APPLICATIONS AND CONSTRUCTION OF FRACTIONALLY REPLICATED DESIGNS*

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ABSTRACT

The concepts and definitions of full factorial treatment design and fractionally replicated treatment designs are discussed. Regular and irregular fractional replicates of a factorial design are presented together with illustrative examples. The diallel crossing design, which is also a paired comparisons design, and response surface designs are treated as fractional replicates. A brief discussion of recent research by the author was presented.

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1. INTRODUCTION AND DEFINITIONS

A complete factorial experiment refers to an experiment in which the treatments in the experiment are composed of n factors at two or more levels for each of the factors. If all n factors have the same number, k, of levels, we call this a k^n factorial design. The design refers to the selection of treatments to be included in the experiment and is therefore denoted as a treatment design. The factorial treatment design may be arranged in the experiment according to any number of experimental designs. To illustrate let us suppose that there are n=3 factors A, B, and C such that factor A is at two levels a_0 and a_1, factor B is at two levels b_0 and b_1, and factor C is at three levels, c_0, c_1, and c_2 to yield $p \times q \times r = 2 \times 2 \times 3 = 2^2 \times 3$ factorial = 12 treatments or combinations as follows:

$a_0b_0c_0 = 000$	$a_0b_0c_1 = 001$	$a_0b_0c_2 = 002$
$a_1b_0c_0 = 100$	$a_1b_0c_1 = 101$	$a_1b_0c_2 = 102$
$a_0b_1c_0 = 010$	$a_0b_1c_1 = 011$	$a_0b_1c_2 = 012$
$a_1b_1c_0 = 110$	$a_1b_1c_1 = 111$	$a_1b_1c_2 = 112$

where the factor level subscripts are used to denote the treatment combination of the three factors. These 12 treatments may be arranged in a completely

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randomized design, a randomized complete block design, a 12 × 12 latin square design, one of the many incomplete block designs, or any one of a number of other experimental designs. Thus, it is obvious that a factorial treatment design should not be called an experimental design even though many writers often do. This loose, imprecise and incorrect usage only leads to confusion for readers of published material.

In the above definition of a factorial treatment design nothing is said about equality of replication on the various combinations in a factorial as long as every combination is present in the treatment design. If not all of the combinations are present in the treatment design, then the treatment design is a fraction or fractional replicate of the complete treatment design. The word replicate in the term fractional replicate has not been defined in statistical literature, but let us define it as meaning that we have r replicates of the factorial treatment design composed of N treatments, say, and that if we omit some or all of the replicates for some of the treatments the result will be a fraction of complete factorial experiment which is composed of rN observations. In almost all of the literature on fractional replication r is taken as unity and the discussion starts from there. As should be obvious there should be no reason why an experimenter could not use a 3/2 replicate, e.g., of a complete factorial for which r=2. For our purpose it will suffice to have rN observations in the complete factorial and often to have r=1 in the ensuing discussion.

In a 2 × 2 and 3 × 3 factorial design we have the following relationship between the observations, Y_{ij}, and the parameters or effects in the factorial.

2 x 2:

$$
\begin{aligned}
M - A/2 - B/2 + AB/2 &= Y_{00} \\
M + A/2 - B/2 - AB/2 &= Y_{10} \\
M - A/2 + B/2 - AB/2 &= Y_{01} \\
M + A/2 + B/2 + AB/2 &= Y_{11}
\end{aligned}
$$

3 x 3:

$$
\begin{aligned}
M - A_L + A_Q - B_L + B_Q + A_L B_L - A_L B_Q - A_Q B_L + A_Q B_Q &= Y_{00} \\
M - 2A_Q - B_L + B_Q + 2A_Q B_L - 2A_Q B_Q &= Y_{10} \\
M + A_L + A_Q - B_L + B_Q - A_L B_L + A_L B_Q - A_Q B_L + A_Q B_Q &= Y_{20} \\
M - A_L + A_Q - 2B_Q + 2A_L B_Q - 2A_Q B_Q &= Y_{01} \\
M - 2A_Q - 2B_Q + 4A_Q B_Q &= Y_{11} \\
M + A_L + A_Q - 2B_Q - 2A_L B_Q - 2A_Q B_Q &= Y_{21} \\
M - A_L + A_Q + B_L + B_Q - A_L B_L - A_L B_Q + A_Q B_L + A_Q B_Q &= Y_{02} \\
M - 2A_Q + B_L + B_Q - 2A_Q B_L - 2A_Q B_Q &= Y_{12} \\
M + A_L + A_Q + B_L + B_Q + A_L B_L + A_L B_Q + A_Q B_L + A_Q B_Q &= Y_{22}
\end{aligned}
$$

Another way of writing these equations is

$$
\begin{bmatrix} + & - & - & + \\ + & + & - & - \\ + & - & + & - \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \end{bmatrix} = \begin{bmatrix} Y_{00} \\ Y_{10} \\ Y_{01} \\ Y_{11} \end{bmatrix}
$$

$$
= \quad X\beta \quad = \quad Y
$$

and

$$\begin{bmatrix} + & - & + & - & + & + & - & - & + \\ + & 0 & -2 & - & + & 0 & 0 & 2 & -2 \\ + & + & + & - & + & - & + & - & + \\ + & - & + & 0 & -2 & 0 & 2 & 0 & -2 \\ + & 0 & -2 & 0 & -2 & 0 & 0 & 0 & 4 \\ + & + & + & 0 & -2 & 0 & -2 & 0 & -2 \\ + & - & + & + & + & - & - & + & + \\ + & 0 & -2 & + & + & 0 & 0 & -2 & -2 \\ + & + & + & + & + & + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A_L \\ A_Q \\ B_L \\ B_Q \\ A_L B_L \\ A_L B_Q \\ A_Q B_L \\ A_Q B_Q \end{bmatrix} = \begin{bmatrix} Y_{00} \\ Y_{10} \\ Y_{20} \\ Y_{01} \\ Y_{11} \\ Y_{21} \\ Y_{02} \\ Y_{12} \\ Y_{22} \end{bmatrix}$$

$$= \qquad X\beta \qquad = \qquad Y$$

where + mean plus one and - mean minus one. Also, if r=2 for the 2 $\times$ 2 factorial we would have the following form for $X\beta = Y$:

$$\begin{bmatrix} + & - & - & + \\ + & + & - & - \\ + & - & + & - \\ + & + & + & + \\ + & - & - & + \\ + & + & - & - \\ + & - & + & - \\ + & + & + & + \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ AB/2 \end{bmatrix} = \begin{bmatrix} Y_{001} \\ Y_{101} \\ Y_{011} \\ Y_{111} \\ Y_{002} \\ Y_{102} \\ Y_{012} \\ Y_{112} \end{bmatrix}$$

Here there are 8 observations. The number of observations does not change the parameters or the effects. <u>Only</u> the theory in the field of experimentation decrees whether or not any of the parameters are equal to zero, and are, therefore, omitted from β. Thus in general form we may write any factorial in the form $X\beta=Y$ where X has N rows and N columns and is called the design matrix, β has one column and N rows of the parameters or single degree of freedom effects in a factorial, and Y consists of the N observations (or treatment totals composed of r observations each).

Now if we take any subset of the observations from Y we have a fractional replicate. Let us denote the observations retained as Y_r and those omitted as Y_0. This corresponds to omitting rows in the design matrix X. Hence we may divide the N equations into two sets, those that correspond to the retained observations and those that correspond to the omitted observations. Let us denote this as follows for p observations retained and for $p < N$:

$$X\beta = \begin{bmatrix} X_r \\ \cdots \\ X_0 \end{bmatrix} \beta = Y = \begin{bmatrix} Y_r \\ \cdots \\ Y_0 \end{bmatrix}$$

or

$$X_r\beta = Y_r$$

and

$$X_0\beta = Y_0$$

It is clear that there are more parameters, N, than there are observations, p. This means that the N parameters <u>cannot</u> be estimated individually, but that combinations of parameters must be estimated. For example, suppose that the first 3 observations (Y_{00}, Y_{10}, and Y_{01}) in the 2×2 factorial with $r=1$ are retained to yield a 3/4 replicate of a 2^2 factorial. The combination of effects that can be estimated are

$$\begin{bmatrix} M + AB/2 \\ A/2 - AB/2 \\ B/2 - AB/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & + & + \\ - & + & 0 \\ - & 0 & + \end{bmatrix} \cdot \begin{bmatrix} Y_{00} \\ Y_{10} \\ Y_{01} \end{bmatrix}$$

Thus, $M + AB/2 = (Y_{10}+Y_{01})/2$, etc. Now if AB were actually (not assumed to be because assumptions don't help) equal to zero, then M, A/2 and B/2 could be estimated from the three observations. This mixing up or confounding of effects accompanies all fractional replicates whenever the fraction is less than one.

If one effect is confounded with a second effect, if the second effect is confounded with a third effect and if the first effect is also confounded with the third effect, then this is called complete confounding of effects. If one effect is partially mixed up with one or more additional effects this is termed partial confounding of effects.

2. TYPES OF FRACTIONAL REPLICATES

There are many types of fractional replicates and many ways of constructing them. Much research work in statistics has been concerned with construction of fractional replicates, with analyses of fractional replicates, and with properties of the various fractions. Almost all of the presentation of fractional replication in statistical texts has been on the type denoted as regular fractional replicates. No precise definition for this type has appeared in published literature. Raktoe and Federer [1965] have given a precise geometrical definition for this type of fraction, but their paper has not yet been published. For our purpose, let us simply take a regular fractional replicate to be one where there is complete confounding of sets of effects. This definition, as well as others, have been shown to be incomplete by Raktoe and Federer [1965], but it is desirable to keep this discussion as nonmathematical as possible. An example of a regular one-half fractional replicate of a 2^3 factorial is:

$$\begin{bmatrix} + & - & - & - & + & + & + & - \\ + & - & + & + & - & - & + & - \\ + & + & - & + & - & + & - & - \\ + & + & + & - & + & - & - & - \end{bmatrix} \cdot \begin{bmatrix} M \\ A/2 \\ B/2 \\ C/2 \\ AB/2 \\ AC/2 \\ BC/2 \\ ABC/2 \end{bmatrix} = \begin{bmatrix} Y_{000} \\ Y_{011} \\ Y_{101} \\ Y_{110} \end{bmatrix}$$

Solving, we obtain

$$\begin{bmatrix} M - ABC/2 \\ A - BC/2 \\ B - AC/2 \\ C - AB/2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} + & + & + & + \\ - & - & + & + \\ - & + & - & + \\ - & + & + & - \end{bmatrix} \cdot \begin{bmatrix} Y_{000} \\ Y_{011} \\ Y_{101} \\ Y_{110} \end{bmatrix}$$

The other four observations (Y_{111}, Y_{100}, Y_{010}, and Y_{001}) from the 2^3 factorial also forms a regular one-half replicate with minus sign changed to plus, i.e., M + ABC/2, etc. We note here that sets of effects are completely confounded.

As might be surmized, there are many regular fractional replicates of a k^m factorial, but they must all be fractions of the type k^{-n} where $n < m$ (see Raktoe and Federer[1965]).

An irregular fractional replicate is simply one that is not regular. A fact which was pointed out by Banerjee [1950] some time ago and which has been utilized (often without their knowing of the existence of this paper) in one form or another by several statisticians is that irregular fractional replicates may be constructed by adding together smaller fractions of regular fractional replicates. For example, if we add three 1/4 fractional replicates together we obtain a 3/4 fractional replicate which is irregular. Likewise we may, as Raktoe and Federer [1965] did, form an irregular 1/2 fractional replicate by adding together two 1/4 fractional replicates which are regular fractions.

Likewise, we may utilize Banerjee's [1950] technique to construct the group of designs known as "response surface designs". Some of the simpler ones in this class are described in standard texts (e.g., Cochran and Cox [1957], chapter 8A). This method of construction was illustrated by Federer [1964] where he constructed the following response surface design by adding two 1/3 replicates and three 1/9 replicates from a 3^2 factorial to obtain the nine treatments:

00	02	11
11	11	11
22	20	11

which in "response surface design jargon" are represented as:

X_1	X_2	X_1	X_2	X_1	X_2
-1	-1	-1	1	0	0
0	0	0	0	0	0
1	1	1	-1	0	0

and the 00 treatment observations are called "center points". Other treatment designs may be handled much as the above.

Federer [1964,1966] has treated the diallel crossing design in genetics as a fractional replicate of a k^2 factorial. The diallel crossing treatment design consists of all combinations of i and j for $i < j$, for $i=1,2,\cdots,k-1$ and for $j=2,3,\cdots,k$. For example, let k=5 then the crosses are

12	13	14	15
-	23	24	25
-	-	34	35
-	-	-	45

This is also the "paired comparisons" treatment design. Other diallel crossing plans are:

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35
41	42	43	44	45
51	52	53	54	55

-	12	13	14	15
21	-	23	24	25
31	32	-	34	35
41	42	43	-	45
51	52	53	54	-

11	12	13	14	15
-	22	23	24	25
-	-	33	34	35
-	-	-	44	45
-	-	-	-	55

-	12	13	14	-
-	-	23	24	25
31	-	-	34	35
41	42	-	-	45
51	52	53	-	-

These are all fractions of a full replicate with the first one being the full k^2 factorial. There are many other types and many other uses for these designs than in genetics (see Federer[1964,1966]).

3. EXAMPLES OF THE TYPE OF RESEARCH A STATISTICIAN DOES IN THE AREA OF FRACTIONAL REPLICATION

For the remaining time allotted, the discussion will be on various aspects of research in fractional replication with which the author was directly or indirectly related in recent years. We shall briefly discuss each of the papers listed in the next section in this context. There are many unsolved problems in fractional replication, but some of them may remain unsolved until there are new developments in group theory aspects of mathematics.

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