 1, 1)$. Recall that

$D(w) =$  , $D_{\boldsymbol{i},\boldsymbol{m}} =$  , so $\tau = 12435 = \tau^{-1}$. Consider the elements $1 \in \mathcal{T}_w(3)$, $1232 \in \mathcal{T}_w(2)$, $1242 \in \mathcal{T}_w(1)$, and $11342 \in \mathcal{T}_w(0)$. The column filling order of each $O(w,r)$ is given by reading τ^{-1} in one-line notation right to left: in the indexing of $D(w)$, fill down column 4, then down column 2, then down column 1. The elements of each set $\mathcal{T}_w(r)$ are column-strict fillings in the corresponding diagrams $O(w,r)$:





Lemma 3.2.11. *Let w be a permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. For each $0 \leq r \leq l$, $O(w,r)$ has the northwest property.*

Definition 3.2.12. A filling T of a diagram D is called *row-flagged* if $T(p,q) \leq p$ for each box $(p,q) \in D$.

Lemma 3.2.13. *For each $0 \leq r \leq l$, the elements of $\mathcal{T}_w(r)$ are row-flagged fillings of $O(w,r)$.*

Proof. Clearly, the singleton $\mathcal{T}_w(l)$ is a row-flagged filling of $O(w,l)$. Assume that for some $l \geq s > 0$, the result holds with $r = s$. We show that the result also holds with $r = s-1$. Let $T \in \mathcal{T}_w(s)$. We must show that for each u, if $f_{i_s}^u(T)$ is defined, then $\widetilde{\omega}_{i_{s-1}}^{m_{s-1}} \oplus f_{i_s}^u(T)$ is a row-flagged filling of $O(w, s-1)$. By the orthodontia construction, $O(w,s)$ is obtained from $O(w,s-1)$ by removing the m_{s-1} columns with no missing tooth, and then switching rows i_s+1 and i_s.

Since T is a row-flagged filling of $O(w,s)$, each box in $O(w,s)$ containing an entry of T equal to i_s lies in a row with index at least i_s. Any box in $O(w,s)$ containing an entry of T equal to i_s and lying in row i_s of $O(w,s)$ will have row index i_s+1 in $O(w,s-1)$. Any box in $O(w,s)$ containing an entry in T equal to i_s and lying in a row $d > i_s$ of $O(w,s)$ will still have row index d in $O(w,s-1)$. Then if $f_{i_s}^{u}(T)$ is defined, $\widetilde{\omega}_{i_{s-1}}^{m_{s-1}} \oplus f_{i_s}^{u}(T)$ will be a row-flagged filling of $O(w,s-1)$. $\square$

3.3 A Sufficient Condition for Zero-Oneness

This section is devoted to giving a sufficient condition on the orthodontic sequence $(\boldsymbol{i},\boldsymbol{m})$ of w for the Schubert polynomial $\mathfrak{S}_w$ to be zero-one. We give such a condition in Theorem 3.3.6. We will see in Theorem 3.6.1 that this condition turns out to also be a necessary condition for $\mathfrak{S}_w$ to be zero-one.

We start with a less ambitious result:

Proposition 3.3.1. *Let $w \in S_n$ and $(\boldsymbol{i},\boldsymbol{m})$ be the orthodontic sequence of w. If $\boldsymbol{i} = (i_1,\dots,i_l)$ has distinct entries, then $\mathfrak{S}_w$ is zero-one.*

Proof. Let $T, T' \in \mathcal{T}_w$ with $\mathrm{wt}(T) = \mathrm{wt}(T')$. By Definition 3.2.4, we can find $p_1,\dots,p_l$ so that

$$T = \widetilde{\omega}_1^{k_1} \oplus \cdots \oplus \widetilde{\omega}_n^{k_n} \oplus f_{i_1}^{p_1}(\widetilde{\omega}_{i_1}^{m_1} \oplus \cdots \oplus f_{i_l}^{p_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots).$$

Then if $e_1,\dots,e_n$ denote the standard basis vectors of $\mathbb{R}^n$,

$$\mathrm{wt}(T) = \sum_{j=1}^{n} \mathrm{wt}(\widetilde{\omega}_j^{k_j}) + \sum_{j=1}^{l} \mathrm{wt}(\widetilde{\omega}_{i_j}^{m_j}) + \sum_{j=1}^{l} p_j(e_{i_j+1} - e_{i_j}).$$

Similarly, we can find $q_1, \ldots, q_l$ so that

$$T' = \widetilde{\omega}_1^{k_1} \oplus \cdots \oplus \widetilde{\omega}_n^{k_n} \oplus f_{i_1}^{q_1}(\widetilde{\omega}_{i_1}^{m_1} \oplus \cdots \oplus f_{i_l}^{q_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots),$$

which implies

$$\mathrm{wt}(T') = \sum_{j=1}^{n} \mathrm{wt}(\widetilde{\omega}_j^{k_j}) + \sum_{j=1}^{l} \mathrm{wt}(\widetilde{\omega}_{i_j}^{m_j}) + \sum_{j=1}^{l} q_j(e_{i_j+1} - e_{i_j}).$$

As $\mathrm{wt}(T) = \mathrm{wt}(T')$,

$$0 = \mathrm{wt}(T) - \mathrm{wt}(T') = (p_1 - q_1)(e_{i_1+1} - e_{i_1}) + \cdots + (p_l - q_l)(e_{i_l+1} - e_{i_l}). \qquad (*)$$

Since the vectors $\{e_{i_j+1} - e_{i_j}\}_{j=1}^{l}$ are independent and $\boldsymbol{i}$ has distinct entries, $p_j = q_j$ for all j. Thus $T = T'$. This shows that all elements of $\mathcal{T}_w$ have distinct weights, so $\mathfrak{S}_w$ is zero-one. □

We now strengthen Proposition 3.3.1 to allow $\boldsymbol{i}$ to not have distinct entries. To do this, we will need a technical definition related to the orthodontic sequence. Recall the construction of the orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$ of a permutation $w \in S_n$ (Definition 3.1.2) and the intermediate diagrams $O(w, r)$ (Definition 3.2.8). Let $\boldsymbol{i} = (i_1, \ldots, i_l)$, and define $O(w, r)_-$ to be the diagram $O(w, r)$ with all columns of the form $[i_r]$ replaced by empty columns.

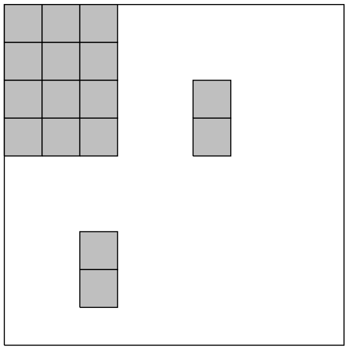
Definition 3.3.2. Define the *orthodontic impact function* $\mathcal{I}_w : [l] \to 2^{[n]}$ by

$$\mathcal{I}_w(j) = \{c \in [n] \mid (i_j + 1, c) \in O(w, j-1)_-\}.$$

That is, $\mathcal{I}_w(j)$ is the set of indices of columns of $O(w, j-1)_-$ that are changed when rows i_j and $i_j + 1$ are swapped to form $O(w, j)$.

Definition 3.3.3. Let $w \in S_n$ have orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. We say w is *multiplicity-free* if for any $r, s \in [l]$ with $r \neq s$ and $i_r = i_s$, we have $\mathcal{I}_w(r) = \mathcal{I}_w(s) = \{c\}$ for some $c \in [n]$.

Example 3.3.4. If $w = 457812693$, then

$D(w) =$  and $\boldsymbol{i} = (6, 5, 7, 6, 2, 1, 3, 2)$.

The only entries of $\boldsymbol{i}$ occurring multiple times are $i_1 = i_4 = 6$ and $i_5 = i_8 = 2$. Their respective impacts are $\mathcal{I}_w(1) = \mathcal{I}_w(4) = \{3\}$ and $\mathcal{I}_w(5) = \mathcal{I}_w(8) = \{6\}$, so w is multiplicity-free.

The proof of the generalization of Proposition 3.3.1 will require the following technical lemma. Before proceeding, recall Lemma 3.2.9 and Lemma 3.2.13: for every $0 \leq j \leq l$, elements of $\mathcal{T}_w(j)$ can be viewed as row-flagged, column-strict fillings of $O(w, j)$ (via the column filling order of $O(w, j)$ specified prior to Lemma 3.2.9). Applying $\widetilde{\omega}_{i_{j-1}}^{m_{j-1}} \oplus f_{i_j}$ to an element of $\mathcal{T}_w(j)$ gives an element of $\mathcal{T}_w(j-1)$, a filling of $O(w, j-1)$. Thus, when we speak below of the application of f_{i_j} to an element $T \in \mathcal{T}_w(j)$ "changing an i_j to an $i_j + 1$ in column c", we specifically mean that when we view T as a filling of $O(w, j)$ and $\widetilde{\omega}_{i_{j-1}}^{m_{j-1}} \oplus f_{i_j}(T)$ as a filling of $O(w, j-1)$, T and $\widetilde{\omega}_{i_{j-1}}^{m_{j-1}} \oplus f_{i_j}(T)$ differ (in the stated way) in their entries in column c.

Lemma 3.3.5. *Let w be a multiplicity-free permutation with orthodontic sequence $(\boldsymbol{i}, \boldsymbol{m})$, $\boldsymbol{i} = (i_1, \ldots, i_l)$. Suppose $i_r = i_s$ with $r < s$ and $\mathcal{I}_w(r) = \mathcal{I}_w(s) = \{c\}$. Then for each j with $r \leq j \leq s$, $\mathcal{I}_w(j) = \{c\}$ and the application of f_{i_j} to an element of $\mathcal{T}_w(j)$ is either undefined or changes an i_j to an $i_j + 1$ in column c.*

Proof. We handle first the case that $j = r$. In the diagram $O(w, r-1)$, column c is the leftmost column containing a missing tooth, and i_r is the smallest missing tooth in column c. Reading column c of $O(w, r-1)$ top-to-bottom, one sees a (possibly empty) sequence of boxes in $O(w, r-1)$, followed by a sequence of boxes not in $O(w, r-1)$. The sequence of boxes not in $O(w, r-1)$ has length at least two since i_r occurs at least twice in $\boldsymbol{i}$, and terminates with the box $(i_r + 1, c) \in O(w, r-1)$. Note that since $(i_r - 1, c), (i_r, c) \notin O(w, r-1)$, the northwest property of $O(w, r-1)$ implies that there can be no box $(i_r - 1, c')$ or (i_r, c') in $O(w, r-1)$ with $c' > c$. Note also that since $\mathcal{I}_w(r) = \{c\}$, we have $(i_r + 1, c') \notin O(w, r-1)$ for each $c' > c$. Lastly, observe that for any $c' > c$ and $d > i_r + 1$, there can be no box $(d, c') \in O(w, r-1)$. Otherwise there would be some $t \in [l]$ with $i_t = i_r$ and $t \neq r$ such that $c' \in \mathcal{I}_w(t)$, violating that w is multiplicity-free.

As a consequence of the previous observations, the largest row index that a column $c' > c$ of $O(w, r-1)$ can contain a box in is $i_r - 2$. In particular, Lemma 3.2.13 implies that the application of f_{i_r} to an element of $\mathcal{T}_w(r)$ either is undefined or changes an i_r to an $i_r + 1$ in column c. This concludes the case that $j = r$.

When $j = s$, an entirely analogous argument works. The only significant difference in the observations is that when column c of $O(w, s-1)$ is read top-to-bottom, the (possibly empty) initial sequence of boxes in $O(w, s-1)$ is followed by a sequence of boxes not in $O(w, s-1)$ with length at least 1, ending with the box $(i_s + 1, c)$. Consequently, the largest row index that a column $c' > c$ of $O(w, s-1)$ can contain a box in is $i_s - 1$. In particular, Lemma 3.2.13 implies that the application of f_{i_s} to an element of $\mathcal{T}_w(s)$ either is undefined or changes an i_s to an $i_s + 1$ in column c. This concludes the case that $j = s$.

Now, let $r < j < s$. Since $\mathcal{I}_w(r) = \mathcal{I}_w(s) = \{c\}$, we have $c \in \mathcal{I}_w(j)$. If i_j occurs multiple times in $\boldsymbol{i}$, then multiplicity-freeness of w implies $\mathcal{I}_w(j) = \{c\}$. In this case, we can find $j' \neq j$ with $i_j = i_{j'}$ and apply the previous argument (with r and s replaced by j and j') to conclude that the application of f_{i_j} to an element of $\mathcal{T}_w(j)$ is either undefined or changes an i_j to an $i_j + 1$ in column c.

Thus, we assume i_j occurs only once in $\boldsymbol{i}$. Recall that it was shown above that $O(w, r-1)$ has no boxes (d, c') with $d > i_r$ and $c' > c$. Read top-to-bottom, let column c of $O(w, r-1)$ have a (possibly empty) initial sequence of boxes ending with a missing box in row u, so clearly $u \leq i_r - 1$. Since the first missing tooth in column c of $O(w, r-1)$ is in row i_r, none of the boxes $(u, c), (u+1, c), \ldots, (i_r, c)$ are in $O(w, r-1)$, but $(i_r + 1, c) \in O(w, r-1)$. Then, the northwest property implies that there is no box in $O(w, r-1)$ in any column $c' > c$ in any of rows $u, u+1, \ldots, i_r$. In particular, the largest row index such that a column $c' > c$ of $O(w, r-1)$ can contain a box in is $u - 1$.

As $r < j < s$ and $\mathcal{I}_w(r) = \mathcal{I}_w(s) = \{c\}$, we have that $c \in \mathcal{I}_w(j)$. Also since $r < j < s$, the leftmost nonempty column in $O(w, j-1)$ is column c, and $i_j \geq u$. Then in $O(w, j-1)$, the maximum row index a box in a column $c' > c$ can have is $u - 1$. In particular, $\mathcal{I}_w(j) = \{c\}$, and Lemma 3.2.13 implies that the application of f_{i_j} to an element of $\mathcal{T}_w(j)$ is either undefined or changes an i_j to an $i_j + 1$ in column c. $\square$

Theorem 3.3.6. *If w is multiplicity-free, then $\mathfrak{S}_w$ is zero-one.*

Proof. Assume $\mathrm{wt}(T) = \mathrm{wt}(T')$ for some $T, T' \in \mathcal{T}_w$. If we can show that $T = T'$, then we can conclude that all elements of $\mathcal{T}_w$ have distinct weights, so $\mathfrak{S}_w$ is zero-one. To begin, write

$$T = \widetilde{\omega}_1^{k_1} \oplus \cdots \oplus \widetilde{\omega}_n^{k_n} \oplus f_{i_1}^{p_1}(\widetilde{\omega}_{i_1}^{m_1} \oplus \cdots \oplus f_{i_l}^{p_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots)$$

and

$$T' = \widetilde{\omega}_1^{k_1} \oplus \cdots \oplus \widetilde{\omega}_n^{k_n} \oplus f_{i_1}^{q_1}(\widetilde{\omega}_{i_1}^{m_1} \oplus \cdots \oplus f_{i_l}^{q_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots),$$

for some $p_1, \ldots, p_l, q_1, \ldots, q_l$. The basic idea of the proof is to show that as T and T' are constructed step-by-step from $\widetilde{\omega}_{i_l}^{m_l}$, the resulting intermediate tableaux are intermittently equal. At termination of the construction, this will imply that $T = T'$.

By the expansion $(*)$ of $\mathrm{wt}(T) - \mathrm{wt}(T')$ used in the proof of Proposition 3.3.1, we observe that $p_u = q_u$ for all u such that i_u occurs only once in $\boldsymbol{i}$. Let s be the largest index such that $p_s \neq q_s$. Suppose $\mathcal{I}_w(s) = \{c\}$. Let r_1 be the smallest index such that i_{r_1} occurs multiple times in $\boldsymbol{i}$ and $\mathcal{I}_w(r_1) = \{c\}$. We know $r_1 < s$, because $(*)$ implies that $p_{s'} \neq q_{s'}$ for another $s' < s$ with $i_{s'} = i_s$, and by multiplicity-freeness $\mathcal{I}_w(s') = \{c\}$. We wish to find an interval $[r, s] \subseteq [r_1, s]$ such that $r < s$ and the following two conditions hold:

(i) If $v \geq r$ and i_v occurs multiple times in $\boldsymbol{i}$, then any other v' with $i_v = i_{v'}$ will satisfy $v' \geq r$.

(ii) For every j with $r < j < s$ and i_j occurring only once in $\boldsymbol{i}$, there are t and u with $r \leq t < j < u \leq s$ such that $i_t = i_u$.

We first show that (i) holds for $[r_1, s]$. Note that if i_v occurs multiple times in $\boldsymbol{i}$ and $r_1 \leq v \leq s$, then it must be that $\mathcal{I}_w(v) = \{c\}$ by the fact that the orthodontia construction records all missing teeth needed to eliminate one column before moving on to the next column. If $i_{v'} = i_v$, then $\mathcal{I}_w(v') = \{c\}$ also, by multiplicity-freeness of w. The choice of r_1 implies $r_1 \leq v'$. If i_v occurs multiple times in $\boldsymbol{i}$ with $s < v$ and $\mathcal{I}_w(v) = \{c\}$, then the choice of r_1 again implies that $r_1 \leq v'$ for any $i_{v'} = i_v$. If i_v occurs multiple times in $\boldsymbol{i}$ with $s < v$ and $\mathcal{I}_w(v) \neq \{c\}$, then the orthodontia construction implies that any v' with $i_v = i_{v'}$ must satisfy $s < v'$. In particular, $r_1 < v'$ as needed. Thus, (i) holds for $[r_1, s]$. If $[r_1, s]$ also satisfies (ii), then we are done.

Otherwise, assume $[r_1, s]$ does not satisfy (ii). Then there is some j with $r_1 < j < s$ such that i_j occurs only once in $\boldsymbol{i}$ and there are no t and u with $r_1 \leq t < j < u \leq s$ and $i_t = i_u$. Consequently for every pair $i_u = i_t$ with $r_1 \leq t < u \leq s$, it must be that either $t < u < j$ or $j < t < u$. Let r_2 be the smallest index such that $j < r_2$ and i_{r_2} occurs multiple times in $\boldsymbol{i}$. By the choice of j, it is clear that the interval $[r_2, s]$ still satisfies (i). If $[r_2, s]$ also satisfies (ii), then we are done.

Otherwise, $[r_2, s]$ satisfies (i) but not (ii), and we can argue exactly as in the case of $[r_1, s]$ to find an r_3 such that $r_2 < r_3 < s$ and $[r_3, s]$ satisfies (i). Continue working in this fashion. We show that this process terminates with an interval $[r, s]$ satisfying $r < s$, (i), and (ii).

As mentioned above, there exists $s' < s$ such that $i_{s'} = i_s$. Let s' be the maximal index less than s with this property. Since all of the intervals $[r_*, s]$ will satisfy (i), it follows that $r_1 < r_2 < \cdots \leq s'$. At worst, the process will terminate after finitely many steps with the interval $[s', s]$. The interval $[s', s]$ will then satisfy (i) since the process reached it, and will trivially satisfy (ii) since $i_s = i_{s'}$.

Hence, we can assume that we have found an interval $[r, s]$ satisfying $r < s$, (i), and (ii). Consider the tableaux

$$T_r = \widetilde{\omega}_{i_{r-1}}^{m_{r-1}} \oplus f_{i_r}^{p_r}(\widetilde{\omega}_{i_r}^{m_r} \oplus \cdots \oplus f_{i_l}^{p_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots), \qquad T_s = \widetilde{\omega}_{i_s}^{m_s} \oplus f_{i_{s+1}}^{p_{s+1}}(\widetilde{\omega}_{i_{s+1}}^{m_{s+1}} \oplus \cdots \oplus f_{i_l}^{p_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots),$$

$$T'_r = \widetilde{\omega}_{i_{r-1}}^{m_{r-1}} \oplus f_{i_r}^{q_r}(\widetilde{\omega}_{i_r}^{m_r} \oplus \cdots \oplus f_{i_l}^{q_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots), \qquad T'_s = \widetilde{\omega}_{i_s}^{m_s} \oplus f_{i_{s+1}}^{q_{s+1}}(\widetilde{\omega}_{i_{s+1}}^{m_{s+1}} \oplus \cdots \oplus f_{i_l}^{q_l}(\widetilde{\omega}_{i_l}^{m_l}) \cdots).$$

By definition, $T_r, T'_r \in \mathcal{T}_w(r-1)$, so we can view T_r and T'_r as fillings of

$O(w, r-1)$. Similarly, $T_s, T'_s \in \mathcal{T}_w(s)$, so we can view T_s and T'_s as fillings of $O(w, s)$. Since we chose s to be the largest index such that $p_s \neq q_s$, it follows that $T_s = T'_s$. By property (i) of $[r, s]$, $i_u \neq i_v$ for any $u < r \leq v$. Hence, it must be that $\mathrm{wt}(T_r) = \mathrm{wt}(T'_r)$. Finally, property (ii) of $[r, s]$ allows us to apply Lemma 3.3.5 and conclude that for any $a_r, a_{r+1}, \ldots, a_s \geq 0$, when $\widetilde{\omega}_{i_{r-1}}^{m_{r-1}} \oplus f_{i_r}^{a_r}(\widetilde{\omega}_{m_r}^{i_r} \oplus \cdots \oplus \widetilde{\omega}_{i_{s-1}}^{m_{s-1}} f_{i_s}^{a_s}(\text{—}) \cdots)$ is applied to an element of $\mathcal{T}_w(s)$, only the entries in column c are affected by the root operators $f_{i_r}^{a_r}, \ldots, f_{i_s}^{a_s}$. Since

$$T_r = \widetilde{\omega}_{i_{r-1}}^{m_{r-1}} \oplus f_{i_r}^{p_r}(\widetilde{\omega}_{m_r}^{i_r} \oplus \cdots \oplus \widetilde{\omega}_{i_{s-1}}^{m_{s-1}} f_{i_s}^{p_s}(T_s) \cdots) \quad \text{and} \quad T'_r = \widetilde{\omega}_{i_{r-1}}^{m_{r-1}} \oplus f_{i_r}^{q_r}(\widetilde{\omega}_{m_r}^{i_r} \oplus \cdots \oplus \widetilde{\omega}_{i_{s-1}}^{m_{s-1}} f_{i_s}^{q_s}(T'_s) \cdots),$$

T_r and T'_r must coincide outside of column c. Since we already deduced that $\mathrm{wt}(T_r) = \mathrm{wt}(T'_r)$, it follows that column c of T_r and T'_r have the same weight. By column-strictness of T_r and T'_r, column c of T_r and T'_r must coincide, so $T_r = T'_r$.

To complete the proof, let $\hat{s}$ be the largest index $\hat{s} < r$ such that $p_{\hat{s}} \neq q_{\hat{s}}$. If no such index exists, then $T = T'$. Otherwise, set $\hat{r}_1$ to be the smallest index such that $i_{\hat{r}_1}$ occurs multiple times in $\boldsymbol{i}$ and $\mathcal{I}_w(\hat{r}_1) = \mathcal{I}_w(\hat{s})$. We have $\hat{r}_1 < \hat{s}$ because some other $\hat{s}'$ distinct from $\hat{s}$ such that $p_{\hat{s}'} \neq q_{\hat{s}'}$ and $i_{\hat{s}'} = i_{\hat{s}}$ must exist as before, and $\hat{s}'$ is also less than r by property (i) of $[r, s]$. Use the previous algorithm to find an interval $[\hat{r}, \hat{s}] \subseteq [\hat{r}_1, \hat{s}]$ satisfying $\hat{r} < \hat{s}$, (i), and (ii). Construct $T_{\hat{r}}, T'_{\hat{r}}, T_{\hat{s}}, T'_{\hat{s}}$, and argue exactly as in the case of $[r, s]$ that $T_{\hat{r}} = T'_{\hat{r}}$.

Continuing in this manner for a finite number of steps will show $T = T'$. □

As we will show in Theorem 3.6.1, it is not only sufficient but also necessary that w be multiplicity-free for the Schubert polynomial $\mathfrak{S}_w$ to be zero-one.

3.4 A Configuration Avoidance Condition for Multiplicity-Freeness

This section is devoted to showing that w being multiplicity-free is equivalent to a certain configuration avoidance condition on $D(w)$. We start with several definitions.

Definition 3.4.1. We say a Rothe diagram $D = D(w)$ contains an instance of configuration A if there are r_1, c_1, r_2, c_2, r_3 such that $1 \leq r_3 < r_1 < r_2$, $1 < c_1 < c_2$, $(r_1, c_1), (r_2, c_2) \in D$, $(r_1, c_2) \notin D$, and $w_{r_3} < c_1$.

Definition 3.4.2. We say a Rothe diagram $D = D(w)$ contains an instance of configuration B if there are $r_1, c_1, r_2, c_2, r_3, r_4$ such that $1 \leq r_4 \neq r_3 < r_1 < r_2$, $2 < c_1 < c_2$, $(r_1, c_1), (r_1, c_2), (r_2, c_2) \in D$, $w_{r_3} < c_1$, and $w_{r_4} < c_2$.

Definition 3.4.3. We say a Rothe diagram $D = D(w)$ contains an instance of configuration B$'$ if there are $r_1, c_1, r_2, c_2, r_3, r_4$ such that $1 \leq r_4 < r_3 < r_1 < r_2$, $2 < c_1 < c_2$, $(r_1, c_1), (r_1, c_2), (r_2, c_1) \in D$, $w_{r_3} < c_1$, and $w_{r_4} < c_1$.

Given a Rothe diagram $D(w)$, we will call a tuple $(r_1, c_1, r_2, c_2, r_3)$ meeting the conditions of Definition 3.4.1 an *instance* of configuration A in $D(w)$. Similarly, we will call a tuple $(r_1, c_1, r_2, c_2, r_3, r_4)$ meeting the conditions of Definition 3.4.2 (resp. 3.4.3) an instance of configuration B (resp. B$'$) in $D(w)$. See Figure 3.1 for examples of these configurations in Rothe diagrams.

Theorem 3.4.4. *If $w \in S_n$ is a permutation such that $D(w)$ does not contain any instance of configuration* A, B, *or* B$'$, *then w is multiplicity-free.*

Theorem 3.6.1 will also imply the converse of this theorem.

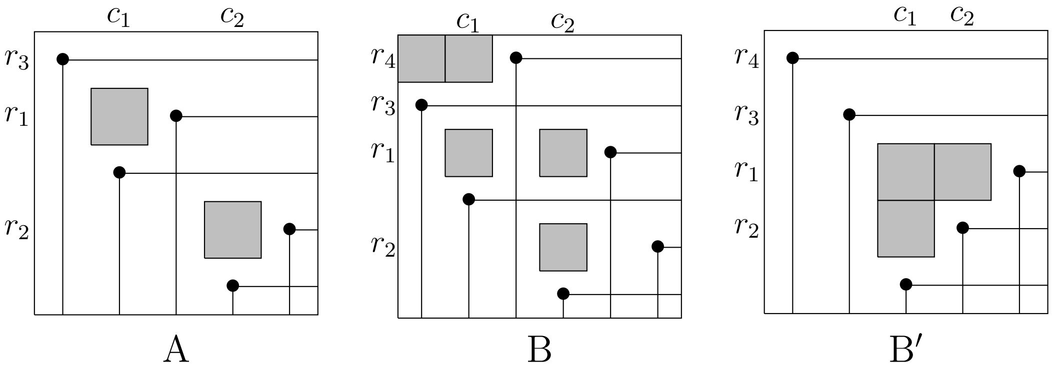


Figure 3.1: Examples of instances of the configurations A, B, and B′ in Rothe diagrams. Both the hooks removed from the $n \times n$ grid to form each Rothe diagram and the remaining boxes are shown.

Proof. We prove the contrapositive. Assume w is not multiplicity-free and let $(\boldsymbol{i}, \boldsymbol{m})$ be the orthodontic sequence of w. Then, we can find entries $i_{p_1} = i_{p_2}$ of $\boldsymbol{i}$ with $p_1 < p_2$ such that either $\mathcal{I}_w(p_1) \neq \mathcal{I}_w(p_2)$, or $\mathcal{I}_w(p_1) = \mathcal{I}_w(p_2)$ with $\#\mathcal{I}_w(p_1) > 1$. We show that $D(w)$ must contain at least one instance of configuration A, B, or B′.

Case 1: Assume that $\mathcal{I}_w(p_1) \not\subseteq \mathcal{I}_w(p_2)$ and $\mathcal{I}_w(p_2) \not\subseteq \mathcal{I}_w(p_1)$. Take $c_1 \in \mathcal{I}_w(p_1) \backslash \mathcal{I}_w(p_2)$ and $c_2 \in \mathcal{I}_w(p_2) \backslash \mathcal{I}_w(p_1)$. We show that columns c_1 and c_2 of $D(w)$ contain an instance of configuration A.

In step p_1 of the orthodontia on $D(w)$, a box in column c_1 is moved (by the missing tooth i_{p_1}) to row i_{p_1}. Let this box originally be in row r_1 of $D(w)$. Analogously, let the box in column c_2 moved to row i_{p_2} in step p_2 of the orthodontia (by the missing tooth i_{p_2}) originally be in row r_2 of $D(w)$. Observe that $r_1 < r_2$. If $c_2 < c_1$, then the northwest property would imply that $(r_1, c_2) \in D(w)$, contradicting that $c_2 \notin \mathcal{I}_w(p_1)$. Thus $c_1 < c_2$. Since $c_2 \notin \mathcal{I}_w(p_1)$, $(r_1, c_2) \notin D(w)$. Lastly, since the box (r_1, c_1) is moved by the orthodontia, there is some box $(r_3, c_1) \notin D(w)$ with $r_3 < r_1$. Consequently, $w_{r_3} < c_1$. Thus, $(r_1, c_1, r_2, c_2, r_3)$ is an instance of configuration A.

Case 2: Assume $\mathcal{I}_w(p_2)$ is a proper subset of $\mathcal{I}_w(p_1)$. Let $c_1 = \max(\mathcal{I}_w(p_2))$ and $c_2 = \min(\mathcal{I}_w(p_1)\backslash\mathcal{I}_w(p_2))$. Let the box in column c_1 moved to row $i_{p_1} = i_{p_2}$ in step p_1 (resp. p_2) of the orthodontia originally be in row r_1 (resp. r_2) of $D(w)$. Observe that $r_1 < r_2$.

Assume first that $c_1 < c_2$. Since $c_1 \in \mathcal{I}_w(p_1) \cap \mathcal{I}_w(p_2)$, the boxes (r_1, c_1) and (r_2, c_2) both move weakly above row i_{p_1} in the orthodontia. Then, we can find indices r_3, r_4 with $r_4 < r_3 < r_1$ such that $(r_3, c_1), (r_4, c_1) \notin D(w)$. Hence, $w_{r_3} < c_1$ and $w_{r_4} < c_1$, so $(r_1, c_1, r_2, c_2, r_3, r_4)$ is an instance of configuration B′.

Otherwise $c_1 > c_2$. Since the box (r_1, c_2) is moved by the orthodontia, we can find $r_3 < r_1$ with $(r_3, c_2) \notin D(w)$. Then $w_{r_3} < c_2$. As we are assuming $c_2 < c_1$, $(r_3, c_1) \notin D(w)$ also. Since the boxes (r_1, c_1) and (r_2, c_1) in $D(w)$ are moved weakly above row i_{p_1} by the orthodontia, we can find some $r_4 < r_1$ with $r_4 \neq r_3$ such that $(r_4, c_1) \notin D(w)$. Then, $w_{r_4} < c_1$, so $(r_1, c_2, r_2, c_1, r_3, r_4)$ is an instance of configuration B.

Case 3: Assume $\mathcal{I}_w(p_1)$ is a proper subset of $\mathcal{I}_w(p_2)$. This case is handled similarly to Case 2. Let $c_1 = \max(\mathcal{I}_w(p_1))$ and $c_2 = \min(\mathcal{I}_w(p_2)\backslash\mathcal{I}_w(p_1))$. Let the box in column c_1 moved to row $i_{p_1} = i_{p_2}$ in step p_1 (resp. p_2) of the orthodontia originally be in row r_1 (resp. r_2) of $D(w)$. Observe that $r_1 < r_2$.

Assume $c_1 < c_2$. Since the boxes (r_1, c_1) and (r_2, c_1) of $D(w)$ are moved weakly above row i_{p_1} by the orthodontia, we can find indices r_3, r_4 with $r_4 < r_3 < r_1$ such that $(r_3, c_1), (r_4, c_1) \notin D(w)$. Then, $w_{r_3} < c_1$ and $w_{r_4} < c_1$. Since $c_2 \notin \mathcal{I}_w(p_1)$, $(r_1, c_2) \notin D(w)$. Then, $(r_1, c_1, r_2, c_2, r_3)$ is an instance of configuration A.

Otherwise $c_1 > c_2$. As $c_2 \notin \mathcal{I}_w(p_1)$, $(r_1, c_2) \notin D(w)$. Since $(r_2, c_2), (r_1, c_1) \in D(w)$, this is a contradiction of the northwest property of $D(w)$.

Case 4: Assume $\mathcal{I}_w(p_1) = \mathcal{I}_w(p_2)$ is not a singleton. Let $c_1, c_2 \in \mathcal{I}_w(p_1)$ with $c_1 < c_2$. Let the box in column c_1 moved to row $i_{p_1} = i_{p_2}$ in step p_1 (resp. p_2) of the orthodontia originally be in row r_1 (resp. r_2) of $D(w)$. Observe that $r_1 < r_2$. Since the boxes (r_1, c_1) and (r_2, c_1) in $D(w)$ are moved weakly above row i_{p_1} by the orthodontia, we can find indices r_3, r_4 with $r_4 < r_3 < r_1$ such that $(r_3, c_1), (r_4, c_1) \notin D(w)$. Then, $w_{r_3} < c_1$ and $w_{r_4} < c_1$. Thus, $(r_1, c_1, r_2, c_2, r_3, r_4)$ is an instance of configuration B′. □

3.5 A Pattern Avoidance Condition for Multiplicity-Freeness

We now relate multiplicity-freeness to pattern avoidance of permutations. We begin by clarifying our pattern avoidance terminology. A *pattern* σ of *length* n is a permutation in S_n. The length n is a crucial part of the data of a pattern; we make no identifications between patterns of different lengths, unlike what is usual when handling permutations in the Schubert calculus. A permutation w *contains* σ if w has n entries $w_{j_1}, \dots, w_{j_n}$ with $j_1 < j_2 < \cdots < j_n$ that are in the same relative order as $\sigma_1, \sigma_2, \dots, \sigma_n$. In this case, the indices $j_1 < j_2 < \cdots < j_n$ are called a *realization* of σ in w. We say that w *avoids* the pattern σ if w does not contain σ. To illustrate the dependence of these definitions on n, note that $w = 154623$ contains the pattern 132, but not the pattern 132456.

The following easy lemma gives a diagrammatic interpretation of pattern avoidance.

Lemma 3.5.1. *Let* $w \in S_n$ *be a permutation and* σ *a pattern of length* m *contained in* w. *Choose a realization* $j_1 < j_2 < \cdots < j_m$ *of* σ *in* w. *Then* $D(\sigma)$ *is obtained from* $D(w)$ *by deleting the rows* $[n]\backslash\{j_1, \ldots, j_m\}$ *and the columns* $[n]\backslash\{w_{j_1}, \ldots, w_{j_m}\}$, *and reindexing the remaining rows and columns by* $[m]$, *preserving their order.*

Definition 3.5.2. The *multiplicitous patterns* are those in the set

$$\text{MPatt}\{12543, 13254, 13524, 13542, 21543, 125364,$$
$$125634, 215364, 215634, 315264, 315624, 315642\}.$$

Theorem 3.5.3. *Let* $w \in S_n$. *Then* $D(w)$ *does not contain any instance of configuration* A, B, *or* B′ *if and only if* w *avoids all of the multiplicitous patterns.*

Proof. It is easy to check (see Figure 3.2) that each of the twelve multiplicitous patterns contains an instance of configuration A, B, or B′. Lemma 3.5.1 implies that if w contains $\sigma \in$ MPatt, then deleting some rows and columns from $D(w)$ yields $D(\sigma)$. Since $D(\sigma)$ contains at least one instance of configuration A, B, or B′, so does $D(w)$.

Conversely, assume $D(w)$ contains at least one instance of configuration A, B, or B′. We must show that w contains some multiplicitous pattern. Let $\tau^1, \tau^2, \ldots, \tau^n$ be the n patterns of length $n-1$ contained in w; say τ^j is realized in w by forgetting w_j. Without loss of generality, we may assume none of $D(\tau^1), \ldots, D(\tau^n)$ contain an instance of configuration A, B, or B′: if $D(\tau^j)$ does contain an instance of one of these configurations, replace w by τ^j and iterate.

For each j, $D(\tau^j)$ is obtained from $D(w)$ by deleting row j and column w_j. Since $D(\tau^j)$ does not contain any instance of any of our three configurations, each cross $\{(j,q) \mid (j,q) \in D(w)\} \cup \{(p,w_j) \mid (p,w_j) \in D(w)\}$ intersects each instance of every configuration contained in $D(w)$. However, an instance of configuration A involves only three rows and two columns, and an instance of B or B′ involves only four rows and two columns. Thus, it must be that $w \in S_n$ for some $n \leq 6$. It can be checked by exhaustion that the only permutations in S_n with $n \leq 6$ that are minimal (with respect to pattern avoidance) among those whose Rothe diagrams contain an instance of configuration A, B, or B′ are the twelve multiplicitous patterns. □

3.6 A Characterization of Zero-One Schubert Polynomials

We are now ready to state and prove our complete characterization of zero-one Schubert polynomials.

Theorem 3.6.1. *The following are equivalent:*

- *(i) The Schubert polynomial* $\mathfrak{S}_w$ *is zero-one.*
- *(ii) The permutation* w *is multiplicity-free,*
- *(iii) The Rothe diagram* $D(w)$ *does not contain any instance of configuration* A, B, *or* B′,
- *(iv) The permutation* w *avoids the multiplicitous patterns, namely* 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, *and* 315642.

Proof. Theorem 3.3.6 shows $(ii) \Rightarrow (i)$. Theorem 3.4.4 shows $(iii) \Rightarrow (ii)$. Theorem 3.5.3 shows $(iii) \Leftrightarrow (iv)$. The final implication $(i) \Rightarrow (iv)$ will follow immediately from Corollary 3.6.3 below, since the Schubert polynomials associated to the permutations 12543, 13254, 13524, 13542, 21543, 125364, 125634, 215364, 215634, 315264, 315624, and 315642 each have a coefficient equal to 2. ☐

To complete the proof of Theorem 3.6.1, we specialize Theorem 2.3.4:

Theorem 3.6.2. *Fix $w \in S_n$ and let $\sigma \in S_{n-1}$ be the pattern with Rothe diagram $D(\sigma)$ obtained by removing row k and column w_k from $D(w)$. Then*

$$\mathfrak{S}_w(x_1, \ldots, x_n) = M(x_1, \ldots, x_n)\mathfrak{S}_\sigma(x_1, \ldots, \widehat{x_k}, \ldots, x_n) + F(x_1, \ldots, x_n), \tag{3.1}$$

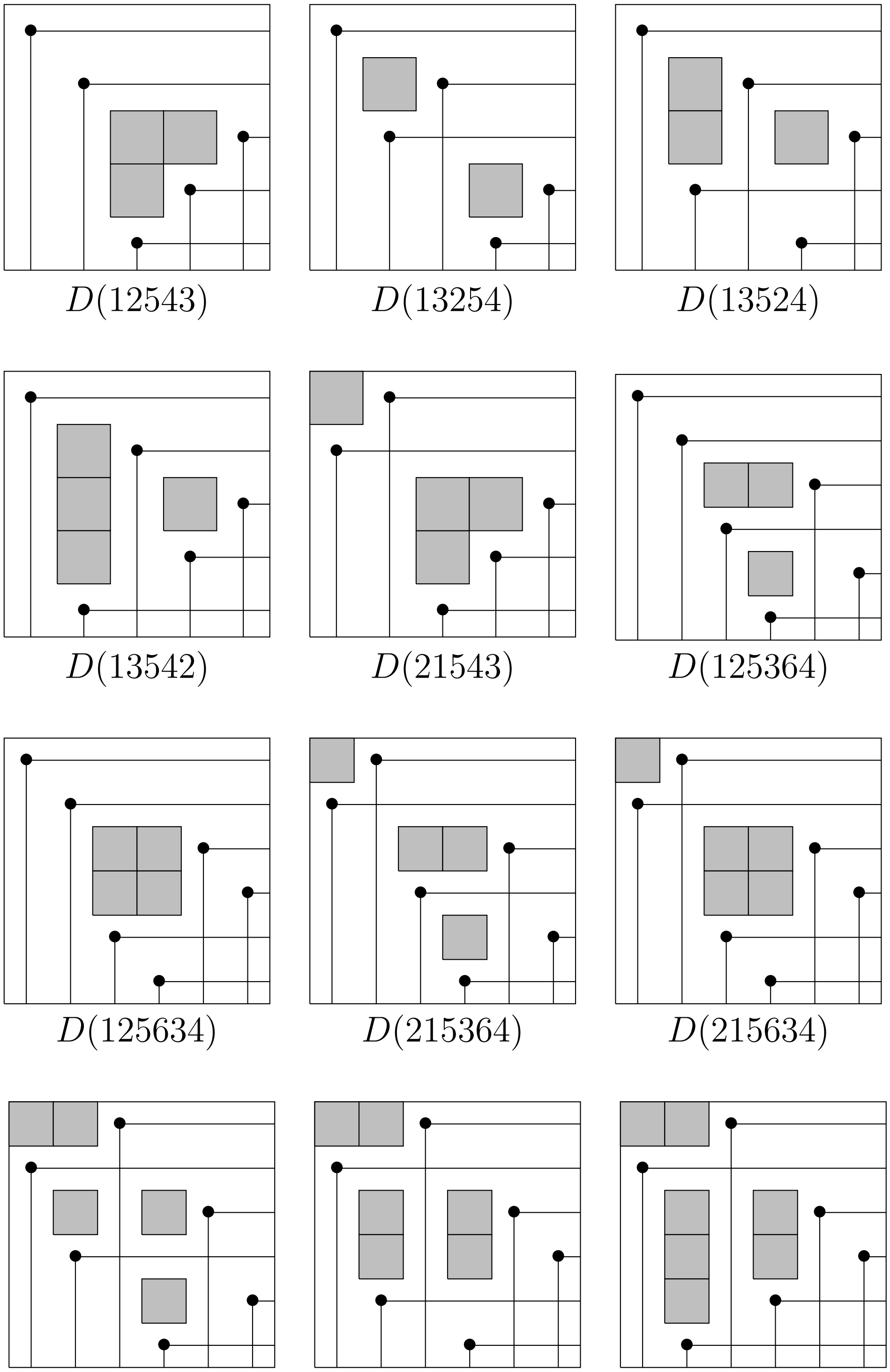
where $F \in \mathbb{N}[x_1, \ldots, x_n]$ and

$$M(x_1, \ldots, x_n) = \left(\prod_{(k,i) \in D(w)} x_k\right)\left(\prod_{(i,w_k) \in D(w)} x_i\right).$$

Proof. Specialize Theorem 2.3.4 to the case that D is a Rothe diagram $D(w)$ and $l = w_k$. The dropping of x_k is due to reindexing, since the entirety of row k and column w_k of $D(w)$ are removed from to obtain $D(\sigma)$, not just the boxes in row k and column w_k. ☐

Corollary 3.6.3. *Fix $w \in S_n$ and let $\sigma \in S_m$ be any pattern contained in w. If c is the coefficient of a monomial in $\mathfrak{S}_\sigma$, then $\mathfrak{S}_w$ has a monomial with coefficient at least c.*

Proof. Immediate consequence of repeated applications of Theorem 3.6.2. ☐



$D(12543)$ $D(13254)$ $D(13524)$

$D(13542)$ $D(21543)$ $D(125364)$

$D(125634)$ $D(215364)$ $D(215634)$

$D(315264)$ $D(315624)$ $D(315642)$

Figure 3.2: The Rothe diagrams of the twelve multiplicitous patterns

CHAPTER 4

LOG-CONCAVITY OF SCHUR AND SCHUBERT POLYNOMIALS

The goal of this chapter is to prove various log-concavity properties of Schur and Schubert polynomials. This is achieved through the vehicle of Lorentzian polynomials. We first give background on Lorentzian polynomials and the subclass of volume polynomials. We then extract from results of Knutson and Miller the Lorentzian property of a family of polynomials similar to Schubert polynomials. We use this to show normalized Schur polynomials are Lorentzian. We then give a variety of partial results and conjectures for other families of polynomials. The results and conjectures of this chapter are from the paper [22] joint with June Huh, Jacob Matherne, and Karola Mészáros.

4.1 Background on Lorentzian Polynomials

Lorentzian polynomials were introduced by Brändén and Huh in [9] as a simultaneous generalization of volume polynomials from algebraic geometry and stable polynomials from optimization theory. They showed that Lorentzian polynomials bridge continuous and discrete log-concavity. We now present their definition and results, beginning with a notion from discrete convexity theory.

A subset $J \subseteq \mathbb{Z}^n$ is called *M-convex* if, for any index $i \in [n]$ and any $\alpha, \beta \in J$ whose i-th coordinates satisfy $\alpha_i > \beta_i$, there is an index $j \in [n]$ satisfying

$$\alpha_j < \beta_j, \quad \alpha - e_i + e_j \in J, \quad \text{and} \quad \beta - e_j + e_i \in J.$$

The notion of M-convexity forms the foundation of discrete convex analysis [39]. The convex hull of an M-convex set is a generalized permutohedron (Definition 1.5.3), and conversely, the set of integer points in an integral generalized permutohedron is an M-convex set [39, Theorem 1.9].

Remark 4.1.1. A homogeneous polynomial f has M-convex support if and only if f has SNP and Newton(f) is a generalized permutahedron.

Fix nonnegative integers d and $e = d - 2$.

Definition 4.1.2. Let $h(x_1, \ldots, x_n)$ be a degree d homogeneous polynomial. We say that h is *Lorentzian* if it satisfies the following conditions:

i) All the coefficients of h are nonnegative.
ii) The support of h is M-convex.
iii) For any $i_1, \ldots, i_e \in [n]$, the quadratic polynomial

$$\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h$$

has at most one positive eigenvalue.

Example 4.1.3. Consider the quadratic forms $f = x_1^2 + x_1x_2 + x_2^2$ and $g = x_1^2 + 2x_1x_2 + x_2^2$. Both have M-convex support, the line segment from $(2, 0)$ to $(0, 2)$ in $\mathbb{Z}^2$. Their respective matrices are

$$\mathrm{Mat}_f = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{Mat}_g = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Mat_f has eigenvalues $1/2$ and $3/2$, so f is not Lorentzian. Mat_g has eigenvalues 1 and 0, so g is Lorentzian.

The following results from [9] demonstrate the connection between Lorentzian polynomials and continuous and discrete log-concavity.

Definition 4.1.4 ([9])**.** A polynomial $h(x_1, \ldots, x_n)$ with nonnegative coefficients is called *strongly log-concave* if for any $i_1, i_2, \ldots \in [n]$ and any positive integer k, the functions

$$h \quad \text{and} \quad \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_k}} h$$

are either identically zero or log-concave on $\mathbb{R}^n_{>0}$.

Theorem 4.1.5 ([9, Theorem 5.3])**.** *A homogeneous polynomial is Lorentzian if and only if it is strongly log-concave.*

Definition 4.1.6. Define the linear *normalization operator* N on polynomials by

$$N(x^\alpha) = \frac{x^\alpha}{\alpha!} = \frac{x_1^{\alpha_1}}{\alpha_1!} \cdots \frac{x_m^{\alpha_m}}{\alpha_m!} \text{ for all } \alpha \in \mathbb{N}^m.$$

Theorem 4.1.7 ([9, Proposition 9.4])**.** *Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$ be a homogeneous polynomial. If $N(f)$ is Lorentzian and nonzero, then for every α and every $i, j \in [n]$,*

$$c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}.$$

Lorentzian polynomials also enjoy the following product properties.

Theorem 4.1.8 ([9, Corollary 6.8])**.** *If $N(f)$ and $N(g)$ are Lorentzian, then $N(fg)$ is Lorentzian.*

Theorem 4.1.9 ([9, Corollary 5.5])**.** *If f and g are Lorentzian, then fg is Lorentzian.*

4.1.1 Volume Polynomials of Projective Varieties

We give a brief introduction to the necessary algebraic geometry behind Theorem 4.3.2. The main tool required is Corollary 4.1.11. We refer to [21, 31] for the algebro-geometric preliminaries and [37] for background on Hilbert polynomials and multigraded Hilbert series. Additionally, see [24, Lemma 2.1] for a proof of the existence of multigraded Hilbert polynomials. We follow the exposition of [9] for volume polynomials.

Let V be a n-dimensional irreducible projective variety (over $\mathbb{C}$). If $D_1, \ldots, D_m$ are Cartier divisors on V, the *intersection product* $(D_1, \ldots, D_m)$ is an integer defined by the following properties:

- $(D_1, \ldots, D_m)$ is symmetric and multilinear as a function of its arguments.
- $(D_1, \ldots, D_m)$ depends only on the linear equivalence classes of the D_i.
- If $D_1, \ldots, D_m$ are effective divisors meeting transversely at smooth points of V, then $(D_1, \ldots, D_m) = \#(D_1 \cap \cdots \cap D_m)$.

Let $D = (D_1, \ldots, D_m)$ be a sequence of Cartier divisors on an irreducible projective variety V of dimension n. The *volume polynomial* of V with respect to D is defined by

$$\begin{aligned}\mathrm{vol}_D(x) &= \frac{1}{n!}(x_1 D_1 + \cdots + x_m D_m)^n \\ &= \sum_{\substack{\alpha \in \mathbb{N}^m \\ |\alpha| = n}} \frac{1}{\alpha!} v_\alpha(D) x^\alpha,\end{aligned}$$

where $v_\alpha(D) = (\underbrace{D_1, \ldots, D_1}_{\alpha_1}, \ldots, \underbrace{D_m, \ldots, D_m}_{\alpha_m})$.

Theorem 4.1.10 ([9, Theorem 10.1]). *If $D = (D_1, \ldots, D_n)$ is a sequence of nef divisors on an irreducible projective variety V, then $\mathrm{vol}_D(x)$ is a Lorentzian polynomial.*

Corollary 4.1.11. *If $V \subseteq \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_m}$ is an irreducible subvariety, then the top homogeneous component of the multigraded Hilbert polynomial of V is Lorentzian.*

Proof. For each $i \in [m]$, let D_i be a divisor corresponding to the pullback of $\mathscr{O}(1)$ on $\mathbb{P}^{k_i}$ via the coordinate projection. Each D_i is nef, and it follows that the multigraded Hilbert polynomial of V is given by

$$x = (x_1, \ldots, x_m) \mapsto \dim H^0(V, \mathscr{O}_V(x_1 D_1 + \cdots + x_m D_m)).$$

By [31, Corollary 1.4.41], if U is an irreducible projective variety of dimension n and D is a nef divisor on U, then

$$\dim H^0(U, \mathscr{O}_U(x_1 D)) = \frac{(D^n)}{n!} x_1^n + O(x_1^{n-1}).$$

Applying [31, Corollary 1.4.41] iteratively m times, we see that

$$\dim H^0(V, \mathscr{O}_V(x_1 D_1 + \cdots + x_m D_m)) = \frac{(x_1 D_1 + \cdots + x_m D_m)^{\dim V}}{(\dim V)!} + \text{lower order terms.}$$

In particular, top homogeneous component of the multigraded Hilbert polynomial of V equals the volume polynomial of V relative to the divisors $D_1, \ldots, D_p$. Applying Theorem 4.1.10 completes the the multigraded Hilbert polynomial is Lorentzian. □

4.2 Matrix Schubert Varieties

To work toward Schur and Schubert polynomials, we need a relevant class of varieties upon which to apply Corollary 4.1.11. To this end, we recall the matrix Schubert varieties. We follow the exposition of [25]. We then state Theorem A of [25], which gives the $\mathcal{K}$-polynomials and multidegrees of matrix Schubert varieties. We conclude this section by using Theorem A to compute the top homogeneous component of the multigraded Hilbert polynomial of any matrix Schubert variety.

We point the reader to [37, Chapter 8] for background on multigraded Hilbert series and multidegrees.

Let M_n be the variety of $n \times n$ matrices over $\mathbb{R}$, with coordinate ring $\mathbb{R}[Z]$ in indeterminates $Z = \{z_{ij}\}_{i,j=1}^{n}$. For any matrix $A \in M_n$ and $1 \leq p, q \leq n$, denote by $A_{p\times q}$ the northwest $p \times q$ submatrix of A. Given a permutation $w \in S_n$, let w^T be the matrix with a 1 in row i and column $w(i)$ for each $i \in [n]$, and zeros elsewhere. Note that

$$\operatorname{rank}(w^T_{p\times q}) = \#\{(i,j) \mid i \leq p,\ j \leq q,\ w(i) = j\},$$

the number of ones in the submatrix $w^T_{p\times q}$.

Definition 4.2.1. Let $w \in S_n$ be a permutation and Z be the generic matrix of indeterminates $(z_{ij})_{i,j=1}^{n}$. The *Schubert determinantal ideal* $I_w \subset \mathbb{R}[Z]$ is generated by all minors in $Z_{p\times q}$ of size $1 + \operatorname{rank}(w^T_{p\times q})$ for all $1 \leq p, q \leq n$.

The subvariety of M_n cut out by I_w is the matrix Schubert variety of w.

Definition 4.2.2. Let $w \in S_n$. The **matrix Schubert variety** $\overline{X}_w \subseteq M_n$ consists of the matrices $A \in M_n$ such that $\operatorname{rank}(A_{p\times q}) \leq \operatorname{rank}(w^T_{p\times q})$ for all $1 \leq p, q \leq n$.

Theorem 4.2.3 ([25, Theorem A])**.** *The Schubert determinantal ideal I_w is prime, so I_w is the ideal $I(\overline{X}_w)$ of the matrix Schubert variety $\overline{X}_w$.*

We embed $\overline{X}_w$ in $(\mathbb{P}^{n-1})^n$ by using each row of a matrix $A \in \overline{X}_w$ as homogeneous components. Denote by Γ_w the multigraded coordinate ring $\mathbb{R}[Z]/I_w$ of $\overline{X}_w$. Note that Γ_w is a $\mathbb{Z}^n$-graded module over $\mathbb{R}[Z]$ via the grading that assigns weight e_i to z_{ij} for each i and j, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{Z}^n$.

To describe the multigraded Hilbert series associated to $\overline{X}_w$, recall the definitions of Schubert and Grothendieck polynomials (Definitions 1.4.5 and 1.4.8 respectively).

Theorem 4.2.4 ([25, Theorem A]). *For any $w \in S_n$, the $\mathbb{Z}^n$-graded $\mathcal{K}$-polynomial $\mathcal{K}(\Gamma_w; t)$ is the Grothendieck polynomial $\mathfrak{G}_w(1-t)$, so the Hilbert series of Γ_w is*

$$\mathrm{HS}(\Gamma_w; t) = \frac{\mathfrak{G}_w(1-t)}{\prod_{i=1}^n (1-t_i)^n}.$$

The $\mathbb{Z}^n$-graded multidegree $\mathcal{C}(\Gamma_w; t)$ is the Schubert polynomial $\mathfrak{S}_w(t)$.

We now compute the Hilbert polynomial of Γ_w and identify its highest-degree homogeneous component. We begin with two simple lemmas to simplify the computation. Recall that $|\alpha|$ denotes the sum of the entries of α and $f(1-t)$ is shorthand for $f(1-t_1, \ldots, 1-t_n)$.

Lemma 4.2.5. *If $f(t) = \sum_{\beta \in \mathbb{N}^n} c_\beta t^\beta$ is a polynomial, then*

$$f(1-t) = \sum_{\alpha \in \mathbb{N}^n} t^\alpha \sum_{\beta \geq \alpha} c_\beta (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n},$$

where $\beta \geq \alpha$ means $\beta_i \geq \alpha_i$ for each $i \in [n]$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Proof. We have

$$\begin{aligned} f(1-t) &= \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n (1-t_i)^{\beta_i} \\ &= \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n \left(\sum_{\alpha_i = 0}^{\beta_i} (-1)^{\alpha_i} \binom{\beta_i}{\alpha_i} t_i^{\alpha_i} \right). \end{aligned}$$

Setting $\alpha = (\alpha_1, \ldots, \alpha_n)$ and letting $|\alpha| = \alpha_1 + \cdots + \alpha_n$, we get

$$\begin{aligned} f(1-t) &= \sum_{\beta \in \mathbb{N}^n} c_\beta \sum_{\alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} t^\alpha \\ &= \sum_{\alpha \in \mathbb{N}^n} t^\alpha \sum_{\beta \geq \alpha} c_\beta (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}. \end{aligned}$$

□

Lemma 4.2.6. *If* $g(t) = \sum_{\beta \in \mathbb{N}^n} c_\beta t^\beta$*, then*

$$[t^\alpha] \frac{g(t)}{\prod_{i=1}^n (1-t_i)^n} = \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n \binom{n+\alpha_i-\beta_i-1}{n-1}.$$

Proof. We have

$$\begin{aligned} [t^\alpha] \frac{g(t)}{\prod_{i=1}^n (1-t_i)^n} &= \sum_{\beta \in \mathbb{N}^n} c_\beta [t^\alpha] \frac{t^\beta}{\prod_{i=1}^n (1-t_i)^n} \\ &= \sum_{\beta \in \mathbb{N}^n} c_\beta [t^{\alpha-\beta}] \frac{1}{\prod_{i=1}^n (1-t_i)^n} \\ &= \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n [t_i^{\alpha_i-\beta_i}] \frac{1}{(1-t_i)^n} \\ &= \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n \binom{n+\alpha_i-\beta_i-1}{n-1}. \end{aligned}$$

□

Theorem 4.2.7. *For* $w \in S_n$*, the top homogeneous component of the Hilbert polynomial* $\mathrm{HP}(\Gamma_w; x)$ *is*

$$N((x_1 \cdots x_n)^{n-1} \mathcal{C}(\Gamma_w; x^{-1})),$$

where $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$ *and* N *is the linear operator taking* x^α *to*

$$\frac{x^\alpha}{\alpha!} = \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}.$$

Proof. If $\mathcal{K}(\Gamma_w; t) = \sum_{\beta \in \mathbb{N}^n} c_\beta t^\beta$, then Lemma 4.2.6 implies

$$\mathrm{HP}(\Gamma_w; x) = [t^x] \frac{\mathcal{K}(\Gamma_w; t)}{\prod_{i=1}^n (1-t_i)^n} = \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^n \binom{n+x_i-\beta_i-1}{n-1}.$$

From the elementary identity

$$\binom{p+q-r}{p} = \sum_{j=0}^r (-1)^j \binom{r}{j} \binom{p+q-j}{p-j},$$

it follows that

$$\begin{aligned}
\mathrm{HP}(\Gamma_w; x) &= \sum_{\beta \in \mathbb{N}^n} c_\beta \prod_{i=1}^{n} \left(\sum_{j=0}^{\beta_i} (-1)^j \binom{\beta_i}{j} \binom{n-1+x_i-j}{n-1-j} \right) \\
&= \sum_{\beta \in \mathbb{N}^n} c_\beta \sum_{\alpha \leq \beta} (-1)^{|\alpha|} \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n} \binom{n-1+x_1-\alpha_1}{n-1-\alpha_1} \cdots \binom{n-1+x_n-\alpha_n}{n-1-\alpha_n} \\
&= \sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} \binom{n-1+x_1-\alpha_1}{n-1-\alpha_1} \cdots \binom{n-1+x_n-\alpha_n}{n-1-\alpha_n} \sum_{\beta \geq \alpha} c_\beta \binom{\beta_n}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}.
\end{aligned}$$

Since the degree of the Hilbert polynomial of a coordinate ring equals the dimension of the corresponding variety, it follows that the degree of $\mathrm{HP}(\Gamma_w; x)$ is the dimension of $\overline{X}_w$, which equals $n(n-1) - \ell(w)$. Thus, the highest-degree homogeneous component of $\mathrm{HP}(\Gamma_w; x)$ is

$$\sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = \ell(w)}} (-1)^{|\alpha|} \frac{x_1^{n-1-\alpha_1} \cdots x_n^{n-1-\alpha_n}}{(n-1-\alpha_1)! \cdots (n-1-\alpha_n)!} \sum_{\beta \geq \alpha} c_\beta \binom{\beta_n}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}.$$

On the other hand, Lemma 4.2.5 implies

$$\mathcal{K}(\Gamma_w; 1-x) = \sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|} x^\alpha \sum_{\beta \geq \alpha} c_\beta \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}.$$

Since the multidegree $\mathcal{C}(\Gamma_w; x) = \mathfrak{S}_w(x)$ equals the lowest-degree homogeneous component of $\mathcal{K}(\Gamma_w; 1-x)$, we obtain

$$\mathcal{C}(\Gamma_w; x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = \ell(w)}} (-1)^{|\alpha|} x^\alpha \sum_{\beta \geq \alpha} c_\beta \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}.$$

This implies $N((x_1 \cdots x_n)^{n-1} \mathcal{C}(\Gamma_w; x^{-1}))$ equals

$$\sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = \ell(w)}} (-1)^{|\alpha|} \frac{x_1^{n-1-\alpha_1} \cdots x_n^{n-1-\alpha_n}}{(n-1-\alpha_1)! \cdots (n-1-\alpha_n)!} \sum_{\beta \geq \alpha} c_\beta \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n},$$

so $N((x_1 \cdots x_n)^{n-1} \mathcal{C}(\Gamma_w; x^{-1}))$ is the top homogeneous component of the Hilbert polynomial of Γ_w. □

Corollary 4.2.8. *For $w \in S_n$, the highest-degree homogeneous component of the Hilbert polynomial of Γ_w equals*

$$N((x_1 \cdots x_n)^{n-1} \mathfrak{S}_w(x_1^{-1} \cdots x_n^{-1})).$$

Proof. By Theorem 4.2.4, $\mathcal{C}(\Gamma_w; x) = \mathfrak{S}_w(x)$. □

4.3 The Case of Schur Polynomials

In this section, we show that normalized Schur polynomials are Lorentzian and explore the consequences. The results of this section are from the joint work [22] with June Huh, Jacob Matherne, and Karola Mészáros.

4.3.1 Normalized Schur Polynomials are Lorentzian

Recall (Definition 1.4.1) that the Schur polynomial of a partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ is given by

$$s_\lambda(x_1, \ldots, x_m) = \sum_T x^{\mathrm{wt}(T)} = \sum_\alpha K_{\lambda\alpha} x^\alpha,$$

where $K_{\lambda\alpha}$ is the Kostka number, the number of SSYT of shape λ and weight α.

We prove that all normalized Schur polynomials are Lorentzian, and state their resulting log-concavity properties. Our starting point is the following Lorentzian polynomials.

Definition 4.3.1. For any $w \in S_n$, define $\mathfrak{S}_w^\vee$ to be the polynomial

$$\mathfrak{S}_w^\vee = N(x_1^{n-1} \cdots x_n^{n-1} \mathfrak{S}_w(x_1^{-1}, \ldots, x_n^{-1})).$$

Theorem 4.3.2. *For any $w \in S_n$, the polynomial $\mathfrak{S}^{\vee}_w(x_1, \ldots, x_n)$ is Lorentzian.*

Proof. Immediate consequence of Corollary 4.2.8 and Corollary 4.1.11. $\square$

We now record a few simple results.

Lemma 4.3.3. *For any $\beta \in \mathbb{N}^n$, there is an equality of operators*

$$\frac{\partial^\beta}{\partial x^\beta} N x^\beta = N.$$

Proof. By linearity, it is enough to prove the equality for monomials x^α. We have

$$\begin{aligned}\frac{\partial^\beta}{\partial x^\beta} N x^\beta x^\alpha &= \frac{\partial^\beta}{\partial x^\beta} N x^{\alpha+\beta} \\ &= \frac{\partial^\beta}{\partial x^\beta} \frac{x^{\alpha+\beta}}{(\alpha+\beta)!} \\ &= \frac{x^\alpha}{\alpha!}. \\ &= N(x^\alpha).\end{aligned}$$

$\square$

Proposition 4.3.4. *If $N(x^\beta f)$ is Lorentzian for some $\beta \in \mathbb{N}^n$, then $N(f)$ is Lorentzian.*

Proof. If $N(x^\beta f)$ is Lorentzian, then $\frac{\partial^\beta}{\partial x^\beta} N(x^\beta f)$ is also Lorentzian. But

$$\frac{\partial^\beta}{\partial x^\beta} N(x^\beta f) = N(f)$$

by Lemma 4.3.3. Thus $N(f)$ is Lorentzian. $\square$

The following lemma is a well-known property of Schur polynomials (see for instance [20, Exercise 15.50] needed for the proof of Theorem 4.3.6. We include a combinatorial proof for completeness.

Lemma 4.3.5. *Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition and fix $m \geq \lambda_1$. If μ denotes the complement of λ in the $r \times m$ rectangle, then*

$$x_1^m \cdots x_r^m s_\lambda(x_1^{-1}, \ldots, x_r^{-1}) = s_\mu(x_1, \ldots, x_r).$$

Proof. Consider $\mathbb{R}^{\binom{r+1}{2}} = \mathbb{R}^r \oplus \mathbb{R}^{r-1} \oplus \cdots \oplus \mathbb{R}$ with the coordinates arranged in the same fashion as Gelfand–Tsetlin patterns. Let $\vec{1}$ denote the vector in $\mathbb{R}^{\binom{r+1}{2}}$ of all ones. Define rev to be the map on $\mathbb{R}^{\binom{r+1}{2}}$ that reverses the rows of its input array. Set P to be the polytope

$$\mathrm{rev}\left(-\mathrm{GT}\left(\lambda\right) + m\vec{1}\right).$$

If $A \in \mathrm{GT}(\lambda)$, then $\mathrm{wt}(\mathrm{rev}(-A + m\vec{1})) = \mathrm{wt}(-A + m\vec{1}) = -\mathrm{wt}(A) + \mathrm{wt}(m\vec{1}) = m - \mathrm{wt}(A)$. Hence,

$$x_1^m \cdots x_r^m s_\lambda(x_1^{-1}, \ldots, x_r^{-1}) = \sum_{A \in P} x^{\mathrm{wt}(A)}.$$

However, it can be checked that $P = \mathrm{rev}\left(-\mathrm{GT}\left(\lambda\right) + m\vec{1}\right) = \mathrm{GT}(\mu)$, so

$$s_\mu(x_1, \ldots, x_r) = \sum_{A \in \mathrm{GT}(\mu)} x^{\mathrm{wt}(A)} = \sum_{A \in P} x^{\mathrm{wt}(A)} = x_1^m \cdots x_r^m s_\lambda(x_1^{-1}, \ldots, x_r^{-1}).$$

□

Theorem 4.3.6. *If $\lambda \in \mathbb{N}^r$ is a partition, then $N(s_\lambda)$ is Lorentzian.*

Proof. Let μ be the complement of λ in the $r \times \lambda_1$ rectangle, i.e. $\mu_i = \lambda_1 - \lambda_{r-i+1}$ for $i \in [r]$. Let $w \in S_n$ be the Grassmannian permutation corresponding to μ (see Theorem 1.4.7 and preceding statements). By Theorem 4.3.2, $\mathfrak{S}_w^\vee(x_1, \ldots, x_n)$ is

Lorentzian. By Theorem 1.4.7 and Lemma 4.3.5, we have

$$\begin{aligned}\mathfrak{S}_w^\vee(x_1,\ldots,x_n) &= (x_1\cdots x_n)^{n-1}\mathfrak{S}_w(x_1^{-1},\ldots,x_n^{-1})\\ &= (x_1\cdots x_n)^{n-1}s_\mu(x_1^{-1},\ldots,x_n^{-1})\\ &= (x_1\cdots x_r)^{n-1-\lambda_1}(x_{r+1}\cdots x_n)^{n-1}(x_1\cdots x_r)^{\lambda_1}s_\mu(x_1^{-1},\ldots,x_n^{-1})\\ &= (x_1\cdots x_r)^{n-1-\lambda_1}(x_{r+1}\cdots x_n)^{n-1}s_\lambda(x_1,\ldots,x_r).\end{aligned}$$

Thus $N((x_1\cdots x_r)^{n-1-\lambda_1}(x_{r+1}\cdots x_n)^{n-1}s_\lambda(x_1,\ldots,x_r))$ is Lorentzian. An application of Proposition 4.3.4 completes the proof. □

In general, if a polynomial h is Lorentzian, then its normalization $N(h)$ is a Lorentzian polynomial [9, Corollary 6.7]. We record here that Schur polynomials without normalization need not be Lorentzian.

Example 4.3.7. The Schur polynomial of the partition $\lambda = (2, 0)$ in two variables is

$$s_\lambda(x_1, x_2) = x_1^2 + x_1x_2 + x_2^2.$$

This quadratic form has eigenvalues $\frac{3}{2}$ and $\frac{1}{2}$, and hence s_λ is not Lorentzian.

A polynomial $f(x_1,\ldots,x_m)$ is called *stable* if f has no zeros in the product of m open upper half planes [48]. Homogeneous stable polynomials with nonnegative coefficients are motivating examples of Lorentzian polynomials [9, Proposition 2.2]. We record here that normalized Schur polynomials, although Lorentzian, need not be stable.

Example 4.3.8. The normalized Schur polynomial of $\lambda = (3, 1, 1, 1, 1)$ in five variables is

$$N(s_\lambda) = \frac{1}{12}x_1x_2x_3x_4x_5\left(\sum_{1\le i<j\le 5} 3x_ix_j + \sum_{1\le i\le 5} 2x_i^2\right).$$

By [48, Lemma 2.4], if $N(s_\lambda)$ is stable, then so is its univariate specialization

$$N(s_\lambda)|_{x_2=x_3=x_4=x_5=1} = \frac{1}{6}x_1\left(x_1^2+6x_1+13\right).$$

However, the displayed cubic has a pair of nonreal zeros, and hence $N(s_\lambda)$ is not stable.

We also have the following corollary.

Corollary 4.3.9. *For any sequence of partitions $\lambda^1,\ldots,\lambda^\ell$ with respectively $m_1,\ldots,m_\ell$ parts, both*

$$\mathrm{N}(\prod_{k=1}^{\ell} s_{\lambda^k}(x_1,\ldots,x_{m_k})) \quad \textit{and} \quad \prod_{k=1}^{\ell}\mathrm{N}(s_{\lambda^k}(x_1,\ldots,x_{m_k}))$$

are Lorentzian.

Proof. These statements are immediate from Theorem 4.3.6 together with respectively Theorem 4.1.8 and Theorem 4.1.9. □

4.3.2 Log-Concavity Consequences

We now state two interesting immediate consequences of the normalized Schur polynomials being Lorentzian (c.f. Theorems 4.1.5 and 4.1.7).

Corollary 4.3.10. *For any partition $\lambda\in\mathbb{N}^m$, the normalized Schur polynomial*

$$N(s_\lambda)=\sum_{\alpha\in\mathbb{N}^m}K_{\lambda\alpha}\frac{x^\alpha}{\alpha!}$$

is either identically zero, or its logarithm is concave on the positive orthant $\mathbb{R}^m_{>0}$.

Let e_i be the i-th standard unit vector in $\mathbb{Z}^m_{\geq 0}$.

Corollary 4.3.11. *For any partition λ, any $\alpha \in \mathbb{N}^m$, and any $i, j \in [m]$, we have*

$$K_{\lambda\alpha}^2 \geq K_{\lambda,\alpha+e_i-e_j} K_{\lambda,\alpha-e_i+e_j}.$$

Corollary 4.3.11 is a discrete log-concavity for the Kostka numbers. It says that as you walk along any line in a root direction within Newton(s_λ), the sequence of coefficients you encounter is always log-concave.

Corollary 4.3.11 is illustrated for $\lambda = (4, 2, 0)$ in Figure 4.1.

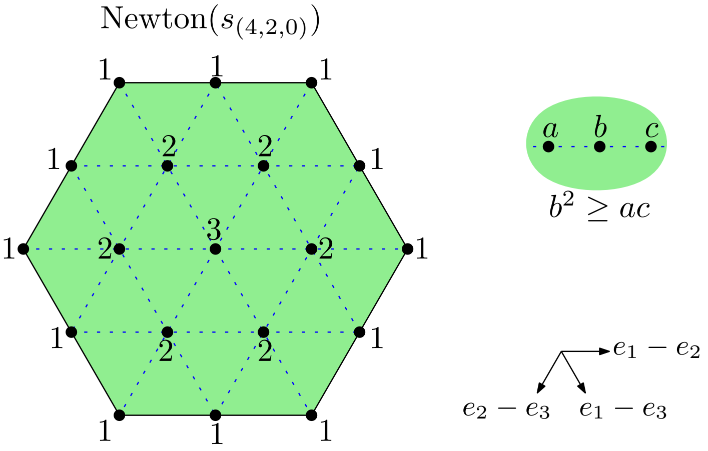


Figure 4.1: Discrete log-concavity of the Kostka numbers

4.3.3 Asymptotic Side Note

We note very briefly that for any fixed λ, the log-concavity of $K_{\lambda\mu}$ along *any* direction is known to hold asymptotically. We refer the reader to the introduction of [22] for an exposition of the relevant geometry behind this fact.

4.3.4 Relation to Okounkov's Conjecture

Schur polynomials are also the characters of finite-dimensional irreducible polynomial representations of the general linear group $\mathrm{GL}_m(\mathbb{C})$. Corresponding to each partition $\lambda \in \mathbb{N}^m$ is the *Schur module* $V(\lambda)$, an irreducible representation of $\mathrm{GL}_m(\mathbb{C})$ with highest weight λ. Each module $V(\lambda)$ has the weight space decomposition

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_\mu \quad \text{with} \quad \dim V(\lambda)_\mu = K_{\lambda\mu},$$

as each weight space $V(\lambda)_\mu$ admits a basis indexed by SSYT of shape λ and weight μ. See [19, Chapter 8] for details.

For partitions ν, κ, λ, the *Littlewood–Richardson coefficient* $c^{\nu}_{\kappa\lambda}$ is given by the decomposition

$$V(\kappa) \otimes V(\lambda) \simeq \bigoplus_{\nu} V(\nu)^{\oplus c^{\nu}_{\kappa\lambda}}.$$

Equivalently, $c^{\nu}_{\kappa\lambda}$ is the coefficient of s_ν in the Schur basis expansion of $s_\kappa s_\nu$.

It is well-known that the Littlewood–Richardson numbers contain the Kostka numbers: when the skew partition ν/κ has at most one box in each column, $c^{\nu}_{\kappa\lambda}$ is the Kostka number $K_{\lambda\mu}$, where $\mu = \nu - \kappa$.

Conversely, for any partition λ and any μ, we have

$$K_{\lambda\mu} = c^{\nu}_{\kappa\lambda},$$

where ν and κ are the partitions given by $\nu_i = \sum_{j=i}^{n} \mu_j$ and $\kappa_i = \sum_{j=i+1}^{n} \mu_j$. Thus Corollary 4.3.11 verifies a special case of Okounkov's conjecture that the discrete function

$$(\nu, \kappa, \lambda) \longmapsto \log c^{\nu}_{\kappa\lambda}$$

is concave [40, Conjecture 1].

4.4 Beyond Schur Polynomials: Results and Conjectures

In this section, we summarize the conjectures made in joint work [22] with June Huh, Jacob Matherne, and Karola Mészáros. We broadly conjecture that the Lorentzian property holds for many interesting families of polynomials. We present partial results in support of these conjectures where available. Code supporting all computations referenced in this section can be found at `https://github.com/avstdi/Lorentzian-Polynomials`.

4.4.1 Multiplicities of Highest Weight Modules

We point to [23] for background on representation theory of semisimple Lie algebras. Let Λ be the integral weight lattice of the Lie algebra $\mathfrak{sl}_m(\mathbb{C})$, let $\varpi_1, \ldots, \varpi_{m-1}$ be the fundamental weights, and let ρ be the sum of the fundamental weights. For $\lambda \in \Lambda$, we write $V(\lambda)$ for the irreducible $\mathfrak{sl}_m(\mathbb{C})$-module with highest weight λ, and consider its decomposition into finite-dimensional weight spaces

$$V(\lambda) = \bigoplus_{\mu} V(\lambda)_\mu.$$

Conjecture 4.4.1. *For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have*

$$(\dim V(\lambda)_\mu)^2 \geq \dim V(\lambda)_{\mu+e_i-e_j} \dim V(\lambda)_{\mu-e_i+e_j} \quad \text{for any } i, j \in [m].$$

When λ is dominant, the dimension of the weight space $V(\lambda)_\mu$ is the Kostka number $K_{\lambda\mu}$, and Corollary 4.3.11 shows that Conjecture 4.4.1 holds in this case. When λ is antidominant [23, Section 4.4], $V(\lambda)$ is the *Verma module* $M(\lambda)$, the universal highest weight module of highest weight λ. We note that Conjecture 4.4.1 holds in this case as well.

Proposition 4.4.2. *For any $\lambda \in \Lambda$ and any $\mu \in \Lambda$, we have*

$$(\dim M(\lambda)_\mu)^2 \geq \dim M(\lambda)_{\mu+e_i-e_j} \dim M(\lambda)_{\mu-e_i+e_j} \quad \textit{for any } i, j \in [m].$$

Proof. The Poincaré–Birkhoff–Witt theorem shows that the dimensions of the weight spaces are given by the Kostant partition function p:

$$\dim M(\lambda)_\mu = p(\mu - \lambda) = \text{number of ways to write } \mu - \lambda \text{ as a sum of negative roots.}$$

Lidskii's volume formula for flow polytopes shows that all Kostant partition function evaluations are mixed volumes of Minkowski sums of polytopes [3]. The Alexandrov–Fenchel inequality for mixed volumes [46, Section 7.3] yields the desired log-concavity property. □

A strengthening of Proposition 4.4.2 using Lorentzian polynomials is given in [22, Proposition 13]. The corresponding strengthening of Conjecture 4.4.1 is [22, Conjecture 12].

The diagram in Figure 4.2 shows some of the weight multiplicities of the irreducible $\mathfrak{sl}_4(\mathbb{C})$-module with highest weight $-2\varpi_1 - 3\varpi_2$. We start from the highlighted vertex $\varpi_1 - 6\varpi_2 - 3\varpi_3$ and walk along negative root directions in the hyperplane spanned by $e_2 - e_1$ and $e_3 - e_2$. In the shown region, the sequence of weight multiplicities along any line is log-concave, as predicted by Conjecture 4.4.1.

We note, however, that a naive analog of Conjecture 4.4.1 does not hold for symplectic Lie algebras. In the weight diagram of the irreducible representation of $\mathfrak{sp}_4(\mathbb{C})$ with highest weight $2\varpi_2$ shown in Figure 4.3, the weight multiplicities along the two diagonals of the square do not form log-concave sequences.

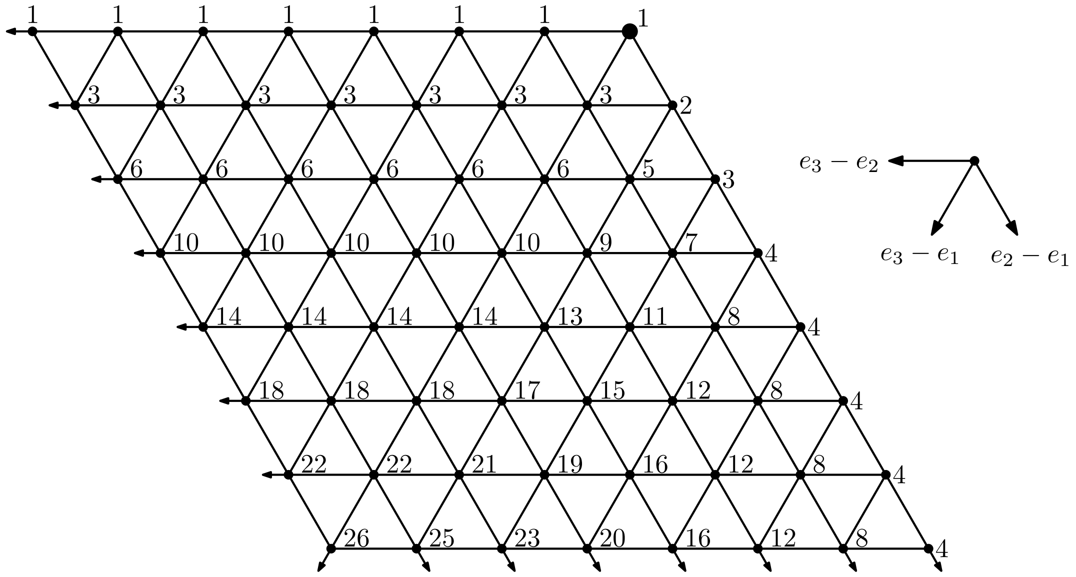


Figure 4.2: An illustration of Conjecture 4.4.1

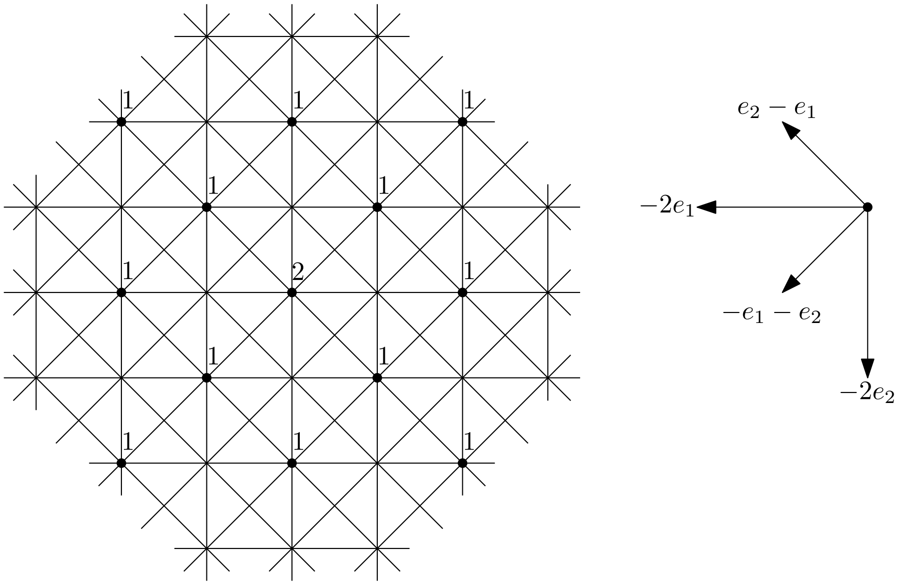


Figure 4.3: The weight diagram of the irreducible $\mathfrak{sp}_4(\mathbb{C})$ with highest weight $2\varpi_2$

4.4.2 Schubert Polynomials

For $w \in S_n$ and $\alpha \in \mathbb{Z}^n$, we define the number $K_{w\alpha}$ by

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{\alpha} K_{w\alpha} x^{\alpha}.$$

We note that Corollary 4.3.11 can be strengthened as follows.

Proposition 4.4.3. *For any $w \in S_n$, any $\alpha \in \mathbb{N}^n$, and any $i, j \in [n]$, we have*

$$K_{w\alpha}^2 \geq K_{w,\alpha+e_i-e_j} K_{w,\alpha-e_i+e_j}$$

Proof. By Theorem 4.3.2, the polynomial $\mathfrak{S}_w^{\vee}$ is Lorentzian. The inequality follows from Theorem 4.1.7 applied to $\mathfrak{S}_w^{\vee}$. $\square$

Are normalized Schubert polynomials Lorentzian? We tested the following statement for all permutations in S_n for $n \leq 8$.

Conjecture 4.4.4. *The polynomial $N(\mathfrak{S}_w)$ is Lorentzian for any $w \in S_n$.*

More generally, we conjecture that, for double Schubert polynomials [37, Section 15.5],

$$N(\mathfrak{S}_w(x_1, \ldots, x_n, -y_1, \ldots, -y_n)) \text{ is Lorentzian for any } w \in S_n.$$

This would imply that the support of any double Schubert polynomial is M-convex, and hence also saturated [38, Conjecture 5.2].

Proposition 4.4.5. *The support of $\mathfrak{S}_w(x_1, \ldots, x_n)$ is M-convex for any $w \in S_n$.*

Proposition 4.4.5 was originally conjectured in [38, Conjecture 5.1]. It is proved in Corollary 2.2.4 using an explicit description of flagged Weyl modules. Here we give a short alternative proof based on Theorem 4.3.2. A similar argument can be used more generally to show that the supports of single quiver polynomials appearing in [37, Section 17.4] are M-convex.

Proof. By Theorem 4.3.2, the support of $\mathfrak{S}_w^\vee$ is M-convex. It is straightforward to check using the definition of M-convexity the general fact that, if the support of $h(x_1, \ldots, x_n)$ is M-convex, then the support of $x^\alpha h(x_1^{-1}, \ldots, x_n^{-1})$ is M-convex for any monomial x^α divisible by all monomials in the support of h. □

Proposition 4.4.6. *Conjecture 4.4.4 holds when $w \in S_n$ avoids the patterns* 1423 *and* 1432.

Proof. By [9, Corollary 6.7], the Lorentzian property of $\mathfrak{S}_w$ implies that of $N(\mathfrak{S}_w)$. We deduce the Lorentzian property of $\mathfrak{S}_w$ from known results on Schubert and Lorentzian polynomials, for permutations avoiding 1423 and 1432.

It is shown in Theorem 2.2.3 that, for any $w \in S_n$, the support of $\mathfrak{S}_w$ is the set of integer points in the Minkowski sum of n matroid polytopes. The set J_w of integer points in the Cartesian product of these matroid polytopes is an M-convex subset of $\mathbb{N}^{n\times n}$, and hence the generating function f_w of J_w is a Lorentzian polynomial in n^2 variables x_{ij} [9, Theorem 7.1]. Since any nonnegative linear change of coordinates preserves the Lorentzian property [9, Theorem 2.10], substituting the variables x_{ij} by x_i in the generating function f_w gives a Lorentzian polynomial. According to Lemma 2.3.1 and [13, Theorem 1.1], this specialization of f_w coincides with $\mathfrak{S}_w$ when w avoids 1423 and 1432. Thus $\mathfrak{S}_w$ is Lorentzian for such permutations. □

We note that the Schubert polynomials $\mathfrak{S}_{1423}$ and $\mathfrak{S}_{1432}$ are not Lorentzian.

4.4.3 Degree Polynomials

Let $w < w(i,j)$ be a covering relation in the Bruhat order of S_n labeled by the transposition of $i < j$ in $[n]$. The *Chevalley multiplicity* is the assignment

$$w < w(i,j) \longmapsto \sum_{i \leq k < j} x_k,$$

where x_k are independent variables. The *degree polynomial* of $w \in S_n$ is the generating function

$$\mathfrak{D}_w(x_1, \ldots, x_{n-1}) = \sum_C m_C(x_1, \ldots, x_{n-1}),$$

where the sum is over all saturated chains C from the identity permutation to w, and m_C is the product of Chevalley multiplicities of the covering relations in C. Degree polynomials were introduced by Bernstein, Gelfand, and Gelfand [5], and were studied from a combinatorial perspective by Postnikov and Stanley [43].

Proposition 4.4.7. *The degree polynomial* $\mathfrak{D}_w(x_1, \ldots, x_{n-1})$ *is Lorentzian for any* $w \in S_n$.

Proof. Let B be the group of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, and let X_w be the closure of the B-orbit of the permutation matrix corresponding to w in the flag variety $\mathrm{GL}_n(\mathbb{C})/B$. By [43, Proposition 4.2], the degree polynomial of w is, up to a normalizing constant, the volume polynomial of X_w with respect to the line bundles associated to the fundamental weights $\varpi_1, \ldots, \varpi_{n-1}$. The conclusion follows from Theorem 4.1.10. $\square$

4.4.4 Skew Schur Polynomials

Let λ/ν be a skew Young diagram. The *skew Schur polynomial* of λ/ν in m variables is the generating function

$$s_{\lambda/\nu}(x_1, \ldots, x_m) = \sum_T x^{\mathrm{wt}(T)},$$

where the sum is over all SSYT T of skew shape λ/ν with entries from $[m]$, and $\mathrm{wt}(T)$ is the vector in $\mathbb{N}^m$ whose ith component equals the number of i's among the entries of T.

Are normalized skew Schur polynomials Lorentzian? We tested the following statement for all partitions λ with at most 12 boxes and at most 6 parts.

Conjecture 4.4.8. *The polynomial $N(s_{\lambda/\nu}(x_1, \ldots, x_m))$ is Lorentzian for any λ/ν.*

Theorem 4.3.6 shows that Conjecture 4.4.8 holds when ν is zero, and Corollary 4.3.9 provides some further evidence. We remark that the M-convexity of the support of any skew Schur polynomial can be deduced from [38, Proposition 2.9].

4.4.5 Schur P-Polynomials

Let λ be a *strict partition*, that is, a decreasing sequence of positive integers. The *Schur P-polynomial* of λ in m variables is the generating function

$$P_\lambda(x_1, \ldots, x_m) = \sum_T x^{\mathrm{wt}(T)},$$

where the sum is over all marked shifted Young tableaux of shape λ with entries from $[m]$. See [33, Chapter III] for this and other equivalent definitions of the polynomial P_λ.

Are normalized Schur P-polynomials Lorentzian? We tested the following statement for all strict partitions λ with $\lambda_1 \leq 12$ and at most 4 parts.

Conjecture 4.4.9. *The polynomial* $N(P_\lambda(x_1, \ldots, x_m))$ *is Lorentzian for any strict partition* λ*.*

The M-convexity of the support of P_λ was observed in [38, Proposition 3.5].

4.4.6 Grothendieck Polynomials

Let $\ell(w)$ be the degree of the Schubert polynomial of w, let $d(w)$ be the degree of the Grothendieck polynomial of w, and let $\mathfrak{G}_w^k$ be the degree $\ell(w)+k$ homogeneous component of the Grothendieck polynomial.

Conjecture 4.4.10. *The polynomial* $(-1)^k N(\mathfrak{G}_w^k(x_1, \ldots, x_n))$ *is Lorentzian for any* $w \in S_n$ *and* $k \in \mathbb{N}$*.*

The M-convexity of the support of $\mathfrak{G}_w^k$ was conjectured in [36, Conjecture 5.1] and proved in [12] when w is a Grassmannian permutation. Conjecture 4.4.10 implies Conjecture 4.4.4 because the degree $\ell(w)$ homogeneous component of $\mathfrak{G}_w$ is the Schubert polynomial $\mathfrak{S}_w$.

We may strengthen Conjecture 4.4.10 in terms of the *homogeneous Grothendieck polynomial*

$$\widetilde{\mathfrak{G}}_w(x_1, \ldots, x_n, z) := \sum_{k=0}^{d(w)-\ell(w)} (-1)^k \mathfrak{G}_w^k(x_1, \ldots, x_n) z^{d(w)-\ell(w)-k},$$

where z is a new variable. Are normalized homogeneous Grothendieck polynomials Lorentzian? We tested the following statement for all permutations in S_n for $n \leq 7$.

Conjecture 4.4.11. *The polynomial* $N(\widetilde{\mathfrak{G}}_w(x_1,\ldots,x_n,z))$ *is Lorentzian for any* $w \in S_n$.

Conjecture 4.4.11 implies Conjecture 4.4.10 because taking partial derivatives and setting a variable equal to zero preserve the Lorentzian property. We expect an analogous Lorentzian property for double Grothendieck polynomials.

4.4.7 Key Polynomials

Recall the key polynomial κ_α of a composition $\alpha \in \mathbb{N}^n$ is defined as follows. When α is a partition, $\kappa_\alpha(x_1,\ldots,x_n) = x^\alpha$. If $\alpha_i < \alpha_{i+1}$ for some i and s_i is the adjacent transposition $(i\ i+1)$, then

$$\kappa_\alpha(x_1,\ldots,x_n) = \partial_i x_i \kappa_\beta, \quad \text{where } \beta = \alpha s_i = (\alpha_1,\ldots,\alpha_{i+1},\alpha_i,\ldots,\alpha_n).$$

Are normalized key polynomials Lorentzian? We tested the following statement for all compositions α with at most 12 boxes and at most 6 parts.

Conjecture 4.4.12. *The polynomial* $N(\kappa_\alpha)$ *is Lorentzian for any* $\alpha \in \mathbb{N}^n$.

Theorem 4.3.6 shows that Conjecture 4.4.12 holds when α is a weakly increasing sequence of nonnegative integers, because in this case the key polynomial of α is a Schur polynomial. The M-convexity of the supports of key polynomials was conjectured in [38, Conjecture 3.13] and proved in Corollary 2.2.4.

As noted in Section 2.1.1, key polynomials and Schubert polynomials are both dual characters of flagged Weyl modules (equivalently characters of flagged Schur modules). It is shown in Theorem 2.2.3 that the dual character of any flagged

Weyl module has M-convex support. Are normalized dual characters of flagged Weyl modules Lorentzian?

BIBLIOGRAPHY

[1] M. Aguiar and F. Ardila. Hopf monoids and generalized permutahedra, 2017. arXiv:1709.07504.

[2] F. Ardila, C. Benedetti, and J. Doker. Matroid polytopes and their volumes. *Discrete & Computational Geometry*, 43(4):841–854, 2010.

[3] W. Baldoni and M. Vergne. Kostant partitions functions and flow polytopes. *Transform. Groups*, 13(3-4):447–469, 2008.

[4] N. Bergeron and S. Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993.

[5] I. N. Bernstein, I. M. Gel'fand, and S. I. Gel'fand. Schubert cells and the cohomology of the spaces G/P. *Russian Math. Surveys*, 28(3):1–26, 1973.

[6] S. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993.

[7] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. M. Ziegler. *Oriented Matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 1999.

[8] A. Borovik, I. Gelfand, and N. White. *Coxeter Matroids*. Progress in Mathematics. Birkhäuser Basel, 1 edition, 2003.

[9] P. Brändén and J. Huh. Lorentzian polynomials, 2019. arXiv:1902.03719.

[10] M. Demazure. Key polynomials and a flagged Littlewood–Richardson rule. *Bull. Sci. Math.*, 98:163–172, 1974.

[11] M. Demazure. Une nouvelle formule des caracteres. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.

[12] L. Escobar and A. Yong. Newton polytopes and symmetric Grothendieck polynomials. *C. R. Math. Acad. Sci. Paris*, 355(8):831–834, 2017.

[13] N. Fan and P. Guo. Upper bounds of Schubert polynomials, 2019. arXiv:1909.07206.

[14] A. Fink, K. Mészáros, and A. St. Dizier. Schubert polynomials as integer point transforms of generalized permutahedra. *Adv. Math.*, 332:465–475, 2018.

[15] A. Fink, K. Mészáros, and A. St. Dizier. Zero-one Schubert polynomials. *Math. Z.*, to appear. arXiv:1903.10332.

[16] S. Fomin, C. Greene, V. Reiner, and M. Shimozono. Balanced labellings and Schubert polynomials. *European J. Combin*, 18:373–389, 1997.

[17] S. Fomin and A. N. Kirillov. The Yang-Baxter equation, symmetric functions, and Schubert polynomials. *Discrete Math.*, 153(1):123–143, 1996. Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics.

[18] S. Fomin and R. P. Stanley. Schubert polynomials and the nilCoxeter algebra. *Adv. in Math.*, 103(2):196 – 207, 1994.

[19] W. Fulton. *Young tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997. With applications to representation theory and geometry.

[20] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course; Readings in Mathematics.

[21] R. Hartshorne. *Algebraic geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[22] J. Huh, J. Matherne, K. Mészáros, and A. St. Dizier. Logarithmic concavity of Schur and related polynomials, 2019. arXiv:1906.09633.

[23] J. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathscr{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[24] L. Katthä, J. Moyano-Fernández, and J. Uliczka. Which series are hilbert series of graded modules over standard multigraded polynomial rings? *Mathematische Nachrichten*, 293(1):129–146, 2020.

[25] A. Knutson and E. Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math.*, 161(3):1245–1318, 2005.

[26] W. Kraśkiewicz and P. Pragacz. Foncteurs de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(9):209–211, 1987.

[27] T. Lam, S. Lee, and M. Shimozono. Back stable Schubert calculus, 2018. arXiv:1806.11233.

[28] A. Lascoux and M.-P. Schützenberger. Polynômes de Schubert. *C. R. Acad. Sci. Paris Sér. I Math.*, 294(13):447–450, 1982.

[29] A. Lascoux and M.-P. Schützenberger. Tableaux and noncommutative Schubert polynomials. *Funct. Anal. Appl.*, 23(3):223–225, 1989.

[30] A. Lascoux and M.-P. Schützenberger. Keys and standard bases. *IMA, Math. Appl.*, 19:125–144, 1990.

[31] R.K. Lazarsfeld. *Positivity in Algebraic Geometry I*, volume 48 of *A Series of Modern Surveys in Mathematics*. Springer-Verlag Berlin Heidelberg, 2004.

[32] C. Lenart. A unified approach to combinatorial formulas for Schubert polynomials. *J. Algebraic Combin.*, 20(3):263–299, 2004.

[33] I. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Classic Texts in the Physical Sciences. The Clarendon Press, Oxford University Press, New York, 2 edition, 2015.

[34] P. Magyar. Schubert polynomials and Bott-Samelson varieties. *Comment. Math. Helv.*, 73(4):603–636, 1998.

[35] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*, volume 6 of *SMF/AMS Texts and Monographs*. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés [Specialized Courses], 3.

[36] K. Mészáros and A. St. Dizier. From generalized permutahedra to Grothendieck polynomials via flow polytopes, 2017. arXiv:1705.02418.

[37] E. Miller and B. Sturmfels. *Combinatorial commutative algebra*, volume 227 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.

[38] C. Monical, N. Tokcan, and A. Yong. Newton polytopes in algebraic combinatorics. *Selecta Math.*, 25(5):66, Oct 2019.

[39] K. Murota. *Discrete convex analysis.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2003.

[40] A. Okounkov. *Why Would Multiplicities be Log-Concave?*, pages 329–347. Birkhäuser Boston, Boston, MA, 2003.

[41] A. Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not. IMRN*, 2009(6):1026–1106, 2009.

[42] A. Postnikov, V. Reiner, and L. Williams. Faces of generalized permutohedra. *Doc. Math.*, 13:207–273, 2008.

[43] A. Postnikov and R. P. Stanley. Chains in the Bruhat order. *J. Algebraic Combin.*, 29(2):133–174, 2009.

[44] R. Rado. An inequality. *J. Lond. Math. Soc.*, 27:1–6, 1952.

[45] V. Reiner and M. Shimozono. Key polynomials and a flagged Littlewood–Richardson rule. *J. Combin. Theory Ser. A*, 70(1):107–143, 1995.

[46] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second expanded edition edition, 2014.

[47] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

[48] D. Wagner. Multivariate stable polynomials: theory and applications. *Bull. Amer. Math. Soc. (N.S.)*, 48(1):53–84, 2011.

[49] A. Weigandt and A. Yong. The prism tableau model for Schubert polynomials. *J. Comb. Theory, Ser. A*, 154:551–582, 2018.

[50] G. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate texts in mathematics*. Springer-Verlag New York, 1995.