# HEAT KERNEL ESTIMATES FOR INNER UNIFORM SUBSETS OF HARNACK-TYPE DIRICHLET SPACES

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The main result of this thesis is the two-sided heat kernel estimates for both Dirichlet and Neumann problem in any inner uniform domain of the Euclidean space $\mathbb{R}^n$. The results of this thesis are shown to hold more generally for any inner uniform domain in many other spaces with Gaussian-type heat kernel estimates. We assume that the heat equation is associated with a local divergence form differential operator, or more generally with a strictly local Dirichlet form on a complete locally compact metric space. Other results include the (parabolic) Harnack inequality and the boundary Harnack principle.

## BIOGRAPHICAL SKETCH

Pavel Gyrya was born in Kharkiv, Ukraine in 1980. He studied in Kharkiv lyceum number 27 until 1997 and spent his school years focusing on mathematical competitions. He then went on to study for two years in Moscow State university and then transferred to the University of Toronto. After graduation from University of Toronto he entered the graduate program in Cornell university, and after two years of studying all the major subjects in mathematics, began his work with Prof. Saloff-Coste with the concentration on Analysis and Probability. After graduation from Cornell University Pavel will switch gears and join American Express company to focus on business decisions based on modelling of financial risk.

## ACKNOWLEDGEMENTS

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# Introduction

## 1.1 Basic setting

To introduce our approach, we focus on the case when $U$ is an unbounded domain in $\mathbb{R}^n$, keeping in mind that our approach will be extended to a much more general setting including in particular manifolds with boundary. This paper is concerned with the problem of obtaining global two-sided heat kernel estimates for the Dirichlet heat semigroup in $U$. That is, we want to estimate the fundamental solution

$$(t, x, y) \mapsto p_U^D(t, x, y)$$

of the PDE problem

$$\begin{cases} (\partial_t + \Delta)u = 0 & \text{in } (a, b) \times U \\ u = 0 & \text{on } (a, b) \times \partial U \end{cases} \tag{1.1}$$

with $a = 0$, $b = \infty$. Here $\Delta = -\sum_1^n \partial_i^2$ is the Laplacian in $\mathbb{R}^n$ and $\partial U$ is the boundary of $U$. Equation (1.1) is the heat equation in $U$ with Dirichlet boundary condition and $p_U^D(t, x, y)$ is the Dirichlet heat kernel in $U$. A classical solution of (1.1) in a cylinder $Q = (a, b) \times U$ is a continuous function on $(a, b) \times \overline{U}$ vanishing on $(a, b) \times \partial U$ which, in $Q$, is twice continuously differentiable in the space variable, once continuously differentiable in the time variable, and satisfies (1.1). Note that such classical solutions do not always exist because $\overline{U} \setminus U$ can contain a polar set where nonnegative solutions of the heat equation cannot vanish (e.g. isolated points, submanifold of dimension at most $n-2$, or other irregular boundary points).

In this generality, estimating the Dirichlet heat kernel is a challenging question with difficulties arising both from the possible lack of regularity of the boundary and from the global geometry of the domain. See Figure 1.1. This problem is well understood but already non-trivial when $U$ is a cone. See, e.g., [9, 22, 67, 68, 69] and the references therein. The case when $U$ is the domain above the graph of a Lipschitz function has been studied intensively, especially from the view point of elliptic theory. See [4, 15] and also [2, 6, 45] for generalizations and further pointers to the literature. Deep results concerning the heat equation are obtained in [28, 29, 52] and in [64, 65, 66] where further references can be found. Interesting global phenomena are studied in [8] in the case where $U$ is the inside of a parabola. Other specific cases such as the exterior of a compact set [36, 70] and horn shaped and twisted domains [23] have also been studied. Further references include [17, 62, 63, 56, 71] among many others.

The goal of the present work is to present a general approach that leads to very good two-sided bounds in cases where the effects of the boundary and of the global geometry of the domain are relatively mild. This can be illustrated by treating two simple but essential examples,

$$U \text{ is the domain above the graph of a Lipschitz function } \Phi : \mathbb{R}^{n-1} \to \mathbb{R} \tag{1.2}$$

$$U = \mathbb{R}^n \setminus V \text{ where } V \text{ is a convex domain.} \tag{1.3}$$

In particular for the case (1.3) our results are new.

To explain our main idea, let us consider another important and perhaps better understood problem which is the study of the Neumann heat kernel $p_U^N(t, x, y)$, that is, the fundamental solution of the heat equation in $U$ with Neumann boundary condition, that is,

$$\begin{cases} (\partial_t + \Delta)u = 0 & \text{in } (a, b) \times U \\ \frac{\partial}{\partial \vec{n}} u = 0 & \text{on } (a, b) \times \partial U \end{cases} \tag{1.4}$$

Here $\vec{n} = \vec{n}(x)$ is the normal vector to $\partial U$ at $x$ and we assume for simplicity here that $\partial U$ is smooth. A classical solution of (1.4) in a cylinder $Q = (a, b) \times U$

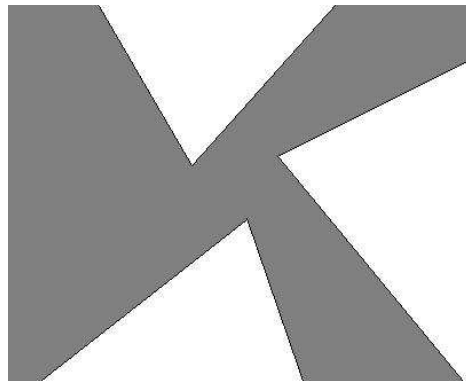

Figure 1.1: The complement of three cones in $\mathbb{R}^2$.

is a continuous function $u : (a, b) \times \overline{U} \to \mathbb{R}$ with continuous first space partial derivatives in $(a, b) \times \overline{U}$ which, in $Q$, is twice continuously differentiable in space, once continuously differentiable in time and satisfies (1.4).

The main reason this is a simpler problem comes from the fact that there is no heat loss, i.e., the heat flow is conservative. Technically, this means that one can make use of most of the tools developed to study the heat equation for divergence form elliptic operators in $\mathbb{R}^n$ and Laplace operators on complete Riemannian manifolds without boundary. Here, we are referring in particular to the celebrated works of Nash and Moser on Harnack inequalities and of Aronson on two-sided Gaussian type heat kernel estimates. See [18, 33, 34, 50, 51] and the references therein. For instance, the techniques of [35, 37] leads to sharp two-sided estimates for the Neumann heat kernel in the region shown in Figure 1.1. We review some of these tools in Chapter 2.6 and illustrate their use by proving two-sided heat kernel estimates for $p_U^N(t, x, y)$ when $U$ is an inner uniform domain. See Theorem 1.3.1 and Definition 3.1.2.

Returning now to the heat equation in $U$ with Dirichlet boundary condition, the main idea we want to apply here is to reduce the problem to one without Dirichlet boundary condition to which the techniques alluded to above can be applied. The crucial first step in this direction is to use the technique of Doob's transform. This is a well-known idea and one of the most relevant references for us in this spirit is [17]. Surprisingly, and despite a rather large literature, e.g., around the notion of intrinsic ultracontractivity [7, 17, 18, 21], this idea has not been developed and used as much as it can to study the Dirichlet heat kernel. Recall that, to any positive function $h$ on $U$ and any semigroup $P_t$ the Doob's transform technique associates another semigroup defined by

$$P_t^h(f) = h^{-1}P_t(hf).$$

If $P_t$ is the heat diffusion semigroup with Dirichlet boundary condition in $U$ and $h$ is harmonic, then $P_t^h$ is a diffusion semigroup to which one may hope to apply the techniques of analysis of local Dirichlet spaces by working on $L^2(U, h^2d\mu)$, $\mu$ being the Lebesgue measure. Moreover, if the harmonic function $h$ vanishes at the boundary, one may hope to show that $P_t^h$ is conservative. In this last case and in probabilistic terms $P_t^h$ is the semigroup associated with Brownian motion conditioned to leave $U$ at infinity. A function $h$ that is positive harmonic in $U$ and vanishes (in the appropriate sense) at the boundary, is called a *réduite* for $U$. The existence and unicity of réduites is discussed in the literature for various specific domains (e.g., [4, 67]). In terms of the notion of Martin boundary, réduites are produced by points at infinity. From the viewpoint taken in this work, the properties of the réduite $h$ are essential for the analysis of $P_t^h$.

One of the aims of this paper is to present a complete implementation of these ideas. However, in order to obtain interesting estimates by analysing the semigroup $P_t^h$ acting on $L^2(U, h^2d\mu)$, one needs to prove some basic results concerning the réduite $h$ and the corresponding Dirichlet space on $L^2(U, h^2d\mu)$. In fact, the hope behind the use of this method is that most of the analysis can be reduced to verifying some basic properties of the réduite $h$. It is thus very important to be able to construct an appropriate réduite $h$ and we will do so in Chapter 5.5. The strategy outlined above is illustrated in this paper by treating inner uniform domains including domains of types (1.2)-(1.3). This strategy is presented in general context in Chapter 1.2. In the well studied case (1.2) of domains above the graph of a Lipschitz function, our analysis makes no use of the many existing results in the literature (e.g., [4, 15, 28, 64]). Instead, we recover some of these results by a different method. In case (1.3), the results we obtain are new. Further examples where the method developed here applies will be discussed in Chapter 3.2. The structure of this paper is discussed at the end of Chapter 1.2.

## 1.2 General approach

Let $X$ be a connected locally compact complete separable metric space, $\mu$ - a positive Borel measure on $X$ with full support. The natural setting for this paper is that of regular strictly local Dirichlet space $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$. Such a Dirichlet form is associated with a self-adjoint nonnegative operator $L$ acting on the domain $\mathcal{D}(L)$ which is a dense subset of $L^2(X, \mu)$. In Chapter 2.2 we will explore a notion of a local weak solution of the elliptic equation

$$Lf = g$$

and of the parabolic heat equation

$$\frac{\partial}{\partial t} f = -Lf$$

on $X$. The heat semigroup $(P_t)_{t>0}$ of contractions on $L^2(X, \mu)$ is defined by the spectral theorem via

$$P_t = e^{-Lt}$$

By the spectral theorem we know that $g(t,x) = P_t f(x)$ is naturally a weak global solution of

$$\begin{cases} \frac{\partial}{\partial t} g = -Lg, & t > 0 \\ g(0,\cdot) = f(\cdot), & t = 0 \end{cases}$$

We are interested in the case when $(X, \mu, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type space (see Chapter 2.5 and Theorem 2.6.1). On such a space there exists a kernel $p(t,x,y)$ of the heat semigroup $(P_t)_{t>0}$. Moreover, $p(t,x,y)$ is Hölder continuous and the two-sided heat kernel estimates (2.40) are satisfied (see the work of Stürm [60]).

For any subset $U \subset X$ we may consider the analogue of a heat equation in $U$ with Dirichlet or Neumann boundary conditions on $\partial U$ by restricting the Dirichlet form $\mathcal{E}$ to certain subsets of $\mathcal{D}(\mathcal{E})$. There are two corresponding heat semigroups $P^N_{U,t}$ and $P^D_{U,t}$. We ask when can we obtain the heat kernel estimates for each of these semigroups. The answer is surprisingly general - the estimates we are obtaining hold for any uniform subset of $X$.

For the Neumann case, our aim is to prove that the Dirichlet space $(\mathcal{E}^N_U, \mathcal{D}(\mathcal{E}^N_U))$ on $L^2(\widetilde{U}, \mu)$ corresponding to the Neumann problem on some completion of $U$ is a Harnack-type space (the set $\widetilde{U}$ denotes a completion of $U$ with respect to the inner geodesic metric $\rho_U$, see Chapter 3). In view of Theorem 2.6.1 this includes proving the doubling property of the measure $\mu$ on balls in $\widetilde{U}$ and the family of Poincaré inequalities for the balls in $\widetilde{U}$. This is done for uniform sets in Chapter 4.

For the Dirichlet case, which is the main focus of this work, the Dirichlet heat semigroup $P^D_{U,t}$ does not preserve the total heat content and so the Dirichlet space $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U), L^2(\widetilde{U}, \mu))$ cannot be Harnack-type. We will see that for inner uniform domains there exists a réduite $h$ which is a global (weak) solution of $Lh = 0$ with weak Dirichlet boundary conditions on $\partial U$. This function $h$ can be used to relate the semigroup $P^D_{U,t}$ via $h$-transform to a conservative semigroup

$$P^D_{U,h,t} f = \frac{1}{h} P^D_{U,t}(hf)$$

acting on $L^2(U, h^2 d\mu)$. Our aim is to prove that the semigroup $P^D_{U,h,t}$ corresponds to a Harnack-type space. In view of Theorem 2.6.1 this requires showing that the measure $h^2 d\mu$ satisfies the doubling condition on $\widetilde{U}$. This also requires proving the family of Poincaré inequalities for the Dirichlet form

$$\mathcal{E}^D_{U,h}(f,g) = \mathcal{E}^D_U(hf, hg)$$

with domain

$$\mathcal{D}(\mathcal{E}^D_{U,h}) = \frac{1}{h}\mathcal{D}(\mathcal{E}^D_U) \subset L^2(U, h^2 d\mu).$$

The structure of this paper is as follows. In Chapter 2 we will introduce the context of Dirichlet forms, the associated metric, Harnack-type spaces and state the main tool for this work. In Chapter 3 we will discuss, with examples, the notions of a uniform and inner uniform domains, for which our estimates of the heat kernel will be proved to hold in this paper. In Chapter 4 we will show that the heat kernel for the Neumann heat equation in $U$ satisfies the two-sided estimates of the same type, provided $U \subset X$ is uniform or even inner uniform.

In Chapter 5.4 we will prove boundary Harnack principle in the context of the uniform subset of $X$ - this is the main tool for the construction and analysis of the réduite function on $U$. In Chapter 5.5 we will construct some réduite function $h$ and we will use the $h$-transform technique to obtain the two-sided estimates on the heat kernel of the Dirichlet heat semigroup in $U$ if $U$ is uniform or even inner uniform.

## 1.3 Statement of results

In this section we state some of the main results proved in this paper. We start with heat kernel estimates for the Neumann heat kernel in inner uniform domains in $\mathbb{R}^n$ and then discuss the Dirichlet heat kernel in domains of types (1.2)–(1.3). The distance

$$\rho = \rho_U$$

used in the statements below is simply the shortest path distance in $U$ (paths must stay in $U$). We call it the inner geodesic distance in $U$. Later we will see how this metric is also associated to both Dirichlet and Neumann diffusion problems in $U$ via Definition 2.1.12. The ball of radius $r$ around $x \in U$ for the distance $\rho_U$ is denoted by

$$B_U(x,r) = \{y \in U : \rho_U(x,y) < r\}.$$

Inner uniform domains are described in Definition 3.1.2 below. These domains include the domains of types (1.2)-(1.3) above and many more. Note however that it does not include all convex domains (e.g., the interior of a parabola is not inner uniform). As Theorem 1.3.1 makes very clear, the condition of inner uniformity is both local (boundary regularity) and global (geometry of the domain).

### 1.3.1 Neumann heat kernel

In the following statements, the boundary of any set is always the boundary in the ambient Euclidean space $\mathbb{R}^n$.

**Theorem 1.3.1** *Let $U$ be an unbounded inner uniform domain in $\mathbb{R}^n$. There exist positive finite constants $c_1, \dots, c_5$ such that the Neumann heat kernel $p_U^N$ in $U$ satisfies*

$$c_1 t^{-n/2} e^{-c_2\rho^2/t} \le p_U^N(t,x,y) \le c_3 t^{-n/2} e^{-c_4\rho^2/t}, \quad \rho = \rho_U(x,y),$$

*for all $x, y \in U$ and all $t > 0$. For any integer $k \geq 0$ there exists a constant $C(k)$ such that the $k$-th time derivative of the Neumann heat kernel satisfies*

$$\left| \left( \frac{\partial}{\partial t} \right)^k p_U^N(t,x,y) \right| \leq C(k) t^{(-k-n/2)} e^{-c_5 \rho^2/t}, \rho = \rho_U(x,y)$$

*for all $x, y \in U$ and all $t > 0$. The constants $c_1, \ldots, c_5$ above depend only on the dimension $n$ and the constants $c_0, c_1$ that appear in Definition 3.1.2, which introduces the notion of inner uniform domain. The constant $C(k)$ depends only on $n, c_0, c_1$ and $k$.*

We will prove two generalizations of this result - Theorem 4.0.5 and Theorem 4.2.7. By [33, 50, 60] (see also Theorem 2.6.1 below), given that the volume of any geodesic ball of radius $r$ in an inner uniform domain grows like $r^n$, the two-sided inequality above is equivalent to saying that the heat equation with Neumann boundary condition in $U$ satisfies a uniform parabolic Harnack principle as stated in the following theorem.

**Theorem 1.3.2** *Let $U$ be an inner uniform domain in $\mathbb{R}^n$. There exists a constant $C$ such that, for any $z \in U$, $r > 0$, and for any non-negative solution $u$ of (1.4) in $Q = (0, 4r^2) \times B_U(z, 2r)$, that is,*

$$\begin{cases} (\partial_t + \Delta) u = 0 & \text{in } (0, 4r^2) \times B_U(z, 2r) \\ \frac{\partial}{\partial \vec{n}} u = 0 & \text{on } (0, 4r^2) \times (\partial B_U(z, 2r) \cap \partial U), \end{cases} \tag{1.5}$$

*we have*

$$\sup_{Q_-} \{u\} \leq C \inf_{Q_+} \{u\} \tag{1.6}$$

*where $Q_- = (r^2, 2r^2) \times B_U(z, r)$, $Q_+ = (3r^2, 4r^2) \times B_U(z, r)$.*

**Remark 1.** Note that the boundaries in $(\partial B_U(x, 2r) \cap \partial U)$ are understood in $\mathbb{R}^n$. To make sense of the explicit boundary condition in (1.5), one needs to assume some minimal regularity of $\partial U$. In fact, we will interpret (1.5) in a weak sense in such a way that no additional regularity assumption is needed. Namely, consider the geodesic closure $\widetilde{U}$ of $U$ obtained by completing $U$ for the distance $\rho_U$ (this is not a subset of $\mathbb{R}^n$). Let $\widetilde{B}(z,r) = \{y \in \widetilde{U} : \rho_U(x,y) < r\}$. Then, a weak solution of the heat equation in $I \times \widetilde{B}(x,r) = I \times B$ is a function $u$ in $L^2(I \to W^1(B))$ with time derivative in the sense of distribution $\partial_t u$ in $L^2(I \to W^1(B)')$ such that

$$\int_I \int_B [(\partial_t u)\phi + \nabla u \cdot \nabla \phi] d\mu = 0$$

for all $\phi \in L^2(I \to W^1(B))$ *with compact support* in $\widetilde{B}(x,r) \subset \widetilde{U}$. Here $W^1(B)$ denotes the Sobolev space in $B$, and $W^1(B)'$ denotes its dual with respect to the inner product in $L^2(U)$. Because the test function $\phi$ is required to have compact support not in $B$ but in $\widetilde{B}$, the equation above implies that $\frac{\partial}{\partial \vec{n}} u = 0$ on the part of the boundary $\partial U$ that touches $B$, assuming that $\frac{\partial}{\partial \vec{n}} u$ makes sense.

**Remark 2.** As we already mentioned, not all convex domains are inner uniform. However, the Harnack inequality stated in Theorem 1.3.2 does hold for any convex domain $U$. Indeed, any geodesic ball $B_U(x,r)$ (geodesic and Euclidean distances coincide) is a convex set of diameter at most $2r$. The necessary Poincaré inequality holds for such sets. The volume $V(x,r)$ of $B_U(x,r)$ can be significantly smaller than $r^n$ but a simple argument shows that the doubling property holds. These properties together with Theorem 2.6.1 (proved in [37, 60]) imply the following two-sided estimates for the Neumann heat kernel

$$c_1 V(x,\sqrt{t})^{-1} e^{-c_2\rho^2/t} \le p_U^N(t,x,y) \le c_3 V(x,\sqrt{t})^{-1} e^{-c_4\rho^2/t}, \quad \rho = \rho_U(x,y),$$

for all $x, y \in U$ and all $t > 0$.

### 1.3.2 Dirichlet heat kernel

To state similar estimates for the Dirichlet heat kernel, we need some notation. Let $h = h_U$ be a réduite of $U$, that is, a positive harmonic function in $U$ which vanishes continuously on the boundary $\partial U$ (in the cases considered below, a posteriori, it turns out that the réduite is unique, up to a positive multiplicative constant). For general domains, in order to deal with the possible existence of irregular boundary points, the more correct requirement is that $h$ is a positive harmonic function on $U$ that vanishes quasi-everywhere on $\partial U$. Set

$$V_{h^2}(x,r) = \int_{B_U(x,r)} h(y)^2 dy.$$

**Theorem 1.3.3** *Let $U$ be an unbounded inner uniform domain in $\mathbb{R}^n$. Let $h$ be a réduite for $U$. There are positive finite constants $c_1, \ldots, c_5$ such that the Dirichlet heat kernel $p_U^D(t,x,y)$ in $U$ satisfies*

$$\frac{c_1 h(x)h(y)e^{-c_2\rho^2/t}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \leq p_U^D(t,x,y) \leq \frac{c_3 h(x)h(y)e^{-c_4\rho^2/t}}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}}, \quad \rho = \rho_U(x,y),$$

*for all $x, y \in U$ and all $t > 0$. For any integer $k \geq 1$ there exists a constant $C(k)$ such that the $k$-th time derivative of the Dirichlet heat kernel satisfies*

$$\left|\left(\frac{\partial}{\partial t}\right)^k p_U^D(t,x,y)\right| \leq \frac{C(k)h(x)h(y)e^{-c_5\rho^2/t}}{t^k\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}}, \quad \rho = \rho_U(x,y)$$

*for all $x, y \in U$ and all $t > 0$. The constants $c_1, \ldots, c_6$ above depend only on the constants $c_0, c_1$ which appear in* Definition 3.1.2*, which introduces the notion of inner uniform domain. The constant $C(k)$ depends only on $n, c_0, c_1$ and $k$.*

**Remark.** In the assumptions of Theorem 1.3.3 there exists a constant $c_6$ depending only on the constants $c_0, c_1$ which appear in Definition 3.1.2, such that for any $x \in U$ and $r > 0$, the volume function $V_{h^2}(x,r)$ can be estimated by

$$c_6{}^{-1}h^2(x_r)\mu(B_U(x,r)) \leq V_{h^2}(x,r) \leq c_6 h^2(x_r)\mu(B_U(x,r))$$

where $x_r$ is any point with $\rho_U(x_r, x) = \frac{r}{4}$ and $\rho_U(x_r, \partial U) \geq \frac{c_1}{8} r$, and $c_1$ is a constant appearing in Definition 3.1.2. Such a point $x_r$ exists by Lemma 4.1.5.

In fact we will prove Theorem 5.0.8 which is a generalization of Theorem 1.3.3 . Our proof also provides a parabolic Harnack inequality which takes the following form.

**Theorem 1.3.4** *Let $U, h$ be as in* Theorem 1.3.3. *There exists a constant $C$ such that if $u$ is a (classical) non-negative solution of $\frac{\partial u}{\partial t} + \Delta u = 0$ in $(0, 4r^2) \times B_U(z, 2r)$, $z \in U$, which vanishes continuously on $(0, 4r^2) \times (\partial B_U(z, 2r) \cap \partial U)$, we have*

$$\sup_{(t,x) \in Q_-} \left\{ \frac{u(t,x)}{h(x)} \right\} \leq C \inf_{(t,x) \in Q_+} \left\{ \frac{u(t,x)}{h(x)} \right\} \tag{1.7}$$

*where $Q_- = (r^2, 2r^2) \times B_U(z, r)$, $Q_+ = (3r^2, 4r^2) \times B_U(z, r)$.*

# Chapter 2

# Dirichlet forms

## 2.1 Dirichlet spaces

The main setting for this paper is that of a regular, strictly local Dirichlet space. Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support. For any open set $U \subset X$, let $C_c(U)$ be the set of all continuous functions with compact support in $U$ and let $C_0(U)$ be the completion of $C_c(U)$ with respect to the supremum norm. Consider a Dirichlet form, i.e., a closed densely defined symmetric Markovian bilinear form $\mathcal{E}$ with domain $\mathcal{D}(\mathcal{E}) \subset L^2(X, \mu)$. For a detailed introduction to Dirichlet forms we refer to [31, Chapter 1]. We recall some important definitions and results.

**Definition 2.1.1 ([31], p.5)** *A function $v$ on $X$ is called a normal contraction of a function $u$ on $X$ if*

$$\forall x, y \in X, |v(x) - v(y)| \leq |u(x) - u(y)| \text{ and } \forall x \in X, |v(x)| \leq |u(x)|.$$

*A function $v \in L^2(X, \mu)$ is called a normal contraction of $u \in L^2(X, \mu)$ if some Borel version of $v$ is a normal contraction of some Borel version of $u$.*

**Lemma 2.1.2 ([31], Theorem 1.4.1)** *Normal contractions operate on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, i.e., whenever $u \in \mathcal{D}(\mathcal{E})$ and $v$ is a normal contraction of $u$, we have $v \in \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(v, v) \leq \mathcal{E}(u, u).$$

**Definition 2.1.3** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a closed form on $L^2(X, \mu)$. For any function $f \in \mathcal{D}(\mathcal{E})$ let*

$$||f||_{\mathcal{D}(\mathcal{E})} = \sqrt{\int_X f^2 d\mu + \mathcal{E}(f, f)} \tag{2.1}$$

*denote the norm of $f$ in the Hilbert space $\mathcal{D}(\mathcal{E})$.*

**Definition 2.1.4 ([31])** *A set $\mathcal{C} \subset \mathcal{D}(\mathcal{E}) \cap C_0(X)$ is called a* ***core*** *for the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ if $\mathcal{C}$ is dense in $\mathcal{D}(\mathcal{E})$ for the norm $(||f||^2_{L^2(X,\mu)} + \mathcal{E}(f,f))^{\frac{1}{2}}$ and dense in $C_0(X)$ for the uniform norm. A Dirichlet form is called* ***regular*** *if it admits a core.*

The following lemma is an important property of the domain $\mathcal{D}(\mathcal{E})$.

**Lemma 2.1.5 ([31], Theorem 1.4.2)** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form. Then the space $\mathcal{D}(\mathcal{E}) \cap L^\infty(X, \mu)$ forms an algebra.*

We will make use of the functions constructed in the following lemma as "cutoff" functions.

**Lemma 2.1.6 ([31], Problem 1.4.1)** *If a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular then it admits a core $\mathcal{C}$ which is a dense subalgebra of $C_0(X)$. Also for any compact set $K$ and relatively compact open set $G$ containing $K$, there exists a nonnegative function $u \in \mathcal{C}$ such that $u \equiv 1$ on $K$ and $u \equiv 0$ on $X \setminus G$.*

**Definition 2.1.7** *A Dirichlet form $\mathcal{D}(\mathcal{E})$ is called* ***strictly local*** *if for any $u, v \subset \mathcal{D}(\mathcal{E})$ such that the supports of $u$ and $v$ are compact and $v$ is constant on the neighborhood of the support of $v$, we have $\mathcal{E}(u,v) = 0$. See [31, p 6] where such Dirichlet forms are called "strong local".*

Any strictly local regular Dirichlet form $\mathcal{E}$ can be written in terms of an "energy measure" $d\Gamma$ so that

$$\mathcal{E}(u,v) = \int_X d\Gamma(u,v)$$

where $d\Gamma(u,v)$ is a signed Radon measure on $X$. The quadratic form $d\Gamma(\cdot,\cdot)$ is defined for $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$ as a Radon measure by

$$\forall \phi \in \mathcal{D}(\mathcal{E}) \cap \mathcal{C}, \int_X \phi d\Gamma(u,u) = \mathcal{E}(u, \phi u) - \frac{1}{2}\mathcal{E}(u^2, \phi) \tag{2.2}$$

and extended to all $u \in \mathcal{D}(\mathcal{E})$ by

$$d\Gamma(u,u) = \sup\{d\Gamma(v,v) |\ v = \min(n, \max(u, -n)), n = 1, 2, \cdots\}.$$

As in [49] we define the measure-valued bilinear form $d\Gamma(\cdot,\cdot)$ on $\mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E})$ by

$$d\Gamma(u,v) = \frac{1}{4}\big(d\Gamma(u+v, u+v) - d\Gamma(u-v, u-v)\big). \tag{2.3}$$

The "energy" form $d\Gamma(\cdot,\cdot)$ is symmetric by definition. Moreover $d\Gamma(\cdot,\cdot)$ satisfies the Leibnitz rule and the chain rule, see [49]. Also the form $d\Gamma$ is strictly local in the sense that for any open $\Omega \subset X$, and any $f_1, f_2, g \in \mathcal{D}(\mathcal{E})$, we have

$$d\Gamma(f_1, g)|_\Omega \equiv d\Gamma(f_2, g)|_\Omega \text{ whenever } f_1 - f_2 \equiv const \ on \ \Omega. \tag{2.4}$$

We now introduce the notion of a local domain of the Dirichlet form.

**Definition 2.1.8** *For any open set $\Omega \subset X$ denote*

$$\mathcal{F}_{loc}(\Omega) = \left\{ f \in L^2_{loc}(\Omega, \mu) : \forall \ compact \ V \subset \Omega, \exists \hat{f} \in \mathcal{D}(\mathcal{E}) : f = \hat{f} \ a.e. \ in \ V \right\}. \tag{2.5}$$

We extend the measure-valued form $d\Gamma(\cdot,\cdot)$ to $\mathcal{F}_{loc}(\Omega) \times \mathcal{F}_{loc}(\Omega)$ in the following way.

**Definition 2.1.9** *In the above context, for any function $f \in \mathcal{F}_{loc}(\Omega)$ define the quadratic form $d\Gamma_\Omega(f,f)$ to be the unique nonnegative Radon measure on $\Omega$ that coincides on $V$ with $d\Gamma(\hat{f},\hat{f})$ for any pair $(V,\hat{f})$ as in* (2.5). *We then define the bilinear form $d\Gamma_\Omega(\cdot,\cdot)$ to be the polarization of the quadratic form $d\Gamma_\Omega(\cdot,\cdot)$ in the sense of* (2.3).

Such a measure exists because the bilinear form $d\Gamma(\cdot,\cdot)$ is local in the sense of (2.4). We will often omit the reference to $\Omega$ from the notation $d\Gamma_\Omega(\cdot,\cdot)$.

We will often assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator in the sense of the following definition.

**Definition 2.1.10 (see e.g. [44])** *The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is said to admit a carré du champ operator if for any $f, g \in \mathcal{D}(\mathcal{E})$, the measure $d\Gamma(f,g)$ is absolutely continuous with respect to $d\mu$. We denote Radon-Nikodym derivative by*

$$\Upsilon(f,g) = \frac{d\Gamma(f,g)}{d\mu} \in L^1(X,\mu), \quad \textit{for} \;\; f,g \in \mathcal{D}(\mathcal{E}).$$

For any open set $\Omega$, the carré du champ operator $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X,\mu)$ can be extended to an operator

$$\begin{aligned} \Upsilon_\Omega \;\; &: \;\; \mathcal{F}_{loc}(\Omega) \times \mathcal{F}_{loc}(\Omega) \to L^1_{loc}(\Omega,\mu) \\ \Upsilon_\Omega(f,g) \;\; &= \;\; \frac{d\Gamma_\Omega(f,g)}{d\mu} \end{aligned} \tag{2.6}$$

because of the following lemma.

**Lemma 2.1.11** *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator in the sense of* Definition2.1.10. *Then for any open set $\Omega \subset X$ and any two functions $u, v \in \mathcal{F}_{loc}(\Omega)$, the measure $d\Gamma_\Omega(u,v)$ on $\Omega$ is absolutely continuous with respect to $d\mu$.*

**Proof.** Take any compact $V \subset \Omega$. By definition there exist functions $\tilde{u}, \tilde{v} \in \mathcal{D}(\mathcal{E})$ coinciding on $V$ with $u$ and $v$ respectively. Therefore $d\Gamma(\tilde{u}, \tilde{v})$ is absolutely continuous w.r.t. $d\mu$ by assumption. It remains to notice that $d\Gamma_\Omega(u,v)$ coincides with $d\Gamma(\tilde{u}, \tilde{v})$ on $V$ by definition. This holds for any compact $V \subset \Omega$, therefore $d\Gamma_\Omega(u,v)$ is absolutely continuous with respect to $d\mu$ as a Radon measure on $\Omega$. □

### 2.1.1 The metric associated with the Dirichlet form

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $L^2(X,\mu)$. In this section we define and explore the properties of the metric and the corresponding length structure associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $X$.

**Definition 2.1.12 (See [60])** *Let $\rho_{\mathcal{E}}$ denote a pseudo-metric associated with the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and given by*

$$\rho(x,y) = \rho_{\mathcal{E}}(x,y) = \sup\left\{u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}) \cap C_0(X), d\Gamma(u,u) \leq d\mu\right\} \quad (2.7)$$

*The condition $d\Gamma(u,u) \leq d\mu$ is understood in the sense of Radon-Nikodym derivative $\frac{d\Gamma(u,u)}{d\mu}$ being less than or equal to one. We will often omit the reference to $\mathcal{E}$ from the notation unless the Dirichlet form is other than the original form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$.*

Note that $\rho_{\mathcal{E}}$ is always a lower semicontinuous function. It is only a pseudo-metric because it might happen that $\rho(x,y) = +\infty$ for some $x, y$. For a careful introduction to this definition and the associated geometry we refer the reader to [61].

For the rest of this paper we will restrict our attention to the case when the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ is local and satisfies two assumptions:

(A1) The pseudo-distance $\rho_{\mathcal{E}}$ is finite everywhere and the topology induced by $\rho_{\mathcal{E}}$ is equivalent to the initial topology on $X$. In particular $x, y \to \rho_{\mathcal{E}}(x,y)$ is a continuous function.

(A2) $(X, \rho_{\mathcal{E}})$ is a complete metric space.

We will state these assumptions again in Chapter 2.1.2. Such Dirichlet forms were studied in [61, 60, 58, 59]. For such Dirichlet forms there is another way to define a metric associated with the Dirichlet form, e.g.,

$$\rho^*(x,y) = \rho^*_{\mathcal{E}}(x,y) = \sup\left\{u(x) - u(y) : u \in \mathcal{F}_{loc}(X) \cap C(X), d\Gamma_X(u,u) \leq d\mu\right\} \quad (2.8)$$

and it is proved in [61] that in the case that is of interest to us here, i.e. under assumptions (A1) and (A2), these associated metrics coincide, i.e. $\rho = \rho^*$.

It is known [61, Corollary 1] that under assumptions (A1) and (A2), the metric $\rho_{\mathcal{E}}$ is a length metric in the sense of Definition 3.0.4, i.e. the distance between two points can be computed by looking at the length (in the metric $\rho_{\mathcal{E}}$) of paths connecting these two points,

$$\rho(x,y) \quad = \quad \inf\left\{L(\gamma) : \gamma \text{ is a continuous path connecting } x \text{ and } y \text{ in } X\right\},$$

where for a path $\gamma : [a,b] \to X$, its length $L(\gamma)$ associated with a metric $\rho$ is given by

$$L(\gamma) = \sup\left\{\sum_{i=1}^{k-1} \rho(\gamma(t_i), \gamma(t_{i+1})) : \; k \in N, t_1 = a, t_k = b, t_i < t_j \text{ for } i < j\right\}$$

Throughout this paper we let $B(x,r)$ denote the open ball in $(X, \rho_{\mathcal{E}})$ of radius $r$ around $x$,

$$B(x,r) = \{y \in X : \rho_{\mathcal{E}}(x,y) < r\}.$$

If $y \in B(x,r)$ then when $\rho$ is a length metric, the distance $\rho(x,y)$ can also be computed by looking only at continuous curves $\gamma$ which stay in $B(x,r)$, because all other curves joining $x$ and $y$ have $L(\gamma) > r$. In other words,

$$\rho(x,y) = \inf\{L(\gamma) : \gamma \text{ is a continuous curve in } B(x,r) \text{ joining } x \text{ and } y\} \qquad (2.9)$$

We prepare the following lemma which shows how the length of a path is related to the energy measure $d\Gamma$. We include the proof found in [61, Theorem 3] for convenience and clarity.

**Lemma 2.1.13 ([61], Theorem 3)** *Assume the conditions (A1) and (A2) are satisfied. Assume that the path $\gamma : [a,b] \to X$ does not have self-intersections. Then*

$$L(\gamma) = \sup\{u(\gamma(a)) - u(\gamma(b)) : Y \textit{ is an open neighborhood of } \gamma([a,b]) \subset X,$$
$$u \in \mathcal{F}_{loc}(Y) \cap C(Y), d\Gamma_Y(u,u) \leq d\mu \textit{ on } Y\} \qquad (2.10)$$

**Proof.** Denote the right hand side of (2.10) by $L^*(\gamma)$. Choose $\epsilon > 0$, an open neighborhood $Y$ of $\gamma([a, b])$ and an admissible function $u$ on $Y$ with

$$u(\gamma(a)) - u(\gamma(b)) \geq L^*(\gamma) - \epsilon$$

(here and below we call a function $u$ on an open set $Y \subset X$ admissible if $u \in \mathcal{F}_{loc}(Y) \cap C(Y)$ with $d\Gamma_Y(u, u) \leq d\mu$ on $Y$).

Let $\delta = \frac{1}{4}\rho(\gamma([a, b]), X \setminus Y)$. By Since both $\gamma([a, b])$ and $X \setminus Y$ are closed, we have $\delta > 0$. Choose $a = t_0 < t_1 < \cdots < t_n = b$ with $\delta_i := \rho(\gamma(t_i), \gamma(t_{i+1})) \leq \delta$. Then for every $i = 1, \ldots, n$ the function $u$ is defined and is admissible on the whole ball $B(\gamma(t_i), 4\delta_i)$. Hence so is the function

$$\tilde{v}_i(x) = \min\left(3\delta_i - \rho(\gamma(t_i), x),\ u(x) - u(\gamma(t_{i-1}))\right).$$

It immediately follows that $\tilde{v}_i \leq 0$ on $B(\gamma(t_i), 4\delta_i) \setminus B(\gamma(t_i), 3\delta_i)$. Hence the function

$$v_i = \begin{cases} \max(\tilde{v}_i, 0), & \text{on } B(\gamma(t_i), 3\delta_i) \\ 0, & \text{else,} \end{cases}$$

is defined, nonnegative and admissible on the whole space $X$. From the Definition 2.1.12 of the metric $\rho$ it follows that

$$v_i(\gamma(t_i)) - v_i(\gamma(t_{i-1})) \leq \rho(\gamma(t_i), \gamma(t_{i-1})) = \delta_i$$

and thus

$$u(\gamma(t_i)) - u(\gamma(t_i)) \leq \delta_i.$$

This implies that

$$\begin{aligned} L^*(\gamma) - \epsilon &\leq u(\gamma(a)) - u(\gamma(b)) = \sum_{i=1}^{n} u(\gamma(y_i)) - u(\gamma(t_{i-1})) \\ &\leq \sum_{i=1}^{n} \rho(\gamma(y_i), \gamma(t_{i-1})) \leq L(\gamma) \end{aligned}$$

This holds for every $\epsilon > 0$, therefore $L^*(\gamma) \leq L(\gamma)$. The opposite inequality follows because the following function is admissible in $Y$

$$u(x) = \inf \{L(\gamma') : \gamma' : [0,1] \to Y \text{ is a curve connecting } \gamma(a) \text{ and } x \text{ in } Y\}, \tag{2.11}$$

and as $Y$ becomes smaller and smaller neighborhood of $\gamma([a,b])$, the right hand side of (2.11) tends to $L(\gamma)$. □

### 2.1.2 The assumptions on the Dirichlet space

We will be interested in Dirichlet forms that satisfy the following properties (see e.g. Theorem 2.6.1).

(A1) The pseudo-distance $\rho_{\mathcal{E}}$ is finite everywhere and the topology induced by $\rho_{\mathcal{E}}$ is equivalent to the initial topology on $X$. In particular $x, y \to \rho_{\mathcal{E}}(x,y)$ is a continuous function.

(A2) $(X, \rho_{\mathcal{E}})$ is a complete metric space.

(A3) The measure $\mu$ on $X$ satisfies doubling condition, i.e for any $x \in X$ and any $R > 0$,

$$\mu(B(x,2R)) \leq c_2 \mu(B(x,R)) \tag{2.12}$$

(A4) The following Poincaré inequality is satisfied for any $x \in X$ and any $R > 0$

$$\inf_{\xi} \int_{B(x,R)} (u-\xi)^2 d\mu \leq c_3 R^2 \int_{B(x,R)} d\Gamma(u,u) \tag{2.13}$$

for any $u \in \mathcal{C}$.

**Remark 1.** The infimum in (2.13) is attained at $\xi = u_{B(x,R)} = \frac{1}{\mu(B(x,R))} \int_{B(x,r)} u d\mu$.

**Remark 2.** The family of Poincaré inequalities (2.13) is equivalent to the same family of inequalities for $u \in \mathcal{D}(\mathcal{E})$ since $\mathcal{C}$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E})$.

## 2.2 Weak solutions

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(X, \mu)$. Let $(L, \mathcal{D}(L))$ be the nonnegative self-adjoint operator associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, implicitly defined using the Riesz representation theorem by

$$\begin{aligned} \mathcal{D}(L) &= \{f \in \mathcal{D}(\mathcal{E}) : \exists C > 0,\ \forall g \in \mathcal{D}(\mathcal{E}), \mathcal{E}(f, g) \leq C ||g||_{L^2(X,\mu)}\} \\ \langle Lf, g \rangle &= \mathcal{E}(f, g) \end{aligned} \tag{2.14}$$

Indeed for each $f \in \mathcal{D}(L)$, the map

$$\mathcal{E}(f, \cdot) : \mathcal{D}(\mathcal{E}) \to \mathbb{R}, \quad g \to \mathcal{E}(f, g)$$

extends to a bounded operator on $L^2(X, \mu)$. The function $Lf$ is then the representation of this map as an element of $L^2(X, \mu)$. Our goal in this section is to introduce the notion of a local solution of the elliptic and parabolic equation involving $L$.

### 2.2.1 Local domains and their properties

Let $X$ be a locally compact separable metric space and let $\mu$ be a Radon measure on $X$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on $L^2(X, \mu)$. Let $\Omega$ be an open subset of $X$. In this section we explore some properties of the domain $\mathcal{F}_{loc}(\Omega)$ and other important function spaces associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

**Lemma 2.2.1** *Let $\Omega \subset X$ be an open set. The space $\mathcal{F}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega, \mu)$ is an algebra. If additionally the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator $\Upsilon$ as in* Definition 2.1.10, *then*

$$\Upsilon_\Omega(gf, gf) \leq 2g^2 \Upsilon_\Omega(f, f) + 2f^2 \Upsilon_\Omega(g, g). \tag{2.15}$$

**Proof.** Let $f, g \in \mathcal{F}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega, \mu)$. Say, $|f|, |g| \le C$ a.e. on $\Omega$. Then $fg \in L^\infty_{loc}(\Omega, \mu) \cap L^2_{loc}(\Omega, \mu)$. To show that $fg \in \mathcal{F}_{loc}(\Omega)$ take any compact subset $V$ of $\Omega$. Let $\hat{f}, \hat{g} \in \mathcal{D}(\mathcal{E})$ be the functions coinciding a.e. on $V$ with $f$ and $g$ respectively. Without loss of generality we can assume that $\hat{f}$ and $\hat{g}$ are in $L^\infty(X, \mu)$, otherwise we may replace them by the functions of type $\hat{f}_1 = \min(\max(\hat{f}, -C), C)$ which is in $\mathcal{D}(\mathcal{E})$ because $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form. By Lemma 2.1.5, $\hat{f}\hat{g} \in \mathcal{D}(\mathcal{E})$, and so the condition (2.5) is satisfied for the function $fg$ and every compact subset $V$ of $\Omega$. Therefore $fg \in \mathcal{F}_{loc}(\Omega)$. We can estimate its local energy measure using the chain rule by

$$\begin{aligned}
\Upsilon_\Omega(gf, gf) &= g^2\Upsilon_\Omega(f,f) + f^2\Upsilon_\Omega(g,g) + 2fg\Upsilon_\Omega(f,g) \\
&\le g^2\Upsilon_\Omega(f,f) + f^2\Upsilon_\Omega(g,g) + 2g\sqrt{\Upsilon_\Omega(f,f)}f\sqrt{\Upsilon_\Omega(g,g)} \\
&\le g^2\Upsilon_\Omega(f,f) + f^2\Upsilon_\Omega(g,g) + g^2\Upsilon_\Omega(f,f) + f^2\Upsilon_\Omega(g,g) \\
&= 2g^2\Upsilon_\Omega(f,f) + 2f^2\Upsilon_\Omega(g,g) \qquad (2.16)
\end{aligned}$$

by Minkovski inequality since $\Upsilon_\Omega(\cdot,\cdot)$ is a nonnegative-definite $L^1_{loc}(\Omega, \mu)$-valued bilinear form. □

Using the quadratic form $d\Gamma_\Omega(\cdot,\cdot)$ we define

$$\begin{aligned}
\mathcal{F}(\Omega) &= \left\{ f \in \mathcal{F}_{loc}(\Omega) \cap L^2(\Omega, \mu) : \int_\Omega d\Gamma_\Omega(f,f) < \infty \right\}, \qquad (2.17) \\
\mathcal{F}_c(\Omega) &= \{ f \in \mathcal{F}(\Omega) : \text{ essential support of } f \text{ is compact in } \Omega \}
\end{aligned}$$

We can extend each function in $\mathcal{F}_c(\Omega)$ by zero outside of $\Omega$ to become a function on $X$, thus we will regard $\mathcal{F}_c(\Omega)$ as a subset of $L^2(X, \mu)$.

**Lemma 2.2.2** *The space $\mathcal{F}_c(\Omega)$ is a subset of $\mathcal{D}(\mathcal{E})$, and*

$$\mathcal{F}_c(\Omega) = \{ f \in \mathcal{D}(\mathcal{E}) : \textit{ essential support of } f \textit{ is a compact subset of } \Omega \}$$

**Proof.** Every $f \in \mathcal{D}(\mathcal{E})$ with essential support being a compact subset of $\Omega$ is trivially in $\mathcal{F}_c(\Omega)$. Take any $f \in \mathcal{F}_c(\Omega)$, and regard it as a function on $X$. Let $V \subset \Omega$ be the essential support of $f$. Let $V'$ be any precompact neighborhood of $V$ in $\Omega$. Then by definition of $\mathcal{F}(U)$ there exists a function $\tilde{f} \in \mathcal{D}(\mathcal{E})$ such that $\tilde{f} \equiv f$ in $\overline{V'}$. Since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Dirichlet form, for every $N > 0$ the function $\tilde{f}_N = \min\{\max\{\tilde{f}, -N\}, N\}$ is in $\mathcal{D}(\mathcal{E}) \cap L^\infty(X)$. Also $\tilde{f}_N \to \tilde{f}$ as $N \to \infty$ in the Hilbert space $\mathcal{D}(\mathcal{E})$, see [31, Theorem 1.4.2].

Pick two intermediate open sets $V''$ and $V'''$ with $V \subset V''$, $\overline{V''} \subset V'''$ and $\overline{V'''} \subset V'$. By Lemma 2.1.6 there exists a bounded nonnegative "cutoff" function $\varphi \in \mathcal{D}(\mathcal{E})$ such that $\varphi \equiv 1$ on $V''$ and $\varphi \equiv 0$ outside $V'''$. Since $\mathcal{D}(\mathcal{E}) \cap L^\infty(X)$ is an algebra, we have $\tilde{f}_N \cdot \varphi \in \mathcal{D}(\mathcal{E})$. Denote $f_N = \min(\max(f, -N), N)$. Then $f_N = \tilde{f}_N \varphi \in \mathcal{D}(\mathcal{E})$.

To show that $f \in \mathcal{D}(\mathcal{E})$ we let $N$ go to $\infty$ and notice that $f_N \to f$ in $L^2(X, \mu)$; to show the convergence is in $\mathcal{D}(\mathcal{E})$ it remains to prove that $f_N$ is a Cauchy sequence in $\mathcal{D}(\mathcal{E})$. We estimate

$$\begin{aligned}
\mathcal{E}(f_M - f_N, f_M - f_N) &= \mathcal{E}((\tilde{f}_M - \tilde{f}_N)\varphi, (\tilde{f}_M - \tilde{f}_N)\varphi) \\
&= \int_X d\Gamma((\tilde{f}_M - \tilde{f}_N)\varphi, (\tilde{f}_M - \tilde{f}_N)\varphi) \\
&= \left[\int_V + \int_{V'\setminus V} + \int_{X\setminus V'}\right] d\Gamma((\tilde{f}_M - \tilde{f}_N)\varphi, (\tilde{f}_M - \tilde{f}_N)\varphi) \\
&= \int_V d\Gamma(\tilde{f}_M - \tilde{f}_N, \tilde{f}_M - \tilde{f}_N) + \int_{V'\setminus V} d\Gamma(0,0) + \int_{X\setminus V'} d\Gamma(0,0)
\end{aligned}$$

by strict local property of $d\Gamma$. Indeed $(\tilde{f}_M - \tilde{f}_N)\varphi \equiv \tilde{f}_M - \tilde{f}_N$ on a neighborhood of $V$; $\tilde{f}_M - \tilde{f}_N \equiv 0$ a.e. on an open set $V' \setminus V$; $\varphi \equiv 0$ on a neighborhood $X \setminus \overline{V'''}$ of $X \setminus V'$. Therefore

$$\mathcal{E}(f_M - f_N, f_M - f_N) \leq \int_X d\Gamma(\tilde{f}_M - \tilde{f}_N, \tilde{f}_M - \tilde{f}_N) = \mathcal{E}(\tilde{f}_M - \tilde{f}_N, \tilde{f}_M - \tilde{f}_N) \to 0$$

as $N \to \infty$ since $\tilde{f}_N \to \tilde{f}$ in $\mathcal{E}$-norm. $\square$

The following Lemma is a weaker version of (2.15) in a more general setting.

**Lemma 2.2.3** *Let $f, g \in \mathcal{F}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega, \mu)$ and assume that $g \in \mathcal{D}(\mathcal{E})$ is a continuous function with compact support in $\Omega$. Then $gf \in \mathcal{F}_c(\Omega) \subset \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(gf, gf) \leq 2\int_\Omega g^2 d\Gamma(f,f) + 2\int_\Omega f^2 d\Gamma(g,g) \tag{2.18}$$

**Proof.** The function $fg$ is compactly supported in $\Omega$ and is in $\mathcal{F}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega, \mu)$ by Lemma 2.2.1. Therefore it is in $\mathcal{F}_c(\mathcal{E})$ and thus in $\mathcal{D}(\mathcal{E})$ by Lemma 2.2.2. Let $D$ denote the difference between the right hand side of (2.18) and the left hand side. We need to prove $D \geq 0$. Using the chain rule we write

$$\begin{aligned} D &= 2\int_\Omega g^2 d\Gamma(f,f) + 2\int_\Omega f^2 d\Gamma(g,g) - \mathcal{E}(gf,gf) \\ &= \int_\Omega g^2 d\Gamma(f,f) + \int_\Omega f^2 d\Gamma(g,g) - 2\int_\Omega fg d\Gamma(f,g) \\ &= \int_V g^2 d\Gamma(f,f) + \int_V f^2 d\Gamma(g,g) - 2\int_V fg d\Gamma(f,g) + \int_{\partial V} f^2 d\Gamma(g,g) \end{aligned} \tag{2.19}$$

where $V$ is an open set of points where $g$ is nonzero. Then $\frac{1}{g} \in \mathcal{F}_{loc}(V) \cap L^\infty_{loc}(V, \mu)$ because each of the functions

$$h_n = \begin{cases} \frac{1}{n^2 g(x)}, & \text{if } |g(x)| \geq \frac{1}{n}; \\ g(x), & \text{if } |g(x)| < \frac{1}{n}. \end{cases}$$

coincides with $\frac{1}{n^2 g}$ on $V_n = \{x \in X : g(x) > \frac{1}{n}\}$, is a normal contraction of $g$, and thus belongs to $\mathcal{D}(\mathcal{E})$. Since both $g$ and $\frac{1}{g}$ are in $\mathcal{F}_{loc}(V) \cap L^\infty_{loc}(V, \mu)$, so is their product by Lemma 2.2.1. By chain rule we know that for any function $h \in \mathcal{F}_{loc}(V)$, we can write the energy measure

$$d\Gamma_V(1,h) = d\Gamma_V\left(g\frac{1}{g}, h\right) = \frac{1}{g} d\Gamma_V(g,h) + g\, d\Gamma_V\left(\frac{1}{g}, h\right). \tag{2.20}$$

Since the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local, $d\Gamma_V(1,h) = 0$ and so (2.20) gives

$$d\Gamma_V\left(\frac{1}{g}, h\right) = -\frac{d\Gamma_V(g,h)}{g^2} \tag{2.21}$$

Since $d\Gamma$ is a nonnegative Radon measure, we can drop the last term in (2.19) and estimate

$$\begin{aligned} D &\geq \int_V g^2 d\Gamma_V(f,f) + \int_V f^2 d\Gamma_V(g,g) - 2\int_V fg d\Gamma_V(f,g) \\ &= \int_V g^4 \left[\frac{1}{g^2} d\Gamma_V(f,f) + f^2 d\Gamma_V\left(\frac{1}{g},\frac{1}{g}\right) + 2f\frac{1}{g} d\Gamma_V\left(\frac{1}{g}, f\right)\right] \\ &= \int_V g^4 d\Gamma_V\left(\frac{f}{g},\frac{f}{g}\right) \geq 0 \end{aligned}$$

because $d\Gamma_V(h,h)$ is a nonnegative Radon measure on $V$ for any $h \in \mathcal{F}_{loc}(V)$. $\square$

### 2.2.2 Weak solutions, elliptic case

We identify $L^2(X,\mu)$ with its dual and let $\mathcal{D}'(\mathcal{E})$ be the dual of $\mathcal{D}(\mathcal{E})$ so that naturally $\mathcal{D}(\mathcal{E}) \subset L^2(X,\mu) \subset \mathcal{D}'(\mathcal{E})$. For and open subset $\Omega$ of $X$ let $\mathcal{F}_c(\Omega)$ be as in (2.17) and let $\mathcal{F}'_c(\Omega)$ denote the dual of $\mathcal{F}_c(\Omega)$ with respect to $L^2(\Omega,\mu)$-norm. Naturally $L^2(\Omega,\mu) \subset \mathcal{D}'(\mathcal{E}) \subset \mathcal{F}'_c(\Omega)$.

**Definition 2.2.4** *Let $\Omega$ be an open subset of $X$. Let $f \in \mathcal{F}'_c(\Omega)$. We say that a function $u : \Omega \to \mathbb{R}$ is a weak solution of*

$$Lu = f$$

*in $\Omega$ if*

(1) $u \in \mathcal{F}_{loc}(\Omega)$

(2) *For any function $\phi \in \mathcal{F}_c(\Omega)$ we have*

$$\int_\Omega d\Gamma_\Omega(\phi,u) = \int_\Omega \phi f \; d\mu \tag{2.22}$$

**Remark.** If $u$ is a weak solution of $Lu = f$ in $\Omega$ and there exists a function $u' \in \mathcal{D}(L)$ such that $u' = u$ a.e. in some subset $\Omega' \subset \Omega$ then $Lu' = f$ a.e. in $\Omega'$ by definition of the operator $L$.

The examples below demonstrate boundary conditions that may be hidden in Definition 2.2.4.

**Examples.** Let $D$ be the open unit ball in $\mathbb{R}^2$, $\mu$ - the Lebesgue measure on $D$ and consider the Dirichlet form associated to the Neumann heat semigroup in $\overline{D}$, given by

$$\begin{aligned} \mathcal{E}_D^N(f,g) &= \int_D \left[\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] d\mu \\ \mathcal{D}(\mathcal{E}_D^N) &= \left\{ f \in L^2(D,\mu) : \text{ distributions } \frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ are in } L^2(D,\mu) \right\} \end{aligned}$$

Let $L_D^N$ be the self-adjoint nonnegative operator associated with this Dirichlet form.

1. A smooth function $u$ is a weak solution of $Lu = f$ in $D$ for some smooth function $f$ if and only if the condition (2) above is satisfied, i.e. for any $\phi \in \mathcal{F}_c(D)$ we have

$$\int_D \left[\frac{\partial \phi}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial u}{\partial y}\right] d\mu = \int_D \phi f \; d\mu \tag{2.23}$$

Since the function $\phi$ is compactly supported in $D$, we can integrate by parts the left hand side to obtain the equivalent equality

$$-\int_D \phi \left[\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial x)^2}\right] d\mu = \int_D \phi f d\mu$$

In other words, $u$ is a smooth weak solution of $Lu = f$ in $D$ if and only if

$$\Delta u = -f \;\text{ in }\; D.$$

2. A smooth function $u$ is a weak solution of $Lu = f$ in $\overline{D}$ for some smooth function $f$ only if for any smooth function $\phi \in \mathcal{F}_c(\overline{D})$ (e.g. any smooth function $\phi$ on $\overline{D}$) the equality (2.23) holds. Since the function $\phi$ is no longer required to be compactly supported in $D$, integrating the left hand side of (2.23) by parts we pick up the boundary term

$$\int_{\partial D} \phi \frac{\partial u}{\partial \vec{n}} d\nu - \int_D \phi \left[\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial x)^2}\right] d\mu = \int_D \phi f d\mu$$

where $\frac{\partial u}{\partial \vec{n}}$ is the normal derivative of $u$ and $\nu$ is the natural measure on $\partial D$. Since $\mu(\partial D) = 0$, the right hand side does not depend on the boundary values of $\phi$. It becomes clear that for $u$ to be a smooth weak solution of $Lu = f$ in $\overline{D}$ it is necessary that

$$\begin{cases} \frac{\partial u}{\partial \vec{n}} = 0 \text{ on } \partial D, \\ \Delta u = f \text{ in } D. \end{cases} \tag{2.24}$$

### 2.2.3 Weak solutions, parabolic case

The next definition introduces the notion of (local) weak solution of the heat equation

$$\frac{\partial u}{\partial t} = -Lu.$$

We need the following notation. Given an open time interval $I$ and a Hilbert space $H$, we let $L^2(I \to H)$ be the Hilbert space of the functions $v : I \to H$ equipped with the natural norm

$$||v||_{L^2(I\to H)} = \left( \int_I ||v(t)||_H^2 dt \right)^{\frac{1}{2}}.$$

We let $W^1(I \to H)$ be the Hilbert space of functions $v : I \to H$ with distributional time derivative $\frac{\partial u}{\partial t}$ that belongs to $L^2(U \to H)$, equipped with its natural norm

$$||v||_{W^1(I\to H)} = \left( \int_I ||v(t)||_H^2 dt + \int_I \left|\left| \frac{\partial v(t)}{\partial t} \right|\right|_H^2 dt \right)^{\frac{1}{2}}.$$

We set

$$\mathcal{F}(I \times X) = L^2(I \to \mathcal{D}(\mathcal{E})) \cap W^1(I \to \mathcal{D}'(\mathcal{E})).$$

Given an open interval $I$ and an open set $\Omega \subset X$, we define $\mathcal{F}_{loc}(I \times \Omega)$ to be the set of all functions $v : I \times \Omega \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in $I$ and any open set $\Omega' \subset \Omega$ relatively compact in $\Omega$ there exists a function $u' \in \mathcal{F}(I \times X)$ such that $u' = u$ a.e. in $I' \times \Omega'$. Finally $\mathcal{F}_c(I \times \Omega)$

is the set of all functions $v$ in $\mathcal{F}(I \times X)$ such that, for a.e. $t \in I$, $v(t, \cdot)$ has compact support in $\Omega$.

**Definition 2.2.5** *Let $I$ be an open time interval. Let $\Omega$ be an open set in $X$ and $Q = I \times \Omega$. We say that a function $u : Q \to \mathbb{R}$ is a weak solution of the heat equation in $Q$ if*

*(1) $u \in \mathcal{F}_{loc}(Q)$*

*(2) For any open interval $J$ relatively compact in $I$ and any function $\phi \in \mathcal{F}_c(Q)$ we have*

$$\int_J \int_\Omega d\Gamma_\Omega(\phi(t, \cdot), u(t, \cdot))dt + \int_J \int_\Omega \phi \frac{\partial}{\partial t} u \; d\mu dt = 0 \tag{2.25}$$

Notice that if $u$ is a weak solution of the heat equation in $I \times \Omega$, then by definition of $\mathcal{F}_{loc}(I \times \Omega)$, for almost all $t \in I$ the distributional derivative $v(t, x) = \frac{\partial}{\partial t} u(t, x)$ is in $\mathcal{D}'(\mathcal{E})$. Letting the function $\phi$ in (2.25) be independent of time, we see that for any bounded interval $J \subset I$ the regularization

$$u_J(x) := \frac{1}{|J|} \int_J u(t, x) dt$$

is a weak solution of the equation

$$Lu_J = -v_J = -\frac{1}{|J|} \int_J \frac{\partial}{\partial t} u(t, x) dt$$

This is similar to saying that for almost all $t > 0$, the function $u(t, \cdot)$ is a weak solution of the equation

$$Lu = -v$$

in the sense of Definition 2.2.4.

The following lemma presents an example of a local weak solution, as well as demonstrates how one could glue together weak local solutions on consecutive time intervals.

**Lemma 2.2.6** *Let $\phi \in \mathcal{D}(\mathcal{E})$. Let $P_t$ be the heat semigroup defined in* Chapter 2.3. *The following two properties hold for the function*

$$\psi(t,x) = \begin{cases} P_t\phi(x), & \text{if } t > 0 \\ \phi(x), & \text{if } t \le 0 \end{cases} \tag{2.26}$$

*(i) The function $\psi$ is in $\mathcal{F}_{loc}(\mathbb{R} \times X)$. For any finite open time interval $I \subset \mathbb{R}$ the function $\psi$ is in $\mathcal{F}(I \times X)$.*

*(ii) If for some open $\Omega \subset X$ the function $\phi(x)$ is constant one in $\Omega$ then the function $\psi(t,x)$ is a weak solution in $\mathbb{R} \times \Omega$ of*

$$\frac{\partial}{\partial t}\psi = -L\psi.$$

**Proof.** To show *(i)* it suffices to prove $\psi \in \mathcal{F}(I \times X)$. To see that $\psi \in L^2(I \to \mathcal{D}(\mathcal{E}))$ it suffices to notice that the functions $\phi$ and $P_t\phi$ are in the Banach space $\mathcal{D}(\mathcal{E})$ and the norm (2.1) of $P_t\phi$ in $\mathcal{D}(\mathcal{E})$ is uniformly bounded by the corresponding norm of $\phi$ since

$$||P_t\phi||^2_{\mathcal{D}(\mathcal{E})} = \mathcal{E}(P_t\phi, P_t\phi) + \int_X (P_t\phi)^2 d\mu \le \mathcal{E}(\phi,\phi) + \int_X \phi^2 d\mu = ||\phi||^2_{\mathcal{D}(\mathcal{E})}$$

To see that $\psi \in W^1(I \to \mathcal{D}'(\mathcal{E}))$ notice that the function

$$\theta(t,x) = \begin{cases} -LP_t\phi(x), & \text{if } t > 0 \\ 0, & \text{if } t \le 0 \end{cases}$$

is the distributional derivative of $\frac{\partial}{\partial t}\psi(t,x)$ by spectral theorem for the self-adjoint nonnegative operator $L$. Indeed for any real numbers $s > 0 \ge r$ we have

$$\begin{aligned} \int_r^s \theta(t,\cdot)dt &= \int_0^s -LP_t\phi dt = \int_0^s \left( \int_0^\infty -\lambda e^{-t\lambda} dE_\lambda(\phi) \right) dt \\ &= \int_0^\infty \left(e^{-s\lambda} - 1\right) dE_\lambda(\phi) = P_s\phi - \phi = \psi(s,\cdot) - \psi(r,\cdot). \end{aligned}$$

For every $t > 0$ the norm of $\theta(t,\cdot)$ in the Hilbert space $\mathcal{D}'(\mathcal{E})$ can be estimated by

$$\begin{aligned}
\left|\left|\frac{\partial}{\partial t}\psi(t,\cdot)\right|\right|_{\mathcal{D}'(\mathcal{E})} &= ||\theta(t,\cdot)||_{\mathcal{D}'(\mathcal{E})} = \sup_{\beta\in\mathcal{D}(\mathcal{E})}\left\{\int_X \theta(t,x)\beta(x)d\mu(x) : ||\beta||_{\mathcal{D}(\mathcal{E})} \leq 1\right\} \\
&= \sup_{\beta\in\mathcal{D}(\mathcal{E})}\left\{\int_X \beta L(P_t\phi)d\mu : ||\beta||_{\mathcal{D}(\mathcal{E})} \leq 1\right\} \\
&= \sup_{\beta\in\mathcal{D}(\mathcal{E})}\left\{\mathcal{E}(P_t\phi,\beta) : ||\beta||_{\mathcal{D}(\mathcal{E})} \leq 1\right\} \\
&\leq \sup_{\beta\in\mathcal{D}(\mathcal{E})}\left\{\sqrt{\mathcal{E}(P_t\phi,P_t\phi)}\cdot\sqrt{\mathcal{E}(\beta,\beta)} : ||\beta||_{\mathcal{D}(\mathcal{E})} \leq 1\right\} \\
&\leq \sqrt{\mathcal{E}(P_t\phi,P_t\phi)} \leq \sqrt{\mathcal{E}(\phi,\phi)}
\end{aligned}$$

uniformly in $t$. Therefore $\psi \in W^1(I \to \mathcal{D}'(\mathcal{E}))$ and thus in $\mathcal{F}(I\times X)$ as desired.

To show *(ii)* according to (2.25) it suffices to check that for almost every $t \in I$

$$\int_\Omega d\Gamma_\Omega(\psi(t,\cdot),q(t,\cdot)) + \int_\Omega q\frac{\partial}{\partial t}\psi\, d\mu = 0 \tag{2.27}$$

for any bounded open interval $I \subset \mathbb{R}$ and any test function $q \in \mathcal{F}_c(I\times\Omega)$. We know that $\theta$ is the distributional derivative $\frac{\partial}{\partial t}\psi$. Notice that for almost every $t \leq 0$, $t \in I$ we have

$$d\Gamma_\Omega(\psi(t,\cdot),q(t,\cdot)) = d\Gamma_\Omega(1,q(t,\cdot)) = 0$$

because the measure $d\Gamma$ is strictly local in the sense of (2.4). Also for $t < 0$, $\theta(t,\cdot) = 0$. Therefore both integrals in (2.27) are zero. For almost all $t \in I$, $t > 0$ we have $\psi(t,\cdot) = P_t\phi$ and $\theta(t,\cdot) = -LP_t(\phi)$ and $q(t,\cdot) \in \mathcal{F}_c(\Omega)$. Therefore the equality (2.27) follows from the integration by parts formula (2.22) and the remark thereafter. □

## 2.3 The heat semigroup and kernel

Fix a Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on $L^2(X,\mu)$. Let $L$ be the nonnegative self-adjoint operator defined on a dense subspace $\mathcal{D}(L) \subset L^2(X,\mu)$ given by (2.14). There exists a unique self-adjoint semigroup $\{P_t\}_{t>0}$ of contractions acting on $L^2(X,\mu)$, having $-L$ as its infinitesimal generator so that $P_t = e^{-tL}$ in the sense of the spectral theorem. Moreover, $\{P_t\}_{t>0}$ is (sub-)Markovian, see [31, section 1].

It has been proved in [60] that with the assumptions of Theorem 2.6.1 the transition function $A \to (P_t 1_A)(x)$ of the semigroup $P_t$ is absolutely continuous with respect to the measure $\mu$ for every $x$, and so there exists a kernel $p(t, x, y)$ of the semigroup $P_t$ relative to the measure $\mu$. The next proposition is a well-known consequence of spectral theory.

**Proposition 2.3.1** *Assume that the heat semigroup $P_t$ associated with any Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ possesses a kernel $p(t, x, y)$ with respect to the measure $d\mu$. Then for any fixed $t_0$ and $y_0$, the function $p(t_0, x, y_0)$, as a function of $x$, belongs to the domain of every power of the operator $L$. In particular, $p(t_0, x, y_0)$ belongs to the domain $\mathcal{D}(\mathcal{E})$. Also for any $t > 0$ and any $x, y_0 \in X$ we have*

$$\frac{\partial}{\partial t} p(t, x, y_0) = -Lp(t, x, y_0). \tag{2.28}$$

**Proof.** As a function of $x$,

$$p(t_0, x, y_0) = \left( P_{\frac{t_0}{2}} p(t_0/2, \cdot, y_0) \right) (x) = P_{\frac{t_0}{2}} f(x) \tag{2.29}$$

with $f(x) = p(t_0/2, x, y_0)$. Also $f \in L^2(X, \mu)$ because by symmetry

$$||p(t_0/2, \cdot, y_0)||^2_{L^2(X,\mu)} = \int_X p(t_0/2, y_0, x) p(t_0/2, x, y_0) d\mu(x) = p(t_0, y_0, y_0) < \infty$$

since the kernel exists. By spectral theorem for $L$, we have $P_{\frac{t_0}{2}} = e^{-\frac{t_0}{2} L}$ and

$$p(t_0, \cdot, y_0) = \int_0^\infty \mathbf{e}^{-\frac{t_0}{2}\lambda} dE_\lambda(f)$$

where $E_\lambda$ is a family of projection operators associated to the self-adjoint nonnegative operator $L$. Therefore $p(t_0, \cdot, y_0)$ belongs to the domain of every power of $L$ by spectral theorem, since for every $n$ the integral

$$\int_0^\infty \lambda^n \mathbf{e}^{-\frac{t_0}{2}\lambda} dE_\lambda(f)$$

is absolutely convergent in $L^2(X, \mu)$. To show (2.28), we notice similarly to (2.29) that for any $t_0 > 0$ and any $t > t_0/2$, $p(t, x, y_0) = P_{t-t_0/2} f(x)$ and therefore satisfies the heat equation (2.28) by definition of the semigroup $P_t$ via spectral theory. □

In particular Proposition 2.3.1 implies that for $t > 0$ the function $v(t, y) = p(t, x, y)$ is a local (weak) solution of

$$\frac{\partial}{\partial t} v(t, y) = -Lv(t, y)$$

and therefore is Hölder continuous in variables $t$ and $y$ by Proposition 2.5.2. The heat kernel $p(t, x, y)$ is also Hölder continuous in $x$ variable by symmetry.

## 2.4 Boundary conditions in open sets

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - a strictly local regular Dirichlet form on $X$. Let $U \subset X$ be an open set. In this section we will define the bilinear forms on $L^2(U, \mu)$ associated with the Neumann and Dirichlet problems in $U$. These bilinear forms will give rise to the Dirichlet and Neumann operators, semigroups and kernels. The definitions below are analogous to the known bilinear forms associated with the Neumann and Dirichlet problems in a smooth open subset of $\mathbb{R}^n$. We start with the Dirichlet problem in $U$.

### 2.4.1 Dirichlet boundary conditions

**Definition 2.4.1** *Let $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ denote the minimal closed extension for the restriction of the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ to the domain $\mathcal{F}_c(U) \subset \mathcal{D}(\mathcal{E})$.*

**Remark.** The domain $\mathcal{D}(\mathcal{E}_U^D)$ is a subset of $\mathcal{D}(\mathcal{E})$. A function $f \in \mathcal{D}(\mathcal{E}) \subset L^2$ belongs to the domain $\mathcal{D}(\mathcal{E}_U^D)$ if and only if there exists a quasi-continuous representative $\tilde{f}$ of $f \in L^2(X)$ such that $\tilde{f} \equiv 0$ quasi-everywhere on $X \setminus U$, see [31, Lemma 2.1.4 and Corollary 2.3.1]. To explain this statement, we recall from [31, section 2.1] that quasi-everywhere means 'everywhere except on a set of 1-capacity zero', where $\lambda$-capacity is defined as follows

$$\begin{aligned} \mathrm{Cap}_\lambda(V) &= \inf \big\{ \lambda ||u||^2_{L^2(X,\mu)} + \mathcal{E}(u,u) : \\ &\qquad u \in \mathcal{D}(\mathcal{E}), u \geq 1 \text{ a.e. on some open } V' \text{ containing } V \big\}. \end{aligned} \tag{2.30}$$

A quasi-everywhere defined function $f$ is called quasi-continuous if for every $\varepsilon > 0$ there exists an open set $V \subset X$ with $\mathrm{Cap}_1(V) < \varepsilon$ such that $f|_{X \setminus V}$ is continuous.

**Remark.** Sets of 1-capacity zero are exactly sets of 0-capacity zero, according to [31, Theorem 2.1.6].

The form $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U))$ is closed by definition. It is straightforward to see that $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U)))$ is regular on $U$ with core $\mathcal{F}_c(U) \cap C_c(U)$ because $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is regular on $X$ with core $\mathcal{F}(U) \cap C_c(X)$. The form $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U))$ is also Markovian because the set $\mathcal{F}_c(U)$ is preserved under normal contractions (see Definition 2.1.1). We denote by $L^D_U$, and $P^D_{U,t}$ the associated nonnegative self-adjoint operator and contractive semigroup on $L^2(U, \mu)$.

As we will see in section 2.7, for any Borel set $A \subset X$ the expression $\big(P^D_{U,t}\chi_A\big)(x)$ - which is called the transition function of the semigroup $P^D_{U,t}$ - is a monotone increasing function of $U$. Also if $U = X$, the domain $\mathcal{D}(\mathcal{E}^D_X)$ is closed and includes a core of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Thus $(\mathcal{E}^D_X, \mathcal{D}(\mathcal{E}^D_X))$ coincides with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and therefore $P^D_{X,t} = P_t$. So if the operator $P_t$ on $L^2(X, \mu)$ possesses a kernel, each of the operators $P^D_{U,t}$ does. Let $p^D_U(t,x,y)$ denote the kernel of $P^D_{U,t}$. Then $p^D_U(t,x,y)$ is a monotone increasing function of the domain $U$.

**Remark.** The form $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U))$ on $U$ does not in general satisfy the condition (A2) stated in Chapter 2.1.2, i.e. $(U, \rho_{\mathcal{E}^D_U})$ is not a complete metric space. If instead we consider the form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ on the geodesic closure $\widetilde{U}$ of $U$ with respect to the metric $\rho_U$, then the condition (A1) will not be satisfied, as the topology given by the metric $\rho_{\mathcal{E}_U^D}$ treats $\partial U$ as one point.

**Definition 2.4.2** *For any Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ possessing a kernel, let $G^{\mathcal{E}}$ denote its Green function,*

$$G^{\mathcal{E}}(x,y) = \int_0^\infty p(t,x,y)dt$$

*The Green function $G^{\mathcal{E}_U^D}$ for the form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ will be denoted by $G_U$.*

The expression $G_U(x,y)$ is then a monotone increasing function of the domain $U$. Notice that the integral does not converge in general unless the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is transient.

**Remark.** If the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ does not possess a kernel, $G^{\mathcal{E}}(x,\cdot)$ must be understood as a measure

$$G^{\mathcal{E}}(x,A) = \int_0^\infty \big(P_t\chi_A\big)(x)dt$$

where $P_t$ is the semigroup on $L^2(X,\mu)$ associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Further properties of the Dirichlet Green function on a precompact domain will be studied in Chapter 5.3.

### 2.4.2 Weak solutions, Dirichlet case

Let $\rho_U$ be the inner metric in $U$ as in Definition 3.0.3. We define by analogy with $\mathcal{F}_{loc}(U)$ the following space of local (weak) solutions (of $Lu = f$) in $V$ with weak Dirichlet boundary conditions on $\partial U$.

**Definition 2.4.3** *Let $V$ be any open subset of $U$. Let*

$$\begin{aligned}\mathcal{F}^0_{loc}(V,U) \;&=\; \{f \in L^2_{loc}(V, \mu|_V) : \forall \textit{ open } \Omega \subset V \textit{ rel. cpt. in } \overline{U} \textit{ with } \rho_U(\Omega, U\setminus V) > 0, \\ &\quad\; \exists \tilde{f} \in \mathcal{D}(\mathcal{E}_U^D) : \tilde{f} \equiv f \quad \mu\textit{-a.e. on } \Omega\}\end{aligned} \tag{2.31}$$

*In case $V = U$, we abbreviate the notation $\mathcal{F}^0_{loc}(U,U)$ to $\mathcal{F}^0_{loc}(U)$.*

**Remark 1.** A space $\mathcal{D}(\mathcal{E}^D_U)$ is clearly a subset of $\mathcal{F}^0_{loc}(U)$ which in turn is a subset of $\mathcal{F}_{loc}(U)$. In view of Definition 2.4.3 and the description of the domain $\mathcal{D}(\mathcal{E}^D_U)$ following Definition 2.4.1, any function in $\mathcal{F}^0_{loc}(U)$ has a quasi-continuous representative - a function on $U$ - which can be extended by zero in a quasi-continuous way to a function in $\mathcal{F}_{loc}(X)$.

**Remark 2.** It is interesting to observe that the space $\mathcal{F}^0_{loc}(V,U)$ in Definition 2.4.3 would not change if we replace $\overline{U}$ by $\widetilde{U}$ in (2.31), see Definition 3.0.3.

Lemma 5.2.3 gives an alternative view on $\mathcal{F}^0_{loc}(V,U)$. Next we define the notion of a local (weak) solution of the elliptic equation $Lu = f$ with weak Dirichlet boundary conditions on $\partial U$.

**Definition 2.4.4** *Let $\Omega$ be an open set in $U$. Let $f \in \mathcal{F}'_c(\Omega)$. We say that a function $u : \Omega \to \mathbb{R}$ is a local (weak) solution of the equation*

$$Lu = f$$

*in $\Omega$ with weak Dirichlet boundary conditions on $\partial U$ if*

*(1) $u \in \mathcal{F}^0_{loc}(\Omega, U)$*

*(2) For any function $\phi \in \mathcal{F}_c(\Omega)$ we have*

$$\int_\Omega d\Gamma_\Omega(\phi, u) = \int_\Omega \phi f d\mu \tag{2.32}$$

Finally for any open subset $V \subset U$, similarly to Chapter 2.2 we will define the notion of a (local) weak solution of the heat equation with Dirichlet boundary conditions on $\partial U$. For any open interval $I$, we set

$$\mathcal{F}^0(I \times U, U) = L^2(I \to \mathcal{D}(\mathcal{E}^D_U)) \cap W^1(I \to \mathcal{D}'(\mathcal{E}^D_U)).$$

Given an open interval $I$ and an open set $V \subset U$, we define $\mathcal{F}^0_{loc}(I \times V, U)$ to be the set of all functions $v : I \times V \to \mathbb{R}$ such that, for any open interval $I' \subset I$ relatively compact in $I$ and any open set $\Omega \subset V$ relatively compact in $\overline{U}$ with $\rho_U(\Omega, U \setminus V) > 0$, there exists a function $u' \in \mathcal{F}^0(I \times U, U)$ such that $u' = u$ a.e. in $I' \times \Omega$.

**Definition 2.4.5** *Let $I$ be an open time interval. Let $\Omega$ be an open set in $U$ and let $Q = I \times \Omega$. We say that a function $u : Q \to \mathbb{R}$ is a weak solution of the heat equation*

$$\frac{\partial}{\partial t} u + Lu = 0$$

*in $Q$ with weak Dirichlet boundary conditions on $\partial U$ if the following two conditions are satisfied*

*(1)* $u \in \mathcal{F}^0_{loc}(Q, U)$

*(2) For any open interval $J$ relatively compact in $I$ and any function $\phi \in \mathcal{F}_c(Q, U)$, we have*

$$\int_J \int_\Omega d\Gamma_\Omega(\phi(t,\cdot), u(t,\cdot))dt + \int_J \int_\Omega \phi \frac{\partial}{\partial t} u \; d\mu dt = 0 \qquad (2.33)$$

### 2.4.3 Neumann boundary conditions

Now we begin defining the Neumann problem in $U$ for an open set $U \subset X$.

**Definition 2.4.6** *Using (2.17) we define the form $(\mathcal{E}^N_U, \mathcal{D}(\mathcal{E}^N_U))$ by*

$$\begin{aligned} \mathcal{D}(\mathcal{E}^N_U) &= \mathcal{F}(U) \subset L^2(U, \mu) \\ \mathcal{E}^N_U(f, g) &= \int_U d\Gamma(f, g) \end{aligned}$$

*where $d\Gamma = d\Gamma_U$ is a measure-valued bilinear form on $\mathcal{F}_{loc}(U) \times \mathcal{F}_{loc}(U)$ as in Definition 2.1.9.*

Notice that the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ on $L^2(U,\mu)$ is strongly local, since $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local on $L^2(X,\mu)$. Normal contractions [31, p.5] clearly operate on $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ since they operate on $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. To show that $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a Dirichlet form it suffices to show that it is closed. We will need the following definitions and lemmas.

**Definition 2.4.7** *Let $V \subset U$ be a compact set. Set*

$$\psi_V(x) = \max\left(1 - \frac{\rho(x,V)}{\frac{1}{2}\rho(\partial U, V)}, 0\right) \tag{2.34}$$

The function $\psi_V$ on $(X,\rho)$ is Lipschitz, identically one in $V$ and is compactly supported in $U$. These functions will be used as cutoff functions thanks to the following Lemma.

**Lemma 2.4.8** *Assume that the metric $\rho_{\mathcal{E}}$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $X$ satisfies the conditions (A1) and (A2) of Chapter 2.1.2. Let $V$ be any compact subset of $U$. Then $\psi_V \in \mathcal{D}(\mathcal{E}) \cap L^\infty(U,\mu)$. For every $u \in \mathcal{F}_{loc}(U)$ the function $\psi_V u$ is in $\mathcal{F}_c(U) \subset \mathcal{D}(\mathcal{E})$ and*

$$\mathcal{E}(\psi_V u, \psi_V u) \leq C\left(\int_{V'} u^2 d\mu + \int_{V'} d\Gamma(u,u)\right) \tag{2.35}$$

*where $V' \subset U$ is the support of $\psi_V$ and the constant $C$ depends only on $U$ and $V$.*

**Proof.** For any compact $V \subset U$, the function $\psi_V u$ is compactly supported in $U$ so in view of Lemma 2.1.6, in order to prove $\psi_V u \in \mathcal{F}_c(U) \subset \mathcal{D}(\mathcal{E})$ it suffices to show (2.35). Using Lemma 2.2.3 we estimate

$$\begin{aligned}
\int_U d\Gamma(\psi_V u, \psi_V u) &\leq 2\int_{V'} u^2 d\Gamma(\psi_V, \psi_V) + 2\int_{V'} \psi_V^2 d\Gamma(u,u) \\
&\leq 2\sup_{V'} \frac{d\Gamma(\psi_V,\psi_V)}{d\mu} \int_{V'} u^2 d\mu + 2\sup_{V'} \psi_V^2 \int_{V'} d\Gamma(u,u)
\end{aligned}$$

So (2.35) follows from the following estimate [58, Lemma 1]

$$d\Gamma(\rho(\cdot,V), \rho(\cdot,V)) \leq d\mu$$

which essentially states that under assumptions (A1) and (A2) of Chapter 2.1.2 the distance function of the regular strictly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is in $\mathcal{F}_{loc}(X)$ and has a weak gradient bounded by one. Finally $\psi_V \in \mathcal{F}_c(U)$ by the above argument with $u \equiv 1 \in \mathcal{F}_{loc}(U)$. $\square$

**Proposition 2.4.9** *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local and regular on $X$. Then for any open subset $U$ of $X$, the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is closed.*

**Outline of the proof.** For the case $X = \mathbb{R}^n$ and $\mathcal{E}(f,f) = \int_{\mathbb{R}^n} |\nabla f|^2 d\mu$, the proof is a simple application of the theory of distributions. Namely, a Cauchy sequence $\{f_i\}_{i=1}^{\infty}$ in the space $\mathcal{D}(\mathcal{E}_U^N)$ is a Cauchy sequence in $L^2(U,\mu)$ such that the sequence of weak gradients $\nabla f_i$ (each of the weak gradients $\nabla f_i$ can be represented by a $n$-dimensional vector of functions in $L^2(U,\mu)$) is a Cauchy sequence in $L^2(U,\mu)$. Since $L^2(U,\mu)$ is complete, there must exists a limit $f$ of the sequence of $f_i$ and the limit $g$ of $\nabla f_i$. The limit is unique in the distribution sense, and therefore $\nabla f = g$ in the sense of distribution, i.e. $\nabla f$ can be represented by an $L^2(U,\mu)$-function, and so

$$\int_U |\nabla f|^2 d\mu < \infty.$$

This shows that in the case $X = \mathbb{R}^n$, the limit $f$ of the Cauchy sequence $\{f_i\}_{i=1}^{\infty}$ is in $\mathcal{D}(\mathcal{E}_U^N)$. $\square$

**Proof of Proposition 2.4.9.** Let $\{u_i\}_{i=1}^{\infty}$ be a Cauchy sequence in $L^2(U,\mu)$ and in $\mathcal{E}_U^N$-sense. First, this sequence converges in $L^2$-sense to some $u \in L^2(U,\mu)$. For any compact subset $V \subset U$, the sequence $\psi_V u_i$ is a Cauchy sequence in $\mathcal{D}(\mathcal{E})$ by Lemma 2.4.8 and therefore converges since the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is closed. Since $\psi_V \equiv 1$ on $V$, we have shown that the sequence $u_i$ converges to $u$ in $\int_V d\Gamma(\cdot,\cdot)$-sense for any compact $V \subset U$. In particular $u \in \mathcal{F}_{loc}(U)$ and for any $f \in \mathcal{F}_{loc}(U)$ the measure $d\Gamma(u,f)$ is well-defined as a measure on $U$.

To prove that the sequence $u_i$ converges to $u$ in $\mathcal{E}_U^N$-sense, we first aim to establish the existence of such limit. Let $\mathscr{M}(U)$ denote the space of signed Radon measures on $U$. Let also $\mathscr{M}^1(U)$ denote the space of finite signed Radon measures on $U$, which is the dual to $C_0(U)$ with supremum norm. The associated norm on $\mathscr{M}^1(U)$ is

$$||\nu||_{\mathscr{M}^1(U)} = \sup_{\sigma \in C_0(U), |\sigma| \leq 1} \int_U \sigma d\nu = \nu_+(U) + \nu_-(U)$$

where $\nu = \nu_+ - \nu_-$ and both $\nu_+$ and $\nu_-$ are nonnegative Borel measures with disjoint supports. The space $\mathscr{M}^1(U)$ is then a Banach space with respect to this norm.

For any function $v \in \mathcal{F}_{loc}(U)$, consider the linear mapping $T_v$,

$$\begin{aligned} T_v : \mathcal{F}_{loc}(U) &\rightarrow \mathscr{M}(U), \\ f &\rightarrow d\Gamma(v, f) \end{aligned}$$

Since $d\Gamma$ is local in the sense of (2.4), the operator $T_v$ is local, i.e. for any open set $V \subset U$

$$T_v(f)|_V \equiv T_v(g)|_V \quad \text{whenever} \quad f \equiv g \text{ a.e. in } V \tag{2.36}$$

Notice that the correspondence $v \rightarrow T_v$ is linear. If $v \in \mathcal{D}(\mathcal{E}_U^N)$, then also

$$T_v : \mathcal{D}(\mathcal{E}_U^N) \rightarrow \mathscr{M}^1(U)$$

Equipping the space $\mathcal{D}(\mathcal{E}_U^N)$ with the seminorm

$$||v||^2 = \mathcal{E}_U^N(v, v) = \int_U d\Gamma(v, v),$$

we see that for $v \in \mathcal{D}(\mathcal{E}_U^N)$, the operator $T_v : \mathcal{D}(\mathcal{E}_U^N) \to \mathscr{M}^1(U)$ is bounded because for every $f \in \mathcal{D}(\mathcal{E}_U^N)$ we have

$$\begin{aligned}||T_v(f)||_{\mathscr{M}^1(U)} &= \big\{\Gamma(v,f)_+(U) + \Gamma(v,f)_-(U)\big\} = \sup_{U'\subset U}\left\{\int_{U'} d\Gamma(v,f) - \int_{U\setminus U'} d\Gamma(v,f)\right\} \\ &\le \sup_{U'\subset U}\left\{\sqrt{\int_{U'} d\Gamma(f,f)}\sqrt{\int_{U'} d\Gamma(v,v)d\mu} + \sqrt{\int_{U\setminus U'} d\Gamma(f,f)}\sqrt{\int_{U\setminus U'} d\Gamma(v,v)d\mu}\right\} \\ &\le \sqrt{\int_U d\Gamma(v,v)}\sqrt{\int_U d\Gamma(f,f)} = ||v||\cdot||f||,\end{aligned}$$

with the equality when $f$ is proportional to $v$. We used that $\sqrt{ab}+\sqrt{cd} \le \sqrt{(a+c)(b+d)}$ for $a,b,c,d \ge 0$. We could apply the Minkovski inequality because for every Borel set $V$, the quadratic form $\int_V d\Gamma(u,u)$ is non-negative definite. Thus

$$||T_v||_{\mathcal{D}(\mathcal{E}_U^N)\to\mathscr{M}^1(U)} = ||v|| = \sqrt{\mathcal{E}_U^N(v,v)} \tag{2.37}$$

Since $u_i$ is a Cauchy sequence in $\mathcal{D}(\mathcal{E}_U^N)$, the sequence of linear operators $T_{u_i} : \mathcal{D}(\mathcal{E}_U^N) \to \mathscr{M}^1(U)$ is a Cauchy sequence by (2.37). Since $\mathscr{M}^1(U)$ is complete, the sequence $T_{u_i}$ converges in the operator norm to some bounded linear operator $T : \mathcal{D}(\mathcal{E}_U^N) \to \mathscr{M}^1(U)$.

We will prove that the operators $T$ and $T_u$ coincide on $\mathcal{F}_{loc}(U)$. Take any $\varphi \in \mathcal{F}_{loc}(U)$. It suffices to compare $T(\varphi)$ to $T_u(\varphi)$ locally, i.e. on any compact subset $V \subset U$. Since both $T$ and $T_u$ are local operators by (2.36) and $\varphi = \varphi\psi_V$ on $V$, it is left to compare $T(\varphi\psi_V)$ and $T_u(\varphi\psi_V)$ as Radon measures in $V$. Let $v = \varphi\psi_V \in \mathcal{F}_c(U)$. Let $V' \subset U$ be the neighborhood of the support of $v$. To prove $T(v) = T_u(v)$ we will show that

$$d\Gamma(u_i, v) \to d\Gamma(u,v)$$

in $\mathscr{M}^1(U)$ as $i \to \infty$. We estimate $\mathscr{M}^1(U)$-norm of the difference

$$\begin{aligned}||d\Gamma(u-u_i,v)||_{\mathscr{M}^1(U)} &= ||d\Gamma((u-u_i)\psi_{V'},v)||_{\mathscr{M}^1(U)} = ||T_v((u-u_i)\psi_{V'})||_{\mathscr{M}^1(U)} \\ &\le ||(u-u_i)\psi_{V'}||\cdot||T_v||_{\mathcal{D}(\mathcal{E}_U^N)\to\mathscr{M}^1(U)} = ||v||\cdot||(u-u_i)\psi_{V'}||\end{aligned}$$

by (2.37). The right hand side tends to zero by the argument in the beginning of this proof.

This holds for any compact $V \subset U$, therefore $T$ coincides with $T_u$ on $\mathcal{F}_{loc}(U) \supset \mathcal{D}(\mathcal{E}_U^N)$, and thus the sequence $T_{u_i}$ converges to $T_u$ in the operator norm. Therefore by (2.37), we have

$$||u - u_i|| = ||T_{u-u_i}||_{\mathcal{D}(\mathcal{E}_U^N)\to\mathscr{M}^1(U)} = ||T_u - T_{u_i}||_{\mathcal{D}(\mathcal{E}_U^N)\to\mathscr{M}^1(U)} \to 0,$$

as $i \to \infty$. Therefore $u_i \to u$ in both $\mathcal{E}_U^N$ and $L^2(U, \mu)$ norms as desired. □

The closed form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is associated with a nonnegative-definite self-adjoint operator and a contractive semigroup, which are denoted $L_U^N$ and $P_{U,t}^N$ respectively. For a general open set $U \subset X$, however, the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is not necessarily regular. This and further properties of these objects will be developed in section 4.2.

## 2.5 Harnack-type forms and Hölder continuity of weak solutions

In this section we introduce the notion of Harnack-type Dirichlet form and begin to introduce their important properties. Let $X$ be a locally compact Hausdorff space equipped with Radon measure $\mu$ with full support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form and let $L$ be the associated nonnegative self-adjoint operator on $\mathcal{D}(L) \subset L^2(X, \mu)$. Let $B(z, r)$ denote a ball in metric space $(X, \rho_{\mathcal{E}})$, centered at $z$.

**Definition 2.5.1** *We say that a regular strictly local Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ is of Harnack type if the distance $\rho_{\mathcal{E}}$ satisfies the conditions (A1) and (A2) of* Chapter 2.1.2, *and the following uniform parabolic Harnack inequality is satisfied with some uniform constant $C$. For any $z \in X$, $r > 0$ and any (weak)* *non-negative solution $u$ of $\frac{\partial u}{\partial t} + Lu = 0$ in $(0, 4r^2) \times B(z, 2r)$, we have*

$$\sup_{(t,x)\in Q_-} u(t,x) \le C \inf_{(t,x)\in Q_+} u(t,x) \tag{2.38}$$

*where $Q_- = (r^2, 2r^2) \times B(z,r)$, $Q_+ = (3r^2, 4r^2) \times B(z,r)$ and both* sup *and* inf *are essential, i.e. computed up to a set of measure zero.*

For any Harnack-type Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ the following elliptic Harnack inequality holds trivially with the same constant $C$ as in (2.38). For any $z \in X$ and $r > 0$ and any (weak) non-negative solution $u$ of the equation $Lu = 0$ in $B(z, 2r)$, we have

$$\sup_{B(z,r)} u \le C \inf_{B(z,r)} u. \tag{2.39}$$

One of the important consequences of the Harnack inequality (2.38) is the following quantitative Hölder continuity estimate found in [51, Theorem 5.4.7].

**Proposition 2.5.2** *Assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a Harnack-type Dirichlet form on $L^2(X, \mu)$. Fix $\tau > 0$. Then there exists $\alpha \in (0,1)$ and $A > 0$ such that any local (weak) solution of $\frac{\partial}{\partial t} u + Lu = 0$ in $Q = (s - \tau r^2, s) \times B(x,r)$, $x \in X$, $r > 0$ has a continuous representative and satisfies*

$$\sup_{(t,y),(t',y')\in Q_-} \left\{ \frac{|u(y,t) - u(y',t')|}{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha} \right\} \le \frac{A}{r^\alpha} \sup_Q |u|.$$

*Here $Q_- = (s - \frac{3}{4}\tau r^2, s - \frac{1}{2}\tau r^2) \times B(x, r/2)$ and $B(x,r)$ is a ball in $(X, \rho_{\mathcal{E}})$ centered at $x$.*

## 2.6 Heat kernel estimates for Dirichlet forms of Harnack type

It turns out that the $L^2$-semigroup associated with each of the Harnack-type Dirichlet forms has a kernel that can be very well estimated from both sides using the associated metric $\rho_{\mathcal{E}}$. Also there are simpler conditions to see if a particular Dirichlet form is of Harnack type. The following theorem is our main tool and is proved in [60].

**Theorem 2.6.1** *Let $X$ be a locally compact Hausdorff space and $\mu$ - a Radon measure on $X$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strictly local regular Dirichlet form on $X$. Assume that the metric $\rho_{\mathcal{E}}$ satisfies the assumptions (A1) and (A2) of* Chapter 2.1.2. *Then the following properties are equivalent:*

- *The form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack type, i.e. the uniform parabolic Harnack inequality (2.38) is satisfied for the (weak) local solutions of $\frac{\partial u}{\partial t} + Lu = 0$.*

- *For any $x \in X$ and $r > 0$ the doubling condition (2.12) for the measure $\mu$ and $L^2$ Poincaré inequality (2.13) are satisfied with some constants.*

- *There exist constants $c_1, c_2, c_3, c_4$ such that the kernel $p(t, x, y)$ of the semigroup $P_t$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ satisfies*

$$\frac{c_1 \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_2 t}\right)}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \leq p(t, x, y) \leq \frac{c_3 \exp\left(-\frac{\rho_{\mathcal{E}}(x,y)^2}{c_4 t}\right)}{\sqrt{\mu(B(x, \sqrt{t}))\mu(B(y, \sqrt{t}))}} \qquad (2.40)$$

  *for all $x, y \in X$ and all $t > 0$.*

In fact the constant $c_4$ in Theorem 2.6.1 can be chosen to be $c_4 = 4 + \varepsilon$ for any $\varepsilon > 0$, see [51]. In the setting above it is possible to use the upper heat kernel estimates to obtain the related upper estimates on the time derivative of the heat kernel using the method presented in [16]. For Harnack-type Dirichlet forms the following is a straightforward corollary of Propositions 2.5.2 and 2.3.1, since the kernel $p(t, x, y)$ exists.

**Corollary 2.6.2** *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is of Harnack type. Then the heat kernel $p(t, x, y)$ is Hölder continuous in $X$ and for every $t, t' > 0$*

*and* $x, y, y' \in X$ *satisfies*

$$|p(t,x,y) - p(t',x,y')| \leq A \left( \frac{\sqrt{|t-t'|} + \rho_{\mathcal{E}}(y,y')}{\sqrt{t}} \right)^{\alpha} p(2t,x,y)$$

*whenever* $|t - t'| < t/2$, $\rho(y,y') \leq \sqrt{t}$.

Also for Dirichlet forms of Harnack type, the semigroup $P_t$ turns out to be conservative, and we present here one of the several ways to show this.

**Lemma 2.6.3** *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a Harnack-type Dirichlet form on* $L^2(X,\mu)$. *For any* $t > 0$ *and any* $x \in X$,

$$\int_X p(t,x,y)d\mu(y) = 1, \tag{2.41}$$

*in other words the semigroup* $P_t$ *is conservative.*

**Proof.** Since the semigroup $P_t$ is Markovian, we have

$$\int_X p(t,x,y)d\mu(y) \leq 1.$$

Fix $z \in X$ and $R > 0$. Let

$$\phi_R(x) = \min(1, \max(0, R + 1 - \rho(x,z))).$$

We know that the function $\phi_R$ is supported in $B(z, R+1)$ and is identically one on $B(z,R)$. Since $\rho(z,\cdot) \in \mathcal{F}_{loc}(X)$ with $d\Gamma(\rho(z,\cdot), \rho(z,\cdot)) \leq d\mu$ by [58, Lemma 1], it follows that $\phi_R \in \mathcal{F}_c(X) \subset \mathcal{D}(\mathcal{E})$ and $d\Gamma(\phi_R, \phi_R) \leq d\mu$ on $X$.

Let $\psi_R$ be the function $\psi$ defined in (2.26) based on the function $\phi_R$. Consider the function

$$v(t,x) = \begin{cases} \int_X p(t,x,y)dy, & \text{if } t > 0 \\ 1, & \text{if } t \leq 0 \end{cases}$$

which is an increasing limit of the functions $\psi_R$ by dominated convergence theorem. Each of the functions $\psi_R$ is a nonnegative weak solution in $\mathbb{R} \times B(z,R)$ of the parabolic equation

$$\frac{\partial}{\partial t}\psi_R = -L\psi_R.$$

Since $0 \leq \psi_R \leq 1$ by Hölder estimates of Proposition 2.5.2, for for any $t, t'$ with $|t - t'| \leq R^2$ and $y, y' \in B(z, \frac{R}{2})$ we have

$$|\psi_R(y,t) - \psi_R(y',t')| \leq A\frac{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha}{R^\alpha}.$$

Taking the limit as $R \to \infty$, we see that for all $y, y' \in X$ and $t, t' \in \mathbb{R}$

$$|v(y,t) - v(y',t')| \leq \limsup_{R\to\infty} \left\{ A\frac{[|t-t'|^{1/2} + \rho_{\mathcal{E}}(y,y')]^\alpha}{R^\alpha} \right\} = 0.$$

□

Let $B = B(z, R)$ be any ball in $(X, \rho_{\mathcal{E}})$. The following theorem summarizes the important estimates of the Dirichlet heat kernel $p_B^D(t,x,y)$ found in [41].

**Theorem 2.6.4** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Harnack type Dirichlet form on $L^2(X, \mu)$. Then the Dirichlet heat kernel $p_B^D(t,x,y)$ in the ball $B = B(z,R)$ satisfies the following estimates*

*(i) There exist constants $\varepsilon, C_1, C_2 > 0$ and $\epsilon \in (0,1)$ such that for any $x, y \in B(z, \epsilon R)$,*

$$\frac{C_1}{\mu(B(z,\sqrt{t}))} \leq p_B^D(t,x,y) \leq \frac{C_2}{\mu(B(z,\sqrt{t}))} \tag{2.42}$$

*whenever $4\rho(x,y)^2 < t \leq (\varepsilon R)^2$.*

*(ii) There exist constants $C, \varepsilon > 0$, and for any $0 < \theta < 1$ there exists a constant $C_\theta$ such that for all $x, y \in B$ we have*

$$\forall t > \theta(\varepsilon R)^2, \quad p_B^D(t,x,y) \leq \frac{C_\theta}{\mu(B(z,\varepsilon R))} \exp\left(-C_3\frac{t}{R^2}\right) \tag{2.43}$$

*(iii) There exist a constant $C_3$, such that for all $x, y \in B$ we have*

$$\forall t > 0, \quad p_B^D(t,x,y) \leq \frac{C_4}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{\rho(x,y)^2}{5t}\right) \tag{2.44}$$

*All the constants above depend only on the constants* $c_2, c_3$ *appearing in* (2.12) *and in* (2.13).

**Proof.** The upper bounds in the estimates *(i)* and *(iii)* follow by comparing the Dirichlet heat kernel to the original heat kernel $p(t, x, y)$ in $X$, as explained in Chapter 2.7. The lower bound in *(i)* follows from [41, Lemma 3.7] and the parabolic Harnack inequality (2.38). The estimate *(ii)* follows by changing notation in [41, Lemma 3.9, part 3]. □

## 2.7 The Markov process and the harmonic measure

Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}), L^2(X, \mu), X, \rho_{\mathcal{E}})$ be a Harnack-type Dirichlet space. Let $P_t$ be the semigroup associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ and for any relatively compact set $V \subset X$ set

$$P(t, x, V) = (P_t \chi_V)(x) \geq 0$$

to be the transition function of the semigroup $P_t$. In view of Theorem 2.6.1 we see that for any $t > 0, x \in X$ the expression $P(t, x, \cdot)$ on $X$ is a Radon measure which is absolutely continuous with respect to $\mu$ with kernel $p(t, x, y)$ - a continuous function of $t, x, y$ which vanishes at infinity. Combining this with the Markovian property of $P_t$ we see that for every $t > 0$ the map

$$f \to g = \int_X f(y) p(t, \cdot, y) d\mu(y)$$

sends the space of bounded function into the space $C(X)$ of bounded continuous functions. This means in other words [25, p.52], that $P(t, x, \cdot)$ is a Feller transition function. The heat kernel estimates from above (2.40) are more than sufficient to apply Theorem 3.5 in [25] which states that there exists a continuous Markov process $\{X_t\}_{t \geq 0}$ with transition function

$$P^x\{X_t \in V\} = P\{X_t \in V | X_0 = x\} = P(t, x, V).$$

This process is then strong Markov by [25, Theorem 3.10], and moreover is strong Feller [26, p.28]. The equation (2.41) is called the stochastic completeness for the semigroup $P_t$. It implies that the process $\{X_t\}_{t\geq 0}$ has almost surely infinite lifetime because for any $t > 0$,

$$P\left\{X_t \text{ alive at time } t|X_0 = x\right\} = P(t,x,X) = \int_X p(t,x,y)d\mu(y) = 1$$

The process $X_t$ has the following characterizing property: $\forall t \geq 0,\ \forall x \in X$,

$$\left(P_t f\right)(x) = E\left[f(X_t)|X_0 = x\right]$$

According to [31, Theorem 4.4.2] the semigroup $P^D_{U,t}$ associated with the Dirichlet problem in an open subset $U$ can be described in terms of this process $X_t$,

$$\left(P^D_{U,t} f\right)(x) = E\left[f(X_t)1_{\{t<\sigma_{X\setminus U}\}}|X_0 = x\right]$$

Here and later $\sigma_{X\setminus U}$ denotes the first hitting time of $X \setminus U$ by the process $X_t$. Because the sample paths of $X_t$ are continuous, the first hitting time of $X \setminus U$ is a monotone nondecreasing (random) function of $U$, and so for any nonnegative $f \in L^2(X,\mu)$, the expression $\left(P^D_{U,t} f\right)(x)$ is a monotone nondecreasing function of $U$.

**Definition 2.7.1** *Let $V$ be an open subset of $X$ and $E \subset \partial V$ be compact. Then*

$$\omega(x,E,V) = P(\sigma_{X\setminus V} < \infty, X_{\sigma_{X\setminus V}} \in E|X_0 = x)$$

*denotes the harmonic measure of a set $V$, as seen from $x \in V$.*

As a function of $x$, $\omega(x,E,V) \in \mathcal{F}_{loc}(V)$ and $L\omega(x,E,V) = 0$ weakly in $V$ by [26, Theorem 12.13]. The strong Markov process $X_t$ has continuous paths, therefore the measure $\omega(x,\cdot,V)$ is supported on $\partial V$.

**Definition 2.7.2 (see [26], p.32 and Theorem 13.1)** *Let $V \subset X$ be a Borel set, and let $\sigma'$ be the first exit time from $V$ after 0. A point $x \in \partial V$ is called regular if $P^x\{\sigma' > 0\} = 0$, i.e.*

$$P\{\exists \varepsilon > 0 \ s.t.\ \forall t < \varepsilon, X_t \in V | X_0 = x\} = 0.$$

**Remark.** An alternative definition of regular points will be presented in Chapter 5.1, see also [10, p.9] and [31] for identification of these notions.

By [26, Theorem 13.1 on p.32] at every regular point $a \in \partial V$ which is an interior point of $E$, we have

$$\lim_{V \ni x \to a} \omega(x, E, V) = 1.$$

Similarly at every regular point $b \in \partial V \setminus E$ we have

$$\lim_{V \ni x \to b} \omega(x, E, V) = 0.$$

The space $X$ is unbounded and satisfies the doubling estimate (2.12). Together with the heat kernel estimates (2.40) this shows that for any bounded open $V \subset X$, the exit time $\tau$ is almost surely finite. Therefore $\omega(x, \partial V, V) = 1$.

# Chapter 3

# The inner metric and uniform sets

Let $(X, \rho)$ be a connected locally compact separable metric space. We can define the associated length function by setting for any path $\gamma : [a, b] \to X$,

$$L(\gamma) = \sup \left\{ \sum_{i=1}^{k-1} \rho(\gamma(t_i), \gamma(t_{i+1})) : \ k \in N, t_1 = a, t_k = b, t_i < t_j \text{ for } i < j \right\} \quad (3.1)$$

**Definition 3.0.3** *Let $U$ be an open subset of a metric space $(X, \rho)$. Define $\rho_U$ to be the geodesic metric in $U$ associated with the length function $L(\gamma)$ given by (3.1),*

$$\rho_U(x, y) \ = \ \inf \{L(\gamma) : \gamma \textit{ is a continuous curve connecting } x \textit{ and } y \textit{ in } U\}. \quad (3.2)$$

*Let $\widetilde{U}$ be the completion of $U$ with respect to the metric $\rho_U$.*

There exists a natural continuous map from $\widetilde{U}$ onto the closure $\overline{U}$ of $U$ in $X$.

**Definition 3.0.4** *We say that the metric $\rho$ on $X$ is a length metric if $\rho = \rho_X$, i.e.*

$$\rho(x, y) = \inf\{L(\gamma) : \gamma \textit{ is a continuous curve joining } x \textit{ and } y\}$$

If $\rho$ is a length metric, then the equality (2.9) says exactly that for $y \in B(x, r)$,

$$\rho(x, y) = \rho_{B(x,r)}(x, y). \quad (3.3)$$

In particular, if $r = \rho_U(x, \widetilde{U} \setminus U)$, then all paths with $L(\gamma) < r$ starting at $x$ must stay in $U$, therefore $B(x, r) \subset U$ and thus for all $y \in B(x, r)$ we have

$$\rho(x, y) = \rho_{B(x,r)}(x, y) = \rho_U(x, y). \quad (3.4)$$

## 3.1 Uniform sets

In this section we will define two related notions of uniform sets which are the main focus of our study.

**Definition 3.1.1** *A metric space $(U, \rho)$ is called uniform with respect to a closed subset $\Gamma$ if any two points $x, y \in U \setminus \Gamma$ can be connected in $U$ by a continuous curve $\gamma$ of length $L(\gamma)$ at most $c_0 \cdot \rho(x, y)$ such that for any $z \in \gamma$,*

$$\rho(z, \Gamma) \geq c_1 \frac{\rho(z, x)\rho(z, y)}{\rho(x, y)} \tag{3.5}$$

*An open subset $U$ of a metric space $(X, \rho)$ is called uniform if $(\overline{U}, \rho)$ is uniform with respect to its subset $\partial U$.*

**Remark.** The condition (3.5) can be replaced by a simpler equivalent condition

$$\rho(z, \Gamma) \geq c \min(\rho(z, x), \rho(z, y))$$

with a new constant $c$. For the sake of not modifying the computations of the following sections, we will keep our current definition.

**Definition 3.1.2** *An open subset $U$ of a metric space $(X, \rho)$ is called inner uniform if $(\widetilde{U}, \rho_U)$ is uniform with respect to $\widetilde{U} \setminus U$, i.e. if any two points $x, y \in U$ can be connected in $U$ by a continuous curve $\gamma$ of length $L(\gamma)$ at most $c_0 \cdot \rho_U(x, y)$ such that for any $z \in \gamma$,*

$$\rho_U(z, \widetilde{U} \setminus U) \geq c_1 \frac{\rho_U(z, x)\rho_U(z, y)}{\rho_U(x, y)} \tag{3.6}$$

Any uniform domain is clearly an inner uniform domain. We are interested in developing the theory of heat kernels for inner uniform domains.

**Definition 3.1.3** *Let $\mathrm{Lip}(\widetilde{U})$ be the space of Lipschitz functions on $(\widetilde{U}, \rho_U)$. Let $\mathrm{Lip}_c(\widetilde{U})$ be the space of Lipschitz functions on $(\widetilde{U}, \rho_U)$ which are compactly supported in $\widetilde{U}$.*

**Lemma 3.1.4** *Assume that the measure $\mu$ on the metric space $(X, \rho)$ satisfies the doubling condition (2.12). Let $U \subset X$ be inner uniform. Then the measure $\mu|_U$ on $(\widetilde{U}, \rho_U)$ satisfies the doubling condition.*

**Proof.** Fix any $x \in U$ and $R > 0$. Without loss of generality assume that the ball $B_U(x, R)$ is not contained in any ball of smaller radius. Then the ball $B_U(x, R)$ contains a point $z$ with $\rho_U(x, z) \geq R/2$. Applying the uniform condition (3.6) we see that there exists a continuous curve $\gamma$ connecting $x$ and $z$ and satisfying (3.6). Take some point $y \in B_U(x, R)$ on the path $\gamma$ such that $\rho_U(x, y) = R/4$. Such a point exists because the distance function $\rho_U(x, \cdot)$ is continuous and $\rho_U(x, z) \geq R/2$. By the uniform condition (3.6) and by triangle inequality, we have

$$\begin{aligned} \rho_U(y, \widetilde{U} \setminus U) \ &\geq \ c_1 \frac{\rho_U(x, y)\rho_U(y, z)}{\rho_U(x, z)} = c_1 R/4 \cdot \frac{\rho_U(y, z)}{\rho_U(x, z)} \geq c_1 R/4 \cdot \frac{\rho_U(x, z) - \rho_U(x, y)}{\rho_U(x, z)} \\ &\geq \ \frac{c_1}{4} R \left(1 - \frac{R/4}{R/2}\right) = \frac{c_1}{8} R. \end{aligned}$$

Therefore the ball $B_U(y, \frac{c_1}{8}R)$ also happens to be the ball $B(y, \frac{c_1}{8}R)$ in $(X, \rho)$. On the other hand the ball $B_U(x, 2R)$ is a subset of $B(y, 4R)$. The doubling property (2.12) of the measure $\mu$ gives

$$\mu(B_U(x, 2R)) \leq \mu(B(y, 4R)) \leq C\mu(B(y, \frac{c_1}{8}R)) \leq C\mu(B_U(x, R))$$

for some constant $C$ depending only on $c_1$ and the constant $c_2$ appearing in (2.12).

□

## 3.2 Examples

In this section we present some examples of uniform and inner uniform domains in $\mathbb{R}^n$. For some of these, the behavior of a réduite function $h$ discussed in the introduction will be studied in the Appendix.

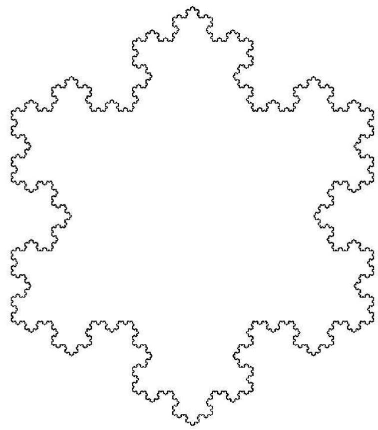

Figure 3.1: Von Koch snowflake - a domain in $\mathbb{R}^2$ with fractal boundary.

Before we proceed to examples in $\mathbb{R}^n$, notice that a wide and natural class of examples of inner uniform subsets of Harnack-type Dirichlet spaces is - inner uniform subsets of complete Riemannian manifolds of nonnegative Ricci curvature, see [50].

**Proposition 3.2.1** *Let $U$ be a domain above the graph of a Lipschitz function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$. Then $U$ is uniform with respect to the usual metric in $\mathbb{R}^n$.*

**Proof.** The proof is in the Appendix. $\square$

**Proposition 3.2.2** *Let $U$ be a domain of the form $U = \mathbb{R}^n \setminus V$ for some closed convex set $V \subset \mathbb{R}^n$. Then $U \subset \mathbb{R}^n$ is inner uniform with $c_0 = 21$, $c_1 = 1/462$.*

**Proof.** This result is not as obvious as it may appear. The proof is in the Appendix. $\square$

For the next example we look at the von Koch snowflake domain. It can be constructed by starting with an equilateral triangle, then recursively altering each line segment via the following procedure:

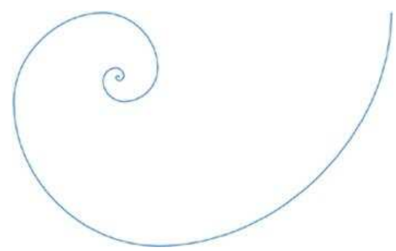

Figure 3.2: The Fibonacci spiral in $\mathbb{R}^2$.

1. Divide the line segment into three segments of equal length.
2. Draw an equilateral triangle that has the middle segment from step 1 as its base.
3. Remove the line segment that is the base of the triangle from step 2.

The von Koch curve is the limit approached as the above steps are followed over and over again. These domains and other domains with fractal boundaries were studied from the point of view of heat equation in [19, 20].

**Proposition 3.2.3** *Both the interior and the exterior of a von Koch snowflake domain of* Figure 3.2 *constructed above are uniform domains in* $\mathbb{R}^2$.

**Proof.** This result is well-known., and we will present the proof in the Appendix. $\square$

We will end this section with the following example without proof.

**Proposition 3.2.4** *The complement in* $\mathbb{C}$ *of the spiral* $S$ *given in the parametric form by* $z(t) = \exp(t + ic\pi t)$ *(see* Figure 3.2*) for some constant* $c > 0$ *is inner uniform.*

# Chapter 4

# Neumann heat kernel

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - a strictly local regular Dirichlet form on $X$. Let $\rho = \rho_{\mathcal{E}}$ be the metric associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $X$, and assume that conditions (A1-A4) stated in Chapter 2.1.2 are satisfied. Let $U$ be an open subset of $X$. Let $\rho_U$ denote the inner geodesic metric in $U$. Let $\widetilde{U}$ be the completion of $U$ with respect to $\rho_U$. Throughout this section let $B_U(x, r)$ denote the open ball in $(\widetilde{U}, \rho_U)$ centered at $x$. Let $V(x, r)$ denote its volume $\mu(B_U(x, r))$.

The goal of this section is to apply Theorem 2.6.1 to obtain the heat kernel estimates for the kernel of the Neumann semigroup $P^N_{U,t}$ in case when $U \subset X$ is inner uniform. We will assume that the energy measure $d\Gamma$ is absolutely continuous with respect to $\mu$ in the sense of Definition 2.1.10.

We will prove the following result that implies Theorem 1.3.1 when $X = \mathbb{R}^n$. In fact, later we will prove another generalization of this result - Theorem 4.2.7.

**Theorem 4.0.5** *Let $(X, \mu)$ be as above. Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the conditions* (A1-A4) *of* Chapter 2.1.2 *and admits a carré du champ operator $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X, \mu)$. Let $U$ be an inner uniform domain in $(X, \rho_{\mathcal{E}})$, see* Definition 3.1.2*. Then the Neumann heat kernel $p^N_U(t, x, y)$ in $U$ exists and satisfies*

$$\frac{c_1 \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{\sqrt{\mu(B_U(x, \sqrt{t}))\mu(B_U(y, \sqrt{t}))}} \le p^N_U(t, x, y) \le \frac{c_3 \exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{\sqrt{\mu(B_U(x, \sqrt{t}))\mu(B_U(y, \sqrt{t}))}} \quad (4.1)$$

*for all $x, y \in U$ and all $t > 0$. For any positive integer $k$ there exists a constant*

*$C(k)$ such that the $k$-th time derivative of the Neumann heat kernel satisfies*

$$\left|\left(\frac{\partial}{\partial t}\right)^k p_U^N(t,x,y)\right| \leq \frac{C(k)\exp\left(-\frac{\rho_U(x,y)^2}{c_5 t}\right)}{t^k\sqrt{\mu(B_U(x,\sqrt{t}))\mu(B_U(y,\sqrt{t}))}} \tag{4.2}$$

*for all $x, y \in U$ and all $t > 0$. Also for arbitrary $z \in U$ every nonnegative (local) weak solution in $(0, 4r^2) \times B_U(z, 2r)$ of the equation*

$$\frac{\partial u}{\partial t} + L_U^N u = 0$$

*satisfies*

$$\sup_{(t,x)\in Q_-} u(t,x) \leq c_6 \inf_{(t,x)\in Q_+} u(t,x) \tag{4.3}$$

*where $Q_- = (r^2, 2r^2) \times B_U(z,r)$, $Q_+ = (3r^2, 4r^2) \times B_U(z,r)$. Here the constants $C(k), c_1, \ldots, c_6$ depend only on $k$ and on the constants $c_0, c_1, c_2$ in* Definition 3.1.2 and (2.12). *In particular the from $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a Harnack-type Dirichlet form on $\widetilde{U}$, see* Definition 2.5.1.

**The plan of the proof.** We learned in Proposition 2.4.9 that $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a closed symmetric form on $L^2(U, \mu)$. In view of Theorem 2.6.1, the following results combined imply this theorem

- Lemma 3.1.4. The doubling condition (2.12) for the measure $\mu|_U$ on $(\widetilde{U}, \rho_U)$.
- The family of Poincaré inequalities (2.13) for the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ on $L^2(\widetilde{U}, \mu|_U)$ with respect to the metric $\rho_U$. This follows from Proposition 4.1.1 applied to the measure $\mu|_U$.
- The form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a regular and strictly local Dirichlet form on $L^2(\widetilde{U}, \mu|_U)$. We will explore these basic properties in Chapter 4.2.
- Lemma 4.2.5. The metric $\rho_{\mathcal{E}_U^N}$ associated with the Dirichlet form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ coincides with $\rho_U$. In particular this implies conditions (A1) and (A2) of Chapter 2.1.2.

- Finally any time derivative of the heat kernel can be estimated from above as in (4.2) by [16, Theorem 4] which uses the estimates of the heat kernel to produce the estimates on its time derivative using the analytic nature of the heat kernel.

$\square$

**Remark.** Theorem 4.0.5 holds more generally if the Dirichlet form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ on $L^2(\widetilde{U}, \mu)$ is replaced by the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ on $L^2(U, vd\mu)$ as will be discussed in Theorem 4.2.7.

The outline above provides the structure for the remainder of this section, where we will complete the analysis of the Neumann heat kernel in $U$. We now focus on proving the Poincaré inequalities for the balls in $(\widetilde{U}, \rho_U)$ in case $U \subset X$ is inner uniform.

## 4.1 Poincaré inequalities for inner uniform subsets

Let $X$ be a locally compact separable metric space. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on $L^2(X, \mu)$. Let $\rho = \rho_{\mathcal{E}}$ be the metric associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ via (2.7). Assume that the conditions (A1) and (A2) of Chapter 2.1.2 are satisfied for the metric $\rho$. Let $\rho_U$ be the inner geodesic metric in $U$. Let $\widetilde{U}$ be the completion of $U$ with respect to the metric $\rho_U$. Throughout this section let $B_U(x, r)$ denote the open ball in $(\widetilde{U}, \rho_U)$ centered at $x$ of radius $r$.

**Proposition 4.1.1** *Let $(X, \rho)$ be a locally compact separable metric space. Let $\mu$ be a positive Radon measure on $X$ with full support. Let $U$ be an inner uniform domain in $(X, \rho)$. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local Dirichlet form on $L^2(X, \mu)$, given by the energy measure,*

$$\mathcal{E}(f, g) = \int_X d\Gamma(f, g), \text{ whenever } f, g \in \mathcal{D}(\mathcal{E}).$$

*Let $\nu$ be any nonnegative Radon measure on $U$ that satisfies the doubling property (2.12) for all balls in $(\widetilde{U}, \rho_U)$.*

*Fix a constant $N > 1$ and assume that there exists a constant $A$ such that for any ball $B = B_U(x, R)$ such that $\rho_U(B, \widetilde{U} \setminus U) \geq NR$, the $L^2$ Poincaré inequality*

$$\forall f \in \mathcal{D}(\mathcal{E}),\ \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^2 d\nu \leq AR^2 \int_B d\Gamma(f, f) \tag{4.4}$$

*holds. Then there is a constant $C$ such that the $L^2$ Poincaré inequality*

$$\forall f \in \mathcal{F}(B), \inf_{\xi \in \mathbb{R}} \int_B |f - \xi|^2 d\nu \leq CR^2 \int_B d\Gamma(f, f) \tag{4.5}$$

*holds for any geodesic ball $B = B_U(x, R)$ in $(\widetilde{U}, \rho_U)$.*

**Remarks.** 1. The main point of this proposition is that the balls involved in the assumption (4.4) are balls in $(X, \rho)$ that happen to be in $U$, whereas the conclusion (4.25) holds for all balls in $(\widetilde{U}, \rho_U)$.

2. Even if the domain $U$ is such that $\rho_U$ is comparable to $\rho$, the Poincaré inequalities (4.25) do not hold true if instead of inner geodesic ball $B$ we consider the trace of a ball in $(X, \rho)$ on $U$ (see figure 4.1), i.e.

$$B'_U(x, r) = \{y \in U : \rho(x, y) < r\}.$$

In this case for these balls, only the weaker inequality (4.7) below holds.

**Outline of the proof.** First, notice that the assumption $f \in \mathcal{D}(\mathcal{E})$ in (4.4) can be relaxed in the following way. For any $\epsilon > 0$ and for any $f \in \mathcal{F}(B_U(x, R + \epsilon))$ there exists a function $\tilde{f} \in \mathcal{D}(\mathcal{E})$ coinciding with $f$ on $B_U(x, R)$. Therefore $d\Gamma_{B_U(x,r)}(\tilde{f}, \tilde{f}) = d\Gamma_{B_U(x,R)}(f, f)$ on $B$ by the local property (2.4) of $d\Gamma$, and hence (4.4) implies that for any ball $B = B_U(x, R)$ such that $\rho_U(B, \widetilde{U} \setminus U) \geq NR$, the $L^2$ Poincaré inequality

$$\forall f \in \mathcal{F}(B_U(x, R + \epsilon)),\ \inf_{\xi \in \mathbb{R}} \int_{B_U(x,R)} |f - \xi|^2 d\nu \leq AR^2 \int_{B_U(x,R)} d\Gamma(f, f) \tag{4.6}$$

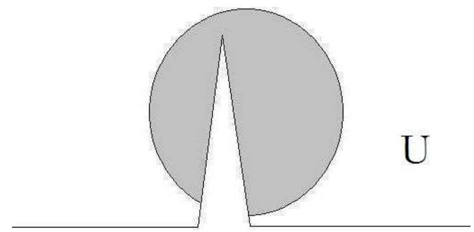


Figure 4.1: A bad Euclidean ball for $U$ (large Poincaré constant)

holds and $d\Gamma$ is understood as $d\Gamma_B$. We prove (4.25) in two stages. First we prove that there exists $k \geq 1$ such that

$$\forall f \in \mathcal{F}(B_U(x, kR)), \ \inf_{\xi \in \mathbb{R}} \int_{B_U(x,R)} |f - \xi|^2 d\nu \leq CR^2 \int_{B_U(x,kR)} d\Gamma(f, f) \tag{4.7}$$

for each ball $B_U(x, R)$, $x \in \widetilde{U}$, $r > 0$. We call this a weak Poincaré inequality because the ball on the right-hand side has been enlarged.

The second step consists of showing that the family of weak $L^2$ Poincaré inequalities (4.7) for $x \in \widetilde{U}$, $R > 0$ and functions $f \in \mathcal{F}(B_U(x, kR))$ implies the standard $L^2$ Poincaré inequality (4.25) for functions in $\mathcal{F}(B)$. This is a well established result, and we will omit the proof. See, e.g. [51, Chapter 5.3.2-5.3.3] and the references therein.

In fact the proof of the second step is very similar to the proof of step one. It is essential that the requirement $f \in \mathcal{D}(\mathcal{E})$ of (4.4) can be relaxed to $f \in \mathcal{F}(B_U(x, kR))$ of (4.7) in step one, and henceforth similarly relaxed to the requirement $f \in \mathcal{F}(B)$ of (4.25) in step two. $\square$

We now focus on proving the weak Poincaré inequality (4.7).

### 4.1.1 Proof of the weak Poincaré inequality (4.7).

In this section we aim to prove (4.7) in the assumptions of Proposition 4.1.1. We will use a Whitney cover of the domain $U$ by the balls in $(U, \rho_U)$ whose distance to the boundary of $U$ is large compared to their radius and for which, by hypothesis, the $L^2$-Poincaré inequality (4.4) is satisfied.

We will need the following notation. On a general metric space, a set can be a ball in more than one way. Thus we follow the convention that a ball $B$ in $(\widetilde{U}, \rho_U)$ is always assumed to be given in the form $B = B_U(x, R)$ with a specified center and a radius $R = r(B)$ which is minimal in the sense that

$$B_U(x, s) \neq B \quad if \quad s < r(B). \tag{4.8}$$

For any ball $B = B_U(x, r)$ with fixed center and radius, define the multiple $kB$ of $B$ by setting

$$kB = B_U(x, kr).$$

**Definition 4.1.2** *A strict $\varepsilon$-Whitney cover of an open set $U$ in a metric space $(X, \rho)$ is any set $\Re$ of **disjoint** balls $A = B(x, r) \subset U$ such that the union of the balls $3A$ cover $U$ and for any $A = B(x, r) \in \Re$:*

$$r(A) = \varepsilon\rho(x, \widetilde{U} \setminus U). \tag{4.9}$$

For $\varepsilon$ small enough, e.g. $\varepsilon \in (0, \frac{1}{3})$ such a cover exists for any open set by a general argument using Zorn's lemma. If, as in the case of interest for us here, the metric space is equipped with a Borel measure satisfying the doubling property, the cover will always be countable.

**Remark.** For a domain in Euclidean space one can use a very neat Whitney covering using cubes instead of balls (see figure 4.2). Consider all the cubes of size length $2^k$ with edges parallel to the coordinate axis and each of the vertices having all coordinates of the form $n2^k$. A given cube $Q$ is included into the covering $\Re$ if and only if its distance to the boundary is at least equal to the fixed desired multiple of its side length and no larger cube has this property.

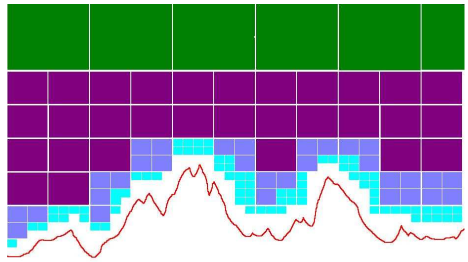

Figure 4.2: Typical cover by Whitney cubes

Whitney covers have the following nice property.

**Lemma 4.1.3 (Finite intersection property)** *Let $\Re$ be a strict $\varepsilon$-Whitney covering of an open set $U$ in some metric space $(X, \rho)$ with $\varepsilon \in (0, \frac{1}{4})$. Assume that $X$ is equipped with a Borel measure having the doubling property (2.12). Then there is a finite constant $a_1$ such that*

$$\forall k < \frac{1}{10\varepsilon}, \sum_{A \in \Re} \chi_{kA} \leq a_1.$$

**Proof.** Pick any point $y \in U$. It belongs to some triple of a Whitney ball $B \in \Re$ with center $z$. If a k-multiple of a given Whitney ball $A = B_U(x, r)$ contains point $y$, then $\rho_U(x, y) \leq kr$. Since

$$\rho_U(x, \widetilde{U} \setminus U) = \frac{r}{\varepsilon}$$

by the Whitney covering condition (4.9), this means that by triangle inequality

$$\frac{r}{\varepsilon} - kr \leq \rho_U(y, \widetilde{U} \setminus U) \leq \frac{r}{\varepsilon} + kr.$$

Applying the Whitney covering condition (4.9) and a triangle inequality,

$$r(B) = \varepsilon\rho(z, \widetilde{U} \setminus U) \leq \varepsilon(3r(B) + \rho(y, \widetilde{U} \setminus U))$$

since $y \in 3B$. Therefore

$$r(B) \leq \frac{\varepsilon}{1-\varepsilon}\rho(y, \widetilde{U} \setminus U) \leq 2\varepsilon\left(\frac{r}{\varepsilon} + kr\right) = (2 + 2k\varepsilon)r \leq 3r(A).$$

Similarly $r(A) \leq 3r(B)$. Hence, $x \in 5kB$, and $A \subset 10kB \subset \frac{1}{\varepsilon}B$. But by doubling condition (2.12), there are only finitely many disjoint Whitney balls of radius at least $r(B)/3$ in the ball $\frac{1}{\varepsilon}B$: their number is uniformly bounded from above by doubling property (2.12). Thus the number of Whitney balls A with the property that a $k$-multiple of it covers $y$, is finite and bounded from above by a constant independent of $y$. $\square$

We return to the proof of (4.7). Set $\varepsilon = 10^{-4}/N$, where the constant $N$ comes from the assumption of the Proposition 4.1.1. Let $\Re$ be a strict $\varepsilon$-Whitney covering of the set $U$ in $(\widetilde{U}, \rho_U)$.

**Definition 4.1.4** *For any ball $B = B_U(x, r)$ in $(U, \rho_U)$ define the collection $\Im(B)$ by*

$$\Im(B) = \{A|\ A \in \Re,\ 3A \cap B \neq \emptyset\} \tag{4.10}$$

Fix a ball $B = B_U(x, R)$ in $(\widetilde{U}, \rho_U)$. Recall that we aim to prove (4.7) for the ball $B$. If $B$ is relatively far from the boundary, i.e. $\rho(B, \widetilde{U} \setminus U) \geq NR$, then the strong $L^2$-Poincaré inequality (4.6) holds. Hence assume that $B = B_U(x, R)$ is relatively large compared with $\rho(B, \widetilde{U} \setminus U)$, namely

$$R \geq \frac{1}{N}\rho(B, \widetilde{U} \setminus U) \tag{4.11}$$

The ball $B$ is covered by the triples of the balls in the collection $\Im(B)$. All the balls $A \in \Re$ are small compared to their distance to the boundary in the sense of (4.9), and the ball $B$ is relatively large by assumption (4.11). Hence it is not hard to see that

$$B \subset \bigcup_{A \in \Im(B)} 3A \subset 2B \tag{4.12}$$

**Lemma 4.1.5** *Let $U$ be inner uniform domain in $(X, \rho)$, and let $\rho_U$ be the geodesic metric in $U$. Then for every ball $B = B_U(x, R)$ in $(\widetilde{U}, \rho_U)$ with $R = r(B)$ being* *the minimal radius of the ball $B$ in the sense of* (4.8)*, there exists a point $y \in B$ with $\rho_U(y, \widetilde{U} \setminus U) \geq \frac{c_1}{8} R$ and $\rho_U(y,x) = R/4$. Here $c_1$ is the constant appearing in* (3.6)*.*

**Proof.** Take some point $z \in B_U(x,R) \setminus B_U(x,R/2)$, which is a nonempty set because by convention the radius $R$ of the ball $B$ is minimal in the sense (4.8). Let $\gamma$ be a path from $x$ to $z$ given by the uniform condition (3.6). Take some point $y \in B$ on the path $\gamma$ such that $\rho_U(x,y) = R/4$. Such a point exists because the distance function $\rho_U(x,\cdot)$ is continuous and $\rho_U(x,z) \geq R/2$. By the uniform condition (3.6) and by triangle inequality, we have

$$\begin{aligned}\rho_U(y, \widetilde{U} \setminus U) &\geq c_1 \frac{\rho_U(x,y)\rho_U(y,z)}{\rho_U(x,z)} = c_1 R/4 \cdot \frac{\rho_U(y,z)}{\rho_U(x,z)} \geq c_1 R/4 \cdot \frac{\rho_U(x,z) - \rho_U(x,y)}{\rho_U(x,z)} \\ &\geq \frac{c_1}{4} R \left(1 - \frac{R/4}{R/2}\right) = \frac{c_1}{8} R\end{aligned}$$

$\square$

**Definition 4.1.6** *Let $B_0$ be a ball from the Whitney cover $\Im(B)$ with the property that the point $y$ constructed in Lemma 4.1.5 is inside $3B_0$. We call the ball $B_0$* ***the central ball*** *in $B$.*

Note that by construction, we have

$$\rho_U(B_0, \widetilde{U} \setminus U) \geq \frac{c_1}{16} R \tag{4.13}$$

We proceed to estimate the left-hand side of (4.7) for any function $f \in \mathcal{F}(kB)$, where the constant $k$ will be chosen later. Choose $\xi = f_{4B_0} = \frac{1}{\nu(4B_0)} \int_{4B_0} f d\nu$ and estimate

$$\begin{aligned}\inf_{\xi} \int_B |f - \xi|^2 d\nu &\leq \sum_{D \in \Im(B)} \int_{3D} |f - f_{4B_0}|^2 d\nu \\ &\leq 2 \sum_{D \in \Im(B)} \left[ \int_{4D} |f_{4D} - f_{4B_0}|^2 d\nu + \int_{4D} |f - f_{4D}|^2 d\nu \right]\end{aligned} \tag{4.14}$$

Estimating the second term is easy since for every $D \in \Im(B)$ we have $4D \subset 2B$, the ball $4D$ is far from the boundary compared to its radius in the sense of (4.9) and thus the Poincaré inequality (4.6) is satisfied on $4D$ for any $f \in \mathcal{F}(3B)$. Thus there exists a universal constant $C$ such that

$$\sum_{D \in \Im(B)} \int_{4D} |f - f_{4D}|^2 d\nu \leq CR^2 \sum_{D \in \Im(B)} \int_{4D} d\Gamma(f,f) \leq CR^2 \int_{2B} \left( \sum_{D \in \Im(B)} \chi_{4D} \right) d\Gamma(f,f) \tag{4.15}$$

The sum of characteristic functions appearing in (4.15) is bounded from above by a universal constant by Lemma 4.1.3.

To estimate the first term of (4.14), we will use the following Lemma which estimates the difference of averages of a function on close Whitney balls via its energy integral.

**Lemma 4.1.7** *Let $\varepsilon \in (0, \frac{1}{100})$ and let $\Re$ be a strict $\varepsilon$-Whitney cover of an open set $U$ in $(X, \rho)$. There exists a constant $a_2$ such that for two neighboring Whitney balls, i.e. any balls $D, E \in \Re$ with $3D \cap 3E \neq \emptyset$, and for any $f \in \mathcal{F}(16D) \cap \mathcal{F}(16E)$ we have*

$$|f_{4D} - f_{4E}| \leq a_2 r(D) \left( \frac{1}{\nu(D)} \int_{16D} d\Gamma(f,f) \right)^{\frac{1}{2}}$$

**Proof.** Using the Poincaré inequality (4.6) we estimate

$$\begin{aligned}
\nu(4D \cap 4E)|f_{4D} - f_{4E}|^2 &= \int_{4D\cap 4E} |f_{4D} - f_{4E}|^2 d\nu \\
&\leq 2 \int_{4D\cap 4E} |f - f_{4D}|^2 d\nu + 2 \int_{4D\cap 4E} |f - f_{4E}|^2 d\nu \\
&\leq 2 \int_{4D} |f - f_{4D}|^2 d\nu + 2 \int_{4E} |f - f_{4E}|^2 d\nu \\
&\leq 2A \cdot r(D)^2 \int_{4D} d\Gamma(f,f) + 2A \cdot r(E)^2 \int_{4E} d\Gamma(f,f)
\end{aligned}$$

As Whitney balls $D, E$ are neighboring, their radii must be approximately equal, up to the multiple of $4/3$, by the Whitney condition (4.9) and the triangle inequality. Therefore the four multiple of $E$ is contained inside the 16 multiple of $D$.

Furthermore, by the doubling property (2.12) for the measure $\nu$ on $\widetilde{U}$, we have

$$\nu(4D \cap 4E) \geq C\nu(D),$$

up to a universal multiplication constant $C$ depending on the doubling constant of $\nu$ appearing in (2.12). The desired inequality follows. $\square$

Next in order to estimate the first term of (4.14), we need the following construction. Recall that for each ball $D \in \Im(B)$, the uniform condition on the domain $U$ produces a path $\gamma$ of length at most $c_0\rho_U(B_0, D)$, connecting the closest points of $B_0$ and $D$. Let's choose a string of distinct balls $\mathbb{S}(D) = \{B_0^D, B_1^D, \ldots B_l^D\}$ of length $l = l(D)$ with the following properties:

1. $\forall j,\ B_j^D \in \Re$
2. $B_0 = B_0^D$ and $B_l^D = D$
3. $3B_j^D \cap 3B_{j-1}^D \neq \emptyset$
4. $3B_j^D \cap \gamma \neq \emptyset$

In other words, connect the two balls $B_0$ and $D$ by Whitney balls along the path given by the uniform condition.

**Lemma 4.1.8** *Let $\Re$ be an $\varepsilon$-Whitney cover of an inner uniform domain $U$ (see 3.6) in $(X, \rho)$. There is a constant $a_3$ such that for any inner geodesic ball $B = B_U(x, R)$ satisfying (4.11) and for any ball $D \in \Im(B)$, the sequence of Whitney balls $\mathbb{S}(D)$ constructed above satisfies for any index $j$*

*(i)* $\rho_U(B_j^D, B_0) \leq c_0\rho_U(B_0, D) < 2c_0R$, *so that* $B_j^D \subset 4c_0B$

*(ii)* $\rho_U(B_j^D, D) \leq \frac{a_3}{2}r(B_j^D)$, *so that* $D \subset a_3B_j^D$.

**Proof.** The first inequality *(i)* follows from the length estimate on the path given by the uniform condition (3.6) and from (4.12). To show *(ii)*, we use the Whitney condition (4.9), the uniform condition and the triangle inequality to obtain

$$\frac{2}{\varepsilon} r(B_j^D) \geq \rho_U(B_j^D, \widetilde{U} \setminus U) \geq c_1 \frac{\rho_U(B_j^D, B_0)\rho_U(B_j^D, D)}{\rho_U(B_0, D)} \tag{4.16}$$

$$\frac{2}{\varepsilon} r(B_j^D) \geq \rho_U(B_j^D, \widetilde{U} \setminus U) \geq \rho_U(B_0, \widetilde{U} \setminus U) - \rho_U(B_j^D, B_0) \tag{4.17}$$

One of these inequalities will give the desired result in each of the two cases below.

(a) Assume $2\rho_U(B_j^D, B_0) > \rho_U(B_0, \widetilde{U} \setminus U)$. Because $\rho_U(B_0, D) \leq 2R$, (4.13) and (4.16) give

$$r(B_j^D) \geq \frac{\varepsilon}{2} c_1 \frac{c_1 R}{32} \cdot \frac{\rho_U(B_j^D, D)}{2R} = C\rho_U(B_j^D, D)$$

for some constant $C = \varepsilon c_1^2/128$.

(b) Assume instead that $2\rho_U(B_j^D, B_0) \leq \rho_U(B_0, \widetilde{U} \setminus U)$, then (4.17) allows us to estimate $r(B_j^D)$ from below by

$$r(B_j^D) \geq \frac{\varepsilon}{2} \cdot \frac{1}{2} \rho_U(B_0, \widetilde{U} \setminus U) \geq \frac{\varepsilon c_1}{64} R \geq \frac{\varepsilon c_1}{128 c_0} \rho_U(B_j^D, D).$$

Here, to obtain the last inequality, we have used (3.6) to see that

$$\rho_U(B_j^D, D) \leq L(\gamma) \leq c_0 \rho_U(B_0, D) \leq 2c_0 R$$

This gives $\rho_U(B_j^D, D) \leq \frac{a_3}{2} r(B_j^D)$ with $a_3 = \min\left(\frac{256}{\varepsilon c_1^2}, \frac{256 c_0}{\varepsilon c_1}\right)$ as desired. $\square$

**Definition 4.1.9** *Given $U$, $\Re$, $B = B_U(x, R)$ and $\Im(B)$ as in (4.10), set*

$$\Im_1(B) = \{B_U(x, r) \in \Re |\ \ B_U(x, r) \in \mathbb{S}(D) \ for\ some\ D \in \Im(B)\}.$$

Note that the first part of Lemma 4.1.8 can be rephrased as

$$\bigcup_{A \in \Im_1(B)} A \subset 4c_0 B \tag{4.18}$$

Returning now to estimating the first term of (4.14) for $f \in \mathcal{F}(kB)$, observe that the doubling property (2.12) gives

$$\sum_{D \in \Im(B)} \int_{4D} |f_{4D} - f_{4B_0}|^2 d\nu \le c_2^2 \int_U \sum_{D \in \Im(B)} |f_{4D} - f_{4B_0}|^2 \chi_D \, d\nu \tag{4.19}$$

Using, for each $D$, the string $\mathbb{S}(D) = \{B_j^D\}_{j=1}^{l(D)}$, write

$$|f_{4D} - f_{4B_0}| \le \sum_{j=1}^{l(D)} |f_{4B_j^D} - f_{4B_{j-1}^D}| \le \sum_{j=1}^{l(D)} a_2 r(B_j^D) \left( \frac{1}{\nu(B_j^D)} \int_{16B_j^D} d\Gamma(f,f) \right)^{\frac{1}{2}}$$

by Lemma 4.1.7. Using Lemma 4.1.8, we see that $\chi_D = \chi_D \chi_{a_3 B_j^D}$, and thus

$$\begin{aligned} |f_{4D} - f_{4B_0}|\chi_D &\le \sum_{i=1}^{l(D)} a_2 r(B_j^D) \left( \frac{1}{\nu(B_j^D)} \int_{16B_j^D} d\Gamma(f,f) \right)^{\frac{1}{2}} \cdot \chi_D \cdot \chi_{a_3 \cdot B_j^D} \\ &\le \sum_{A \in \Im_1} a_2 r(A) \left( \frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right)^{\frac{1}{2}} \cdot \chi_D \cdot \chi_{a_3 \cdot A} \end{aligned} \tag{4.20}$$

where we have extended the summation from the collection $\mathbb{S}(D)$ to the collection $\Im_1$.

We will need the following result which is a special case of [51, Lemma 5.3.12].

**Lemma 4.1.10 ([51], Lemma 5.3.12)** *Assume that the doubling condition (2.12) is satisfied for the balls in $(U, \rho_U)$ with respect to the measure $\nu$. Fix $K \ge 1$. There exist a constant $C = C(K)$ such that for any (possibly infinite) sequence of balls $B_i = B_U(x_i, r_i)$ in $(U, \rho_U)$ and any sequence of non-negative numbers $b_i$, we have*

$$\int_U \left( \sum_i b_i \chi_{KB_i} \right)^2 d\nu \le C \int_U \left( \sum_i b_i \chi_{B_i} \right)^2 d\nu. \tag{4.21}$$

To complete the estimation of the first term in (4.14), we continue the estimate (4.19) using the inequality (4.20) and Lemma 4.1.10 with $K = a_3$ to get

$$\begin{aligned}
\sum_{D\in\Im(B)} \int_{4D} |f_{4D} - f_{4B_0}|^2 d\nu &\leq c_2^2 \int \sum_{D\in\Im(B)} |f_{4D} - f_{4B_0}|^2 \chi_D d\nu \\
&\leq c_2^2 \int \sum_{D\in\Im(B)} \left( \sum_{A\in\Im_1(B)} a_2 r(A) \left( \frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right)^{\frac{1}{2}} \cdot \chi_D \cdot \chi_{a_3 A} \right)^2 d\nu \\
&= c_2^2 \int \left( \sum_{D\in\Im(B)} \chi_D \right) \cdot \left( \sum_{A\in\Im_1(B)} a_2 r(A) \left( \frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right)^{\frac{1}{2}} \chi_{a_3 A} \right)^2 d\nu \\
&\leq c_2^2 a_2^2 C(a_3) \int \left( \sum_{A\in\Im_1(B)} r(A) \left( \frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right)^{\frac{1}{2}} \chi_A \right)^2 d\nu \\
&= c_2^2 a_2^2 C(a_3) \sum_{A\in\Im_1(B)} r(A)^2 \left( \frac{1}{\nu(A)} \int_{16A} d\Gamma(f,f) \right) \nu(A) \\
&\leq c_2^2 a_2^2 C(a_3) \cdot R^2 \int \left( \sum_{A\in\Im_1(B)} \chi_{16A} \right) d\Gamma(f,f) \leq a_1 c_2^2 a_2^2 C(a_3) \cdot R^2 \int_{64c_0 B} d\Gamma(f,f)
\end{aligned}$$

We used that the balls $D$ are disjoint to see that $\left(\sum_{D\in\Im(B)} \chi_D\right) \leq 1$, and, for the last inequality, Lemma 4.1.3 and the fact that if $A \in \Im_1(B)$ then $A \subset 4c_0 B$ by (4.18).

This completes the analysis of the first term in (4.14) and together with (4.15) establishes the weak Poincaré inequality (4.7) with $k = 64c_0$. $\square$

This completes the proof of Proposition 4.1.1, in view of the outline presented after the statement of Proposition 4.1.1. To complete the proof of Theorem 4.0.5 it remains to establish some properties of the Neumann Dirichlet form and the associated metric, which we complete in Chapter 4.2. Before we focus on those, we will explore how Proposition 4.1.1 establishes the family of Poincaré inequalities for a symmetric form obtained from the original form by a simple change of measure.

### 4.1.2 Neumann type Dirichlet forms obtained by the change of measure

Assume that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X, \mu)$. Let $U \subset X$ be an open set and let $v \in L^\infty_{loc}(U, \mu)$ be a locally uniformly positive and locally bounded measurable function on $U$. Set

$$\begin{aligned} \mathcal{E}_U^{N,v}(f,f) &= \int_U v d\Gamma(f,f) = \int_U \Upsilon(f,f) v d\mu \\ \mathcal{D}(\mathcal{E}_U^{N,v}) &= \mathcal{F}^v(U) = \left\{ f \in \mathcal{F}_{loc}(U) \cap L^2(U, vd\mu) : \int_U \Upsilon(f,f) v d\mu < \infty \right\} \end{aligned} \tag{4.22}$$

to be a symmetric form on $L^2(U, vd\mu)$.

**Remark.** If we take the function $v$ to be constant one, the form defined in (4.22) becomes $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$.

Notice that because of the special structure of this form, the normal contractions operate on $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$. The form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is symmetric and densely defined in $L^2(U, vd\mu)$ since compactly supported in $U$ functions which are Lipschitz with respect to the metric $\rho$ are in $\mathcal{D}(\mathcal{E}_U^{N,v})$. It is also closed by the proof of Proposition 2.4.9. So we see that the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is a Dirichlet form. It is also strongly local because the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is. So each of the forms $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is associated with the nonnegative self-adjoint operator $L_U^{N,v}$ and a self-adjoint semigroup $P_{U,t}^{N,v}$ on $L^2(U, vd\mu)$. It is straightforward to see that the energy measure associated with the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ on $L^2(U, vd\mu)$ by (2.2) is simply

$$d\Gamma^v(f,g) = v d\Gamma(f,g) = \Upsilon(f,g) v d\mu.$$

and so the Radon-Nikodym derivative of $d\Gamma^v$ with respect to the reference measure $vd\mu$ is

$$\Upsilon^v(f,g) = \frac{d\Gamma^v(f,g)}{vd\mu} = \Upsilon(f,g) \tag{4.23}$$

The following straightforward corollary of Proposition 4.1.1 is important to proving the heat kernel estimates for the heat semigroup associated with $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$.

**Corollary 4.1.11** *Let $X$ be a locally compact separable metric space. Let $\mu$ be a positive Radon measure on $X$ with full support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local Dirichlet form on $L^2(X,\mu)$. Let $\rho = \rho_{\mathcal{E}}$ and assume that the conditions* (A1-A4) *of* Chapter 2.1.2 *are satisfied. Let $U$ be an inner uniform domain in $(X,\rho)$. Let $v \in L^\infty_{loc}(U,\mu)$ be a locally bounded measurable function on $U$. Assume that the measure $vd\mu$ on $U$ satisfies the doubling condition (2.12). Assume also that there exist positive constants $C$ and $N$ such that the function $v$ satisfies the Harnack inequality*

$$\sup_B v \le C \inf_B v \tag{4.24}$$

*on any ball $B = B_U(x,R)$ with $\rho_U(B, \widetilde{U} \setminus U) \ge NR$. Then for any geodesic ball $B = B_U(x,R)$ in $(\widetilde{U}, \rho_U)$, we have*

$$\forall f \in \mathcal{F}^v(B),\ \inf_{\xi\in\mathbb{R}} \int_B |f-\xi|^2 v d\mu \le CR^2 \int_B v d\Gamma(f,f), \tag{4.25}$$

*i.e. the family of $L^2$ Poincaré inequalities for the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ with reference measure $vd\mu$ holds on $\widetilde{U}$.*

**Proof.** The idea is simply to apply Proposition 4.1.1 for the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ with replacing $X$ by $\widetilde{U}$, $\mu$ by $vd\mu$ and $\rho$ by $\rho_U$. Notice that the condition (4.4) translates to

$$\forall f \in \mathcal{F}^v(U),\ \inf_{\xi\in\mathbb{R}} \int_B |f-\xi|^2 v d\mu \le AR^2 \int_B v d\Gamma(f,f)$$

and is satisfied for any ball $B_U(x,R) = B(x,R)$ with $\rho_U(B, \widetilde{U} \setminus U) \ge NR$ by the assumption (A4) of Chapter 2.1.2 together with the assumption (4.24). $\square$

**Remark.** An example of a function $v$ satisfying the conditions of Corollary 4.1.11 is any positive power of distance to the boundary,

$$v(x) = \delta_U(x)^\alpha, \quad \text{where} \quad \delta_U(x) = \rho_U(x, \widetilde{U} \setminus U)$$

The heat kernel estimates for the forms $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ will be used for obtaining the heat kernel estimates for the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ in Chapter 5.

## 4.2 Properties of Neumann type Dirichlet forms

We first aim to prove that the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is regular on some superset of $U$. Recall that the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a special case of the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ when the function $v$ is taken to be constant one. Thus the following proposition is interesting.

**Proposition 4.2.1** *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies conditions* (A1-A2) *of* Chapter 2.1.2 *and admits a carré du champ operator $\Upsilon$, as in* Definition 2.1.10. *Let $U \subset X$ be an open subspace of $X$ and let $\epsilon$ be any positive number. Let $v$ be a locally bounded measurable function on $\widetilde{U}$ which is locally uniformly positive on $U$. Assume that the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ on $\widetilde{U}$ satisfies the following family of Poincaré inequalities with respect to the metric $\rho_U$*

$$\forall x \in \widetilde{U}, 0 < R < \epsilon, \quad \inf_{\xi} \int_{B_U(x,R)} (u-\xi)^2 v d\mu \leq C' R^2 \int_{B_U(x,R)} \Upsilon(u,u) v d\mu. \quad (4.26)$$

*for any $f \in \mathcal{F}^v(B_U(x,R))$. Assume that the measure $vd\mu|_U$ satisfies the following doubling condition on $\widetilde{U}$ with respect to the metric $\rho_U$,*

$$\forall x \in \widetilde{U}, 0 < R < \epsilon, \quad \int_{B_U(x,2R)} v d\mu \leq C \int_{B_U(x,R)} v d\mu. \quad (4.27)$$

*Then the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ on $L^2(\widetilde{U}, vd\mu|_U)$ is regular with core* $\mathrm{Lip}_c(\widetilde{U})$.

In order to prove Proposition 4.2.1 we will need the following description of Lipschitz functions on $X$, given in [44, Corollary 3.6].

**Proposition 4.2.2** *Assume that the Dirichlet form* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *satisfies conditions* (A1-A2) *of* Chapter 2.1.2 *and admits a carré du champ operator* $\Upsilon$*, as in* Definition 2.1.10. *Then every Lipschitz function on* $(X, \rho_{\mathcal{E}})$ *with Lipschitz constant* $k$ *is in* $\mathcal{F}_{loc}(X)$ *and satisfies*

$$k = \sup_X \sqrt{\Upsilon(f, f)}$$

**Corollary 4.2.3** *Let* $U$ *be an open subset of* $X$*. In the setting of Proposition 4.2.2, every function on* $U$ *which is Lipschitz with respect to* $\rho_U$ *with Lipschitz constant* $k$ *is in* $\mathcal{F}_{loc}(U)$ *and satisfies*

$$k \geq \sup_U \sqrt{\Upsilon(f, f)} \tag{4.28}$$

**Proof.** For any open ball $B = B(x, r)$ in $(X, \rho_{\mathcal{E}})$ which happens to be in $U$ and such that $\rho(B, \partial U) \geq 2r$, the restriction $f|_B$ is Lipschitz with respect to $\rho_U$ and thus with respect to $\rho$ since $\rho = \rho_X$ is a length metric (see [61]). Therefore we can extend $f|_U$ to some compactly supported Lipschitz function $f'$ on $(X, \rho_{\mathcal{E}})$ with the same Lipschitz constant. We have $f \equiv f'$ in $B$. Using Proposition 4.2.2 and the local property (2.4) of $d\Gamma$ we see that $f' \in \mathcal{D}(\mathcal{E})$, $f \in \mathcal{F}(B)$ and the Lipschitz constant $k$ of $f'$ satisfies

$$k = \sup_X \sqrt{\Upsilon(f', f')} \geq \sup_B \sqrt{\Upsilon(f, f)}.$$

This holds for any open ball $B = B(x, r)$ in $(X, \rho_{\mathcal{E}})$ which is in $U$ such that $\rho(B, \partial U) \geq 2r$, therefore $f \in \mathcal{F}_{loc}(U)$. Also this shows that $f$ is locally Lipschitz in $(U, \rho)$. Since $\rho_U$ is the inner geodesic metric in $U$ based on $\rho$, the function $f$ is Lipschitz in $(U, \rho_U)$ with Lipschitz constant $k$ satisfying the desired estimate (4.28). $\square$

**Proof of Proposition 4.2.1.** The space $\mathrm{Lip}_c(\widetilde{U})$ is dense in $C_0(\widetilde{U})$ with supremum norm by [42, Theorem 6.8]. To see that $\mathrm{Lip}_c(\widetilde{U})$ is dense in $\mathcal{D}(\mathcal{E}_U^{N,v})$, we follow [39, page 205], [40, page 13] and [38, Lemma 10]. Let

$$d\nu = v d\mu$$

denote the reference measure on $U$. Let $g$ be any function in $\mathcal{D}(\mathcal{E}_U^{N,v}) = \mathcal{F}^v(U)$. We aim to prove that $g$ can be approximated by functions in $\mathrm{Lip}_c(\widetilde{U})$. Because we can approximate the function g in $\mathcal{D}(\mathcal{E}_U^{N,v})$ by bounded functions $g_n = \min(\max(g, -n), n)$, without loss of generality we can assume that the function $g$ is bounded. Set

$$\phi_R(x) = R^{-1} \min\{(2R - \rho_U(x, \widetilde{U} \setminus U))_+, (2R - \rho_U(x, x_0))_+, R\}$$

where $x_0$ is a fixed point in $U$ and $(t)_+ = \min\{0, t\}$. Since $v$ is locally finite on $\widetilde{U}$, these compactly supported 'cut-off' functions $\phi_R$ are in $\mathcal{F}^v(U) \cap L^\infty(U, vd\mu)$. Since $g \in \mathcal{F}^v(U) \cap L^\infty(U, vd\mu)$, we have $g\phi_R \in \mathcal{F}^v(U)$ by the energy estimate of Lemma 2.2.1. It is easy to see that $\phi_R g$ tends to $g$ in $\mathcal{E}_U^{N,v}$-norm and in $L^2(\widetilde{U}, d\nu)$ when $R$ tends to infinity. Thus, in the rest of the proof, we assume that $g$ is a function in $\mathcal{D}(\mathcal{E}_U^{N,v})$ with compact support in $\widetilde{U}$.

For any $r > 0$, set

$$g_r(y) = \frac{1}{\nu\left(B_U(y, r)\right)} \int_{B_U(y,r)} g d\nu$$

where $B_U(y, r)$ denotes the ball in $(\widetilde{U}, \rho_U)$ centered at $y$. Fix $r \in (0, \epsilon)$ and set $r_i = 2^{-i} r$, $B_i = B_U(x, r_i)$. We say that $x \in X$ is a Lebesgue point of $g$ if

$$\lim_{i \to \infty} g_{r_i}(x) = g(x)$$

It is known that for every $g \in L^2(X, \nu)$, the points in $X$ that are not Lebesgue for a function $g$ form a set of $\nu$-measure zero.

For every Lebesgue point $x \in U$, using the Jensen's inequality we can write

$$\begin{aligned}
|g(x) - g_r(x)| &\leq \sum_{i=0}^{\infty} |g_{r_i} - g_{r_{i+1}}| \leq \sum_{i=0}^{\infty} \left( \frac{1}{\nu(B_{i+1})} \int_{B_i} |g(y) - g_{r_i}(x)|^2 d\nu(y) \right)^{\frac{1}{2}} \\
&\leq \sum_{i=0}^{\infty} r_i \left( \frac{CC'}{\nu(B_i)} \int_{B_i} \Upsilon(g,g) d\nu \right)^{\frac{1}{2}} \leq \sqrt{CC'} \sum_{i=0}^{\infty} r_i \sqrt{\mathcal{M}(\Upsilon(g,g))(x)} \\
&= r\sqrt{CC'}\sqrt{\mathcal{M}\Upsilon(g,g)(x)}. \qquad (4.29)
\end{aligned}$$

by Poincaré inequality and the doubling property of the measure $\nu$ on $(\widetilde{U}, \rho_U)$. Here $\mathcal{M}(f)$ is the $2\epsilon$-maximal function of $f$, i.e.

$$\mathcal{M}f(x) = \mathcal{M}_{2\epsilon} f(x) = \sup_{0<s<2\epsilon} \frac{1}{\nu(B_U(x,s))} \int_{B_U(x,s)} f d\nu.$$

Similarly for any Lebesgue points $x, y \in U$ with $\rho(x,y) \leq r$, the doubling property of $\nu|_U$ and the Poincaré inequality (4.26) yield

$$\begin{aligned}
|g_r(x) - g_r(y)| &\leq |g_r(x) - g_{2r}(x)| + |g_r(y) - g_{2r}(x)| \\
&\leq 2 \left( \frac{C}{\nu(B_U(x,2r))} \int_{B_U(x,2r)} |g(z) - g_{2r}(x)|^2 d\nu(z) \right)^{\frac{1}{2}} \\
&\leq 2r \left( \frac{CC'}{\nu(B_U(x,2r))} \int_{B_U(x,2r)} \Upsilon_U(g,g) d\nu \right)^{\frac{1}{2}} \\
&\leq (2\sqrt{CC'}r)\sqrt{\mathcal{M}\Upsilon(g,g)(x)}. \qquad (4.30)
\end{aligned}$$

Combining (4.29) and (4.30) we see that for any Lebesgue points $x, y \in U$ with $\rho_U(x,y) \leq r$, there exists another constant $C$ such that

$$|g(x) - g(y)| \leq Cr \left[ \sqrt{\mathcal{M}\Upsilon(g,g)(x)} + \sqrt{\mathcal{M}\Upsilon(g,g)(y)} \right]. \qquad (4.31)$$

For any $\lambda > 0$, set

$$\begin{aligned}
E_\lambda &= \left\{ x \in U : \ x \text{ is a Lebesgue point of } g,\ g(x)^2 \leq \lambda^2 \text{ and } \mathcal{M}\Upsilon(g,g)(x) \leq \lambda^2 \right\} \\
F_\lambda &= U \setminus E_\lambda
\end{aligned}$$

Note that $F_\lambda$ is precompact in $\widetilde{U}$ for $\lambda$ large enough, say $\lambda \geq \lambda_0$, because $g$ has compact support in $\widetilde{U}$. Furthermore, the restriction $g|_{E_\lambda}$ of $g$ to $E_\lambda$ is Lipschitz with constant $2C\lambda$ on $E_\lambda$ by (4.31). Let $f_\lambda$ be some Lipschitz extension of $g$ from $E_\lambda$ to $\widetilde{U}$ with the same Lipschitz constant (see, e.g., [42, Theorem 6.2]). Let $\lambda \geq \lambda_0$. For such $\lambda$, $F_\lambda$ is precompact in $\widetilde{U}$. As $f_\lambda = g$ in $E_\lambda$, it follows that $f_\lambda$ is a bounded function in $\widetilde{U}$ with compact support in $\widetilde{U}$ and with $\|f_\lambda\|_\infty \leq \lambda(1+2CR_0)$ where $R_0$ is the diameter of $F_{\lambda_0}$. Moreover, $f_\lambda$ has compact support. Since $g \in \mathcal{D}(\mathcal{E}_U^{N,v})$, we have $g \in L^2(U,\nu)$, and $\int_U \Upsilon(g,g)d\nu < \infty$. It is known that the maximal function $\mathcal{M}\Upsilon(g,g)$ is in weak $L^1(U,\nu)$, i.e.

$$N\nu\left\{x \in U : \mathcal{M}\Upsilon(g,g) > N\right\} \to 0$$

as $N \to \infty$, see [48, Theorem 2.19]. Also, we have

$$\int_{F_\lambda} |g|^2 d\nu \to 0, \ \ as\ \lambda \to \infty.$$

Since non-Lebesgue points of $g$ form a set of measure zero,

$$\begin{aligned} \lambda^2\nu\{F_\lambda\} &\leq \lambda^2\nu\{x \in U : \mathcal{M}\Upsilon(g,g) > \lambda^2\} + \lambda^2\nu\{x \in U : g(x)^2 > \lambda^2\} \\ &\leq \lambda^2\nu\{x \in U : \mathcal{M}\Upsilon(g,g) > \lambda^2\} + \int_{\{g^2>\lambda^2\}} |g|^2 d\nu \to 0 \qquad (4.32) \end{aligned}$$

as $\lambda \to \infty$. The function $f_\lambda$ is bounded by $\lambda(1+2CR_0)$ and Lipschitz with respect to $\rho_U$ with Lipschitz constant $2C\lambda$. Therefore $\Upsilon(f,f) \leq 4C^2\lambda^2$ by Corollary 4.2.3. Inequality (4.32) gives

$$\int_{F_\lambda} \left(|f_\lambda|^2 + \Upsilon(f_\lambda, f_\lambda)\right) d\nu \leq \lambda^2 \left((1+2CR_0)^2 + 4C^2\right) \nu\{F_\lambda\} \to 0$$

as $\lambda \to \infty$. Now, since $f_\lambda = g$ on $E_\lambda$, we have

$$\int_U \left(|g - f_\lambda|^2 + \Upsilon(g - f_\lambda, g - f_\lambda)\right) d\nu \leq 2 \int_{F_\lambda} \left(|g|^2 + |f_\lambda|^2 + \Upsilon(g,g) + \Upsilon(f_\lambda, f_\lambda)\right) d\nu$$

and the right-hand side tends to 0 as $\lambda$ tends to infinity. Thus $f_\lambda$ tends to $g$ in Hilbert space $\mathcal{D}(\mathcal{E}_U^{N,v})$, as desired. $\square$

**Corollary 4.2.4** *In the context of Proposition 4.2.1, the form* $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ *is a strongly local regular Dirichlet form on* $L^2(\widetilde{U}, v d\mu)$.

**Lemma 4.2.5** *Let* $v \in L^\infty_{loc}(U, \mu)$ *be a locally uniformly positive and locally bounded measurable function on* $U$*. In the context of Proposition 4.2.1, the metric* $\rho_{\mathcal{E}_U^{N,v}}$ *on* $\widetilde{U}$ *coincides with the geodesic metric* $\rho_U$.

**Proof.** For any $x, y \in \widetilde{U}$ we have

$$\rho_{\mathcal{E}_U^{N,v}}(x,y) = \sup\left\{u(x) - u(y) : u \in \mathcal{D}(\mathcal{E}_U^{N,v}) \cap C_0(\widetilde{U}), \Upsilon(u,u) \leq 1 \text{ a.e. on } U\right\}. \tag{4.33}$$

To show $\rho_{\mathcal{E}_U^{N,v}}(x,y) \geq \rho_U(x,y)$ it suffices to notice that the function $\max(\rho_U(x,y) - \rho_U(x,\cdot), 0)$ is a compactly supported Lipschitz function on $(\widetilde{U}, \rho_U)$ with $\Upsilon(u,u) \leq 1$ a.e. on $U$.

To show the opposite inequality, we first focus on the case when $x, y \in U$. Let $\gamma : [0,1] \to U$ be any continuous curve without self-intersections connecting $x$ and $y$. In view of Lemma 2.1.13,

$$\begin{aligned} L(\gamma) = \sup\{u(\gamma(1)) - u(\gamma(0)) : Y &\text{ is an open neighborhood of } \gamma([0,1]) \subset X, \\ &u \in \mathcal{F}_{loc}(Y) \cap C(Y), \Upsilon(u,u) \leq 1 \text{ a.e. on } Y\} \end{aligned}$$

which is greater than $\rho_{\mathcal{E}_U^{N,v}}(x,y)$ because we can choose $Y$ to be $U$. Therefore the distance $\rho_{\mathcal{E}_U^{N,v}}(x,y)$ can be estimated by

$$\rho_{\mathcal{E}_U^{N,v}}(x,y) \leq \inf_{\gamma:[0,1]\to U} L(\gamma) = \rho_U(x,y) \tag{4.34}$$

where the infimum above is taken over all continuous curves which are not self-intersecting.

To show $\rho_{\mathcal{E}_U^{N,v}}(x,y) \leq \rho_U(x,y)$ in case at least one of the points $x, y$ belongs to $\widetilde{U} \setminus U$, choose a sequence $\{x_i\}_{i=1}^\infty$ of points in $U$ approximating $x \in \widetilde{U}$ and a sequence $\{y_i\}_{i=1}^{\infty}$ of points in $U$ approximating $y \in \widetilde{U}$. For any continuous function $u$ on $\widetilde{U}$ satisfying conditions (4.33), we estimate

$$\begin{aligned}|u(x) \quad - \quad u(y)| &\leq \inf_i \left[|u(x_i) - u(y_i)| + |u(x) - u(x_i)| + |u(y) - u(y_i)|\right] \\ &\leq \liminf_{i\to\infty} |u(x_i) - u(y_i)| \leq \liminf_{i\to\infty} \rho_{\mathcal{E}_U^{N,v}}(x_i, y_i) = \liminf_{i\to\infty} \rho_U(x_i, y_i) = \rho_U(x, y)\end{aligned}$$

□

**Corollary 4.2.6** *In the context of Proposition 4.2.1, the metric $\rho_{\mathcal{E}_U^{N,v}}$ is everywhere finite and the topology given by this metric coincides with the original topology on $\widetilde{U}$, i.e. the assumptions (A1) and (A2) of Chapter 2.1.2 are satisfied for the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$.*

Since all the results used to prove Theorem 4.0.5 were extended to be true for the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ under some conditions for the function $v$, in fact we have shown a stronger result.

**Theorem 4.2.7** *In the assumptions of* Theorem 4.0.5, *let $v \in L^{\infty}_{loc}(U, \mu)$ be a locally uniformly positive and locally bounded measurable function on $U$. Assume that the measure $v d\mu$ on $U$ satisfies the doubling condition (2.12). Assume also that there exist positive constants $C$ and $N$ such that the function $v$ satisfies the Harnack inequality*

$$\sup_B v \leq C \inf_B v \tag{4.35}$$

*on any ball $B = B_U(x, R)$ with $\rho_U(B, \widetilde{U} \setminus U) \geq NR$. Then there exists a kernel $p_U^{N,v}(t, x, y)$ of the semigroup $P_{U,t}^{N,v}$ on $L^2(U, v d\mu)$ and it satisfies*

$$\frac{c_1 \exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{\sqrt{V_v(x, \sqrt{t}) V_v(y, \sqrt{t})}} \leq p_U^{N,v}(t, x, y) \leq \frac{c_3 \exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{\sqrt{V_v(x, \sqrt{t}) V_v(y, \sqrt{t})}} \tag{4.36}$$

*for all $x, y \in U$ and all $t > 0$. Here $V_v$ denotes the volume*

$$V_v(x, r) = \int_{B_U(x,r)} v d\mu$$

*For any positive integer $k$ there exists a constant $C(k)$ such that the $k$-th time derivative of the heat kernel $p_U^{N,v}(t,x,y)$ satisfies*

$$\left| \left(\frac{\partial}{\partial t}\right)^k p_U^{N,v}(t,x,y) \right| \leq \frac{C(k)\exp\left(-\frac{\rho_U(x,y)^2}{c_5 t}\right)}{t^k \sqrt{V_v(x,\sqrt{t})V_v(y,\sqrt{t})}} \tag{4.37}$$

*for all $x, y \in U$ and all $t > 0$. Also for arbitrary $z \in U$ every nonnegative (local) weak solution in $(0, 4r^2) \times B_U(z, 2r)$ of the equation*

$$\frac{\partial u}{\partial t} + L_U^{N,v} u = 0$$

*satisfies*

$$\sup_{(t,x)\in Q_-} u(t,x) \leq c_6 \inf_{(t,x)\in Q_+} u(t,x) \tag{4.38}$$

*where $Q_- = (r^2, 2r^2) \times B_U(z,r)$, $Q_+ = (3r^2, 4r^2) \times B_U(z,r)$. Here the constants $c_1, \ldots, c_6$ and $C(k)$ depend only on $k, N, C$ and the constants $c_0, c_1, c_2$ appearing in* (2.12) *and in* Definition 3.1.2. *In particular the form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is also a Harnack type regular Dirichlet form on $\widetilde{U}$, see* Definition 2.5.1.

**Proof.** Similarly to Proposition 2.4.9 we know that the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a closed strictly local symmetric form on $L^2(U,\mu)$. In view of Theorem 2.6.1, the following results combined imply this theorem

- The assumption that the measure $v d\mu$ satisfies the doubling condition (2.12) on $(\widetilde{U}, \rho_U)$.

- The family of Poincaré inequalities proved in Corollary 4.1.11.

- Proposition 4.2.1 shows that the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is regular on $L^2(\widetilde{U}, v d\mu)$.

- Lemma 4.2.5 shows that the metric $\rho_{\mathcal{E}_U^{N,v}}$ associated with the Dirichlet form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ coincides with $\rho_U$. In particular this implies conditions (A1) and (A2) of Chapter 2.1.2.

- Finally any time derivative of the heat kernel can be estimated from above as in (4.37) by [16, Theorem 4] which uses the estimates of the heat kernel to produce the estimates on its time derivative using the analytic nature of the heat kernel.

$\square$

This completes the analysis of the Neumann heat kernel, and the proof of Theorem 4.0.5 and Theorem 1.3.1. These results will be reused for the analysis of the Dirichlet heat kernel.

# Chapter 5

# Dirichlet heat kernel

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - a strictly local regular Dirichlet form on $X$ satisfying the conditions (A1-A4) of Chapter 2.1.2. Let $U$ be an open subset of $X$. Let $\rho = \rho_{\mathcal{E}}$ be the metric associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, and let $\rho_U$ denote the inner geodesic metric in $U$, see (3.2). Let $\widetilde{U}$ be the completion of $U$ with respect to $\rho_U$. Throughout this section let $B_U(x,r)$ denote the open ball in $(\widetilde{U}, \rho_U)$ centered at $x$, unless otherwise specified.

The goal of this section is to obtain the heat kernel estimates for the Dirichlet semigroup $P^D_{U,t}$ in case when $U \subset X$ is unbounded inner uniform. We will use the technique of $h$-transform to prove the following result that implies Theorem 1.3.3 when $X = \mathbb{R}^n$.

**Theorem 5.0.8** *Let $(X, \mu)$ be as above and assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X, \mu)$, as in* Definition 2.1.10. *Assume that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ satisfies the conditions* (A1-A4) *of* Chapter 2.1.2. *Let $U$ be an unbounded inner uniform domain in $X$, see* Definition 3.1.2. *Then there exists a nonnegative local (weak) solution $h \in \mathcal{F}^0_{loc}(U)$ of $Lh = 0$ in $U$ with weak Dirichlet boundary conditions on $\partial U$. For any such function $h$, the Dirichlet heat kernel $p^D_U(t,x,y)$ in $U$ satisfies*

$$\frac{c_1 h(x)h(y)\exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \le p^D_U(t,x,y) \le \frac{c_3 h(x)h(y)\exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{\sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \tag{5.1}$$

*for all $x, y \in U$ and all $t > 0$. Here*

$$V_{h^2}(x,r) = \int_{B_U(x,r)} h^2 d\mu \tag{5.2}$$

*is the volume of* $B_U(x,r)$ *with respect to the measure* $h^2 d\mu$*. For any positive integer* $k$ *there exists a constant* $C(k)$ *such that the* $k$*-th time derivative of the Dirichlet heat kernel in* $U$ *satisfies*

$$\left| \left(\frac{\partial}{\partial t}\right)^k p_U^D(t,x,y) \right| \leq \frac{C(k)h(x)h(y)}{t^k \sqrt{V_{h^2}(x,\sqrt{t})V_{h^2}(y,\sqrt{t})}} \exp\left(-\frac{\rho_U(x,y)^2}{c_5 t}\right) \tag{5.3}$$

*for all* $x,y \in U$ *and all* $t > 0$*. For any* $z \in \widetilde{U}$ *every nonnegative weak solution of the heat equation in* $(0,4r^2) \times B_U(z,2r)$ *with weak Dirichlet boundary conditions on* $\partial U$ *satisfies*

$$\sup_{(t,x)\in Q_-} \left(\frac{u(t,x)}{h(x)}\right) \leq c_6 \inf_{(t,x)\in Q_+} \left(\frac{u(t,x)}{h(x)}\right) \tag{5.4}$$

*where* $Q_- = (r^2, 2r^2) \times B_U(z,r)$*,* $Q_+ = (3r^2, 4r^2) \times B_U(z,r)$*. The constants* $c_1, \ldots, c_7$ *depend only on the constants* $c_0, c_1, c_2, c_3$ *appearing in* Definition 3.1.2*,* (2.12) *and* (2.13)*.*

**Remark 1.** In the context of Theorem 5.0.8 the volume function appearing in (5.2) can be estimated by

$$c_7^{-1} h^2(x_r)\mu(B_U(x,R)) \leq V_{h^2}(x,r) \leq c_7 h^2(x_r)\mu(B_U(x,R)),$$

where $x_r$ is any point with $\rho_U(x_r,x) = \frac{r}{4}$ and $\rho_U(x_r, \widetilde{U}\setminus U) \geq \frac{c_1}{8}r$. Such a point $x_r$ exists by Lemma 4.1.5. The constant $c_7$ depends only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12), (2.13) and in Definition 3.1.2.

**Remark 2.** In the context of Theorem 5.0.8 using the heat kernel estimates (5.1) we can see [35] that the quotient $\frac{h(x)}{h(x_{\sqrt{t}})}$ is comparable to

$$\frac{h(x)}{h(x_{\sqrt{t}})} \asymp P_{U,t}^D 1_U(x),$$

which is the probability that the process $X_t$ started at $x$ stays in $U$ for the duration of time $t$, or the total heat content after time $t$ of the diffusion system with original heat distribution given by a delta mass at $x$. We will denote this probability by $P(t,x)$. This implies in particular that the Dirichlet heat kernel can be estimated by

$$\frac{c_1 P(t,x)P(t,y)\exp\left(-\frac{\rho_U(x,y)^2}{c_2 t}\right)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}} \leq p_U^D(t,x,y) \leq \frac{c_3 P(t,x)P(t,y)\exp\left(-\frac{\rho_U(x,y)^2}{c_4 t}\right)}{\sqrt{V(x,\sqrt{t})V(y,\sqrt{t})}}.$$

Before embarking on the proof of Theorem 5.0.8 we focus on developing some tools.

## 5.1 Application of the axiomatic potential theory, the harmonic measure and the maximum principle

The setting of this section is that of a Harnack-type Dirichlet space $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ on a locally compact separable metric measure space $(X,\mu)$. The aim of this section is to provide the basis for the axiomatic potential theory as described in [14]. We will state a theorem that uses the method of Perron-Wiener-Brelot to construct the harmonic measure from the point of view of potential theory, rather than the theory of Markov processes, as in Chapter 2.7. Even though the two notions coincide, throughout the rest of this paper we will work only with the potential theoretic notion of the harmonic measure. We will need the following notation.

**Definition 5.1.1** *Let $\mathcal{H}$ denote the sheaf of* ***harmonic*** *functions on $X$, i.e. for any open set $V \subset X$ let $\mathcal{H}(V)$ denote the set of local weak solutions in $V$ of $Lu = 0$.*

Harmonic functions are Hölder continuous according to (2.5.2). The elliptic version of the Harnack inequality (2.38) is satisfied for every function in $\mathcal{H}(V)$. The sheaf $\mathcal{H}$ coincides with the sheaf of harmonic functions with respect to the process $X_t$ defined in Chapter 2.7, see [31]. The space $X$ together with a harmonic sheaf $\mathcal{H}$ is a Brelot space, and even a $\mathcal{P}$-Brelot space, see [10, Chapter 2.5 and Theorem

3.1.1]. Such spaces possess a rich potential theory and we refer to [10, 14] for the terminology and results, some of which we recall here.

**Definition 5.1.2** *An open relatively compact subset $V$ of $X$ is called regular if for every continuous function $\phi$ on $\partial V$ there exists a unique harmonic function $H_\phi^V$ on $V$ which is a continuous extension of $\phi$ to $\overline{V}$.*

We recall that by definition, Brelot spaces are such that the regular sets form a base of the topology for $X$. For any regular set $V$ and any point $x \in V$, the map

$$H_\cdot^V(x) : C(\partial V) \to \mathbb{R}, \quad \phi \to H_\phi^V(x)$$

is then associated with a measure which will be denoted by $\omega(x, \cdot, V)$, so that

$$H_\phi^V(x) = \int_{\partial V} \phi(y)\omega(x, dy, V).$$

The measure $\omega(x, \cdot, V)$ is called the **harmonic measure** of $V$. The probabilistic approach of Chapter 2.7 allowed us to construct such a measure (and the function $H_\phi^V$) via the process $X_t$.

Next we will extend the harmonic measure to more general sets. We will need the following definitions.

**Definition 5.1.3** *A lower semicontinuous function $f$ with values in $\mathbb{R} \cup \{+\infty\}$ is called **hyperharmonic** in $V$ if for any $x_0 \in V$, there exists a neighborhood $V' \subset V$ of $x$ such that for any regular set $V''$ with $\overline{V''} \subset V'$, we have*

$$\int_{\partial V''} f(y)\omega(x, dy, V'') \leq f(x), \text{ for any } x \in V''$$

*Let $\mathcal{U}$ denote the sheaf of hyperharmonic functions on $X$, so that $\mathcal{U}(V)$ denotes the convex cone of hyperharmonic functions on $V$.*

**Definition 5.1.4** *An upper semicontinuous function $f$ with values in $\mathbb{R}\cup\{-\infty\}$ is called **hypoharmonic** if $-f$ is hyperharmonic in $V$. Let $\mathcal{L}$ denote the sheaf of hypoharmonic functions on $X$, so that $\mathcal{L}(V)$ denote the convex cone of hypoharmonic functions on $V$.*

We will now introduce the harmonic measure for non-regular open sets. Let $V$ be any relatively compact open subset of $X$. Let $f$ be a function on $\partial V$. As in [14, p.18], we define the 'upper class' of hyperharmonic functions associated with $f$ by

$$\begin{aligned} \mathcal{U}_f^V \quad = \quad & \Big\{ u \in \mathcal{U}(V) : u \text{ is bounded from below on } V, \\ & \text{non-negative outside a compact subset of } V \\ & \text{and } \forall y \in \partial V, \quad \liminf_{V \ni x \to y} u(x) \geq f(y) \Big\} \end{aligned}$$

Similarly we define the 'lower class' of hypoharmonic functions by

$$\begin{aligned} \mathcal{L}_f^V \quad = \quad & \Big\{ u \in \mathcal{L}(V) : u \text{ is bounded from above on } V, \\ & \text{non-positive outside a compact subset of } V \\ & \text{and } \forall y \in \partial V, \quad \limsup_{V \ni x \to y} u(x) \leq f(y) \Big\} \end{aligned}$$

We define the upper and lower solutions of the Dirichlet problem in $V$ with boundary conditions $f$ by

$$\overline{H}_f^V(x) = \inf \Big\{ u(x) : u \in \mathcal{U}_f^V \Big\}, \quad \underline{H}_f^V(x) = \sup \Big\{ u(x) : u \in \mathcal{L}_f^V \Big\}, \tag{5.5}$$

If the class $\mathcal{U}_f^V$ (resp. $\mathcal{L}_f^V$) is empty, then $\overline{H}_f^V$ (resp. $\underline{H}_f^V$) is identically $+\infty$ (resp.$-\infty$). A simple argument shows that $\underline{H}_f^V \leq \overline{H}_f^V$ on $V$.

**Definition 5.1.5 ([14], p.19)** *An open set $V \subset X$ is called resolutive if for any finite continuous function $\phi$ with compact support on $\partial V$, the upper and lower solutions $\overline{H}_\phi^V$ and $\underline{H}_\phi^V$ on $V$ coincide and are harmonic in $V$.*

On resolutive sets we will set $H_\phi^V = \overline{H}_\phi^V = \underline{H}_\phi^V$. The function $H_\phi^V$ can be represented as

$$H_\phi^V(x) = \int_{\partial V} \phi(y)\omega(x, dy, V)$$

for some measure $\omega(x, \cdot, V)$ on $\partial V$. This completes the extension of the harmonic measure to resolutive sets. We recall the following important result.

**Theorem 5.1.6 (see [14], Theorem 2.4.2)** *Any open relatively compact subset $V$ of $X$ is resolutive.*

**Definition 5.1.7 ([14], §2.2)** *A hyperharmonic function $f \in \mathcal{U}(V)$ is called **superharmonic** in an open subset $V$ of $X$ if for any relatively compact open subset $V' \subset V$, the function*

$$\int_{\partial V'} f(y)\omega(x, dy, V')$$

*is harmonic in $V'$. A hypoharmonic function $f \in \mathcal{L}(V)$ is called **subharmonic** in $V$ if $-f$ is superharmonic in $V$.*

**Definition 5.1.8 ([14], §2.2)** *A positive superharmonic function $p$ on $V$ is called a **potential** on $V$ if no positive harmonic function $u$ on $V$ satisfies $u \leq p$ on $V$. For any potential $p$ on $V$, we denote the **harmonic support** $S(p)$ of $p$ to be the set where $p$ is not harmonic.*

**Definition 5.1.9 ([14], §6.2)** *A bounded set $K \subset X$ is called **polar** if there exists a covering of $K$ by a family $\mathcal{B}$ of open subsets $W \in \mathcal{B}$ of $X$ for every $W \in \mathcal{B}$ there exists a positive hyperharmonic function $f$ on $W$ which is finite on $W \setminus K$ and identically $+\infty$ on $W \cap K$.*

In our context the following axioms are satisfied for the harmonic sheaf $\mathcal{H}$ and the hyperharmonic sheaf $\mathcal{U}$.

**Proposition 5.1.10 (Axiom of Proportionality, see Theorem 3.1.1 in [10])** *Any two potentials which are harmonic outside a given point are proportional.*

**Proposition 5.1.11 (Axiom of Domination, see §9.2 in [14])** *For any open relatively compact subset $V$ of some regular subset of $X$ and for any bounded hyperharmonic function $u$ defined in a neighborhood of $\overline{V}$, $H_u^V$ is the greatest harmonic minorant of the restriction $u|_V$.*

**Proof.** See Theorem 9.2.1 in [14] and Theorem 4.12 in [13]. □

According to [14, Theorem 9.2.1] the Axiom of Domination in our setting implies the following equivalent facts.

(i) Any locally bounded potential $p$ on $X$ is continuous if its restriction to the set $S(p)$ is continuous.

(ii) For any locally bounded potential $p$ on $X$ and any positive hyperharmonic function $u$ on $X$, we have

$$u \geq p \text{ on } S(p) \Rightarrow u \geq p \text{ on } X.$$

**Proposition 5.1.12 (Axiom of Polarity, see §9.1 in [14])** *For any open set $V$ of $X$ and for any family $\mathcal{S}$ of positive hyperharmonic functions on $V$, the set*

$$\{x \in V | \widehat{\inf_{u \in \mathcal{S}} u(x)} < \inf_{u \in \mathcal{S}} u(x)\}$$

*is polar. Here $\widehat{f}$ denotes the lower semi-continuous regularization of $f$.*

**Proof.** In our setting this proposition follows from [31, Theorems 4.1.2 and 4.1.3] together with [14, Theorem 9.1.1]. Alternatively, the axiom of domination is known to imply the axiom of polarity. □

In our context polar sets are exactly the sets of one-capacity zero, see e.g. [31, Theorem 4.2.1, Theorem 4.1.2, Theorem 2.1.6]. Therefore the notion 'quasi-everywhere' introduced through capacity in Chapter 2.4.1 also means 'except on a polar set'.

We now move on to prepare a version of maximum principle.

**Definition 5.1.13** *A point $y \in \partial V$ is called regular if for every continuous function $\phi$ on $\partial V$ we have*

$$\lim_{V \ni x \to y} H_\phi^V(x) = \phi(y).$$

*a point $y \in \partial V$ is called irregular if it is not regular.*

It is known that the set of irregular points of $\partial V$ form a set of capacity zero, see [12, VII.4.2] and [14, Theorem 9.1.1 (i)] together with [31, Theorem ] where it is proved that such sets are polar. It is also known that the measure $\omega(x, \cdot, V)$ does not charge subsets of $\partial V$ of capacity zero. Also both $\overline{H}_f^V$ and $\underline{H}_f^V$ do not change if we alter the function $f$ on a set of capacity zero. These facts are shown in [14, Chapter 2] and are sufficient to imply the following maximum principle.

**Proposition 5.1.14 (Maximum principle)** *Let $V$ be a relatively compact subset of $X$. Let $u$ be any bounded from above subharmonic function in $V$. Assume that for some constant $C$, we have*

$$\limsup_{V \ni x \to y} u(x) \leq C$$

*for quasi every $y \in \partial V$. Then $u \leq C$ in $V$. Moreover if we also assume that for some $D \leq C$ we have*

$$\limsup_{V \ni x \to y} u(x) \leq D$$

*for all $y \in E \subset \partial V$, then*

$$u \leq D + (C - D)\omega(\cdot, \partial V \setminus E, V) \quad \textit{on } V.$$

**Proof.** Define $\phi : \partial V \to \mathbb{R}$ by

$$\phi(y) = \limsup_{V \ni x \to y} u(x).$$

The function $u$ belongs to the lower class $\mathcal{L}_\phi^V$. Therefore since $\underline{H}_\phi^V \leq \overline{H}_\phi^V$, we have $u \leq \overline{H}_\phi^V$ on $V$. The first part of this proposition follows from the fact that $\overline{H}_\phi^V$ does not change if we alter the function $\phi$ on a set of capacity zero. Therefore,

$$u \leq \overline{H}_\phi^V \leq \overline{H}_C^V = C$$

on $V$. To show the second estimate of this proposition, consider any continuous function $\varphi$ on $\partial V$ which is at least $C$ on $\partial V \setminus E$ and at least $D$ on $E$. Then we have $\varphi \geq \phi$ quasi everywhere on $\partial V$, and therefore for any $x \in V$, we have

$$u(x) \leq \overline{H}_\phi^V(x) \leq \overline{H}_\varphi^V(x) = H_\varphi^V(x) = \int_{\partial V} \varphi(y)\omega(x, dy, V)$$

on $V$. Taking the infimum over all such continuous functions $\varphi$, we obtain the desired inequality

$$u \leq D + (C - D)\omega(\cdot, \partial V \setminus E, V) \quad \text{on } V.$$

$\square$

It follows that in order to compare two bounded local solutions of $Lu = 0$ in an open set $V$ it suffices to compare their limit behavior around quasi every point of $\partial V$.

## 5.2 Local solutions, Dirichlet case, revisited

In the context of a Harnack-type Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$, we are now ready to give an alternative view on $\mathcal{F}_{loc}^0(V, U)$, see Definition 2.4.3.

**Definition 5.2.1** *For any set $V \subset U$ let $V^{\#}$ denote the interior of the closure of $V$ in $\widetilde{U}$.*

**Lemma 5.2.2** *Let $V$ be any open subset of $U$. A function $f$ belongs to the space $\mathcal{F}^0_{loc}(V,U)$ of* Definition 2.4.3*, if and only if for any open set $\Omega \subset V$ which is relatively compact in $V^{\#}$, the function $f|_{\Omega}$ has an extension to a function in $\mathcal{D}(\mathcal{E}^D_U)$.*

**Proof.** Indeed, the sets $\Omega$ considered in Definition 2.4.3 are exactly the open sets at a positive $\rho_U$-distance from $U \setminus V$ which are relatively compact in $U$, or equivalently, relatively compact in $\widetilde{U}$. The relatively compact in $\widetilde{U}$ set $\Omega$ is relatively compact in $V^{\#}$ if and only if $\rho_U(\Omega, U \setminus V) > 0$. □

**Lemma 5.2.3** *Let $V$ be an open subset of $U$. A function $f \in \mathcal{F}_{loc}(V)$ is in $\mathcal{F}^0_{loc}(V,U)$ if and only if for every bounded function $\phi \in \mathcal{F}(U)$ with some compact support $\Omega \subset \widetilde{U}$ such that $\frac{d\Gamma(\phi,\phi)}{d\mu}$ is bounded on $U$ and $\rho_U(\Omega, U \setminus V) > 0$, we have $\phi f \in \mathcal{D}(\mathcal{E}^D_U)$.*

**Proof.** To prove the 'if' implication, pick any open $\Omega \subset V$ relatively compact in $\overline{U}$ (equivalently, $\Omega$ is relatively compact in $\widetilde{U}$) with $\rho_U(\Omega, U \setminus V) > 0$. Denote $\varepsilon = \rho_U(U \setminus V, \Omega) > 0$. Let $\Omega'$ be an $\frac{\varepsilon}{2}$-neighborhood of $\Omega$ in $(U, \rho_U)$. Then $\rho_U(\Omega', U \setminus V) \geq \frac{\varepsilon}{2} > 0$. Consider a compactly supported in some $\Omega'' \subset \widetilde{U}$ cutoff function of the form

$$\phi(x) = \max\left(0, 1 - \frac{\rho_U(x,\Omega)}{\rho_U(U \setminus \Omega', \Omega)}\right).$$

The support $\Omega''$ of $\phi$ is a subset of the closure of $\Omega'$ in $(\widetilde{U}, \rho_U)$, and therefore

$$\rho_U(\Omega'', U \setminus V) \geq \rho_U(\Omega', U \setminus V) > 0.$$

We have $\phi f \equiv f$ on $\Omega$. So the function $\tilde{f} = \phi f$ satisfies the conditions of Definition 2.4.3 and is in $\mathcal{D}(\mathcal{E}^D_U)$ by assumption.

To show 'only if' pick any bounded function $\phi \in \mathcal{F}(U)$ with some compact support $\Omega \subset \widetilde{U}$ such that $\rho_U(\Omega, U \setminus V) > 0$ and $\frac{d\Gamma_U(\phi,\phi)}{d\mu}$ is bounded on $U$. By definition of $\mathcal{F}^0_{loc}(V,U)$ there exists a function $\tilde{f} \in \mathcal{D}(\mathcal{E}^D_U)$ coinciding with $f$ on $\Omega \cap U$. By definition of $\mathcal{D}(\mathcal{E}^D_U)$, $\tilde{f}$ can be approximated in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$ by functions $f_n \in \mathcal{F}_c(U)$. Since $(\mathcal{E}^D_U, \mathcal{D}(\mathcal{E}^D_U))$ is a Dirichlet form, each of the functions $f_n$ can be approximated in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$ by bounded functions $h^n_m = \min(\max(f_n, -m), m) \in \mathcal{F}_c(U) \cap L^\infty(U,\mu)$. By the standard argument there exists a sequence in the family $\{h^n_m\}^\infty_{m,n=1}$ that converges to $\tilde{f}$ in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$. Therefore w.l.o.g. we can assume that each of the functions $f_n$ is in $\mathcal{F}_c(U) \cap L^\infty(U,\mu)$.

Let $\tilde{\phi}_n \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$ be any function coinciding with $\phi$ on the support of $f_n$. Since both $\tilde{\phi}_n$ and $f_n$ are in $\mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$, we have

$$\phi f_n = \tilde{\phi}_n f_n \in \mathcal{D}(\mathcal{E}) \cap L^\infty(X,\mu)$$

by Lemma 2.1.5. Since the function $f_n$ is compactly supported in $U$, so is $\phi f_n$ and therefore $\phi f_n \in \mathcal{F}_c(U) \subset \mathcal{D}(\mathcal{E}^D_U)$ by Lemma 2.2.2.

Since $\phi$ is bounded, $\phi f_n \to \phi \tilde{f}$ in $L^2(U, \mu|_U)$. To prove that $\phi f_n \to \phi\tilde{f}$ in $\mathcal{E}^D_U$-norm it suffices to show that the sequence $\phi f_n$ in $\mathcal{D}(\mathcal{E}^D_U)$ is Cauchy. Using Lemma 2.2.1 and the chain rule we estimate

$$\begin{aligned}\mathcal{E}^D_U(\phi f_n - \phi f_m, \phi f_n - \phi f_m) &\le 2\int_U \phi^2 d\Gamma_U(f_n - f_m, f_n - f_m) + 2\int_U d\Gamma_U(\phi,\phi)(f_n - f_m)^2 \\ &\le 2\sup_U \phi^2 \int_U d\Gamma_U(f_n - f_m, f_n - f_m) + 2\sup_U \frac{d\Gamma_U(\phi,\phi)}{d\mu}\int_U (f_n - f_m)^2 d\mu \to 0,\end{aligned}$$

as $m, n \to \infty$ because $f_n$ is a Cauchy sequence in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$ and both $\phi^2$ and $\frac{d\Gamma_U(\phi,\phi)}{d\mu}$ are bounded on $U$ by assumption. $\square$

## 5.3 Green function as a tool

The context of this section is that of a Theorem 5.0.8, i.e. of a Harnack type strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, \mu)$ and an open subset $U$ of $X$. Throughout the section let $B_U(x, R)$ denote a ball in $(\widetilde{U}, \rho_U)$ centered at $x$, and let $B(x, R)$ denote a ball in $(X, \rho)$. Let $\xi \in \widetilde{U} \setminus U$ and $R > 0$. For any open subset $V \subset X$ let $G_V$ denote the Green function of the Dirichlet form $(\mathcal{E}^D_V, \mathcal{D}(\mathcal{E}^D_V))$ as in Definition 2.4.2. Let $G_R = G_{U_R}$ denote the Green function for the Dirichlet form $(\mathcal{E}^D_{U_R}, \mathcal{D}(\mathcal{E}^D_{U_R}))$, in $U_R = U \cap B(\xi, R)$, i.e. a Dirichlet Green function for the domain $U_R$. As a local (weak) solution of $LG_R(\cdot, y) = 0$ in $U_R$, the Green function $G_R(\cdot, y)$ satisfies the elliptic version of the Harnack inequality (2.38) by Theorem 2.6.1. From Chapter 2.5 we know that $G_R(x, y)$ is Hölder continuous locally in $U_R$. Also the Green function $G_R(x, y)$ is symmetric since the semigroup $P^D_{U_R,t}$ is.

The following definition introduces the notion of Green capacity.

**Definition 5.3.1** *Let $V$ be an open subset of $X$ with Green function $G_V$. Define the Green capacity $Cap_V(E)$ for a Borel set $E \subset V$ by*

$$Cap_V(E) = \sup\{\mu(E) : G_V\mu \leq 1 \textit{ on } V,\ \mu \textit{ is a Borel measure supported on } E\}$$

*In the usual way $Cap_V(E)$ extends to a general set $E \subset V$.*

**Remark.** It turns out that if a set $E$ has capacity zero relatively to one open set $V$, then it has capacity zero relative to any open set $V$ containing $E$. In other words, the property of having capacity zero does not depend on the set $V$. Also it is known that sets of Green capacity zero in this definition are exactly sets of 0-capacity zero, see (2.30) and [31, Chapter 2.2 and Theorem 2.1.6].

For any $x \in U$ let $\delta_U(x)$ denote the distance $\rho_U(x, \widetilde{U} \setminus U) = \rho(x, X \setminus U)$. We will make extensive use of the following Green function estimate.

**Lemma 5.3.2** *In the context above for any constant $\varepsilon < 1$ there exist constants $C_1, C_2$ such that for any $x, y \in U \cap B(\xi, R)$ with $\rho(x,y) \geq \varepsilon R$, we have*

$$G_R(x,y) \leq C_1 \frac{R^2}{\mu(B(y,R))} \tag{5.6}$$

*Moreover if $U$ is a uniform domain in $X$ then whenever $x, y \in B(\xi, \frac{R}{4c_0})$ and*

$$\delta_U(x), \delta_U(y), \rho(x,y) \geq \varepsilon R$$

*we also have*

$$G_R(x,y) \geq C_2 \frac{R^2}{\mu(B(y,R)))}$$

*where the constants $C_1$ and $C_2$ depend only on $\varepsilon$ and the constants $c_0, c_1, c_2, c_3$ appearing in* Definition 3.1.1, (2.12) *and* (2.13).

**Proof.** We will use the representation of the Green function $G_R$ via the heat kernel of the corresponding Dirichlet heat semigroup in $U \cap B(\xi, R)$,

$$G_R(x,y) = \int_0^\infty p^D_{U \cap B(\xi,R)}(t,x,y)dt \leq \int_0^\infty p^D_{B(\xi,R)}(t,x,y)dt$$

For the upper bound (5.6), we use doubling (2.12) together with Theorem 2.6.4 to estimate the Dirichlet heat kernel in the ball $B(\xi, R)$ by

$$p^D_{B(\xi,R)}(t,x,y) \leq \begin{cases} \frac{C}{\mu(B(x,\sqrt{t}))} \exp\left(-\frac{\rho(x,y)^2}{5t}\right), & \text{if } t \leq R^2; \\ \frac{C}{\mu(B(x,R))} \exp\left(-C_3 \frac{t}{R^2}\right), & \text{if } t > R^2, \end{cases}$$

where the constants $A, C, C_3$ depend only on the constants $c_0, c_1, c_2, c_3$ from Definition 3.1.1 (2.12) and in (2.13). Integrating over $t$ and making use of the doubling condition (2.12) gives the estimate (5.6).

To prove the corresponding lower estimate in case $\delta_U(x), \delta_U(y), \rho_U(x,y) \in (\varepsilon R, \frac{R}{2c_0})$, notice that by a simple geometric argument, we also have $\rho(x,y) \geq \varepsilon R$. We first estimate $G_R(z,y)$ for some $z$ close to $y$. Let $r = \delta_U(y) \in (\varepsilon R, \frac{R}{2c_0})$. When

$z \in B(y,r) \subset U \cap B(\xi, R)$, we can use the comparison of Dirichlet heat kernels in $U \cap B(\xi, R)$ and in the ball $B_U(y,r) = B(y,r)$ to estimate

$$G_R(z,y) \geq \int_0^\infty p^D_{B(y,r)}(t,z,y)dt$$

Using the doubling condition (2.12) and Theorem 2.6.4 to estimate the Dirichlet heat kernel in the ball $B(y, \delta_U(y))$, we see that for $t$ such that $4\rho(z,y)^2 \leq t \leq (\epsilon_1 r)^2$,

$$\forall z \in B(y, \epsilon_2 r), \quad p^D_{B(y,r)}(t,y,z) \geq \frac{\epsilon_3}{\mu(B(y,\sqrt{t}))} \geq \frac{\epsilon_3}{\mu(B(y,r))}$$

for some constants $\epsilon_1, \epsilon_2, \epsilon_3$ depending only on $c_0, c_1, c_2, c_3$. Let $\epsilon_4 = \min(\epsilon_2, \frac{1}{4}\epsilon_1)$. Then for every $z \in B(y, \epsilon_4 r)$ we can integrate the inequality above over $t$ from $\frac{(\epsilon_1 r)^2}{4}$ to $(\epsilon_1 r)^2$ to get

$$G_R(z,y) \geq \int_{(\epsilon_1 r)^2/4}^{(\epsilon_1 r)^2} p^D_{B(y,r)}(t,y,z)dt \geq \frac{3}{4}\epsilon_3 \epsilon_1^2 \frac{r^2}{\mu(B(y,r))}$$

Using the doubling condition (2.12) we obtain

$$G_R(z,y) \geq \epsilon_5 \frac{R^2}{\mu(B(x,R))},$$

for some constant $\epsilon_5 > 0$ depending only on the constants $c_0, c_1, c_2, c_3$. Assume w.l.o.g. that $x \notin B(y, \epsilon_4 r)$. In order to compare $G_R(x,y)$ with $G_R(z,y)$ for some $z \in \partial B(y, \epsilon_4 r)$, we make use of the uniform property of $U$ together with the Harnack principle. Since $(\widetilde{U}, \rho_U)$ is uniform and $\delta_U(x), \delta_U(y), \rho_U(x,y) \in (\varepsilon R, \frac{1}{2c_0}R)$, there exists a path $\gamma : [0,1] \to U$ between $x$ and $y$ of length at most $c_0 \rho_U(x,y) \leq c_0 \frac{R}{2c_0} = \frac{R}{2}$. The path $\gamma$ stays in $B(\xi, \frac{3}{4}R)$. Also the path $\gamma$ satisfies $\forall t$, $\rho_U(\gamma(t), \widetilde{U} \setminus U) \geq \epsilon_6 R$ for some small positive constant $\epsilon_6$ depending only on $c_0, c_1, c_2, c_3$. The constant $\epsilon_6$ will be assumed to be less than $\varepsilon\epsilon_4$, without loss of generality. Let $z$ be the last point on the way along the path $\gamma$ from $y$ to $x$ that is in the closure of $B(y, \epsilon_6 R)$. Such a point exist because $\epsilon_6 R \leq \epsilon_4 \varepsilon R \leq \epsilon_4 r$ and so $x \notin B(y, \epsilon_6 R)$ by assumption above. The whole segment of the path $\gamma$ from $z$ to $x$ stays inside

$B(\xi, R) \cap U$ at a distance at least $\epsilon_6 R$ away from $y$, from $\partial U$ and from $\partial B(\xi, R)$. In view of the Harnack inequality for the function $G_R(\cdot, y)$, there is a constant $\epsilon_7$ such that

$$G_R(x, y) \geq \epsilon_7 G_R(z, y) \geq \epsilon_5 \epsilon_7 \frac{R^2}{\mu(B(x, R))},$$

and all the constants introduced in this lemma depend only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12), (2.13) and in Definition 3.1.1. □

**Proposition 5.3.3** *Let $V$ be a bounded domain in $X$. Then for any $y_0 \in V$, the function $G_V(y_0, \cdot)$ belongs to the space $\mathcal{F}^0_{loc}(V \setminus \{y_0\}, V)$.*

**Proof.** Fix any $y_0 \in V$. Applying Lemma 2.3.1 to the Dirichlet form $(\mathcal{E}^D_V, \mathcal{D}(\mathcal{E}^D_V))$ on $L^2(V, \mu)$, we see that for every $t > 0$ the Dirichlet heat kernel $p^D_V(t, \cdot, y_0)$, belongs to $\mathcal{D}(\mathcal{E}^D_V)$ as a function of the second variable.

Next we choose any nonnegative bounded function $\phi \in \mathcal{D}(\mathcal{E})$ with compact support $K \subset X$ such that $y_0 \notin K$ and $\frac{d\Gamma(\phi,\phi)}{d\mu}$ is bounded on $X$. We will show that the convergence of the integral

$$\phi(x) G_V(y_0, x) = \phi(x) \lim_{N \to \infty} \int_0^N p^D_V(t, y_0, x) dt \tag{5.7}$$

is in $L^2(V, \mu)$. By dominated convergence theorem this would follow from the fact that $\phi(x) G_V(y_0, x) \in L^2(V, \mu)$ as a function of $x$. Let

$$I_N(x) = \int_0^N p^D_V(t, y_0, x) dt.$$

Let $R$ be the diameter of $V$ with respect to $\rho$. By Theorem 2.6.4 together with the doubling condition (2.12), the heat kernel for the Dirichlet problem in the ball $B(y_0, R) \subset X$ satisfies

$$p^D_{B(y_0,R)}(t, y_0, x) \leq \begin{cases} \frac{C}{\mu(B(y_0,\sqrt{t}))} \exp\left(-\frac{\rho(x,y_0)^2}{5t}\right), & \text{if } t \leq R^2; \\ \frac{C}{\mu(B(y_0,R))} \exp\left(-C_3 \frac{t}{R^2}\right), & \text{if } t > R^2. \end{cases} \tag{5.8}$$

Integrating over $t$ and using doubling (2.12), we see that the Green function $G_{B(y_0,R)}(y_0,\cdot)$ is uniformly bounded on $K$. Now $V \subset B(y_0,R)$, therefore

$$G_V(y_0,x) \le G_{B(y_0,R)}(y_0,x)$$

is also uniformly bounded on $K$ as a function of $x$ and so since $\phi$ is supported on $K$ indeed $\phi(x)G_V(y_0,x) \in L^2(V,\mu)$. Therefore the convergence in (5.7) is in $L^2(V,\mu)$.

Finally we will show that the convergence in (5.7) is in $\mathcal{E}_V^D$ - norm by showing that the sequence $\phi I_N$ is Cauchy in Hilbert space $\mathcal{D}(\mathcal{E}_V^D)$. Take any $M \ge N$ and let $f(x) = I_M - I_N \ge 0$. We know $f \in \mathcal{D}(\mathcal{E}_V^D) \cap L^\infty(V,\mu)$. We estimate the energy using the chain rule

$$\begin{aligned}
\mathcal{E}_V^D(\phi(I_M - I_N) \quad &, \quad \phi(I_M - I_N))) = \int_V d\Gamma(\phi f, \phi f) \\
&= \int_V f^2 d\Gamma(\phi,\phi) + 2\int_V \phi f d\Gamma(\phi, f) + \int_V \phi^2 d\Gamma(f,f) \\
&= \int_V f^2 d\Gamma(\phi,\phi) + \int_V d\Gamma(f, \phi^2 f) \\
&\le \sup_K \frac{d\Gamma(\phi,\phi)}{d\mu} \int_{V\cap K} f^2 d\mu + \int_V \phi^2 f L_V^D f d\mu \\
&= \sup_K \frac{d\Gamma(\phi,\phi)}{d\mu} \int_{K\cap V} f^2 d\mu + \int_{K\cap V} \phi^2 f \left( \int_N^M \frac{\partial}{\partial t} p_V^D(t,x,y_0) dt \right) d\mu \\
&\le \sup_K \frac{d\Gamma(\phi,\phi)}{d\mu} \int_{K\cap V} f^2 d\mu + \sup_{K\cap V} \left[ \phi^2 p_V^D(M,x,y_0) \right] \int_{K\cap V} f d\mu
\end{aligned}$$

The first term tends to zero as $M, N \to \infty$ since $\frac{d\Gamma(\phi,\phi)}{d\mu}$ is bounded on $X$ and $I_M(x) \to G_V(y_0,x)$ in $L^2(K \cap V, \mu)$, so that $f \to 0$ in $L^2(V \cap K, \mu)$. Hence $f \to 0$ in $L^1(K \cap V, \mu)$ since $\mu(V \cap K) < \infty$. Therefore the second term tends to zero as $M, N \to \infty$ because both $\varphi$ and the Dirichlet heat kernel $p_V^D(t,x,y_0)$ are bounded from above for $x \in V \cap K$, using (5.8).

So the sequence $\phi I_N$ is Cauchy in $\mathcal{D}(\mathcal{E}_V^D)$. Since the form $(\mathcal{E}_V^D, \mathcal{D}(\mathcal{E}_V^D))$ is closed, the function $\phi(x)G_V^D(y_0,x)$, which is the $L^2(V,\mu)$-limit of $\phi I_N$, must be in $\mathcal{D}(\mathcal{E}_V^D)$.

This holds for all nonnegative bounded $\phi \in \mathcal{D}(\mathcal{E})$ with compact support $K \subset X$ such that $y_0 \notin K$ and $\frac{d\Gamma(\phi,\phi)}{d\mu}$ is bounded on $X$; taking for various integers $n$,

$$\phi = \min(1, \max(n\rho(\cdot, y_0) - 1, 0)),$$

we see that by definition $G_V(y_0, \cdot) \in \mathcal{F}^0_{loc}(V \setminus \{y_0\}, V)$. $\square$

This shows that the Green's function $G_V(y_0, \cdot)$, as a measurable function, has a quasi-continuous representative which is zero quasi-everywhere on $\partial V$. In our context of a Harnack-type Dirichlet space, the Green function $G_V(y_0, \cdot)$ is a Hölder continuous function in $U$, is uniquely determined (at least up to a constant multiple) by the property that it is a potential with harmonic support $y_0$, see Chapter 5.1 and the equivalent notion of potential in [31, Lemma 2.2.6]. Similar to [24, Chapter VII.4] it is known that in our context the Green function $G_V(y_0, \cdot)$ vanishes at every regular point of $\partial V$, i.e. $G_V(y_0, \cdot)$ vanishes q.e. at $\partial V$, see Chapter 5.1. There are examples where $G_V(y_0, \cdot)$ does not vanish at some points of $\partial V$.

## 5.4 Boundary Harnack Principle on a uniform subset

We will make extensive use of the boundary Harnack principle which we will prove in this section following the ideas in [2]. First, it is useful in constructing a harmonic function $h$ which plays a central role in the $h$-transform - our approach to solving the Dirichlet heat diffusion problem in $U$. Second, we will use it to prove that the measure $d\nu = h^2 d\mu$ satisfies the doubling condition (2.12).

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - a strictly local regular Dirichlet form on $X$ satisfying the conditions (A1-A4) of Chapter 2.1.2. Let $\rho = \rho_{\mathcal{E}}$ be the metric on $X$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Let $U \subset X$ be an open subset of $X$.

For any $x \in U$ let $\delta_U(x)$ denote the distance $\rho_U(x, \widetilde{U} \setminus U) = \rho(x, X \setminus U)$.

**Definition 5.4.1** *In this section we say that two functions $f$ and $g$ are comparable on a set $V \subset U$ (write $f \asymp g$ on $V$) if there exists a constant $A$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in* Definition 3.1.1 *such that*

$$\frac{1}{A} g \leq f \leq Ag \text{ on } V$$

**Theorem 5.4.2** *Let $(X, \rho, \mu)$ be a metric measure space with a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on a Hilbert space $L^2(X, \mu)$. Assume that the doubling property (2.12) and the $L^2$ Poincaré inequality (2.13) are satisfied for all $x \in X$ and $R > 0$. Let $U \subset X$ be an unbounded uniform domain in $(X, \rho)$. Then there exists a constant $A_0 > 1$ depending only on $c_0, c_1, c_2, c_3$ such that for any $\xi \in \partial U$ and any $R > 0$, the following boundary Harnack principle holds. Suppose $u$ and $v$ are positive local solutions of $Lu = 0$ in $U \cap B(\xi, A_0 R)$, bounded on $U \cap B(\xi, A_0 R)$ and vanishing q.e. on $\partial U \cap B(\xi, A_0 R)$. Then*

$$\frac{u(x)}{u(x')} \asymp \frac{v(x)}{v(x')} \text{ uniformly for } x, x' \in U \cap B(\xi, R)$$

*where the constant of comparison depends only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in* Definition 3.1.1.

We follow the proof of H.Aikawa [2] making use of the Dirichlet Green function estimates proved in Lemma 5.3.2.

**Definition 5.4.3** *Let $0 < \eta < 1$. For $V \subset X$ we define the capacitary width $w_\eta(V)$ by*

$$w_\eta(V) = \inf \left\{ r > 0 : \frac{Cap_{B(x,2r)}(B(x,r) \setminus V)}{Cap_{B(x,2r)}(B(x,r))} \geq \eta \text{ for all } x \in V \right\}.$$

**Proposition 5.4.4** *In the setting of Theorem 5.4.2 there exist constants $A_1, \eta > 0$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12),* (2.13) *and in* Definition 3.1.1 *such that for any $r > 0$ we have*

$$w_\eta(\{x \in U : \delta_U(x) \leq r\}) \leq A_1 r \tag{5.9}$$

**Proof.** Let $V = \{x \in U : \delta_U(x) \leq r\}$. Using the uniform property of $U$, for every $y \in V$, there exists a point $z \in B(y, \frac{2}{c_1}r) \cap U$ with $\rho(z, \partial U) \geq 2r$. Let $A = \frac{2}{c_1} + 1$. Then $B(z, r) \subset B(y, Ar) \setminus V$, and therefore

$$\mathrm{Cap}_{B(y,2Ar)}(B(z, Ar) \setminus V) \geq \mathrm{Cap}_{B(y,2Ar)}(B(z, r)) \geq \mathrm{Cap}_{B(z,3Ar)}(B(z, r))$$

It remains to prove that

$$\mathrm{Cap}_{B(z,3Ar)}(B(z, r)) \asymp \mathrm{Cap}_{B(y,2Ar)}(B(y, Ar))$$

This follows from the following capacity estimate for $X$ proven in our setting in [36]

$$\mathrm{Cap}_{B(y,R)}(B(y, r)) \asymp \int_r^R \frac{s}{\mu(B(y, s))} ds$$

which together with the volume doubling condition (2.12) implies

$$\begin{aligned} \mathrm{Cap}_{B(z,3Ar)}(B(z, r)) &\asymp (3Ar - r)\frac{r}{\mu(B(z, 2Ar))} \\ \mathrm{Cap}_{B(y,2Ar)}(B(y, Ar)) &\asymp (2Ar - Ar)\frac{Ar}{\mu(B(y, Ar))} \end{aligned}$$

and the right hand sides are comparable. $\square$

The following Lemma relates harmonic measure to capacitary width, and for the proof we closely follow [2].

**Lemma 5.4.5** *In the setting of Theorem 5.4.2 there is a positive constant $A_2$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in*

Definition 3.1.1 *such that for any nonempty open subset $V$ of $X$, any $x \in V$ and any $R > 0$, we have*

$$\omega(x, V \cap \partial B(x,R), V \cap B(x,R)) \leq \exp\left(2 - A_2 \frac{R}{w_\eta(V)}\right) \tag{5.10}$$

**Proof.** For any $\varepsilon > 0$ we can choose $r$ with $w_\eta(V) \leq r < w_\eta(V) + \varepsilon$, such that

$$\frac{\mathrm{Cap}_{B(y,2r)}(B(y,r) \setminus V)}{\mathrm{Cap}_{B(y,2r)}(B(y,r))} \geq \eta \quad \text{for all } y \in V \tag{5.11}$$

For a moment we fix $y \in V$. Let $E = B(y,r) \setminus V$ and let $G_B$ be the Green function $G_{B(y,2r)}$. Let $\mu_E$ be the capacitary measure of $E$, i.e.

$$\mu_E \text{ is supported on } \overline{E} \subset X,$$
$$||\mu_E|| = \mathrm{Cap}_{B(y,2r)}(E),$$
$$G_B \mu_E = 1 \text{ q.e. on } E.$$

The existence of such a measure can be established in the general context in a way similar to [31, (2.2.13)]. We claim

$$G_B \mu_E \geq \epsilon\eta \text{ on } B(y,r) \tag{5.12}$$

for some constant $\epsilon$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12), (2.13) and in Definition 3.1.1. To this end let $\nu$ be the capacitary measure of $B(y,r)$. Then $\nu$ is supported on $B(y,r)$ and $||\nu|| = \mathrm{Cap}_{B(y,2r)}(B(y,r))$. By Harnack principle,

$$G_B(\cdot, x) \asymp G_B(\cdot, y) \text{ on } \partial B\left(y, \frac{3}{2} r\right)$$

uniformly for $x \in B(y,r)$. Hence

$$G_B \mu_E(z) = \int_{\overline{E}} G_B(z,x) d\mu_E(x) \asymp G_B(z,y) ||\mu_E||,$$
$$G_B \nu(z) = \int_{\overline{E}} G_B(z,x) d\nu(x) \asymp G_B(z,y) ||\nu||$$

uniformly for $z \in \partial B(y, \frac{3}{2}r)$. Since $G_B\nu \asymp 1$ on $\partial B(y, \frac{3}{2}r)$, it follows from (5.11) that on $\partial B(y, \frac{3}{2}r)$,

$$G_B\mu_E \asymp \frac{G_B\mu_E}{G_B\nu} \asymp \frac{||\mu_E||}{||\nu||} = \frac{\mathrm{Cap}_{B(y,2r)}(E)}{\mathrm{Cap}_{B(y,2r)}(B(y,r))} \geq \eta.$$

By the maximum principle of Proposition 5.1.14 applied to the function $-G_B\mu_E$, (5.12) follows.

Now let us move on to the proof of Lemma 5.4.5. For simplicity write $\Omega$ for $\omega(\cdot, V \cap \partial B(x,R), V \cap B(x,R))$. Because of the factor of $e^2$ on the right hand side of the desired estimate (5.10), without loss of generality we may assume that $R/w_\eta(V) > 2$ and let k be the positive integer such that $2kw_\eta(V) < R < 2(k+1)w_\eta(V)$. Take $r > w_\eta(V)$ so close to $w_\eta(V)$ that $2kr < R$. We claim

$$\sup_{V \cap B(x,R-2jr)} \Omega \leq (1-\epsilon\eta)^j \tag{5.13}$$

for $j = 0, 1, \ldots, k$. Since $k \approx \frac{R}{2w_\eta(V)}$, (5.13) implies

$$\Omega(x) \leq (1-\epsilon\eta)^k \leq \exp\left(-A_2 \frac{R}{w_\eta(V)}\right)$$

where $A_2 \approx -\frac{1}{2}\log(1-\epsilon\eta) > 0$.

To prove (5.13) by induction, we start with the obvious estimate (5.13) for $j = 0$. Assume that (5.13) holds for $j-1$ and we shall prove (5.13) for $j$. In view of the maximum principle of Proposition 5.1.14, it is sufficient to show that

$$\sup_{V \cap \partial B(x,R-2jr)} \Omega \leq (1-\epsilon\eta)^j. \tag{5.14}$$

Let $y \in V \cap \partial B(x, R-2jr)$. Then $B(y,2r) \subset B(x, R-2(j-1)r)$, so that (5.13) for $j-1$ implies

$$\Omega \leq (1-\epsilon\eta)^{j-1} \text{ on } V \cap B(y,2r).$$

Since $\Omega$ vanishes q.e. on $\partial V \cap B(x,R) \supset \partial V \cap B(y,2r)$, the maximum principle of Proposition 5.1.14 implies

$$\Omega \leq (1-\epsilon\eta)^{j-1}\omega(\cdot, V \cap \partial B(y,2r), V \cap B(y,2r)) \text{ on } V \cap B(y,2r). \tag{5.15}$$

Let us compare $\omega(\cdot, V \cap \partial B(y, 2r), V \cap B(y, 2r))$ and $1 - G_B\mu_E$ where $\mu_E$ is as in (5.12). By the maximum principle of Proposition 5.1.14, we have

$$\omega(\cdot, V \cap \partial B(y, 2r), V \cap B(y, 2r)) \leq 1 - G_B\mu_E \text{ on } V \cap B(y, 2r),$$

because this inequality holds q.e. in the limit sense on $\partial(V \cap B(y, 2r))$ and both functions are harmonic inside. In particular

$$\omega(y, V \cap \partial B(y, 2r), V \cap B(y, 2r)) \leq 1 - G_B\mu_E(y) \leq 1 - \epsilon\eta$$

by (5.12). Substituting this into (5.15), we obtain $\Omega(y) \leq (1 - \epsilon\eta)^j$ for any point $y \in V \cap \partial B(x, R - 2jr)$. Hence (5.14) and (5.13) follows. □

**Lemma 5.4.6** *For any point $\xi \in \partial U$ and any $R > 0$ there exists a point $\xi_R \in U$ with*

$$\rho(\xi, \xi_R) = 4R, \quad \textit{and} \quad \delta_U(\xi_R) = \rho(\xi_R, \partial U) \geq 4c_1 R \tag{5.16}$$

**Proof.** Choose $\xi \in \partial U \subset X$. Choose an integer $i > 0$. Applying the uniform condition (3.5) to some point $\xi_i \in U$ with $\rho(\xi, \xi_i) = 1/i$ and some other point $\xi_i' \in U$ with $\rho(\xi, \xi_i') = i$, we obtain a path $\gamma$ connecting $\xi_i$ and $\xi_i'$ satisfying the condition in Definition 3.1.1. For any $R > 0$ let $\xi_{i,R}$ be a point on this path with $\rho(\xi_i, \xi_{i,R}) = 4R$. Then the uniform condition (3.5) together with a triangle inequality gives

$$\rho(\xi_{i,R}, \partial U) \geq 4c_1 R \left( \frac{\rho(\xi_{i,R}, \xi_i')}{\rho(\xi_i, \xi_i')} \right) \geq 4c_1 R \left( 1 - \frac{\rho(\xi_{i,R}, \xi_i)}{\rho(\xi_i, \xi_i')} \right) \geq 4c_1 R \left( 1 - \frac{4R}{i - 1/i} \right)$$

letting $i$ go to $\infty$ we obtain a sequence of points $\xi_{i,R}$ in $B(\xi, 5R)$. Since the balls in $X$ are compact, we can choose a subsequence converging to some point $\xi_R$ with $\rho(\xi, \xi_R) = 4R$ and $\rho(\xi_R, \partial U) \geq 4c_1 R$, as desired. □

For any $r > 0$ we let $U_r = B(x, r) \cap U$ and we let $G_r$ to be the Green function for the Dirichlet form $(\mathcal{E}^D_{U_r}, \mathcal{D}(\mathcal{E}^D_{U_r}))$, i.e. the Dirichlet Green function in $U_r$. This function has been studied in Chapter 5.3.

**Lemma 5.4.7** *In the setting of Theorem 5.4.2 there exists a positive constant $A_3$ and $A_5$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in* Definition 3.1.1, *such that for any $\xi \in \partial U$, any $R > 0$, and any $k \geq A_3$ we have*

$$\omega(\cdot, U \cap \partial B(\xi, R), U \cap B(\xi, R)) \leq A_5 \frac{\mu(B(\xi, R))}{R^2} G_{kR}(\cdot, \xi_R) \text{ on } U \cap B(\xi, R) \tag{5.17}$$

*where $\xi_R$ is any point in $U$ that satisfies $\rho(\xi_R, \xi) = 4R$ and $4c_1 R \leq \delta_U(\xi_R) \leq 4R$, e.g. a point produced in* Lemma 5.4.6.

**Proof.** We follow the structure of the proof in the paper of H.Aikawa [2]. Choose $A_3 = 100c_0$ large enough so that in particular all the paths given by the uniform condition (3.5) connecting points in $B(\xi, 10R)$ must stay in $U_{A_3 R/2} = U \cap B(\xi, \frac{A_3}{2} R)$. By the monotonicity of the Green function on the domain it suffices to prove the lemma with $k = A_3$. Since

$$B(\xi_R, \frac{1}{2}\delta_U(\xi_R)) \subset U \cap B(\xi, 6R) \setminus B(\xi, 2R) \subset U_{A_3 R} \setminus B(\xi, 2R),$$

it follows from the maximum principle of Proposition 5.1.14 that

$$G_{A_3 R}(\cdot, \xi_R) \leq \sup_{y \in \partial B(\xi_R, \frac{1}{2}\delta_U(\xi_R))} G_{A_3 R}(y, \xi_R) \text{ on } U \cap B(\xi, 2R)$$

The right hand side is comparable to $\frac{R^2}{\mu(B(\xi,R))}$ by Lemma 5.3.2 since both $y$ and $\xi_R$ are in $B_U(\xi, \frac{A_3 R}{4c_0})$. Hence we can find $\epsilon_1 > 0$ such that

$$\epsilon_1 \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(\cdot, \xi_R) < \exp(-1)$$

on $U \cap B(\xi, 2R)$. Then

$$U \cap B(\xi, 2R) = \bigcup_{j \geq 0} U_j \cap B(\xi, 2R), \tag{5.18}$$

where

$$U_j = \left\{ x \in U : \exp(-2^{j+1}) \leq \epsilon_1 \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R) < \exp(-2^j) \right\}.$$

Let $V_j = \left( \bigcup_{k \geq j} U_k \right) \cap B(\xi, 2R)$. We claim that

$$w_\eta(V_j) \leq AR \exp\left( -\frac{2^j}{\lambda} \right) \tag{5.19}$$

with some constants $A, \lambda$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12), (2.13) and in Definition 3.1.1. Suppose $x \in V_j$. Observe that for $z \in \partial B(\xi_R, \frac{1}{2}\delta_U(\xi_R))$, by the uniform condition (3.5), the length of the Harnack chain of balls in $U_{A_3 R} \setminus \{\xi_R\}$ connecting $x$ to $z$ is at most $\epsilon_2 \log\left( \epsilon_3 \frac{R}{\delta_U(x)} \right)$ for some constants $\epsilon_2, \epsilon_3$ depending only on $c_0, c_1, c_2, c_3$, and therefore

$$\begin{aligned} \exp(-2^j) &> \epsilon_1 \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R) \\ &\geq \epsilon_4 \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(z, \xi_R) \left( \frac{\delta_U(x)}{\epsilon_3 R} \right)^\lambda \geq \left( \frac{\delta_U(x)}{\epsilon_5 R} \right)^\lambda \end{aligned}$$

by Lemma 5.3.2 for some positive constants $\epsilon_4, \epsilon_5, \lambda$ depending only on the constants $c_0, c_1, c_2, c_3$. To apply Lemma 5.3.2 we have used that both $z$ and $\xi_R$ are in $B_U(\xi, \frac{A_3 R}{4c_0})$ and that $\rho(z, \xi_R) \geq 2R$. Therefore for any $x \in V_j$ we have

$$\delta_U(x) \leq \epsilon_5 R \exp\left( \frac{-2^j}{\lambda} \right)$$

This together with (5.9) yields (5.19).

We proceed by induction. Let $R_0 = 2R$ and

$$R_j = \left( 2 - \frac{6}{\pi^2} \sum_{k=1}^{j} \frac{1}{k^2} \right) R$$

for $j \geq 1$. Then $R_j \downarrow R$ and

$$\sum_{j=1}^{\infty} \exp\left( 2^{j+1} - \frac{A_2 \left( R_{j-1} - R_j \right)}{AR \exp(-2^j/\lambda)} \right) \tag{5.20}$$

$$= \sum_{j=1}^{\infty} \exp\left( 2^{j+1} - \frac{6A_2}{A\pi^2} j^{-2} \exp\left( \frac{2^j}{\lambda} \right) \right) < C < \infty.$$

where the constant $C$ is independent of $R$. Let $\omega_0 = \omega(\cdot, U \cap \partial B(\xi, 2R), U \cap B(\xi, 2R))$ and

$$d_j = \begin{cases} \sup_{x \in U_j \cap B(\xi, R_j)} \frac{R^2 \omega_0(x)}{\mu(B(\xi,R)) G_{A_3 R}(x, \xi_R)}, & \text{if } U_j \cap B(\xi, R_j) \neq \emptyset, \\ 0, & \text{if } U_j \cap B(\xi, R_j) = \emptyset. \end{cases}$$

It is sufficient to show that there exists a constant $C$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in (2.12), (2.13) and in Definition 3.1.1, such that

$$\sup_{j \geq 0} d_j \leq C < \infty$$

Since $\omega_0 < 1$, we have by definition of $U_0$,

$$d_0 = \sup_{U_0 \cap B(\xi, 2R)} \frac{R^2 \omega_0(x)}{\mu(B(\xi, R)) G_{A_3 R}(x, \xi_R)} \leq \epsilon_1 e^2$$

Let $j > 0$. For $x \in U_{j-1} \cap B(\xi, R_{j-1})$ we have

$$\omega_0(x) \leq d_{j-1} \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R)$$

Also $\omega_0 \leq 1$. Therefore the maximum principle of Proposition 5.1.14 yields that

$$\omega_0(x) \leq \omega(x, V_j \cap \partial B(\xi, R_{j-1}), V_j \cap B(\xi, R_{j-1})) + d_{j-1} \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R) \tag{5.21}$$

for $x \in V_j \cap B(\xi, R_{j-1})$. If $x \in U \cap B(\xi, R_j)$, then $B(x, R_{j-1} - R_j) \cap \partial B(\xi, R_{j-1}) = \emptyset$, so that the first term on the right hand side is not greater than

$$\begin{aligned} \omega(x, V_j \cap \partial B(x, R_{j-1} - R_j), V_j \cap B(x, R_{j-1} - R_j)) &\leq \exp\left(2 - A_2 \frac{R_{j-1} - R_j}{w_\eta(V_j)}\right) \\ \leq \exp\left(2 - \frac{A_2}{A} \exp\left(\frac{2^j}{\lambda}\right) \frac{R_{j-1} - R_j}{R}\right) &= \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) \end{aligned}$$

by Lemma 5.4.5 and (5.19). Here $\epsilon_6 = \frac{6A_2}{\pi^2 A}$. Moreover, $\epsilon_1 \frac{\mu(B(\xi,R))}{R^2} G_{A_3 R}(x, \xi_R) \geq \exp(-2^{j+1})$ for $x \in U_j$ by definition. Hence (5.21) becomes

$$\begin{aligned} \omega_0(x) &\leq \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1} \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R) \\ &\leq \left(\epsilon_1 \exp\left(2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}\right) \frac{\mu(B(\xi, R))}{R^2} G_{A_3 R}(x, \xi_R) \end{aligned}$$

Dividing both sides by $\frac{\mu(B(\xi,R))}{R^2} G_{A_3R}(x,\xi_R)$ and taking the supremum over $x \in U_j \cap B(\xi, R_j)$, we obtain

$$d_j \le \epsilon_1 \exp\left(2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) + d_{j-1}$$

and hence for every integer $i > 0$

$$d_i \le \epsilon_1 \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{6A_2}{\pi^2 A} j^{-2} \exp\left(\frac{2^j}{\lambda}\right)\right) < \infty$$

by (5.20). $\square$

Let $A_3$ be the constant appearing in Lemma 5.4.7. The next lemma is a version of a boundary Harnack estimate for Green's functions. For the proof we follow H. Aikawa [2].

**Lemma 5.4.8** *In the setting of Theorem 5.4.2, there exists a constant $A_4$ depending only on the constants $c_0, c_1, c_2, c_3$ of* (2.12), (2.13) *and* Definition 3.1.1 *with $A_4 \ge A_3 + 7$, such that for any $\xi \in \partial U$ and any $R > 0$, we have*

$$\frac{G_{A_4R}(x,y)}{G_{A_4R}(x',y)} \asymp \frac{G_{A_4R}(x,y')}{G_{A_4R}(x',y')} \text{ for } x, x' \in B(\xi, R) \text{ and } y, y' \in U \cap \partial B(\xi, 6R) \quad (5.22)$$

*with the constant of comparison depending only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in* Definition 3.1.1.

**Proof.** Set $A_4 = A_3 + 7 \ge 100c_0 + 7$ so that in particular all the paths given by the uniform condition (3.5) connecting points in $B(\xi, 10R)$ must stay in $U_{A_4R/2} = U \cap B(\xi, \frac{A_4}{2}R)$. Let us take $x^* \in U \cap \partial B(\xi, R)$ and $y^* \in U \cap \partial B(\xi, 6R)$ such that $c_1 R \le \delta_U(x^*) \le R$ and $6c_1 R \le \delta_U(y^*) \le 6R$. It is sufficient to show

$$G_{A_4R}(x,y) \asymp \frac{G_{A_4R}(x^*,y)}{G_{A_4R}(x^*,y^*)} G_{A_4R}(x,y^*) \quad (5.23)$$

for $x \in U \cap B(\xi, R)$ and $y \in U \cap \partial B(\xi, 6R)$.

First we show that the left hand side of (5.23) is not less than the right hand side of (5.23) up to a multiplicative constant. To this end we fix $y \in U \cap \partial B(\xi, 6R)$ and observe that

- $u(x) = G_{A_4R}(x,y)$ is a positive harmonic function on $U_{A_4R} \setminus \{y\}$ vanishing q.e. on $\partial U_{A_4R}$;

- $v(x) = \frac{G_{A_4R}(x^*,y)}{G_{A_4R}(x^*,y^*)} G_{A_4R}(x,y^*)$ is a positive harmonic function on $U_{A_4R} \setminus y^*$ vanishing q.e. on $\partial U_{A_4R}$.

Since $y^* \in U \cap \partial B(\xi, 6R)$ and $6c_1R \leq \delta_U(y^*) \leq 6R$, it follows that the ball $B(y^*, 3c_1R) \subset U \cap B(\xi, 9R) \setminus B(\xi, 3R) \subset U$.

Let us prove that $u \geq Av$ on $\partial B(y^*, c_1R)$. Take $z \in \partial B(y^*, c_1R)$. Then by repeated application of the Harnack inequality,

$$\begin{aligned} v(z) &= \frac{G_{A_4R}(x^*,y)}{G_{A_4R}(x^*,y^*)} G_{A_4R}(z,y^*) \asymp \frac{G_{A_4R}(x^*,y)}{G_{A_4R}(x^*,y^*)} G_{A_4R}(x^*,y^*) \\ &= G_{A_4R}(x^*,y) \leq C_1 \frac{R^2}{\mu(B(\xi,R))}. \end{aligned} \tag{5.24}$$

by Lemma 5.3.2.

If $y \in B(y^*, 2c_1R)$, then $u(z) = G_{A_4R}(z,y) \geq C_2 \frac{R^2}{\mu(B(\xi,R))}$ by Lemma 5.3.2, so that $u(z) \geq Av(z)$ for some constant $A$ independent of $R$ and $\xi$. If $y \in U \setminus B(y^*, 2c_1R)$, then $z$ and $x^*$ can be connected by a Harnack chain in $U_{A_4R} \setminus \{y\}$ of fixed length, and so

$$v(z) \asymp G_{A_4R}(x^*,y) \asymp G_{A_4R}(z,y) = u(z)$$

by (5.24). Hence we have $u \geq Av$ on $\partial B(y^*, c_1R)$ in any case. By the maximum principle of Proposition 5.1.14, $u \geq Av$ on $U_{A_4R} \setminus B(y^*, c_1R)$ which includes $U \cap B(\xi, R)$.

For the opposite estimate we make use of Lemma 5.4.7. For $x \in U \cap B(\xi, 2R)$ and $z \in U \cap B(\xi, 9R) \setminus B(\xi, 3R)$ we have

$$G_{A_4R}(x,z) \leq C_1 \frac{R^2}{\mu(B(\xi,R))}$$

by Lemma 5.3.2. Regarding $G_{A_4R}(x,z)$ as a harmonic function of $x$, we obtain from the maximum principle of Proposition 5.1.14 that

$$G_{A_4R}(\cdot,z) \le C_1 \frac{R^2}{\mu(B(\xi,R))}\omega(\cdot, U\cap\partial B(\xi,2R), U\cap B(\xi,2R)) \quad \text{on} \quad U\cap B(\xi,2R)$$

We obtain from Lemma 5.4.7 and the Harnack inequality that

$$G_{A_4R}(x,z) \le C_1 \frac{R^2}{\mu(B(\xi,R))} A_5 \frac{\mu(B(\xi,R))}{R^2} G_{A_4R}(x,\xi_R) \le \epsilon_1 G_{A_4R}(x,y^*) \qquad (5.25)$$

for $x \in U\cap B(\xi,R)$ and $z \in U\cap B(\xi,9R)\setminus B(\xi,3R)$ and some constant $\epsilon_1 > 0$ independent of $\xi, R$. Now fix $x \in U\cap B(\xi,R)$ and $y \in U\cap\partial B(\xi,6R)$. If $\delta_U(y) \ge \frac{1}{2}c_1R$, then $G_{A_4R}(x,y) \asymp G_{A_4R}(x,y^*)$ and $G_{A_4R}(x^*,y) \asymp G_{A_4R}(x^*,y^*)$ by the Harnack inequality, so that (5.23) follows. Hence we can assume that $\delta_U(y) < \frac{1}{2}c_1R$. Then we can find a point $\xi' \in \partial U$ such that $\rho_U(\xi',y) < \frac{1}{2}c_1R$. Observe that $y \in U\cap B(\xi',\frac{1}{2}c_1R) \subset U\cap B(\xi,R)$ since without loss of generality $c_1 < 1$. Also

$$U\cap B(\xi',2R) \subset U\cap B(y,3R) \subset U\cap B(\xi,9R)\setminus B(\xi,3R).$$

Hence (5.25) implies $G_{A_4R}(x,z) \le \epsilon_1 G_{A_4R}(x,y^*)$ for $z \in U\cap B(\xi,2R)$, so that

$$G_{A_4R}(x,y) \le \epsilon_1 G_{A_4R}(x,y^*)\omega(y, U\cap\partial B(\xi',2R), U\cap B(\xi',2R)). \qquad (5.26)$$

Let us invoke Lemma 5.4.7 with replacing $\xi$ by $\xi'$. Since $\rho(\xi,\xi') \le \rho(\xi,y) + \rho(y,\xi') \le 7R$, it follows that $U\cap B(\xi',A_3R)$ is a subset of $U\cap B(\xi,(A_3+7)R) = U_{A_4R}$. Hence

$$\begin{aligned}\omega(y, U\cap\partial B(\xi',2R), U\cap B(\xi',2R)) &\le A_5 \frac{\mu(B(\xi',R))}{R^2} G_{U\cap B(\xi',A_3R)}(y,\xi'_R) \\ &\le A_5 \frac{\mu(B(\xi',R))}{R^2} G_{A_4R}(y,\xi'_R) \asymp \frac{\mu(B(\xi,R))}{R^2} G_{A_4R}(\xi'_R,y) \quad (5.27)\end{aligned}$$

with $\xi'_R \in U\cap\partial B(\xi',4R)$ such that $4c_1R \le \delta_U(\xi'_R) \le 4R$. Here we have used the symmetry of Green function and the doubling condition (2.12). Hence (5.26) and (5.27) give

$$G_{A_4R}(x,y) \le \epsilon_2 G_{A_4R}(x,y^*)\frac{\mu(B(\xi,R))}{R^2} G_{A_4R}(\xi'_R,y).$$

for some constant $\epsilon_2 > 0$ independent of $\xi$ and $R$. Observe that since w.l.o.g. $c_1 < 1$, we have

$$\begin{aligned}\rho(\xi'_R, y) &\geq \rho(\xi'_R, \xi') - \rho(\xi', y) \geq 4R - \frac{1}{2}c_1 R \geq 2R,\\ \rho(x^*, y) &\geq \rho(\xi, y) - \rho(x, \xi) = 6R - R = 5R.\end{aligned}$$

Therefore using the uniform property of $U$ we can connect $x^*$ and $\xi'_R$ by a fixed length chain of balls $B(x_i, \epsilon_3 R)$ in $U \setminus \{y\}$ so that $B(x_i, 2\epsilon_3 R) \subset U \setminus \{y\}$ and $B(x_i, \epsilon_3 R) \cap B(x_{i+1}, \epsilon_3 R) \neq \emptyset$. Here the constant $\epsilon_3$ depends only on the constants $c_0, c_1, c_2, c_3$. Then by Harnack principle $G_{A_4R}(\xi'_R, y) \asymp G_{A_4R}(x^*, y)$. Since $G_{A_4R}(x^*, y^*) \asymp \frac{R^2}{\mu(B(\xi,R))}$ by Lemma 5.3.2, it follows that

$$G_{A_4R}(x, y) \leq \epsilon_4 \frac{G_{A_4R}(x^*, y)}{G_{A_4R}(x^*, y^*)} G_{A_4R}(x, y^*)$$

for some constant $\epsilon_4$ which depends only on the constants $c_0, c_2, c_3, c_4$ appearing in (2.12), (2.13) and in Definition 3.1.1. This completes the proof of the upper estimate in (5.23) and thus the proof of this lemma. $\square$

In order to prove Theorem 5.4.2 we represent $u$ and $v$ as regularized reduced functions and then as Green potentials. In general let $E$ be a subset of $U_{A_4R}$ and let $u$ be a positive harmonic function on $U_{A_4R}$. Let $\Phi_u^E$ be he family of all positive superharmonic functions $v$ on $U_{A_4R}$ such that $v \geq u$ on $E$ and let

$$R_u^E(x) = \inf\{v(x) : x \in \Phi_u^E\}$$

The lower regularization $\hat{R}_u^E$ is called the regularized reduced function of $u$ to $E$ relative to $U_{A_4R}$. It is known that $\hat{R}_u^E \leq u$ in $U_{A_4R}$, $\hat{R}_u^E = u$ q.e. on $E$ and that $\hat{R}_u^E$ is superharmonic on $U_{A_4R}$ and harmonic on $U_{A_4R} \setminus \bar{E}$, see [14, §5.3]. The global positivity and superharmonicity of $u$ over $U_{A_4R}$ is essential.

**Proof of Theorem 5.4.2.** Let $u, v$ be positive harmonic functions as in Theorem 5.4.2. Then $\hat{R}_u^{U\cap\partial B(\xi,6R)}$ is a lower semicontinuous superharmonic function on $U_{A_4R}$ such that $\hat{R}_u^{U\cap\partial B(\xi,6R)} = u$ q.e. on $U \cap \partial B(\xi, 6R)$ and harmonic on $U \cap B(\xi, 6R)$. Moreover $0 \leq \hat{R}_u^{U\cap\partial B(\xi,6R)} \leq u$ and $u$ vanishes q.e. on $\partial U_{A_4R} \cap \partial U_{6R}$ by assumption. Hence $\hat{R}_u^{U\cap\partial B(\xi,6R)} = u$ on $U \cap B(\xi, 6R)$ by the maximum principle of Proposition 5.1.14. By [14, Proposition 5.3.5], $\hat{R}_u^{U\cap\partial B(\xi,6R)}$ is a Green potential of some Borel measure $\mu$ supported on $U \cap \partial B(\xi, 6R)$, we have

$$u(x) = \int_{U\cap\partial B(\xi,6R)} G_{A_4R}(x, y)d\mu(y) \text{ for } x \in U \cap B(\xi, 6R)$$

Choose any $y' \in U \cap \partial B(\xi, 6R)$. Using Lemma 5.4.8, we can write

$$u(x) \asymp G_{A_4R}(x, y') \frac{\int_{U\cap\partial B(\xi,6R)} G_{A_4R}(x', y)d\mu(y)}{G_{A_4R}(x', y')} \text{ for } x, x' \in U \cap B(\xi, R)$$

Therefore

$$\frac{u(x)}{u(x')} \asymp \frac{G_{A_4R}(x, y')}{G_{A_4R}(x', y')} \asymp \frac{v(x)}{v(x')} \text{ for any } x, x' \in U \cap B(\xi, R)$$

□

## 5.5 Construction and properties of a réduite

Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support. Throughout this section we assume that $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is a strictly local regular Dirichlet form on $L^2(X, \mu)$ satisfying the conditions (A1-A4) of Chapter 2. Let $U$ be an unbounded domain in $X$.

In this section we will construct a réduite, i.e. a local (weak) solution $h \in \mathcal{F}^0_{loc}(U)$ of the equation $Lh = 0$ in $U$, see Definition 2.4.3. As a consequence of the remark following Definition 2.4.3, the harmonic function $h$ that we are looking for will equal to zero on $\partial U$ in quasi-continuous sense. For our construction we will assume that the following local boundary Harnack principle is satisfied for local (weak) solutions of $Lu = 0$ in $U$ with the Dirichlet boundary conditions on $\partial U$, see [6].

**Definition 5.5.1** *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a strongly local regular Dirichlet form on $L^2(X, \mu)$. Let $U \subset X$ be an open subset of $X$. We say that a local boundary Harnack principle is satisfied for the set $U$ if for any $\xi \in \partial U$ there exists exist constants $A > 1$, $C > 0$ and $R > 0$, such that the following boundary Harnack principle holds. Suppose $u$ and $v$ are positive local solutions of $Lu = 0$ in $U \cap B(\xi, AR)$, bounded on $U \cap B(\xi, AR)$ and vanishing q.e. on $\partial U \cap B(\xi, AR)$. Then*

$$\frac{u(x)}{u(x')} \leq C \frac{v(x)}{v(x')} \text{ for all } x, x' \in U \cap B(\xi, R) \tag{5.28}$$

**Remark.** In case when $U$ is a uniform domain in $(X, \rho)$, a stronger boundary Harnack principle holds by Theorem 5.4.2.

Fix a point $y \in U$. Let $\{r_i\}_{i=1}^{\infty}$ be an increasing sequence of radii, $r_i \to \infty$ as $i \to \infty$. Let $\{B_i\}_{i=1}^{\infty}$, $B_i = B_U(y, r_i)$ be a sequence of balls in $(U, \rho_U)$. Let $x_i \in B_U(y, r_i/2), i = 1, 2, \ldots$ be a sequence of points converging to a point at infinity of the one-point compactification of $X$. Consider the sequence of functions

$$h_i(x) = \frac{G_{B_i}(x_i, x)}{G_{B_i}(x_i, y)}.$$

We will construct a réduite function $h$ as a limit of some subsequence of $\{h_i\}_{i=1}^{\infty}$. We prepare a sequence of lemmas in the above context.

**Lemma 5.5.2** *There exists a subsequence of $\{h_i\}_{i=1}^{\infty}$ that converges uniformly on $K$ and in $L^2(K, \mu)$ for every compact subset $K \subset U$.*

**Proof.** For every index $i$, we have $h_i(y) = 1$. Therefore, the sequence $h_i$ is bounded on every compact subset of $U$ by Harnack inequality that follows from

Theorem 2.6.1. We aim to apply the Arzela-Ascoli theorem to show that there exists a convergent subsequence for $\{h_i\}_{i=1}^{\infty}$. We need to show that this sequence is equicontinuous, i.e. $\forall x \in U, \forall \varepsilon > 0$ there exists an open neighborhood $V$ of $x$ in $U$ such that whenever $z \in V$,

$$|h_i(x) - h_i(z)| \leq \varepsilon$$

for $i$ large enough. This estimate follows from the Hölder continuity estimates for local (weak) solutions of $Lu = 0$ in $X$, see Chapter 2.5. Indeed for every $\delta > 0$ and any open ball $B_U(x, R) \subset U$ we can choose small enough radius $r$ such that for any $z \in B_U(x, r)$ we have

$$|h_i(x) - h_i(z)| \leq \delta \left[ \sup_{B_U(x,R)} h_i - \inf_{B_U(x,R)} h_i \right] \leq \delta \sup_{B_U(x,R)} h_i \leq C\delta h_i(y) = C\delta$$

for $i$ large enough by Harnack inequality. It remans to choose $\delta = \varepsilon / C$.

The Arzela-Ascoli theorem implies that there exists a subsequence converging pointwise to some function $h$. Moreover the convergence is uniform on compact subsets of $U$. The convergence is then in $L^2(K, \mu)$ for any compact subset $K \subset U$ since $\mu(K) < \infty$. $\square$

Without loss of generality we assume that the subsequence chosen in Lemma 5.5.2 is the sequence $h_i$ itself, and it converges almost everywhere on $U$ to some function $h$.

**Lemma 5.5.3** *The subsequence $\{h_i\}_{i=1}^{\infty}$ constructed in Lemma 5.5.2 converges to $h$ in $L^2_{loc}(\widetilde{U}, \mu|_U)$.*

**Proof.** Let $h_i$ be the subsequence constructed in Lemma 5.5.2, and let $h$ be its limit. Choose any compact subset $V \subset X$, and let $V' = V \cap U$. It suffices to show that for any such $V$, the sequence $h_i$ is Cauchy in $L^2(V',\ \mu|_U)$. For every $\varepsilon > 0$

we define a compact set

$$V_\varepsilon = \{x \in V \cap U : \rho(x, \partial U) \geq \varepsilon\}$$

Choose any point $z \in \partial U$. Let $R$ be twice the diameter of $V' \cup \{z\}$. Let $g$ be any positive local weak solution of $Lg = 0$ in $U \cap B(z, AR)$ vanishing q.e. on $\partial U \cap B(z, AR)$, e.g., the Dirichlet Green function. For large enough $i$ and $j$ we can use the local boundary Harnack principle (5.28) to estimate the difference $|h_i - h_j|$ in $V$ by $|2Cg|$ and therefore

$$\begin{aligned} ||h_i - h_j||^2_{L^2(V',\ \mu|_U)} &= \int_{V_\varepsilon} |h_i - h_j|^2 d\mu + \int_{V' \setminus V_\varepsilon} |h_i - h_j|^2 d\mu \\ &\leq ||h_i - h_j||^2_{L^2(V_\varepsilon,\ \mu)} + \int_{V' \setminus V_\varepsilon} |2Cg|^2 d\mu \end{aligned}$$

Since the sets $V_\varepsilon$ exhaust $V'$ and $h_1 \in L^2(V', \mu|_U)$, we can choose $\varepsilon$ small enough so that the second term in the estimate above becomes arbitrarily small. The first term tends to zero as $i, j \to \infty$ for any $\varepsilon > 0$ because $h_i \to h$ in $L^2(V_\varepsilon, \mu)$ by Lemma 5.5.2. $\square$

**Proposition 5.5.4** *The subsequence $\{h_i\}_{i=1}^\infty$ constructed in Lemma 5.5.2 converges to $h$ in the Hilbert space $\mathcal{F}(V)$ for every open set $V \subset U$ relatively compact in $X$. The limit function $h$ belongs to the space $\mathcal{F}^0_{loc}(U)$ and is a local (weak) solution of $Lh = 0$ in $U$ with weak Dirichlet boundary conditions on $\partial U$. Also the function $h$ vanishes quasi everywhere on $\partial U$.*

**Proof.** Let $h_i$ be the subsequence constructed in Lemma 5.5.2. Let $V \subset U$ be an open set in $U$ which is relatively compact in $X$, i.e. the closure $\overline{V}$ of $V$ in $X$ is compact. To show that the convergence $h_i \to h$ is in $\mathcal{F}(V)$, we set

$$\phi(x) = (1 - \rho(x, V))_+ = \max(1 - \rho(x, V), 0).$$

We know by [58, Lemma 1] that $\phi \in \mathcal{F}_{loc}(X)$ with $d\Gamma(\phi,\phi) \le d\mu$, thus $\phi \in \mathcal{F}_c(X)$. Let $V'$ be the support of $\phi$ in $X$ and let $V'' = V' \cap U$. Since $h_i \in \mathcal{F}^0_{loc}(B_i \setminus \{x_i\}, B_i)$, for large enough $i$ we know that $x_i \notin V''$ and $V'' \subset B_i$. Thus $V'' \subset B_i \setminus \{x_i\}$ and so $\phi h_i \in \mathcal{D}(\mathcal{E}^D_{B_i}) \subset \mathcal{D}(\mathcal{E}^D_U)$ by Proposition 5.3.3. Let $V'''$ be some neighborhood of $V''$ in $U$. Assume without loss of generality that $\rho_U(x_i, V''') > 0$.

If we prove that the sequence $\phi h_i$ is Cauchy in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$ then we would know that $h_i \to h$ in $\mathcal{F}(V)$, $\phi h \in \mathcal{D}(\mathcal{E}^D_U)$ and thus $h \in \mathcal{F}^0_{loc}(U)$ by Definition 2.4.3. Since the sequence $\phi h_i$ converges to $\phi h$ in $L^2(\widetilde{U}, \mu|_U)$, it is left to estimate the energy

$$
\begin{aligned}
\mathcal{E}^D_U(\phi(h_j - h_i), \phi(h_j - h_i)) &= \int_U d\Gamma(\phi(h_j - h_i), \phi(h_j - h_i)) \\
= \int_U (h_j - h_i)^2 d\Gamma(\phi,\phi) &+ 2\int_U \phi(h_j - h_i) d\Gamma(\phi, h_j - h_i) + \int_U \phi^2 d\Gamma(h_j - h_i, h_j - h_i) \\
= \int_U (h_j - h_i)^2 d\Gamma(\phi,\phi) &+ \int_U d\Gamma(h_j - h_i, \phi^2(h_j - h_i))
\end{aligned}
$$

Let's integrate by parts to get rid of the second term in the last line. Integrating by parts works because the function $h_j - h_i$ is a weak solution of $Lu = 0$ in $V'''$ with Dirichlet boundary conditions on $\partial U$. Also the function $\phi^2(h_j - h_i)$, which is zero in the open set $U \setminus V''$ is in $\mathcal{D}(\mathcal{E}^D_U)$ by Proposition 5.3.3, and therefore in $\mathcal{D}(\mathcal{E}^D_{V'''})$ thus can be approximated in $\mathcal{D}(\mathcal{E})$ by functions in $\mathcal{F}_c(V''')$. Therefore,

$$
\mathcal{E}^D_U(\phi(h_j - h_i), \phi(h_j - h_i)) \le \int_{V''} (h_j - h_i)^2 d\Gamma(\phi,\phi) \le \int_{V''} (h_j - h_i)^2 d\mu \to 0,
$$

as $i, j \to \infty$ because the sequence $h_i$ converges to $h$ in $L^2_{loc}(\widetilde{U}, \mu_U)$ and $V''$ is relatively compact in $\widetilde{U}$. Here we have used the inequality $d\Gamma(\phi,\phi) \le d\mu$. Therefore the sequence $\phi h_i$ is indeed Cauchy in the Hilbert space $\mathcal{D}(\mathcal{E}^D_U)$. In particular $h \in \mathcal{F}^0_{loc}(U)$ by Definition 2.4.3.

To show that $h$ is a weak solution of $Lh = 0$ in $U$ with weak Dirichlet boundary conditions on $\partial U$, take any test function $\phi \in \mathcal{F}_c(U)$ with support in any compact

$K \subset U$, let $W$ be a relatively compact in $U$ neighborhood of $K$ and write

$$\int_U d\Gamma_U(h,\phi)d\mu = \int_W d\Gamma_U(\lim_{i\to\infty} h_i, \phi) = \lim_{i\to\infty} \int_W d\Gamma_U(h_i,\phi) = \lim_{i\to\infty} \int_U d\Gamma_U(h_i,\phi) = 0$$

because each of the functions $h_i$ is a weak solution of $Lh_i = 0$ in $B_i \setminus \{x_i\}$. Here we have used that the sequence $h_i$ converges to $h$ in $\mathcal{F}(W)$ to interchange the operations of taking the limit and integration.

Choose any point $z \in \partial U$. To show that the function $h$ vanishes quasi everywhere on $\partial U \cap B(z,1)$ it remains to notice that each of the functions $h_i$ used to approximate $h$ does so in a controlled way. More specifically, let $g$ be any positive local weak solution of $Lg = 0$ in $U \cap B(z,A)$ vanishing q.e. on $\partial U \cap B(z,A)$, e.g., the Dirichlet Green function for the set $U \cap B(z,2A)$. For large enough $i$ and $j$ we can use the local boundary Harnack principle (5.28) to estimate $h_i$ in $B(z,1)$ by $cg$ for some positive constant $c$. We get $h_i \le cg$ for large enough $i$ and therefore $h \le cg$ and so the function $h$ vanishes q.e. on $U \cap B(z,1)$. This holds for any $z \in \partial U$ as desired. □

The next Lemma is an interesting result which could be used to alternatively show that $h \in \mathcal{F}^0_{loc}(U)$.

**Lemma 5.5.5** *Let $U$ be an open subset of $X$. Let $U_i$ be an exhaustion of the set $U$ and let $\{f_i\}_{i=1}^{\infty}$ be a sequence of functions such that $f_i \in \mathcal{F}_{loc}(U_i)$. Extend each of $f_i$ to all of $U$ by zero. Assume that for some bounded compactly supported in $\widetilde{U}$ function $\phi$ there exist constant $C_\phi$ and $N_\phi$ such that $\phi f_i \in \mathcal{D}(\mathcal{E}^D_U)$ and*

$$\mathcal{E}^D_U(\phi f_i, \phi f_i) \le C_\phi.$$

*for every index $i \ge N_\phi$. Assume that $f_i$ converges in $L^2_{loc}(\widetilde{U})$ to some function $f$. Then $f \in \mathcal{F}_{loc}(U)$, $\phi f \in \mathcal{D}(\mathcal{E}^D_U)$ and*

$$\mathcal{E}^D_U(\phi f, \phi f) \le C_\phi. \tag{5.29}$$

**Proof.** According to the spectral theorem for the nonnegative self-adjoint operator $L_U^D$ associated with the form $\mathcal{E}_U^D$, for any function $f \in \mathcal{D}(L_U^D)$ we have

$$L_U^D f = \int_0^\infty \lambda dE_\lambda(f)$$

where $E_{-\infty} = 0$, $E_\infty = Id$ and for every $\lambda_1, \lambda_2 \in [-\infty, +\infty]$ with $\lambda_1 > \lambda_2$, the expression $E_{\lambda_1} - E_{\lambda_2}$ is a bounded linear orthogonal projection operator on $L^2(U, \mu)$. In particular $E_\lambda$ is a self-adjoint operator of orthogonal projection.

Let $\langle \cdot, \cdot \rangle$ denote the inner product on $L^2(U, \mu)$. As in [31, (1.3.8)], for any two functions $f, g \in \mathcal{D}(\mathcal{E}_U^D)$ we can express $\mathcal{E}_U^D$ as a Lebesgue-Stiltjes integral

$$\begin{aligned} \mathcal{E}_U^D(f,g) &= \int_0^\infty \lambda d\langle E_\lambda(f), g\rangle \\ \mathcal{D}(\mathcal{E}_U^D) &= \left\{ f \in L^2(U,\mu) : \int_0^\infty \lambda d\langle E_\lambda(f), f\rangle < \infty \right\}. \end{aligned} \tag{5.30}$$

It suffices to prove (5.29). For any $f \in L^2(U, \mu)$ the quadratic form

$$R_\lambda(f) := \langle E_\lambda(f), f\rangle \leq \langle E_\lambda(f), f\rangle \leq \langle f, f\rangle \tag{5.31}$$

is a nonnegative nondecreasing function of $\lambda$ because for $\lambda_1 > \lambda_2$, the difference $\langle E_{\lambda_1}(f) - E_{\lambda_2}(f), f\rangle > 0$ is an inner product of $f$ and its orthogonal projection. Therefore $R_\lambda(f)$ is almost everywhere continuous function of $\lambda$. Also for fixed $\lambda \in \mathbb{R}$, $R_\lambda(\cdot) : L^2(U, \mu) \to \mathbb{R}$ is a continuous functional. Therefore

$$\begin{aligned} \mathcal{E}_U^D(\phi f, \phi f) &= \int_0^\infty \lambda d\langle E_\lambda(\phi f), \phi f\rangle = \limsup_{N\to\infty} \int_0^N \lambda dR_\lambda(\phi f) \\ &= \limsup_{N\to\infty} \left[ NR_N(\phi f) - \int_0^N R_\lambda(\phi f) d\lambda \right] \\ &= \limsup_{N\to\infty} \left[ N \lim_{i\to\infty} R_N(\phi f_i) - \int_0^N \lim_{i\to\infty} R_\lambda(\phi f_i) d\lambda \right] \end{aligned}$$

By (5.31), $0 \leq R_\lambda(\phi f_i) \leq ||\phi f_i||_2^2$, which is of bounded integral on $[0, N]$. Therefore by the dominated convergence theorem, we can continue

$$\begin{aligned}
\mathcal{E}_U^D(\phi f, \phi f) &= \limsup_{N\to\infty} \left[ N \lim_{i\to\infty} R_N(\phi f_i) - \lim_{i\to\infty} \int_0^N R_\lambda(\phi f_i) d\lambda \right] \\
&= \limsup_{N\to\infty} \lim_{i\to\infty} \int_0^N \lambda d R_\lambda(\phi f_i) \le \sup_{i>0, N>0} \int_0^N \lambda d R_\lambda(\phi f_i) \\
&= \sup_{i>0} \int_0^\infty \lambda d R_\lambda(\phi f_i) = \sup_{i>0} \mathcal{E}_U^D(\phi f_i, \phi f_i) \le C_\phi
\end{aligned}$$

in particular $\phi f \in \mathcal{D}(\mathcal{E}_U^D)$ by (5.30). □

**Lemma 5.5.6** *Let $X$ be a connected locally compact separable metric space, $\mu$ - a positive Radon measure on $X$ with full support and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - a strictly local regular Dirichlet form on $L^2(X, \mu)$ satisfying the conditions* (A1-A4) *of* Chapter 2*. Let $U$ be a uniform domain in $X$. Let $h$ be a function constructed in Lemma 5.5.2. Then the measure $h^2 d\mu$ on $U$ satisfies the following volume estimate*

$$V_{h^2}(x, R) = \int_{U \cap B(x,R)} h^2 d\mu \asymp h^2(x_R) \mu(B(x, R)) \tag{5.32}$$

*for any $x \in \overline{U}$, any $R > 0$ and any point $x_R$ with $\rho(x_R, x) = \frac{R}{4}$ and $\rho(x_R, \partial U) \ge \frac{c_1}{8} R$. The following doubling condition*

$$\forall x \in X, \forall R > 0, \quad V_{h^2}(x, 2R) \le C V_{h^2}(x, R) \tag{5.33}$$

*holds for some constant $C$ depending only on the constants $c_0, c_1, c_2, c_3$ appearing in* (2.12), (2.13) *and in* Definition 3.1.1.

**Proof.** Fix $x \in \overline{U}$ and $R > 0$. Let $x_R \in U$ be a point with $\rho(x_R, x) = \frac{R}{4}$ and $\rho(x_R, \partial U) \ge \frac{c_1}{8} R$ given by Lemma 4.1.5. We know by Proposition 5.5.4 that $h$ is a local weak solution in $U$ of $Lh = 0$. It suffices to prove (5.32) because the volume doubling condition (5.33) would follow from the doubling condition (2.12) for the measure $\mu$ and by comparing $h(x_R)$ to $h(x_{2R})$ using the Harnack principle and the curve $\gamma$ between $x_R$ and $x_{2R}$ given by the uniform condition (3.5).

Assume first that $\delta_U(x) = \rho(x, \partial U) > 2R$. Then the doubling condition (5.33) follows from the Harnack inequality for the function $h$ and the doubling condition for the measure $\mu$.

Assume now that $\delta_U(x) \leq 2R$ and choose a point $\xi \in \partial U$ with $\rho(x, \xi) \leq 2R$. Let $A_0$ be a constant appearing in Theorem 5.4.2. Let $\xi_{A_0 R}$ be a point in $U$ with $\rho(\xi, \xi_R) = 4A_0 R$ given by Lemma 5.4.6. For every $R > 0$ let $U_r$ denote $U \cap B(\xi, r)$ and let $G_r$ denote the Dirichlet Green function in $U_r$. Let $k = 20A_0 c_0$ where $c_0$ is a constant appearing in (3.5).

Both $h$ and $G_{kR}(\cdot, \xi_R)$ are in $\mathcal{F}^0_{loc}(U_{4A_0R}, U)$, both of these function are nonnegative weak solutions of $Lu = 0$ in $U_{4A_0R}$ and have a quasi-continuous representative that is vanishing quasi-everywhere on $\partial U \cap B(x, 4A_0 R)$. Since $h$ is vanishing q.e. on $\partial U$ by Proposition 5.5.4, we can use the boundary Harnack principle of Theorem 5.4.2 to see that

$$\frac{h(\cdot)}{h(x_R)} \asymp \frac{G_{kR}(\cdot, \xi_R)}{G_{kR}(x_R, \xi_R)} \text{ on } B(\xi, 5c_0 R)$$

which includes $B(\xi, 4R)$ and therefore includes $B(x, 2R)$. So if we denote $\varepsilon_1 = h(x_R)/G_{kR}(x_R, \xi_R)$ then we obtain

$$h(\cdot) \asymp \epsilon_1 G_{kR}(\cdot, \xi_R) \tag{5.34}$$

on $B(\xi, 4R)$. The lower estimate of (5.32),

$$\int_{U \cap B(x,R)} h^2 d\mu \geq \epsilon_2 h^2(x_R) \mu(B(x, R))$$

follows by the doubling condition (2.12) for the measure $\mu$ and the Harnack estimate for the function $h$ since $B(x_R, \frac{c_1}{16}R) \subset U \cap B(x, R)$ and $h$ is a positive weak solution of $Lh = 0$ in $B(x_R, \frac{c_1}{8}R) \subset U$. The upper estimate of (5.32),

$$\int_{U \cap B(x,R)} h^2 d\mu \leq \epsilon_3^2 h^2(x_R) \mu(B(x, R))$$

follows from the estimate $h(\cdot) \leq \epsilon_3 h(x_R)$ on $B(\xi, 4R)$. The latter estimate is true because of (5.34) and Lemma 5.3.2 which estimates the supremum of the Green's function $G_{kR}(\cdot, \xi_R)$ in $B(\xi, 4R)$ by its value at $x_R$. Here we have used that both $\xi_R$ and $x_R$ are in the ball $B(\xi, \frac{kR}{4c_0}) = B(\xi, 5A_0R)$. □

### 5.5.1 Dirichlet type Dirichlet forms obtained by the change of measure

Assume that the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ admits a carré du champ operator $\Upsilon : \mathcal{D}(\mathcal{E}) \times \mathcal{D}(\mathcal{E}) \to L^1(X, \mu)$. Let $U \subset X$ be an open set and let $v \in L^\infty_{loc}(U, \mu)$ be a locally uniformly positive and locally bounded measurable function on $U$. Similarly to the Neumann type forms of (4.22) we define the Dirichlet type form associated with the function $v$ in the following way

**Definition 5.5.7** *We set $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ to be the closure of a symmetric form*

$$\mathcal{E}_U^{D,v}(f,f) = \int_U v d\Gamma_U(f,f) = \int_U \Upsilon_U(f,f) v d\mu \tag{5.35}$$

*on $L^2(U, vd\mu)$ with initial domain $\mathcal{F}_c(U)$.*

Such a form is indeed closable since $\mathcal{F}_c(U) \subset \mathcal{D}(\mathcal{E}_U^{N,v})$ and the Neumann type form $(\mathcal{E}_U^{N,v}, \mathcal{D}(\mathcal{E}_U^{N,v}))$ is closed by the proof of Proposition 2.4.9. In particular

$$\mathcal{D}(\mathcal{E}_U^{D,v}) \subset \mathcal{D}(\mathcal{E}_U^{N,v}).$$

If we take the function $v$ to be constant one, the form we defined in (5.35) becomes $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$.

Notice that because of the special structure of this form, the normal contractions operate on $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$. The form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ is symmetric and densely defined in $L^2(U, vd\mu)$ since compactly supported in $U$ functions which are Lipschitz with respect to the metric $\rho$ are in $\mathcal{D}(\mathcal{E}_U^{D,v})$. It is also closed by definition, and so the form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ is Dirichlet. It is also strongly local because the form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is. So each of the forms $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ is associated with the nonnegative self-adjoint operator $L_U^{D,v}$ and a self-adjoint semigroup $P_{U,t}^{D,v}$ on $L^2(U, vd\mu)$. It is straightforward to see that the energy measure associated with the form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ on $L^2(U, vd\mu)$ by (2.2) is simply

$$d\Gamma^v(f,g) = vd\Gamma(f,g) = \Upsilon(f,g)vd\mu.$$

and so the Radon-Nikodym derivative of $d\Gamma^v$ with respect to the reference measure $vd\mu$ is

$$\Upsilon^v(f,g) = \frac{d\Gamma^v(f,g)}{vd\mu} = \Upsilon(f,g) \tag{5.36}$$

In view of the main equivalence Theorem 2.6.1 notice that the volume doubling condition (2.12) for the measure $vd\mu$ and the Poincaré inequalities for the Dirichlet form $(\mathcal{E}_U^{D,v}, \mathcal{D}(\mathcal{E}_U^{D,v}))$ follow from the same estimate for the Neumann type form $(\mathcal{E}_U^{N,h^2}, \mathcal{D}(\mathcal{E}_U^{N,h^2}))$ on $L^2(U, vd\mu)$. We will use this fact to obtain the heat kernel estimates for the Dirichlet form $(\mathcal{E}_U^{D}, \mathcal{D}(\mathcal{E}_U^{D}))$. Specifically, in the next section we will explore the technique of $h$-transform which, if $h$ is a harmonic function, will in fact produce the form $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2}))$.

## 5.6 $h$-transform

In this section we will develop the technique of $h$-transform that allows one to construct a family of symmetric forms associated with a Dirichlet form. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a Dirichlet form on $L^2(X, \mu)$ and let $h$ be a measurable positive function on $X$.

**Definition 5.6.1** *Let $H$ be a multiplication by $h$, as a unitary map:*

$$H : L^2(X, h^2 d\mu) \to L^2(X, \mu), f \to hf$$

*and let $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$, $L_h$ and $P_{h,t}$ be the pulled-back form, operator and semigroup* *on $L^2(X, h^2 d\mu)$ defined by*

$$\begin{aligned} \mathcal{E}_h(f,g) &= \mathcal{E}(hf, hg), \qquad \mathcal{D}(\mathcal{E}_h) = H^{-1}\mathcal{D}(\mathcal{E}) \\ L_h &= H^{-1} \circ L \circ H, \quad \mathcal{D}(L_h) = H^{-1}\mathcal{D}(L) \\ P_{h,t}(f) &= H^{-1} \circ P_t \circ H \end{aligned} \tag{5.37}$$

The form $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$ is a closed symmetric densely defined form on $L^2(X, h^2 d\mu)$ by the unitary nature of the map $H$. This form corresponds to the semigroup $P_{h,t}$ and the operator $L_h$ on $L^2(X, h^2 d\mu)$ in the usual way. The form $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$ is not, however, Dirichlet for general function $h$ because it is usually not Markovian, i.e. normal contractions do not operate on $(\mathcal{E}_h, \mathcal{D}(\mathcal{E}_h))$. It is Markovian if and only if the semigroup $P_{h,t}$ is Markovian, i.e. if and only if $P_{h,t}1 \leq 1$ a.e. in $X$. This happens if and only if $P_t h \leq h$ a.e. in $X$. Here $P_{h,t}$ and $P_t$ are understood as integral operators, initially defined on $L^2(X, h^2 d\mu)$ and $L^2(X, d\mu)$ respectively.

The following statements are immediate from Definition 5.6.1.

**Lemma 5.6.2** *Assume that the linear space $W$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E})$. Then the set $H^{-1}(W)$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}_h)$.*

**Lemma 5.6.3** *If the semigroup $P_t$ possesses a kernel $p(t,x,y)$ with respect to the measure $\mu$, then the semigroup $P_{h,t}$ also possesses a kernel with respect to the measure $h^2 d\mu$. This kernel $p_h(t,x,y)$ is related to the kernel of the semigroup $p(t,x,y)$ by*

$$p(t,x,y) = p_h(t,x,y)h(x)h(y) \tag{5.38}$$

**Proof.** By definition for any function $f \in L^2(X, \mu)$, we have

$$P_{h,t}f(x) = \frac{1}{h}P_t(hf) = \frac{1}{h(x)}\int_X p(t,x,y)f(y)h(y)d\mu(y) = \int_X \frac{p(t,x,y)}{h(x)h(y)}f(y)h^2(y)d\mu(y)$$

and therefore the function

$$\frac{p(t,x,y)}{h(x)h(y)}$$

is the kernel of the semigroup $P_{h,t}$ with respect to the measure $h^2 d\mu$. $\square$

Let us now focus on the $h$-transform of the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$, which will be denoted by $(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$.

**Lemma 5.6.4** *Assume that the function $h \in \mathcal{F}_{loc}(U)$ is locally finite and locally uniformly positive on $U$. Then the set $H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U,\mu))$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}_{U,h}^D)$, and*

$$H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U,\mu)) = \mathcal{F}_c(U) \cap L^\infty(U, h^2 d\mu) \tag{5.39}$$

**Proof.** The set $H^{-1}(\mathcal{F}_c(U) \cap L^\infty(U,\mu))$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}_{U,h}^D)$ because the linear operator is unitary and the set $\mathcal{F}_c(U) \cap L^\infty(U,\mu)$ is dense in the Hilbert space $\mathcal{D}(\mathcal{E}_U^D)$. Since both $h, \frac{1}{h} \in \mathcal{F}_{loc}(U) \cap L^\infty_{loc}(U,\mu)$, the equality (5.39) follows because the space $\mathcal{F}_{loc}(U) \cap L^\infty_{loc}(U,\mu)$ is an algebra by Lemma 2.2.1. $\square$

It turns out that if $h$ is a weak solution of $Lh = 0$ in $U$ then the form $(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$ is a Dirichlet form because it coincides with the form $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2}))$ obtained from $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ by the change of measure.

**Proposition 5.6.5** *Assume that $h$ is a weak local solution of $Lh = 0$ in $U$. Then the form $(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$ coincides with the form $(\mathcal{E}_U^{D,h^2}, \mathcal{D}(\mathcal{E}_U^{D,h^2}))$ defined in* (5.35).

**Proof.** For both forms, the space $\mathcal{F}_c(U) \cap L^\infty(U, d\mu)$ is a dense subset of the domain by Lemma 5.6.4 and by Definition 5.5.7. It remains to compare these forms on this space. For any $g \in \mathcal{F}_c(U) \cap L^\infty(U,\mu)$, by Lemma 2.2.1 we know that the functions $g, g^2, gh, g^2h$ belong to the space $\mathcal{F}_{loc}(U)$ and thus to the space $\mathcal{F}_c(U)$ since they are compactly supported in $U$. Using the chain rule we have

$$\begin{aligned}
\mathcal{E}_{U,h}^D(g,g) &= \int_U d\Gamma_U(hg,hg) = \int_U g^2 d\Gamma_U(h,h) + 2\int_U gh d\Gamma_U(g,h) + \int_U h^2 d\Gamma_U(g,g) \\
&= \int_U d\Gamma_U(h, g^2h) + \int_U h^2 d\Gamma_U(g,g) = \int_U h^2 d\Gamma_U(g,g) = \mathcal{E}_U^{D,h^2}(g,g)
\end{aligned}$$

because $g^2h \in \mathcal{F}_c(U)$ and $h$ is a weak solution in $U$ of $Lh = 0$ by assumption. $\square$

## 5.7 Proof of Theorem 5.0.8

In this section we will prove Theorem 5.0.8, so the context of this section is that of a Harnack-type Dirichlet space $(X, \mu, \rho, \mathcal{E}, \mathcal{D}(\mathcal{E}))$ together with an inner uniform subset $U \subset X$. Let $L$ be a nonnegative self-adjoint operator on $L^2(U, \mu)$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. We are interested in the the heat kernel associated with the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$.

Notice that the form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ can also be obtained from the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ on $\widetilde{U}$ by considering Dirichlet boundary value diffusion problem in $U$, i.e.

$$\begin{aligned} \mathcal{D}(\mathcal{E}_U^D) &= \mathcal{D}((\mathcal{E}_U^N)_U^D) \\ \mathcal{E}_U^D(f,g) &= (\mathcal{E}_U^N)_U^D(f,g) = \int_U d\Gamma_U(f,g), \text{ whenever } f, g \in \mathcal{D}(\mathcal{E}_U^D) \end{aligned}$$

The advantage of this approach is that now $U$ is a uniform domain in $(\widetilde{U}, \rho_U)$, rather than only an inner uniform domain - and the theory developed in Chapter 5 applies, because according to Theorem 4.0.5 the form $(\mathcal{E}_U^N, \mathcal{D}(\mathcal{E}_U^N))$ is a strongly local regular Dirichlet form on $\widetilde{U}$ of Harnack type, see Definition 2.5.1.

The boundary Harnack principle proved in Theorem 5.4.2 provides the basis for the construction of a réduite function $h$ on $U$ that is carried out in section 5.5. The function $h$ is in $\mathcal{F}^0_{loc}(U)$ and is a nonnegative local (weak) solution in $U$ of $Lh = 0$ with weak Dirichlet boundary conditions on $\partial U$ by Proposition 5.5.4. In fact the function $h$ is positive on $U$ by Harnack inequality in any compact subset of $U$, because $h(y) = 1$ for the point $y$ chosen in Chapter 5.5.

Let $(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$ denote the $h$-transform of the Dirichlet form $(\mathcal{E}_U^D, \mathcal{D}(\mathcal{E}_U^D))$ on $L^2(U, \mu)$. The following important lemma relates the closed form $(\mathcal{E}_{U,h}^D, \mathcal{D}(\mathcal{E}_{U,h}^D))$ to the Dirichlet form $(\mathcal{E}_U^{N,h^2}, \mathcal{D}(\mathcal{E}_U^{N,h^2}))$ defined in (4.22).

**Proposition 5.7.1** *Let $h$ be a positive local (weak) solution of $Lh = 0$ in $U$ with weak Dirichlet boundary conditions on $\partial U$. Assume that the measure $h^2 d\mu$ satisfies* *the doubling condition* (4.27) *for some* $\epsilon > 0$*. Then the closed form* $(\mathcal{E}^D_{U,h}, \mathcal{D}(\mathcal{E}^D_{U,h}))$ *coincides with the regular Dirichlet form* $(\mathcal{E}^{N,h^2}_U, \mathcal{D}(\mathcal{E}^{N,h^2}_U))$ *on* $\widetilde{U}$ *defined by* (4.22)*.*

**Proof.** Since the forms in question are closed, it suffices to compare their cores and the values of these form on each of the functions in their cores. A space $\mathrm{Lip}_c(\widetilde{U})$ is a core for the form $(\mathcal{E}^{N,h^2}_U, \mathcal{D}(\mathcal{E}^{N,h^2}_U))$, by Proposition 4.2.1, while $\mathcal{F}_c(U) \cap L^\infty(U, h^2\mu)$ is a core for the form $(\mathcal{E}^D_{U,h}, \mathcal{D}(\mathcal{E}^D_{U,h}))$ by Lemma 5.6.4. For any $f \in \mathrm{Lip}_c(\widetilde{U})$, we have $f \in \mathcal{D}(\mathcal{E}^N_U)$ and by Lemma 5.2.3, applied to the Dirichlet form $(\mathcal{E}^N_U, \mathcal{D}(\mathcal{E}^N_U))$ instead of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$, we have $hf \in \mathcal{D}(\mathcal{E}^D_U)$. Therefore by Definition 5.6.1, $f \in \mathcal{D}(\mathcal{E}^D_{U,h})$. So

$$\mathrm{Lip}_c(\widetilde{U}) \subset \mathcal{D}(\mathcal{E}^D_{U,h})$$

and therefore it suffices to check that the two forms in question coincide on $\mathcal{F}_c(U) \cap L^\infty(U, h^2\mu)$, which is a core for the form $(\mathcal{E}^D_{U,h}, \mathcal{D}(\mathcal{E}^D_{U,h}))$. For any $g \in \mathcal{F}_c(U) \cap L^\infty(U, h^2\mu)$, by Lemma 2.2.1 we know that the functions $g, g^2, gh, g^2h$ belong to the space $\mathcal{F}_{loc}(U)$ and thus to the space $\mathcal{F}_c(U)$ since they are compactly supported in $U$. Using the chain rule we have

$$\begin{aligned}\mathcal{E}^D_{U,h}(g,g) &= \int_U d\Gamma_U(hg, hg) = \int_U g^2 d\Gamma_U(h,h) + 2\int_U gh d\Gamma_U(g,h) + \int_U h^2 d\Gamma_U(g,g) \\ &= \int_U d\Gamma_U(h, g^2h) + \int_U h^2 d\Gamma_U(g,g) = \int_U h^2 d\Gamma_U(g,g) = \mathcal{E}^{N,h^2}_U(g,g)\end{aligned}$$

because $g^2h \in \mathcal{F}_c(U)$ and $h$ is a weak solution in $U$ of $Lh = 0$ by assumption. $\square$

Theorem 5.0.8 follows as the exact translation of results from Theorem 4.2.7 using the relationships between different Dirichlet forms, kernels, semigroups and self-adjoint operators established in Lemma 5.6.3 and in Proposition 5.7.1. This completes the proof of Theorem 5.0.8 and Theorem 1.3.3.

Theorem 1.3.4 follows from the parabolic Harnack estimate of Theorem 4.2.7 using the following relationship between the classical solutions and the weak solutions of the heat equation with Dirichlet boundary conditions that we present in the last proposition of this section. We will need the following notation.

For any set $V \subset \widetilde{U}$ we denote by $C^\infty(V)$ the set of smooth functions $f$ on $V \cap U$ such that for any $y \in V \setminus U$ and any integer $k \geq 0$, the limit

$$\lim_{U \cap V \ni x \to y} f^{(k)} \text{ exists .}$$

**Proposition 5.7.2** *Let $U$ be a domain in $\mathbb{R}^n$. Let $h$ be a positive harmonic function in $U$ that belongs to $\mathcal{F}^0_{\mathrm{loc}}(\widetilde{U})$. Set $d\nu = h^2 d\mu$. Let $I$ be an open time interval, $\Omega$ be an open set in $\widetilde{U}$. Set $Q = I \times \Omega$. Let $u$ be a continuous function on $Q$ which vanishes on $I \times (\Omega \cap (\widetilde{U} \setminus U))$, is once continuously differentiable in time, twice continuously differentiable in space and satisfies $\partial_t u + \Delta u = 0$ in $\Omega \cap U$. Then $v = u/h$ is a weak solution of the heat equation in $I \times \Omega$ in the sense of* Definition 2.2.4 *for the Dirichlet form $(\mathcal{E}^{N,h^2}_U, \mathcal{D}(\mathcal{E}^{N,h^2}_U))$ on $L^2(\widetilde{U}, d\nu)$.*

**Proof.** This is essentially well known. For instance, [30, Corollary 2.3] is a very similar (essentially equivalent) statement. However, we do not know of a proper reference making use as we do here of the set $\widetilde{U}$. Since this is an important technical result, we give a complete proof. Without loss of generality, we can assume that $u$ is bounded on $Q$ (simply replace $Q$ by an arbitrary $Q' = I' \times \Omega'$ relatively compact in $I \times \Omega$). For every $\epsilon \in (0,1)$, let $G_\epsilon$ be a smooth function of one real variable such that $G_\epsilon, G'_\epsilon, G''_\epsilon \geq 0$, $G_\epsilon$ vanishes on $(-\infty, \epsilon]$ and $G'_\epsilon \equiv 1$ on $(3\epsilon, \infty)$. Given $u$ as above, set $u_\epsilon = G_\epsilon(\sqrt{u^2 + \epsilon^2} - \epsilon)$ on $Q$. This function has the same smoothness property as $u$ and vanishes on $\{u^2 \leq 3\epsilon^2\}$. Moreover, a simple computation shows that $\frac{\partial}{\partial t} u_\epsilon + \Delta u_\epsilon \leq 0$ on $\Omega \cap U$. Let $\phi \in C^\infty(\widetilde{U})$ with compact support in $\Omega$ and $0 \leq \phi \leq 1$. Note that $\phi u_\epsilon$ has compact support in $\Omega \cap \{u^2 > 3\epsilon^2\} \subset U$. Now, using the inequality satisfied by $u_\epsilon$ and integrating by parts, we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega |\phi u_\epsilon|^2 d\mu\right) + \int |\nabla(\phi u_\epsilon)|^2 d\mu &\le -\int_\Omega \phi^2 u_\epsilon \Delta u_\epsilon + \int \phi u_\epsilon \Delta(\phi u_\epsilon) d\mu \\
&= \int_\Omega u_\epsilon^2 \phi \Delta \phi d\mu - \int_\Omega \phi u_\epsilon \nabla\phi \cdot \nabla u_\epsilon d\mu \\
&= \int_\Omega u_\epsilon^2 \phi \Delta\phi d\mu - \int_\Omega u_\epsilon \nabla\phi \cdot \nabla(\phi u_\epsilon) d\mu + \int_\Omega u_\epsilon^2 |\nabla\phi|^2 d\mu \\
&\le C_\phi \int_\Omega u_\epsilon^2 d\mu + \frac{1}{2}\int_\Omega |\nabla(\phi u_\epsilon)|^2 d\mu.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t}\int_\Omega |\phi u_\epsilon|^2 d\mu + \int |\nabla(\phi u_\epsilon)|^2 d\mu \le 2C_\phi \int_\Omega u_\epsilon^2 d\mu.$$

Multiplying $\phi u_\epsilon$ by an appropriate cutoff function in time and integrating in time yields (after some simple manipulations)

$$\sup_{I'}\left[\int_{\Omega'} |u_\epsilon|^2 d\mu\right] + \int_{Q'} |\nabla u_\epsilon|^2 dt d\mu \le C(Q')\int_Q u_\epsilon^2 dt d\mu \tag{5.40}$$

for any $Q' = I' \times \Omega'$ relatively compact in $Q$. Next, observe that $\operatorname{sgn}(u)u_\epsilon$ tends to $u$ in $L^2(Q)$ and that $|\nabla u_\epsilon|$ tends to $|\nabla u|$ pointwise in $Q'$. Hence we also have

$$\sup_{I'}\int_{\Omega'} |u|^2 d\mu + \int_{Q'} |\nabla u|^2 dt d\mu \le C(Q')\int_Q |u|^2 dt d\mu \tag{5.41}$$

By straightforward variant of Lemma 5.5.5 for functions of time and space it follows that, for any function $\phi \in C^\infty(\widetilde{U})$ with compact support in $\Omega$, the function $w = \phi u(t, \cdot)$ is in $\mathcal{D}(\mathcal{E}_U^D)$ for a.e. $t \in I'$ and satisfies

$$\int_{Q'} |w|^2 + |\nabla w|^2 dt d\mu \le C(\phi, Q')\int_Q |u|^2 dt d\mu.$$

Moreover, for any $\psi \in \mathcal{D}(\mathcal{E}_U^D)$ and a.e. $t \in I'$, we have

$$\begin{aligned}
\left|\int_U \psi \frac{\partial}{\partial t} w d\mu\right| &= \left|\int_U \psi\phi\Delta u d\mu\right| \\
&\le \left|\int_U \psi\Delta(\phi u) d\mu\right| + \int_U |\psi|[|u\Delta\phi| + |\nabla u \cdot \nabla\phi|] d\mu \\
&\le \int_U |\nabla\psi \cdot \nabla(\phi u)| d\mu + \int_U |\psi|[|u\Delta\phi| + |\nabla u \cdot \nabla\phi|] d\mu \\
&\le C_1(\phi, Q')\left(\int_{\Omega'} |u|^2 + |\nabla u|^2 d\mu\right)^{1/2} \|\psi\|_{\mathcal{D}(\mathcal{E}_U^D)}.
\end{aligned}$$

for some constant $C_1$ depending on $\phi$ and $Q'$. It follows that $\frac{\partial}{\partial t}w$ belongs to the dual $\mathcal{D}'(\mathcal{E}_U^D)$ of $\mathcal{D}(\mathcal{E}_U^D)$ and that

$$\int_{I'} \|\partial_t w\|^2_{\mathcal{D}'(\mathcal{E}_U^D)} dt \leq C_2(\phi, Q') \int_Q |u|^2 dt d\mu.$$

We want to show that the function $v = u/h$ is in the space $\mathcal{F}_{\text{loc}}(I \times \Omega)$ used in Definition 2.2.4. For this it suffices to show that, for any $\phi \in C^\infty(\widetilde{U})$ with compact support in $\Omega'$, we have $\phi v \in \mathcal{F}(I' \times U)$, that is,

$$\phi v \in L^2(I' \to \mathcal{D}(\mathcal{E}_U^{N,h^2}))$$

and

$$\frac{\partial}{\partial t}(\phi v) \in L^2(I' \to \mathcal{D}'(\mathcal{E}_U^{N,h^2})).$$

By Proposition 5.7.1, $\mathcal{D}(\mathcal{E}_U^{N,h^2}) = h^{-1}\mathcal{D}(\mathcal{E}_U^D)$. Therefore, the two desired conclusions for $\phi v$ follow directly from the estimates of $w = \phi u$ given above. $\square$

# Appendix A

# Uniform domains

**Proposition A.0.3 (postponed from Chapter 3.2)** *Let $U$ be a domain of the form $U = \mathbb{R}^n \setminus V$ for some closed convex set $V \subset \mathbb{R}^n$. Then*

*(1) The domain $U \subset \mathbb{R}^n$ is inner uniform with $c_0 = 21$, $c_1 = 1/462$.*

*(2) For any $x, y \in U$ there exists $z \in U$ such that $\rho_U(x,z) + \rho_U(z,y) \leq 4\rho(x,y)$ and $\rho_U(x,z) \leq 4|x-z|$, $\rho_U(z,y) \leq 4|y-z|$.*

**Proof of Proposition A.0.3** This result is not as obvious as it may first appear and the proof is somewhat technical. We need some notation. For any $x \in U$, let $z(x)$ be the closest point of $V$. Set $\vec{u}(x) = (x - z(x))/|x - z(x)|$. Both $z(x)$ and $u(x)$ are continuous functions of $x$. See, e.g., [32, pages 11–12].

**Claim.** *For any two points $x, y \in U$ with $\min\{\rho_U(x,V), \rho_U(y,V)\} = r > 0$, there exists an absolutely continuous curve $\gamma \subset U$ joining $x$ to $y$, of length at most $4(\rho_U(x,y) + 2r)$ such that $\rho_U(\gamma, V) \geq r$.*

Proposition A.0.3(1) easily follows form this claim. Indeed, let $x, y$ be points in $U$ with $R = \rho_U(x,y)$, $r = \min\{\rho_U(x,V), \rho_U(y,V)\}$. If $R \leq r$ the straight line segment $[x,y]$ from $x$ to $y$ is contained in $U$. Moreover, $[x,y]$ is contained in a half-space $E$ contained in $U$ (to see this, consider a point $\xi$ of $[x,y]$ such that $\rho_U([x,y],V) = \rho_U(\xi,V)$). The semi-circle with diameter $[x,y]$ contained in $E$ and orthogonal to the hyperplane bounding $E$ yields a curve of length $\pi\rho_U(x,y) = \pi|x-y|$ such that

$$\rho_U(z,V) \geq \frac{|z-x||z-y|}{|x-y|} = \frac{\rho_U(z,x)\rho_U(z,y)}{\rho_U(x,y)}.$$

Consider now the case where $R > r$. Let $x_R = x + (R/2)\vec{u}(x)$, $y_R = y +$

$(R/2)\vec{u}(y)$ and let $\gamma'$ be the curve joining $x_R$ to $y_R$ given by the claim. Note that

$$\min\{\rho_U(x_R, V), \rho_U(y_R, V)\} \in (R/2, 3R/2).$$

Hence, $\rho_U(\gamma', V) \geq R/2$ and $\gamma'$ has length at most $4(\rho_U(x_R, y_R) + 3R) \leq 20R$. Let $\gamma$ be the absolutely continuous curve that goes straight from $x$ to $x_R$, then from $x_R$ to $y_R$ following $\gamma'$, and finally straight from $y_R$ to $y$. By construction, the length of $\gamma$ is at most $21R$ and for any point $z$ on $\gamma$,

$$\rho_U(z, V) \geq \begin{cases} \rho_U(z, x) & \text{if } z \in [x, x_R] \\ R/2 & \text{if } z \in \gamma' \\ \rho_U(z, y) & \text{if } z \in [y_R, y]. \end{cases}$$

If $z \in [x, x_R]$ (resp. $z \in [y_R, y]$) then we have $\rho_U(z, y) \leq 3R/2$ (resp. $\rho_U(z, x) \leq 3R/2$) and thus

$$\rho(z, V) \geq \frac{2}{3} \frac{\rho_U(z, x)\rho_U(z, y)}{\rho_U(x, y)}.$$

If $z \in \gamma$ then $\rho_U(z, x)\rho_U(z, y) \leq 231R^2$ and thus

$$\rho(z, V) \geq \frac{1}{462} \frac{\rho_U(z, x)\rho_U(z, y)}{\rho_U(x, y)}.$$

To finish the proof of Proposition A.0.3(1) we are now left with the task of proving the claim made above.

For any $x \in U$, let $H_x$ be the linear hyperplane orthogonal to $\vec{u}(x)$. By construction $V$ is contained in the half-space $\{\xi : (\xi - z(x)) \cdot \vec{u}(x) \leq 0\}$ and we have

$$\rho_U((x + H_x), V) = \rho_U(x, V).$$

Fix two points $x, y \in U$ with $\min\{\rho_U(x, V), \rho_U(y, V)\} = r > 0$ and set

$$\alpha = \alpha(x, y) = \vec{u}(x) \cdot \vec{u}(y).$$

If $\alpha = 1$ we must have $H_x = H_y$ and it follows that the straight line segment $[x, y]$ satisfies the conditions required in the claim. Assume next that $\alpha \in (-\sqrt{2}/2, 1)$ and let $P$ be the $(n-2)$ dimensonal vector space $H_x \cap H_y$. The unit vectors

$$\vec{v}(y) = (1-\alpha^2)^{-1/2}(\vec{u}(x) - \alpha\vec{u}(y)) \in H_y, \;\; \vec{v}(x) = (1-\alpha^2)^{-1/2}(\vec{u}(y) - \alpha\vec{u}(x)) \in H_x$$

are orthogonal to $P$ and have scalar product

$$\vec{v}(x) \cdot \vec{v}(y) = (1-\alpha^2)^{-1}(-\alpha + \alpha^3) = -\alpha.$$

We can write (uniquely)

$$y - x = \vec{p} + a\vec{v}(x) + b\vec{v}(y), \;\; p \in P, a, b \in \mathbb{R}.$$

Since $x, y \in U$, we must have $(x-y) \cdot \vec{u}(y) > 0$ and $(y-x) \cdot \vec{u}(x) > 0$, that is, $a < 0$, $b > 0$. Thus if $\alpha \geq -1/2$,

$$|y-x|^2 = |p|^2 + a^2 + b^2 - 2ab\alpha \geq |p|^2 + \frac{1}{4}(a^2 + b^2).$$

Consider the curve $\gamma$ made of the three straight line segments

$$\begin{cases} [x, x + a\vec{v}(x)] & \subset x + H_x, \\ [x + a\vec{v}(x), y - b\vec{v}(y)] & \subset x + a\vec{v}(x) + P \subset (x + H_x) \cap (y + H_y), \\ [y - b\vec{v}(y), y] & \subset y + H_y. \end{cases}$$

Its length is

$$|p| + |a| + |b| \leq \sqrt{3}\sqrt{|p|^2 + a^2 + b^2} \leq 2\sqrt{3}|y - x| \leq 2\sqrt{3}\rho_U(x, y)$$

and

$$\rho_U(\gamma, V) = \min\{\rho_U(x, V), \rho_U(y, V)\}.$$

Thus $\gamma$ satisfies the conditions required in the claim.

Finally, consider the case when $\alpha \in [-1, -\sqrt{2}/2]$. Let $\gamma'$ be an absolutely continuous path in $U$ from $x$ to $y$ of length $\lambda\rho_U(x, y)$ for some arbitrary $\lambda > 1$. Let $[0, T] \ni t \mapsto \gamma(t)$ be the arclength parametrization of $\gamma$. and let $\alpha(t) = \vec{u}(x) \cdot \vec{u}(\gamma(t))$. The function $t \mapsto \alpha(t)$ is continuous and varies from $\alpha(0) = |\vec{u}(x)|^2 = 1$ to $\alpha(T) = \vec{u}(x) \cdot \vec{u}(y) = \alpha \in [-1, -1/2]$. Hence there exists $t_0 \in [0, T]$ such that $\alpha(t_0) = 0$. Let $x_0 = \gamma(t_0)$. As the unit vectors $, \vec{u}(x), \vec{u}(x_0), \vec{u}(y)$ satisfy $\vec{u}(x) \cdot \vec{u}(x_0) = 0$, $\vec{u}(x) \cdot \vec{u}(y) < -\sqrt{2}/2$, we must have $|\vec{u}(x_0) \cdot \vec{u}(y)| \le \sqrt{2}/2$. Observe however that $x_0$ may be closer to $V$ than $x$ and $y$. Thus, let $x_0' = x_0 + r\vec{u}(x_0)$ so that

$$\vec{u}(x_0') = \vec{u}(x_0), \quad \rho_U(x_0', V) > r, \quad \rho_U(x, x'0) + \rho_U(x_0', y) \le 2r + \lambda\rho_U(x, y)$$

As $\vec{u}(x) \cdot \vec{u}(x_0') = 0$ and $\vec{u}(x_0') \cdot \vec{u}(y) \ge -\sqrt{2}/2$, the argument above yields curves $\gamma_1, \gamma_2$ from $x$ to $x_0'$ and $x_0'$ to $y$ which stay at least distance $r$ away from $V$ and have length at most $2\sqrt{3}\rho_U(x, x_0')$, $2\sqrt{3}\rho_U(x_0', y)$, respectively. Putting $\gamma_1, \gamma_2$ together we obtain a curve from $x$ to $y$ that stays at distance at least $r$ away from $V$ and has length at most $2\sqrt{3}(\lambda\rho_U(x, y) + 2r)$. Picking $\lambda > 1$ close enough to 1 proves the claim. This finishes the proof of Proposition A.0.3(1). In addition, the argument above also proves Proposition A.0.3(2). Indeed, if $\vec{u}(x) \cdot \vec{u}(y) \ge -\sqrt{2}/2$, take $z = y$ and if $\vec{u}(x) \cdot \vec{u}(y) < -\sqrt{2}/2$, take $z = x_0$. $\square$

**Proposition A.0.4 (postponed from Chapter 3.2)** *Let $U$ be a domain in $\mathbb{R}^n$ above the graph of a Lipschitz function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ with Lipschitz constant $k$. Then $U$ is $(c_0, c_1)$-uniform with respect to the usual metric in $\mathbb{R}^n$, with $c_0 = 4k + 3$ and $c_1 = (2k + 2)^{-2}$.*

**Proof.** Given any two points $x, y \in U$, let $R = \rho(x, y)$ be the Euclidean distance between $x$ and $y$. Let $\vec{e}_n$ be the unit vector pointing 'up', in relationship to the graph of the function $\Phi$. Consider the path $\gamma$ consisting of three line segments:

$$(x, x'),\ (x', y') \text{ and } (y', y),$$

where

$$x' = x + (2k+1)R\vec{e}_n, \text{ and } y' = y + (2k+1)R\vec{e}_n$$

We have $\rho(x', \partial U) \geq 2R$ and $\rho(y', \partial U) \geq 2R$, while $\rho(x', y') = R$, so the second segment of the curve $\gamma$ is at least $R$ away from $\partial U$. The length of the path $\gamma$ is at most $(4k+3)R$. It remains to confirm that on the first segment of the path $\gamma$, for $z = x + t\vec{e}_n$ with $t \leq (2k+1)R$,

$$\rho(z, \partial U) \geq c_1 t \frac{\rho(z, y)}{R}$$

Using the Lipschitz nature of the function $\Phi$, after a simple trigonometry exercise we obtain

$$\rho(x + t\vec{e}_n, \partial U) \geq \frac{t}{\sqrt{k^2+1}} \geq \frac{t}{k+1} \geq t \frac{\rho(z, y)}{(k+1)(2k+2)R} \geq c_1 t \frac{\rho(z, y)}{R}$$

□

**Proposition A.0.5 (postponed from Chapter 3.2)** *The interior and the exterior of von Koch snowflake discussed in* Chapter 3.2 *is a uniform domain in* $\mathbb{R}^2$.

**Proof.** Let $U$ denote the interior of von Koch snowflake. Let $x$ and $y$ be any two points in the interior of von Koch snowflake. Then both $x$ and $y$ belong to one of the triangles that were part of the iterative construction. Say, $x \in T_0$ for some triangle $T_0$ which was constructed on the $n$-th iteration. Consider the sequence $\{T_i\}_{i=1}^k$ of triangles constructed in the following way. Let $T_1$ be the triangle which side serves as a base $b(T_0)$ of the triangle $T_0$, let $T_2$ be the triangle which side serves as a base $b(T_1)$ of the triangle $T_1$, etc., until $T_k$ is the main triangle $T$ of the von Koch snowflake. Let $\{T_i'\}_{i=1}^l$ be a similar sequence for the point $y \in T_0'$.

Let 1 be the side length of the main triangle in the von Koch snowflake, and let $R = \rho(x, y)$ be the Euclidean distance between $x$ and $y$. Without loss of generality we can assume that $T_{k-1} \neq T_{l-1}'$, or otherwise we can zoom in and consider the triangle $T_{k-1}$ as the main triangle of von Koch snowflake.

Since $x$ and $y$ are located in different triangles and since the Euclidean distance $\rho$ is comparable to the inner geodesic distance $\rho_U$ in the interior of von Koch snowflake, we know that

$$\rho_U(x, b(T_{k-1})) \leq CR, \quad \text{and} \quad \rho_U(y, b(T_{l-1}')) \leq CR$$

for some positive universal constant $C$. Let $\gamma'$ be the geodesic curve in $U$ connecting $x$ to the base $b(T_{k-1})$ and let $x_i' = \gamma' \cap b(T_i)$, $i = 0, \ldots, k-1$ . Let $|T_i|$ denote the length of the edge of the triangle $T_i$. Let $x_i$ be the closest point in the base $b(T_i)$ to $x_i'$ with

$$\rho_U(x_i, \partial U) \geq \min\left(\frac{R}{8}, \frac{|T_i|}{4}\right), \tag{A.1}$$

so that

$$\rho_U(x_i, x_i') \leq \min\left(\frac{R}{4}, \frac{|T_i|}{2}\right)$$

and the sequence $\{x_i\}_{i=0}^{k-1}$ of points $x_i \in b(T_i)$ satisfies

$$\begin{aligned} \sum_{j=1}^{k-1} \rho_U(x_i, x_{i-1}) &\leq L(\gamma') + \sum_{i=0}^{k-1} 2\rho_U(x_i, x_i') \\ &\leq CR + \sum_{i\,:\,|T_i| \leq R/2} |T_i| + \sum_{i\,:\,|T_i| > R/2} R/2 \\ &\leq CR + \frac{R}{2}\left(1 + \frac{1}{3} + \frac{1}{9} + \ldots\right) + \frac{R}{2} \cdot N \end{aligned} \tag{A.2}$$

where $N$ is the number of triangles in the family $\{T_i\}_{i=0}^{k-1}$ with $|T_i| > \frac{R}{2}$. The diameters of the triangles in the sequence $\{T_i\}_{i=0}^{k-1}$ are growing at least exponentially and there is at most one triangle in this sequence with $|T_i| > \rho_U(x, y)$ because for any index $i < k-1$, we have

$$\rho_U(x, y) \geq \rho_U(b(T_i), b(T_{i+1})) \geq |T_i|.$$

Therefore the constant $N$ in (A.2) is uniformly bounded, and so there exists a constant $C'$ such that

$$\sum_{j=1}^{k-1} \rho_U(y_i, y_{i-1}) \leq C'R.$$

Similarly consider a sequence $\{y_j\}_{j=0}^{l-1}$ of points $y_j \in b(T'_j)$ in the base of the triangle $T'_j$ with

$$\begin{aligned} \rho_U(y_j, \partial U) &\geq \min\left(\frac{R}{8}, \frac{|T'_j|}{4}\right), \\ \sum_{j=1}^{l-1} \rho_U(y_i, y_{i-1}) &\leq C'R. \end{aligned}$$

Let $z$ be the point in $T_k = T'_l$ with $\rho_U(z, \partial U) \geq \frac{R}{8}$, $\rho_U(z, x_{k-1}) \leq 2CR$ and $\rho_U(z, y_{l-1}) \leq 2CR$. The path $\gamma$ consisting of line segments connecting the points

$$x, x_0, x_1, \ldots, x_{k-1}, z, y_{l-1}, y_{l-2}, \ldots, y_0, y$$

in this order is a desired path satisfying the uniform condition (3.5).

Similarly we can prove that the exterior of von Koch snowflake is a uniform domain in $\mathbb{R}^2$, because it can be represented as a union of countably many triangles constructed via similar procedure. $\square$

# Appendix B

# Behavior of a réduite $h$

In this section we will focus on examples of the domains in $\mathbb{R}^n$ where the function $h$ constructed in Chapter 5.5 is known.

**Proposition B.0.6 (see [9])** *Let $U = \mathbb{R}_+ \times \Omega \subset \mathbb{R}^n$ be a cone in $\mathbb{R}^n$ based on the spherical domain $\Omega \subset S^{n-1}$, where a sphere $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$. Let $\phi$ be the first Dirichlet eigenfunction of the spherical Laplacian with eigenvalue $\lambda$. Then in polar coordinates,*

$$h(x) = |x|^\alpha \phi(x/|x|)$$

*with*

$$\alpha = \frac{\sqrt{(n-2)^2 + 4\lambda} - (n-2)}{2} > 0$$

*(so that $\alpha(\alpha + n - 2) = \lambda$) is a positive harmonic function in $U$ vanishing on $\partial U$.*

**Proof.** This result follows from the positivity of the first Dirichlet eigenfunction and the representation of $\Delta$ in polar coordinates via spherical Laplacian $L_{S^{n-1}}$,

$$\Delta = \frac{1}{r^2} L_{S^{n-1}} + \frac{1}{r^{n-1}} \frac{\partial}{\partial r} \left( r^{n-1} \frac{\partial}{\partial r} \right) \tag{B.1}$$

□

**Proposition B.0.7** *Let $U = \mathbb{R}^n \setminus H$ be the exterior in $\mathbb{R}^n$ of the half hyperplane*

$$H = \{(x_1, \ldots, x_n) : x_1 \leq 0, x_2 = 0\}$$

*Then the function*

$$h(\vec{x}) = \mathrm{Re}\sqrt{\mathrm{x_1 + ix_2}}$$

*is a harmonic function in $U$ vanishing on $\partial U$. Here $\sqrt{z}$ is taken to be an analytic function on $\mathbb{C} \setminus \mathbb{R}_-$, i.e. outside the set of non-positive reals.*

**Proof.** The function $h$ is the real part of the conformal map from $\mathbb{C} \setminus \mathbb{R}_-$ to the set of complex numbers with positive real part. Therefore $h$ is harmonic, positive and vanishes on $\partial U$. $\square$

Notice that for $n = 2$, the level sets of the function $h$ defined in Proposition B.0.7 are parabolas, therefore the function $h$ is given by a similar formula for the exterior of the parabola in $\mathbb{R}^2$.

**Proposition B.0.8** *Let $U \subset \mathbb{R}^n$ be the exterior of the cylinder,*

$$U = \{\vec{x} = (x_1, \dots, x_n) : x_1^2 + x_2^2 + \dots + x_{n-1}^2 \geq 1\}$$

*and let $r(\vec{x}) = \sqrt{x_1^2 + x_2^2 + \dots + x_{n-1}^2}$. Then for $n \geq 4$ the function*

$$h(\vec{x}) = 1 - r(\vec{x})^{3-n}$$

*is a harmonic function in $U$ vanishing on $\partial U$. For $n = 3$ the function*

$$h(\vec{x}) = \log r(\vec{x})$$

*is a harmonic function in $U$ vanishing on $\partial U$.*

**Proof.** We look for the function $h(\vec{x})$ among the functions independent of $x_n$, thus reducing the problem to dimension $n-1$. It remains to use the representation (B.1) of $\Delta$ in polar coordinates to check that $h$ is harmonic. $\square$

**Proposition B.0.9** *Let $U \subset \mathbb{R}^2 = \mathbb{C}$ be the complement of the infinitely winding spiral $S$ (see* Figure 3.2 of Chapter 3.2*) given in the parametric form by $z(t) =$* $\exp(t + ic\pi t)$ *for some constant $c > 0$. Then the function*

$$h(\vec{x}) = \mathrm{Im}\left[\exp\left(\frac{1 - \mathrm{i}c\pi}{2}\log(\mathrm{x}_1 + \mathrm{i}c\mathrm{x}_2)\right)\right]$$

*is a harmonic function in $U$ vanishing on $\partial U$. Here the function* log *is considered to be any branch of a complex logarithm function in the simply connected domain $\mathbb{C} \setminus S$.*

**Proof.** For this result we constructed the function $h$ as the imaginary part of the combination of conformal maps,

$$h = \mathrm{Im} \circ \phi^{-1} \circ \psi,$$

where

$$\phi : \{z \in \mathbb{C} : 0 \leq \mathrm{Im(z)} \leq \frac{2\pi}{1 + \mathrm{c}^2\pi^2}\} \to \mathbb{C} \setminus \mathrm{S}, \quad \mathrm{z} \to \exp\left(\mathrm{z} + \mathrm{i}\mathrm{c}\pi\mathrm{z}\right)$$

and

$$\psi : \{z \in \mathbb{C} : 0 \leq \mathrm{Im(z)} \leq \frac{2\pi}{1 + \mathrm{c}^2\pi^2}\} \to \mathbb{H}, \quad \mathrm{z} \to \exp\left(\frac{1 + \mathrm{c}^2\pi^2}{2}\mathrm{z}\right).$$

The function $h$ is the imaginary part of the conformal map from $\mathbb{C} \setminus S$ to the set $\mathbb{H}$ of complex numbers with positive real part. Therefore $h$ is harmonic, positive and vanishes on $\partial U$. $\square$

**Remark.** For points in the complement of the spiral $S$ of the form $\vec{x} = \exp(t + ic\pi t - \theta)$ with fixed $\theta \in (0, \frac{2}{c})$, we have

$$h(\vec{x}) = \mathrm{Im}\left[\exp\left(\frac{1 - \mathrm{i}\mathrm{c}\pi}{2}(\mathrm{t} + \mathrm{i}\mathrm{c}\pi\mathrm{t} - \theta)\right)\right] = \exp\left(\frac{1 + \mathrm{c}^2\pi^2}{2}\mathrm{t}\right)\mathrm{e}^{-\frac{\theta}{2}}\sin\left(\frac{\theta\mathrm{c}\pi}{2}\right) \asymp |\tilde{\mathrm{x}}|^{\frac{1+\mathrm{c}^2\pi^2}{2}}.$$

This shows the growth of the function $h$ in $\mathbb{C} \setminus S$ resembles that in the cone with angle $\frac{2\pi}{1+c^2\pi^2}$.

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