

# FOLIATIONS FOR QUASI-FUCHSIAN 3-MANIFOLDS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Biao Wang

August 2009

© 2009 Biao Wang

ALL RIGHTS RESERVED

# FOLIATIONS FOR QUASI-FUCHSIAN 3-MANIFOLDS

Biao Wang, Ph.D.

Cornell University 2009

In this paper, we prove that if a quasi-Fuchsian 3-manifold contains a minimal surface whose principal curvatures are in  $(-1, 1)$ , then it admits a foliation such that each leaf is a surface of constant mean curvature. The key method that we use here is volume preserving mean curvature flow.

## BIOGRAPHICAL SKETCH

Biao Wang was born in August 1970 in Shanghai, China. From 1988 to 1992, he studied System Engineering at Shanghai Institute of Mechanical Engineering (SIME), and got his Bachelor degree in the summer of 1992. Then he found his first job at Shanghai Aerospace Bureau as an assistant engineer. In 1994, he had a chance to study mathematics at East China Normal University (ECNU), and got his Master degree on Mathematics in 1997. Then he got his second job in the Department of Mathematics at East China University of Science and Technology (ECUST) as a lecturer. In 2001, he got an offer from University of California at Davis so that he could study mathematics PHD program in America. In 2003, he transferred to Cornell University to continue his PHD program.

This paper is dedicated to my parents, my wife and my son.  
It is also dedicated to my sister Jue, who must be very happy in another world.

## ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Profesoor Bill Thurston for his help, stimulating suggestions and encouragement in the past several years. This paper would not have been possible without his guidance and his encouragement.

I would like to thank Professor John Hubbard for his suggestion, questions and encouragement. He often asked me some fundamental questions which really help me to understand mathematics better.

I also thank Professor Allen Hatcher and Professor Xiaodong Cao for their helps and suggestions.

## TABLE OF CONTENTS

Biographical Sketch . . . . .	iii
Dedication . . . . .	iv
Acknowledgements . . . . .	v
Table of Contents . . . . .	vi
List of Figures . . . . .	vii
<b>1 Introduction</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>3</b>
2.1 Quasifuchsian groups . . . . .	3
2.2 Geometry of submanifolds . . . . .	4
<b>3 Volume preserving mean curvature flow</b>	<b>9</b>
3.1 Evolution equations . . . . .	9
3.2 Existence of the long time solution . . . . .	12
3.3 Exponential convergence to CMC surfaces . . . . .	20
<b>4 Existence of CMC foliation</b>	<b>24</b>
<b>5 A counterexample</b>	<b>28</b>
5.1 Existence of the surfaces with CMC . . . . .	28
5.2 Detail description of the counterexample . . . . .	30
<b>Bibliography</b>	<b>34</b>

## LIST OF FIGURES

5.1	Construction of limit set . . . . .	31
5.2	Two minimal surfaces . . . . .	33

CHAPTER 1  
INTRODUCTION

A codimension one foliation  $\mathcal{F}$  of a Riemannian manifold is called a *CMC foliation*, if each leaf of the foliation is a hypersurface of constant mean curvature. A quasi-Fuchsian group  $\Gamma$  is a Kleinian group which is obtained by a quasiconformal deformation a Fuchsian group, its limit set is a closed Jordan curve dividing the domain of discontinuity  $\Omega$  on  $S_\infty^2$  into two simply connected, invariant component. Topologically,  $(\mathbb{H}^3 \cup \Omega)/\Gamma = S \times [0, 1]$ , where  $S$  is a closed surface with  $\pi_1(S) = \Gamma$ . In this paper, we always assume that  $S$  is a *closed* Riemann surface with genus  $\geq 2$ .

Suppose  $M$  is a 3-dimensional quasi-Fuchsian hyperbolic manifold, Mazzeo and Pacard proved that each end of  $M$  admits a unique CMC foliation (cf. [24]). Next we may ask if the whole quasi-Fuchsian manifold  $M$  admits a CMC foliation? If  $M$  admits a CMC foliation  $\mathcal{F}$ , then the foliation  $\mathcal{F}$  must contain a leaf  $L$  whose mean curvature is zero, i.e.  $L$  is a minimal surface in  $M$ . Therefore we need to know whether  $M$  contains a minimal surface at first. There are several ways to prove that  $M$  contains a least area minimal surface  $\Sigma$  with  $\pi_1(M) \cong \pi_1(\Sigma)$  (cf. [1, 25, 28, 30]).

In this paper, we will prove the following theorem.

**Theorem 1.1.** *Suppose that  $M$  is a quasi-Fuchsian 3-manifold, which contains a closed immersed minimal surface  $\Sigma$  with genus  $\geq 2$  such that  $\pi_1(M) \cong \pi_1(\Sigma)$ , if the principal curvatures  $\Sigma$  are in  $(-1, 1)$ , then  $M$  admits a unique CMC foliation.*

We will use the volume preserving mean curvature flow developed by G. Huisken (cf. [14, 16]) to prove Theorem 1.1 in chapter 4. This idea is inspired by Ecker

and Huisken's paper [10]. Furthermore, we will show that  $M$  doesn't admit a CMC foliation if the principal curvatures of  $\Sigma$  are very large in chapter 5, where the idea of using infinite minimal catenoids as barrier surfaces contributes to Bill Thurston.

This paper is organized as follows. In chapter 2, we give some definitions and basic properties about quasi-Fuchsian groups and submanifolds. In chapter 3, we discuss the volume preserving mean curvature flow and prove the existence of the long time solution. In chapter 4, we will prove Theorem 1.1. In chapter 5, we will give a counterexample.

CHAPTER 2  
PRELIMINARIES

In this chapter, we review some basic facts on quasi-Fuchsian 3-manifolds and geometry of submanifolds.

## 2.1 Quasifuchsian groups

A subgroup  $\Gamma$  of  $\text{Isom}(\mathbb{H}^3)$  is called a *Kleinian groups* if  $\Gamma$  acts on  $\mathbb{H}^3$  properly discontinuously. For any Kleinian group  $\Gamma$ ,  $\forall p \in \mathbb{H}^3$ , the orbit set

$$\Gamma(p) = \{\gamma(p) \mid \gamma \in \Gamma\}$$

has accumulation points on  $S_\infty^2 = \partial\mathbb{H}^3$ , these points are called the *limit points* of  $\Gamma$ , and the closed set of all these points is called the *limit set* of  $\Gamma$ , which is denoted by  $\Lambda_\Gamma$ . The complement of the limit set, i.e.,

$$\Omega_\Gamma = S_\infty^2 \setminus \Lambda_\Gamma ,$$

is called the *region of discontinuity*. If  $\Omega_\Gamma = \emptyset$ ,  $\Gamma$  is called a Kleinian group of the first kind, and otherwise of the second kind.

Suppose  $\Gamma$  is a finitely generated torsion free Kleinian group which has more than two limit points, we call  $\Gamma$  *quasi-Fuchsian* if its limit set  $\Lambda_\Gamma$  is a closed Jordan curve and both components  $\Omega_1$  and  $\Omega_2$  of its region of discontinuity are invariant under  $\Gamma$ . The limit set  $\Lambda_\Gamma$  of the quasi-Fuchsian group  $\Gamma$  is either a (standard) circle or a closed Jordan curve which fails to have a tangent on an everywhere dense set (cf. [19, Theorem 4.2]. When  $\Lambda_\Gamma$  is a circle, we call  $\Gamma$  a Fuchsian group. Of course,  $\Lambda_\Gamma$  is invariant under  $\Gamma$  too. The following statement about quasi-Fuchsian groups can be found in [7, page 8].

**Proposition 2.1** (Maskit [23], Thurston [29]). *If  $\Gamma$  is a finitely generated, torsion-free Kleinian group, then the following conditions are equivalent:*

- (i)  $\Gamma$  is quasi-Fuchsian.
- (ii)  $\Omega_\Gamma$  has exactly two components, each of which is invariant under  $\Gamma$ .
- (iii) There exist a Fuchsian group  $G$  and a quasiconformal homeomorphism  $w : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$  such that  $\Gamma = w \circ G \circ w^{-1}$ .

For a finitely generated, torsion free quasi-Fuchsian group  $\Gamma$  with invariant components  $\Omega_1, \Omega_2$  of  $\Omega_\Gamma$ , Albert Marden (cf. [22]) proved that  $\Gamma$  has the following properties:

- Each of  $S_1 = \Omega_1/\Gamma$  and  $S_2 = \Omega_2/\Gamma$  is a finitely punctured Riemann surface.
- $M_\Gamma = \mathbb{H}^3/\Gamma$  is diffeomorphic to  $(\Omega_1/\Gamma) \times (0, 1)$ , and  $\overline{M}_\Gamma = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$  is diffeomorphic to  $(\Omega_1/\Gamma) \times [0, 1]$ .

We will call  $M_\Gamma$  a *quasi-Fuchsian 3-manifold*. In this paper we write  $M_\Gamma = S \times \mathbb{R}$ , where  $S$  is a closed surface with genus  $\geq 2$ .

## 2.2 Geometry of submanifolds

In this section, we rephrase some materials from [30] for convenience. Let  $(M, \bar{g}_{\alpha\beta})$  be a quasi-Fuchsian 3-manifold, and let  $\Sigma$  be a immersed minimal surface in  $M$ . Suppose the coordinate system on  $\Sigma \equiv \Sigma \times \{0\}$  is isothermal so that the induced metric  $g = (g_{ij})_{2 \times 2}$  on  $\Sigma$  can be written in the form

$$g(x, 0) = \{g_{ij}(x, 0)\}_{1 \leq i, j \leq 2} = e^{2v(x)} \mathbf{I}$$

where  $\mathbf{I}$  is a  $2 \times 2$  unit matrix, and let

$$A(x) \equiv A(x, 0) = \{h_{ij}(x, 0)\}$$

be the second fundamental form of  $\Sigma$ .

In a collar neighborhood of  $\Sigma$  in  $M$ , there exists normal coordinates induced by  $\exp : T^\perp \Sigma \rightarrow M$  in a neighborhood on which

$$\Sigma \times (-\varepsilon, \varepsilon) \subset T^\perp \Sigma \rightarrow M$$

is a (local) diffeomorphism. If coordinates  $(x^1, x^2)$  are introduced on  $\Sigma$ , then

$$\exp((x^1, x^2), x^3) = (x^1, x^2, x^3)$$

induces a coordinate patch in  $M$ . Choose  $p = (x^1, x^2, x^3) = (x, r)$  the local coordinate system in a neighborhood of  $\Sigma$  so that  $\Sigma = \{(x, r) \in M \mid r = 0\}$ . Let  $N_0$  be the unit normal vector field on  $\Sigma$ , and let

$$\Sigma(r) = \{\exp_x r N_0 \mid x \in \Sigma\} \tag{2.1}$$

for a small positive constant  $r$ . For  $(x, r) \in \Sigma \times (-\varepsilon, \varepsilon) \subset T^\perp \Sigma$ , it's well known that the pullback metric has the form

$$\bar{g}(x, r) = \begin{pmatrix} g(x, r) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{11}(x, r) & g_{12}(x, r) & 0 \\ g_{21}(x, r) & g_{22}(x, r) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

where  $g(x, r)$  is the induced metric on  $\Sigma(r)$ .

The second fundamental form  $A = (h_{ij})$  of  $\Sigma(r)$  is a  $2 \times 2$  matrix defined by

$$h_{ij} = \langle \bar{\nabla}_{e_i} e_3, e_j \rangle, \quad 1 \leq i, j \leq 2, \tag{2.3}$$

where  $\bar{\nabla}$  is the covariant differentiation in  $M$ , and  $\{e_1, e_2, e_3\}$  is the local frame for  $M$  such that  $e_3$  is the unit normal vector of  $\Sigma(r)$  and  $e_1, e_2$  are two unit vectors in

the tangent plane of  $\Sigma(r)$ . Direct computation shows that the second fundamental forms  $A(x, r) = \{h_{ij}(x, r)\}$  on  $\Sigma(r)$  are given by

$$h_{ij}(x, r) = \frac{1}{2} \frac{\partial}{\partial r} g_{ij}(x, r) , \quad 1 \leq i, j \leq 2 . \quad (2.4)$$

Note that the sectional curvature of  $M$  is  $-1$ , there are three curvature equations of the form

$$\bar{R}_{i3j3} = -(\bar{g}_{33}\bar{g}_{ij} - \bar{g}_{i3}\bar{g}_{3j}) = -g_{ij} , \quad 1 \leq i, j \leq 2 , \quad (2.5)$$

where the Riemann curvature tensor is given by

$$\bar{R}(X, Y)Z = -\bar{\nabla}_X \bar{\nabla}_Y Z + \bar{\nabla}_Y \bar{\nabla}_X Z + \bar{\nabla}_{[X, Y]} Z$$

for  $X, Y, Z \in \mathfrak{X}(M)$ . Direct computation shows that the curvature forms are given by

$$\bar{R}_{i3j3} = -\frac{1}{2} \frac{\partial^2 g_{ij}}{\partial r^2} + \frac{1}{4} g^{kl} \frac{\partial g_{il}}{\partial r} \frac{\partial g_{jk}}{\partial r} , \quad 1 \leq i, j \leq 2 . \quad (2.6)$$

From (2.5) and (2.6), we get partial differential equations

$$-g_{ij} = -\frac{1}{2} \frac{\partial^2 g_{ij}}{\partial r^2} + \frac{1}{4} g^{kl} \frac{\partial g_{il}}{\partial r} \frac{\partial g_{jk}}{\partial r} , \quad (2.7)$$

whose solutions can be written in the form

$$g(x, r) = e^{2v(x)} [\cosh r \mathbf{I} + \sinh r e^{-2v(x)} A(x)]^2 \quad (2.8)$$

for all  $x \equiv (x, 0) \in \Sigma$  and  $-\epsilon < r < \epsilon$ . This metric is nonsingular in a collar neighborhood of  $\Sigma$  in any case. If the principal curvatures  $\pm\lambda(x)$  of  $\Sigma \subset M$  satisfy

$$|\lambda(x)| = \sqrt{-\det [A(x)e^{-2v(x)}]} < 1 ,$$

then the metric (2.8) is non-singular for all  $r \in \mathbb{R}$ .

**Proposition 2.2.** *The mean curvature of  $\Sigma(r)$  is given by*

$$H(x, r) = \frac{2(1 - \lambda^2(x)) \tanh r}{1 - \lambda^2(x) \tanh^2 r} , \quad \forall x \in \Sigma , \quad (2.9)$$

here the normal vector on  $\Sigma(r)$  points to the minimal surface  $\Sigma$ .

**Proof.** In order to compute the mean curvature  $H$ , we need to find the eigenvalues of the second fundamental form  $A(x, r)$ . In other words, we need solve the equation

$$\det [h_{ij} - \mu g_{ij}] = 0 ,$$

which is equivalent to the equation

$$\det [(\sinh r \mathbf{I} + \cosh r e^{-2v(x)} A(x)) - \mu(\cosh r \mathbf{I} + \sinh r e^{-2v(x)} A(x))] = 0 .$$

Solve the above equation, we get two eigenvalues:

$$\mu_1 = \frac{\tanh r - \lambda(x)}{1 - \lambda(x) \tanh r} \quad \text{and} \quad \mu_2 = \frac{\tanh r + \lambda(x)}{1 + \lambda(x) \tanh r} .$$

Since  $H = \mu_1 + \mu_2$ , the statement follows.  $\square$

It's easy to check that  $H(x, r)$  defined in (2.9) is a monotonically increasing function with respect to  $r$ , i.e.  $H(x, r_1) \leq H(x, r_2)$  if  $r_1 \leq r_2$ . In fact, we have

$$\frac{\partial}{\partial r} H(x, r) = \frac{2(1 - \lambda^2(x))[1 + \lambda^2(x) \tanh^2 r]}{[1 - \lambda^2(x) \tanh^2 r]^2 \cosh^2 r} \geq 0 , \quad \forall x \in \Sigma .$$

As  $r \rightarrow \pm\infty$ ,  $H \rightarrow \pm 2$ , and as  $r \rightarrow 0$ ,  $H \rightarrow 0$ .

**Theorem 2.3 (Uhlenbeck [30]).** *If  $M$  is a complete, hyperbolic manifold and  $\Sigma$  is a minimal surface in  $M$  with principal curvatures in  $(-1, 1)$ , then*

- (i)  $\exp T^\perp \Sigma \cong \widetilde{M} \rightarrow M$ , where  $\widetilde{M}$  is the cover of  $M$  corresponding to  $\pi_1(\Sigma) \subset \pi_1(M)$ .
- (ii)  $\widetilde{M}$  is quasi-Fuchsian.
- (iii)  $\Sigma \subset M$  is area minimizing;  $\Sigma \subset \widetilde{M}$  is the only closed minimal surface of any type in  $\widetilde{M}$ .
- (iv)  $\Sigma \subset \widetilde{M}$  is embedded.

(v)  $\Sigma \subset M$  is totally geodesic if and only if  $\widetilde{M}$  is Fuchsian.

**Corollary 2.4 (Uhlenbeck [30]).** *Suppose  $\Sigma$  is an immersed minimal surface in a quasi-Fuchsian 3-manifold  $M$  which is homotopic to  $\Sigma$ , if the principal curvatures of  $\Sigma$  are in  $(-1, 1)$ , then*

- $\Sigma$  is the unique minimal surface which is embedded in  $M$ ,
- the metric  $\bar{g}_{\alpha\beta}$  on  $M = \Sigma \times \mathbb{R}$  is given by (2.2) and (2.8), and
- $M$  can be foliated by either the geodesics perpendicular to the minimal surface  $\Sigma$  or the equidistant surfaces  $\{\Sigma(r)\}_{-\infty < r < \infty}$  defined by (2.1).

## CHAPTER 3

### VOLUME PRESERVING MEAN CURVATURE FLOW

In this chapter, we will discuss the volume preserving mean curvature flow developed by G. Huisken and others. A good reference for mean curvature flow is the book written by Xi-Ping Zhu (cf. [31]).

By the discussion in chapter 2,  $(M, \bar{g}_{\alpha\beta})$  can be foliated either by the geodesics which are perpendicular to the minimal surface  $\Sigma$  or by the surfaces  $\Sigma(r)$  for all  $r \in \mathbb{R}$ , where  $\Sigma(r)$  is defined by (2.1). Denote by  $N$  the unit tangent vector field on the geodesics, which is a well defined vector field on  $M$ .

For any tensor field  $\Phi$  on  $(M, \bar{g}_{\alpha\beta})$  we define the supremum norms by

$$\|\Phi\| = \sup_{x \in M} |\Phi(x)|_{\bar{g}_{\alpha\beta}} \quad \text{and} \quad \|\Phi\|_k = \sum_{j=0}^k \|\bar{\nabla}^j \Phi\| .$$

### 3.1 Evolution equations

Let  $S$  be a smooth surface which is diffeomorphic to the minimal surface  $\Sigma \subset M$ , and let  $F_0 : S \rightarrow M$  be the immersion of  $S$  in  $M$  such that  $S_0 = F_0(S)$  is contained in the positive side of  $\Sigma$  and is a graph over  $\Sigma$  with respect to  $N$ , i.e.  $\langle N, \nu \rangle \geq C > 0$ , here  $\nu$  is the normal vector on  $S_0$  and  $C$  is a constant depending only on  $S_0$ . Next we consider a family of smoothly immersed surfaces in  $M$ ,

$$F : S \times [0, T) \rightarrow M , \quad 0 \leq T \leq \infty$$

with  $F(\cdot, 0) = F_0$ . For each  $t \in [0, T)$ , write

$$S_t = \{F(x, t) \in M \mid x \in S\} .$$

We need define some quantities and operators on  $S_t$ :

- the induced metric of  $S_t$  is denoted by  $g = \{g_{ij}\}$ ,
- the second fundamental form of  $S_t$  is denoted by  $A = \{h_{ij}\}$ ,
- the mean curvature of  $S_t$  with respect to the normal pointing to the minimal surface  $\Sigma$  is given by  $H = g^{ij}h_{ij}$ ,
- the square norm of the second fundamental form of  $S_t$  is given by

$$|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl} ,$$

- the covariant derivative of  $S_t$  is denoted by  $\nabla$ ,
- the Laplacian on  $S_t$  is given by  $\Delta = g^{ij}\nabla_i\nabla_j$ .

Each quantity or operator with respect to  $(M, \bar{g}_{\alpha\beta})$  will be added a bar on its top.

The curvature operator  $\overline{\text{Rm}}$  on  $(M, \bar{g}_{\alpha\beta})$  is given by

$$\overline{R}_{\alpha\beta\gamma\delta} = -(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}) , \quad 1 \leq \alpha, \beta, \gamma, \delta \leq 3 . \quad (3.1)$$

We consider the volume preserving mean curvature flow (cf. [16]):

$$\begin{cases} \frac{\partial}{\partial t} F(x, t) = [h(t) - H(x, t)]\nu(x, t) , & x \in S , 0 \leq t < T , \\ F(\cdot, 0) = F_0 , \end{cases} \quad (3.2)$$

where

$$h(t) = \int_{S_t} H d\mu = \frac{1}{\text{Area}(S_t)} \int_{S_t} H d\mu$$

is the *average mean curvature* of  $S_t$ , and  $\nu$  is the normal on  $S_t$  so that  $-\nu$  points to the minimal surface  $\Sigma$ . It's easy to verify that the volume of the domain bounded by  $\Sigma$  and  $S_t$  is independent of time. In [15, 16], Huisken proved the following theorem.

**Theorem 3.1 (Huisken).** *If the initial surface  $S_0$  is smooth, then (3.2) has a smooth solution on some maximal open time interval  $0 \leq t < T$ , where  $0 < T \leq \infty$ .*

*If  $T < \infty$ , then*

$$|A|_{\max}(t) \equiv \max_{x \in S} |A|(x, t) \rightarrow \infty, \quad \text{as } t \rightarrow T. \quad (3.3)$$

In this chapter, we will prove the following theorem.

**Theorem 3.2.** *For any fixed  $r > 0$ , the evolution equation (3.2) has a unique long time solution (i.e.  $T = \infty$ ). As  $t \rightarrow \infty$ , the surfaces  $\{S_t\}$  converge exponentially fast to a smooth surface  $S_\infty$  of constant mean curvature.*

For this aim, we assume  $T < \infty$  at the very beginning, if we can prove that there exist constants  $\{C(m)\}_{m=0,1,2,\dots}$  independent of time such that the estimates

$$|\nabla^m A|^2 \leq C(m), \quad m = 0, 1, 2, \dots, \quad (3.4)$$

are uniformly on  $S_t$  for  $0 \leq t < T$ , then we can derive that the limit surface  $S_T = \lim_{t \rightarrow T} S_t$  is a smooth surface, so we can extend  $T$  a little bit further by Theorem 3.1, this is contradicted by the hypothesis that  $T$  is maximal.

To obtain in the next step a priori estimate for  $|A|^2$ , we need evolution equations for the metric and the second fundamental form on  $S_t$ .

**Lemma 3.3 (Huisken–Yau [17]).** *We have the following evolution equations:*

- (i)  $\frac{\partial}{\partial t} g_{ij} = 2(h - H)h_{ij},$
- (ii)  $\frac{\partial}{\partial t} h_{ij} = \nabla_i \nabla_j H + (h - H)h_{il}g^{kl}h_{kj} + (h - H)g_{ij},$
- (iii)  $\frac{\partial}{\partial t} \nu = \nabla H,$
- (iv)  $\frac{\partial}{\partial t} \mu = H(h - H)\mu, \text{ where } \mu \text{ is the measure on } S_t.$

Since  $(M, \bar{g}_{\alpha\beta})$  is a 3-manifold with constant sectional curvature, we have  $\bar{\nabla}_m \bar{R}_{ijkl} \equiv 0$ ,  $\bar{\text{Ric}}(\nu, \nu) = -2$ , and

$$h_{ij}h_{jl}\bar{R}_{lmml} - h_{ij}h_{lm}\bar{R}_{limj} = -(\lambda_1 - \lambda_2)^2 = H^2 - 2|A|^2.$$

Together with Simons' identity (cf. [17, Lemma 1.3(i)]), we obtain the following additional evolution equations.

**Lemma 3.4 (Huisken–Yau [17]).** *Under the evolution equation (3.2), the second fundamental form satisfies the evolution equations*

- (i)  $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} + (h - 2H)h_{il}g^{lk}h_{kj} + (|A|^2 + 2)h_{ij} + (h - 2H)g_{ij}$ ,
- (ii)  $\frac{\partial}{\partial t} H = \Delta H + (H - h)(|A|^2 - 2)$ ,
- (iii)  $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4 - 2h \operatorname{tr} A^3 + 4|A|^2 + 2H(h - 2H)$ , where  $\operatorname{tr} A^3 = \frac{H}{2}(3|A|^2 - H^2)$ .

## 3.2 Existence of the long time solution

Define a function  $\ell : M \rightarrow \mathbb{R}$  by

$$\ell(p) = \operatorname{dist}(p, \Sigma) = \min\{\operatorname{dist}(p, p') \mid p' \in \Sigma\}$$

for all  $p \in M$ , where  $\operatorname{dist}(\cdot, \cdot)$  is the distance function on  $(M, \bar{g}_{\alpha\beta})$ . By Corollary 2.4, every point  $p \in M$  has the form  $p = (p', r)$  for some point  $p' \in \Sigma$ , where  $r = \ell(p)$ . Let

$$u = \ell|_{S_t} \quad \text{and} \quad \Theta = \langle N|_{S_t}, \nu \rangle$$

be the height function and the gradient function of  $S_t$  respectively. Obviously  $S_t$  is a graph over the minimal surface  $\Sigma$  if  $\Theta > 0$  on  $S_t$ . The evolution equations of

$u$  and  $\Theta$  can be derived as follows (cf. [10]),

$$\frac{\partial u}{\partial t} = \left\langle \frac{\partial F}{\partial t}, N \right\rangle = (h - H)\Theta \quad (3.5)$$

and

$$\frac{\partial \Theta}{\partial t} = \langle N, \nabla H \rangle + (h - H)\langle \bar{\nabla}_\nu N, \nu \rangle . \quad (3.6)$$

**Lemma 3.5 (Ecker–Huisken [10]).** *The height function  $u$  on  $S_t$  also satisfies*

$$\frac{\partial}{\partial t} u = \Delta u - \operatorname{div}(\bar{\nabla} \ell) + h\Theta , \quad (3.7)$$

where  $\operatorname{div}$  is the divergence on  $S_t$  and  $\bar{\nabla}$  is the gradient on  $M$ .

**Proof.** Since  $u = \ell|_{S_t}$ , we have  $\nabla u = (\bar{\nabla} \ell)^\# = \bar{\nabla} \ell - \Theta \nu$ , then we obtain

$$\Delta u = \operatorname{div} \nabla u = \operatorname{div}(\bar{\nabla} \ell) - (\operatorname{div} \nu)\Theta = \operatorname{div}(\bar{\nabla} \ell) - H\Theta .$$

Plugin the above identity to (3.5), we get (3.7).  $\square$

**Lemma 3.6 (Bartnik [3]).** *The gradient function  $\Theta$  on  $S_t$  satisfies*

$$\Delta \Theta = -(|A|^2 + \bar{\operatorname{Ric}}(\nu, \nu))\Theta + \langle N, \nabla H \rangle - N(H_N) , \quad (3.8)$$

where  $N(H_N)$  is the variation of mean curvature of  $S_t$  under the deformation vector field  $N$ , which satisfies

$$\begin{aligned} N(H_N) &= \frac{1}{2}(\bar{\nabla}_\nu \mathcal{L}_N \bar{g})(e_i, e_i) - (\bar{\nabla}_{e_i} \mathcal{L}_N \bar{g})(\nu, e_i) - \frac{1}{2} H \mathcal{L}_N \bar{g}(\nu, \nu) \\ &\quad - \mathcal{L}_N \bar{g}(e_i, e_j) \cdot A(e_i, e_j) , \end{aligned} \quad (3.9)$$

here  $\mathcal{L}$  denotes the Lie derivative.

By (3.6) and (3.8), we have the following evolution for the gradient function.

**Corollary 3.7 (Ecker–Huisken [10]).**  $\Theta$  satisfies the following evolution equation

$$\frac{\partial \Theta}{\partial t} = \Delta \Theta + (|A|^2 + \overline{\text{Ric}}(\nu, \nu))\Theta + N(H_N) + (h - H)\langle \overline{\nabla}_\nu N, \nu \rangle, \quad (3.10)$$

where  $\Delta$  is the Laplacian on  $S_t$ .

Next we will prove that  $\{S_t\}_{0 \leq t < T}$  are contained in a bounded domain of  $M$  for all  $T > 0$ , i.e the height function is uniformly bounded. This result is very important for us to prove Theorem 3.2. At first, we need the well known maximum principle.

**Lemma 3.8 (Maximum Principle).** *Let  $\Sigma_1$  and  $\Sigma_2$  be two hypersurfaces in a Riemannian manifold, and intersect at a common point tangentially. If  $\Sigma_2$  lies in positive side of  $\Sigma_1$  around the common point, then  $H_1 < H_2$ , where  $H_i$  is the mean curvature of  $\Sigma_i$  at the common point for  $i = 1, 2$ .*

*In particular, suppose  $\Sigma_i$  is a hypersurface with constant mean curvature  $H_i$ ,  $i = 1, 2$ , in a Riemannian manifold. If whole  $\Sigma_2$  lies in positive side of  $\Sigma_1$ , and they intersect at a common point tangentially, such that  $H_2 \leq H_1$ , then  $\Sigma_1$  and  $\Sigma_2$  are the same surface.*

**Proposition 3.9.** *Suppose the volume preserving mean curvature flow (3.2) has a family of solutions on  $[0, T)$ ,  $0 < T \leq \infty$ , then  $u$  is uniformly bounded on  $S \times [0, T)$ , i.e.,*

$$0 < C_1 \leq u(x, t) \leq C_2 < \infty, \quad \forall (x, t) \in S \times [0, T),$$

where  $C_1$  and  $C_2$  are two constants depending only on the initial data  $S_0$ .

**Proof.** At each time  $t \in [0, T)$ , let  $x(t) \in S$  be the point such that

$$u_{\max}(t) \equiv \max_{x \in S} u(x, t) = u(x(t), t),$$

and let  $y(t) \in S$  be the point such that

$$u_{\min}(t) \equiv \min_{y \in S} u(y, t) = u(y(t), t) .$$

Since  $\Theta = \langle N, \nu \rangle = 1$  at  $F(x(t), t)$ , we have

$$0 \leq \frac{\partial u}{\partial t} = h - H .$$

By Lemma 3.8, we have

$$h(t) \geq H(x(t), t) \geq \frac{2(1 - \Lambda_+) \tanh(u_{\max}(t))}{1 - \Lambda_+ \tanh^2(u_{\max}(t))} ,$$

where  $\Lambda_+ = \max_{p' \in \Sigma} \lambda^2(p')$ . Similarly, at the point  $F(y(t), t)$ , we have

$$h(t) \leq H(y(t), t) \leq \frac{2(1 - \Lambda_-) \tanh(u_{\min}(t))}{1 - \Lambda_- \tanh^2(u_{\min}(t))} ,$$

where  $\Lambda_- = \min_{p' \in \Sigma} \lambda^2(p')$ . Therefore, we have the inequality

$$\frac{2(1 - \Lambda_-) \tanh(u_{\min}(t))}{1 - \Lambda_- \tanh^2(u_{\min}(t))} \geq h(t) \geq \frac{2(1 - \Lambda_+) \tanh(u_{\max}(t))}{1 - \Lambda_+ \tanh^2(u_{\max}(t))} .$$

As  $t \rightarrow T$ , we have five cases:

- (i)  $u_{\min}(t) \rightarrow 0$  and  $u_{\max}(t) \rightarrow 0$ ;
- (ii)  $u_{\min}(t) \rightarrow +\infty$  and  $u_{\max}(t) \rightarrow +\infty$ ;
- (iii)  $u_{\min}(t) \rightarrow 0$  and  $u_{\max}(t) \rightarrow +\infty$ ;
- (iv)  $u_{\min}(t)$  is uniformly bounded, while  $u_{\max}(t) \rightarrow +\infty$ ;
- (v)  $u_{\min}(t) \rightarrow 0$ , while  $u_{\max}(t)$  is uniformly bounded.

Case (i) and (ii) could not happen, since the mean curvature flow is volume preserving. Case (iii) could not happen, otherwise we would get  $0 \geq 2$ , a contradiction. Similarly, Case (iv) and (v) could not happen.

So the mean curvature flow is uniformly bounded by two surfaces  $\Sigma(r_1)$  and  $\Sigma(r_2)$  with  $0 < r_1 \leq r_2 < +\infty$  on the time interval  $[0, T)$ .  $\square$

The proof in Proposition 3.9 actually contains the following statement.

**Corollary 3.10.** *The average mean curvature  $h$  is uniformly bounded on  $[0, T)$ , i.e.*

$$0 < \frac{2(1 - \Lambda_+) \tanh r_1}{1 - \Lambda_+ \tanh^2 r_1} \leq h(t) \leq \frac{2(1 - \Lambda_-) \tanh r_2}{1 - \Lambda_- \tanh^2 r_2} < 2 .$$

Next we will prove that the gradient function  $\Theta$  is uniformly bounded from below and  $|\nabla\Theta|$  is uniformly bounded from above on  $S_t$  for  $t \in [0, T)$ .

**Proposition 3.11.** *Suppose the volume preserving mean curvature flow (3.2) has a solution on  $[0, T)$ ,  $0 < T \leq \infty$ , then there exists constants  $0 < \Theta_0 < 1$  and  $0 < C_3 < \infty$  depending only on  $S_0$  such that*

$$\Theta \geq \Theta_0 \quad \text{and} \quad |\nabla\Theta|^2 \leq C_3$$

on  $S_t$  for  $0 \leq t < T$ .

**Proof.** By assumption  $S_0$  is a graph over  $\Sigma$ , so  $\Theta(\cdot, 0) \geq C > 0$ , then we may assume that  $\Theta > 0$  for a short time. For any point  $p \in S_t$ , we may write

$$p = (p', u) = (p_1, p_2, u) ,$$

where  $p' = (p_1, p_2) \in \Sigma$  and  $u$  is the height function on  $S_t$ . Consider the Gaussian coordinates in  $U \times \mathbb{R} \subset M$ , where  $U \subset \Sigma$  is a neighborhood of  $p'$ . The unit normal  $\nu$  to  $S_t$  is given by (cf. [15, Lemma 3.2])

$$\nu = \frac{1}{\sqrt{1 + |\nabla u|^2}} \left( -\frac{\partial u}{\partial p_1}, -\frac{\partial u}{\partial p_2}, 1 \right) ,$$

and then the gradient function  $\Theta$  is given by

$$\Theta = \langle N, \nu \rangle = \frac{1}{\sqrt{1 + |\nabla u|^2}} , \tag{3.11}$$

where  $N = (0, 0, 1)$ . We can see that  $|\nabla u| = \infty$  if and only if  $\Theta = 0$ .

Next, we consider the quasi-linear parabolic equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \operatorname{div}(\overline{\nabla} \ell) + h\Theta \\ u(0) = u_0, \end{cases} \quad (3.12)$$

where  $u_0$  is the height function on  $S_0$ . By our hypothesis, (3.12) has a solution for  $t \in [0, T)$ . Equation (3.12) is a second order quasi-linear parabolic equation if we write the induced metric on  $S_t$  in terms of  $u$  and its derivatives. By Proposition 3.9,  $u$  is uniformly bounded for  $t \in [0, T)$ . By the standard regularity theory of quasi-linear parabolic equation (cf. [20] or [18, Chapter 6]), there exist constants  $K_l < \infty$  depending only on  $l$  and the initial surface  $S_0$  such that

$$|\nabla^l u| \leq K_l, \quad l = 1, 2, \dots,$$

for  $t \in [0, T)$ .

Using (3.11), these estimates imply that  $\Theta$  is uniformly bounded from below and  $|\nabla\Theta|^2$  is uniformly from above for  $t \in [0, T)$ .  $\square$

**Proposition 3.12.** *Suppose the volume preserving mean curvature flow (3.2) has a family of solutions on  $[0, T)$ ,  $0 < T \leq \infty$ , then there exists a constant  $C_0 < \infty$  depending only on  $S_0$  such that*

$$|A|^2 \leq C_0 < \infty$$

on  $S_t$  for  $0 \leq t < T$ .

**Proof.** We will show that  $|A|^2$  is uniformly bounded by contradiction. Let  $f_\sigma =$

$\frac{|A|^2}{\Theta^{2+\sigma}}$ , where  $\sigma > 0$  is a small constant. The evolution equation of  $f_\sigma$  is given by

$$\begin{aligned} \frac{\partial f_\sigma}{\partial t} = & \Delta f_\sigma + \frac{2(2+\sigma)}{\Theta} \langle \nabla f_\sigma, \nabla \Theta \rangle - \frac{2}{\Theta^{2+\sigma}} |\nabla A|^2 \\ & + \frac{(1+\sigma)(2+\sigma)|A|^2}{\Theta^{4+\sigma}} |\nabla \Theta|^2 \\ & + \frac{1}{\Theta^{2+\sigma}} \left\{ -\sigma |A|^2 (|A|^2 - 2) - 2h \operatorname{tr} A^3 + 8|A|^2 + 2H(h - 2H) \right. \\ & \left. - \frac{(2+\sigma)|A|^2}{\Theta} N(H_N) + \frac{(2+\sigma)|A|^2(h-H)}{\Theta} \langle \bar{\nabla}_\nu N, \nu \rangle \right\} . \end{aligned}$$

Recall that the restriction to  $TS_t$  of any tensor field  $\Phi$  of order  $m$  on  $M$  can be estimated by

$$\|\Phi|_{TS_t}(x)\| \leq \Theta^m(x) \|\Phi(x)\| ,$$

where  $\|\Phi(x)\| = |\Phi(x)|_{\bar{g}_{\alpha\beta}}$  (cf. [10]). By using (3.9) we estimate the expression  $N(H_N)$  in the evolution equation (3.8) by

$$|N(H_N)| \leq C_4(\Theta^3 + \Theta^2|A|) . \quad (3.13)$$

Here  $C_3$  depends on  $\|\mathcal{L}_N \bar{g}\|_1$  where  $\mathcal{L}_N \bar{g}$  is the Lie derivative of the metric with respect to  $N$  whose  $C^1$ -norm can be controlled in terms of  $\|N\|_2$  (cf. [9]). Besides we also have the following estimate

$$|\langle \bar{\nabla}_\nu N, \nu \rangle| \leq C_5 \Theta^2 , \quad (3.14)$$

where  $C_5 = \|\bar{\nabla} N\|$ . Since  $\{S_t\}_{0 \leq t < T}$  are contained in a bounded domain whose boundary is  $\Sigma(r_1) \cup \Sigma(r_2)$ , the constants  $C_4$  and  $C_5$  only depend on  $S_0$ .

Now assume  $|A|_{\max}(t) \rightarrow \infty$  as  $t \rightarrow T$ . Let

$$f_{\max}(t) = \max_{S_t} f_\sigma , \quad \forall t \in [0, T) . \quad (3.15)$$

Obviously  $f_{\max}(t) \geq |A|_{\max}^2(t)$ , so  $f_{\max}(t) \rightarrow \infty$  as  $t \rightarrow \infty$ . There exists  $T_0 \in (0, T)$

such that when  $t > T_0$  we have the estimate

$$\begin{aligned} \frac{d}{dt} f_{\max} &\leq -\sigma \Theta_0^{2+\sigma} f_{\max}^2 + (4\sqrt{2} + (2+\sigma)(C_4 + \sqrt{2}C_5))\Theta_0^{1+\sigma/2} f_{\max}^{3/2} \\ &\quad + \left( 2\sigma + 8 + (2+\sigma)(C_4 + 2C_5) + \frac{(1+\sigma)(2+\sigma)C_3}{\Theta_0^2} \right) f_{\max} \\ &\leq -\frac{\sigma \Theta_0^{2+\sigma}}{2} f_{\max}^2 . \end{aligned}$$

This is a contradiction since  $df_{\max}/dt \geq 0$ . Therefore  $f_\sigma$  must be uniformly bounded, which implies that  $|A|^2$  must be uniformly bounded.  $\square$

**Proposition 3.13 (Huisken [16, §4]).** *For every natural number  $m$ , we have the following evolution equation:*

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^m A|^2 &= \Delta |\nabla^m A|^2 - 2|\nabla^{m+1} A|^2 + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A \\ &\quad + h \sum_{i+j=m} \nabla^i A * \nabla^j A * \nabla^m A . \end{aligned} \tag{3.16}$$

Furthermore, there exists constant  $\{C(m)\}_{m=1,2,\dots}$  depending only on  $m$  and  $S_0$  such that

$$|\nabla^m A|^2 \leq C(m) , \quad m = 1, 2, \dots , \tag{3.17}$$

are uniformly on  $S_t$  for  $0 \leq t < T$ .

By the above discussion, the constants in Proposition 3.9 and Proposition 3.11–3.13 are independent of time. Now we can prove part one of Theorem 3.2.

**Proof of Theorem 3.2 (Part one).** (cf. [14, 16]) Assume that  $T < \infty$ . Let

$$S_T = \lim_{t \rightarrow T} S_t = \left\{ \lim_{t \rightarrow T} F(x, t) \mid x \in S \right\} . \tag{3.18}$$

We claim that  $S_T$  is a smooth surface which is homeomorphic to  $S$ .

In fact, by Proposition 3.9, the height function  $u$  is uniformly bounded on  $S_t$  for  $t \in [0, T)$ . So (3.18) is well defined. Since  $|A|^2$  is uniformly bounded for  $t \in [0, T)$ , we have

$$\int_0^T \max_{S_t} \left| \frac{\partial}{\partial t} g_{ij} \right| dt \leq C < \infty ,$$

so  $S_T$  is a well defined surface by Lemma 14.2 in [13]. Since  $|\nabla^m A|^2$ ,  $m = 1, 2, \dots$ , are uniformly bounded for  $t \in [0, T)$ ,  $S_T$  is smooth.

Now we consider a new volume preserving mean curvature flow

$$\frac{\partial F}{\partial t} = (h - H)\nu$$

with initial data  $S_T$ . This flow has a short time solution for  $t \in [T, T_1)$ , where  $T_1 > T$ , the detail can be found in [8, §6.7]. This contradicts to the assumption that  $T$  is maximal. Therefore the maximal time  $T$  of the volume preserving mean curvature flow (3.2) must be infinite.  $\square$

### 3.3 Exponential convergence to CMC surfaces

We have proved that the volume preserving mean curvature flow (3.2) has a long time solution. Let

$$S_\infty = \lim_{t \rightarrow \infty} S_t \tag{3.19}$$

be the limiting surface. Obviously  $S_\infty$  has the following properties:

- (i) It is well defined since  $\{S_t\}_{0 \leq t < \infty}$  are contained in a bounded domain of  $M$ .
- (ii) It's also a smooth surface since  $|\nabla^m A|^2$ ,  $m = 0, 1, 2, \dots$ , are uniformly bounded for  $t \in [0, \infty)$ .
- (iii) It's a graph over  $\Sigma$  since  $\Theta$  is uniformly bounded from below for  $t \in [0, \infty)$ .

In this section, we will show that the solution surface  $S_t$  converges exponentially fast to  $S_\infty$  (cf. [6, 16, 17]), although we don't need this fact to prove the existence of the CMC foliation of  $M$ .

**Proposition 3.14.** *Suppose  $(S_t, g(t))$  is a solution to the mean curvature flow (3.2) for  $t \in [0, \infty)$ , then*

$$\limsup_{t \rightarrow \infty} \int_{S_t} |H - h| = 0 . \quad (3.20)$$

Therefore  $S_\infty$  is a surface of constant mean curvature.

**Proof.** Since

$$\frac{d}{dt} |S_t| = - \int_{S_t} (H - h)^2 d\mu < 0 ,$$

where  $|S_t|$  denotes the area of  $S_t$  with respect to the metric  $g(t)$ , then we have

$$\int_0^\infty \int_{S_t} (H - h)^2 d\mu dt \leq |S_0| .$$

On the other hand, by Lemma 3.3 and Lemma 3.4, we have

$$\begin{aligned} \frac{d}{dt} \int_{S_t} (H - h)^2 d\mu &= 2 \int_{S_t} (H - h) \frac{d}{dt} (H - h) d\mu - \int_{S_t} H (H - h)^3 d\mu \\ &= 2 \int_{S_t} (H - h) [\Delta H + (H - h)(|A|^2 - 2)] d\mu \\ &\quad - \int_{S_t} H (H - h)^3 d\mu \\ &= -2 \int_{S_t} |\nabla H|^2 d\mu + 2 \int_{S_t} (H - h)^2 (|A|^2 - 2) d\mu \\ &\quad - \int_{S_t} H (H - h)^3 d\mu , \end{aligned}$$

here we use the identity  $\int_{S_t} (H - h) d\mu = 0$ . By Proposition 3.13 and the inequalities  $|\nabla H| \leq \sqrt{2} |\nabla A|$ , there is a constant  $C_6 < \infty$  depending only on  $S_0$  such that

$$\left| \frac{d}{dt} \int_{S_t} (H - h)^2 d\mu \right| \leq C_6 \quad (3.21)$$

is uniformly for  $t \in [0, \infty)$ . So we have

$$\lim_{t \rightarrow \infty} \int_{S_t} (H - h)^2 d\mu = 0. \quad (3.22)$$

Then for any  $p > 2$ , by the interpolation arguments (cf. [6, §5] for detail), the inequality  $|\nabla^2 H| \leq \sqrt{2} |\nabla^2 A|$  and Proposition 3.13, we have

$$\begin{aligned} \sup_{S_t} |H - h| &\leq C \|\nabla^2 H\|_2^{1/p} \|H - h\|_2^{1/p} \\ &\leq C \left( \int_{S_t} (H - h)^2 d\mu \right)^{1/(2p)} \\ &\rightarrow 0 \quad (\text{as } t \rightarrow \infty). \end{aligned}$$

where  $\|\cdot\|_2 = \|\cdot\|_{L^2(S_t)}$ . So the proposition follows.  $\square$

We say that a surface  $S$  with constant mean curvature is (strictly) *stable* if volume preserving variations of  $S$  in  $M$  increase the area, or equivalently if the second variation operator on  $S$ ,

$$L\phi = -\Delta\phi - (|A|^2 + \overline{\text{Ric}}(\nu, \nu))\phi$$

has only strictly positive eigenvalues when restricted to functions  $\phi$  with  $\int_S \phi d\mu = 0$ .

**Lemma 3.15.** *The limit surface  $S_\infty$  to the volume preserving mean curvature flow (3.2) is strictly stable surface of constant mean curvature.*

The proof can be found in Corollary 4.2 in section 4, but at first we need show that  $S_\infty$  is a leaf of the CMC foliation  $\mathcal{F}$ , then we just choose  $S'$  to be a small volume preserving variation of  $S_\infty$  along the normal vector field on  $S_\infty$  so that  $S'$  is also a graph over  $\Sigma$ .

**Proof of Theorem 3.2 (Part two).** Since  $S_\infty$  is stable, the lowest eigenvalue  $\lambda_\infty$  of the Jacobi operator  $L_\infty$  on  $S_\infty$  is positive, where

$$L_\infty\phi = -\Delta_\infty\phi - (|A_\infty|^2 - 2)\phi ,$$

here  $\Delta_\infty$  is the Laplacian on  $S_\infty$  and  $A_\infty$  is the second fundamental form of  $S_\infty$ . Let  $\lambda_t$  be the lowest eigenvalue of the Jacobi operator  $L$  on  $S_t$ . Then  $\lambda_t \rightarrow \lambda_\infty$  as  $t \rightarrow \infty$ . For any  $0 < \varepsilon < \frac{2}{3}\lambda_\infty$ , there exists  $T > 0$  such that for any  $t > T$  we have

$$|\lambda_\infty - \lambda_t| < \varepsilon \quad \text{and} \quad \sup_{S_t} |H(H - h)| \leq \varepsilon .$$

Therefore, when  $t > T$  we have

$$\frac{d}{dt} \int_{S_t} (H - h)^2 d\mu \leq - (2\lambda_\infty - 3\varepsilon) \int_{S_t} (H - h)^2 d\mu ,$$

which implies

$$\int_{S_t} (H - h)^2 d\mu \leq \left( \int_{S_T} (H - h)^2 d\mu \right) e^{-(2\lambda_\infty - 3\varepsilon)t} .$$

By the same interpolation arguments as above, we know that  $\sup |H - h|$  converges exponentially to zero. Since

$$\left| \frac{\partial F}{\partial t} \right| = |h - H| ,$$

we obtain that  $S_t$  converges exponentially to the limiting surface which has constant mean curvature. So Part two of Theorem 3.2 is proved.  $\square$

## CHAPTER 4

### EXISTENCE OF CMC FOLIATION

We need a lemma of Mazzeo and Pacard which will be useful for proving the uniqueness of the CMC foliation of  $M$ .

**Lemma 4.1 (Mazzeo–Pacard [24]).** *Suppose that  $\mathcal{F}$  is a monotonically increasing CMC foliation in  $(M, \bar{g}_{\alpha\beta})$ , then  $\mathcal{F}$  is unique amongst all CMC foliations whose leaves are diffeomorphic to  $\Sigma$ .*

**Proof of Theorem 1.1.** (1) In the first part, we will prove the existence of the CMC foliation of  $M$ . At first, we can foliate the quasi-Fuchsian 3-manifold  $M$  by the surfaces  $\Sigma(r)$ ,  $r \in \mathbb{R}$ . All of these surfaces, except  $\Sigma \equiv \Sigma(0)$  (the minimal surface), are not surfaces of constant mean curvature. But for each  $r > 0$ , we consider the mean curvature flow (3.2) with initial condition  $S_0(r) = \Sigma(r)$ . By Theorem 3.2, we have a long time solution of (3.2) such that the limiting surface, denoted by  $S_\infty(r)$ , is a smooth surface of (positive) constant mean curvature. For these surfaces  $\Sigma(r)$  with  $r < 0$ , we have the surfaces with (negative) constant mean curvature. We need three steps to prove that the limiting surfaces  $S_\infty(r)$ ,  $r \in \mathbb{R}$ , form a CMC foliation of  $M$ .

**Step 1:** *The limiting surfaces are embedded.* This is obviously since each surface  $S_\infty(r)$  is a graph over the minimal surface  $\Sigma$ .

**Step 2:** *The limiting surfaces are disjoint.* Assume that  $0 < r_1 < r_2$ , we will show that  $S_\infty(r_1) \cap S_\infty(r_2) = \emptyset$ . Consider two volume preserving mean curvature flows (3.2) with initial data  $\Sigma(r_1)$  and  $\Sigma(r_2)$  respectively. Let  $u_1$  and  $u_2$  be the height functions of the surfaces  $S_t(r_1)$  and  $S_t(r_2)$  respectively, then we have

$u_1(x, 0) < u_2(x, 0)$  for all  $x \in S$ . Now we assume that two surfaces  $S_t(r_1)$  and  $S_t(r_2)$  touch for the first time at  $T_0 \in (0, \infty)$  and  $p_0 \in M$ . Recall that the height functions satisfy the evolution equation (3.7). Let  $w = u_2 - u_1$ , then  $w \geq 0$ , and around  $p_0$  we have

$$0 > Lw = \Delta w + \langle \cdot, \nabla w \rangle - \frac{\partial w}{\partial t} ,$$

here we use the fact that  $h_1(t) < h_2(t)$  since  $H(S_t(r_1)) < H(S_t(r_2))$  pointwise, where  $h_1(t)$  and  $h_2(t)$  are the average mean curvature of  $S_t(r_1)$  and  $S_t(r_2)$  respectively. By the strong maximum principle (cf. [11, 27]), this is impossible unless  $w \equiv 0$ . But  $w \equiv 0$  implies  $u_1 \equiv u_2$ , which is also impossible since the flows preserve volume.

This means that  $S_t(r_1)$  and  $S_t(r_2)$  are disjoint all the time, and we claim that  $S_\infty(r_1)$  and  $S_\infty(r_2)$  are disjoint. In fact, if  $S_\infty(r_1)$  and  $S_\infty(r_2)$  are not disjoint, they must be tangent at some point  $p \in S_\infty(r_1) \cap S_\infty(r_2)$ , and  $S_\infty(r_1)$  is below  $S_\infty(r_2)$  by the above discussion, then by Lemma 3.8, we have  $H(S_\infty(r_1)) \geq H(S_\infty(r_2))$ . On the other hand, since  $H$  satisfies the (strictly) parabolic equation

$$\frac{\partial H}{\partial t} = \Delta H + (H - h)(|A|^2 - 2) , \quad (4.1)$$

and  $H(\Sigma(r_1)) < H(\Sigma(r_2))$  pointwise, then by the comparison principle for quasi-linear parabolic equations (cf. [20, Theorem 9.7]), we have  $H(S_t(r_1)) < H(S_t(r_2))$  pointwise for  $t \in [0, \infty)$ . As  $t \rightarrow \infty$ , we get  $H(S_\infty(r_1)) \leq H(S_\infty(r_2))$ . But if  $H(S_\infty(r_1)) = H(S_\infty(r_2))$ , then  $S_\infty(r_1)$  and  $S_\infty(r_2)$  must be coincide by Lemma 3.8. This is impossible, since the volume of the region bounded by  $\Sigma$  and  $S_\infty(r_1)$  is not equal to the volume of the region bounded by  $\Sigma$  and  $S_\infty(r_2)$ . So  $S_\infty(r_1)$  and  $S_\infty(r_2)$  are disjoint.

**Step 3:** *We claim*

$$M = \bigcup_{r \in \mathbb{R}} S_\infty(r) .$$

In fact, according to the proof of Proposition 3.11, for each  $r \neq 0$ ,  $\Sigma \cap S_\infty(r) = \emptyset$ . Let  $Q(r)$  be the domain bounded by  $\Sigma$  and  $S_\infty(r)$ . Since  $\{\Sigma(r)\}_{r \in \mathbb{R}}$  foliate  $M$  and each  $S_\infty(r)$  is the limiting surface of the volume preserving mean curvature flow with initial data  $\Sigma(r)$ , the volume of  $Q(r)$  is a continuous function with respect to  $r$ . Together with the facts that the limiting surfaces are embedded and disjoint, Step 3 is proved.

Therefore  $\mathcal{F} = \{S_\infty(r)\}_{r \in \mathbb{R}}$  forms a CMC foliation of  $M$ .

(2) In the second part, we will prove the uniqueness of the CMC foliation  $\mathcal{F}$  of  $M$ . For this aim, we need prove that the mean curvature of the foliation  $\mathcal{F}$  is monotonically increasing, that is,  $H(S_\infty(r_1)) < H(S_\infty(r_2))$  if  $r_1 < r_2$ .

At first, from the discussion in the Step 2, we know that the mean curvature  $H(S_\infty(r))$  is a nondecreasing function with respect to  $r$ , i.e.  $H(S_\infty(r_1)) \leq H(S_\infty(r_2))$  for  $r_1 < r_2$ . But if they are equal, say  $H(S_\infty(r_1)) = H(S_\infty(r_2)) = \mathcal{H}$ , then the region bounded by  $S_\infty(r_1)$  and  $S_\infty(r_2)$  is foliated by  $\{S_\infty(r)\}_{r_1 \leq r \leq r_2}$  such that each leaf has the same constant mean curvature  $\mathcal{H}$ . We will show this is impossible. In fact, choose  $r_0 \in (r_1, r_2)$  and consider a variation  $X : S \times (-\varepsilon, \varepsilon) \rightarrow M$  of  $S_\infty(r_0)$  along  $N$  so that  $X(\cdot, t) = S_\infty(r_t)$  for each  $t \in (-\varepsilon, \varepsilon)$ . Now let  $A(t)$  denote the area of  $S_\infty(r_t)$  and let  $V(t)$  denote the volume of the domain bounded by  $\Sigma$  and  $S_\infty(r_t)$ , then define a functional  $J : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$  by

$$J(t) = A(t) + \mathcal{H}V(t) .$$

By a standard result for CMC surfaces (see [2, Proposition 2.3]), we must have  $J'(t) \equiv 0$  which implies that  $J(t)$  is a constant, so  $A(t) + \mathcal{H}V(t)$  is a constant for  $t \in (-\varepsilon, \varepsilon)$ . But this is impossible since  $A(t)$  is nondecreasing (recall that  $\text{Area}(\Sigma(r_t))$  is monotonically increasing) and  $V(t)$  is monotonically increasing. By Lemma 4.1, we prove the uniqueness of the CMC foliation  $\mathcal{F}$ .  $\square$

**Corollary 4.2.** *Each leaf of the CMC foliation  $\mathcal{F}$  is a strictly stable CMC surface.*

**Proof.** We claim that for each  $r$  the CMC surface  $S_\infty(r)$  is the least area surface among all the volume preserving variation of  $\Sigma(r)$  along  $N$ . In fact, suppose  $S'$  is a volume preserving variation of  $\Sigma(r)$  along  $N$  and  $\text{Area}(S') \leq \text{Area}(S_\infty(r))$ . Consider the volume preserving mean curvature flow (3.2) with initial surface  $S'$ . By Theorem 3.2, there is a long time solution to this volume preserving mean curvature flow. Let  $S'_\infty$  be the limiting surface, then it is a graph over  $\Sigma$  whose mean curvature is a constant and  $\text{Area}(S'_\infty) \leq \text{Area}(S_\infty(r))$ . We have proved that  $\mathcal{F} = \{S_\infty(r)\}_{r \in \mathbb{R}}$  is a CMC foliation of  $M$ , so there are two surfaces  $S_\infty(r_1)$  and  $S_\infty(r_2)$ , where  $r_1 < r_2$ , which touch  $S'_\infty$  from the below and from the above for the first time respectively. By Lemma 3.8, we have

$$H(S_\infty(r_2)) \leq H(S'_\infty) \leq H(S_\infty(r_1)) .$$

On the other hand, since  $r_1 < r_2$ ,  $H(S_\infty(r_1)) \leq H(S_\infty(r_2))$ . Therefore, we must have

$$H(S'_\infty) = H(S_\infty(r_1)) = H(S_\infty(r_2)),$$

which implies that  $S'_\infty \in \mathcal{F}$ , and therefore  $S'_\infty = S_\infty(r)$ . This also implies that each leaf  $S_\infty(r) \in \mathcal{F}$  is a stable CMC surface. Thus the corollary follows.  $\square$

CHAPTER 5  
A COUNTEREXAMPLE

In this chapter, we will show that Theorem 1.1 is not true for the quasi-Fuchsian 3-manifolds containing minimal surfaces with big principal curvature.

### 5.1 Existence of the surfaces with CMC

We need some results of J. Gomes and R. López (cf. [12, 21]). Let  $\mathbb{H}^3$  be a three-dimensional hyperbolic space of constant sectional curvature  $-1$ . We will work in the Poincaré model of  $\mathbb{H}^3$ , i.e.,

$$\mathbb{H}^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$$

equipped with metric

$$ds^2 = \frac{4(dx^2 + dy^2 + dz^2)}{(1 - r^2)^2},$$

where  $r = \sqrt{x^2 + y^2 + z^2}$ . The hyperbolic space  $\mathbb{H}^3$  has a natural compactification  $\overline{\mathbb{H}^3} = \mathbb{H}^3 \cup S_\infty^2$ , where  $S_\infty^2 = \widehat{\mathbb{C}}$  is the Riemann sphere. Suppose  $X$  is a subset of  $\mathbb{H}^3$ , we call the set  $\partial_\infty X$  defined by

$$\partial_\infty X = \overline{X} \cap S_\infty^2,$$

the *asymptotic boundary* of  $X$ , where  $\overline{X}$  is the closure of  $X$  in  $\overline{\mathbb{H}^3}$ .

Suppose  $G$  is a subgroup  $\text{Isom}(\mathbb{H}^3)$  which leaves a geodesic  $\gamma \subset \mathbb{H}^3$  pointwise fixed. We call  $G$  the spherical group of  $\mathbb{H}^3$  and  $\gamma$  the rotation axis of  $G$ . A surface in  $\mathbb{H}^3$  invariant by  $G$  is called a *spherical surface*. For two circles  $C_1$  and  $C_2$  in  $\mathbb{H}^3$ , if there is a geodesic  $\gamma$  such that each of  $C_1$  and  $C_2$  is invariant by the group of rotations that fixes  $\gamma$  pointwise, then  $C_1$  and  $C_2$  are said to be *coaxial*, and  $\gamma$  is called the *rotation axis* of  $C_1$  and  $C_2$ .

Let  $P_1$  and  $P_2$  be two disjoint geodesic plane in  $\mathbb{H}^3$ . Then  $P_1 \cup P_2$  divides  $\mathbb{H}^3$  in three components. Let  $X_1$  and  $X_2$  be the two of them with  $\partial X_i = P_i$  for  $i = 1, 2$ . Given two subsets  $A_1$  and  $A_2$  of  $\overline{\mathbb{H}^3}$ , we say  $P_1$  and  $P_2$  *separate*  $A_1$  and  $A_2$  if one of the following cases occurs (cf. [21]):

- (i) if  $A_1, A_2 \subset \mathbb{H}^3$ , then  $A_i \subset X_i$  for  $i = 1, 2$ ;
- (ii) if  $A_1 \subset \mathbb{H}^3$  and  $A_2 \subset S_\infty^2$ , then  $A_1 \subset X_1$  and  $A_2 \subset \partial_\infty X_2$ ;
- (iii) if  $A_1, A_2 \subset S_\infty^2$ , then  $A_i \subset \partial_\infty X_i$  for  $i = 1, 2$ .

Then we may define the distance between  $A_1$  and  $A_2$  by

$$d(A_1, A_2) = \sup\{\text{dist}(P_1, P_2) \mid P_1 \text{ and } P_2 \text{ separate } A_1 \text{ and } A_2\}, \quad (5.1)$$

where  $\text{dist}(P_1, P_2)$  is the hyperbolic distance between  $P_1$  and  $P_2$ .

**Lemma 5.1 (Gomes [12]).** *There exists a finite constant  $d_0 > 0$  such that for two disjoint circles  $C_1, C_2 \subset S_\infty^2$ , if  $d(C_1, C_2) \leq d_0$ , then there exists a minimal surface  $\Pi$  which is a surface of revolution and whose asymptotic boundary is  $C_1 \cup C_2$ .*

Let  $C_1$  and  $C_2$  be two disjoint circles on  $S_\infty^2$ , and let  $P_1$  and  $P_2$  be two geodesic planes whose asymptotic boundaries are  $C_1$  and  $C_2$  respectively. Suppose  $C'_1 \subset P_1$  and  $C'_2 \subset P_2$  so that  $C'_1$  and  $C'_2$  are two coaxial circles with respect to the rotation axis of  $C_1$  and  $C_2$ .

**Lemma 5.2 (López [21]).** *Given  $H \in (-1, 1)$ , there exists a constant  $d_H$  depending only on  $H$  such that if  $d(C_1, C_2) \leq d_H$ , then there exists a surface  $\Pi$  contained in the domain bounded by  $P_1$  and  $P_2$  such that*

- $\Pi$  is a surface of revolution whose boundary is  $C'_1 \cup C'_2$ , and

- $\Pi$  is a surface whose mean curvature is equal to  $H$  with respect to the normal pointing to the domain containing the rotation axis of  $C_1$  and  $C_2$ .

**Remark.** In Lemma 5.2, when  $H < 0$ , then there is no such a surface  $\Pi$  if we replace  $C'_i$  by  $C_i$  for  $i = 1, 2$  (cf. [26]).

## 5.2 Detail description of the counterexample

Now we choose four circles  $\{C_i\}_{i=1,\dots,4}$  on  $S_\infty^2$  such that  $d(C_1, C_2)$  and  $d(C_3, C_4)$  are sufficiently small, where  $d(\cdot, \cdot)$  is the distance defined by (5.1). Let  $D_i$  be the geodesic plane in  $\mathbb{H}^3$  such that  $\partial_\infty D_i = C_i$  for  $i = 1, \dots, 4$ . By some Möbius transformation, we may assume that the middle point of the geodesic segment which is perpendicular to both  $D_1$  and  $D_2$  passes through the origin.

For any circle  $C \subset S_\infty^2$ , we may define the distance between the origin  $O$  (or any fixed point) and the circle  $C$  to be the hyperbolic distance between  $O$  and the geodesic plane whose asymptotic boundary is  $C$ . Because of this definition, we may say that the radius of the circle  $C$  is big or small if the distance between  $O$  and  $C$  is small or big.

Let  $\Lambda$  be a closed smooth curve on  $S_\infty^2$ , then cover  $\Lambda$  by finite disks  $\{B_l \subset S_\infty^2\}_{l=1,\dots,N}$  with small radii such that

- each circle  $\partial B_l$  is invariant under the rotation along the geodesic connecting the origin  $O$  and the center of the disk  $B_l$ , which locates at  $\Lambda$ ,
- the radii of disks are small enough so that  $B_l \cap C_i = \emptyset$  for  $l = 1, \dots, N$  and  $i = 1, \dots, 4$ , and

- for each  $l \equiv 1 \pmod{N}$ ,  $\partial B_l$  intersects both  $\partial B_{l-1}$  and  $\partial B_{l+1}$  and no other circle,

then we get a quasi-Fuchsian group  $\Gamma$  which is the subgroup of orientation preserving transformations in the group generated by  $N$  reflections about the circles  $\partial B_1, \dots, \partial B_N$  (cf. [4, Page 263] or [5, Page 149]). The limit set of the quasi-Fuchsian group  $\Gamma$ , denoted by  $\Lambda_\Gamma$ , is around the curve  $\Lambda$ . Let  $S_\infty^2 \setminus \Lambda_\Gamma = \Omega_1 \cup \Omega_2$ , where  $\Omega_1$  contains  $C_1$  and  $C_2$ , while  $\Omega_2$  contains  $C_3$  and  $C_4$ . See Figure 1.

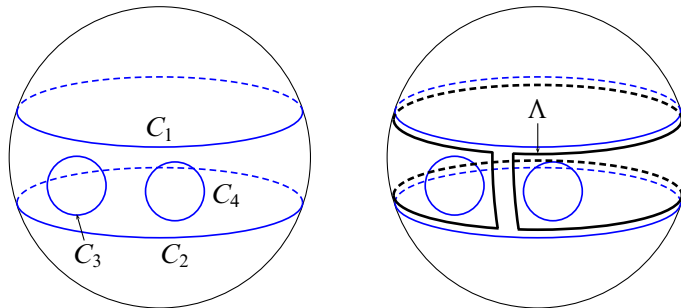


Figure 5.1: Construction of limit set

**Claim:** *The quasi-Fuchsian 3-manifold  $\mathbb{H}^3/\Gamma$  constructed above can not be foliated by surfaces of constant mean curvature.*

Let  $\varepsilon > 0$  be sufficiently small, and let  $H_0 = 2 \tanh \varepsilon$ . Let  $d_0$  and  $d_{H_0}$  be two constants given in Lemma 5.1 and Lemma 5.2, and suppose  $d(C_1, C_2) = 2\varepsilon \ll d_0$  and  $d(C_3, C_4) \ll \min\{d_{H_0}, d_0\}$ .

Now assume that  $\mathbb{H}^3/\Gamma$  is foliated by surfaces of constant mean curvature, where each surface is closed and is homotopic to  $\mathbb{H}^3/\Gamma$ . Lift the foliation to the universal covering space  $\mathbb{H}^3$ , then there should exist a foliation of  $\mathbb{H}^3$  so that each leaf is a disk with constant mean curvature and with the same asymptotic boundary  $\Lambda_\Gamma$ . Notice that any disk type surface in  $\mathbb{H}^3$  with asymptotic boundary  $\Lambda_\Gamma$  divides  $\overline{\mathbb{H}^3}$

into two parts, one of them contains  $C_1$  and  $C_2$ , while the other contains  $C_3$  and  $C_4$ . We choose a normal vector field on the disk type surface so that each normal vector points to the domain containing  $C_1$  and  $C_2$ . Assume that there is a CMC foliation  $\mathcal{F} = \{L_t\}$  with a parameter  $t \in (-\infty, \infty)$  such that

- the leaves are convergent to  $\Omega_1$  as  $t \rightarrow -\infty$  and
- the leaves are convergent to  $\Omega_2$  as  $t \rightarrow \infty$ .

In other words, we have

$$\lim_{t \rightarrow \pm\infty} H(L_t) = \pm 2, \quad (5.2)$$

where  $H(L_t)$  denotes the mean curvature of the leaf  $L_t$  with respect to the normal vector pointing to the domain containing  $C_1$  and  $C_2$ .

Since  $d(C_3, C_4)$  is very small, there exists a minimal surface with asymptotic boundary  $C_3 \cup C_4$  by Lemma 5.1. Consider the leaf  $L_{t'} \in \mathcal{F}$  which touches the minimal surface for the first time, then the mean curvature of  $L_{t'}$  must be positive by the maximal principle. Because of (5.2), there exists  $-\infty < t_1 < t'$  such that the mean curvature of  $L_{t_1}$  is zero, i.e. the leaf  $L_{t_1}$  is a disk type minimal surface. Similarly, we have another leaf  $L_{t_2} \in \mathcal{F}$  which is a disk type minimal surface with asymptotic boundary  $\Lambda$ . See Figure 2.

Let  $X \subset \mathbb{H}^3$  be the domain bounded by  $L_{t_1}$  and  $L_{t_2}$ , then by assumption  $X$  is foliated by  $\{L_t\}_{t_1 \leq t \leq t_2}$ , i.e.

$$X = \bigcup_{t_1 \leq t \leq t_2} L_t.$$

Notice that  $D_3$  and  $D_4$  are disjoint from  $X$ . We choose two circles  $C'_3 \subset D_3$  and  $C'_4 \subset D_4$  so that  $C'_3$  and  $C'_4$  are coaxial with respect to the rotation axis of  $C_3$  and  $C_4$ , by Lemma 5.2 there is a surface  $\Pi_0$  with constant mean curvature  $-H_0$  with

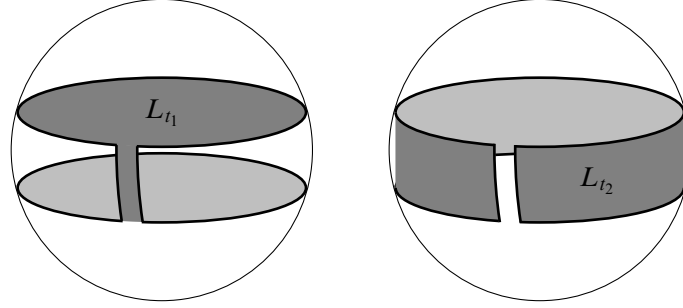


Figure 5.2: Two minimal surfaces

respect to the normal pointing to the domain containing the rotation axis of  $C'_3$  and  $C'_4$ . Obviously  $\Pi_0$  is disjoint from  $L_{t_1}$  but intersects  $L_{t_2}$ . Let  $\Pi'_0 = \Pi_0 \cap X$ . Consider the leaf

$$L_{t''} \in \{L_t \mid t_1 \leq t \leq t_2\}$$

which touches  $\Pi'_0$  for the first time, then  $H(L_{t''}) > H_0$  by the maximal principle. So there exists  $t_3 \in (t_1, t_2)$  such that  $H(L_{t_3}) = H_0$ . We claim that the leaf  $L_{t_3}$  must self-intersect.

Let  $D_1(\varepsilon)$  be the disk bounded by  $C_1$  with  $H(D_1(\varepsilon)) = H_0$  with respect to the normal vector pointing to domain not containing  $C_2$ , and similarly let  $D_2(\varepsilon)$  be the disk bounded by  $C_2$  with  $H(D_2(\varepsilon)) = H_0$  with respect to the normal vector pointing to domain not containing  $C_1$ . Then  $D_1(\varepsilon) \cap D_2(\varepsilon) = \{O\}$ , where  $O \in \mathbb{H}^3$  is the origin. By maximal principle, both  $D_1(\varepsilon)$  and  $D_2(\varepsilon)$  don't intersect  $L_{t_3}$ , so  $L_{t_3}$  must self intersect. This implies that there is no CMC foliation on  $\mathbb{H}^3/\Gamma$ . The claim follows.

Therefore, there exists a quasi-Fuchsian 3-manifold which does not admit CMC foliations.

## BIBLIOGRAPHY

- [1] Michael T. Anderson. Complete minimal hypersurfaces in hyperbolic  $n$ -manifolds. *Comment. Math. Helv.*, 58(2):264–290, 1983.
- [2] J. Lucas Barbosa, Manfredo do Carmo, and Jost Eschenburg. Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. *Math. Z.*, 197(1):123–138, 1988.
- [3] Robert Bartnik. Existence of maximal surfaces in asymptotically flat spacetimes. *Comm. Math. Phys.*, 94(2):155–175, 1984.
- [4] Lipman Bers. Uniformization, moduli, and Kleinian groups. *Bull. London Math. Soc.*, 4:257–300, 1972.
- [5] Lipman Bers. Finite-dimensional Teichmüller spaces and generalizations. *Bull. Amer. Math. Soc. (N.S.)*, 5(2):131–172, 1981.
- [6] Esther Cabezas-Rivas and Vicente Miquel. Volume preserving mean curvature flow in the hyperbolic space. *Indiana Univ. Math. J.*, 56(5):2061–2086, 2007.
- [7] R. D. Canary, D. B. A. Epstein, and P. L. Green. Notes on notes of Thurston. In *Fundamentals of hyperbolic geometry: selected expositions*, volume 328 of *London Math. Soc. Lecture Note Ser.*, pages 1–115. Cambridge Univ. Press, Cambridge, 2006. With a new foreword by Canary.
- [8] Bennett Chow and Dan Knopf. *The Ricci flow: an introduction*, volume 110 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2004.
- [9] Klaus Ecker. Mean curvature flow of spacelike hypersurfaces near null initial data. *Comm. Anal. Geom.*, 11(2):181–205, 2003.
- [10] Klaus Ecker and Gerhard Huisken. Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes. *Comm. Math. Phys.*, 135(3):595–613, 1991.
- [11] Avner Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.
- [12] Jonas de Miranda Gomes. Spherical surfaces with constant mean curvature in hyperbolic space. *Bol. Soc. Brasil. Mat.*, 18(2):49–73, 1987.

- [13] Richard S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.
- [14] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.
- [15] Gerhard Huisken. Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature. *Invent. Math.*, 84(3):463–480, 1986.
- [16] Gerhard Huisken. The volume preserving mean curvature flow. *J. Reine Angew. Math.*, 382:35–48, 1987.
- [17] Gerhard Huisken and Shing-Tung Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Invent. Math.*, 124(1-3):281–311, 1996.
- [18] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva. *Linear and quasi-linear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.
- [19] Olli Lehto. *Univalent functions and Teichmüller spaces*, volume 109 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1987.
- [20] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.
- [21] Rafael López. Hypersurfaces with constant mean curvature in hyperbolic space. *Hokkaido Math. J.*, 29(2):229–245, 2000.
- [22] Albert Marden. The geometry of finitely generated kleinian groups. *Ann. of Math. (2)*, 99:383–462, 1974.
- [23] Bernard Maskit. On boundaries of Teichmüller spaces and on Kleinian groups. II. *Ann. of Math. (2)*, 91:607–639, 1970.
- [24] Rafe Mazzeo and Frank Pacard. Constant curvature foliations on asymptotically hyperbolic spaces, 2007. Available at [arXiv.org:0710.2298](https://arxiv.org/abs/0710.2298).
- [25] William Meeks, III, Leon Simon, and Shing Tung Yau. Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature. *Ann. of Math. (2)*, 116(3):621–659, 1982.

- [26] Oscar Palmas. Complete rotation hypersurfaces with  $H_k$  constant in space forms. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(2):139–161, 1999.
- [27] Murray H. Protter and Hans F. Weinberger. *Maximum principles in differential equations*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1967.
- [28] R. Schoen and Shing Tung Yau. Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature. *Ann. of Math. (2)*, 110(1):127–142, 1979.
- [29] William P. Thurston. *The geometry and topology of three-manifolds*, 1980. Available at <http://www.msri.org/publications/books/gt3m/>.
- [30] Karen K. Uhlenbeck. Closed minimal surfaces in hyperbolic 3-manifolds. In *Seminar on minimal submanifolds*, volume 103 of *Ann. of Math. Stud.*, pages 147–168. Princeton Univ. Press, Princeton, NJ, 1983.
- [31] Xi-Ping Zhu. *Lectures on mean curvature flows*, volume 32 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI, 2002.