STABILIZING BINOMIAL n ESTIMATORS

BU-844-M

June 1984

by

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Abstract

The maximum likelihood estimator of the binomial parameter n can be highly unstable, resulting in huge fluctuations in the estimate when the data are only slightly perturbed. A method of assessing this sensitivity, based on perturbing the log likelihood function, is proposed and examined. It is also pointed out that various stabilized n-estimators are sometimes too stable, and are prone to underestimation. Some alternative procedures are suggested.

KEY WORDS: Maximum likelihood; Method of moments; Unstable estimators.

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1. INTRODUCTION

In some situations, data may be modeled as success counts from a binomial population. In such cases, there is often interest in estimating not only p, the success probability, but also n, the number of Bernoulli trials. The problem of estimating n is much more difficult than that of estimating p. A natural choice, the maximum likelihood estimator, is unsatisfactory because of its over-sensitivity to small perturbations in the data.

The problem of estimating n has a long history, dating back to the works of Student, Fisher, and Haldane. This history is described in detail in Olkin *et al.* (1981), so it will not be repeated here.

Several authors have addressed the problem of obtaining stable estimators of n. Olkin *et al.* (1981) demonstrated the instability of both the maximum likelihood estimator (MLE) and the method of moments estimator (MME), and proposed stable versions of each. More recently, Carroll and Lombard (1983) have investigated the stability of an estimator based on maximizing an integrated maximum likelihood. They demonstrated that their approach results in an estimator whose stability is comparable to those of Olkin *et al.*

Let $x_1, \cdots, x_k$ be observed independent success counts from a binomial(n,p) distribution, and define

$$\bar{x} = \frac{1}{k} \Sigma x_i \quad , \qquad \hat{\sigma}^2 = \frac{1}{k} \Sigma (x_i - \bar{x})^2 \quad . \tag{1.1}$$

As pointed out by Olkin *et al.*, the instability of the MLE and MME occur when $\bar{x}/\hat{\sigma}^2 < 1$. Based on this fact, they construct a stabilized version of the MME, MME:S, given by

$$\text{MME:S} = \max\{\hat{\sigma}^2\varphi^2/(\varphi-1), x_{max}\}$$

where (1.2)

$$\varphi = \begin{cases} \bar{x}/\hat{\sigma}^2 & \text{if} \quad \bar{x}/\hat{\sigma}^2 \geq 1 + 1/\sqrt{2} \\ \max[(x_{max}-\bar{x})/\hat{\sigma}^2,\ 1+\sqrt{2}] & \text{if} \quad \bar{x}/\hat{\sigma}^2 < 1 + 1/\sqrt{2} \end{cases} .$$

Other stabilized estimators were discussed by Olkin *et al.*, but in terms of performance and computation simplicity MME:S is the preferred choice.

Carroll and Lombard take a different approach, using an estimator based on an integrated likelihood. The likelihood function is

$$L(n,p) = \prod_{i=1}^{k} \binom{n}{x_i} p^{\Sigma x_i}(1-p)^{kn-\Sigma x_i} . \tag{1.3}$$

Let p have a density f(p), proportional to $p^a(1-p)^b$, and calculate the integrated likelihood

$$L(n) = \int L(n,p)f(p)dp = \prod_{i=1}^{n} \binom{n}{x_i} \left[(kn+a+b+1)\binom{kn+a+b}{a+\Sigma x_i}\right]^{-1} \tag{1.4}$$

for $n > x_{max}$. The estimate of n, MB(a,b), is obtained by maximizing L(n) as a function of n. Carroll and Lombard investigate (a,b) = (0,0),(1,1).

MME:S, MB(0,0), and MB(1,1) are stable estimators of n: they show no tendency to "blow up," even for small p and small $\bar{x}/\hat{\sigma}^2$. Their performance is very similar for small values of p, with MB(0,0) having a slight edge for larger values of p. All of these estimators, however, suffer from the same deficiency: a distinct tendency to underestimate n, especially when p is small. A heuristic explanation for this phenomenon is the following: since binomial variance is proportional to p(1-p), smaller variance occurs when p is near 0 or 1. If this is the case, the variance of the sample $x_1, \cdots, x_k$ will tend to be small, i.e., the values will be close together. Given such a sample, the estimator must "decide" if the sample represents a population with small p and large n, or one with large p and small n. In almost all cases these estimators decide in favor of large p and small n, resulting in underestimation.

Most performance comparisons are done in terms of relative mean squared error, $E[(\hat{n}-n)/n]^2$. Although this function is probably the most appropriate loss function for this estimation problem, it is quite forgiving of underestimation of large n. Hence, relative mean squared errors can remain quite low even if there is consistent underestimation.

The approach taken here is to obtain a better characterization of the stability of the estimation problem, and to use this information to help choose an appropriate point estimator. To characterize the stability, the log likelihood function is subjected to a systematic perturbation, and the values of the MLE are examined over a range of perturbations. This information can be summarized in a graphical display, and will indicate whether the situation is stable (implying that the MLE itself is a reasonable estimator), or the situation is unstable (implying that a stabilized estimator is more appropriate.

In Section 2 the method of perturbing the log likelihood function is derived, and its applications are described in Section 3. Section 4 contains evaluations of some point estimators of n, and Section 5 is a summary.

2. AN APPROXIMATE LIKELIHOOD FASHION

If $x_1, \cdots, x_k$ are observed independent success counts from a binomial(n,p) population, the log likelihood function is given by

$$\ell(n,p) = \sum_{i=1}^{k} \ell og\binom{n}{x_i} + k\bar{x}\ell ogp + k(n-\bar{x})\ell og(1-p) \quad , \tag{2.1}$$

where $\bar{x} = (1/k)\Sigma x_i$. The instability of $\ell(n,p)$, or its sensitivity to small perturbations in the data, is mainly caused by the first term in (2.1). The approach here is to perturb the first term of (2.1) in a systematic manner, and examine the range of the resulting maximum likelihood estimators.

The function $\ell ogy!$ can be bounded above and below (Feller, 1968) by

$$y\ell ogy - y \le \ell ogy! \le (y+1)\ell og(y+1) - y \tag{2.2}$$

Therefore, it follows that for some α, $0 < \alpha < 1$, we have

$$\ell ogy! \approx (1-\alpha)y\ell ogy + \alpha(y+1)\ell og(y+1) \quad . \tag{2.3}$$

In fact, for α near 1/2, (2.3) is a reasonably accurate approximation even for small values of y. We will use (2.3) to get a family of approximations of $\ell(n,p)$, indexed by α.

For notational convenience, define the function $h_\alpha(y)$ by

$$h_\alpha(y) = (1-\alpha)y\ell ogy + \alpha(y+1)\ell og(y+1) \quad . \tag{2.4}$$

Now, in (2.1), substitute $h_\alpha(n)$ for $\ell ogn!$ and $h_{1-\alpha}(n-x_i)$ for $\ell og(n-x_i)!$ to obtain

$$\ell_\alpha(n,p) = kh_\alpha(n) - \sum_{i=1}^{k} h_{1-\alpha}(n-x_i) - \sum_{i=1}^{n} \ell ogx_i! + k\bar{x}\ell ogp$$
$$+ k(n-\bar{x})\ell og(1-p) \quad , \tag{2.5}$$

the approximate likelihood function. Note that, as α varies from 0 to 1, $\ell_\alpha(n,p)$ strictly increases in α, moving from below $\ell(n,p)$ to above $\ell(n,p)$. Thus, we have an envelope of functions surrounding $\ell(n,p)$.

For α near 1/2, $\ell_\alpha(n,p)$ is very close to $\ell(n,p)$. Thus, by examining $\ell_\alpha(n,p)$ for a range of α values near 1/2, we can examine maximum likelihood estimators for a range of slightly perturbed likelihood functions. Note that the α-perturbation affects not only n and x_i, but also the form of the likelihood, but at all times results in a function that is close to $\ell(n,p)$. Thus, although the manner in which the sample is being perturbed is not explicit, the perturbed statistical problem is close to the original.

If we now treat $\ell_\alpha(n,p)$ as a likelihood function, we can obtain $\hat{n}_\alpha$ and $\hat{p}_\alpha$, the maximum likelihood estimators based on $\ell_\alpha(n,p)$. It is easy to see that $\hat{p}_\alpha = \bar{x}/\hat{n}_\alpha$, and differentiating $\ell_\alpha(n,p)$ with respect to n and doing some algebra will verify that $\hat{n}_\alpha$ is the solution to

$$\frac{\partial}{\partial n}\ell_\alpha(n,p) = \ell og\left[\frac{n^{(1-\alpha)k}(n+1)^{\alpha k}[\Sigma(n-x_i)/kn]^k}{\Pi(n-x_i)^\alpha \Pi(n-x_i+1)^{1-\alpha}}\right] = 0 \quad . \tag{2.6}$$

Before going further, we first investigate the conditions for a solution to (2.6). A little more algebra will show that

$$\frac{\partial}{\partial n}\ell_\alpha(n,p) = \alpha k\ell og\left[\left(\frac{n+1}{n}\right)^2\frac{\Pi(n-x_i+1)^{1/k}}{\Pi(n-x_i)^{1/k}}\right] + k\ell og\left(\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i+1)^{1/k}}\right) \quad , \tag{2.7}$$

from which it can be seen that $(\partial/\partial n)\ell_\alpha(n,p)$ is a linear, increasing function of α, so for each n, there is either a unique α for which (2.6) is satisfied, or else there is no α satisfying (2.6). It then follows that for fixed α, $\ell_\alpha(n,p)$ either has a unique finite root or no finite root.

It is clear that the first term in (2.7) is positive for $\alpha > 0$ and $n > x_{max}$, so for each n, $(\partial/\partial n)\ell_\alpha(n,p)$ will have a root if and only if

$$\ell og\left[\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i+1)^{1/k}}\right] < 0$$

and (2.8)

$$\ell og\left[\left(\frac{n+1}{n}\right)\frac{\Pi(n-x_i+1)^{1/k}}{\Pi(n-x_i)^{1/k}}\right] > \ell og\left[\frac{\Pi(n-x_i+1)^{1/k}}{\Sigma(n-x_i)/k}\right] .$$

The second inequality in (2.8) can be rearranged to

$$\ell og\left[\left(\frac{n+1}{n}\right)\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i)^{1/k}}\right] > 0 . \tag{2.9}$$

From the arithmetic-geometric mean inequality, it follows that (2.9) is always satisfied, so the first condition in (2.8) is necessary and sufficient for a finite root. The first two derivatives of this function are

$$\frac{\partial}{\partial n}\ \ell og\left[\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i+1)^{1/k}}\right] = \frac{k}{\Sigma(n-x_i)} - \frac{1}{k}\ \Sigma(n-x_i+1)^{-1}$$

(2.10)

$$\frac{\partial^2}{\partial n^2}\ell og\left[\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i+1)^{1/k}}\right] = \frac{1}{k}\ \Sigma(n-x_i+1)^{-2}\ \frac{-k^2}{[\Sigma(n-x_i)]^2} .$$

Evaluating the second derivative at the zeros of the first derivative yields

$$\frac{\partial^2}{\partial n^2}\ \ell og\left[\frac{\Sigma(n-x_i)/k}{\Pi(n-x_i+1)^{1/k}}\right] = \frac{1}{k}\left[\Sigma(n-x_i+1)^{-2} - \frac{1}{k}\left(\Sigma(n-x_i+1)^{-1}\right)^2\right] > 0 .$$

Thus, this function has a unique minimum and, since its limit is 0 as $n \to \infty$, once it crosses the axis it remains negative. The function may be positive as $n \to x_{max}$, but it must cross the axis at some point.

Summing up, we can conclude that for each n, (2.6), considered as a function of α, either has a unique finite root or no root. If there is no root, then $(\partial/\partial n)\ell og_{\alpha}(n,p)$ is increasing in n. Therefore, for fixed α, there is a unique root $\hat{n}_{\alpha}$ of (2.6).

Now that the behavior of the approximate likelihood function is known, the plan is to examine the variation in $\hat{n}_{\alpha}$ for a range of values of α. In doing this we can get some idea of the stability of the MLE, and also examine a range of estimates of n.

3. ASSESSING STABILITY

An easy method of assessing the stability of the estimation problem is to examine the estimates $\hat{n}_\alpha$, the solutions to (2.6), for a range of α values. Informally speaking, we would consider the problem to be stable if $\hat{n}_\alpha$ was not overly sensitive to changes in α.

For $\alpha = \frac{1}{2}$, the approximate likelihood function, given in (2.5), is reasonably close to the likelihood function in (2.1), and $\hat{n}_{\frac{1}{2}}$ is reasonably close to $\hat{n}$. This statement is mainly based on empirical evidence, and is supported by Table 1, which gives values of $\ell ogn!$ and $\frac{1}{2}[n\ell ogn + (n+1)\ell og(n+1)]$ for a range of values of n.

To take advantage of the relationship between α and n obtained by setting (2.7) equal to zero, which gives α explicitly as a function of n, the plots are constructed in the following manner: First, solve for $\hat{n}_{\frac{1}{2}}$, then take a range of values of n surrounding $n_{\frac{1}{2}}$. For each of these values of n, solve for α in (2.7), and then plot α and n. Constructing the graph in this manner is not only computationally easier, but also solves the problem of choosing increments for α. Since $\hat{n}_\alpha$ may change very rapidly with α, one may often choose α increments that are too large to be of practical value.

In Figures 1 and 2 we present a series of plots demonstrating how the stability of the maximum likelihood estimator can be portrayed. In Figure 1a, a plot is given for the sample (16,18,22,25,27), one of the samples considered by Olkin *et al.* (1981). The extreme instability of the sample is illustrated by the nearly vertical slope of the graph for $\alpha \approx \frac{1}{2}$. The ratio of the sample mean $\bar{x}$, to the sample variance, $\hat{\sigma}^2 = (1/k)\Sigma(x_i - \bar{x})^2$, is 1.267, also indicating instability by the criterion of Olkin *et al.*, who consider a sample unstable if $\bar{x}/\hat{\sigma}^2 > 1 + (1/2)^{\frac{1}{2}} \approx 1.707$. If we perturb the sample as in Olkin *et al.*, by adding 1 to the largest observation, we obtain Figure 1b, which displays even greater instability than the original sample in showing a much steeper slope ($\bar{x}/\hat{\sigma}^2=1.126$). Thus, in each of these cases the MLE is extremely unstable, allowing small perturbations to result in large changes.

The sample of Figure 1a, (16,18,22,25,27) was generated from a binomial population with n=75 and p=.32. To get some idea of the behavior of the plots and their relationship to the quantity $\bar{x}/\hat{\sigma}^2$, more samples were generated from this binomial population. Plots for three of these samples are shown in Figures 2a-2c in order of increasing stability, according to both the graphical criterion and $\bar{x}/\hat{\sigma}^2$. Two distinct points emerge from these plots:

1. Samples with $\bar{x}/\hat{\sigma}^2 < 1 + (1/2)^{\frac{1}{2}}$ can be stable, as illustrated in Figure 2a, where $\bar{x}/\hat{\sigma}^2 = 1.513$.
2. As the sample becomes more stable, there is a tendency for the MLE, $\hat{n}$, to underestimate n. The intuition behind this is simple: A stable sample will have observations relatively close together, and the MLE will tend to treat such a situation as one of small n and large p, resulting in an underestimate of n.

To further investigate the question of whether or not samples with $\bar{x}/\hat{\sigma}^2 < 1 + (1/2)^{\frac{1}{2}}$ are stable, samples of size 10 were drawn from the same binomial population (n=75, p=.32). Graphs of $\hat{n}_\alpha$, for two of these samples, are shown in Figure 3. In both cases, even though $\bar{x}/\hat{\sigma}^2 < 1 + (1/\sqrt{2})^{\frac{1}{2}}$, the MLE is reasonably stable. Thus, the Olkin *et* *al.* criterion will, as the sample size increases, tend to classify stable samples as unstable. This could lead to a systematic underestimation of n.

Olkin *et al.* recognize that their criterion is deficient in not taking the sample size into account, but offer no alternatives. The graphs of $\hat{n}_\alpha$ vs. α are an alternative. Although the graphs do not provide a "0-1" classification as "stable" or "unstable," they do take the sample size into account, and offer an assessment of the worth of the MLE.

4. POINT ESTIMATION OF n

In one sense, it is ludicrous to attempt to provide a point estimate of n. Maybe one should just concede that the problem is too hard, and be satisfied with choosing a range of n values from the plots of Section 3. This is, perhaps, begging the question, but in practice it might prove to be most useful.

A point estimate of n, or a range of point estimates, can be read off the plots of Section 3 (*a la* ridge trace), or one may take a weighted average of values of $\hat{n}_\alpha$ for α near $\frac{1}{2}$. This second method was explored in a Monte Carlo study. Specifically, we calculated

$$\hat{n}_{avg} = \sum_\alpha |\tfrac{1}{2}-\alpha|\hat{n}_\alpha \Big/ \sum_\alpha |\tfrac{1}{2}-\alpha| \qquad (4.1)$$

for 10 values of α on either side of $\alpha = \frac{1}{2}$. The performance of this estimator was quite similar to those of Olkin *et al.* and Carrol and Lombard. Performance, in terms of relative mean squared error, is summarized in Table 2, which summarizes the results of 500 simulations. (The simulations were carried out in the same manner as Olkin *et al.*: n, p, and k were all generated from uniform distributions, $1 \le n \le 100$, $0 < p < 1$, $3 \le k \le 22$.)

In terms of relative mean squared error, there is little to separate the estimators of Table 2. Indeed, in all performance characteristics these estimators seem remarkably similar, so one may be inclined to choose MME:S based on ease of calculation.

However, all of these estimators suffer from an underestimation problem: In general, for large values of n, these estimators will consistently underestimate the true value, with the problem becoming worse as p becomes smaller.

In terms of relative mean squared error, $(\hat{n}-n)^2/n^2$, underestimation of large n is not severely penalized. Thus, it is possible for an estimator to perform well against this loss while consistently underestimating the parameter.

To better understand this problem, Figure 4 displays the relative error, $(\hat{n}-n)/n$ of MME:S. Thus, the sign of the error is preserved, and the bias in the estimator is clearly evident. (Equivalent figures for $\hat{n}_{avg}$ and MB(0,0) are similar, and are not presented.) It is evident that in stabilizing the n-estimators, the stabilization has perhaps been taken too far. The result is an estimator which seems to be too stable.

There are many ways around this problem, i.e., many ways to "destabilize" these estimators. One may alter the cutoff points of MME:S, or choose a different weighting scheme for $\hat{n}_{avg}$, one that gives more weight to larger values. This second method was investigated (through Monte Carlo trials) and although satisfactory results were obtained, the resulting estimator seemed to be too *ad hoc* – there was no way of justifying it statistically.

The same argument is true for MME:S; not so, however, for the family of estimators MB(a,b) of Carroll and Lombard. Since the parameters (a,b) have meaning in terms of prior knowledge of p, this family of estimators seemed to be the most statistically pleasing.

Carroll and Lombard argue that by downweighting smaller values of p, a stable estimator is obtained. While this is true, one shouldn't forget that MB(a,b) has a Bayesian interpretation, and the values of (a,b) can be chosen to reflect outside knowledge of p. In particular, if it is felt that p is small (the case where underestimation is most prevalent) then the values of (a,b) can be chosen to reflect this knowledge and, effectively, destabilize the estimator.

Thus, a search through values of (a,b) was undertaken to find values that would both i) put more emphasis on smaller values of p and ii) eliminate some of the underestimation problem. The value selected was (a,b) = (-1,0) which, as Carroll and Lombard point out, is equivalent to maximizing the conditional likelihood for n given Σx_i. Although one may call this estimator unstable, it is far more stable, for example, than the MLE. It also, to a certain degree, cures the underestimation problem. Figure 5 illustrates the performance of MB(-1,0) in terms of relative error, and Table 3 gives relative mean squared error comparisons with MME:S.

5. SUMMARY

The problem of obtaining reasonable n-estimators for a binomial population can be divided into two parts:

1. Identification of stable vs. unstable cases.
2. Selection of a reasonable point estimator.

The first half of this procedure is effectively dealt with by the plots of Section 3. These plots are superior to the cutoff method of Olkin *et al.* (1981) because the sample size is also considered. They suffer somewhat in that they are subject to interpretation, but they provide a range of estimates which result from slightly perturbed problems, with the interpretation that the smaller the range, the more stable the problem.

If the problem is identified as stable, then all the estimators described here perform equally well. In unstable (small p) cases, however, stabilized n-estimators tend to underestimate n, so less stable alternatives, such as MB(-1,0), are suggested.

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Table 1. Values of ℓogn! (Exact) and $\frac{1}{2}$[nℓogn+(n+1)ℓog(n+1)] (Approximate).

n	Exact / Approx.	n	Exact / Approx.	n	Exact / Approx.	n	Exact / Approx.
1	0.0	15	27.9	70	230.4	200	863.2
	-.3		27.5		230.0		862.8
2	.7	20	42.3	80	273.7	300	1414.9
	.3		41.9		273.3		1414.5
3	1.8	25	58.0	90	318.2	400	2000.5
	1.4		57.6		317.7		2000.1
4	3.2	30	74.7	100	363.7	500	2611.3
	2.8		74.2		363.3		2610.9
5	4.8	35	92.1	110	410.3	600	3242.3
	4.4		91.7		409.9		3241.9
6	6.6	40	110.3	120	457.8	700	3890.0
	6.2		109.9		457.4		3889.5
7	8.5	45	129.1	130	506.1	800	4552.0
	8.1		128.7		505.7		4551.5
8	10.6	50	148.5	140	555.2	900	5226.5
	10.2		148.1		554.8		5226.1
9	12.8	55	168.3	150	605.0	1000	5912.1
	12.4		167.9		604.6		5911.7
10	15.1	60	188.6	160	655.5	1100	6607.8
	14.7		188.2		655.1		6607.4

Table 2: Square Root of Relative Mean Squared Errors.

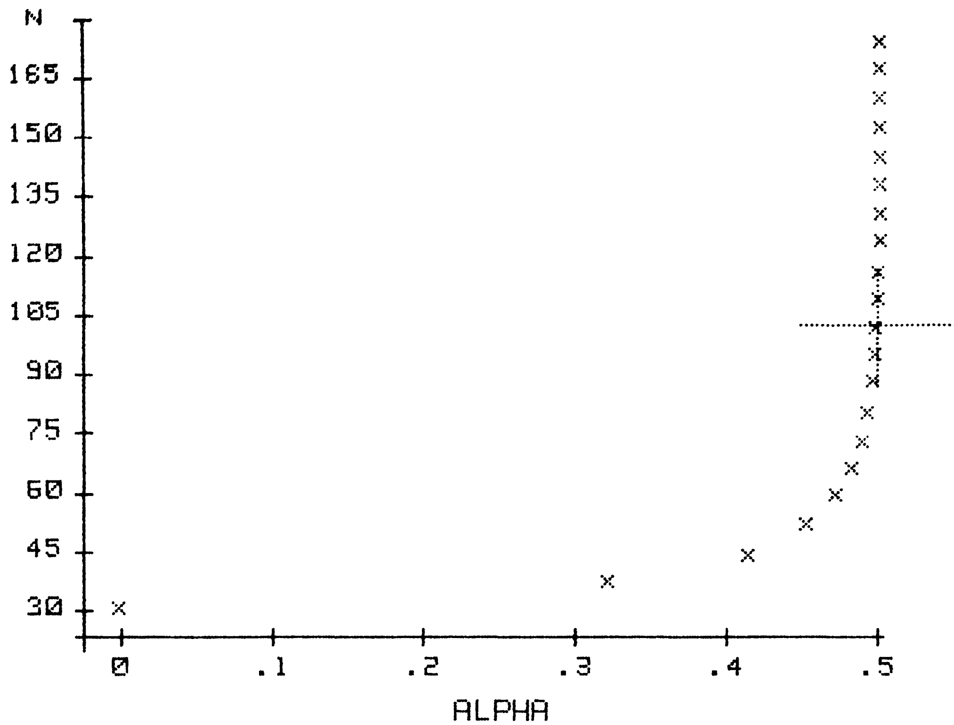
(A case is classified as unstable if $\bar{x}/\hat{\sigma}^2 < 1 + 1/\sqrt{2}$.)

		$\hat{n}_{avg}$	MB(0,0)	MME:S
Stable Cases	369 (74%)	.349	.342	.343
Unstable Cases	131 (26%)	.632	.540	.582
Total Cases	500	.441	.403	.419

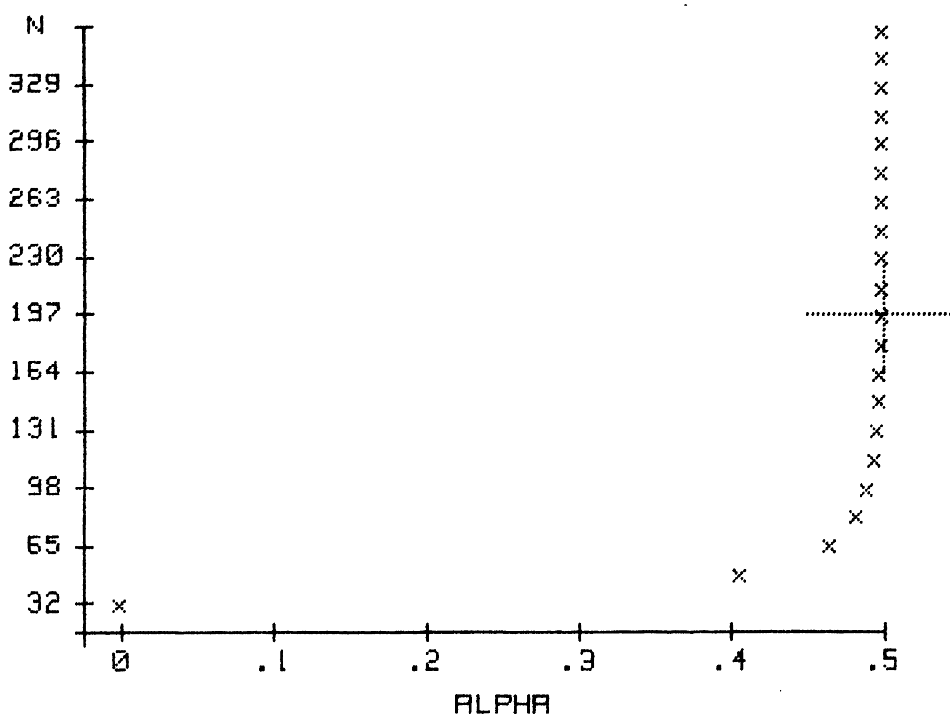
Table 3. Square Root of Relative Mean Squared Errors.

(A case is classified as unstable if $\bar{x}/\hat{\sigma}^2 < 1 + 1/\sqrt{2}$.)

		MB(-1,0)	MME:S
Stable Cases	240 (69%)	.323	.299
Unstable Cases	110 (31%)	2.553	.675
Total Cases	350	1.431	.452

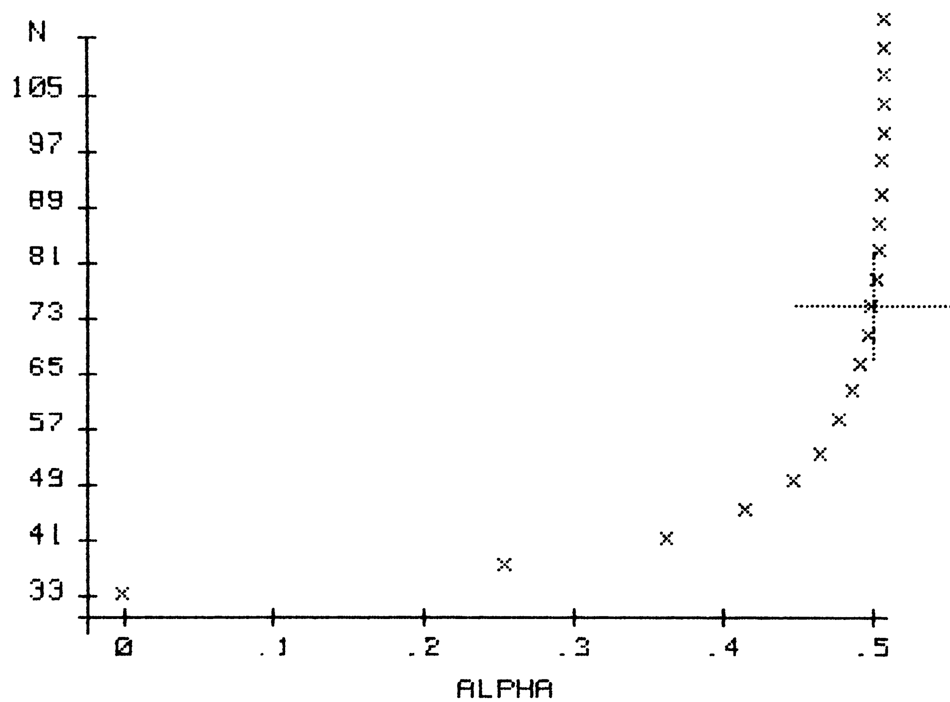


a. Sample = (16,18,22,25,27), $\bar{x}/\hat{\sigma}^2 = 1.267$

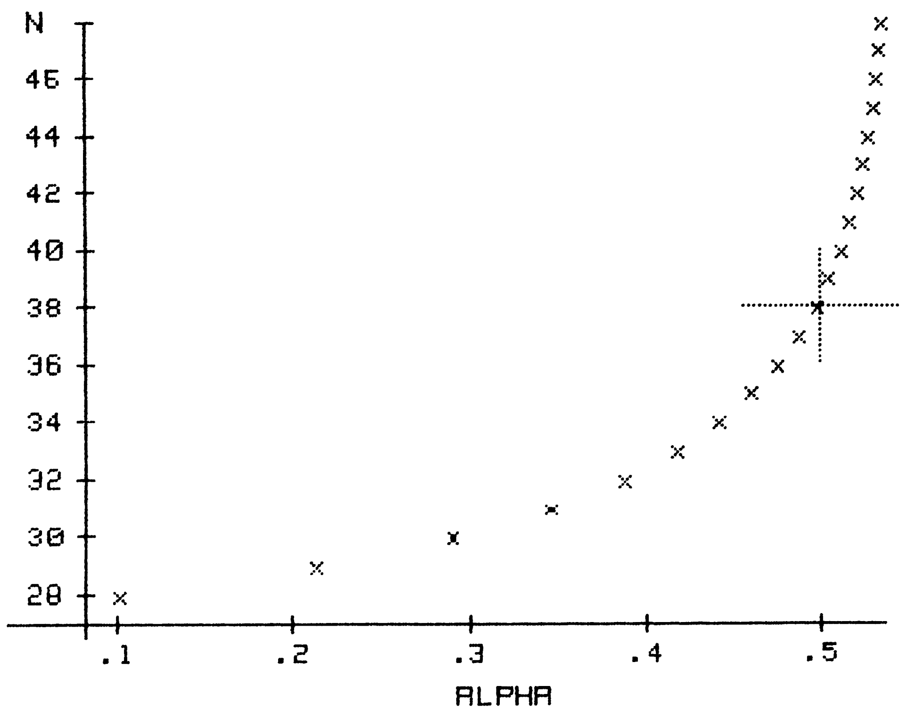


b. Sample = (16,18,22,25,28), $\bar{x}/\hat{\sigma}^2 = 1.126$

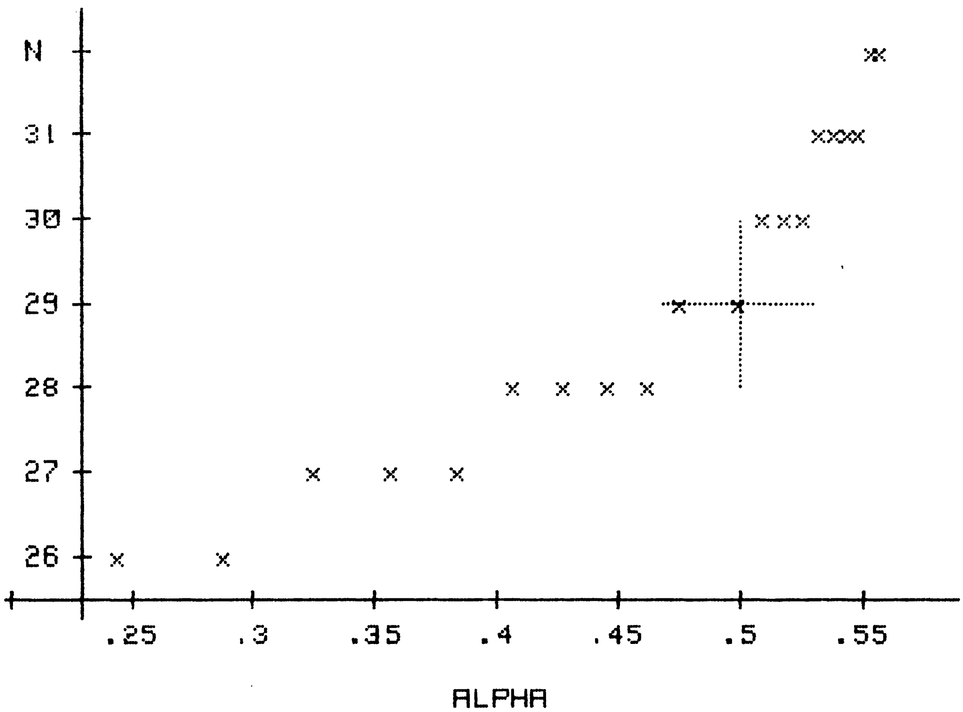
Figure 1. Plots of $\hat{n}_\alpha$ vs. α, $\hat{n}_{\frac{1}{2}}$ (approximate maximum likelihood estimate) marked by cross. Samples from binomial population with n=75, p=.32.



a. Sample = (18, 22, 22, 23, 30), $\bar{x}/\hat{\sigma}^2 = 1.513$

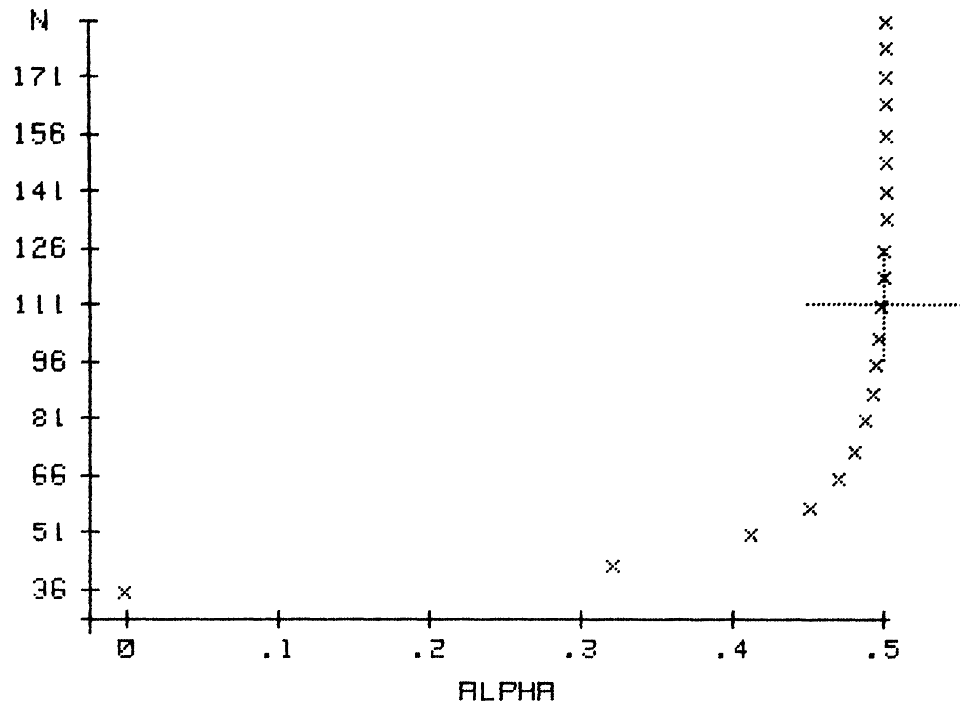


b. Sample = (18, 23, 24, 26, 27), $\bar{x}/\hat{\sigma}^2 = 2.398$

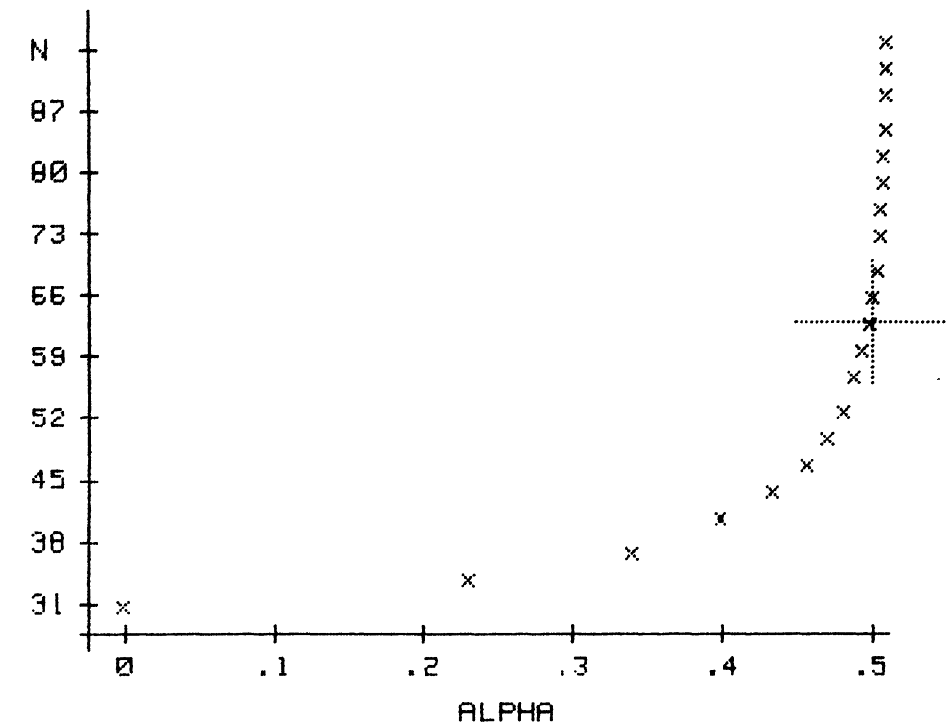


c. Sample = (20, 20, 24, 25, 25), $\bar{x}/\hat{\sigma}^2 = 4.254$

Figure 2. Plots of n_α vs. α, $n_{\frac{1}{2}}$ (approximate maximum likelihood estimate) marked by cross. Samples from binomial population with n=75, p=.32.



a. Sample = (18,18,21,21,23,24,25,25,27,33), $\bar{x}/\hat{\sigma}^2 = 1.302$



b. Sample = (18,18,19,22,23,24,24,25,29,29), $\bar{x}/\hat{\sigma}^2 = 1.594$

Figure 3. Plots of $\hat{n}_\alpha$ vs. α for samples of size 10 drawn from a binomial population with n=75, p=.32.

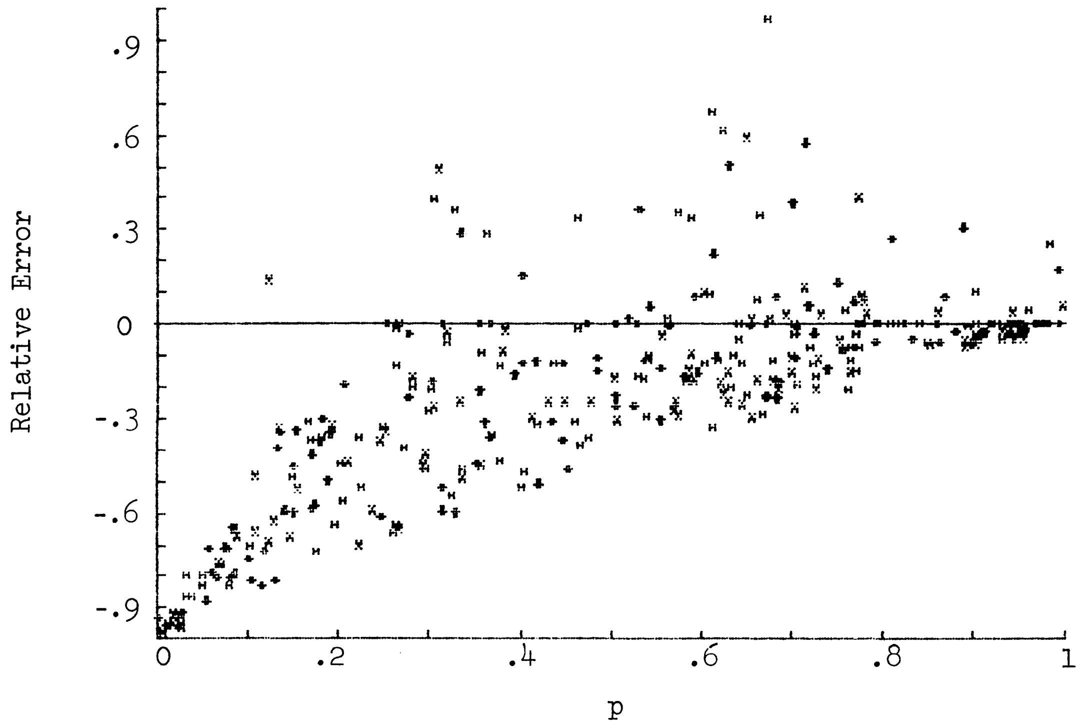


Figure 4a. Relative Error of MME:S vs. p, 350 cases

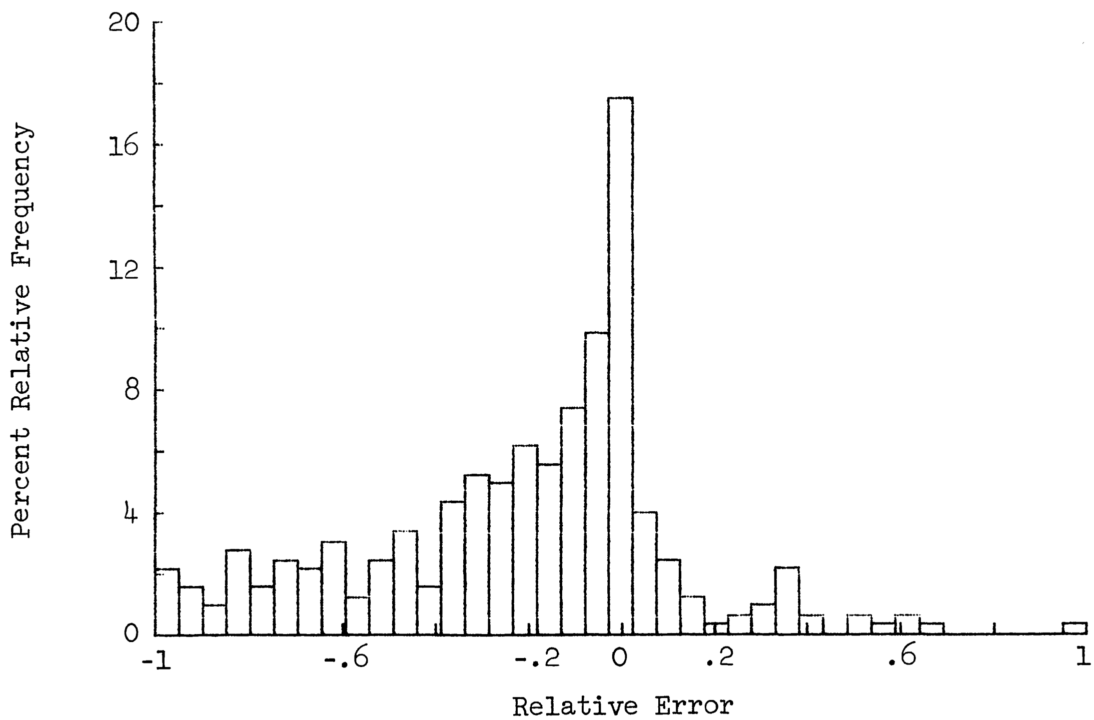


Figure 4b. Distribution of Relative Error of MME:S, 350 cases

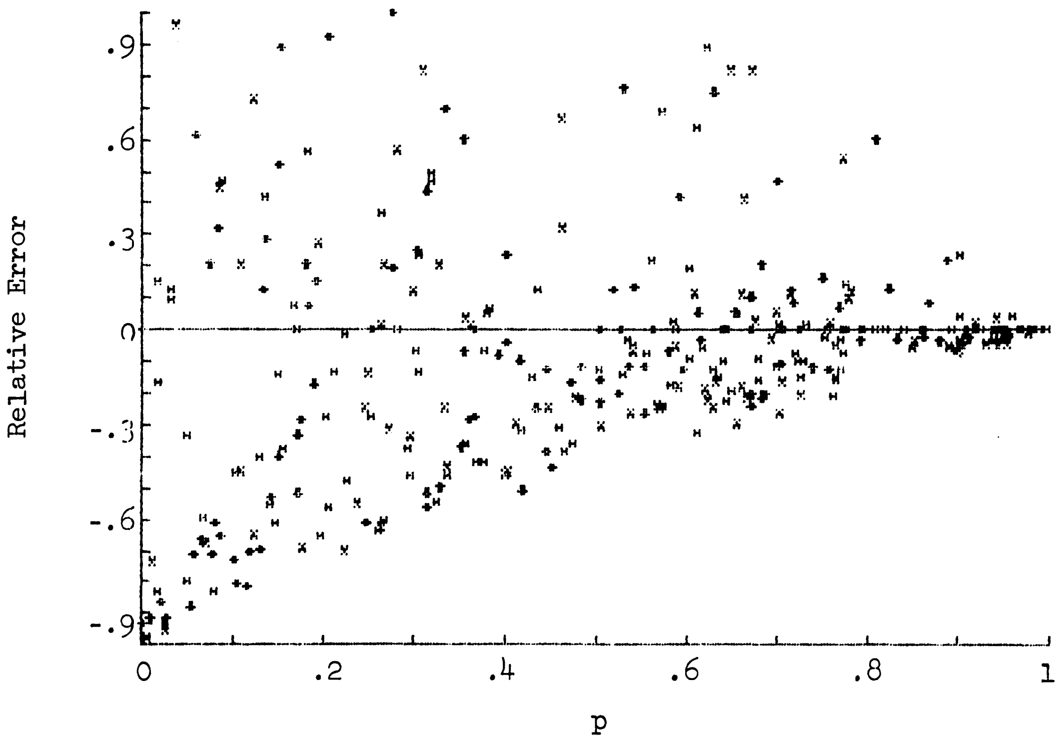


Figure 5a. Relative Error of MB(-1,0) vs. p, 350 cases

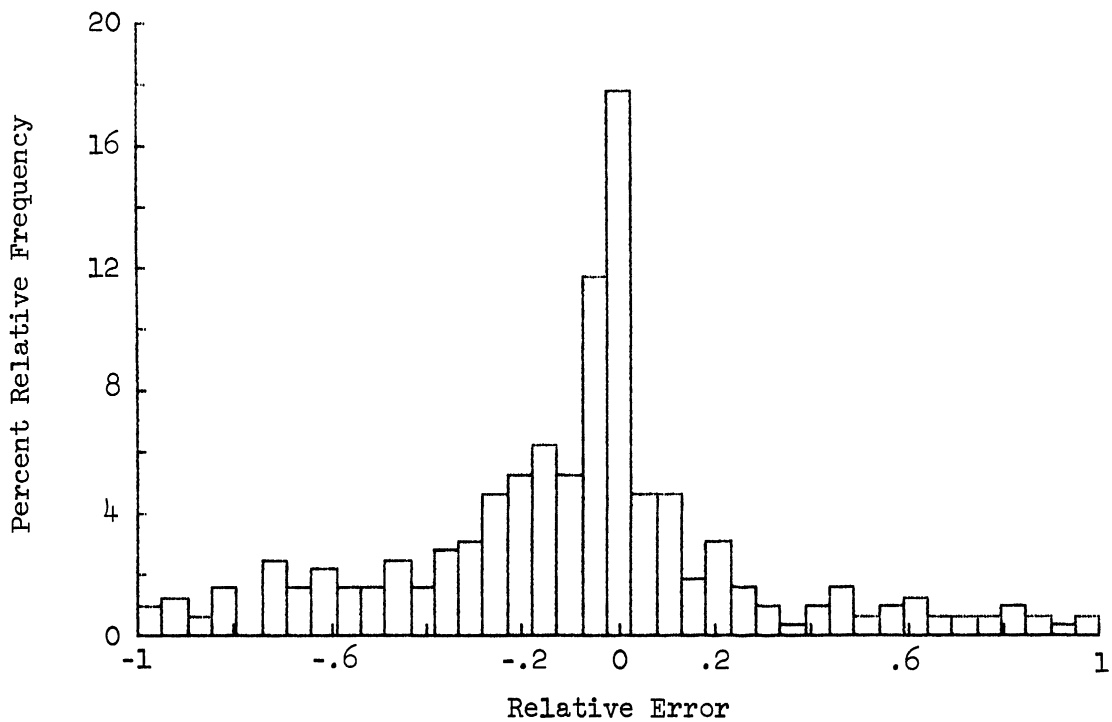


Figure 5b. Distribution of Relative Error of MB(-1,0), 350 cases