FULL EXCEPTIONAL COLLECTIONS OF VECTOR BUNDLES ON LINEAR GIT QUOTIENTS

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Given a reductive group G, a linear G-representation X, and a choice of G-linearized line bundle on X, Geometric Invariant Theory (GIT) produces an open subset $X^{\mathrm{ss}} \subset X$ and a quotient X^{ss}/G. The key motivation of the work in this thesis is to determine when the derived category of coherent sheaves on the GIT quotient X^{ss}/G admits a full exceptional collection consisting of vector bundles.

A full exceptional collection is an important structure on a derived category with many valuable implications. For instance, such a collection produces a basis for the Grothendieck group and the Hochschild Homology of the derived category.

Using ideas from local cohomology and equivariant geometry, we produce a large class of linear GIT quotients with G of rank 2 that admit a full exceptional collection. These vector bundles will come from irreducible G-representations whose weights lie in a particular "window" in the weight space of G. When G has higher rank, we produce a finite list of tautological vector bundles that generate the derived category. These vector bundles do not form a full exceptional collection, but their classes still generate the Grothendieck group.

BIOGRAPHICAL SKETCH

Kimoi Kemboi was born and raised in Baringo, Kenya. She obtained her bachelor's degree in Mathematics from the University of Texas at Arlington and later completed her doctoral studies in Mathematics at Cornell University. Her research focuses on Algebraic Geometry, specifically the intersection of derived categories, equivariant geometry, and moduli theory.

I dedicate this thesis to my parents, Jepkorir and Kipkemoi, for their unwavering dedication and sacrifices that enabled me to get an education.

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CHAPTER 1

INTRODUCTION

The bounded derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ of an algebraic variety X is the category obtained from bounded complexes of coherent sheaves by formally inverting those morphisms that induce an isomorphism on cohomology. While the derived category was initially developed as a book-keeping machine for the homological algebra of sheaves, it has become an essential invariant. Indeed, Bondal and Orlov [BO2] showed that one can reconstruct any smooth projective variety from its derived category when its canonical divisor is ample or anti-ample. Derived categories can also illuminate many unexpected links in the geometry of varieties. One example illustrating this is the D-equivalence conjecture [BO1, K2], which predicts an equivalence between the derived categories of any two Calabi-Yau varieties related by a flop. This conjecture was inspired by Kontsevich's formulation of homological mirror symmetry [K5], and it has led to many fruitful connections between derived categories and, for instance, physics, symplectic geometry, and birational geometry. Thus derived categories have become important objects of study.

One of the ways to study $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is via full exceptional collections. A full exceptional collection on a variety X is an ordered collection $E_1, \ldots, E_n$ of so-called "exceptional" objects in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ (see Definition 2.6.3) that satisfy the vanishing condition $\mathrm{Ext}^m(E_i, E_j) = 0$ for all m whenever $i > j$, and generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, i.e., every object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ can be built out of the E_i's by taking direct sums, cones, and shifts of complexes.

If it exists, this structure has many striking implications $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Indeed, an immediate consequence of having a full exceptional collection is that the classes

$[E_1], \ldots, [E_n]$ give a basis of the Grothendieck group of coherent sheaves on X. If, in addition, the collection is *strong*, i.e., $\mathrm{Ext}^m(E_j, E_i) = 0$ for $m \neq 0$ whenever $i > j$, and the object $\mathcal{E} := E_1 \oplus \ldots \oplus E_n$ is a vector bundle, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is equivalent to the derived category of finitely generated modules over the finite-dimensional algebra $\mathrm{End}(\mathcal{E})$, a significant simplification.

The first example of such a collection came from Beilinson [B] who showed that the line bundles $\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)$ form a full strong exceptional collection on $\mathbb{P}^n$. Beilinson's example led to many questions connecting the existence of full exceptional collections to the geometry of the underlying space. For example, Orlov asked whether a full exceptional collection on a smooth projective variety implied its rationality. Over a general field, this has been verified in the toric case [BDLM]. Dubrovin [D] also used Beilinson's collection to show the semi-simplicity of quantum cohomology of $\mathbb{P}^n$ and conjectured therein that a smooth projective Fano variety has a full exceptional collection if and only if its quantum cohomology is generically semi-simple. The semi-simplicity of the big quantum cohomology is fundamental to the study of genus-zero Gromov-Witten invariants [BM]. In the homological mirror symmetry picture, full exceptional collections consisting of line bundles on toric Fano surfaces are shown in [J] to correspond to Landau-Ginzburg models under the mirror symmetry functor for toric Fano varieties [A]. The implications of these questions highlight the importance of finding smooth varieties (and, more generally, smooth orbifolds) that admit a full exceptional collection.

1.1 A landscape of examples

Beilinson's collection on $\mathbb{P}^n$ inspired many generalizations. For instance, thinking of $\mathbb{P}^n$ as a homogeneous space leads naturally to the question of which homogeneous spaces of semi-simple algebraic groups admit a full (strong) exceptional collection. In this direction, Kapranov [K1] extended the methods of [B] to show that on the Grassmannian $\mathrm{Gr}(r,n)$ of r-planes in $\mathbb{C}^n$, the vector bundles

$$\left\{ \Sigma^\alpha \mathcal{U}^* : \begin{array}{c} \mathcal{U}^* \text{ is the tautological bundle}, \alpha \text{ is a Young diagram} \\ \text{with at most } r \text{ rows and at most } n-r \text{ columns,} \\ \text{and } \Sigma^\alpha \text{ is the Schur functor associated to } \alpha. \end{array} \right\}$$

are a full strong exceptional collection. Kapranov also produced therein a full strong exceptional collection on quadrics.

More examples of homogeneous spaces with full (not necessarily strong) exceptional collections are provided in [PS, F, S1] for Lagrangian Grassmannians, in [K6, KP] for Grassmannians of isotropic planes, in [M1, FM] for some homogeneous spaces of exceptional type, and in [AAGZ] for certain homogeneous spaces from rank 2 groups.

The perspective we consider is that of $\mathbb{P}^n$ as the simplest example of a linear geometric invariant theory (GIT) quotient (see Section 2.7), whence Beilinson's theorem inspires the following question:

Question 1.1.1. *When does a proper linear GIT quotient $X^{\mathrm{ss}}/\!\!/G$ by a reductive group G admit a full (strong) exceptional collection of vector bundles?*

This question has been explored extensively in the context of King's conjecture [K4], which had predicted that smooth projective toric varieties have a full strong exceptional collection consisting of line bundles. This conjecture has many counterexamples. The first counterexample came from Hille and Perling [HP], who showed that the surface obtained by iteratively blowing up the Hirzerbruch surface $\mathbb{F}_2$ three times cannot have an exceptional collection of line bundles of length 7, which is the rank of its Grothendieck group. In [M2], Manivel constructs an infinite family of non-Fano counterexamples, and Efimiov constructs in [E] an infinite family of smooth projective toric Fano varieties of Picard number 3 where King's conjecture fails.

There are also a large class of toric varieties that admit a full exceptional collection consisting of line bundles. Costa and Miró-Roig show in [CMR] that King's conjecture is true for smooth projective toric variety with Picard number at most 2 and also for those with a splitting fan. In [BT], Bernardi and Tirabassi use Frobenius pushforward methods to show that King's conjecture also holds for smooth projective toric Fano threefolds. Analogous results are obtained by Uehara in [U] using birational geometry. In [PN], Prabhu-Naik extends the list to smooth projective toric Fano fourfolds. Castravet and Tevelev show in [CT] that for n odd, the GIT quotient $(\mathbb{P}^1)^n /\!\!/ \mathbb{G}_m$ with respect to the linearization $\mathcal{O}_{\mathbb{P}^1}(1)^{\boxtimes n}$, which is a toric Fano variety of dimension $n-1$ and hence a linear GIT quotient by a torus, has a full exceptional collection consisting of line bundles.

These lists of examples and counterexamples shift the goals of Question 1.1.1 towards understanding the nature and the landscape of the examples where this question has an affirmative answer.

In general, the group G will not act freely on the semi-stable locus X^{ss}, so the good quotient $X^{\mathrm{ss}} /\!\!/ G$ will be singular and will therefore not have a full strong exceptional collection (Corollary 2.6.6). Thus, it is natural to expand the context of Question 1.1.1 to the quotient stack X^{ss}/G, which we do in this thesis. The term "GIT quotient" will henceforth refer to the smooth algebraic stack X^{ss}/G instead of the good quotient $X^{\mathrm{ss}} /\!\!/ G$.

Question 1.1.2. *Consider a triple (G, X, ℓ), where G is a split reductive group* *over a field k, X is the total space of a G-representation, and $\ell \in \mathrm{Pic}_G(X)$ is G-linearized line bundle. Assume that points in $X^{\mathrm{ss}}(\ell)$ have finite G-stabilizers (so that the GIT quotient is an orbifold), and that the GIT quotient $X^{\mathrm{ss}}(\ell)/G$ is proper. Does $X^{\mathrm{ss}}(\ell)/G$ admit a full exceptional collection of vector bundles?*

In this general context, Kawamata [K3] proved that smooth toric orbifolds always admit a full exceptional collection; however, the objects are arbitrary complexes of sheaves. For toric Fano orbifolds with either Picard number or dimension at most 2, Borisov and Hua [BH] showed that they have a full exceptional collection of line bundles.

1.2 The main results

In this thesis, we will work under the following key assumption:

Hypotheses 1.2.1. Let k be a field of characteristic 0 and let G be a split-reductive (possibly non-connected) group over k. Let X be the total space of a G-representation. Denote the weights of X by $\beta_1, \ldots, \beta_n$. Assume there exists a central cocharacter λ_0 of G such that $\langle \lambda_0, \beta_i \rangle < 0$ for all i, with the canonical pairing between weights and cocharacters. Let $\omega^* := \det(X) \otimes \det(\mathfrak{g})^{-1}$ be the anticanonical character of X.

The assumption on the weight pairing is important but mild; in fact, many examples, such as Grassmannians and weighted Grassmannians, come from representations that satisfy this condition. Moreover, the weight pairing condition of Hypotheses 1.2.1 implies $\mathcal{O}_X^G = k$, thus the resulting GIT quotients will be projective over k.

In Chapter 3, we will discuss the following result, which produces a large class of triples (G, X, ℓ) with G of rank two such that Question 1.1.2 is affirmative. This is joint work with Halpern-Leistner [HLK].

Theorem 1.2.2 (Proposition 3.3.1, Theorem 3.5.1, Theorem 3.7.1)**.** *Let (G, X, ℓ) be a triple as in Question 1.1.2 such that X satisfies Hypotheses 1.2.1. We construct a Weyl-invariant subset Ω of the weight space of G (that we call the "barrel window") such that the following G-equivariant vector bundles on $X^{\mathrm{ss}}(\ell)$*

$$\mathcal{E}(\Omega) := \left\{ \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \textit{ irreducible with} \\ \textit{weights in } \Omega \end{array} \right\} \tag{1.1}$$

form a strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$. Moreover, this collection is full when ℓ is close (in the sense of variation of GIT) to the anticanonical character ω^*, $X^{\mathrm{ss}}(\ell)$ has finite G-stabilizers, and G has rank 2.

In Chapter 4, we will discuss a general combinatorial algorithm for producing a finite list of tautological vector bundles that generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$ whenever $X^{\mathrm{ss}}(\ell)$ has finite G-stabilizers. This algorithm is a modification of the algorithm of Špenko and Van den Bergh [ŠVdB1] used to construct noncommutative crepant resolutions.

Theorem 1.2.3 (Theorem 1.2.3)**.** *Let (G, X, ℓ) be a triple as in Question 1.1.2 such that X satisfies Hypotheses 1.2.1. We construct a bounded Weyl-invariant subset $\widetilde{\Omega}$ of the weight space of G such that if $X^{\mathrm{ss}}(\ell)$ has finite G-stabilizers, then the following G-equivariant vector bundles on $X^{\mathrm{ss}}(\ell)$*

$$\mathcal{E}(\widetilde{\Omega}) := \left\{ \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \textit{ irreducible with} \\ \textit{character in } \widetilde{\Omega} \end{array} \right\} \tag{1.2}$$

generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$.

For a large class of representations, this list of vector bundles contain the strong exceptional collection from Theorem 1.2.2. It is an interesting future problem to determine triples (G, X, ℓ) for which the vector bundles in Theorem 1.2.3 can be reduced to a full strong exceptional collection.

1.3 The molecular perspective

A key perspective of our work in [HLK] is that many triples (G, X, ℓ) for which Question 1.1.2 is affirmative can be built out of a smaller list of basic "elemental" examples (atoms) admitting full exceptional collections of vector bundles. The idea is inspired by the work of Craw [C], who constructed full strong exceptional collections of vector bundles on quiver flag varieties by realizing them as iterated fiber bundles where each fiber is a Grassmannian. Since one has an equivariant full exceptional collection on Grassmannians by Kapranov [K1], a full exceptional collection on the quiver flag variety is then deduced. We generalize this observation as follows:

Theorem 1.3.1. *Let Q be a directed acyclic graph with vertices Q_0 and edges Q_1. Suppose each vertex $i \in Q_0$ has a triple (G_i, V_i, ℓ_i) as above satisfying the weight pairing condition in Hypotheses 1.2.1, and a "framing" dimension $w_i \geq 0$. Consider the representation $\operatorname{Rep}(Q)$ of $G_Q := \prod_{i \in Q_0} G_i$ given by taking the product of the following: a copy of $\operatorname{Hom}(V_i, V_j)$ for each edge $i \to j$ in Q_1 and a copy of $V_i^{w_i}$ for each vertex $i \in Q_0$. Suppose that each triple (G_i, V_i, ℓ_i) has a set of irreducible representations Ω_i of G_i such that the vector bundles*

$$\{\mathcal{O}_{V_i^{\mathrm{ss}}(\ell_i)} \otimes U : U \in \Omega_i\}$$

are a full strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(V_i^{\mathrm{ss}}(\ell_i)/G_i)$. Then there is a canonical

G_Q-equivariant line bundle ℓ_Q of G_Q and a set Ω_Q of irreducible G_Q-representations such that $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Rep}(Q)^{\mathrm{ss}}(\ell_Q)/G_Q)$ also admits a full strong exceptional collection of vector bundles of the form $\mathcal{O}_{\mathrm{Rep}(Q)^{\mathrm{ss}}} \otimes U$ for $U \in \Omega_Q$.

Theorem 1.3.1 recovers Craw's result for quiver flag varieties by assigning each vertex a triple whose corresponding GIT quotient is a Grassmannian. Other elemental examples include the previously discussed examples of toric varieties with full exceptional collections of line bundles. From this molecular perspective, Theorem 1.2.2 significantly extends this periodic table of elements.

1.4 A summary of the key techniques

For a convex subset Ω in the weight space of G, the set $\mathcal{E}(\Omega)$ as in Theorem 1.2.2 gives a full exceptional collection on $X^{\mathrm{ss}}(\ell)/G$ if it satisfies:

(i) the functor given by restriction along $X^{\mathrm{ss}}(\ell) \subset X$

$$\mathrm{res} : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{X/G}) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{X}^{\mathrm{ss}}(\ell)/\mathrm{G})$$

becomes an equivalence of categories when restricted to the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$ generated by the vector bundles

$$\left\{ \mathcal{O}_X \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \text{ irreducible with} \\ \text{character in } \Omega \end{array} \right\}, \text{ and}$$

(ii) there is an ordering of the vector bundles in $\mathcal{E}(\Omega)$ so that they form an exceptional collection.

The weight pairing assumption in Hypotheses 1.2.1 gives a canonical way to order the vector bundles $\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U$ using the λ_0-weight of the representation U. Thus the main technical challenge is to produce an Ω that satisfies (i). The key ideas that go into this are the following:

(a) Filtrations of local cohomology with support in the unstable locus.

Fully faithfulness in (i) i.e., obtaining an isomorphism

$$\mathrm{RHom}_{X/G}\left(\mathcal{O}_X \otimes U_1, \mathcal{O}_X \otimes U_2\right) \cong \mathrm{RHom}_{X^{\mathrm{ss}}(\ell)/G}\left(\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U_1, \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U_2\right),$$

is equivalent to the vanishing of the local cohomology object $R\Gamma_{X^{\mathrm{us}}(\ell)}(\mathcal{O}_X \otimes U_2 \otimes U_1^*)^G$ with support in the unstable locus $X^{\mathrm{us}}(\ell) := X \setminus X^{\mathrm{ss}}(\ell)$. We use G-equivariant filtrations of $R\Gamma_{X^{\mathrm{us}}(\ell)}(\mathcal{O}_X)$ given by Van den Bergh [VdB] to give explicit conditions on the weights of a representation U that guarantee the vanishing of the local cohomology $R\Gamma_{X^{\mathrm{us}}(\ell)}(\mathcal{O}_X \otimes U)^G$. These weight conditions define the convex region Ω where fully-faithfulness holds.

This technique uses ideas from Teleman's quantization theorem [T] and the generalization to derived categories in [HL]. The region Ω determines how the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$ sits as a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$, namely as a "window" category in the sense of Halpern-Leistner [HL] and Ballard-Favero-Katzarkov [BFK].

(b) Minimal G-equivariant complexes and combinatorial generation algorithms.

Essential surjectivity in (i) begins with the observation that the functor res is essentially surjective. As X is a vector space, the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$ is generated by all vector bundles of the form $\mathcal{O}_X \otimes W$, where W is an irreducible G-representation. Thus for essential surjectivity in (i), it suffices to show that restrictions of all these vector bundles to the semi-stable locus lie in the smallest subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$ generated by $\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U$, where U is irreducible and has character in Ω.

The unstable locus $X^{\mathrm{us}}(\ell)$ has a concrete description by the Hilbert-Mumford criterion in terms of the G-orbit of attracting loci of certain destabilizing one-parameter subgroups (see Section 2.7.1). To a destabilizing one-parameter subgroup, one associates a particular minimal G-equivariant complex of locally free sheaves with support in $X^{\mathrm{us}}(\ell)$ (see Section 3.5.1). These complexes become exact when restricted to $X^{\mathrm{ss}}(\ell)/G$, and one can obtain explicit descriptions of the weights of the representations appearing in each term. One then uses these minimal complexes in an inductive procedure to build any vector bundle $\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes W$ from the irreducible representations coming from the region Ω obtained in (a).

Combinatorial generation algorithms of this flavor come from the work of Špenko and Van den Bergh [ŠVdB1], where they use such algorithms to construct noncommutative crepant resolutions for certain quotient singularities. They have also appeared in [HLS, ŠVdB2].

Let us illustrate these ideas in the case of orbifold weighted projective spaces.

1.4.1 Example: full strong exceptional collections of line bundles on weighted projective spaces

Let $X = \mathbb{C}^{n+1}$ and let $G = \mathbb{G}_m$ over $\mathbb{C}$. Suppose G acts on X by scaling with strictly positive weights $\beta_i \in \mathbb{Z}$ for $i = 0, \ldots, n$. The anti-canonical character is $\omega^* = \beta_0 + \ldots + \beta_n$, $X^{\mathrm{us}}(\omega^*) = \{0\}$, and $X^{\mathrm{ss}}(\omega^*)/G$ is the orbifold weighted projective space $\mathbb{P}(\beta_0, \dots, \beta_n)$. For ease of notation, we will use X^{us} and X^{ss} to denote the unstable and the semi-stable locus respectively.

A vanishing criterion for local cohomology supported in the unstable locus:

For a commutative ring R and an ideal I, the local cohomology functor $R\Gamma_I(-)$ is the right derived functor for the endofunctor on R-modules defined by

$$\Gamma_I(M) := \{m \in M : I^s m = 0 \text{ for some } s > 0\}.$$

The functor $\Gamma_I(-)$ is left-exact and acyclic for injective modules, thus

$$R\Gamma_I(M) \simeq \varinjlim_{s>0} \operatorname{Ext}^\bullet_R(R/I^s, M).$$

In our case, $R = k[t_0, t_1, \dots, t_n]$ and $I = (t_0, t_1, \dots, t_n)$. Let M be an R-module and let $\widetilde{M}$ denote the corresponding quasi-coherent sheaf on X. Then $R\Gamma_{X^{\mathrm{us}}}(\widetilde{M}) \cong R\Gamma_I(M)$. This complex has a more explicit description in terms of "stable" Koszul complexes (see [L, Prop. 3.1.2]). Namely,

$$R\Gamma_I(M) \simeq M \otimes_R \mathcal{K}(\mathbf{t}),$$

where $\mathcal{K}(\mathbf{t}) := \mathcal{K}(t_0) \otimes_R \mathcal{K}(t_1) \otimes_R \dots \otimes_R \mathcal{K}(t_n)$ and $\mathcal{K}(t_j)$ is the complex $\dots \to 0 \to R \to R_{t_j} \to 0 \to \dots$ such that R in degree 0, R_{t_j} is in degree 1 and is the localization at powers of t_j, and $R \to R_{t_j}$ is the localization map. Note that $\mathcal{K}(\mathbf{t})$ is precisely the complex

$$0 \to R \to \bigoplus_{i=0}^{n} R_{t_i} \to \bigoplus_{i<j} R_{t_i t_j} \to \dots \to R_{t_0 t_1 \dots t_n}, \tag{1.3}$$

which is exact everywhere except in degree $n+1$, where the cohomology is

$$\operatorname{Span}_k \{t_0^{a_0} \dots t_n^{a_n} : a_i \text{ is strictly negative } \forall i\}. \tag{1.4}$$

This description of local cohomology above also holds in the equivariant setup because quasi-coherent sheaves on X/G correspond to graded R-modules and the arrows in (1.3) respect the grading. Thus, (1.4) says that $\mathrm{Span}_k\left\{t_0^{-1}t_1^{-1}\dots t_n^{-1}\right\}$ is the G-representation appearing in $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$ that has the highest weight, and this highest weight is $-\beta_0 - \dots - \beta_n = -\omega^*$. In particular, if U is an irreducible G-representation with weight χ such that $-\omega^* < \chi < \omega^*$, then

$$\begin{aligned} R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X \otimes U)^G &= (U \otimes_R \mathcal{K}(\mathbf{t}))^{\text{weight 0 part}} \\ &= 0, \end{aligned}$$

where the vanishing follows because U^* has weight $-\omega^* < -\chi < \omega^*$, so U^* does not appear in (1.4).

Exceptional collections from vanishing of local cohomology:

For $\chi \in \mathbb{Z}$, let V_χ denote the irreducible G-representation with weight χ. An immediate consequence of the vanishing criterion above is that if $-\omega^* < \chi_2 - \chi_1 < \omega^*$, then

$$\begin{aligned} \mathrm{RHom}_{X/G}\left(\mathcal{O}_X \otimes V_{\chi_1}, \mathcal{O}_X \otimes V_{\chi_2}\right) &= R\Gamma\left(X, \mathcal{O}_X \otimes V_{\chi_2-\chi_1}\right)^G \\ &\cong R\Gamma\left(X^{\mathrm{ss}}, \mathcal{O}_{X^{\mathrm{ss}}} \otimes V_{\chi_2-\chi_1}\right)^G \\ &= \mathrm{RHom}_{X^{\mathrm{ss}}/G}\left(\mathcal{O}_{X^{\mathrm{ss}}} \otimes V_{\chi_1}, \mathcal{O}_{X^{\mathrm{ss}}} \otimes V_{\chi_2}\right). \end{aligned}$$

The isomorphism above implies that for any $\chi \in \mathbb{Z}$, the restriction functor $\mathrm{res} : \mathrm{D^b_{coh}(X/G)} \to \mathrm{D^b_{coh}(X^{ss}(\ell)/G)}$ on the line bundles

$$\Omega := \{\mathcal{O}_X \otimes V_{\chi+\omega^*-1}, \mathcal{O}_X \otimes V_{\chi+\omega^*-2}, \dots, \mathcal{O}_X \otimes V_{\chi+1}, \mathcal{O}_X \otimes V_\chi\}$$

gives a strong exceptional collection on $\mathbb{P}(\beta_0, \dots, \beta_n)$. Indeed, if $-\omega^* < j - i < \omega^*$, then

$$\begin{aligned} \mathrm{RHom}_{X^{\mathrm{ss}}/G}\left(\mathcal{O}_{X^{\mathrm{ss}}} \otimes V_{\chi+i}, \mathcal{O}_{X^{\mathrm{ss}}} \otimes V_{\chi+j}\right) &\cong R\Gamma\left(X, \mathcal{O}_X \otimes V_{j-i}\right)^G \\ &\cong \left(\mathrm{Sym}(X^*) \otimes V_{j-i}\right)^G[0]. \end{aligned} \tag{1.5}$$

When $-\omega^* < j-i < 0$, (1.5) vanishes because the weights of X are strictly positive. When $0 < j-i < \omega^*$, (1.5) is $\mathrm{Sym}(X^*)^{\text{weight } i-j \text{ piece}} \otimes V_{j-i}$ concentrated in degree zero.

Fullness of the exceptional collection:

To verify that Ω restricts to a full collection on X^{ss}/G, it suffices to show that any line bundle $\mathcal{O}_{X^{\mathrm{ss}}} \otimes V_\mu$ with $\mu \in \mathbb{Z}$ is in the smallest triangulated category generated by the restriction of Ω. This can be achieved inductively using the G-equivariant Koszul resolution of the structure sheaf of the origin

$$\mathcal{K} : 0 \to \mathcal{O}_X \otimes V_{-\omega^*} \to \ldots \to \bigoplus_{i_1<i_2} \mathcal{O}_X \otimes V_{-\beta_{i_1}-\beta_{i_2}} \to \bigoplus_{i=1}^{n} \mathcal{O}_X \otimes V_{-\beta_i} \to \mathcal{O}_X \to 0.$$

Now, if $\mu < \chi$, then $\mathcal{K} \otimes V_{\mu+\omega^*}$ restricts to an exact complex in X^{ss}. It has $\mathcal{O}_{X^{\mathrm{ss}}} \otimes V_\mu$ as the line bundle with lowest weight and all other line bundles have weights closer to the list $\chi, \chi+1, \ldots, \chi+\omega^*-1$. On the other hand, if $\mu > \chi+\omega^*-1$, then the restriction of $\mathcal{K} \otimes V_\mu$ has $\mathcal{O}_{X^{\mathrm{ss}}} \otimes V_\mu$ as the line bundle with highest weight and all other line bundles have weights that are closer to the list $\chi, \chi+1, \ldots, \chi+\omega^*-1$. By induction, it follows that the restriction of Ω generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$.

1.5 Notation

Unless otherwise stated, k will be a field of characteristic 0. A variety will be a reduced scheme of finite type over k. For a ringed space X, by "a sheaf on X" will mean a sheaf of $\mathcal{O}_X$-modules.

All group schemes will be smooth and affine. For a split-reductive group G over k, we will use the following standard notation:

(1) $T \subset B \subset G$ is a fixed choice of split maximal torus and Borel subgroup.

(2) M is the character lattice of T, i.e., the weight lattice of G, and N is the cocharacter lattice of T. We denote $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

(3) We denote the canonical pairing between $\lambda \in N_{\mathbb{R}}$ and $\chi \in M_{\mathbb{R}}$ by $\langle \lambda, \chi \rangle$.

(4) W is the Weyl group of G and w_0 is the longest element of W.

(5) The roots of B will be the *negative* roots and ρ will denote the half sum of positive roots.

(6) We say that $\lambda \in N_{\mathbb{R}}$ is *anti-dominant* if it pairs non-negatively with all negative roots.

(7) $G^\circ \subset G$ denotes the identity component.

We will also use the following, less standard notation:

Definition 1.5.1. If G is a split reductive k-group and $\ell \in M_{\mathbb{R}}$ we denote by $\ell^\perp \subset N_{\mathbb{R}}$ the subset consisting of cocharacters that pair zero with ℓ. If $\ell \in M_{\mathbb{Q}}^W$ is non-zero, we let $G_\ell^\circ \subset G$ be the identity component of the kernel of the character $P\ell : G \to \mathbb{G}_m$, where $P \in \mathbb{Z}_{>0}$ is sufficiently divisible so that $P\ell \in M$. We fix a choice of maximal torus and Borel subgroup $T_\ell \subset B_\ell \subset G_\ell^\circ$ such that $T_\ell \subset T$ and $B_\ell \subset B$.

In this definition $\ker(P\ell)$ depends on the choice of P, but $\ker(P\ell) \subset \ker(PQ\ell)$ is a finite index subgroup for any $Q > 0$, so the identity component G_ℓ° is independent of P.

CHAPTER 2

PRELIMINARIES

2.1 Derived categories of abelian categories

Let $\mathcal{A}$ be an abelian category.

Definition 2.1.1. The (bounded, bounded below, bounded above) derived category of $\mathcal{A}$, denoted $\mathrm{D}(\mathcal{A})$ (respectively $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, $\mathrm{D}^{+}(\mathcal{A})$, $\mathrm{D}^{-}(\mathcal{A})$), is the category whose objects are (bounded, bounded below, bounded above) complexes of objects in $\mathcal{A}$, and where morphisms that induce an isomorphism on cohomology are inverted. For a precise definition, see for instance [S2, Tag 05RR].

Definition 2.1.2. If $\mathcal{B} \subset \mathcal{A}$ is a full subcategory, define $\mathrm{D}_{\mathcal{B}}(\mathcal{A})$ to be the category whose objects are $\mathcal{F} \in \mathrm{D}(\mathcal{A})$ such that $\mathrm{H}^n(\mathcal{F}) \in \mathcal{B}$ for all cohomology objects. If $\mathrm{D}^?_{\mathcal{B}}(\mathcal{A})$ is decorated with $? = b, +, -$, then one requires the complexes to have bounded (respectively bounded below, bounded above) cohomology.

Recall that a nonempty full subcategory $\mathcal{B} \subset \mathcal{A}$ is said to be *weakly Serre* if whenever $B_0 \to B_1 \to B_2 \to B_3 \to B_4$ is an exact sequence in $\mathcal{A}$ and $B_0, B_1, B_3, B_4 \in \mathcal{B}$, then $B_2 \in \mathcal{B}$. It is a *Serre* subcategory if whenever whenever $B_1 \to B_2 \to B_3$ is an exact sequence in $\mathcal{A}$ and $B_1, B_3 \in \mathcal{B}$, then $B_2 \in \mathcal{B}$. Serre and weakly Serre subcategories are abelian and the inclusion functor is exact [S2, Tag 02MN].

If $\mathcal{B} \subset \mathcal{A}$ is a weakly Serre subcategory, then $\mathrm{D}^?_{\mathcal{B}}(\mathcal{A}) \subset \mathrm{D}(\mathcal{A})$ is full triangulated subcategory for $? = \emptyset, +, -, b$ [S2, Tag 06UQ]. Moreover, the triangulated functor $\mathrm{D}^?(\mathcal{B}) \to \mathrm{D}(\mathcal{A})$ factors through $\mathrm{D}^?_{\mathcal{B}}(\mathcal{A})$. If $\mathcal{B} \subset \mathcal{A}$ is a Serre subcategory satisfying the following condition:

($\star$) for every surjection $A \twoheadrightarrow B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$, there is a subobject $A' \subset A$ with $A' \in \mathcal{B}$ and $A' \twoheadrightarrow B$,

then the induced functor $\mathrm{D}^-(\mathcal{B}) \to \mathrm{D}^-_{\mathcal{B}}(\mathcal{A})$ is an equivalence [S2, Tag 0FCL].

Lemma 2.1.3. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{C}, \mathcal{B}$ be weak Serre subcategories. Assume $\mathcal{C}$ is a Serre subcategory of $\mathcal{B}$ satisfying ($\star$) and consider the following commutative diagram of triangulated functors*

$$\begin{tikzcd}
\mathrm{D}^-(\mathcal{C}) \arrow[r, "\Phi_{\mathcal{C}}"] \arrow[d] & \mathrm{D}^-_{\mathcal{C}}(\mathcal{A}) \arrow[d] \\
\mathrm{D}^-(\mathcal{B}) \arrow[r, "\Phi_{\mathcal{B}}"] & \mathrm{D}^-_{\mathcal{B}}(\mathcal{A})
\end{tikzcd}.$$

If $\Phi_{\mathcal{B}}$ is an equivalence, then so is $\Phi_{\mathcal{C}}$. In this case, the induced map $\mathrm{D}^{\mathrm{b}}(\mathcal{C}) \to \mathrm{D}^{\mathrm{b}}_{\mathcal{C}}(\mathcal{A})$ is also an equivalence.

Proof. The vertical arrows are fully faithful because $\mathcal{C} \subset \mathcal{B}$ is a full abelian subcategory, hence $\Phi_{\mathcal{C}}$ is fully faithful. Essential surjectivity follows from the fact that the essential image of the left vertical arrow is $\mathrm{D}^-_{\mathcal{C}}(\mathcal{B})$. Indeed, the left vertical arrow factors through $\mathrm{D}^-(\mathcal{C}) \to \mathrm{D}^-_{\mathcal{C}}(\mathcal{B})$, which is an exact equivalence because $\mathcal{C} \subset \mathcal{B}$ is a Serre subcategory satisfying ($\star$). Thus, if $\mathcal{F}$ is a complex in $D^-_{\mathcal{C}}(\mathcal{A})$, then under the equivalence $\Phi_{\mathcal{B}}$, it corresponds to an object in $D^-_{\mathcal{C}}(\mathcal{B})$, which is in the essential image of $D^-(\mathcal{C})$. ☐

2.2 The derived category of quasi-coherent sheaves

Definition 2.2.1. For X a scheme or an algebraic stack, the derived category of quasi-coherent sheaves on X, denoted $\mathrm{D}_{\mathrm{qc}}(X)$, is the triangulated category whose objects are complexes of sheaves on X with quasi-coherent cohomology sheaves.

For X an algebraic stack, we consider sheaves in the lisse-étale topology so that $\mathrm{QCoh}(X) \subset \mathrm{Mod}(X_{\text{lisse-étale}})$ is a weakly Serre subcategory [S2, Tag 07B4].

2.2.1 Perfect complexes

If X is a quasi-compact and quasi-separated scheme, the category $\mathrm{D}_{\mathrm{qc}}(X)$ has arbitrary direct sums. Recall that an object $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X)$ is compact if $\mathrm{Hom}_{\mathrm{D}_{\mathrm{qc}}(X)}(\mathcal{F}, -)$ commutes with direct sums.

For quasi-compact quasi-separated schemes, the compact objects in $\mathrm{D}_{\mathrm{qc}}(X)$ are precisely the perfect complexes (namely, those that are Zariski locally quasi-isomorphic to a bounded complex of locally free sheaves) and $\mathrm{D}_{\mathrm{qc}}(X)$ is generated by one perfect complex [BVdB, Thm. 3.1.1]. We denote by $\mathrm{Perf}(X)$ the full triangulated subcategory of perfect complexes.

If, in addition, X has the *resolution property*, i.e. every quasi-coherent sheaf on X admits a surjection from a locally free sheaf of finite rank, then an object in $\mathrm{D}_{\mathrm{qc}}(X)$ is a perfect complex if and only if it is quasi-isomorphic to a bounded complex of finite locally free sheaves [S2, Tag 0F8E].

2.2.2 $\mathrm{D}_{\mathrm{qc}}(X)$ and $\mathrm{D}(\mathrm{QCoh}(\mathrm{X}))$

For X a Noetherian algebraic stack over a field k of characteristic 0, the induced functor on $\mathrm{D}(\mathrm{QCoh}(X)) \to \mathrm{D}_{\mathrm{qc}}(X)$ is an exact equivalence if X has quasi-affine diagonal [H, Thm. B]. If, in addition, X is smooth over k, then

$\mathrm{D}_{\mathrm{qc}}(X) \cong \mathrm{D}(\mathrm{QCoh}(X))$ is compactly generated and the subcategory of compact objects in $\mathrm{D}_{\mathrm{qc}}(X)$ is precisely $\mathrm{D}^b(\mathrm{Coh}(X))$ [H, Thm. A].

2.3 The bounded derived category of coherent sheaves

Definition 2.3.1. For X a scheme or an algebraic stack, the bounded derived category of coherent sheaves $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is the full triangulated subcategory of $\mathrm{D}_{\mathrm{qc}}(X)$ consisting of complexes with bounded coherent cohomology sheaves.

2.3.1 $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathrm{X}))$

For a locally Noetherian scheme X, the category $\mathrm{Coh}(X)$ of coherent sheaves on X is a weakly Serre subcategory of $\mathcal{O}_X$-Mod and a Serre subcategory of $\mathrm{QCoh}(X)$, with the inclusions being exact. Thus, there is an exact functor $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathrm{X})) \to \mathrm{D}^{\mathrm{b}}(\mathcal{O}_{\mathrm{X}}\text{-Mod})$ that factors through $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

When X is Noetherian, any quasi-coherent sheaf is the filtered colimit of its coherent subsheaves; in particular, $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ satisfies condition $(\star)$ above. Moreover, $\mathrm{D}^{-}(\mathrm{QCoh}(\mathrm{X})) \to \mathrm{D}^{-}_{\mathrm{QCoh}(\mathrm{X})}(\mathcal{O}_{\mathrm{X}}\text{-Mod})$ is an isomorphism [S2, Tag 09T4]. Thus, Lemma 2.1.3 implies that the induced functor $\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathrm{X})) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{X})$ is an exact equivalence.

If X is a scheme with the resolution property (2.2.1), then $\mathrm{Perf}(X)$ is a full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$; they are equivalent if X is Noetherian and regular [S2, Tag 0FDC].

2.4 Equivariant quasi-coherent sheaves

Let k be a field. Let G be a smooth affine group scheme over k and let X be a scheme of finite type over k. Assume that G acts on X via the map $a : G \times X \to X$ of schemes over k. Consider the diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, shift left=2, "p_{12}"] \arrow[r, "p_{23}" description] \arrow[r, shift right=2, "p_{13}"'] & G \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "p_2"'] & X,
\end{tikzcd}$$

where p_2 is the projection onto the second component, $p_{12} = (\mathrm{id}, a)$, p_{23} is the projection onto the second and third component, and $p_{13} = (m, \mathrm{id})$ where m is the group multiplication.

Definition 2.4.1. A G-equivariant structure on a quasi-coherent sheaf $\mathcal{F}$ on X is the data of an isomorphism $\phi : a^*\mathcal{F} \to p_2^*\mathcal{F}$ satisfying the cocycle condition, i.e. the diagram

$$\begin{tikzcd}
 & p_{12}^* a^* \mathcal{F} \cong p_{13}^* a^* \mathcal{F} \arrow[dl, "p_{12}^*\phi"'] \arrow[dr, "p_{13}^*\phi"] & \\
p_{12}^* p_2^* \mathcal{F} \cong p_{23}^* a^* \mathcal{F} \arrow[rr, "p_{23}^*\phi"'] & & p_{23}^* p_2^* \mathcal{F} \cong p_{13}^* p_2^* \mathcal{F}
\end{tikzcd}$$

commutes. A G-equivariant (quasi-)coherent sheaf on X is a (quasi-)coherent sheaf equipped with a G-equivariant structure.

Let $\mathrm{Coh}(X)^G$ (respectively $\mathrm{QCoh}(X)^G$) denote the category of G-equivariant (quasi-)coherent sheaves. A G-equivariant morphism between G-equivariant sheaves $(\mathcal{F}, \phi)$ and $(\mathcal{G}, \psi)$ is a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$ such that the diagram

$$\begin{tikzcd}
a^*\mathcal{F} \arrow[r, "\phi"] \arrow[d, "a^*f"'] & p_2^*\mathcal{F} \arrow[d, "p_2^*f"] \\
a^*\mathcal{G} \arrow[r, "\psi"'] & p_2^*\mathcal{G}
\end{tikzcd}$$

commutes. It follows that $\mathrm{Coh}(X)^G \subset \mathrm{QCoh}(X)^G$ is a full subcategory. The category $\mathrm{QCoh}(X)^G$ is abelian. Moreover, since X is locally Noetherian, $\mathrm{Coh}(X)^G \subset \mathrm{QCoh}(X)^G$ is a Serre subcategory and the inclusion is exact.

Lemma 2.4.2. *Let k be a field of characteristic 0. Let G be a smooth affine group scheme over k and let X be a Noetherian scheme over k with a G-action. Let $\mathfrak{X}$ denote the quotient stack $[X/G]$. Assume that X has quasi-affine diagonal, then there are equivalences of triangulated categories $\mathrm{D}_{\mathrm{qc}}(\mathfrak{X}) \cong D(\mathrm{QCoh}(X)^G)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X}) \cong D^b(\mathrm{Coh}(X)^G)$.*

Proof. As G is a smooth affine group scheme, $\mathfrak{X}$ is an algebraic stack, in particular, Noetherian over k. As X has quasi-affine diagonal, so does $\mathfrak{X}$. Thus, there is an equivalence $\mathrm{D}_{\mathrm{qc}}(\mathfrak{X}) \cong \mathrm{D}(\mathrm{QCoh}(\mathfrak{X})$ of triangulated categories by (2.2.2).

Since $\mathfrak{X}$ is Noetherian, $\mathrm{Coh}(\mathfrak{X}) \subset \mathrm{QCoh}(\mathfrak{X})$ is a Serre subcategory [S2, Tag 0GRB]; moreover, every quasi-coherent module is a filtered colimit of coherent submodules [S2, Tag 0GRE], hence condition $(\star)$ above holds. Lemma 2.1.3 then implies that $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X}) \to \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}(\mathfrak{X}))$ is an equivalence of triangulated categories.

The equivalences $\mathrm{D}_{\mathrm{qc}}(\mathfrak{X}) \cong \mathrm{D}(\mathrm{QCoh}(X)^G)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X}) \cong \mathrm{D}^b(\mathrm{Coh}(X)^G)$ then follow from the characterization of equivariant objects in a fibered category over X given in [V, Sect. 3.8, Prop. 3.49], which implies $\mathrm{QCoh}(\mathfrak{X}) = \mathrm{QCoh}(X)^G$ and $\mathrm{Coh}(\mathfrak{X}) = \mathrm{Coh}(X)^G$. □

2.5 Notions of generation of a triangulated category

Let $\mathcal{T}$ be a triangulated category. Let C be a set of objects in $\mathcal{T}$.

Definition 2.5.1 (Generation)**.** The set C is said to *generate* $\mathcal{T}$ if $\mathcal{T}$ is the smallest triangulated subcategory containing all objects in C.

Definition 2.5.2 (Split generation/ classical generation)**.** The set C is said to *split-generate* $\mathcal{T}$ if $\mathcal{T}$ is the smallest thick triangulated subcategory containing C. We say that a subcategory is thick if it is closed under taking direct summands.

Definition 2.5.3 (Compact generation)**.** Assume that $\mathcal{T}$ has arbitrary direct sums. An object $t \in \mathcal{T}$ is compact if $\operatorname{Hom}(t, -)$ commutes with direct sums. The set C is said to *compactly generate* $\mathcal{T}$ if (i) every object in C is compact and (ii) if $\operatorname{Hom}_{\mathcal{T}}(c[m], t) = 0$ for all $c \in C$ and $m \in \mathbb{Z}$ implies $t = 0$.

It is immediate that if C split-generates $\mathcal{T}$, then it generates $\mathcal{T}$. The following result more deeply relates these notions of generation; it is a corollary of Neeman's generalization of Thomason's localization theorem [N, Thm. 2.1].

Theorem 2.5.4. *Let $\mathcal{T}$ be a triangulated category with arbitrary direct sums and let $\mathcal{T}^c$ denote the full (thick) subcategory of compact objects. Assume that $\mathcal{T}$ is compactly generated. Let C be a set of objects in $\mathcal{T}$. Then C generates $\mathcal{T}$ if and only if C split-generates $\mathcal{T}^c$.*

Definition 2.5.5 (Strong generation, [BVdB])**.** Let E be an object in $\mathcal{T}$ and define for $n \in \mathbb{Z}_{>0}$ the full triangulated subcategory $\langle E \rangle_n$ consisting of objects that can be obtained from E by taking direct sums, summands, and at most $n-1$ cones. An object E is said to be a *strong generator* if $\mathcal{T} = \langle E \rangle_n$ for some n.

Triangulated categories with a strong generator are significant from the perspective of noncommutative geometry [BVdB, O], and it can capture properties such as the regularity of a variety.

Theorem 2.5.6 ([O, Thm. 3.27])**.** *Let X be a separated scheme of finite type over a field k. Then X is regular if and only if* $\mathrm{Perf}(X)$ *has a strong generator.*

2.6 Semi-orthogonal decompositions and full exceptional collections

Let $\mathcal{T}$ be a triangulated category.

Definition 2.6.1. A semi-orthogonal decomposition on $\mathcal{T}$, denoted $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$, is a sequence of full triangulated subcategories satisfying the following conditions:

(i) Semi-orthogonality: if $i > j$ then $\mathrm{Hom}_{\mathcal{T}}(s, t) = 0$ for all $s \in \mathcal{T}_i$ and $t \in \mathcal{T}_j$.

(ii) Generation: $\mathcal{T}$ is the smallest triangulated subcategory containing $\mathcal{T}_1, \ldots, \mathcal{T}_n$.

Remark 2.6.2. The conditions (i) and (ii) imply that any object t in $\mathcal{T}$ admits a unique decreasing filtration $0 = t_n \to t_{n-1} \to \ldots \to t_2 \to t_1 \to t_0 = t$ such that $\mathrm{Cone}(t_i \to t_{i-1}) \in \mathcal{T}_i$ for all $i = 1, \ldots, n$.

The decreasing filtration in the remark implies that semi-orthogonal decompositions “split” additive invariants. For example, if $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$ is a semi-orthogonal decomposition, the Grothendieck group of $\mathcal{T}$ is a direct sum of the Grothendieck groups of the semi-orthogonal components.

Definition 2.6.3. Let $\mathcal{T}$ be a k-linear triangulated category over a field k. An object $E \in \mathcal{T}$ is exceptional if

$$\mathrm{Hom}_{\mathcal{T}}(E, E[m]) = \begin{cases} k & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}.$$

A collection of exceptional objects $E_1, \ldots, E_n$ is an exceptional collection if $\mathrm{Hom}_{\mathcal{T}}(E_i, E_j[m]) = 0$ for all m whenever $i > j$.

An exceptional collection $E_1, \ldots, E_n$ is said to be

(i) *full* if $E_1, \ldots, E_n$ generates $\mathcal{T}$, and

(ii) *strong* if $\mathrm{Hom}_{\mathcal{T}}(E_j, E_i[m]) = 0$ for $m > 0$ whenever $i > j$.

Definition 2.6.4. A full triangulated subcategory $\iota : \mathcal{A} \subset \mathcal{T}$ is admissible if ι has both a left and a right adjoint.

For an exceptional object E in $\mathcal{T}$, the triangulated subcategory of $\mathcal{T}$ generated by E is an admissible subcategory that is equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$. Indeed, given an object E in $\mathcal{T}$, one can consider the functor

$$- \otimes E : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k)) \to \mathcal{T}$$

taking an object $\mathcal{F} = \bigoplus_i F_i[-i] \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$ to $\mathcal{F} \otimes E := \bigoplus_i (F_i \otimes E)[-i]$, where $F_i \otimes E := E^{\dim F_i}$. The functor $- \otimes E$ is fully faithful if and only if E is an exceptional object, and in this case, $- \otimes E$ admits a right adjoint given by $\mathcal{G} \mapsto \mathrm{RHom}_{\mathcal{T}}(E, \mathcal{G})$. This object is in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$ because $\mathcal{T}$ is k-linear.

Thus, a full exceptional collection on a k-linear triangulated category $\mathcal{T}$ corresponds to a semi-orthogonal decomposition where each semi-orthogonal component is an admissible subcategory that is equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$. In particular, if $E_1, \ldots, E_n$ is a full exceptional collection on $\mathcal{T}$, then the classes represented by these objects form a basis of the Grothendieck group of $\mathcal{T}$.

If X is a quasi-projective scheme, it has an ample invertible sheaf and hence has the resolution property. Thus, perfect complexes are globally quasi-isomorphic to a bounded complex of locally free sheaves by (2.2.1). In particular, $\mathrm{Perf}(X) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is a full subcategory. The following result due to Kuznetsov relates semi-orthogonal decompositions in these categories.

Proposition 2.6.5 ([K7, Prop. 4.1])**.** *Let X be a scheme over a field k. Assume X is quasi-projective. Then*

(1) $\mathrm{Perf}(X) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *is a full triangulated subcategory, and*

(2) *if* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ *is a semi-orthogonal decomposition, where each $\mathcal{A}_i$ is an admissible subcategory of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$*, then the triangulated subcategories*

$$\widetilde{\mathcal{A}}_i := \mathcal{A}_i \cap \mathrm{Perf}(X)$$

induce a semi-orthogonal decomposition $\mathrm{Perf}(X) = \langle \widetilde{\mathcal{A}}_1, \dots, \widetilde{\mathcal{A}}_m \rangle$ *that is compatible with* $\mathrm{Perf}(X) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$.

Corollary 2.6.6. *Let X be a scheme over a field k. Assume X is quasi-projective. If* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ *admits a full exceptional collection, then so does* $\mathrm{Perf}(X)$*. In particular, X is regular.*

Proof. A full exceptional collection yields a semi-orthogonal decomposition on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ with each semi-orthogonal component being an admissible subcategory equivalent to $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$, hence indecomposable. By Proposition 2.6.5, it induces a semi-orthogonal decomposition in $\mathrm{Perf}(X)$ which is compatible $\mathrm{Perf}(X) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. This semi-orthogonal decomposition must necessarily arise from a full exceptional collection on $\mathrm{Perf}(X)$.

Now, any triangulated subcategory of $\mathrm{Perf}(X)$ generated by an exceptional object has a strong generator because $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(k))$ is strongly generated by k. It follows from [O, Thm. 3.20] that $\mathrm{Perf}(X)$ has a strong generator, so X is regular by Theorem 2.5.6. □

2.7 GIT quotients

Let k be field of characteristic 0. Let G be a split-reductive group over k. Let X be a projective-over-affine variety (i.e. a closed subvariety of $\mathbb{A}_k^m \times \mathbb{P}_k^n$) with a G-action and let $\mathcal{L}$ be a G-equivariant ample line bundle on X.

The line bundle $\mathcal{L}$ defines a G-invariant open subset

$$X^{\mathrm{ss}}(\mathcal{L}) := \left\{ x \in X : \exists \text{ a section } s \in \Gamma(X, \mathcal{L}^{\otimes m})^G \text{ for some } m > 0 \text{ such that } s(x) \neq 0 \right\}$$

called the *semi-stable* locus.

While one may not always have a reasonable space parametrizing G-orbits in X, Mumford's GIT produces a scheme $X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$ that is a good quotient. Namely, it comes with an affine surjective G-invariant morphism $X^{\mathrm{ss}}(\mathcal{L}) \to X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$ that distinguishes closures of images of disjoint G-invariant closed subsets and furnishes regular sections on an open subset $U \subset X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$ as the G-invariant regular sections on the preimage of U.

One can also consider the quotient stack $X^{\mathrm{ss}}(\mathcal{L})/G$. As G is smooth and affine, it is an algebraic stack, and it is Deligne-Mumford (DM) if $X^{\mathrm{ss}}(\mathcal{L})$ has finite G-stabilizers. As mentioned in the introduction, we will use the term "GIT quotient" to refer to the stack $X^{\mathrm{ss}}(\mathcal{L})/G$ as opposed to the good quotient $X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$. If the G-action on $X^{\mathrm{ss}}(\mathcal{L})$ is free, then $X^{\mathrm{ss}}(\mathcal{L})/G = X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$. Often X will be smooth, and while the scheme $X^{\mathrm{ss}}(\mathcal{L}) /\!\!/ G$ may be singular, $X^{\mathrm{ss}}(\mathcal{L})/G$ will be a smooth algebraic stack.

2.7.1 The Hilbert-Mumford criterion

Let $\lambda : \mathbb{G}_m \to G$ be a one-parameter subgroup. Let $x \in X$ be a point such that $x_0 := \lim_{t\to 0} \lambda(t) \cdot x$ exists. Then λ fixes x_0, and therefore acts on the fiber $\mathcal{L}_{x_0}$ of $\mathcal{L}$ at x_0 via an integer weight that we denote by $\mathrm{wt}_\lambda(\mathcal{L}_{x_0})$.

The Hilbert-Mumford criterion says that $x \in X^{\mathrm{ss}}(\mathcal{L})$ if and only if for all one-parameter subgroups λ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists, $\mathrm{wt}_\lambda(\mathcal{L}_{x_0}) \geq 0$.

2.7.2 KN stratifications

The Hilbert-Mumford criterion furnishes a stratification of the unstable locus $X^{\mathrm{us}}(\mathcal{L}) := X \setminus X^{\mathrm{ss}}(\mathcal{L})$, called a Kempf-Ness (KN) stratification. The stratification comes from pairs (Z, λ) consisting of a one-parameter subgroup λ and $Z \subset X^\lambda$ a connected component of the fixed locus X^λ of λ. To such a pair one associates

(i) the locally closed subset

$$Y_{Z,\lambda} := \left\{ x \in X : \lim_{t\to 0} \lambda(t) \cdot x \in Z \right\},$$

called the *blade* of (Z, λ) or the *attracting locus* of λ with center Z. The limit gives an affine map $\pi : Y_{Z,\lambda} \to Z$ that is a fiber bundle of affine spaces whenever X is smooth.

(ii) the parabolic subgroups

$$P_\lambda = \left\{ g \in G : \lim_{t\to 0} \lambda(t) \cdot g \cdot \lambda(t)^{-1} \text{ exists } \right\}, \text{ and}$$

$$P_{Z,\lambda} = \left\{ p \in P_\lambda : l(Z) \subset Z, \text{ where } l = \lim_{t\to 0} \lambda(t) \cdot p \cdot \lambda(t)^{-1} \right\}.$$

The Levi subgroup $L_{Z,\lambda}$ of $P_{Z,\lambda}$ is given by

$$L_{Z,\lambda} = \left\{ l \in G : l(Z) \subset Z \text{ and } l = \lambda(t) \cdot l \cdot \lambda(t)^{-1} \text{ for all } t \right\}.$$

Taking the limit as $t \to 0$ under the conjugation action by λ gives a map $P_{Z,\lambda} \to L_{Z,\lambda}$ that admits a section. By definition, Z has a trivial $L_{Z,\lambda}$-action, $P_{Z,\lambda}$ acts on $Y_{Z,\lambda}$, and the map $\pi : Y_{Z,\lambda} \to Z$ is equivariant with respect to $P_{Z,\lambda} \to L_{Z,\lambda}$.

(iii) the weight

$$\mu(Z, \lambda) := \frac{-1}{|\lambda|} \operatorname{wt}_\lambda(\mathcal{L}|_Z),$$

where $| - |$ is some fixed norm on the space of one-parameter subgroups of G. By the Hilbert-Mumford criterion, $\mu(Z, \lambda) > 0$ implies $Y_{Z,\lambda} \subset X^{\mathrm{us}}(\mathcal{L})$.

A KN-stratification (see [HL, Sect. 2.1]) is constructed inductively as follows: Take $(Z'_\alpha, \lambda_\alpha)$ such that the weight $\mu(Z'_\alpha, \lambda_\alpha)$ is strictly positive and maximal among pairs (Z, λ) with Z not contained in previous strata. Then one considers the locus $Z_\alpha \subset Z'_\alpha$ consisting of points not contained in any previous strata. Set $Y_\alpha := \pi^{-1}(Z_\alpha) \subset Y_{Z'_\alpha,\lambda}$ and define the new stratum as $S_\alpha := G \cdot Y_\alpha$. The strata with strictly positive μ-weight stratify the unstable locus.

The data of a KN stratum is organized as in [HL, Def. 2.2] into the following useful diagram

$$\begin{tikzcd} Y_\alpha \arrow[r, hook] \arrow[d, "\pi_\alpha"'] & S_\alpha = G \cdot Y_\alpha \arrow[r, "j_\alpha"] & X \\ Z_\alpha, \arrow[u, bend right, "\sigma_\alpha"'] & & \end{tikzcd} \tag{2.1}$$

where σ_α and j_α are inclusions. The stratum S_α is a locally closed subset of X, and the morphism $G \times^{P_\alpha} Y_\alpha \to G \cdot Y_\alpha = S_\alpha$ induced by the group action is an isomorphism. Moreover, π_α is still affine and is equivariant under $P_\alpha \to L_\alpha$, where $P_\alpha := P_{Z_\alpha,\lambda}$ and L_α is the Levi subgroup of P_α.

We will drop the α subscripts when working with a single KN stratum.

G-equivariant sheaves on a KN stratum

We will be interested in understanding the structure of G-equivariant sheaves supported in the unstable locus. It turns out that G-equivariant sheaves on a KN stratum $S = G \cdot Y$ correspond to P-equivariant sheaves on Y. Indeed, the canonical morphism $Y \to G \times^P Y$ is equivariant with respect to the inclusion $P \subset G$, thus it induces an isomorphism of quotient stacks $Y/P \cong (P \times^G Y)/G$. But the latter is isomorphic to S/G because KN strata satisfy $P \times^G Y \cong G \cdot Y = S$.

2.7.3 The case of linear GIT quotients

Suppose $X = \mathrm{Spec}(\mathrm{Sym}(V))$, where V be a vector space over k with a G-action. With notation as in Section 1.5, let $\beta_1, \ldots, \beta_n \in M$ denote the weights of X with respect to the fixed maximal torus $T \subset G$.

Since X is a vector space, the group $\mathrm{Pic}(X)^G$ of G-linearized line bundles corresponds (contravariantly) to the characters of G, the latter of which determines a Weyl-invariant character of T. Thus in the sequel, we will identify a GIT parameter $\mathcal{L}$ with the corresponding character $\ell \in M^W$.

The attracting locus of a one-parameter subgroup $\lambda : \mathbb{G}_m \to T$ is

$$\begin{aligned} X^{\lambda \geq 0} &= \{x \in X : \lim_{t \to 0} \lambda(t) \cdot x \text{ exists}\} \\ &= \mathrm{Span}\{\beta_i : \langle \lambda, \beta_i \rangle \geq 0\} \end{aligned}.$$

We will use similar notation $X^{\lambda > 0}$, $X^{\lambda \leq 0}$, etc., to denote the linear subspace of X spanned by the weights whose pairing with λ is > 0, ≤ 0, etc.. Note that if $\lambda \in N_{\mathbb{R}}$

is anti-dominant, then $X^{\lambda\geq 0}$ and $X^{\lambda>0}$ are naturally linear representations of B.

The Hilbert-Mumford criterion implies that the unstable locus is precisely

$$X^{\mathrm{us}}(\ell) = \bigcup_{\lambda\in N,\ \text{s.t. } \langle\lambda,\ell\rangle<0} G\cdot X^{\lambda\geq 0},$$

where N is the space of one-parameter subgroups of T. In fact, only finitely many subspaces arise as $X^{\lambda\geq 0}$ for some $\lambda\in N$, so finitely many λ suffice to cover the unstable locus.

We will make use of some standard results in the theory of variation of GIT quotients, developed in [DH]. First, we define $X^{\mathrm{ss}}(\ell)$ more generally for $\ell\in M_{\mathbb{R}}^W$ using the fact that $X^{\mathrm{ss}}(\ell)$ for $\ell\in M^W$ depends only on the cell in which ℓ lies with respect to a decomposition of $M_{\mathbb{R}}^W$ into a finite union of rational polyhedral cones. We can thus think about varying ℓ continuously. If $\ell_t\in M_{\mathbb{R}}^W$ for $t\in[0,1]$ is a continuous family of real characters, then $X^{\mathrm{ss}}(\ell_t)$ is constant for $0<t\ll 1$, and $X^{\mathrm{ss}}(\ell_t)\subset X^{\mathrm{ss}}(\ell_0)$ for $0<t\ll 1$ with equality if points of $X^{\mathrm{ss}}(\ell_0)$ have finite stabilizers.

Example 2.7.1. Under Hypotheses 1.2.1, we will consider $X^{\mathrm{ss}}(\omega^*)/G$. If points of $X^{\mathrm{ss}}(\omega^*)$ already have finite stabilizers, then $X^{\mathrm{ss}}(\omega^*+\epsilon\ell)=X^{\mathrm{ss}}(\omega^*)$ for any $\ell\in M_{\mathbb{R}}^W$ and $0<\epsilon\ll 1$. However, if $X^{\mathrm{ss}}(\omega^*)$ has points with positive stabilizers, then $X^{\mathrm{ss}}(\omega^*+\epsilon\ell)$ for $0<\epsilon\ll 1$ will depend on ℓ. If $V\subset M_{\mathbb{R}}^W$ is the linear subspace spanned by the smallest cone in the wall-and-chamber decomposition of $M_{\mathbb{R}}^W$ coming from GIT, then $M_{\mathbb{R}}^W/V$ inherits a wall-and-chamber decomposition into a finite union of rational polyhedral cones such that $X^{\mathrm{ss}}(\omega^*+\epsilon\ell)$ is constant for all ℓ whose image in $M_{\mathbb{R}}^W$ lies in one of these cones.

2.8 Window Categories

Let G be a split-reductive group over k, let X be a projective-over-affine variety with a G-action, and let $\mathcal{L}$ be a G-linearized ample line bundle on X. For ease of notation, we will denote $\mathfrak{X}^{\mathrm{ss}} := X^{\mathrm{ss}}(\mathcal{L})/G$, $\mathfrak{X} := X/G$, and $X^{\mathrm{us}} := X^{\mathrm{us}}(\mathcal{L})$.

Window categories (see [T, HLS, BFK]) arise from the following circle of ideas:

(1) Let $\mathcal{E}$ be a locally free sheaf on $\mathfrak{X}^{\mathrm{ss}}$ and let $\widetilde{\mathcal{E}}$ be a lift of $\mathcal{E}$ via pullback along the open immersion $\mathfrak{X}^{\mathrm{ss}} \hookrightarrow \mathfrak{X}$. It is easier to compute the cohomology of $\widetilde{\mathcal{E}}$, so one wants to understand when the cohomology of $\mathcal{E}$ can be deduced from that of $\widetilde{\mathcal{E}}$. If the local cohomology with support in the unstable locus vanishes, i.e, $R\Gamma_{X^{\mathrm{us}}}(\widetilde{\mathcal{E}})^G = 0$, then there is an isomorphism $R\Gamma(\mathfrak{X}^{\mathrm{ss}}, \mathcal{E}) \cong R\Gamma(\mathfrak{X}, \widetilde{\mathcal{E}})$. This follows from the canonical triangle of G-equivariant complexes

$$R\Gamma_{X^{\mathrm{us}}}(\widetilde{\mathcal{E}}) \to R\Gamma(X, \widetilde{\mathcal{E}}) \to R\Gamma(X^{\mathrm{ss}}, \widetilde{\mathcal{E}}|_{X^{\mathrm{ss}}}) \xrightarrow{+}$$

and the fact that G is reductive, so taking G-invariants is exact.

(2) The local cohomology object $R\Gamma_{X^{\mathrm{us}}}(\widetilde{\mathcal{E}})$ admits G-equivariant filtrations. By characterizing the weights of all irreducible representations that can appear in the associated graded components of a given filtration, one obtains precise weight conditions that determine when $R\Gamma_{X^{\mathrm{us}}}(\widetilde{\mathcal{E}})^G = 0$. These weight conditions cut out a region $\Omega \subset M_{\mathbb{R}}$ in the weight space of G, i.e., a "window", such that the subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X})$

$$\mathcal{W}(\Omega) := \left\langle \mathcal{O}_X \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \text{ irreducible with} \\ \text{character in } \Omega \end{array} \right\rangle,$$

where $\langle - \rangle$ indicates the full triangulated category generated by $-$, restricts to a full subcategory $\mathrm{res} : \mathcal{W}(\Omega) \hookrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X}^{\mathrm{ss}})$. If res is also essentially surjective,

$\mathcal{W}(\Omega)$ is said to be a *window category*. Thus, window categories describe precisely how $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X}^{\mathrm{ss}})$ sits in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathfrak{X})$.

We now discuss some criteria for vanishing of local cohomology.

2.8.1 Vanishing of local cohomology from KN stratifications

Let $S = G \cdot Y$ be a KN stratum associated to a pair (Z, λ) (see Section 2.7.2). Let P be the parabolic of λ and let $L \subset P$ be the Levi subgroup fixing Z.

Assume that S is a closed KN stratum, i.e. $j : S \hookrightarrow X$ is a closed immersion, and assume X is smooth. Let $\mathcal{I}$ be the ideal sheaf of S.

Let $\mathcal{F}$ be a locally free sheaf on X/G, which we think of as a G-equivariant locally free sheaf on X. The local cohomology of $\mathcal{F}$ with support in S is given by

$$R\underline{\Gamma}_S(\mathcal{F}) = \varinjlim_{m>0} R\underline{Hom}_{X/G}\left(\mathcal{O}_X/I^m, \mathcal{F}\right),$$

where the underline indicates the sheafy functor. The object $R\underline{\Gamma}_S(\mathcal{F})$ is not in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$, but it admits a G-equivariant filtration whose associated graded components are the coherent sheaves

$$\mathrm{gr}_m(\mathcal{F}) := \underline{Hom}_{X/G}\left(\mathcal{I}^m/\mathcal{I}^{m+1}, \mathcal{F}\right) \cong \underline{Hom}_{X/G}\left(j_* \operatorname{Sym}^m(N_S X)^\vee, \mathcal{F}\right),$$

where $(N_S X)^\vee$ is the conormal bundle of the regular immersion $S \hookrightarrow X$. It follows that $R\Gamma(X/G, \mathrm{gr}_m(\mathcal{F})) = R\Gamma(X, \mathrm{gr}_m(\mathcal{F}))^G = 0$ for all m implies the vanishing of $R\Gamma_S(\mathcal{F})^G$. Now,

$$\begin{aligned}
\underline{Hom}_{X/G}\left(j_* \operatorname{Sym}^m(N_S X)^\vee, \mathcal{F}\right) &\cong j_* \underline{Hom}_{S/G}\left(\operatorname{Sym}^m(N_S X)^\vee, j^! \mathcal{F}\right) \\
&\cong j_* \underline{Hom}_{S/G}\left(\operatorname{Sym}^m(N_S X)^\vee, j^* \mathcal{F} \otimes_{\mathcal{O}_S} j^! \mathcal{O}_X\right) \\
&\cong j_*\left(\operatorname{Sym}^m(N_S X) \otimes_{\mathcal{O}_S} j^* \mathcal{F} \otimes_{\mathcal{O}_S} \det(N_S X)[-\operatorname{codim} S]\right),
\end{aligned}$$

where $j^!\mathcal{O}_X = \det(N_S X)[-\text{codim } S]$ is computed directly from the Koszul resolution of $j_*\mathcal{O}_S$. Using the notation from (2.1), we have

$$\begin{aligned} R\Gamma(X, \mathrm{gr}_m(\mathcal{F}))^G &\cong R\Gamma(S, (\mathrm{Sym}^m(N_S X) \otimes_{\mathcal{O}_S} j^*\mathcal{F} \otimes_{\mathcal{O}_S} \det(N_S X)[-\text{codim } S]))^G \\ &\cong R\Gamma(Y, (\mathrm{Sym}^m(N_S X) \otimes_{\mathcal{O}_S} j^*\mathcal{F} \otimes_{\mathcal{O}_S} \det(N_S X)[-\text{codim } S]))^P \\ &\cong R\Gamma(Z, \pi_* \left(\mathrm{Sym}^m(N_S X) \otimes_{\mathcal{O}_S} j^*\mathcal{F} \otimes_{\mathcal{O}_S} \det(N_S X)[-\text{codim } S]\right))^L, \end{aligned}$$

where we have used in the second equivalence that P-equivariant sheaves on Y correspond to G-equivariant sheaves on S (see Section 2.7.2).

Since λ fixes Z, it acts on $\det(N_S X)|_Z$ via a weight $-\eta_\lambda < 0$, which is strictly negative by construction of a KN stratum (see [HL, Sect. 2.1]). It follows that $-\eta_\lambda$ is the highest λ-weight that can appear in $\pi_* \left(\mathrm{Sym}^\bullet(N_S X) \otimes_{\mathcal{O}_S} \det(N_S X)\right)$. Thus, if U is a G-representation and all λ-weights of U are strictly less that η_λ, then

$$R\Gamma(X, \mathrm{gr}_m(\mathcal{O}_X \otimes U))^G \cong R\Gamma(Z, \pi_* \left(\mathrm{Sym}^m(N_S X) \otimes_{\mathcal{O}_S} \det(N_S X) \otimes U\right))^L = 0$$

for all $m \geq 0$. These weight conditions on closed KN strata are used in [HL] to construct canonical semi-orthogonal decomposition on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$ and window categories. The windows constructed in Chapter 3 will use the quantities η_λ even when they do not arise from a closed KN stratum.

2.8.2 Vanishing of local cohomology in linear GIT

In addition to the setup of Section 2.8, assume that G is connected, k is algebraically closed, and $X = \mathrm{Spec}(\mathrm{Sym}\, V)$ for a G-representation V.

Let U be an irreducible G-representation. By [VdB, Lem. 1.5], the Cohen-Macaulayness of the module of coinvariants $(\mathrm{Sym}(V) \otimes U)^G$ is determined by whether the character of U^* appears in $R\Gamma_{X^{\mathrm{us}}}(X, \mathcal{O}_X)$. Here X^{us} is defined to be the G-orbit of the following subset

$$\bigcup_{\lambda : B \subset P_\lambda} X_\lambda,$$

where λ is a cocharacter whose associated parabolic subgroup contains the Borel B and

$$X_\lambda := \left\{ x \in X : \lim_{t \to 0} \lambda(t) \cdot x = 0 \right\}.$$

Motivated by the questions of Cohen-Macaulayness, Van den Bergh uses a refined stratification of X^{us} to give explicit descriptions of the highest weights of irreducible representations appearing in $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$.

Proposition 2.8.1 ([VdB, Cor. 6.8])**.** *Suppose X has a point with closed orbit and finite G-stabilizer. Let U be an irreducible G-representation occurring in $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$. Then the highest weight of U has the form*

$$\chi' + \sum_{\alpha \in \Phi} \alpha,$$

where Φ is a subset of the set of roots of G and χ' is a character occurring in the T-representation $R\Gamma_{X_\lambda}(\mathcal{O}_X)$ for some one-parameter subgroup λ of T.

In the context of linear GIT quotients satisfying Hypotheses 1.2.1, we will use this result crucially in Section 3.2 to establish weight conditions on a G-representation U that guarantee the vanishing of the local cohomology object $R\Gamma_{X^{\mathrm{us}(\ell)}}(\mathcal{O}_X \otimes U)^G$.

CHAPTER 3

FULL EXCEPTIONAL COLLECTIONS OF VECTOR BUNDLES ON RANK-TWO LINEAR GIT QUOTIENTS

3.1 The barrel window

Here we introduce some subsets of $M_{\mathbb{R}}$, which we refer to as the "cylinder" and "barrel" windows, that are relevant for our main results. We work under Hypotheses 1.2.1.

Definition 3.1.1. Associate to any cocharacter $\lambda \in N_{\mathbb{R}}$ the T-weight

$$\zeta_\lambda := -\det\left(X^{\lambda \leq 0}\right) + \det\left(\mathfrak{g}^{\lambda < 0}\right),$$

where $\mathfrak{g}$ is the Lie algebra of G and the expression $\det(U) := \wedge^{\dim U} U$ is the sum of the weights (with multiplicity) appearing in U. Define

$$\eta_\lambda := \langle \lambda, \zeta_\lambda \rangle .$$

More concretely, $\eta_\lambda = -\sum_{\beta \in \Psi} \min(0, \langle \lambda, \beta \rangle) + \sum_{\alpha \in \Phi} \min(0, \langle \lambda, \alpha \rangle)$, where Ψ is the set of weights of X and Φ the set of roots of G.

If λ is a one-parameter subgroup associated to a KN-stratum S_λ with center Z_λ in the sense of [T], η_λ is the total λ-weight of the conormal bundle of S_λ in X restricted to Z_λ (see Section 2.8.1).

Definition 3.1.2 (λ-strip)**.** For any $\lambda \in N_{\mathbb{R}}$, we refer to the closed subset

$$B_\lambda := \{\chi \in M_{\mathbb{R}} : |\langle \lambda, \chi \rangle| \leq \eta_\lambda / 2\} \subset M_{\mathbb{R}}$$

as the λ-*strip*. Note that either $B_\lambda \subset B_{-\lambda}$ or $B_{-\lambda} \subset B_\lambda$, with equality if and only if $\eta_{-\lambda} = -\eta_\lambda$.

Definition 3.1.3 (Barrel window)**.** Let $\lambda_0 \in N_{\mathbb{R}}^W$ be a cocharacter that pairs strictly negatively with all weights of X. For any $\ell \in M_{\mathbb{R}}^W$ with $\langle \lambda_0, \ell \rangle \neq 0$, we define the *cylinder window* to be the closed subset

$$\overline{\nabla}_\ell := \left\{ \chi \in B_{\lambda_0} : |\langle \lambda', \chi \rangle| \leq \frac{\eta_{\lambda'}}{2} \text{ for all } \lambda' \in \ell^\perp \right\},$$

and we define the *barrel window* to be the non-closed subset

$$\nabla_\ell := \left\{ \chi \in B_{\lambda_0} : \forall \lambda' \in \ell^\perp, \left\{ \begin{array}{l} |\langle \lambda', \chi \rangle| < \frac{\eta_{\lambda'}}{2}, \text{ or} \\ \langle \lambda', \chi \rangle = \frac{\eta_{\lambda'}}{2} \text{ and } \langle \lambda_0, \chi \rangle \leq \frac{\langle \lambda_0, \zeta_{\lambda'} \rangle}{2}, \text{ or} \\ \langle \lambda', \chi \rangle = \frac{-\eta_{\lambda'}}{2} \text{ and } \langle \lambda_0, \chi \rangle \geq \frac{-\langle \lambda_0, \zeta_{\lambda'} \rangle}{2} \end{array} \right. \right\}.$$

When $\ell = \omega^*$ and $\lambda_0 \in N_{\mathbb{R}}^W$ is a cocharacter that satisfies the condition in Hypotheses 1.2.1, we will simply use $\overline{\nabla} := \overline{\nabla}_{\omega^*}$ and $\nabla := \nabla_{\omega^*}$.

Remark 3.1.4. The cylinder window is the intersection of the λ_0-strip with the preimage of a polytope $\overline{\nabla}' \subset M_{\mathbb{R}}/\mathbb{R}\ell$. If we consider X as a representation of $\ker(\ell) \subset G$, then under the identification of $M_{\mathbb{R}}/\mathbb{R}\ell$ with the weight space of $\ker(\ell)$, the polytope $\overline{\nabla}'$ corresponds to the one studied in [HLS] and it is related to the zonotope studied in [ŠVdB1] if X is quasi-symmetric (e.g. self-dual) as a representation of $\ker(\ell)$.

Example 3.1.5. Let $G = \mathrm{GL}_2$ over the field k and consider the G-linear representation $X = (k^2)^{\oplus 6}$. The weights of X are $(1,0), (0,1) \in M \cong \mathbb{Z}^2$ each with multiplicity 6. The roots of G are $(1,-1), (-1,1)$, where we take $(1,-1)$ to be the positive root. The subset $(\omega^*)^\perp \subset N_{\mathbb{R}}$ consists of positive multiples of the cocharacters $\lambda_1' = (-1,1)$ and $\lambda_2' = (1,-1)$. Take $\lambda_0 = (-1,-1) \in N_{\mathbb{R}}^W$. A direct computation yields $\zeta_{\lambda_0} = (-6,-6)$, $\zeta_{\lambda_1'} = (-5,-1)$, and $\zeta_{\lambda_2'} = (-1,-5)$, thus $\eta_{\lambda_0} = 12$ and $\eta_{\lambda_1'} = \eta_{\lambda_2'} = 4$. The resulting cylinder and barrel windows are described in Figure 3.1.

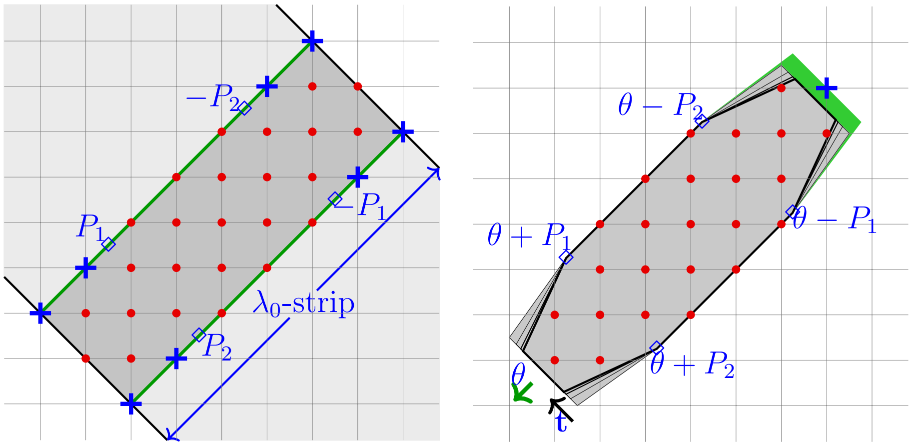

FIGURE 3.1. The diagram to the left is the cylinder window $\overline{\nabla}$ for Example 3.1.5. The T-weights indicated by blue crosses are those in the cylinder window that are excluded from the barrel window, thus the T-weights indicated by red dots are precisely those in the barrel window ∇. We have marked the special weights $P_i := \zeta_{\lambda_i}/2$ for $i = 1, 2$ that become relevant in the passage from the cylinder window to the barrel window as is depicted in the diagram to the right. This diagram indicates a perturbation by some $\theta \in M_{\mathbb{R}}^W$ of the regions $\overline{\nabla}_{\ell,t}$ defined in Remark 3.1.7 for selected values of $0 < t \ll 1$. In this particular example, $\theta = s\omega^*$, where $-s$ is a small positive number. Here the weight indicated by a blue cross is excluded in the θ-perturbed barrel window $\theta + \nabla$.

Definition 3.1.6 (λ_0-generic)**.** Given a pair $(\lambda_0, \ell) \in N_{\mathbb{R}}^W \times M_{\mathbb{R}}^W$ such that $\langle \lambda_0, \ell \rangle \neq 0$, we say that $\theta \in M_{\mathbb{R}}^W$ is λ_0-*generic* with respect to ℓ if $(\theta + H) \cap M = \emptyset$ for every affine hyperplane H of the form $\{\chi \in M_{\mathbb{R}} : \langle \lambda_0, \chi - \zeta/2 \rangle = 0\}$, where $\zeta = \zeta_{\lambda'}$ for some $\lambda' \in \ell^\perp$ (including $\lambda' = 0$). We will say θ is λ_0-generic if it is λ_0-generic with respect to ω^*.

The notion of λ_0-generic is equivalent to θ avoiding all of the hyperplanes $\chi + \zeta_{\lambda'}/2 + \lambda_0^\perp$ for all $\chi \in M$ and $\lambda' \in \ell^\perp$. If λ_0 is rational, then this hyperplane arrangement is locally finite. In fact, the image of the lattice $M \subset M_{\mathbb{Q}}$ under the linear map $\langle \lambda_0, - \rangle : M^{\mathbb{Q}} \to \mathbb{Q}$ will be a free subgroup of rank 1, $\Gamma \subset \mathbb{Q}$, and the hyperplane arrangement is contained in the preimage of $\frac{1}{2}\Gamma$ under this projection.

Remark 3.1.7 (A characterization of the barrel window)**.** Choose a W-invariant norm on $N_{\mathbb{Q}}$. Let us rescale λ_0 so that $|\lambda_0| = 1$. For $t \geq 0$, define the closed subset

$$\overline{\nabla}_{\ell,t} = \left\{ \chi \in \overline{\nabla}_\ell : |\langle \lambda' + t\lambda_0, \chi \rangle| \leq \frac{\langle \lambda' + t\lambda_0, \zeta_{\lambda'} \rangle}{2} \ \forall \lambda' \in \ell^\perp \text{ with } |\lambda'| = 1 \right\}.$$

One can show that if $\theta \in M_{\mathbb{R}}^W$ is λ_0-generic with respect to ℓ, then

$$\left(\theta + \overline{\nabla}_{\ell,t}\right) \cap M = (\theta + \nabla_\ell) \cap M \text{ for all } 0 < t \ll 1.$$

For t small, the region $\overline{\nabla}_{\ell,t}$ represents a slight narrowing of the cylinder window $\overline{\nabla}_\ell$ at the top and bottom (see Figure 3.1), which is why we refer to ∇_ℓ as the "barrel" window.

Proof. Let S denote the unit sphere in $\ell^\perp$. For each $\chi \in M_{\mathbb{R}}$, define the function $F_{\chi-\theta} : [0, \infty) \times S \to \mathbb{R}$ as

$$F_{\chi-\theta}(t, \lambda') = |\langle \lambda' + t\lambda_0, \chi - \theta \rangle| - \langle \lambda' + t\lambda_0, \zeta_{\lambda'}/2 \rangle.$$

Note that $F_{\chi-\theta}$ is continuous in both t and λ.

A direct computation yields $\left(\theta + \overline{\nabla}_{\ell,t}\right) \cap M \subset (\theta + \nabla_\ell) \cap M$ for any $t > 0$ and any $\theta \in M_{\mathbb{R}}^W$. The result then follows from the following observations

(1) For $\chi \in \theta + \overline{\nabla}_\ell$ and any $\lambda' \in S$, if $0 < t \leq s$, then $F_{\chi-\theta}(t, \lambda') > 0$ implies $F_{\chi-\theta}(s, \lambda') > 0$.

(2) If θ is λ_0-generic with respect to ℓ, then the containment

$$\left(\theta + \bigcup\nolimits_{t>0} \overline{\nabla}_{\ell,t}\right) \cap M \subset (\theta + \nabla_\ell) \cap M \tag{3.1}$$

is equality.

Indeed, assuming (1) holds implies that $\theta + \overline{\nabla}_{\ell,s} \subset \theta + \overline{\nabla}_{\ell,t}$ if $0 < t \leq s$. There are finitely many T-weights in $\theta + \bigcup_{t>0} \overline{\nabla}_{t,\ell}$, so one can find a $\tau > 0$ with

$\left(\theta + \bigcup_{t>0} \overline{\nabla}_{\ell,t}\right) \cap M = \left(\theta + \overline{\nabla}_{\ell,s}\right) \cap M$ for all $s \in (0, \tau]$. The result then follows if (2) holds.

Proof of (1) and (2):

The first claim follows by a direct calculation. For the second claim, let χ be a T-weight in $\theta + \left(\nabla_\ell \setminus \bigcup_{t>0} \overline{\nabla}_{\ell,t}\right)$. Observe that $\chi \notin \theta + \mathrm{Int}\overline{\nabla}_\ell$. Indeed, $F_{\chi-\theta}(t, \lambda')$ is continuous in both t and λ and by definition, $\chi - \theta \in \mathrm{Int}\overline{\nabla}_\ell$ if and only if $F_{\chi-\theta}(0, \lambda') < 0$ for all $\lambda' \in S$. In particular, if $\chi - \theta \in \mathrm{Int}\overline{\nabla}_\ell$, then because S is compact, $F_{\chi-\theta}(t, \lambda') < 0$ for all $0 \le t \ll 1$, so $\chi - \theta \in \overline{\nabla}_{\ell,t}$ for t sufficiently small resulting in a contradiction.

The fact that $\chi - \theta \notin \mathrm{Int}\overline{\nabla}_\ell$ implies that the set

$$S_{\chi-\theta} := \{\lambda' \in S : |\langle \lambda', \chi - \theta\rangle| - \langle \lambda', \zeta_{\lambda'}/2\rangle = 0\}$$

is non-empty. We claim that there is a cocharacter $\lambda' \in S_{\chi-\theta}$ satisfying

$$|\langle \lambda_0, \chi - \theta\rangle| - \langle \lambda_0, \zeta_{\lambda'}/2\rangle = 0. \tag{3.2}$$

Because χ is a T-weight in $\theta + \left(\nabla_\ell \setminus \bigcup_{t>0} \overline{\nabla}_{\ell,t}\right)$ and because (1) holds, one has that for all $t > 0$, there is a $\lambda' \in S$ (possibly depending on t) satisfying $F_{\chi-\theta}(s, \lambda') > 0$ for all $s \ge t$. Let $\{t_i\}_{i=1}^{\infty} \subset (0, 1]$ be a sequence converging to 0 and for each t_i, let $\lambda'_i \in S$ be a cocharacter satisfying $F_{\chi-\theta}(s, \lambda'_i) > 0$ for all $s \ge t_i$. Let λ' denote the limit of a convergent subsequence of $\{\lambda'_i\}$. Then λ' satisfies

$$F_{\chi-\theta}(s, \lambda') \ge 0 \text{ for all } s \ge 0.$$

This, together with the fact that $|\langle \lambda', \chi - \theta\rangle| \le \langle \lambda', \zeta_{\lambda'}/2\rangle$ (because $\chi - \theta \in \overline{\nabla}_\ell$ and $\lambda' \in S$) implies (a) $|\langle \lambda', \chi - \theta\rangle| = \langle \lambda', \zeta_{\lambda'}/2\rangle$ and (b) $|\langle \lambda_0, \chi - \theta\rangle| \ge \langle \lambda_0, \zeta_{\lambda'}/2\rangle$. By assumption, $\chi - \theta \in \nabla_\ell$, so (a) implies $|\langle \lambda_0, \chi - \theta\rangle| \le \langle \lambda_0, \zeta_{\lambda'}/2\rangle$, which in addition to (b) implies λ' satisfies (3.2).

The existence of a cocharacter $\lambda' \in S_{\chi-\theta}$ satisfying (3.2) implies that the T-weight χ lies in one of the hyperplanes $\theta \pm \zeta_{\lambda'}/2 + \lambda_0^{\perp}$ and this is ruled out by the genericity assumption on θ. Thus, there are no T-weights in $\theta + \left(\nabla_\ell \setminus \bigcup_{t>0} \overline{\nabla}_{\ell,t}\right)$. $\square$

3.2 A vanishing criterion for local cohomology

Here we use the results of [VdB] to state a vanishing criterion for local cohomology. Recall from Definition 3.1.1 the canonical weight ζ_λ associated to a coweight $\lambda \in N$.

Proposition 3.2.1. *Assume Hypotheses 1.2.1. Let $\ell \in M_{\mathbb{Q}}^W$ be such that $\langle \lambda_0, \ell \rangle < 0$ and let U be a representation of G. If for every weight χ appearing in U and for all cocharacters $\lambda' : \mathbb{G}_m \to T_\ell$, there is a $\tau > 0$ such that for all $0 < t < \tau$,*

$$\langle \lambda' + t\lambda_0, \chi - \zeta_{\lambda'} \rangle < 0,$$

then $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X \otimes U)^G = 0$, where $X^{\mathrm{us}} = X^{\mathrm{us}}(\ell)$ is the ℓ-unstable locus. Furthermore, if G is connected and U is irreducible, it suffices to check these conditions for the lowest weight χ appearing in U, and to verify the condition only for $\lambda' : \mathbb{G}_m \to T_\ell$ that act with weights ≤ 0 on the Lie algebra of B.

Remark 3.2.2. Because λ_0 is central, the condition on χ and λ' can be rephrased as saying that $0 \geq \langle \lambda', \chi - \zeta_{\lambda'} \rangle = \langle \lambda', \chi + \det(X^{\lambda'<0}) + \det(\mathfrak{g}^{\lambda'>0}) \rangle$, and if equality holds then $0 > \langle \lambda_0, \chi - \zeta_{\lambda'} \rangle = \langle \lambda_0, \chi + \det(X^{\lambda' \leq 0}) \rangle$. In the special case where λ' is the trivial cocharacter, the condition simplifies to $\langle \lambda_0, \chi + \det(X) \rangle < 0$.

The proof depends on a slight generalization of the classical characterization of semistability for a point in $\mathbb{P}^n$ under the action of a reductive group.

Lemma 3.2.3. *Assume Hypotheses 1.2.1 and let $\ell \in M_{\mathbb{R}}^W$ be such that $\langle \lambda_0, \ell \rangle < 0$, then $x \in X$ is ℓ-unstable if and only if $0 \in \overline{\{G_\ell^\circ \cdot x\}}$.*

Proof. By the Hilbert-Mumford criterion, it suffices to show that ℓ-instability is equivalent to the existence of a cocharacter $\lambda' : \mathbb{G}_m \to G_\ell^\circ$ such that $\lim_{t\to 0} \lambda(t) \cdot x = 0$. In one direction, if there is such a cocharacter $\lambda' : \mathbb{G}_m \to G_\ell^\circ$, then for $N \gg 0$, $\lim_{t\to 0} \lambda'(t)^N \lambda_0(t)x = 0$ as well, and it is destabilizing with respect to ℓ by hypothesis.

Conversely, after a positive rescaling, any ℓ-destabilizing cocharacter for x has the form $\lambda = \lambda' + a\lambda_0$, where λ' is a cocharacter of G_ℓ° and $a > 0$. We can decompose x into joint eigenvectors

$$x = \sum_{v,w\in\mathbb{Z},w<0} x_{v,w},$$

where $x_{v,w}$ has weight v for λ' and weight w for λ_0. Then the fact that $\lim_{t\to 0} \lambda(t) \cdot x$ exists means that $v + aw \geq 0$ whenever $x_{v,w} \neq 0$, and hence that $v \geq -aw > 0$. It follows that $\lim_{t\to 0} \lambda'(t) \cdot x = 0$. $\square$

Proof of Proposition 3.2.1. Let $G^\circ \subset G$ denote the identity component. The homomorphism $\mathbb{G}_m \times G_\ell^\circ \to G^\circ$ taking $(z, g) \mapsto \lambda_0(z)g$ is surjective. For any G-representation M, we have

$$M^G = (M^{G^\circ})^{G/G^\circ} = (M^{\mathbb{G}_m \times G_\ell^\circ})^{G/G^\circ}.$$

In addition, the unstable locus $X^{\mathrm{us}}(\ell)$ is unaffected by isogenies of G, so it suffices to show the vanishing of $(R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X \otimes U))^{\mathbb{G}_m \times G_\ell^\circ}$. The criterion of the proposition is also unaffected by isogenies, so we may therefore replace G with $\mathbb{G}_m \times G_\ell^\circ$ for the remainder of the proof. We may also assume $T = \mathbb{G}_m \times T_\ell$ and $B = \mathbb{G}_m \times B_\ell$.

We use the methods of [VdB] to bound the weights of $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$ as a G-representation.

First of all, Lemma 3.2.3 implies that the ℓ-unstable locus for the action of G on X, in the sense of GIT, agrees with the unstable locus for the action of

G_ℓ° on X as studied in [VdB]. Then Proposition 2.8.1 says that every irreducible G_ℓ°-representation that occurs in $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$ has a highest weight of the form

$$\chi_{hi} = \chi' + \sum_{\alpha \in S \subset \Phi} \alpha \tag{3.3}$$

where $S \subset \Phi$ is a subset of the roots of G_ℓ°, and χ' is some character occurring in the T_ℓ-representation $R\Gamma_{X^{\lambda'>0}}(\mathcal{O}_X)$ for some cocharacter λ' of T_ℓ. The proof of this claim uses the spectral sequence associated to a certain double complex, constructed in [VdB, Lem. 5.2.2], whose differentials are equivariant for the larger group G, not just for G_ℓ°. As a result, the arguments used to establish [VdB, Cor. 6.8] apply verbatim to conclude that every irreducible G-representation occurring in $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X)$ has highest weight of the form (3.3), where χ' is a character for the maximal torus $T \subset G$ that occurs in the T-representation $R\Gamma_{X^{\lambda'>0}}(\mathcal{O}_X)$ for some cocharacter λ' of T_ℓ, and we regard roots of G_ℓ° canonically as weights for $T = \mathbb{G}_m \times T_\ell$ by letting the first factor of $\mathbb{G}_m$ act trivially.

If U is irreducible with lowest weight χ, then to show that $R\Gamma_{X^{\mathrm{us}}}(\mathcal{O}_X \otimes U)^G = 0$, it suffices to verify that the highest weight of U^*, which is $-\chi$, can not have the form (3.3) discussed in the previous paragraph. For any closed subscheme $i : S \hookrightarrow X$ defined by an ideal $I_S \subset \mathcal{O}_X$, the formula $R\Gamma_S \mathcal{O}_X \cong \operatorname{colim} \mathrm{RHom}(\mathcal{O}_X / I_S^n, \mathcal{O}_X)$ gives a convergent non-negative filtration whose n^{th} associated graded complex is $\mathrm{RHom}(i_*(I_S^n / I_S^{n+1}), \mathcal{O}_X)$ for $n \geq 0$. The closed immersion $X^{\lambda'>0} \hookrightarrow X$ is regular, so Grothendieck's formula for $i^!(\mathcal{O}_X)$ identifies the associated graded complex of $R\Gamma_{X^{\lambda'>0}}(\mathcal{O}_X)$ with

$$\mathcal{O}_{X^{\lambda'>0}} \otimes \mathrm{Sym}(X^{\lambda' \leq 0}) \otimes \det(X^{\lambda' \leq 0})[-\dim(X^{\lambda' \leq 0})].$$

It therefore suffices to show that $-\chi$ does not appear as a weight in the T-representation

$$\mathrm{Sym}\left((X^{\lambda'>0})^* \oplus X^{\lambda' \leq 0}\right) \otimes \det(X^{\lambda' \leq 0}) \otimes \wedge^* \mathfrak{g} \tag{3.4}$$

for any cocharacter λ' of T_ℓ (including $\lambda' = 0$). The highest λ'-weight appearing in (3.4) is $-\eta_{\lambda'} = \langle \lambda', \det(X^{\lambda' \leq 0}) + \det(\mathfrak{g}^{\lambda' > 0}) \rangle$, and the summand with this λ'-weight is

$$\operatorname{Sym}\left(X^{\lambda'=0}\right) \otimes \det(X^{\lambda' \leq 0}) \otimes \det(\mathfrak{g}^{\lambda'>0}) \otimes \wedge^*(\mathfrak{g}^{\lambda'=0}). \tag{3.5}$$

The highest λ_0-weight in (3.5) is $\langle \lambda_0, \det(X^{\lambda' \leq 0}) \rangle < 0$. It follows that if either 1) $\langle \lambda', \chi \rangle < \eta_{\lambda'}$ or 2) $\langle \lambda', \chi \rangle = \eta_{\lambda'}$ and $\langle \lambda_0, \chi \rangle < -\langle \lambda_0, \det(X^{\lambda' \leq 0}) \rangle$, then $-\chi$ does not appear as a weight of (3.4).

To complete the proof we must show that if G is connected and U is irreducible with lowest weight χ, then all of the weights appearing in U satisfy the condition with respect to any cocharacter of T'. By the Weyl character formula, all of the weights of U lie in the convex hull of $W \cdot \chi$,[1] so it suffices to verify the inequalities for the characters $w\chi$ with $w \in W$. Because $w\lambda_0 = \lambda_0$ and $w\zeta_{\lambda'} = \zeta_{w\lambda'}$, the inequality $\langle w\lambda' + t\lambda_0, \chi - \zeta_{w\lambda'} \rangle < 0$ is equivalent to $\langle \lambda' + t\lambda_0, w^{-1}\chi - \zeta_{\lambda'} \rangle < 0$, so we may assume λ' pairs non-negatively with the Lie algebra of B. In that case, $\langle \lambda', w\chi \rangle \leq \langle \lambda', \chi \rangle$ and $\langle \lambda_0, w\chi \rangle = \langle \lambda_0, \chi \rangle$, so the inequality for the characters $w\chi$ follows from that for χ. $\square$

Example 3.2.4. Consider the GL_2 representation $X = \operatorname{Sym}^3 \mathbb{C}^2$ and take $\lambda_0 = (-1,-1)$. The subspace $(\omega^*)^\perp \subset N_\mathbb{R}$ consists of zero and positive multiples of $\lambda_1^{\omega^*} = (-1,1)$ and $\lambda_2^{\omega^*} = (1,-1)$, so these cocharacters suffice to describe a region in $M_\mathbb{R}$ consisting of weights that satisfy the inequality in Proposition 3.2.1. Any GL_2-representation whose weights lie in this region gives a vector bundle that has vanishing local cohomology with respect to X^{us}. By Remark 3.2.2, this "allowable"

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[1] The Weyl character formula says that $\mathrm{Ch}(U) \cdot \sum_{w \in W} \varepsilon(w) w(\rho) = \sum_{w \in W} \varepsilon(w) w(\chi + \rho)$, where ρ is the half-sum of negative roots. If one chooses a generic coweight λ and keeps only the terms with the lowest pairing with λ, one gets $\mathrm{Ch}(U)^{\lambda-\min} \cdot \varepsilon(w)w(\rho) = \varepsilon(w)w(\chi+\rho)$, where $w \in W$ is the unique element such that $w(\lambda)$ pairs positively with all dominant weights. So for a dense set of coweights λ, the minimum λ weight in U is always $\geq \langle \lambda, w(\chi) \rangle$ for some $w \in W$. This implies that the convex hull of the weights in $\mathrm{Ch}(U)$ lies in the convex hull of $W \cdot \chi$.

region is

$$\left\{ \chi \in M_{\mathbb{R}} : \text{for } \lambda' = 0, \lambda_1^{\omega^*}, \lambda_2^{\omega^*}, \begin{cases} \langle \lambda', \chi - \zeta_{\lambda'} \rangle < 0, \text{ or} \\ \langle \lambda', \chi - \zeta_{\lambda'} \rangle = 0 \text{ and } \langle \lambda_0, \chi - \zeta_{\lambda'} \rangle < 0 \end{cases} \right\}.$$

In this particular example, $\zeta_{\lambda_1^{\omega^*}} = (-4, -2)$ and $\zeta_{\lambda_2^{\omega^*}} = (-2, -4)$. The resulting region is shown in Figure 3.2.

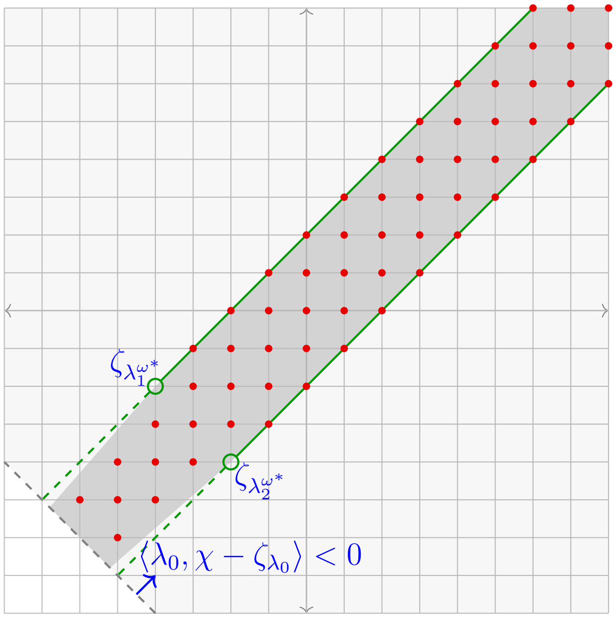

FIGURE 3.2. This diagram illustrates the "allowable" region for Example 3.2.4. The red circles indicate the T-weights that satisfy the inequality in Proposition 3.2.1.

Corollary 3.2.5 (Fully-faithfulness). *Assume Hypotheses 1.2.1, and let $\ell \in M_{\mathbb{R}}^W$ be such that $\langle \lambda_0, \ell \rangle < 0$. Let $\theta \in M_{\mathbb{R}}^W$ be λ_0-generic with respect to ℓ. If $V, W \in \operatorname{Rep}(G)$ are two representations whose weights are contained in $\theta + \nabla_\ell \subset M_{\mathbb{R}}$, then restriction gives a quasi-isomorphism*

$$\operatorname{RHom}_X(\mathcal{O}_X \otimes V, \mathcal{O}_X \otimes W)^G \cong \operatorname{RHom}_{X^{\mathrm{ss}}(\ell)}(\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes V, \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes W)^G.$$

Proof. This is equivalent to showing that $R\Gamma_{X^{\mathrm{us}}(\ell)}(\mathcal{O}_X \otimes W \otimes V^*)^G = 0$. Any weight χ of $W \otimes V^*$ has the form $\chi_2 - \chi_1$ for some $\chi_1, \chi_2 \in (\theta + \nabla_\ell) \cap M$. So it suffices to show that any such χ satisfies the inequalities of Proposition 3.2.1. We must show that for any $\lambda' \in \ell^{\perp}$, one has

$$\langle \lambda', \chi_2 - \chi_1 \rangle \leq \eta_{\lambda'},$$

and if equality holds then $\langle \lambda_0, \chi_2 - \chi_1 - \zeta_{\lambda'} \rangle < 0$.

The first inequality follows immediately from Definition 3.1.3, because $\langle \lambda', \chi_2 \rangle - \langle \lambda', \chi_1 \rangle \leq |\langle \lambda', \chi_2 - \theta \rangle| + |\langle \lambda', \chi_1 - \theta \rangle|$. Equality holds if and only if $\langle \lambda', \chi_2 - \theta \rangle = \eta_{\lambda'}/2$ and $\langle \lambda', \chi_1 - \theta \rangle = -\eta_{\lambda'}/2$. If this is the case, then Definition 3.1.3 implies that $\langle \lambda_0, \chi_2 - \theta \rangle \leq \langle \lambda_0, \zeta_{\lambda'} \rangle / 2$ and $\langle \lambda_0, \chi_1 - \theta \rangle \geq -\langle \lambda_0, \zeta_{\lambda'} \rangle / 2$. Subtracting the second inequality from the first gives $\langle \lambda', \chi_2 - \chi_1 \rangle \leq \langle \lambda_0, \zeta_{\lambda'} \rangle$, and equality holds if and only if $\langle \lambda_0, \chi_2 - \theta \rangle = \langle \lambda_0, \zeta_{\lambda'} \rangle / 2$ and $\langle \lambda_0, \chi_1 - \theta \rangle = -\langle \lambda_0, \zeta_{\lambda'} \rangle / 2$. Because θ is λ_0-generic, there is no weight $\chi_2 \in M$ such that $\langle \lambda_0, \chi_2 - \theta - \zeta_{\lambda'}/2 \rangle = 0$, so $\langle \lambda', \chi_2 - \chi_1 \rangle < \langle \lambda_0, \zeta_{\lambda'} \rangle$. □

3.3 Strong exceptional collections

For an irreducible G-representation U, the central cocharacter λ_0 acts with constant weight on U, which we will denote $\mathrm{wt}_{\lambda_0}(U)$.

Proposition 3.3.1. *Assume Hypotheses 1.2.1, let $\ell \in M_{\mathbb{R}}^W$ be such that $\langle \lambda_0, \ell \rangle < 0$, and let $\theta \in M_{\mathbb{R}}^W$ be λ_0-generic with respect to ℓ (Definition 3.1.6). Let $U_1, \ldots, U_N$ denote the irreducible representations of G whose weights lie in $\theta + \nabla_\ell \subset M_{\mathbb{R}}$, indexed so that $\mathrm{wt}_{\lambda_0}(U_j) \leq \mathrm{wt}_{\lambda_0}(U_i)$ for all $i < j$. Then the G-equivariant locally free sheaves $\mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U_1, \ldots, \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U_N$ are a strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$.*

Proof. This follows immediately from Corollary 3.2.5 and Lemma 3.3.2 below. □

Lemma 3.3.2. *One can choose a total ordering on the set of irreducible representations of G such that $U \prec W$ implies $\mathrm{wt}_{\lambda_0}(W) \leq \mathrm{wt}_{\lambda_0}(U)$. With this ordering, the locally free sheaves $\mathcal{O}_X \otimes U$, where U ranges over all irreducible representations of G, form a full strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$.*

Proof. The fact that such a total ordering exists is elementary – one can use a lexicographic ordering first by $-\mathrm{wt}_{\lambda_0}(-)$, and then break ties using any other total ordering of irreducible representations. The fact that this forms a full exceptional collection follows from the computation for two irreducible representations U and W

$$\mathrm{RHom}_{X/G}(\mathcal{O}_X \otimes U, \mathcal{O}_X \otimes W) \cong (\mathrm{Sym}(X^*) \otimes W \otimes U^*)^G. \tag{3.6}$$

If $\mathrm{wt}_{\lambda_0}(W) > \mathrm{wt}_{\lambda_0}(U)$ then (3.6) vanishes, because the λ_0 weights of $W \otimes U^*$ are strictly positive and the λ_0-weights of $\mathrm{Sym}(X^*)$ are non-negative by hypothesis. If $\mathrm{wt}_{\lambda_0}(U) = \mathrm{wt}_{\lambda_0}(W)$ then (3.6) is isomorphic to $(W \otimes U^*)^G \cong \mathrm{Hom}_{\mathrm{Rep}(G)}(U, W)$, which vanishes if $U \ncong W$ and is isomorphic to k if $U \cong W$. □

3.4 Decorated quiver varieties and iterated fiber bundles

Let Z be a smooth and proper Deligne-Mumford stack with an action of a group G. Then for any Deligne-Mumford stack Y with a principal G-bundle $P \to Y$, one can consider the associated fiber bundle $P \times^G Z \to Y$, whose fiber is Z.[2] We will assume that we have an exceptional collection $E_1, \ldots, E_n \in \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z/G)$ that restricts to a full exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$. When G is a connected reductive group, $\pi_1(G)$ is free, and Z is a projective scheme, any full exceptional

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[2] We take the definition of a G-action on Z to be the data of a morphism of algebraic stacks $\mathcal{Z} \to BG$ along with an isomorphism $\mathrm{pt} \times_{BG} \mathcal{Z} \cong Z$. Then by definition $P \times^G Z := Y \times_{BG} \mathcal{Z}$.

collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z)$ is the restriction of an exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Z/G)$ [P, Lemma 2.2].

Lemma 3.4.1. *For any Deligne-Mumford stack Y and any G-bundle $P \to Y$, the associated fiber bundle $P \times^G Z$ has a semiorthogonal decomposition*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(P \times^G Z) = \langle \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \boxtimes^G E_1, \ldots, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \boxtimes^G E_n \rangle,$$

where $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \boxtimes^G E_i$ denotes the essential image of the fully faithful functor $p_1^*(-) \otimes p_2^*(E_i) : \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) \mapsto \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(P \times^G Z)$, and $p_1 : P \times^G Z \to Y$ and $p_2 : P \times^G Z \to Z/G$ denote the canonical projections.

Proof. By the conservative descent theorem of [BS, Thm. B], it suffices to prove the claim after base change along a faithfully flat morphism $Y' \to Y$. We may therefore assume that $Y = \mathrm{Spec}(R)$ is a smooth affine scheme and the G-bundle P is trivial, so $P \times^G Z \cong \mathrm{Spec}(R) \times Z$. The projection formula implies that $\mathrm{RHom}_{\mathrm{Spec}(R)\times Z}(R \otimes_k E_i, R \otimes_k E_j) \cong R \otimes_k \mathrm{RHom}_Z(E_i, E_j)$, which vanishes if $j > i$ and is isomorphic to R if $i = j$. The fact that R split-generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(R)$ thus implies that the functors $(-) \otimes_k E_i$ are fully faithful and that their essential images are semiorthogonal. To complete the proof, it suffices to show that $R \otimes E_1, \ldots, R \otimes E_n$ split generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathrm{Spec}(R) \times Z)$. This follows from the fact that $\mathrm{Spec}(R) \times Z \to Z$ is an affine morphism, because $F \in \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(R) \times Z)$ is zero if and only if its pushforward to Z is zero. □

As a consequence of Lemma 3.4.1, if Y has a full exceptional collection $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y) = \langle F_1, \ldots, F_m \rangle$, then the objects $F_i \boxtimes^G E_j$, with indices (i, j) given the reverse lexicographic ordering, form a full exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(P \times^G Y)$. If $\{E_i\}$ and $\{F_j\}$ are strongly exceptional, or consist of locally free sheaves, then the same holds for $E_i \boxtimes^G F_j$.

Our next goal is to illustrate a large class of examples in which one can apply Lemma 3.4.1 to construct full exceptional collections.

Definition 3.4.2. A *quiver* consists of a finite set Q_0 of "vertices" and a finite set Q_1 of "arrows," along with source and target maps $s, t : Q_1 \to Q_0$. A *decorated quiver* consists of a quiver Q along with a triple (G_i, V_i, w_i) for all $i \in Q_0$, where G_i is a reductive group, V_i is a G_i-representation, and $w_i \in \mathbb{Z}$ with $w_i \geq 0$. The *representation space* of a decorated quiver is the vector space

$$\mathrm{Rep}(Q) := \prod_{\alpha \in Q_1} \mathrm{Hom}(V_{s(\alpha)}, V_{t(\alpha)}) \times \prod_{i \in Q_0} V_i^{w_i},$$

which is naturally a representation of the group $G_Q := \prod_{i \in Q_0} G_i$. We will sometimes denote a decorated quiver $(Q_1, Q_0, \{(G_i, V_i, w_i\})$ simply by Q, when the context is clear.

Let $Q = (Q_1, Q_0, \{(G_i, V_i, w_i\})$ be a decorated quiver. We assume the underlying quiver has no oriented cycles, so we may choose an identification $Q_0 \cong \{1, \ldots, N\}$ such that arrows only point from i to j for $j > i$. For each $i \in Q_0$, let

$$\mathrm{Rep}(Q)_i := \mathrm{Hom}(k^{w_i} \oplus \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} V_{s(\alpha)}, V_i),$$

$$n_i := w_i + \sum_{\substack{\alpha \in Q_1 \\ t(\alpha) = i}} \dim(V_{s(\alpha)}).$$

Note that $\mathrm{Rep}(Q)_p$ is equipped with a natural action of $G_{\leq p} := \prod_{1 \leq i \leq p} G_i$, and $\mathrm{Rep}(Q)_i \cong V_i^{n_i}$ as a G_i-representation. The natural forgetful maps $\mathrm{Rep}(Q) \to \mathrm{Rep}(Q)_p$ induce an isomorphism $\mathrm{Rep}(Q) \cong \prod_p \mathrm{Rep}(Q)_p$.

Proposition 3.4.3. *Let Q be a decorated quiver with no oriented cycles, and vertices ordered as above, and assume that each V_i has strictly negative weights with respect to some central cocharacter of G_i. For all $i = 1, \ldots, N$, let ℓ_i be a character of G_i*

such that $(V_i^{n_i})^{G_i-\mathrm{ss}}(\ell_i)$ *has finite stabilizers in* G_i*, and let* $\ell = a_1\ell_1 + \cdots + a_N\ell_N$ *for real numbers* a_i *satisfying* $0 < a_N \ll a_{N-1} \ll \cdots \ll a_1$.

Then $\rho \in \mathrm{Rep}(Q)$ *is* ℓ*-semistable with respect to the* G_Q*-action if and only if for every* $i \in Q_0$ *the image of* ρ *in* $\mathrm{Rep}(Q)_i$ *is* ℓ_i*-semistable for the action of* G_i*. Points in* $\mathrm{Rep}(Q)^{\mathrm{ss}}(\ell)$ *have finite stabilizer groups in* G_Q*. If* $Q_{\leq p}$ *denotes the full decorated sub-quiver containing vertices* $\{1, \ldots, p\}$*, then the forgetful map*

$$\mathrm{Rep}(Q_{\leq p})^{G_{\leq p}-\mathrm{ss}}/G_{\leq p} \to \mathrm{Rep}(Q_{\leq p-1})^{G_{\leq p-1}-\mathrm{ss}}/G_{\leq p-1},$$

where semistability is always defined with respect to the restriction of ℓ *to the relevant subgroup, is the fiber bundle associated to the principal* $G_{\leq p-1}$*-bundle*

$$\mathrm{Rep}(Q_{\leq p-1})^{G_{\leq p-1}-\mathrm{ss}} \to \mathrm{Rep}(Q_{\leq p-1})^{G_{\leq p-1}-\mathrm{ss}}(\ell_{\leq p-1})/G_{\leq p-1},$$

whose fiber is the Deligne-Mumford stack $\mathrm{Rep}(Q)_p^{G_p-\mathrm{ss}}/G_p$ *with* $G_{\leq p-1}$*-action coming from the natural* $G_{\leq p}$ *action on* $\mathrm{Rep}(Q)_p$.

Proof. The claims of this proposition are formal consequences of the claim that the canonical isomorphism $\mathrm{Rep}(Q_{\leq p}) \cong \mathrm{Rep}(Q_{\leq p-1}) \times \mathrm{Rep}(G)_p$ induces an isomorphism

$$\mathrm{Rep}(Q_{\leq p})^{G_{\leq p}-\mathrm{ss}} \cong \mathrm{Rep}(Q_{\leq p-1})^{G_{\leq p-1}-\mathrm{ss}} \times \mathrm{Rep}(G)_p^{G_p-\mathrm{ss}}.$$

To show this, represent $\rho \in \mathrm{Rep}(Q_{\leq p})$ as a pair $(\rho_{\leq p-1}, \rho_p)$. If ρ_p is destabilized by a cocharacter λ_p of G_p, then the cocharacter $(0, \lambda_p)$ destabilizes ρ. On the other hand, if ρ_p is semistable, and $\lambda_{\leq p-1}$ is a cocharacter of $G_{\leq p-1}$ that destabilizes $\rho_{\leq p-1}$, then the fact that $\mathrm{Rep}(Q)_p^{G_p-\mathrm{ss}}/G_p$ is a proper Deligne-Mumford stack implies that there is some cocharacter λ_p of G_p such that $(\rho_{\leq p-1}, \rho_p)$ has a limit under $(\lambda_{\leq p-1}, \lambda_p)$. Assuming $a_1, \ldots, a_{p-1}$ fixed, if we choose $a_p > 0$ sufficiently small, then

$$\langle \lambda_{\leq p-1}, a_1\ell_1 + \cdots + a_{p-1}\ell_{p-1} \rangle + a_p\langle \lambda_p, \ell_p \rangle < 0,$$

hence $(\lambda_{\leq p-1}, \lambda_p)$ destabilizes ρ with respect to $a_1\ell_1 + \cdots + a_p\ell_p$. Because finitely many cocharacters of $G_{\leq p-1}$ suffice to destabilize every unstable point of $\operatorname{Rep}(Q_{\leq p-1})$, this shows that we can choose $a_p > 0$ sufficiently small so that if $\rho_{\leq p-1}$ is unstable and ρ_p is semistable, then ρ is unstable. Taken together, this shows that ρ is unstable if either ρ_p or $\rho_{\leq p-1}$ is unstable, and hence

$$\operatorname{Rep}(Q_{\leq p})^{G_{\leq p}-\mathrm{ss}} \subset \operatorname{Rep}(Q_{\leq p-1})^{G_{\leq p-1}-\mathrm{ss}} \times \operatorname{Rep}(Q)_p^{G_p-\mathrm{ss}}.$$

Our hypotheses, and the inductive hypothesis, guarantees that the quotient of the right-hand side by $G_{\leq p}$ is a proper Deligne-Mumford stack. But it is a general fact in GIT that the coarse moduli space of the open substack $\operatorname{Rep}(Q_{\leq p})^{\mathrm{ss}}/G_{\leq p}$ is proper over $\operatorname{Spec}(\mathcal{O}_{\operatorname{Rep}(Q_{\leq p})}^{G_{\leq p}}) = \operatorname{Spec}(k)$. It follows that the open immersion above must be equality. $\square$

We now combine the previous results to show how, given as basic building blocks some examples of linear GIT quotients that admit full strong exceptional collections, one can construct many more examples.

Proposition 3.4.4. *Under the hypotheses of Proposition 3.4.3, assume that for each $i = 1, \ldots, N$ we are given a collection of irreducible G_i-representations $U_{i,j}$ for $j = 1, \ldots, m_i$ such that $\mathcal{O}_{V_i^{n_i}} \otimes_k U_{i,1}, \ldots, \mathcal{O}_{V_i^{n_i}} \otimes U_{i,m_i}$ restricts to a full strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}((V_i^{n_i})^{\mathrm{ss}}(\ell_i)/G_i)$. Then the locally free sheaves*

$$\mathcal{O}_{\operatorname{Rep}(Q)} \otimes (U_{1,j_1} \boxtimes \cdots \boxtimes U_{N,j_N})$$

associated to the G_Q-representations $U_{1,j_1} \boxtimes \cdots \boxtimes U_{N,j_N}$ restrict to a full strong exceptional collection in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(\operatorname{Rep}(Q)^{G_Q-\mathrm{ss}}(\ell)/G_Q)$, using the reverse lexicographic ordering on the indexing tuples $(j_1, \ldots, j_N)$.

Proof. The claim is proved inductively for $\operatorname{Rep}(Q_{\leq p})^{\mathrm{ss}}(\ell_{\leq p})/G_{\leq p}$, for $p = 1, \ldots, N$. The inductive step is an application of Lemma 3.4.1 to the fiber bundle

$\mathrm{Rep}(Q_{\leq p})^{\mathrm{ss}}/G_{\leq p} \to \mathrm{Rep}(Q_{\leq p-1})^{\mathrm{ss}}/G_{\leq p-1}$ of $\mathrm{Rep}(Q)^{\mathrm{ss}}_i(\ell_i)/G_i$ that is constructed in Proposition 3.4.3. $\square$

Example 3.4.5. When $G_i = \mathrm{GL}_{v_i}$ for some integer $v_i \geq 1$, and V_i is the standard representation, then ℓ-stability of $\rho \in \mathrm{Rep}(Q)$ corresponds to the condition that each map

$$\rho_i \oplus \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} \rho_\alpha : k^{w_i} \oplus \bigoplus_{\substack{\alpha \in Q_1 \\ t(\alpha)=i}} V_{s(\alpha)} \to V_i$$

is surjective. Thus $\mathrm{Rep}(Q)^{\mathrm{ss}}(\ell)/G_Q$ is a quiver flag variety, as studied in [C, Thm. 2.2], and Proposition 3.4.3 recovers the main result [C, Thm. 1.1].

Example 3.4.6. Theorem 3.5.1 and Theorem 3.7.1 provide many examples of G_i and representations V_i such that $(V_i^{n_i})^{\mathrm{ss}}/G_i$ admits a full strong exceptional collection of the form $\mathcal{O}_{V_i^{n_i}} \otimes U_{i,j}$, which greatly increases the scope of Proposition 3.4.4.

3.5 Fano GIT quotients by G of rank 2

The goal of this section is to show that the strong exceptional collection from Proposition 3.3.1 is full whenever the rank of the group G is 2, and the GIT parameter is the anticanonical character ω^* with points in the semistable locus having finite G-stabilizers. In this case, the resulting GIT quotients are Fano.

Theorem 3.5.1. *Let G be a split reductive group of rank 2 over k and let X be a linear representation of G satisfying Hypotheses 1.2.1, which we regard as an affine G-scheme. Assume that the weights of X span $M_{\mathbb{R}}$. Then G acts with finite stabilizers on $X^{\mathrm{ss}}(\omega^*)$ if and only if no weights of X are proportional to ω^*, and in* *this case, the vector bundles*

$$\left\{\mathcal{O}_{X^{\mathrm{ss}}(\omega^*)} \otimes U : U \in \mathrm{Rep}(G) \text{ irreducible with weights lying in } \theta + \nabla\right\}$$

split-generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\omega^*)/G)$*. In particular, if* $\theta \in M^W_{\mathbb{R}}$ *is* λ_0*-generic, then the strong exceptional collection from Proposition 3.3.1 is full.*

We begin with a recollection of certain minimal locally free resolutions of equivariant sheaves.

3.5.1 Minimal resolutions of equivariant sheaves

For an anti-dominant cocharacter $\lambda \in N$, that is $\langle \lambda, \alpha \rangle \geq 0$ for all negative roots α, the λ-attracting locus $X^{\lambda \geq 0}$ is a B-module and $G \times^B X^{\lambda \geq 0}$ is a vector bundle over G/B. Consider the diagram of G-equivariant maps

$$\begin{tikzcd} G \times^B X^{\lambda\geq 0} \arrow[r, hook, "j"] \arrow[d] & G/B \times X \arrow[r, "p"] \arrow[d, "\pi"] & G/B \\ \pi(Z(\lambda)) \arrow[r, hook] & X & \end{tikzcd},$$

where $Z(\lambda)$ denotes the image $j(G \times^B X^{\lambda \geq 0})$, the maps π and p are usual projections, and $p \circ j$ is the bundle map $G \times^B X^{\lambda \geq 0} \to G/B$.

Let $\xi(\lambda)$ be the dual of the sheaf of sections of $G \times^B X^{\lambda < 0}$ over G/B. By [W, Prop. 5.1.1], there is a Koszul resolution of $\mathcal{O}_{Z(\lambda)}$ as an $\mathcal{O}_{G/B \times X}$-module

$$\mathcal{K}(\lambda)_\bullet : 0 \to \bigwedge\nolimits^r (p^*\xi(\lambda)) \to \cdots \to \bigwedge\nolimits^2 (p^*\xi(\lambda)) \to p^*\xi(\lambda) \to \mathcal{O}_{G/B \times X}, \tag{3.7}$$

where $r = \operatorname{rank} \xi(\lambda)$.

Definition 3.5.2. For any character $\chi \in M$, define $\mathcal{L}(\chi)$ to be the associated sheaf of sections of the line bundle $G \times^B k\langle \chi \rangle$ over G/B, where $k\langle \chi \rangle$ is the one dimensional B-representation associated to χ.

Let $\chi \in M$ be any character. The complex

$$\mathcal{K}(\lambda, \mathcal{L}(\chi))_\bullet := \mathcal{K}(\lambda)_\bullet \otimes_{\mathcal{O}_{G/B\times X}} p^*\mathcal{L}(\chi)$$

is a locally free resolution of $\mathcal{O}_{Z(\lambda)} \otimes_{\mathcal{O}_{G/B\times X}} p^*\mathcal{L}(\chi)$. The results in [W, Thm. 5.1.2, Thm. 5.4.1] apply to $\mathcal{K}(\lambda, \mathcal{L}(\chi))_\bullet$ yielding the following.

Proposition 3.5.3. *The complex $R\pi_*(\mathcal{O}_{Z(\lambda)} \otimes_{\mathcal{O}_{G/B\times X}} p^*\mathcal{L}(\chi))$ is quasi isomorphic to a minimal G-equivariant complex $C_{\lambda,\chi}$ of free graded $\mathcal{O}_X$-modules with*

$$(C_{\lambda,\chi})_i = \bigoplus_{j\in\mathbb{Z}_{\geq 0}} H^j(G/B, \mathcal{L}(\chi) \otimes \textstyle\bigwedge^{i+j} \xi(\lambda)) \otimes_k \mathcal{O}_X. \tag{3.8}$$

In addition, $C_{\lambda,\chi}$ has homology supported in $G \cdot X^{\lambda\geq 0}$. Since λ is anti-dominant, then so is $-w_0\lambda$ and we define the complex

$$D_{\lambda,\chi} := C_{-w_0\lambda,-w_0\chi}. \tag{3.9}$$

Filtrations of $C_{\lambda,\chi}$ and $D^\vee_{\lambda,\chi}$

Recall that the Weyl group W induces a $*$-action on the weight lattice defined as follows: given $\mu \in M$ and $w \in W$,

$$w * \mu := w(\mu + \rho) - \rho, \tag{3.10}$$

where ρ is the half sum of positive roots. If the stabilizer of $\mu + \rho$ in W is trivial, then there is a unique $w \in W$ such that $w * \mu$ is dominant.

Definition 3.5.4. If $\mathrm{Stab}(\mu + \rho)$ is trivial, define μ^+ to be $w * \mu$, where w is the unique element of the Weyl group such that $w * \mu$ is dominant. Otherwise, we say μ^+ does not exist. When G is connected and μ^+ exists,

$V(\mu^+)$ denotes the irreducible representation with highest weight μ^+.

We will use the convention that $V(\mu^+) = 0$ if μ^+ does not exist.

Recollection 3.5.5 (The Borel-Weil-Bott Theorem)**.** Suppose G is connected and let $\mu \in M$. If $\mu+\rho$ has a non-trivial stabilizer in W, then all the cohomology groups $H^i\left(G/B, \mathcal{L}(\mu)\right)$ vanish. Otherwise, there exists a unique non-vanishing cohomology group $H^{\ell(w)}\left(G/B, \mathcal{L}(\mu)\right)$, where $\ell(w)$ is the length of the unique element $w \in W$ such that $w * \mu$ is dominant. In this case, $H^{\ell(w)}\left(G/B, \mathcal{L}(\mu)\right) \cong V\left(\mu^+\right)$.

The lemmas below are proved in [HLS, Prop. 3.8, Prop. 3.9] but we restate them here for ease of reference.

Lemma 3.5.6. *Assuming G is connected, the complex $C_{\lambda,\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and the remaining terms have G-equivariant filtrations whose associated graded pieces are locally free sheaves of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda, \beta_{i_j} \rangle < 0$ for all $j = 1, \ldots, p$.*

Proof. Note that $\mathcal{L}(\chi) \otimes \bigwedge^\bullet \xi(\lambda)$ admits a G-equivariant filtration by locally free sheaves of the form $\mathcal{L}\left(\chi - \beta_{i_1} - \beta_{i_2} - \ldots - \beta_{i_p}\right)$, where $i_1, \cdots, i_p$ are distinct and $\left\langle \lambda, \beta_{i_j} \right\rangle < 0$. The result then follows by applying Borel-Weil-Bott. $\square$

Lemma 3.5.7. *Assuming G is connected, the complex $D^\vee_{\lambda,\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and the remaining terms have G-equivariant filtrations whose associated graded pieces are locally free sheaves of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi + \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda, \beta_{i_j} \rangle > 0$ for all $j = 1, \ldots, p$.*

Proof. Note that $\mathcal{L}(-w_0\chi) \otimes \bigwedge^\bullet \xi(-w_0\lambda)$ admits a G-equivariant filtration by locally free sheaves of the form $\mathcal{L}\left(-w_0\chi - \beta_{i_1} - \beta_{i_2} - \ldots - \beta_{i_p}\right)$, where $\left\langle -w_0\lambda, \beta_{i_j} \right\rangle < 0$

and $i_1, \cdots, i_p$ are distinct. The result then follows by applying Borel-Weil-Bott and the isomorphism $V((-w_0\alpha)^+)^\vee \cong V(\alpha^+)$ ([HLS, Prop. 3.7]), which holds for any weight $\alpha \in M$ for which α^+ is defined. $\square$

A criterion for $C_{\lambda,\chi}$ and $D^\vee_{\lambda,\chi}$ to be unstably supported

The lemma below implies that when G has rank 2 and $\lambda \in \ell^\perp$, the complexes $C_{\lambda,\chi}$ and $D^\vee_{\lambda,\chi}$ have homology supported in $X^{\mathrm{us}}(\ell)$.

Lemma 3.5.8. *Let X be a linear representation of a reductive group G of rank 2 that satisfies Hypotheses 1.2.1 and let $\ell \in M^W_{\mathbb{R}}$ with $\langle \lambda_0, \ell \rangle < 0$. Then G acts with finite stabilizers on $X^{\mathrm{ss}}(\ell)$ if and only if the weights of X span $M_{\mathbb{R}}$ and no weight of X is proportional to ℓ. In this case,*

$$G \cdot X^{\lambda' \geq 0} \subset X^{\mathrm{us}}(\ell)$$

for any non-zero cocharacter $\lambda' \in N_{\mathbb{R}}$ such that $\langle \lambda', \ell \rangle = 0$.

Proof. Let $\beta_1, \ldots, \beta_n$ denote the weights of X written with multiplicity. Then a point $x \in X$ has a decomposition

$$x = x_1 + \ldots + x_n,$$

where for $i = 1, \ldots, n$, x_i is in the eigenspace corresponding to the weight β_i.

Assume there is a non-zero weight β that is proportional to ℓ. Let x be a non-zero point whose only non-zero component in the decomposition above is in the eigenspace corresponding to β. We claim that x is semistable and has infinite stabilizer group. Indeed, for any cocharacter $\lambda \in N$, $\lim_{t\to 0} \lambda(t) \cdot x$ exists if and only if $\langle \lambda, \beta \rangle \geq 0$. The latter is equivalent to $\langle \lambda, \ell \rangle \geq 0$ because under Hypotheses 1.2.1

the assumption that $\langle \lambda_0, \ell \rangle < 0$ guarantees that β is positively proportional to ℓ. Thus x is semistable by the Hilbert-Mumford criterion. Moreover, x is fixed by any non-trivial cocharacter of the subgroup $G_\ell^\circ = \ker(\ell)^\circ$, so it is strictly semistable and it also has infinite G-stabilizers.

On the other hand, suppose there is a point in $X^{\mathrm{ss}}(\ell)$ with positive dimensional stabilizer. If one chooses $x \in X^{\mathrm{ss}}(\ell)$ to be a point whose orbit has minimal dimension, and hence whose stabilizer has maximal dimension, then $G \cdot x \subset X^{\mathrm{ss}}(\ell)$ is closed. Such a point has a reductive stabilizer group, which is positive dimensional by hypothesis. By replacing x with $g \cdot x$ for some $g \in G$, we may assume that a maximal torus of $\mathrm{Stab}(x)$ is contained in T, thus x is fixed by a non-trivial cocharacter $\lambda \in N$.

Now x has a decomposition $x = x_{i_1} + \ldots + x_{i_p}$, where for $j = 1, \ldots, p$, $x_{i_j} \neq 0$ and $\langle \lambda, \beta_{i_j} \rangle = 0$. Since x is semistable, this cocharacter must necessarily pair non-negatively with ℓ by the Hilbert-Mumford criterion. In fact we must have $\langle \lambda, \ell \rangle = 0$, otherwise $-\lambda$ would be a destabilizing cocharacter. Since M has rank 2, the weights $\beta_{i_1}, \ldots, \beta_{i_p}$ must lie in the span of ℓ.

When X has no weights in the span of ℓ, if one starts with a cocharacter λ' with $\langle \lambda', \ell \rangle = 0$, one can perturb λ' slightly to obtain a $\lambda \in N$ with $\langle \lambda, \ell \rangle < 0$ and $X^{\lambda' \geq 0} = X^{\lambda \geq 0}$. The latter lies in the unstable locus by the Hilbert-Mumford criterion, thus $G \cdot X^{\lambda' \geq 0} \subset X^{\mathrm{us}}(\ell)$. $\square$

3.5.2 Proof of Theorem 3.5.1

We begin with the following observation.

Lemma 3.5.9. *It suffices to prove Theorem 3.5.1 for connected groups G.*

Proof. Let G be a split reductive group and G° the connected component containing the identity. Note that $T \subset G^\circ$, so the windows defined in Section 3.1 for G and G° coincide. We claim that $X^{G-\mathrm{ss}} = X^{G^\circ-\mathrm{ss}}$. Under the isomorphism $X/G^\circ \cong (X \times (G/G^\circ))/G$, $X^{G^\circ-\mathrm{ss}}$ corresponds to the G-semistable locus of $X \times (G/G^\circ)$. Therefore, if we consider the G-equivariant finite surjective étale morphism $f : X \times (G/G^\circ) \to X$, which corresponds to the canonical morphism $X/G^\circ \to X/G$, we must show that $(X \times G/G^\circ)^{G-\mathrm{ss}} = f^{-1}(X^{G-\mathrm{ss}})$. This follows from the Hilbert-Mumford criterion for affine G-schemes, because the finiteness of G/G° implies that if $x \in X \times (G/G^\circ)$ and λ is a cocharacter in G, then $\lim_{t\to 0} \lambda(t) \cdot x$ exists if and only if $\lim_{t\to 0} \lambda(t) \cdot f(x)$ exists.

We continue to use the notation f for the finite surjective étale morphism $X^{\mathrm{ss}}/G^\circ \cong (X^{\mathrm{ss}} \times (G/G^\circ))/G \to X^{\mathrm{ss}}/G$. We claim that for any collection $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ of vector bundles on X^{ss}/G° that split-generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G^\circ)$, the collection $\{f_*(V_\alpha)\}_{\alpha\in\mathcal{A}}$ split-generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$. Because $\mathrm{D}_{\mathrm{qc}}(X^{\mathrm{ss}}/G^\circ)$ is compactly generated by $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G^\circ)$, the fact that $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ split generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G^\circ)$ is equivalent to the fact that for any $\mathcal{F} \in \mathrm{D}_{\mathrm{qc}}(X^{\mathrm{ss}}/G^\circ)$, $\mathrm{Hom}_{X^{\mathrm{ss}}/G^\circ}(V_\alpha, \mathcal{F}) = 0$ for all $\alpha \in \mathcal{A}$ implies $\mathcal{F} = 0$. The same holds for split generation of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$.

Now let $\mathcal{H} \in \mathrm{D}_{\mathrm{qc}}(X^{\mathrm{ss}}/G)$ be an object satisfying $\mathrm{Hom}_{X^{\mathrm{ss}}/G}(f_*V_\alpha, \mathcal{H}) = 0$ for all $\alpha \in \mathcal{A}$. Then

$$\begin{aligned} 0 &= \mathrm{Hom}_{X^{\mathrm{ss}}/G}(f_*V_\alpha, \mathcal{H}) \\ &\cong \mathrm{Hom}_{X^{\mathrm{ss}}/G^\circ}(V_\alpha, f^!\mathcal{H}) \\ &\cong \mathrm{Hom}_{X^{\mathrm{ss}}/G^\circ}(V_\alpha, f^*\mathcal{H}om(f_*\mathcal{O}_{X^{\mathrm{ss}}/G^\circ}, \mathcal{H})), \end{aligned}$$

where the second isomorphism follows from the description of $f^!$ for finite flat morphisms. This implies $f^*\mathcal{H}om(f_*\mathcal{O}_{X^{\mathrm{ss}}/G^\circ}, \mathcal{H}) \cong 0$. Because f is surjective and étale, we have $\mathcal{H}om(f_*\mathcal{O}_{X^{\mathrm{ss}}/G^\circ}, \mathcal{H}) = 0$. Moreover, f is finite and étale, so $f_*\mathcal{O}_{X^{\mathrm{ss}}/G^\circ}$ is locally free, and this implies $\mathcal{H} = 0$. It follows that $\{f_*(V_\alpha)\}_{\alpha\in\mathcal{A}}$ split generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$.

If $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G^\circ)$ is split-generated by vector bundles of the form $\mathcal{O}_{X^{\mathrm{ss}}} \otimes U$, where U is an irreducible G°-representation whose weights lie in $\theta + \nabla$, then we have seen that the collection $f_*(\mathcal{O}_{X^{\mathrm{ss}}} \otimes U)$ will split generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$. We compute

$$f_*\left(\mathcal{O}_{X^{\mathrm{ss}}} \otimes U\right) \cong \mathcal{O}_{X^{\mathrm{ss}}} \otimes U \otimes k[G/G^\circ].$$

Because T acts trivially on $k[G/G^\circ]$, the weights of the G-representation $U \otimes k[G/G^\circ]$ lie in $\theta + \nabla$ if and only if the weights of U lie in $\theta + \nabla$. □

Proof of Theorem 3.5.1. That $X^{\mathrm{ss}}(\omega^*)$ has finite G-stabilizers if and only if X has no weights proportional to ω^* follows from Lemma 3.5.8 and the fact that Hypotheses 1.2.1 implies zero is not a weight of X.

Recall that for χ a dominant character, $V(\chi)$ denotes the irreducible representation of G with highest weight χ. The restriction functor $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$ is essentially surjective because $X^{\mathrm{ss}}/G \subset X/G$ is an open substack. Moreover, X is affine, so $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$ is split-generated by vector bundles of the form $\mathcal{O}_X \otimes U$, where U is an irreducible G-representation. Thus, because all irreducible representations of a connected G are highest weight representations $V(\chi)$, all vector bundles of the form $\mathcal{O}_{X^{\mathrm{ss}}} \otimes V(\chi)$ with χ dominant split-generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}/G)$.

As observed in the proof of Proposition 3.2.1, the weights of $V(\chi)$ are contained in the convex hull of $W \cdot \chi$. Because $\theta + \nabla$ is W-invariant and convex, the weights of $V(\chi)$ lies in $\theta + \nabla$ if and only if $\chi \in \theta + \nabla$. Therefore by Lemma 3.5.9, it suffices to show that for G connected and for any $\chi \in M^+$, the vector bundle

$\mathcal{O}_X \otimes V(\chi)$ lies in the smallest triangulated subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G)$ that contains i) complexes that are set-theoretically supported on the unstable locus, and ii) the vector bundles $\mathcal{O}_X \otimes V(\mu)$ for $\mu \in M^+ \cap (\theta + \nabla)$.

For connected G, it will be convenient to introduce the following notation, for any subset $S \subset M_{\mathbb{R}}$:

$$\mathcal{C}(S) := \left\{ \begin{array}{c} \text{smallest full triangulated subcategory of } \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/G) \\ \text{containing all unstably supported complexes and the} \\ \text{locally free sheaves } \mathcal{O}_X \otimes V(\chi) \text{ for all } \chi \in S \cap M^+ \end{array} \right\}.$$

Thus we must show $\mathcal{C}(\theta + \nabla) = \mathcal{C}(M_{\mathbb{R}})$. We do this in three steps.

(1) *Reduction to the λ_0-strip*: $\mathcal{C}(M_{\mathbb{R}}) = \mathcal{C}(\theta + B_{\lambda_0})$.

(2) *Reduction to the cylinder window*: $\mathcal{C}(\theta + B_{\lambda_0}) = \mathcal{C}(\theta + \overline{\nabla})$.

(3) *Reduction to the barrel window*: $\mathcal{C}(\theta + \overline{\nabla}) = \mathcal{C}(\theta + \nabla)$.

The λ_0-strip, the cylinder window, and the barrel window are as defined in Section 3.1. Because G has rank 2 and $\omega^* \neq 0$, the subset $(\omega^*)^\perp \subset N_{\mathbb{R}}$ is one-dimensional. In this case, the barrel window has a more concrete description as follows.

Let λ' be any cocharacter that spans $(\omega^*)^\perp$. The rank 2 hypothesis implies $\eta_{\lambda'} = \eta_{-\lambda'}$. Thus the barrel window $\theta + \nabla$ consists of those $\chi \in M_{\mathbb{R}}$ such that $\chi - \theta \in B_{\lambda_0}$ and satisfying either

(i) $|\langle \lambda', \chi - \theta \rangle| < \frac{1}{2}\eta_{\lambda'}$, or

(ii) $\langle \lambda', \chi - \theta \rangle = \frac{1}{2}\eta_{\lambda'}$ and

$$\frac{1}{2}\langle \lambda_0, \det X^{-\lambda' \leq 0} \rangle \leq \langle \lambda_0, \chi - \theta \rangle \leq \frac{1}{2}\langle \lambda_0, -\det X^{\lambda' \leq 0} \rangle, \text{ or}$$

(iii) $\langle \lambda', \chi - \theta \rangle = -\frac{1}{2}\eta_{\lambda'}$ and

$$\frac{1}{2}\langle \lambda_0, \det X^{\lambda' \leq 0} \rangle \leq \langle \lambda_0, \chi - \theta \rangle \leq \frac{1}{2}\langle \lambda_0, -\det X^{-\lambda' \leq 0} \rangle.$$

Note that the terms involving $\mathfrak{g}^{\lambda' < 0}$ do not appear in the inequalities in (ii) and (iii) because λ_0 is a central cocharacter.

The aforementioned reduction steps will be carried out inductively using the following quantities attached to each $\chi \in M^+$:

$$\alpha_\chi := 2|\langle \lambda_0, \chi - \theta \rangle|/\eta_{\lambda_0}, \text{ and} \tag{3.11}$$

$$r_\chi := \frac{|\langle \lambda', \chi - \theta \rangle| - \langle \lambda', \rho \rangle}{\langle \lambda', -\det X^{\lambda' \leq 0} \rangle}, \tag{3.12}$$

where ρ is one half of positive roots and $\lambda' \in (\omega^*)^\perp$ is an anti-dominant cocharacter satisfying $\langle \lambda', \chi - \theta \rangle \leq 0$. The latter is possible because if the Weyl group is non-trivial, there is a unique anti-dominant λ' up to positive scaling that must pair non-negatively with any dominant character, and if the Weyl group is trivial, then both λ' and $-\lambda'$ are anti-dominant. We also observe that r_χ is well-defined and non-negative. Indeed, $\langle \lambda', -\det X^{\lambda' \leq 0} \rangle > 0$ because the weights of X span $M_\mathbb{R}$ and $\langle \lambda', \rho \rangle \leq 0$ because λ' is anti-dominant.

Remark 3.5.10. The quantity r_χ is a modification of the induction parameter used in the combinatorial generation algorithms in [ŠVdB1, HLS, ŠVdB2] that is suitable for our particular window.

Lemma 3.5.11 (Reduction to the λ_0-strip)**.** $\mathcal{C}(M_\mathbb{R}) = \mathcal{C}(\theta + B_{\lambda_0})$.

Proof of Lemma 3.5.11. Let $\chi \notin \theta + B_{\lambda_0}$ be a dominant T-weight. We wish to show that the tautological vector bundle $\mathcal{O}_X \otimes V(\chi)$ is in $\mathcal{C}(\theta + B_{\lambda_0})$. We proceed by induction on the quantity α_χ in (3.11). Note that $\chi \notin \theta + B_{\lambda_0}$ if and only if $\alpha_\chi > 1$.

If $\langle \lambda_0, \chi - \theta \rangle = \alpha_\chi \eta_{\lambda_0}/2$, consider the G-equivariant complex $C_{\lambda_0,\chi}$. The assumption that λ_0 pairs strictly negatively with all weights of X (Hypotheses 1.2.1) implies that the subset $X^{\lambda_0 \geq 0}$ coincides with the stratum of $X^{\mathrm{us}}(\omega^*)$ consisting only of the origin, so $C_{\lambda_0,\chi}$ is unstably supported. Lemma 3.5.6 says that $C_{\lambda_0,\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that are direct sums of vector bundles of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi - \beta_{i_1} - \cdots - \beta_{i_p}$ with $p > 0$ and $i_1, \ldots, i_p$ distinct. Because $\alpha_\chi > 1$, one can check that the weights μ^+ lie in the interior of the strip $\theta + \alpha_\chi B_{\lambda_0}$.

Otherwise, $\langle \lambda_0, \chi - \theta \rangle = -\alpha_\chi \eta_{\lambda_0}/2$ and one takes the complex $C_{\lambda_0,\chi+\det X}$. It is an unstably supported complex that, by Lemma 3.5.6, relates $\mathcal{O}_X \otimes V(\chi)$ with sheaves of the form $\mathcal{O}_X \otimes V(\mu^+)$ for $\mu = \chi + \beta_{i_1} + \cdots + \beta_{i_p}$ with $p > 0$ and $i_1, \ldots, i_p$ distinct. As in the previous case, these weights lie in the interior of the strip $\theta + \alpha_\chi B_{\lambda_0}$.

Because the set of possible α_χ that arise above is discrete, we can apply induction to conclude that $\mathcal{O}_X \otimes V(\chi) \in \mathcal{C}(\theta + B_{\lambda_0})$ for any $\chi \in M^+$, hence $\mathcal{C}(M_\mathbb{R}) = \mathcal{C}(\theta + B_{\lambda_0})$. Figure 3.3 illustrates an example of this reduction. □

Lemma 3.5.12 (Reduction to the cylinder window)**.** $\mathcal{C}(\theta + B_{\lambda_0}) = \mathcal{C}(\theta + \overline{\nabla})$.

Proof of Lemma 3.5.12. Let χ be a dominant T-weight in $\theta + (B_{\lambda_0} \setminus \overline{\nabla})$. We would like to show that the tautological vector bundle $\mathcal{O}_X \otimes V(\chi)$ is in $\mathcal{C}(\theta + \overline{\nabla})$. We will make use of the quantity r_χ in (3.12), thus we choose $\lambda' \in (\omega^*)^\perp$ to be anti-dominant such that $\langle \lambda', \chi - \theta \rangle \leq 0$.

Note that the inequality $r_\chi \leq 1/2$ is equivalent to

$$|\langle \lambda', \chi - \theta \rangle| \leq \langle \lambda', -\det X^{\lambda' \leq 0} \rangle / 2 + \langle \lambda', \rho \rangle = \eta_{\lambda'}/2,$$

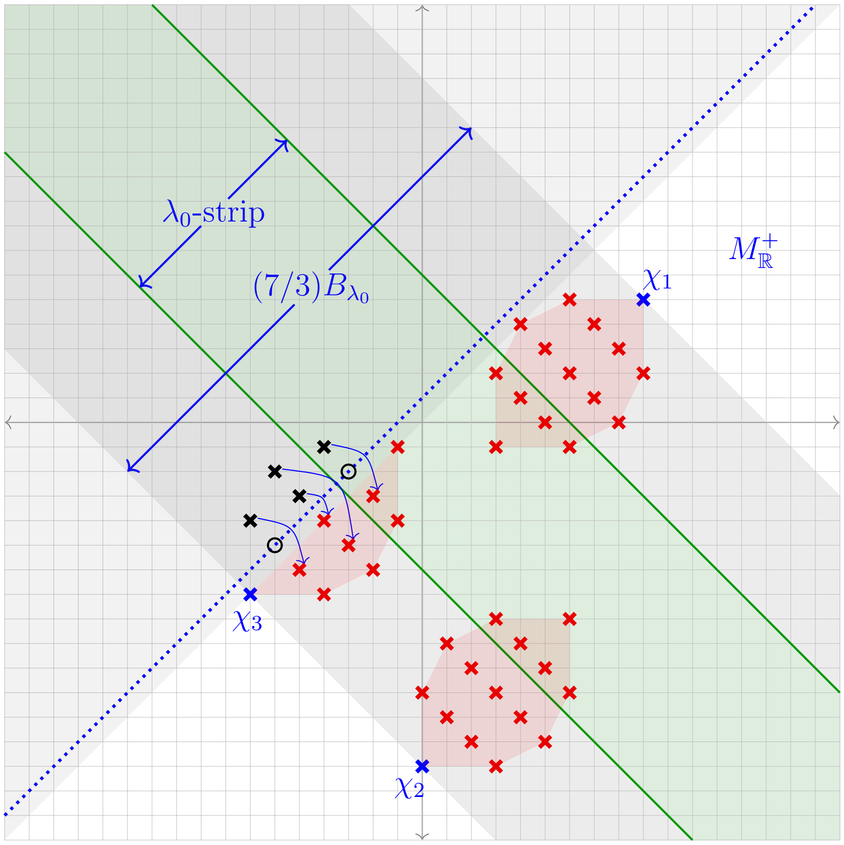

FIGURE 3.3. This illustrates the reduction to the λ_0-strip for the GL_2 representation $X = \mathrm{Sym}^3 \mathbb{C}^2$ with $\lambda_0 = (-1,-1)$ and $\theta = 0$. The roots of G are $(1,-1), (-1,1)$, where we take $(1,-1)$ to be the positive root. The weights of X are $(0,3), (1,2), (2,1), (3,0)$, so $\eta_{\lambda_0} = 12$. Here we are using the standard identification $M_{\mathbb{R}} \cong N_{\mathbb{R}} \cong \mathbb{R}^2$. The T-weights indicated by blue crosses satisfy $\langle \lambda_0, \chi \rangle = \pm \alpha_\chi \eta_{\lambda_0}/2$, where $\alpha_\chi = 7/3$. In all three cases, the red crosses indicate the weights μ^+ appearing in the respective complexes as described in Lemma 3.5.11. For χ_1, the complex is C_{λ_0,χ_1} and all μ are dominant, so $\mu^+ = \mu$. For χ_3, the complex is $C_{\lambda_0,\chi_3+\det X}$ and some of the μ are not dominant. These weights are indicated by the black crosses when μ^+ exists and by the black circles otherwise. The ρ-shifted Weyl group action $(-)^+$ is reflection along the dotted blue line as indicated by the arrows. One sees that all the resulting μ^+ lie in a strictly smaller λ_0-strip.

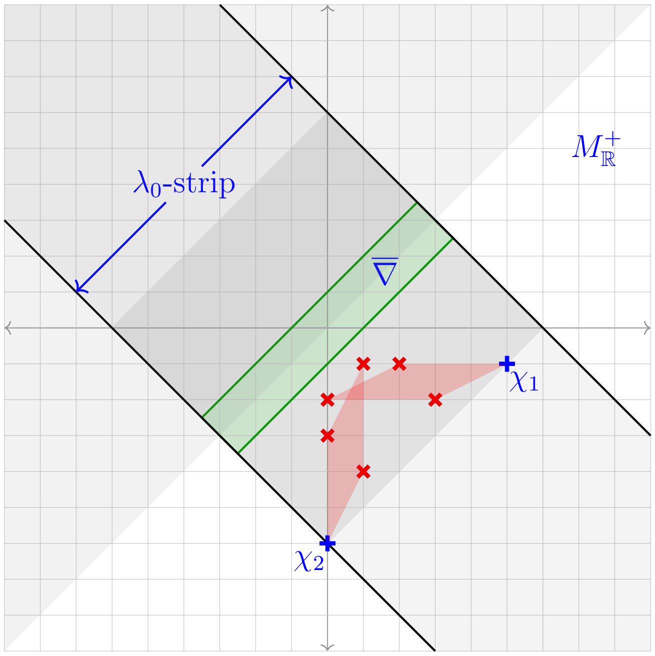

FIGURE 3.4. This diagram illustrates the reduction to the cylinder window $\overline{\nabla}$ for the GL_2 representation $X = \mathrm{Sym}^3 \mathbb{C}^2$ with $\lambda_0 = (-1,-1)$, $\lambda' = (-1,1)$, and $\theta = 0$. As in Figure 3.3, $\eta_{\lambda_0} = 12$ and $\eta_{\lambda'} = 2$. In this example, $Q_{\lambda'} = 0$. The character χ_1 falls into Case 2a and one selects the complex C_{λ',χ_1}. On the other hand, χ_2 falls into Case 2b and one selects the complex $D^{\vee}_{\lambda',\chi_2}$. In both cases, the resulting weights μ^+ are indicated by the red crosses.

where the equality used the identification $2\rho = \det \mathfrak{g}^{\lambda'<0}$, which holds because λ' is anti-dominant. Thus for $\chi \in \theta + B_{\lambda_0}$, $\chi \in \theta + \overline{\nabla}$ if and only if $r_\chi \leq 1/2$.

The complexes $D^{\vee}_{\lambda',\chi}$ or $C_{\lambda',\chi}$ from Proposition 3.5.3 are both unstably supported by Lemma 3.5.8. Thus the key idea is to choose either $D^{\vee}_{\lambda',\chi}$ or $C_{\lambda',\chi}$ in such a way that the corresponding weights μ^+ from Lemma 3.5.6 and Lemma 3.5.7 still lie in the λ_0-strip and have $r_{\mu^+} < r_\chi$.

The number

$$Q_{\lambda'} := \frac{1}{2}\eta_{\lambda_0} + \langle \lambda_0, \det X^{\lambda'<0} \rangle. \tag{3.13}$$

will allow us to decide the appropriate complex. Note that $Q_{\lambda'} = 0$ when the Weyl group is non-trivial. There are two cases to consider.

2a. The case $-\eta_{\lambda_0}/2 \le \langle \lambda_0, \chi - \theta \rangle \le Q_{\lambda'}$:

Take the complex $C_{\lambda',\chi}$. Lemma 3.5.6 says that $C_{\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda', \beta_{i_j} \rangle < 0$ for all $j = 1, \ldots, p$.

Claim 3.5.13. *For* μ *as in case 2a, and assuming* $r_\chi \ge 1/2$*, if* μ^+ *exists it satisfies the following.*

(1) μ^+ *is in the* λ_0*-strip* $\theta + B_{\lambda_0}$*, and*

(2) $r_{\mu^+} \le r_\chi$ *with strict inequality if* $r_\chi > 1/2$.

2b. The case $Q_{\lambda'} < \langle \lambda_0, \chi - \theta \rangle \le \eta_{\lambda_0}/2$:

Take the complex $D^\vee_{\lambda',\chi}$. Lemma 3.5.7 says that $D^\vee_{\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi + \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda', \beta_{i_j} \rangle > 0$ for all $j = 1, \ldots, p$.

Claim 3.5.14. *For* μ *as in case 2b, and assuming* $r_\chi \ge 1/2$*, if* μ^+ *exists it satisfies the following.*

(1) μ^+ *is in the* λ_0*-strip* $\theta + B_{\lambda_0}$*, and*

(2) $r_{\mu^+} \le r_\chi$ *with strict inequality if* $r_\chi > 1/2$.

With these choices of complexes, Claim 3.5.13 and Claim 3.5.14 together with the fact that the set of r_χ that arise is discrete imply by induction that $\mathcal{O}_X \otimes V(\chi) \in$

$\mathcal{C}(\theta + \overline{\nabla})$ for any $\chi \in M^+ \cap (\theta + B_{\lambda_0})$. Hence $\mathcal{C}(\theta + B_{\lambda_0}) = \mathcal{C}(\theta + \overline{\nabla})$. Figure 3.4 illustrates an example of this reduction.

We now verify Claim 3.5.13 and Claim 3.5.14.

Proof of Claim 3.5.13. For the first claim, note that $\langle \lambda_0, \mu^+ \rangle = \langle \lambda_0, \mu \rangle$ because λ_0 is central, thus it suffices to show μ is in the λ_0-strip. From the description of μ, we have the inequalities

$$\langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \mu - \theta \rangle \leq \langle \lambda_0, \chi - \theta \rangle + \langle \lambda_0, -\det X^{\lambda' < 0} \rangle, \tag{3.14}$$

which together with the assumed bounds $-\eta_{\lambda_0}/2 \leq \langle \lambda_0, \chi - \theta \rangle \leq Q_{\lambda'}$ imply that $\mu \in \theta + B_{\lambda_0}$. The strict inequality on the left holds because in the description of μ above, $p > 0$ and all weights of X are assumed to pair strictly negatively with λ_0.

For the second claim, we observe from the description of μ that

$$\langle \lambda', \chi - \theta \rangle \leq \langle \lambda', \mu - \theta \rangle \leq \langle \lambda', \chi - \theta \rangle - \langle \lambda', \det(X^{\lambda' < 0}) \rangle.$$

Using the fact that $\langle \lambda', \chi - \theta \rangle \leq 0$, we have $\langle \lambda', \chi - \theta \rangle = -r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle$, so we can rewrite the bounds on $\langle \lambda', \mu - \theta \rangle$ as

$$\begin{aligned} -r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle < \langle \lambda', \mu - \theta \rangle & \\ \leq (1 - r_\chi) \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle & \\ \leq r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle & \end{aligned} \tag{3.15}$$

For the last inequality, we have used the assumption $r_\chi \geq 1/2$, and this inequality is strict if $r_\chi > 1/2$.

If $\rho = 0$, then $\mu^+ = \mu$ and it is immediate from (3.15) that $r_{\mu^+} \leq r_\chi$ with strict inequality if $r_\chi > 1/2$.

If $\rho \neq 0$, then there is a single non-trivial element of the Weyl group w_0, namely the reflection along $\mathrm{Span}(\omega^*)$. Thus θ must be in the span of ω^*. In this case, if $\mu^+ = \mu$, then μ is dominant and because λ' is anti-dominant, we must have $\langle \lambda', \mu - \theta \rangle \leq 0$. Applying this to (3.15), we have

$$|\langle \lambda', \mu - \theta \rangle| < r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle + \langle \lambda', \rho \rangle,$$

and it implies $r_{\mu^+} < r_\chi$. Otherwise, $\mu^+ = w_0(\mu + \rho) - \rho$, and

$$\langle \lambda', \mu^+ \rangle = \langle \lambda', -\mu - 2\rho \rangle.$$

Applying this to (3.15), we have

$$-r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle \leq \langle \lambda', \mu^+ + \theta \rangle < r_\chi \langle \lambda', -\det X^{\lambda' \leq 0} \rangle - \langle \lambda', \rho \rangle \quad (3.16)$$

with strict inequality on the left if $r_\chi > 1/2$. Since $\theta \in \mathrm{Span}(\omega^*)$, the pairing $\langle \lambda', \theta \rangle$ is zero, and since λ' is anti-dominant $\langle \lambda', \rho \rangle < 0$. Combining these with $\langle \lambda', \mu^+ \rangle = \langle \lambda', -\mu - 2\rho \rangle$ and the triangle inequality, we obtain the middle inequality in the following

$$r_{\mu^+} := \frac{|\langle \lambda', \mu^+ - \theta \rangle| - \langle \lambda', \rho \rangle}{\langle \lambda', -\det X^{\lambda' \leq 0} \rangle} \leq \frac{|\langle \lambda', \mu^+ + \theta \rangle + \langle \lambda', \rho \rangle|}{\langle \lambda', -\det X^{\lambda' \leq 0} \rangle} \leq r_\chi.$$

The rightmost inequality then follows from (3.16), thus it is strict if $r_\chi > 1/2$. □

Proof of Claim 3.5.14. For the first claim, it suffices to show μ is in the λ_0-strip. The key observation here is

$$-\frac{1}{2}\eta_{\lambda_0} - \langle \lambda_0, \det X^{\lambda' > 0} \rangle \leq Q_{\lambda'}, \quad (3.17)$$

which follows from $-\eta_{\lambda_0} = \langle \lambda_0, \det X \rangle \leq \langle \lambda_0, \det X^{\lambda' > 0} + \det X^{\lambda' < 0} \rangle$. In fact, the inequality above is an equality by the assumption that there are no weights of X in the span of ω^*. Now from the description of μ, we have

$$\langle \lambda_0, \chi - \theta \rangle + \langle \lambda_0, \det X^{\lambda' > 0} \rangle \leq \langle \lambda_0, \mu - \theta \rangle < \langle \lambda_0, \chi - \theta \rangle,$$

which together with the assumed bounds for $\langle \lambda_0, \chi - \theta \rangle$ and (3.17) imply that $\mu \in \theta + B_{\lambda_0}$.

For the second claim, we observe that $\langle \lambda', \det X^{\lambda' \geq 0} \rangle = \langle \lambda', -\det X^{\lambda' \leq 0} \rangle$. With this identification, one obtains the inequalities (3.15) for these μ and the rest of the proof follows exactly as that of Claim 3.5.13. ☐

☐

Lemma 3.5.15 (Reduction to the barrel window)**.** $\mathcal{C}(\theta + \overline{\nabla}) = \mathcal{C}(\theta + \nabla)$.

Proof of Lemma 3.5.15. Recall the quantity α_χ in (3.11) and let us choose an indexing of the finite set $M^+ \cap (\theta + (\overline{\nabla} \setminus \nabla)) = \{\chi_1, \dots, \chi_m\}$ such that $\alpha_{\chi_i} \geq \alpha_{\chi_j}$ for $i < j$. As in previous reductions, we will produce for each χ_i an unstably supported complex that relates $\mathcal{O}_X \otimes V(\chi_i)$ with sheaves that admit a G-equivariant filtration with associated graded pieces of the form $\mathcal{O}_X \otimes V(\mu^+)$ with either $r_{\mu^+} < r_{\chi_i}$ (in which case $\mu^+ \in \theta + \nabla$) or $r_{\mu^+} = 1/2$ and $\alpha_{\mu^+} < \alpha_\chi$ (in which case $\mu^+ = \chi_j$ for some $j > i$). It will then follow by induction that for any n,

$$\mathcal{C}(\theta + \overline{\nabla}) = \mathcal{C}((\theta + \nabla) \cup \{\chi_{n+1}, \dots, \chi_m\}).$$

The case $m = n$ would give $\mathcal{C}(\theta + \nabla) = \mathcal{C}(\theta + \overline{\nabla})$.

Let $\chi \in \{\chi_1, \dots, \chi_m\}$. By definition of the barrel window, $|\langle \lambda', \chi - \theta \rangle| = \eta_{\lambda'}/2$, thus $r_\chi = 1/2$, where we again take $\lambda' \in (\omega^*)^\perp$ to be an anti-dominant cocharacter such that $\langle \lambda', \chi - \theta \rangle \leq 0$. By the rank-two description of the barrel window, one of the following cases (3a) or (3b) must hold.

3a. The case $-\eta_{\lambda_0}/2 \leq \langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \det X^{\lambda' \leq 0} \rangle / 2$:

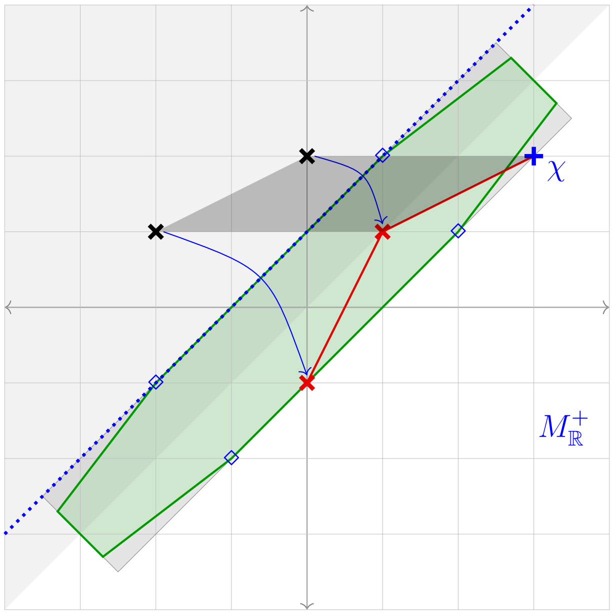

FIGURE 3.5. The diagram illustrates the reduction from the cylinder window to the barrel window for our running GL_2 representation $X = \mathrm{Sym}^3 \mathbb{C}^2$ from Figure 3.3 and Figure 3.4. Here $\lambda' = (-1, 1)$ and $Q_{\lambda'} = 0$. We have indicated by the blue diamonds the special weights $\pm\zeta_{\lambda'}/2 = \pm(-2, -1)$ and $\pm\zeta_{-\lambda'}/2 = \pm(-1, -2)$ that are relevant in the passage from the cylinder to the barrel window. The character χ lies in $M^{+} \cap (\overline{\nabla} \setminus \nabla)$ and it falls into Case 3a. Thus, one takes the complex $C_{\lambda', \chi}$. The weights μ appearing as in Lemma 3.5.6 are not all dominant and we have indicated these by the black crosses. As in Figure 3.3, the ρ-shifted action $(-)^{+}$ for these weight is the reflection along the dotted line as shown by the blue arrows. The resulting μ^{+} are indicated by the red crosses.

Take the complex $C_{\lambda',\chi}$. It is unstably supported by Lemma 3.5.8. Moreover, Lemma 3.5.6 says that $C_{\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda', \beta_{i_j} \rangle < 0$ for all $j = 1, \ldots, p$.

Using the fact that $\eta_{\lambda_0} = \langle \lambda_0, -\det X \rangle$ and the definition of $Q_{\lambda'}$ from (3.13), we observe that $Q_{\lambda'} > \langle \lambda_0, \det X^{\lambda' \leq 0} \rangle / 2$, so that we are in the case (2a) of Lemma 3.5.12 above. We may then apply Claim 3.5.13 to show that μ^+ is still in the λ_0-strip and $r_{\mu^+} \leq 1/2$. Moreover, we can use (3.14) and the assumed bound $\langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \det X^{\lambda' \leq 0} \rangle / 2$ to obtain

$$\begin{aligned} \langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \mu - \theta \rangle &< \langle \lambda_0, \det X^{\lambda' \leq 0} \rangle / 2 + \langle \lambda_0, -\det X^{\lambda' < 0} \rangle \\ &= \langle \lambda_0, -\det X^{\lambda' \leq 0} / 2 \rangle < -\langle \lambda_0, \chi - \theta \rangle, \end{aligned}$$

where we have used the assumption that $X^{\lambda' = 0} = 0$ for the equality. Thus, $\alpha_{\mu^+} = \alpha_\mu < \alpha_\chi$.

3b. The case $\langle \lambda_0, -\det X^{\lambda' \geq 0} \rangle / 2 < \langle \lambda_0, \chi - \theta \rangle \leq \eta_{\lambda_0} / 2$:

Take the complex $D^\vee_{\lambda',\chi}$. It is unstably supported by Lemma 3.5.8. Moreover, Lemma 3.5.7 says that $D^\vee_{\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu^+)$, where $\mu = \chi + \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda', \beta_{i_j} \rangle > 0$ for all $j = 1, \ldots, p$.

Using the fact that $\eta_{\lambda_0} = \langle \lambda_0, -\det X \rangle$ and the definition of $Q_{\lambda'}$, we observe that $Q_{\lambda'} < \langle \lambda_0, -\det X^{\lambda' \geq 0} \rangle / 2$, so that we are in the case (2b) of Lemma 3.5.12 above. We may then apply Claim 3.5.14 to show that μ^+ is still in the λ_0-strip and $r_{\mu^+} \leq 1/2$.

Moreover, we can use (3.14), the assumed bound $\langle \lambda_0, -\det X^{\lambda' \geq 0} \rangle /2 < \langle \lambda_0, \chi - \theta \rangle$, and the hypothesis $X^{\lambda'=0} = 0$ to obtain

$$\begin{aligned} \langle \lambda_0, \chi - \theta \rangle > \langle \lambda_0, \mu - \theta \rangle &> \langle \lambda_0, -\det X^{\lambda' \geq 0} \rangle /2 + \langle \lambda_0, \det X^{\lambda' > 0} \rangle \\ &= \langle \lambda_0, \det X^{\lambda' \geq 0}/2 \rangle > -\langle \lambda_0, \chi - \theta \rangle, \end{aligned}$$

as in the case (3a). Thus, $\alpha_{\mu^+} = \alpha_\mu < \alpha_\chi$. □

Lemma 3.5.11, Lemma 3.5.12, and Lemma 3.5.15 imply $\mathcal{C}(\theta + \nabla) = \mathcal{C}(M_{\mathbb{R}})$, and Theorem 3.5.1 follows.

□

3.6 Example: representations of GL_2

Recall that any finite dimensional representation X of GL_2 has the form

$$X \cong \bigoplus_{i=1}^{P} \mathrm{Sym}^{n_i}(k^2) \otimes_k \det{}^{m_i}, \tag{3.18}$$

where n_i and m_i are integers and $n_i \geq 0$. $N_{\mathbb{R}}^W$ is one-dimensional, so there are, up to scaling by positive real numbers, only two choices for a central cocharacter λ_0 to which Hypotheses 1.2.1 applies. Without any loss of generality, let us assume $\lambda_0 = (-1, -1)$. Then λ_0 pairs strictly negatively with the weights of X if and only if $m_i + n_i/2 > 0$ for all i. For representations of this kind, ω^* is a positive multiple of the determinant character. $X^{\mathrm{ss}}(\omega^*)$ has finite stabilizers if and only if n_i is odd for all i, and in this case the unstable locus is $\mathrm{GL}_2 \cdot X^{\lambda' \geq 0}$ for $\lambda' = (-1, 1)$. Thus Theorem 3.5.1 gives the following:

Corollary 3.6.1. *Let X be a representation of GL_2 whose corresponding decomposition (3.18) satisfies, for all $i = 1, \ldots, P$, the conditions*

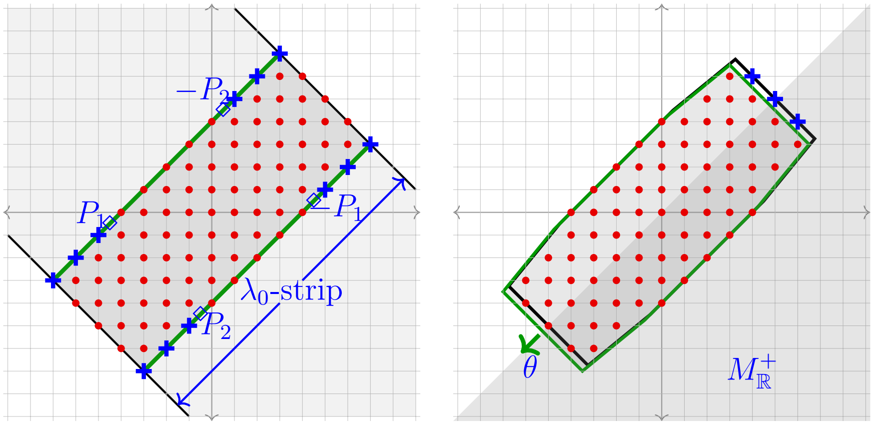

FIGURE 3.6. This is a description of the windows for the representation $X = (k^2)^{\oplus 10}$ discussed in Example 3.6.2. To the left is the cylinder window in the case $\theta = 0$ with the blue crosses indicating the T-weights in $\overline{\nabla} \setminus \nabla$. As in Figure 3.1, we have marked the special weights $P_i := \zeta_{\lambda_i}/2$ for $i = 1, 2$ that become relevant in the passage from the cylinder window to the barrel window. To the right is a presentation of the perturbation of the barrel window by a parameter $\theta = s\omega^*$. Here we take $-s$ to be a small positive real number. The weights indicated by blue crosses are those that are excluded from ∇ by the θ perturbation.

(1) $m_i + n_i/2 > 0$, *and*

(2) n_i *is odd.*

Then $X^{\mathrm{ss}}(\omega^*)$ *is non-empty and has finite stabilizers, and* $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\omega^*)/\,\mathrm{GL}_2)$ *has a full strong exceptional collection consisting of vector bundles.*

The GL_2 representations satisfying the conditions of Corollary 3.6.1 provide a large class of examples for Theorem 3.5.1. For such a representation, we will generally take $\lambda_0 = (-1, -1)$, although any positive multiple of it also works. The subset $(\omega^*)^\perp \subset N_{\mathbb{R}}$ consists of zero and positive multiples of the cocharacters $\lambda_1^{\omega^*} := (-1, 1)$ and $\lambda_2^{\omega^*} := (1, -1)$. Therefore, $\lambda_0, \lambda_1^{\omega^*}, \lambda_2^{\omega^*}$ suffice to describe the cylinder and barrel windows (see Definition 3.1.3).

Example 3.6.2. Consider the linear representation $X = (k^2)^{\oplus N}$ of GL_2, whose corresponding GIT quotient $X^{\mathrm{ss}}(\omega^*)/\,\mathrm{GL}_2$ is the Grassmannian of planes in k^N.

Then $\zeta_{\lambda_0} = (-N,-N), \zeta_{\lambda_1^{\omega^*}} = (-N+1,-1)$, and $\zeta_{\lambda_2^{\omega^*}} = (-1,-N+1)$, so $\eta_{\lambda_0} = 2N$ and $\eta_{\lambda_1^{\omega^*}} = \eta_{\lambda_2^{\omega^*}} = N-2$. Proposition 3.3.1 and Theorem 3.5.1 imply that the set of T-weights in $M^+ \cap (\theta + \nabla)$ produce a full strong exceptional collection of vector bundles in $\mathrm{Gr}(2,10)$ for any $\theta \in M_{\mathbb{R}}^W$ that is λ_0-generic. The barrel window in the case $N = 10$ is described in Figure 3.6 below. One can check that the number of dominant T-weights in $\theta + \nabla$ is 45. This full strong exceptional collection on $\mathrm{Gr}(2,10)$ differs from that of [K1].

3.7 Nef-Fano GIT quotients by a rank-2 torus

The goal of this section is to extend the methods of Section 3.5 to cover GIT quotients by a rank-two torus with a GIT parameter that is close to the anticanonical character ω^* in the sense of variation of GIT, thus the resulting GIT quotients are nef-Fano. We prove the following theorem and discuss how it extends the results of [BH] from the Fano case to the nef-Fano case.

Theorem 3.7.1. *Let $G = T$ be a split torus of rank 2 over k and let X be an affine scheme with a linear T-action satisfying Hypotheses 1.2.1. Assume that the weights of X span $M_{\mathbb{R}}$. Let $\theta \in M_{\mathbb{R}}$ be a character such that $\theta + \overline{\nabla}$ contains no weights on its boundary. If some weight of X is proportional to ω^*, then for some generic character $\ell \in M_{\mathbb{R}}$ and $0 < \epsilon \ll 1$, $X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ has finite T-stabilizers, and the line bundles*

$$\left\{\mathcal{O}_{X^{\mathrm{ss}}(\omega^*+\epsilon\ell)} \otimes U : U \in \mathrm{Rep}(T) \text{ a character lying in } \theta + \nabla_{\omega^*+\epsilon\ell}\right\}$$

split-generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\omega^* + \epsilon\ell)/T)$. In particular, the exceptional collection from Proposition 3.3.1 is full.

Proof. That $X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ has finite T-stabilizers for some generic character ℓ and $0 < \epsilon \ll 1$ is a consequence the theory of variation of GIT (see Example 2.7.1), so we need to only prove fullness of the collection.

The "genericity" assumption that $\theta + \overline{\nabla}$ contains no weights on its boundary implies the equality of sets

$$M \cap \left(\theta + \overline{\nabla}\right) = M \cap \left(\theta + \overline{\nabla}_{\omega^*+\epsilon\ell}\right)$$

for all $0 \leq \epsilon \ll 1$. Indeed, $(\omega^*)^\perp$ is spanned by a single cocharacter, say λ', so

$$\lambda'_\epsilon = \lambda' - \epsilon \frac{\langle \lambda', \ell \rangle}{\langle \lambda_0, \omega^* + \epsilon\ell \rangle} \lambda_0 \tag{3.19}$$

spans $(\omega^* + \epsilon\ell)^\perp$. Moreover, there are finitely many T-weights in $\theta + \mathrm{Int}\overline{\nabla}$ and for such χ, ϵ can be made sufficiently small so that $|\langle \lambda', \chi - \theta \rangle| < \eta_{\lambda'}/2$ if and only if $|\langle \lambda'_\epsilon, \chi - \theta \rangle| < \eta_{\lambda'_\epsilon}/2$.

The same genericity assumption together with the definition of the barrel window imply the equality $M \cap \left(\theta + \overline{\nabla}\right) = M \cap (\theta + \nabla)$. Thus, adopting the notation $\mathcal{C}(S) \subset \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X/T)$ for $S \subset M_{\mathbb{R}}$ from the proof of Theorem 3.5.1, we wish to show that for any $\chi \in M$, $\mathcal{O}_X \otimes V(\chi) \in \mathcal{C}(\theta + \overline{\nabla})$. This is equivalent to showing $\mathcal{C}(M_{\mathbb{R}}) = \mathcal{C}(\theta + \overline{\nabla})$. The first step, that $\mathcal{C}(M_{\mathbb{R}}) = \mathcal{C}(\theta + B_{\lambda_0})$, is identical to the proof of Lemma 3.5.11. To complete this proof, we will redo the reduction to the cylinder window in the proof of Theorem 3.5.1, showing $\mathcal{C}(\theta + B_{\lambda_0}) = \mathcal{C}(\theta + \overline{\nabla})$, with modifications accounting for the existence of a weight that is proportional to ω^* and from having a trivial Weyl group, i.e., $M_{\mathbb{R}}^W = M_{\mathbb{R}}$.

Reduction to the cylinder window.

Let λ' be a cocharacter that spans $(\omega^*)^\perp$ and satisfies $\langle \lambda', \ell \rangle < 0$. This pairing cannot equal zero because ℓ cannot be proportional to ω^*. Define for $R \geq 1$ the enlarged cylinder window

$$\overline{\nabla}^R := \{\chi \in B_{\lambda_0} : |\langle \lambda', \chi \rangle| \leq R\eta_{\lambda'}/2\}\,.$$

Take $Q_{\lambda'}$ as in (3.13) and define the following subset of the boundary of $\overline{\nabla}^R$.

$$\Psi_R := \left\{ \chi \in \overline{\nabla}^R : \left\{ \begin{array}{l} \langle \lambda', \chi \rangle = -R\eta_{\lambda'}/2 \text{ and } Q_{\lambda'} < \langle \lambda_0, \chi \rangle \leq \eta_{\lambda_0}/2, \text{ or} \\ \langle \lambda', \chi \rangle = R\eta_{\lambda'}/2 \text{ and } -\eta_{\lambda_0}/2 \leq \langle \lambda_0, \chi \rangle < -Q_{\lambda'}. \end{array} \right. \right\}$$

Claim 3.7.2. *If $R > 1$, then for any T-weight $\chi \in \theta + \Psi_R$, $\mathcal{O}_X \otimes V(\chi) \in \mathcal{C}(\theta + (\overline{\nabla}^R \setminus \Psi_R))$.*

Proof. There are two cases:

Case 1: $\langle \lambda', \chi - \theta \rangle = -R\eta_{\lambda'}/2$ and $Q_{\lambda'} < \langle \lambda_0, \chi - \theta \rangle \leq \eta_{\lambda_0}/2$.

We will prove the claim for the finite set of weights satisfying these properties using induction on $\langle \lambda_0, \chi - \theta \rangle$. Recall the cocharacter λ'_ϵ from (3.19), and consider the complex $D^\vee_{\lambda'_\epsilon}, \chi$ for $0 < \epsilon \ll 1$. It has homology supported in $G \cdot X^{-\lambda'_\epsilon \geq 0}$, which lies in $X^{\mathrm{us}}(\omega^* + \epsilon\ell)$ by Lemma 3.5.8. Lemma 3.5.7 says that $D^\vee_{\lambda'_\epsilon,\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu)$, where $\mu = \chi + \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda'_\epsilon, \beta_{i_j} \rangle > 0$ for all $j = 1, \ldots, p$.

That μ remains in the λ_0-strip follows from the assumed bounds on $\langle \lambda', \chi - \theta \rangle$ and the inequality

$$Q_{\lambda'} = -\frac{\eta_{\lambda_0}}{2} + \langle \lambda_0, -\det X^{\lambda' \geq 0} \rangle > -\frac{\eta_{\lambda_0}}{2}, \tag{3.20}$$

where the first equality is obtained from $-\eta_{\lambda_0} = \langle \lambda_0, \det X \rangle$.

Note that the weights β_{i_j} that appear in μ are independent of ϵ for $\epsilon > 0$ sufficiently small. Since λ' satisfies $\langle \lambda', \ell \rangle < 0$ and we can take ϵ to be arbitrarily

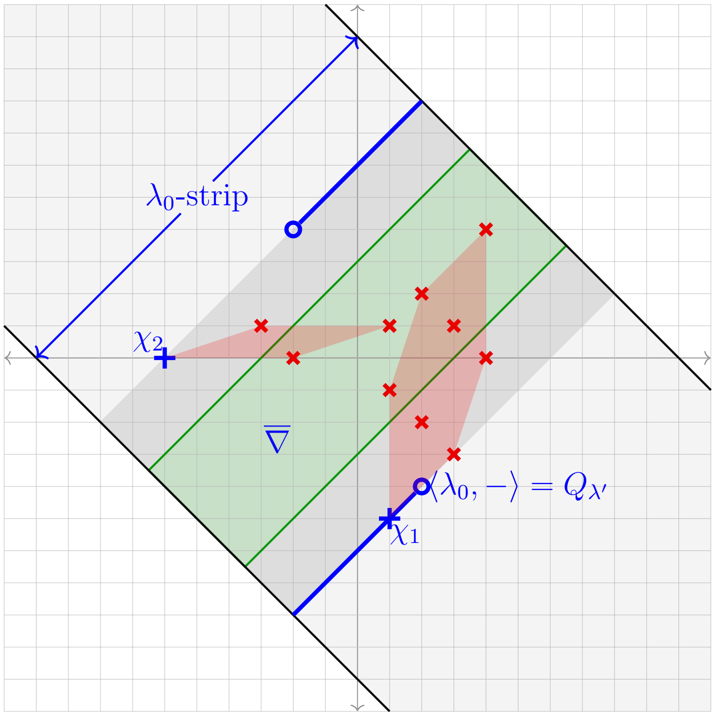

FIGURE 3.7. This diagram illustrates the reduction to the cylinder window $\overline{\nabla}$ for the $\mathbb{G}_m^2$ representation $X = \mathrm{Sym}^4\mathbb{C}^2$ with $\lambda_0 = (-1,-1)$, $\theta = 0$, $\ell = (1,-1)$, and $\lambda' = (-1,1)$. The weights of this representation are $(0,4),(1,3),(2,2),(3,1),(4,0)$. Thus, $\eta_{\lambda_0} = 20$, $\eta_{\lambda'} = 6$, and the green shaded region is the cylinder window. For each T-weight we pick an appropriate complex depending on a comparison of its λ_0-pairing with $Q_{\lambda'} = 2$. In this example, the gray shaded rectangle is the enlarged cylinder window for $R = 2$. The blue circles on the right and left side facets indicate the weights on the boundary of $\overline{\nabla}^R$ whose λ_0-pairing is $Q_{\lambda'}$ and $-Q_{\lambda'}$ respectively, thus the blue lines on the boundary of the enlarged cylinder indicate the region Ψ_R. The weight χ_1 is in Ψ_R and this falls into Case 1 of Claim 3.7.2. The resulting weights μ are indicated by the red crosses and one can see that these lie either in the interior of $\overline{\nabla}^R$ or they have a smaller pairing with λ_0. The weight χ_2 lies in $\overline{\nabla}^R \setminus \Psi_R$ and it falls into the case 2 of Claim 3.7.3. Again, the resulting weights μ are indicated by the red crosses and one can see that these lie entirely in the interior of $\overline{\nabla}^R$.

small, the weights β_{i_j} appearing in μ are precisely those for which $\langle \lambda', \beta_{i_j} \rangle \geq 0$. Therefore we have

$$\langle \lambda', \mu - \theta \rangle \leq -R\eta_{\lambda'}/2 + \langle \lambda', \det X^{\lambda' \geq 0} \rangle = (-R/2 + 1)\eta_{\lambda'} < R\eta_{\lambda'}/2.$$

Here we have used $\langle \lambda', \det X^{\lambda' \geq 0} \rangle = \eta_{-\lambda'} = \eta_{\lambda'}$ to deduce the equality and the assumption $R > 1$ for the last inequality. If there is one β_{i_j} in μ with $\langle \lambda', \beta_{i_j} \rangle > 0$, we also have $\langle \lambda', \mu - \theta \rangle > \langle \lambda', \chi - \theta \rangle = -R\eta_{\lambda'}/2$, and hence $\mu - \theta \in \overline{\nabla}^S \subset \overline{\nabla}^R \setminus \Psi_R$ for some $S < R$.

Otherwise, all β_{i_j} appearing in μ pair zero with λ', so $\langle \lambda', \mu - \theta \rangle = -R\eta_{\lambda'}/2$. But since $p > 0$, we have $\langle \lambda_0, \chi - \theta \rangle > \langle \lambda_0, \mu - \theta \rangle$, so by the inductive hypothesis we know $\mathcal{O}_X \otimes V(\mu) \in \mathcal{C}(\theta + (\overline{\nabla}^R \setminus \Psi_R))$. The claim follows.

Case 2: $\langle \lambda', \chi - \theta \rangle = R\eta_{\lambda'}/2$ *and* $-\eta_{\lambda_0}/2 \leq \langle \lambda_0, \chi - \theta \rangle < -Q_{\lambda'}$.

We will prove the claim for the finite set of weights satisfying these properties using induction on $-\langle \lambda_0, \chi - \theta \rangle$. Take the complex $C_{-\lambda'_\epsilon}, \chi$ with λ'_ϵ as in (3.19) and $0 < \epsilon \ll 1$. It has homology supported in $G \cdot X^{-\lambda'_\epsilon \geq 0}$, which lies in $X^{\mathrm{us}}(\omega^* + \epsilon\ell)$ by Lemma 3.5.8. Lemma 3.5.6 says that $C_{-\lambda'_\epsilon, \chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu)$, where $\mu = \chi - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle -\lambda'_\epsilon, \beta_{i_j} \rangle < 0$ for all $j = 1, \ldots, p$. As in the previous case, (3.20) and the assumed bounds for $\langle \lambda_0, \chi - \theta \rangle$ imply that μ remains in the λ_0-strip.

As in the previous case, the facts that the β_{i_j} appearing in μ are constant for small $\epsilon < 0$ and that $\langle \lambda', \ell \rangle < 0$ imply that the weights β_{i_j} appearing in μ are precisely those for which $\langle \lambda', \beta_{i_j} \rangle \geq 0$. This implies that $\langle \lambda', \mu - \theta \rangle \geq R\eta_{\lambda'}/2 - \langle \lambda', \det X^{\lambda' \geq 0} \rangle > -R\eta_{\lambda'}/2$, where we have used $\eta_{-\lambda'} = \eta_{\lambda'}$ and the assumption $R > 1$ to deduce the last inequality. If one of the β_{i_j} satisfies $\langle \lambda', \beta_{i_j} \rangle > 0$, then $\langle \lambda', \mu - \theta \rangle < \langle \lambda', \chi - \theta \rangle = R\eta_{\lambda'}/2$, and hence $\mu - \theta \in \overline{\nabla}^S \subset \overline{\nabla}^R \setminus \Psi_R$ for some $S < R$.

On the other hand, if all β_{i_j} appearing in μ pair zero with λ', then $\langle \lambda', \mu - \theta \rangle = R\eta_{\lambda'}/2$. But since $p > 0$, we have $\langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \mu - \theta \rangle$. Thus, by the inductive hypothesis we know $\mathcal{O}_X \otimes V(\mu) \in \mathcal{C}(\theta + (\overline{\nabla}^R \setminus \Psi_R))$. The claim follows.

□

Claim 3.7.3. *If $R > 1$, then for any T-weight $\chi \in \theta + (\overline{\nabla}^R \setminus \Psi_R)$, $\mathcal{O}_X \otimes V(\chi) \in \mathcal{C}(\theta + \overline{\nabla}^S)$ for some $S < R$.*

Proof. There are two cases:

Case 1: $\langle \lambda', \chi - \theta \rangle = -R\eta_{\lambda'}/2$ and $-\eta_{\lambda_0}/2 \leq \langle \lambda_0, \chi - \theta \rangle \leq Q_{\lambda'}$.

Take the complex $C_{\lambda',\chi}$. It has homology supported in $G \cdot X^{\lambda' \geq 0}$, which lies in $X^{\mathrm{us}}(\omega^* + \epsilon\ell)$ by the Hilbert-Mumford criterion and the assumption $\langle \lambda', \ell \rangle < 0$. Lemma 3.5.6 says that $C_{\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu)$, where $\mu = \chi - \beta_{i_1} - \beta_{i_2} - \cdots - \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle \lambda', \beta_{i_j} \rangle < 0$ for all $j = 1, \ldots, p$.

As in the proof of Claim 3.7.2, the assumed bounds for $\langle \lambda_0, \chi - \theta \rangle$ imply that μ remains in the λ_0-strip $\theta + B_{\lambda_0}$. We also have

$$\langle \lambda', \chi - \theta \rangle < \langle \lambda', \mu - \theta \rangle \leq (-R/2 + 1)\eta_{\lambda'} < R\eta_{\lambda'}/2,$$

where the first inequality holds because $p > 0$ and the last inequality holds because $R > 1$. Thus, $\mu - \theta \in \overline{\nabla}^S$ with $S < R$.

Case 2: $\langle \lambda', \chi - \theta \rangle = R\eta_{\lambda'}/2$ *and* $-Q_{\lambda'} \leq \langle \lambda_0, \chi - \theta \rangle \leq \eta_{\lambda_0}/2$.

Take the complex $D^\vee_{-\lambda',\chi}$. It has homology supported in $G \cdot X^{\lambda' \geq 0}$, which lies in $X^{\mathrm{us}}(\omega^* + \epsilon\ell)$ by the Hilbert-Mumford criterion and the assumption $\langle \lambda', \ell \rangle < 0$. Lemma 3.5.7 says that $D^\vee_{-\lambda',\chi}$ has one term of the form $\mathcal{O}_X \otimes V(\chi)$ and other terms that have a filtration whose associated graded pieces are of the form $\mathcal{O}_X \otimes V(\mu)$, where $\mu = \chi + \beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_p}$ with $p > 0$, $i_1, \ldots, i_p$ distinct, and $\langle -\lambda', \beta_{i_j} \rangle > 0$ for all $j = 1, \ldots, p$.

As before, the assumed bounds for $\langle \lambda_0, \chi - \theta \rangle$ imply that μ remains in the λ_0-strip $\theta + B_{\lambda_0}$. We also have

$$\langle \lambda', \chi - \theta \rangle > \langle \lambda', \mu - \theta \rangle \geq (R/2 - 1)\eta_{\lambda'} > -R\eta_{\lambda'}/2$$

where the first inequality holds because $p > 0$, the second inequality holds because $\eta_{\lambda'} = \eta_{-\lambda'}$, and the last inequality holds because $R > 1$. Thus, $\mu - \theta \in \overline{\nabla}^S$ with $S < R$. $\square$

The set of R for which $|\langle \lambda', \chi - \theta \rangle| = R\eta_{\lambda'}/2$ for some $\chi \in M$ is discrete, so one can perform induction on R, using Claim 3.7.2 and Claim 3.7.3, to deduce that for any $\chi \in (\theta + B_{\lambda_0}) \cap M$, $\mathcal{O}_X \otimes V(\chi) \in \mathcal{C}(\theta + \overline{\nabla})$. Figure 3.7 illustrates an example of this inductive argument. $\square$

3.7.1 Example: toric Deligne-Mumford stacks of Picard rank two

Smooth toric Deligne-Mumford (DM) stacks are introduced in [BCS]. They are smooth DM stacks associated to some combinatorial data, called a *stacky fan*, consisting of a triple $(A, \Sigma, \{v_1, \ldots, v_n\})$ of a finitely generated abelian group A, a complete rational simplicial fan $\Sigma \subset A_{\mathbb{R}}$, and a choice of non-torsion elements v_i in each one-dimensional cone of Σ. The associated toric DM stack is denoted $\mathbb{P}_\Sigma$.

For a smooth toric Fano DM stack $\mathbb{P}_\Sigma$ with Picard rank at most 2, the existence of a full strong exceptional collection consisting of line bundles was established by Borisov and Hua in [BH]. Their technique produces a particular window $P \subset \mathrm{Pic}(\mathbb{P}_\Sigma) \otimes \mathbb{R}$ from which the exceptional collection is exhibited as the set of line bundles whose image in $\mathrm{Pic}(\mathbb{P}_\Sigma) \otimes \mathbb{R}$ lies in $\theta + P$, where θ is any generic point. We will show below that whenever M is spanned by the weights of X and $G = \mathbb{G}_m^2$, then the GIT quotients appearing in Theorem 3.5.1 and Theorem 3.7.1 that have Picard rank 2 are smooth toric DM stacks. In particular, Theorem 3.7.1 can be viewed as providing a large class of examples where the result of [BH] for Picard rank 2 can be extended to the case of smooth nef-Fano toric DM stacks.

Linear GIT quotients by $\mathbb{G}_m^2$ as toric stacks

Lemma 3.7.4. *Let X be a linear representation of $T = \mathbb{G}_m^2$ over $\mathbb{C}$ satisfying Hypotheses 1.2.1. Assume*

1. *the weights of X span M,*
2. *$X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ has finite stabilizers in T for some $\ell \in M_{\mathbb{R}}$ and all $0 < \epsilon \ll 1$, and*
3. *the GIT quotient $X^{\mathrm{ss}}(\omega^* + \epsilon\ell)/T$ has Picard rank 2,*

then $X^{\mathrm{ss}}(\omega^* + \epsilon\ell)/T$ is a smooth toric DM stack.

Proof. By definition of toric DM stacks [BCS, Section 3] it will suffice to produce a stacky fan $(A, \Sigma, \{v_1, \ldots, v_n\})$ such that $X^{\mathrm{us}}(\omega^* + \epsilon\ell)$ coincides with the complement of the vanishing set of the monomial ideal

$$I_\Sigma := \left(\prod_{v_i \not\subset \sigma} z_i : \sigma \text{ is a cone in } \Sigma \right) \tag{3.21}$$

of $\Gamma(X, \mathcal{O}_X) = \mathbb{C}[z_1, \ldots, z_n]$. We will use the toric GIT constructions in [CLS, Chapter 14]. For a character $\chi \in M$, the semistable locus therein is described in terms of the sheaf of sections $\mathcal{L}(\chi)$ of the trivial line bundle linearized with respect to T by the character χ, thus $X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ means $X^{\mathrm{ss}}_{-(\omega^* + \epsilon\ell)}$ in the notation of [CLS, Section 14.1]. We are using here that $R(\omega^* + \epsilon\ell) \in M$ for some $0 < \epsilon \ll 1$ and some integer $R \gg 0$.

Let $\beta_1, \ldots, \beta_n$ be the weights of X and consider the dual short exact sequences of abelian groups

$$0 \to A^* \xrightarrow{\varrho} \mathbb{Z}^n \xrightarrow{\varphi} M \to 0, \quad \text{and} \quad 0 \to N \xrightarrow{\varphi^*} \mathbb{Z}^n \xrightarrow{\varrho^*} A \to 0, \tag{3.22}$$

where $\varphi : \mathbb{Z}^n \to M$ sends the i^{th} standard vector in $\mathbb{Z}^n$ to the weight β_i and $A^* := \ker(\varphi)$. The map φ is surjective by the assumption that the weights of X span M, so one obtains an inclusion $T \subset \mathbb{G}_m^n$ by applying $\mathrm{Hom}(-, \mathbb{G}_m)$ to φ. Here we are identifying the characters of $\mathbb{G}_m^n$ with $\mathbb{Z}^n$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ be a lift of $-R(\omega^* + \epsilon\ell)$ via φ. We may assume that $\alpha_i < 0$ for all $i = 1, \ldots, n$ because $(-1, \ldots, -1)$ is a lift of $-\omega^* = -\det X \in M$ and $0 < \epsilon \ll 1$. Let $v_1, \ldots, v_n \in A$ be the image of the standard basis of $\mathbb{Z}^n$ under the map ϱ^*. Then one can associate to α the polyhedron

$$\Delta = \{p \in A^*_{\mathbb{R}} : p \cdot v_i \geq -\alpha_i \text{ for all } i = 1, \ldots, n\}$$

that defines a (possibly degenerate) normal fan $\Sigma \subset A^*_{\mathbb{R}}$ as done in [CLS, Prop. 14.2.10]. Here $\cdot$ is the standard pairing between $A^*_{\mathbb{R}}$ and $A_{\mathbb{R}}$. Then

[CLS, Thm. 14.2.13] says that the toric variety corresponding to Σ coincides with the good quotient $Y := X^{\mathrm{ss}}(\omega^* + \epsilon\ell)//T$. This correspondence implies the following:

(a) Σ *is a (non degenerate) fan.*

By [CLS, Prop. 14.2.10], it suffices to show that Δ is full dimensional. We assume that Δ is not full dimensional and derive a contradiction.

We first show that there is a non-trivial cocharacter of $T' := \mathbb{G}_m^n/T \cong \mathbb{G}_m^{n-2}$ that acts trivially on Y. Let $\mathcal{L}(\alpha)$ be the sheaf of sections of the trivial line bundle on X with $\mathbb{G}_m^n$ linearization by α, then for some integer $d \gg 0$, $\mathcal{L}(\alpha)^{\otimes d}$ descends to a very ample line bundle on Y, and $U := \Gamma\left(Y, \mathcal{L}(\alpha)^{\otimes d}\right)$ is the T' representation with weights $d\Delta \cap A^*$. Here we are identifying A^* with the characters of T'. Thus Y has a T'-equivariant embedding into $\mathbb{P}(U)$. If Δ is not of full dimension, then it is possible to shift the weights of U by a single character so that they lie in a linear subspace of $\mathbb{A}^*_{\mathbb{Q}}$. Thus every character in $d\Delta \cap A^*$ pairs zero with some $\lambda \in A_{\mathbb{Q}}$. In particular, a sufficiently large integer multiple of λ is a cocharacter of T' that acts trivially on $\mathbb{P}(U)$ and hence on Y.

Now the existence of this cocharacter implies that the dimension of the dense T' orbit in Y is strictly smaller than $\dim T' = n - 2$. This contradicts the fact that $\dim Y = \dim X - \dim T = n - 2$, which holds because $X^{\mathrm{ss}}(\omega^*)$ is assumed to have finite stabilizers and thus Y is a geometric quotient.

(b) Σ *is complete and simplicial:*

Note that Y is projective over $\mathbb{C}$ because the morphism $Y \to \mathrm{Spec}(\mathcal{O}_X^T)$ is projective and $\mathcal{O}_X^T \cong \mathbb{C}$ under Hypotheses 1.2.1. Furthermore, the assumption on finite stabilizers implies that Y only has finite quotient singularities. Thus

[CLS, Thm. 3.1.19] implies Σ is simplicial and complete.

(c) Σ *has exactly* n *one dimensional cones.*

Because Σ is complete and simplicial, [CLS, Thm. 5.1.11] implies Y is isomorphic to a geometric quotient of the form $U//G$, where U is a particular open subset of $\mathbb{C}^{|\Sigma(1)|}$, $\Sigma(1)$ is the set of one dimensional cones of Σ, and $G \subset \mathbb{G}_m^{|\Sigma(1)|}$ is a subgroup. If Σ had fewer than n one dimensional cones, then because Y is of dimension $n-2$, the above construction would imply that Y has Picard rank strictly less than 2. This would contradict the assumption on the Picard rank.

Completing the proof:

Note that the facets of Δ occur among the following (possibly empty) subsets $F_1, \ldots, F_n$, where

$$F_i := \{p \in \Delta : p \cdot v_i = -\alpha_i\}.$$

We have shown that Σ has exactly n one dimensional cones, thus $F_1, \ldots F_n$ are all the facets of Δ. From the definition of Σ (see [CLS, Prop. 14.2.10]), the one dimensional cones are $\sigma_1, \ldots, \sigma_n$, where

$$\sigma_i = \{q \in A_{\mathbb{R}} : p \cdot q \geq 0 \text{ for all } p \in F_i\}.$$

Because we chose α so that $\alpha_i < 0$ for all $i = 1, \ldots, n$, it follows that $v_i \in \sigma_i$ and is in fact a generator. It follows from (a), (b), and (c) that $(A, \Sigma, \{v_1, \ldots, v_n\})$ is a stacky fan.

Now [CLS, Prop. 14.2.21] says that

($\star$) $(x_1, \cdots, x_n) \in X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ if and only if $\bigcap_{i \in B_x} F_i \neq \emptyset$, where $B_x = \{i : x_i = 0\}$.

The condition $\bigcap_{i\in B_x} F_i \neq \emptyset$ is equivalent to the condition that there is a cone $\sigma \in \Sigma$ containing all σ_i for $i \in B_x$. We have already seen that v_i is a generator for $\sigma_i \subset A_{\mathbb{R}}$, so the latter condition is equivalent to having a cone $\sigma \in \Sigma$ containing all v_i for $i \in B_x$. Using $(\star)$ we obtain that $x \in X^{\mathrm{ss}}(\omega^* + \epsilon\ell)$ if and only if x lies in the complement of the vanishing set of the ideal (3.21). $\square$

The Borisov-Hua window

Let $(A, \Sigma, \{v_1, \ldots, v_n\})$ be a stacky fan.

(1) Let $a_1, \ldots, a_n$ be a collection of non-zero rational numbers such that $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i v_i = 0$. Let I_+ be the set of indices where $a_i > 0$. For Fano DM toric stacks, the existence and uniqueness up to scaling of such a tuple is shown in [BH, Prop. 5.4].

(2) Let $r_1, \ldots, r_n$ be another collection of positive real numbers r_i such that $\sum_{i=1}^n r_i = 1$ and $\sum_{i=1}^n r_i v_i = 0$. The existence of such a tuple also follows from the proof of [BH, Prop. 5.4].

Let $E_1, \ldots, E_n$ be the standard basis of $\mathbb{Z}^n$. The collection $r_1, \ldots, r_n$ defines a linear functional f on $\mathbb{Z}^n$ sending E_i to r_i. Analogously, there is a functional ψ sending E_i to a_i. By [BH, Prop. 3.3], the Picard group of the toric DM stack $\mathbb{P}_\Sigma$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the subgroup of elements of the form $\sum_{i=1}^n (p \cdot v_i) E_i$, for all $p \in A^*$. With this description, it follows that ψ and f descend to $\mathrm{Pic}(\mathbb{P}_\Sigma)$ because (a_i) and (v_i) satisfy $\sum_{i=1}^n a_i v_i = 0$ and $\sum_{i=1}^n r_i v_i = 0$ respectively.

The Borisov-Hua window $P \subset \mathrm{Pic}(\mathbb{P}_\Sigma) \otimes \mathbb{R}$ is then defined to be the region cut out by the inequalities

$$|f(-)| \leq \frac{1}{2}, \qquad |\psi(-)| \leq \frac{1}{2}\sum_{i\in I_+} a_i. \tag{3.23}$$

A comparison of the barrel window to the Borisov-Hua window

Under Hypotheses 1.2.1, assume that $X^{\mathrm{ss}}(\omega^*)$ has finite stabilizers, the weights of X span M, and X^{ss}/T has Picard rank 2. Then by Lemma 3.7.4 we can consider $X^{\mathrm{ss}}(\omega^*)/T$ as a toric DM stack with the stacky fan $(A, \Sigma, \{v_1, \ldots, v_n\})$ constructed above using (3.22), namely the ray generators v_i are the images of the standard vectors in $\mathbb{Z}^n$ under ϱ^*. In this context, the subgroup of elements of the form $\sum_{i=1}^n (p \cdot v_i)E_i$ for $p \in A^*$ is precisely the image of the map $\varrho : A^* \hookrightarrow \mathbb{Z}^n$ from the first exact sequence in (3.22). Thus $\mathrm{Pic}(X^{\mathrm{ss}}(\omega^*)/T)$ is canonically identified with M via the map φ that sends the i^{th} standard vector to the weight β_i. In particular, if we find a suitable choice of tuples (a_i) and (r_i), then the Borisov-Hua window P is the image in $M_{\mathbb{R}}$ of the region in $\mathbb{R}^n$ cut out by (3.23).

We claim that there is a particular choice of tuples $(a_1, \ldots, a_n)$ and $(r_1, \ldots, r_n)$ in the construction of the Borisov-Hua window such that the parallelogram P in $M_{\mathbb{R}}$ defined by these tuples coincides with the cylinder window $\overline{\nabla}$.

(1) *The choice of* $\{a_1, \ldots, a_n\}$:

Let $\lambda' \in N_{\mathbb{R}} \setminus 0$ be any cocharacter satisfying $\langle \lambda', \omega^* \rangle = 0$. For each $i = 1, \ldots, n$, define

$$a_i := -\langle \lambda', \beta_i \rangle.$$

We show that the collection $\{a_1, \cdots, a_n\}$ satisfies the conditions for 3.7.1, i.e., all $a_i \neq 0$, $\sum_i a_i = 0$, and $\sum_i a_i v_i = 0$.

The assumption that $X^{\mathrm{ss}}(\omega^*)$ has finite stabilizers together with Lemma 3.5.8 implies that no weights of X lie in the span of ω^*, so $a_i \neq 0$ for all i, and $\det X^{\lambda' \leq 0} = \det X^{\lambda' < 0}$. We then have

$$\sum_{i \in I_+} a_i = \langle \lambda', -\det X^{\lambda' < 0} \rangle = \eta_{\lambda'} \qquad \sum_{i \in I \setminus I_+} a_i = \langle \lambda', -\det X^{\lambda' > 0} \rangle = -\eta_{\lambda'},$$

which implies that $\sum_i a_i = 0$.

For the second condition, we observe that for every cocharacter $\lambda \in N = M^*$, the expression $\sum_{i=1}^n \langle \lambda, \beta_i \rangle v_i$ is the image of λ under the composition $M^* \overset{\varphi^*}{\hookrightarrow} \mathbb{Z}^n \overset{\varrho^*}{\twoheadrightarrow} A$, so $\sum_{i=1}^n a_i v_i = 0$.

(2) *The choice of* $\{r_1, \ldots, r_n\}$:

Recall the cocharacter λ_0 from Hypotheses 1.2.1 that pairs strictly negatively with respect to all weights of X. For each $i = 1, \ldots, n$, define

$$r_i := -\langle \lambda_0, \beta_i \rangle / \eta_{\lambda_0},$$

where $\eta_{\lambda_0} = \langle \lambda_0, -\det X \rangle$. Thus $\sum_i^n r_i = 1$ and all r_i are strictly positive. That $\sum_{i=1}^n r_i v_i = 0$ follows as in the case above.

The Borisov-Hua window for these choices of tuples (a_i) and (r_i) is the image under φ of

$$\left\{ (y_1, \ldots, y_n) \in \mathbb{R}^n : \left| \sum_{i=1}^n -y_i \langle \lambda_0, \beta_i \rangle \right| \leq \frac{\eta_{\lambda_0}}{2}, \left| \sum_{i=1}^n -y_i \langle \lambda', \beta_i \rangle \right| \leq \frac{1}{2} \sum_{i \in I_+} a_i \right\},$$

which is the cylinder window $\overline{\nabla}$ because $\sum_{i \in I_+} a_i = \eta_{\lambda'}$. Hence for these choices of (a_i) and (r_i), and $\theta \in M_{\mathbb{R}}$, $\theta + \overline{\nabla} = \theta + P$.

Let $\theta \in M_{\mathbb{R}}$ be a character such that $\theta + \partial\overline{\nabla}$ does not contain any points in $M_{\mathbb{Q}}$. This is the genericity condition for θ in [BH, Thm. 5.11]. The condition also guarantees that $(\theta+\partial\overline{\nabla})\cap M=\emptyset$, hence $(\theta+\overline{\nabla})\cap M=(\theta+\nabla)\cap M$. In particular, Proposition 3.3.1 and Theorem 3.5.1 recover the Borisov-Hua exceptional collection on $X^{\mathrm{ss}}(\omega^*)/T$ under the hypotheses of Lemma 3.7.4.

CHAPTER 4

GENERATORS OF THE DERIVED CATEGORY OF LINEAR GIT QUOTIENTS

In this chapter, we describe a general algorithm for finding a class of vector bundles that generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$ whenever $X^{\mathrm{ss}}(\ell)$ has finite G-stabilizers. This class of vector bundles will be constructed using the algorithm used by Špenko and Van den Bergh [ŠVdB1] to construct non-commutative resolutions. It turns out that these vector bundles contain the strong exceptional collection from Proposition 3.3.1 – it is thus an interesting future problem to establish when one can reduce this list of generating vector bundles to a full exceptional collection.

4.1 Combinatorial generation algorithms

Recall the λ_0-strip B_{λ_0} and the barrel window ∇_ℓ from Section 3.1.

Definition 4.1.1. Let G be a split-reductive connected group over k and let $X = \mathrm{Spec}(\mathrm{Sym}\, V)$ be the total space of a linear G-representation V. With notation as in Section 1.5, let $\beta_1, \ldots, \beta_n \in M$ denote the weights of X. Let $\ell \in M_{\mathbb{R}}^W$ be a Weyl-invariant character and define

$$M_{\mathbb{R}}^{\ell} := M_{\mathbb{R}}/\mathbb{R}\ell.$$

For a character $\nu \in M_{\mathbb{R}}$, we will use ν' to denote its image under $\pi_\ell : M_{\mathbb{R}} \twoheadrightarrow M_{\mathbb{R}}^{\ell}$, and define

$$\overline{\Sigma}' := \left\{ \sum_{i=1}^{n} a_i \beta_i' : a_i \in [-1, 0] \right\} \subset M_{\mathbb{R}}^{\ell},$$

$$\overline{\Sigma}_\ell := \left\{ \chi \in B_{\lambda_0} : \chi' \in -\rho' + \overline{\Sigma}' \right\} \subset M_{\mathbb{R}}.$$

Proposition 4.1.2. *Let G be a split-reductive group over k and let X be a linear G-representation satisfying Hypotheses 1.2.1, i.e. there is a central cocharacter λ_0 pairing strictly negatively with all weights of X. Let $\ell, \theta \in M_{\mathbb{R}}^W$ be Weyl-invariant characters with $\langle \lambda_0, \ell \rangle < 0$ and θ being λ_0-generic with respect to ℓ (Definition 3.1.6). Assume $X^{\mathrm{ss}}(\ell)$ has finite G-stabilizers. Then the tautological vector bundles*

$$\left\{ \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \textit{ irreducible with} \\ \textit{character in } \theta + \overline{\Sigma}_\ell \end{array} \right\}$$

generate $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$. Furthermore, $M^+ \cap (\theta + \nabla_\ell) \subset M^+ \cap (\theta + \overline{\Sigma}_\ell)$ if $\langle \lambda', -\det X^{\lambda' \le 0} \rangle > 0$ for all $\lambda' \in \ell^\perp \setminus 0$. In this case, the vector bundles above contain the strong exceptional collection from Proposition 3.3.1.

Proof. Let us first introduce the following parameters associated to a character $\chi \in M_{\mathbb{R}}$.

$$r_\chi := \min \left\{ r \ge 0 : \chi' \in -\rho' + r\overline{\Sigma}' + \theta' \right\}, \text{ and} \tag{4.1}$$

$$\begin{array}{c} p_\chi := \text{ the minimal number of } a_i \text{ which are equal to } -r_\chi \text{ among} \\ \text{all the ways of writing } \chi' = -\rho' + \sum_i a_i \beta_i' + \theta' \text{ for } a_i \in [-r_\chi, 0]. \end{array} \tag{4.2}$$

These parameters are slight modifications of those appearing in [ŠVdB1, HLS]. Note that because ℓ is Weyl-invariant, $\overline{\Sigma}' + \theta'$ is still Weyl-invariant. In particular, r_χ and p_χ are invariant under the ρ-shifted action of W on $M_{\mathbb{R}}$ (3.10).

As in the proof of Theorem 3.5.1, we will use the following notation for any subset $S \subset M_{\mathbb{R}}$:

$$\mathcal{C}(S) := \left\{ \begin{array}{c} \text{smallest full triangulated subcategory of } \mathrm{D}_{\mathrm{qc}}(X/G) \text{ containing} \\ \text{all complexes supported in } X^{\mathrm{us}}(\ell) \text{ and the locally} \\ \text{free sheaves } \mathcal{O}_X \otimes V(\nu) \text{ for all } \nu \in S \cap M^+ \end{array} \right\}.$$

In this notation, the the proposition amounts to showing $\mathcal{C}(\theta + \overline{\Sigma}_\ell) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$. By definition, $\theta + \overline{\Sigma}_\ell \subset \theta + B_{\lambda_0}$, and we prove the equality above in two steps:

(1) $\mathcal{C}(\theta + B_{\lambda_0}) = \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{\mathrm{ss}}(\ell)/G)$, and

(2) $\mathcal{C}(\theta + \overline{\Sigma}_\ell) = \mathcal{C}(\theta + B_{\lambda_0})$.

The argument in Lemma 3.5.12 applies directly to this case to conclude (1).

Proof of (2):

By definition of r_χ, χ' lies on the boundary of $-\rho' + r_\chi \overline{\Sigma}' + \theta'$, so there exists a cocharacter $\lambda' \in \ell^\perp \setminus 0$ such that the functional $\langle \lambda', - \rangle$ is maximized on a facet of $-\rho' + r_\chi \overline{\Sigma}' + \theta'$ containing χ'. Then λ' can be modified to an anti-dominant cocharacter in $\ell^\perp$ satisfying $\langle \lambda', \chi' + \rho' \rangle \leq \langle \lambda', \mu \rangle$ for all $\mu \in -\rho' + r_\chi \overline{\Sigma}' + \theta'$. Indeed, let $w \in W$ be any element of the Weyl group such that $w\lambda'$ is dominant, then [ŠVdB1, Cor. 11.12] says that $\langle w\lambda', \chi' + \rho' \rangle \geq \langle \lambda', \chi + \rho' \rangle$. Because $r_\chi \overline{\Sigma}' + \theta'$ is Weyl-invariant, $-w\lambda'$ is the cocharacter we seek.

Let $\lambda' \in \ell^\perp \setminus 0$ be the minimizing anti-dominant cocharacter from the previous paragraph. Using the quantity

$$Q_{\lambda'} := \frac{1}{2}\eta_{\lambda_0} + \langle \lambda_0, \det X^{\lambda' < 0} \rangle,$$

define the complex

$$E^\bullet_\chi := \begin{cases} C_{\lambda',\chi}, & \text{if } Q_{\lambda'} < \langle \lambda_0, \chi \rangle \leq \frac{\eta_{\lambda_0}}{2} \\ D^\vee_{\lambda',\chi}, & \text{if } -\frac{\eta_{\lambda_0}}{2} \leq \langle \lambda_0, \chi \rangle \leq Q_{\lambda'} \end{cases},$$

where the complexes $C_{\lambda',\chi}$, $D^\vee_{\lambda',\chi}$ are those from Proposition 3.5.3.

We claim that the complex $E^\bullet_\chi$ is unstably supported. By construction, $E^\bullet_\chi$ has homology supported on $G \cdot X^{\lambda' \geq 0}$ with $\lambda' \in \ell^\perp$ nontrivial, so it suffices to show that $X^{\lambda' \geq 0} \subset X^{\mathrm{us}}(\ell)$. For this, we note that Lemma 3.2.3 says that x is ℓ-semi-stable if and only if $0 \notin \overline{\{G^\circ_\ell \cdot x\}}$, where G°_ℓ is the connected component of the identity of the subgroup $\ker(\ell) \subset G$. Thus, if $\{G_\ell^\circ \cdot x\}$ is closed, then x is semi-stable. Conversely, if $\{G_\ell^\circ \cdot x\}$ is not closed, then the Hilbert-Mumford criterion implies that there is a non-trivial cocharacter $\lambda : \mathbb{G}_m \to G_\ell^\circ$ such that $x_0 := \lim_{t\to 0} \lambda(t) \cdot x$ exists and does not lie in $G_\ell^\circ \cdot x$. Because x_0 is fixed by λ, it has a positive-dimensional stabilizer, so $x_0 \in X^{G-\mathrm{us}}(\ell)$ by hypothesis. Applying Lemma 3.2.3 again, we see that $0 \in \overline{\{G_\ell^\circ \cdot x_0\}}$. The fact that x_0 lies in the G_ℓ° orbit-closure of x and 0 lies in the G_ℓ° orbit-closure of x_0 implies that 0 lies in the G_ℓ° orbit-closure of x as well, so x is unstable for the action of G by Lemma 3.2.3. This implies that $X^{\lambda' \geq 0} \subset X^{\mathrm{us}}(\ell)$ for any non-trivial $\lambda' \in \ell^\perp$, because every point in that subspace has either a non-closed G_ℓ°-orbit or a positive dimensional stabilizer (containing λ').

Moreover, Lemma 3.5.6 and Lemma 3.5.7 imply that one term of $E_\chi^\bullet$ is $\mathcal{O}_X \otimes V(\chi)$ and other terms have a G-equivariant filtration whose associated graded pieces are direct sums of vector bundles $\mathcal{O}_X \otimes V(\mu^+)$, with $(-)^+$ indicating the ρ-shift action of the Weyl group (3.10) and μ has the form

$$\mu = \begin{cases} \chi - \beta_{i_1} - \ldots - \beta_{i_p} : \langle \lambda', \beta_{i_j} \rangle < 0 \quad \forall j = 1, \ldots, p & \text{if } E_\chi = C_{\lambda',\chi} \\ \chi + \beta_{i_1} + \ldots + \beta_{i_p} : \langle \lambda', \beta_{i_j} \rangle < 0 \quad \forall j = 1, \ldots, p & \text{if } E_\chi = D^\vee_{\lambda',\chi} \end{cases} . \tag{4.3}$$

In both expressions of μ, $i_1, \ldots, i_p$ are distinct.

Part (2) will follow by induction on the quantities r_χ and p_χ and Claim 4.1.3 below. Indeed, let $R > 1$ be a real number and consider the following set of dominant T-weights

$$\Delta_R := \bigcup_P \Delta(R, P), \text{ where } \Delta(R,P) := \left\{\chi \in M^+ \cap (\theta + B_{\lambda_0}) : r_\chi = R \text{ and } p_\chi = P\right\}.$$

There are finitely many real numbers P for which $\Delta(R,P) \neq \emptyset$. Let us enumerate them in decreasing order $P_1 > \ldots > P_q$. Suppose

$$\mathcal{S}(\theta + B_{\lambda_0}) = \mathcal{S}\left(\bigcup_{S<R} \Delta_S \cup \bigcup_{P_s \leq P_i} \Delta(R, P_s)\right)$$

If $\chi \in \Delta(R, P_i)$ and $\langle \lambda_0, \chi \rangle \in (Q_{\lambda'}, \eta_{\lambda_0}/2]$, then μ has the form of the first expression of (4.3), so $\langle \lambda_0, \mu^+ \rangle > \langle \lambda_0, \chi \rangle$. Part (i) and (ii) of Claim 4.1.3 imply that μ^+ is either in Δ_S with $S < R$, or in $\Delta(R, P_i)$. But since μ^+ has a larger λ_0-weight, we conclude by induction that

$$\mathcal{S}(\theta+B_{\lambda_0}) = \mathcal{S}\left(\bigcup_{S<R} \Delta_S \cup \bigcup_{P_s<P_i} \Delta(R, P_s) \cup \{\chi \in \Delta(R, P_i) \text{ and } \langle \lambda_0, \chi \rangle \in [-\eta_{\lambda_0}/2, Q_{\lambda'}]\}\right)$$

Now, if $\chi \in \Delta(R, P_i)$ has $\langle \lambda_0, \chi \rangle \in [-\eta_{\lambda_0}/2, Q_{\lambda'}]$, then part (i) and (iii) of Claim 4.1.3 imply that μ^+ is either in Δ_S with $S < R$, or in $\Delta(R, P_j)$ with $P_j < P_i$. Thus,

$$\mathcal{S}(\theta + B_{\lambda_0}) = \mathcal{S}\left(\bigcup_{S<R} \Delta_S \cup \bigcup_{P_s<P_i} \Delta(R, P_s)\right),$$

and (2) follows by induction.

Claim 4.1.3. *The weights μ^+ appearing in the complex $E^\bullet_\chi$ satisfy*

(i) μ^+ is also in the λ_0-strip $\theta + B_{\lambda_0}$,

(ii) if $r_\chi > 1$ and $E^\bullet_\chi = C_{\lambda',\chi}$, then either $r_{\mu^+} < r_\chi$, or $r_{\mu^+} = r_\chi$ and $p_{\mu^+} = p_\chi$, and

(ii) if $r_\chi > 1$ and $E^\bullet_\chi = D^\vee_{\lambda',\chi}$, then either $r_{\mu^+} < r_\chi$, or $r_{\mu^+} = r_\chi$ and $p_{\mu^+} < p_\chi$.

Proof. Since λ_0 is a central cocharacter, $\langle \lambda_0, \mu^+ \rangle = \langle \lambda_0, \mu \rangle$; so, for (i), it suffices to show $\mu \in \theta + B_{\lambda_0}$.

Suppose $\mu = \chi - \beta_{i_1} - \ldots - \beta_{i_p}$ in (4.3). Then

$$\langle \lambda_0, \chi - \theta \rangle < \langle \lambda_0, \mu - \theta \rangle \leq \langle \lambda_0, \chi - \theta \rangle + \langle \lambda_0, -\det X^{\lambda'<0} \rangle,$$

where we have used in both inequalities the assumption that all weights of X pair strictly negatively with λ_0. It follows from these bounds and the definition of $Q_{\lambda'}$ that $\mu \in \theta + B_{\lambda_0}$.

Analogously, if $\mu = \chi + \beta_{i_1} + \ldots + \beta_{i_p}$ in (4.3), then

$$\langle \lambda_0, \chi - \theta\rangle + \langle \lambda_0, \det X^{\lambda' > 0}\rangle \leq \langle \lambda_0, \mu - \theta\rangle < \langle \lambda_0, \chi - \theta\rangle.$$

Applying Hypotheses 1.2.1 again, we see that $-\eta_{\lambda_0}/2 - \langle \lambda_0, \det X^{\lambda' > 0}\rangle \leq Q_{\lambda'}$, which together with the inequalities above imply $\mu \in \theta + B_{\lambda_0}$.

The proof of (ii) and (iii) follow from the Špenko-Van den Bergh algorithm [ŠVdB1, Sect. 11.1] as follows. The quantities r_χ and p_χ are Weyl-invariant, so it suffices to show that either $r_\mu < r_\chi$, or $r_\mu = r_\chi$ and $p_\mu < p_\chi$.

By construction, the functional $\langle \lambda', - \rangle$ is minimized on a facet of $-\rho' + r_\chi \overline{\Sigma}' + \theta'$ containing χ'. Thus

$$\chi' = -\rho + \sum_i a_i \beta_i' + \theta', \text{ where } a_i = \begin{cases} -r_\chi & \text{if } i \in T \\ 0 & \text{if } i \in U \\ [-r_\chi, 0] & \text{if } i \in Q \end{cases}, \tag{4.4}$$

where the sets T, U, Q are defined as follows

$$T := \{i \in \{1, \ldots, n\} : \langle \lambda', \beta_i' \rangle > 0\},$$

$$U := \{i \in \{1, \ldots, n\} : \langle \lambda', \beta_i' \rangle < 0\},$$

$$Q := \{i \in \{1, \ldots, n\} : \langle \lambda', \beta_i' \rangle = 0\}.$$

We may assume that there are precisely p_χ many a_i that are equal to $-r_\chi$ in this expression.

Then we may rewrite χ' as follows

$$\chi' = -\rho' + \sum_{L \subset M_{\mathbb{R}}'} \left(\sum_{\beta_i' \in L} a_i \beta_i' \right) + \theta',$$

where L runs over all one dimensional subspaces of $M'_{\mathbb{R}}$. For μ as in (4.3), set

$$S_\mu := \{i_1, \dots, i_p\}.$$

Note that by definition, either $S_\mu \subset T$ or $S_\mu \subset U$. With this notation, we may write μ' as

$$\mu' = -\rho' + \sum_{L \subset M'_{\mathbb{R}}} \left(\sum_{\beta'_i \in L} \widetilde{a}_i \beta'_i \right) + \theta',$$

where $\sum_{\beta'_i \in L} \widetilde{a}_i \beta'_i = \sum_{\beta'_i \in L} a_i \beta'_i$ if $\beta'_i \notin L$ for all $i \in S_\mu$, otherwise

$$\sum_{\beta'_i \in L} \widetilde{a}_i \beta'_i = \sum_{\beta'_i \in L \text{ and } i \in S_\mu} (-r_\chi + 1)\beta'_i + \sum_{\beta'_i \in L \text{ and } i \in T \setminus S_\mu} -r_\chi \beta'_i, \qquad \text{if } S_\mu \subset T, \tag{4.5}$$

$$\sum_{\beta'_i \in L} \widetilde{a}_i \beta'_i = \sum_{\beta'_i \in L \text{ and } i \in S_\mu} -\beta'_i + \sum_{\beta'_i \in L \text{ and } i \in T} -r_\chi \beta'_i, \qquad \text{if } S_\mu \subset U. \tag{4.6}$$

Since $r_\chi > 1$, the coefficients of (4.5) and (4.6) lie in the interval $(-r_\chi, 0)$. It follows that $r_\mu \leq r_\chi$. If $r_\mu = r_\chi$ and $S_\mu \subset T$ then $p_\mu < p_\chi$, and if $r_\mu = r_\chi$ and $S_\mu \subset U$ then $p_\mu = p_\chi$. $\square$

The generating vector bundles contain an exceptional collection:

For the second claim, we begin with the observation that the cylinder window $\overline{\nabla}_\ell$ is the intersection of the λ_0-strip with the preimage under π_ℓ of the region

$$\overline{\nabla}' = \left\{ \nu \in M^\ell_{\mathbb{R}} : \begin{array}{c} \text{for all } \lambda \in (M^\ell_{\mathbb{R}})^* \cong \ell^\perp \subset N_{\mathbb{R}}, \\ |\langle \lambda, \nu \rangle| \leq \eta_\lambda / 2 \end{array} \right\}.$$

Thus, it suffices to show that if $\chi \in M$ is a dominant T-weight with $\chi' \in \overline{\nabla}' + \theta'$, then $\chi' \in -\rho' + \overline{\Sigma}' + \theta'$, i.e. that $r_\chi \leq 1$. If this holds, it follows that the exceptional collection from Proposition 3.3.1 is contained in the list of vector bundles in the statement of the proposition.

In fact, we will show that if $\eta_{\lambda'} > 0$ for all $\lambda' \in \ell^\perp \setminus 0$, then $r_\chi \leq 1/2$. By minimality of r_χ, χ' lies on the boundary of $-\rho' + r_\chi \overline{\Sigma}' + \theta'$, so there exists a cocharacter $\lambda' \in \ell^\perp$ satisfying $\langle \lambda', \chi' + \rho' \rangle \geq \langle \lambda', \mu \rangle$ for all $\mu \in r_\chi \overline{\Sigma}' + \theta'$. Let $w \in W$ be any element of the Weyl group such that $w\lambda'$ is dominant, then [ŠVdB1, Cor. 11.12] implies that $\langle w\lambda', \chi' + \rho' \rangle \geq \langle \lambda', \chi + \rho' \rangle$. Because $r_\chi \overline{\Sigma}' + \theta'$ is Weyl-invariant, we may replace λ' with $w\lambda'$ so that we have a dominant cocharacter λ' satisfying $\langle \lambda', \chi' + \rho' \rangle \geq r_\chi \max\{\langle \lambda', \mu \rangle : \mu \in \overline{\Sigma}'\} + \langle \lambda', \theta' \rangle$. We obtain

$$r_\chi \max\{\langle \lambda', \mu \rangle : \mu \in \overline{\Sigma}'\} + \langle \lambda', \theta' \rangle \leq \langle \lambda', \chi' + \rho' \rangle \leq \frac{1}{2} \max\{\langle \lambda', \mu \rangle : \mu \in \overline{\Sigma}'\} + \langle \lambda', \theta' \rangle,$$

where the second inequality holds because $\chi' - \theta' \in \overline{\nabla}'$, hence $\langle \lambda', \chi' - \theta' \rangle \leq \eta_{\lambda'}/2$, and $\eta_{\lambda'} = \max\{\langle \lambda', \mu \rangle : \mu \in \overline{\Sigma}'\} - 2\langle \lambda', \rho' \rangle$ since λ' is dominant. Note that $\max\{\langle \lambda', \mu \rangle : \mu \in \overline{\Sigma}'\}$ is strictly positive because $\langle \lambda', -\det X^{\lambda' \leq 0} \rangle > 0$ by assumption. It follows that $r_\chi \leq 1/2$. □

4.2 Example: Projectivizations of self-dual representations

Let us fix the ground field to be $\mathbb{C}$. Let $G' = \mathbb{G}_m^2$ and consider the following self-dual G'-representation $X = \mathrm{Sym}^3 \mathbb{C}^2 \otimes_{\mathbb{C}} (\mathrm{Sym}^3 \mathbb{C}^2)^*$. The weight lattice of G' is $\mathbb{Z}^2$ and X has weights $\beta'_1 = (0,3)$, $\beta'_2 = (1,2)$, $\beta'_3 = (2,1)$, $\beta'_4 = (3,0)$, $\beta'_5 = (0,-3)$, $\beta'_6 = (-1,-2)$, $\beta'_7 = (-2,-1)$, $\beta'_8 = (-3,0)$.

Let $G = G' \times \mathbb{G}_m$ act on X with $\mathbb{G}_m$ acting with weight 1. The weights of X as a G-representation are $\beta_1 = (0,3,1)$, $\beta_2 = (1,2,1)$, $\beta_3 = (2,1,1)$, $\beta_4 = (3,0,1)$, $\beta_5 = (0,-3,1)$, $\beta_6 = (-1,-2,1)$, $\beta_7 = (-2,-1,1)$, $\beta_8 = (-3,0,1)$. Then X satisfies Hypotheses 1.2.1 as a G-representation and we may take $\lambda_0 = (0,0,-1) \in (\mathbb{R}^3)^*$, the anti-canonical character is $\omega^* = (0,0,8)$, and

$$X^{G-\mathrm{ss}}(\omega^*)/G \cong \mathbb{P}(X)^{G'-\mathrm{ss}}(\omega^*)/G'.$$

The cylinder window $\overline{\nabla} \subset \mathbb{R}^3$ has the following description

$$\overline{\nabla} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : |\langle \lambda, (x_1, x_2) \rangle| \leq \eta_\lambda / 2 \text{ for all } \lambda \in (\mathbb{R}^2)^* \text{ and } |x_3| \leq 4 \right\},$$

and the zonotope $\overline{\Sigma}_{\omega^*}$ has base in $\mathbb{R}^2$ given by the zonotope of the weights $\beta_1', \ldots, \beta_8'$. Since X is self-dual as a representation of G', it follows that $\eta_\lambda > 0$ for all $\lambda \in \mathbb{R}^2 \setminus \{0\} = (\omega^*)^\perp \setminus \{0\}$, thus by Proposition 4.1.2 it follows that for any λ_0-generic character θ, the set of G-equivariant line bundles with weights in the set $\mathbb{Z}^2 \cap (\theta + \overline{\Sigma}_{\omega^*})$ generates $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{G-\mathrm{ss}}(\omega^*)/G)$. Furthermore, this set contains $\mathbb{Z}^2 \cap (\theta + \nabla)$ which define a strong exceptional collection of line bundles.

Naturally, we are interested in whether the strong exceptional collection

$$\Omega := \left\{ \mathcal{O}_{X^{\mathrm{ss}}(\ell)} \otimes U : \begin{array}{c} U \in \mathrm{Rep}(G) \text{ irreducible with} \\ \text{weight in } M^+ \cap (\theta + \nabla) \end{array} \right\}$$

is full. Namely, whether a G-equivariant line bundle $\mathcal{O}_X \otimes V(\chi)$ with $\chi \in \mathbb{Z}^2 \cap (\theta + \overline{\Sigma}_{\omega^*} \setminus \nabla)$ is in the smallest triangulated subcategory generated by Ω. While we do not establish whether this collection is full, one can still implement the combinatorial generation algorithm above in a computer to check whether it implies fullness of the collection.

We use the following pseudo-code:

```
Input:
    $L_1 := M^+ \cap (\theta + \nabla)$
    $L_2 := M^+ \cap (\theta + \overline{\Sigma}_{\omega^*})$
    A set $S := \{(S_i, \chi_i)\}$, where $S_i$ is a set of weights appearing in an unstably supported complex of the form Proposition 3.5.3 and $\chi_i$ is either the highest or the lowest weight of the particular complex.
```

```
Output:
    “win” means $\mathcal{S}(L_1) = \mathcal{S}(L_2)$, or
    “lose” means this algorithm goes not show fullness, so it is unlikely that
    $\mathcal{S}(L_1) = \mathcal{S}(L_2)$.
Algorithm
    For each $\nu \in L_1$ and $(S_i, \chi_i) \in S$:
    If $\nu + S_i \subset L_1$ and $\nu + \chi_i \in L_2$
        Add $\nu + \chi_i$ to $L_1$ and remove it from $L_2$
    If no points have been added to $L_1$:
        If $L_2$ is empty:
            Return “win”
        If $L_2$ is not empty:
            Return “lose
```

This pseudo code is implemented in python as follows. I thank Eleanor Goh for coding help.

```
import numpy as np

def check_interior(x, L1, L2, Si):
    rules, distinguished_vector = Si
    for L1_vector in L1:
        for j in range(len(rules)):
            if list(np.array(x) + np.array(rules[j])) in L1:
```

```
                if list(np.array(x) + np.array(distinguished_vector)) in L2:
                    return True
    return False

def algorithm(L1, L2, S):
    while len(L2) != 0:
        added = False
        for L1_vector in L1:
            for Si in S:
                if check_interior(L1_vector, L1, L2, Si):
                    added = True
                    new_vector = list(np.array(L1_vector) + np.array(Si[1]))
                    L1.append(new_vector)
                    L2.remove(new_vector)
        if not added:
            break
     return "win" if len(L2) == 0 else "lose"
```

Applying this algorithm to the example above returns “lose”, thus it is unlikely that the set Ω gives a full exceptional collection on $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X^{G-\mathrm{ss}}(\omega^*)/G)$.

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