STUDY OF BI-PARAMETER FLAG PARAPRODUCTS AND BI-PARAMETER STOPPING-TIME ALGORITHMS

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Yujia Zhai

May 2019

STUDY OF BI-PARAMETER FLAG PARAPRODUCTS AND BI-PARAMETER STOPPING-TIME ALGORITHMS

Yujia Zhai, Ph.D.

Cornell University 2019

We prove full range of estimates for bi-parameter flag paraproducts, including end-point estimates, on restricted function spaces. The machinery invented to treat the multi-parameter objects is a robust stopping-time argument which incorporates information on subspaces to derive estimates on the entire space.

BIOGRAPHICAL SKETCH

Yujia Zhai was born in Shanghai, China. She grew up with her parents and her extended family which consists of more than thirty people. She left her hometown and came to the US for college. Upon completion of her bachelor degree, she continued her study of mathematics at Cornell University.

This document is dedicated to my parents and my family.

ACKNOWLEDGEMENTS

First I would like to thank my advisor, Camil Muscalu, for his guidance and support throughout the program. His proposal of natural curiosities in time-frequency analysis gave birth to the project. His interests and emphasis on the robust understanding of problems, instead of results themselves, have great impact on me, which can be seen in the project. I am also grateful to his encouragement and insightful suggestions whenever I encounter challenges or experience frustrations. His value of pursuing one's own passion, both in mathematics and in general, have become a part of my life philosophy as well.

I would also like to thank Prof. Strichartz, Prof. Saloff-Coste, Gennady Uraltsev and Philippe Sosoe for sharing their knowledge and insights through classes, formal and informal conversations.

I am thankful to my undergraduate professors Alex Iosevich, Steve Gonek and Naomi Jochnowitz who have profound influence on my career path. Enlightened by classes with Steve during the freshman year, I decided to choose mathematics as my major. Naomi has been my role model because of her passion and dedication for mathematics and teaching. Alex introduced me to the world of mathematical researches and the field of harmonic analysis. I appreciate his unconditional support and friendship over the years.

I would like to thank my academic sisters and brothers, Cristina Benea, Robert Kesler, Pok Wai Fong, Eyvindur Palsson, Itamar Oliveira and Alex Niepla for their advice and support. I am also grateful to my friends Daoji Huang, Jun Le Goh, Wai Kit Yeung, Alex Fok, Jingbo Liu, Tinyi Chu, Juan Wang and Daqian Sun.

Last but not least, I would like to thank my family for everything they have done or have been doing to support my career.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Background

The bi-parameter flag paraproduct is a multilinear operator defined as

$$T_m(F_1, F_2, F_3)(x, y) := \int m(\vec{\zeta}_1, \ldots, \vec{\zeta}_n)\widehat{F_1}(\vec{\zeta}_1) \ldots \widehat{F_n}(\vec{\zeta}_n)e^{2\pi i(x,y)\cdot(\vec{\zeta}_1+\ldots+\vec{\zeta}_n)}d\vec{\zeta}_1 \ldots d\vec{\zeta}_n$$

where $\zeta_i = (\xi_i, \eta_i) \in \mathbb{R}^2$ and

$$m(\vec{\zeta}_1, \ldots, \vec{\zeta}_n) := \prod_{S \subseteq \{1,\ldots,n\}} m_S(\vec{\zeta}_S). \tag{1.1.1}$$

with m_S being a symbol in $\mathbb{R}^{\text{card}(S)} \times \mathbb{R}^{\text{card}(S)}$ that is bounded, smooth away from the subspaces $\{(\xi_i)_{i\in S} = 0\} \cup \{(\eta_i)_{i\in S} = 0\}$ and satisfying the Marcinkiewitz condition

$$\left|\partial^{\vec{\alpha}}_{(\xi_i)_{i\in S}} \partial^{\vec{\beta}}_{(\eta_i)_{i\in \S}} m_S\right| \lesssim \frac{1}{|(\xi_i)_{i\in S}|^{|\vec{\alpha}|}|(\eta_i)_{i\in S}|^{|\vec{\beta}|}}$$

where $\vec{\alpha}$ and $\vec{\beta}$ denote multi-indices with nonnegative entries.

The subject of our study is the above multiplier operator with a special class of symbol m with

$$m(\xi) = a((\xi_1, \eta_1), (\xi_2, \eta_2))b((\xi_1, \eta_1), (\xi_2, \eta_2), (\xi_3, \eta_3))$$

where a, b are symbols defined as m_S in (1.1.1) with $S = \{1, 2\}, \{1, 2, 3\}$ respectively. We further restrict our function spaces in the sense that

$$F_1 = f_1 \otimes g_1$$

$$F_2 = f_2 \otimes g_2$$

More precisely, our goal is to study

$$T_{ab} := \int a((\xi_1,\eta_1),(\xi_2,\eta_2))b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3))\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{g_1}(\eta_1)\widehat{g_2}(\eta_2)\widehat{h}(\xi_3,\eta_3)$$
$$e^{2\pi ix(\xi_1+\xi_2+\xi_3)}e^{2\pi iy(\eta_1+\eta_2+\eta_3)}d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3 \qquad (1.1.2)$$

The single-parameter variant of the multiplier operator defined in (2.1.1) takes the form

$$T_{a_1b_1}(f_1,f_2,f_3)(x) := \int a_1(\xi_1,\xi_2)b_1(\xi_1,\xi_2,\xi_3)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)e^{2\pi ix(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3 \qquad (1.1.3)$$

where $a_1 \in \mathcal{M}(\mathbb{R}^2)$, $b_1 \in \mathcal{M}(\mathbb{R}^3)$ are Coifman-Meyer symbols. The operator (1.1.3) was studied by Muscalu [9] and reproved by Miyachi and Tomita [7]. It is closely related to various nonlinear partial differential equations, including water wave equations and nonlinear Schrodinger equations, which was discovered by Germain, Masmoudi and Shatah [6]. As can be seen, (1.1.3) is a special case of (2.1.1) when

$$a((\xi_1,\eta_1),(\xi_2,\eta_2)) = a_1(\xi_1,\xi_2)$$

$$b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3)) = b_1(\xi_1,\xi_2,\xi_3)$$

Another multilinear operator related to (2.1.1) with wide applications in PDEs, as explored in [1], is the single-parameter paraproduct:

$$T_{b_1}(f_1,f_2)(x) := \int b_1(\xi_1,\xi_2,\xi_3)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{f_3}(\xi_3)e^{2\pi ix(\xi_1+\xi_2+\xi_3)}d\xi_1 d\xi_2 d\xi_3 \qquad (1.1.4)$$

where $b_1(\xi_1,\xi_2,\xi_3) \in \mathcal{M}(\mathbb{R}^3)$ is a Coifman-Meyer symbol. Its boundedness is proved by Coifman-Meyer's theorem on paraproducts [2]. One may notice that (1.1.4) can be deduced from (2.1.1) by choosing

$$a((\xi_1,\eta_1),(\xi_2,\eta_2)) = 1$$

$$b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3)) = b_1(\xi_1,\xi_2,\xi_3)$$

The bi-parameter variant of (1.1.4) is called bi-parameter paraproducts, which was studied by Muscalu, Pipher, Tao and Thiele [10]. It also appeared naturally in nonlinear PDEs, such as Kadomtsev-Petviashvili equations studied by Kenig [4]. It is defined as

$$T_b(F_1,F_2,F_3)(x,y) := \int b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3))\widehat{F_1}(\xi_1,\eta_1)\widehat{F_2}(\xi_2,\eta_2)\widehat{F_3}(\xi_3,\eta_3)$$
$$e^{2\pi i x(\xi_1+\xi_2+\xi_3)}e^{2\pi i y(\eta_1+\eta_2+\eta_3)}d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3 \tag{1.1.5}$$

where $b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3))$. Suppose that one again restricts the function space to tensor-product space, it is not difficult to observe that (1.1.5) is a subcase of (2.1.1) when

$$a((\xi_1,\eta_1),(\xi_2,\eta_2)) = 1$$

1.2 Methodology

While the object of study has implications and connections with other multilinear operators, the machinery implemented in the proof has interests of its own. It involves various stopping-time decompositions:

1. **Two-dimensional tensor-type stopping-time decompositions** refer to algorithms that first perform one-dimensional stopping-time decompositions for each variable and then combine information for different variables to obtain estimates for operators involving several variables
2. **Two-dimensional Level-Sets Stopping-Time Decompositions** refer to algorithms to partition the collection of dyadic rectangles such that the dyadic rectangles in each sub-collection intersect with a certain level set non-trivially.

The machinery outlined above is considered to be robust in the sense that it captures all local behaviors of the operator. The robustness may also be verified by the entire range of estimates obtained.

After closer inspection of the machinery, it would be not surprising that the machinery gives estimates involving L^∞-norms, even mixed-norm estimates involving L^∞-norms. In particular, the tensor-type stopping-time decompositions process information on each subspaces independently. As a consequence, when some function defined on some subspace lies in L^∞, one simply "forgets" about that function and glues the information from subspaces in an intelligent way specified later.

1.3 Structure

The paper is organized as follows: main theorems are stated in Chapter 2 followed by necessary definitions introduced in Chapter 3. Chapter 4 describes how to reduce the multilinear operator to discrete model operators and estimates one needs to obtain for the model operators. Chapter 5 gives the definition and estimates for the building blocks in the argument - sizes and energies. Chapter 6 and 7 are devoted to estimates for the multilinear operator in the Walsh case. Chapter 6 develops all the L^p-estimates with $p < \infty$ whereas Chapter 7 focuses on estimates involving L^∞-norms. Chapter 6 and 7 are separated into sections where each section describes a discrete model operator and starts with a specification of the stopping-time decompositions used. Chapter 13 extends all the estimates in Walsh case to the general Fourier case.

It is also important to notice that Section 6.1 develops an argument for one of the simpler model operators with emphasis on the key geometric feature implied by a stopping-time decomposition, that is the sparsity condition. Section 6.2 focuses on a more complicated model which requires not only sparsity condition, but also a Fubini-type argument which is discussed in details. In Sections 6.3 and 6.4, those two key ingredients are used again, though with some modifications in the implementations, in the argument for some other models. Section 7.1 - 7.3 prove estimates involving L^∞-norms. The argument for those cases is similar to the ones in Section 6.1, in the sense that only the sparsity condition is necessary to obtain the desired estimates.

CHAPTER 2

MAIN RESULTS

2.1 Bi-parameter Flag Paraproduct On Restricted Function Spaces

Theorem 2.1.1. *Suppose $a \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, $b \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2)$. a and b are smooth away from $\{(\xi_1, \xi_2) = 0\} \cup \{(\eta_1, \eta_2) = 0\}$ and $\{(\xi_1, \xi_2, \xi_3) = 0\} \cup \{(\eta_1, \eta_2, \eta_3) = 0\}$ respectively and satisfying the Marcinkiewitz condition. For $f_1, f_2, g_1, g_2 \in \mathcal{S}(\mathbb{R})$ and $h \in \mathcal{S}(\mathbb{R}^2)$, define*

$$
\begin{aligned}
T_{ab}(f_1^x \otimes g_1^y, f_2^x \otimes g_2^y, h^{x,y}) := \int_{\mathbb{R}^6} & a((\xi_1,\eta_1),(\xi_2,\eta_2)) b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3)) \\
& \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)\hat{g}_1(\eta_1)\hat{g}_2(\eta_2)\hat{h}(\xi_3,\eta_3) \\
& e^{2\pi i x(\xi_1+\xi_2+\xi_3)} e^{2\pi i y(\eta_1+\eta_2+\eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3 \quad (2.1.1)
\end{aligned}
$$

Then for $1 < p, q, s < \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = \frac{1}{r}$,

$$T_{ab} : L^p \times L^q \times L^s \to L^r$$

Theorem 2.1.2. *Let T_{ab} be defined as (2.1.1). Then for $1 < p, s < \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = \frac{1}{r}$,*

$$
\begin{aligned}
T_{ab} : & L^p \times L^\infty \times L^s \to L^r \\
& L^\infty \times L^p \times L^s \to L^r \\
& L^p_x(L^\infty_y) \times L^\infty_x(L^p_y) \times L^s \to L^r \\
& L^\infty_x(L^p_y) \times L^p_x(L^\infty_y) \times L^s \to L^r \\
& L^p_x(L^\infty_y) \times L^p_x(L^\infty_y) \times L^s \to L^r \\
& L^\infty_x(L^p_y) \times L^\infty_x(L^p_y) \times L^s \to L^r
\end{aligned}
$$

2.2 Application - Leibniz Rule

A direct corollary of Theorem 2.1.1 is the following Leibniz rule:

Theorem 2.2.1. *Suppose $f_1 \otimes g_1, f_2 \otimes g_2, h \in \mathcal{S}(\mathbb{R}^2)$. Is it true that for $\beta_1, \beta_2, \alpha_1, \alpha_2 \geq 0$ sufficiently large and $1 < p_i, q_i, s_i < \infty$, $\frac{1}{p_i} + \frac{1}{q_i} + \frac{1}{s_i} = \frac{1}{r}$, $i = 1, \ldots, 16$,*

$$
\begin{aligned}
& \|D_1^{\beta_1} D_2^{\beta_2}(D_1^{\alpha_1} D_2^{\alpha_1}(f_1^x \otimes g_1^y f_2^x \otimes g_2^y) h^{x,y})\|_{L^r(\mathbb{R}^2)} \\
\lesssim \; & \textit{sum of} \;\; 16 \;\; \textit{terms in the forms:} \\
& \|D_1^{\alpha_1+\beta_1} f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{q_1}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_1}(\mathbb{R})} \|g_2\|_{L^{q_1}(\mathbb{R})} \|h\|_{L^{s_1}(\mathbb{R}^2)} + \\
& \|f_1\|_{L^{p_2}(\mathbb{R})} \|D_1^{\alpha+\beta_1} f_2\|_{L^{q_2}(\mathbb{R})} \|D_2^{\alpha_2+\beta_2} g_1\|_{L^{p_2}(\mathbb{R})} \|g_2\|_{L^{q_2}(\mathbb{R})} \|h\|_{L^{s_2}(\mathbb{R}^2)} + \\
& \|D_1^{\alpha+\beta_1} f_1\|_{L^{p_3}(\mathbb{R})} \|f_2\|_{L^{q_3}(\mathbb{R})} \|D_2^{\alpha_2} g_1\|_{L^{p_3}(\mathbb{R})} \|g_2\|_{L^{q_3}(\mathbb{R})} \|D_2^{\beta_2} h\|_{L^{s_3}(\mathbb{R}^2)} + \ldots
\end{aligned}
$$

where the partial derivative is defined as

$$
D_1^{\gamma_1} D_2^{\gamma_2} F := \mathcal{F}^{-1}(|\xi_1|^{\gamma_1} |\xi_2|^{\gamma_2} \widehat{F}(\xi_1, \xi_2)).
$$

CHAPTER 3

PRELIMINARIES

3.1 Terminology

We will first introduce some notations which will be useful throughout the paper.

Definition 3.1.1. Suppose $I \in \mathbb{R}$ is an interval. Then we say a smooth function ϕ is *adapted to I* if

$$\phi^{(l)}(x) \leq C_l C_M \frac{1}{|I|^l} \frac{1}{(1 + \frac{|x-x_I|}{|I|})^M}$$

for sufficiently many derivatives l, where x_I denotes the center of the interval I.

Definition 3.1.2. Supoose $\mathcal{I}$ is a collection of dyadic intervals. Then a family of L^2-nomralized bump functions $(\phi_I)_{I \in \mathcal{I}}$ is *non-lacunary* if and only if for every $I \in \mathcal{I}$,

$$\text{supp } \widehat{\phi_I} \subseteq [-4|I|^{-1}, 4|I|^{-1}]$$

A family of L^2-nomralized bump functions $(\phi_I)_{I \in \mathcal{I}}$ is *lacunary* if and only if for every $I \in \mathcal{I}$,

$$\text{supp } \widehat{\phi_I} \subseteq [-4|I|^{-1}, \frac{1}{4}|I|^{-1}] \cup [\frac{1}{4}|I|^{-1}, 4|I|^{-1}]$$

We usually denote bump functions in non-lacunary family by $(\varphi_I)_I$ and those in lacunary family by $(\psi_I)_I$.

3.2 Useful Operators - Definitions and Theorems

We also give explicit definitions for some operators that will appear naturally in the argument.

Definition 3.2.1. The *Hardy-Littlewood maximal operator* M is defined as

$$Mf(\vec{x}) = \sup_{\vec{x}\in B} \int_B |f(\vec{u})| d\vec{u}$$

where where the supremum is taken over all open balls in $B \subset \mathbb{R}^d$ containing $\vec{x}$.

Definition 3.2.2. Suppose $\mathcal{I}$ is a finite family of dyadic intervals and $(\psi_I)_I$ a lacunary family of L^2-normalized bump functions. The *discretized Littlewood-Paley square function operator* S is defined as

$$Sf(x) = \Big(\sum_{I\in\mathcal{I}} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \chi_I(x) \Big)^{\frac{1}{2}}$$

Definition 3.2.3. Suppose $\mathcal{R}$ is a finite collection of dyadic rectangles. Let $(\phi_R)_{R\in\mathcal{R}}$ denote the family of L^2-normalized bump functions with $\phi_R = \phi_I \otimes \phi_J$ where $R = I \times J$.

1. the *double square function operator* SS is defined as

$$SSh(x, y) = \Big(\sum_{I\times J} \frac{|\langle h, \psi_I \otimes \psi_J \rangle|^2}{|I||J|} \chi_{I\times J}(x, y) \Big)^{\frac{1}{2}}$$

2. the *hybrid maximal-square operator* MS is defined as

$$MSh(x, y) = \sup_I \frac{1}{|I|^{\frac{1}{2}}} \Big(\sum_J \frac{|\langle h, \varphi_I \otimes \psi_J \rangle|^2}{|J|} \chi_J(y) \Big)^{\frac{1}{2}} \chi_I(x)$$

3. the *hybrid square-maximal operator* SM is defined as

$$SMh(x, y) = \Big(\sum_I \frac{(\sup_J \frac{|\langle h, \psi_I \otimes \varphi_J \rangle|}{|J|} \chi_J(y))}{|I|} \chi_I(x) \Big)^{\frac{1}{2}}$$

4. the *double maximal function* MM is defined as

$$MMh(x, y) = \sup_{(x,y)\in R} \frac{1}{|R|} \int_R |h(s, t)| ds dt$$

where the supremum is taken over all dyadic rectangles in $\mathcal{R}$ containing (x, y).

The following theorem about the operators defined above is used frequently in the argument.

Theorem 3.2.4. *1. M is bounded in $L^p(\mathbb{R}^d)$ for $1 < p \leq \infty$ and $M : L^1 \longrightarrow L^{1,\infty}$*

2. S is bounded in $L^p(\mathbb{R})$ for $1 < p < \infty$

3. The hybrid operators SS, MS, SM, MM are bounded in $L^p(\mathbb{R}^2)$ for $1 < p < \infty$.

3.3 Restricted Weak Type Estimates

By multilinear interpolation [8], we can reduce the desired estimates specified in Theorem 2.1.1 and Theorem 2.1.2 to the following restricted weak type estimates:

Theorem 3.3.1. *Let T_{ab} and range of p, q, s, r be as specified in Theorem 2.1.1. Then for any $|f_1(x)| \leq \chi_{F_1}(x)$, $|f_2(x)| \leq \chi_{F_2}(x)$, $|g_1(y)| \leq \chi_{G_1}(y)$, $|f_2(y)| \leq \chi_{F_2}(y)$, $|h(x,y)| \leq \chi_H(x,y)$, where $F_1, F_2 \subset \mathbb{R}_x$, $G_1, G_2 \subset \mathbb{R}_y$, $H, E \subset \mathbb{R}^2$ with $|F_1|, |F_2|, |G_1|, |G_2|, |H|, |E| < \infty$, there exists $E' \subseteq E$ with $|E'| > |E|/2$ such that the linear form associated with T_{ab} satisfies*

$$|\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'})| \lesssim |F_1|^{\frac{1}{p}} |G_1|^{\frac{1}{p}} |F_2|^{\frac{1}{q}} |G_2|^{\frac{1}{q}} |H|^{\frac{1}{s}} |E|^{\frac{1}{r'}}.$$

Theorem 3.3.2. *Let T_{ab} and range of p, q, s, r be as specified in Theorem 2.1.2. Then for any $|f_1(x)| \leq \chi_{F_1}(x)$, $|f_2(x)| \leq \chi_{F_2}(x)$, $|g_1(y)| \leq \chi_{G_1}(y)$, $|f_2(y)| \leq \chi_{F_2}(y)$, $|h(x,y)| \leq \chi_H(x,y)$, where $F_1, F_2 \subset \mathbb{R}_x$, $G_1, G_2 \subset \mathbb{R}_y$, $H, E \subset \mathbb{R}^2$ with $|F_1|, |F_2|, |G_1|, |G_2|, |H|, |E| < \infty$, there exists $E' \subseteq E$ with $|E'| > |E|/2$ such that the linear form associated with T_{ab} satisfies*

$$|\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'})| \lesssim |F_1|^{\frac{1}{p}} |G_1|^{\frac{1}{p}} |F_2|^{\frac{1}{q}} |G_2|^{\frac{1}{q}} |H|^{\frac{1}{s}} |E|^{\frac{1}{r'}}.$$

Remark 3.3.3. The theorems hint the necessity of localization and the major subset E' of E is constructed based on the philosophy to localize the operator where it is well-behaved.

CHAPTER 4

REDUCTION TO MODEL OPERATORS

In this chapter, we will introduce the discrete model operators whose boundedness implies the estimates specified in Theorem 3.3.1 and Theorem 3.3.2. The model operators are usually more desirable because they are more "localizable".

4.1 Models for Bi-Parameter Flag Paraproduct

Theorem 4.1.1. *Suppose $\mathcal{I}, \mathcal{J}, \mathcal{K}$ are finite collections of dyadic intervals. Suppose $(\phi_I^i)_{I\in\mathcal{I}}$, $(\phi_J^j)_{J\in\mathcal{J}}$, $(\phi_K^k)_{K\in\mathcal{K}}, i, j, k = 1, 2, 3$ are families of L^2-normalized bump functions adapted to I, J, K respectively. We further assume that for at least two families of $(\phi_I^i)_{I\in\mathcal{I}}, i = 1, 2, 3$, are lacunary. Same conditions are assumed for families $(\phi_J^j)_{J\in\mathcal{J}}$ and $(\phi_K^k)_{K\in\mathcal{K}}$. In some models, we specify the lacunary and non-lacunary families by explicitly denoting the functions in lacunary family as ψ and those in non-lacunary family as φ. Let*

1.
$$\Pi_{flag^0\otimes paraproduct}(f_1^x\otimes g_1^y, f_2^x\otimes g_2^y, h^{x,y})$$
$$:= \sum_{I\times J\in\mathcal{I}\times\mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle g_1,\phi_J^1\rangle\langle g_2,\phi_J^2\rangle\langle h,\psi_I^2\otimes\phi_J^2\rangle\psi_I^3\otimes\phi_J^3$$
where
$$B_I(f_1,f_2)(x) := \sum_{K\in\mathcal{K}:|K|>|I|}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\phi_K^3(x).$$

2.
$$\Pi_{flag^{\#1}\otimes paraproduct}(f_1^x\otimes g_1^y, f_2^x\otimes g_2^y, h^{x,y})$$

$$:= \sum_{I\times J\in\mathcal{I}\times\mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|}\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle\langle g_1,\phi_J^1\rangle\langle g_2,\phi_J^2\rangle\langle h,\psi_I^2\otimes\phi_J^2\rangle\psi_I^3\otimes\phi_J^3$$

where

$$B_I^{\#_1}(f_1,f_2)(x) := \sum_{K\in\mathcal{K}:|K|\sim 2^{\#_1}|I|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\phi_K^3(x).$$

3.

$$\Pi_{flag^0\otimes flag^0}(f_1^x\otimes g_1^y, f_2^x\otimes g_2^y, h^{x,y})$$
$$:= \sum_{I\times J\in\mathcal{I}\times\mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\psi_I^3\otimes\psi_J^3$$

where

$$B_I(f_1,f_2)(x) := \sum_{K\in\mathcal{K}:|K|>|I|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\phi_K^3(x),$$
$$\tilde{B}_J(g_1,g_2)(y) := \sum_{L\in\mathcal{L}:|L|>|J|} \frac{1}{|L|^{\frac{1}{2}}}\langle g_1,\phi_L^1\rangle\langle g_2,\phi_L^2\rangle\phi_L^3(y).$$

4.

$$\Pi_{flag^0\otimes flag^{\#_2}}(f_1^x\otimes g_1^y, f_2^x\otimes g_2^y, h^{x,y})$$
$$:= \sum_{I\times J\in\mathcal{I}\times\mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\psi_I^3\otimes\psi_J^3$$

where

$$B_I(f_1,f_2)(x) := \sum_{K\in\mathcal{K}:|K|>|I|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\phi_K^3(x),$$
$$\tilde{B}_J^{\#_2}(g_1,g_2)(y) := \sum_{L\in\mathcal{L}:|L|\sim 2^{\#_2}|J|} \frac{1}{|L|^{\frac{1}{2}}}\langle g_1,\phi_L^1\rangle\langle g_2,\phi_L^2\rangle\phi_L^3(y).$$

5.

$$\Pi_{flag^{\#_1}\otimes flag^{\#_2}}(f_1^x\otimes g_1^y, f_2^x\otimes g_2^y, h^{x,y})$$
$$:= \sum_{I\times J\in\mathcal{I}\times\mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\psi_I^3\otimes\psi_J^3$$

where

$$B_I^{\#_1}(f_1,f_2)(x) := \sum_{K\in\mathcal{K}:|K|\sim 2^{\#_1}|I|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\phi_K^3(x),$$

$$\tilde{B}_J^{\#_2}(g_1, g_2)(y) := \sum_{L \in \mathcal{L}: |L| \sim 2^{\#_2}|J|} \frac{1}{|L|^{\frac{1}{2}}} \langle g_1, \phi_L^1 \rangle \langle g_2, \phi_L^2 \rangle \phi_L^3(y).$$

Then $\Pi_{flag^0 \otimes paraproduct}$, $\Pi_{flag^{\#_1} \otimes paraproduct}$, $\Pi_{flag^0 \otimes flag^0}$, $\Pi_{flag^0 \otimes flag^{\#_2}}$ and $\Pi_{flag^{\#_1} \otimes flag^{\#_2}}$ satisfy the mapping property specified in Theorem (3.3.1), where the constants are independent of $\#_1, \#_2$ and the cardinality of the collection of dyadic rectangles.

Theorem 4.1.2. *Let $\Pi_{flag^0 \otimes paraproduct}$, $\Pi_{flag^{\#_1} \otimes paraproduct}$, $\Pi_{flag^0 \otimes flag^0}$, $\Pi_{flag^0 \otimes flag^{\#_2}}$ and $\Pi_{flag^{\#_1} \otimes flag^{\#_2}}$ be defined as in Theorem 4.1.1. Then all the operators listed satisfy the mapping property specified in Theorem 3.3.2, where the constants are independent of $\#_1, \#_2$ and the cardinality of the collection of dyadic rectangles.*

We will further specify some models for $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ generated from different lacunary and non-lacunary positions. The main difference of the following two models are that $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$ has $(\phi_K^3)_K$ as a lacunary family whereas $\Pi^0_{\text{flag}^0 \otimes \text{flag}^0}$ has $(\phi_K^3)_K$ as a non-lacunary family. Our plan would be to prove estimates for $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$ and modify the argument to study $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$. It turns out this strategy can be applied to all other models with $(\phi_K^3)_K$ being specified as lacunary or non-lacunary family.

Model 1:

$$\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}(f_1^x \otimes g_1^y, f_2^x \otimes g_2^y, h^{x,y})$$
$$:= \sum_{I \times J \in \mathcal{I} \times \mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \langle B_I(f_1, f_2), \varphi_I^1 \rangle \langle \tilde{B}_J(g_1, g_2), \varphi_J^1 \rangle \langle h, \psi_I^2 \otimes \psi_J^2 \rangle \psi_I^3 \otimes \psi_J^3$$

where

$$B_I(f_1, f_2)(x) := \sum_{K \in \mathcal{K}: |K| > |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \varphi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \psi_K^3(x),$$

$$\tilde{B}_J(g_1, g_2)(y) := \sum_{L \in \mathcal{L}: |L| > |J|} \frac{1}{|L|^{\frac{1}{2}}} \langle g_1, \varphi_L^1 \rangle \langle g_2, \psi_L^2 \rangle \psi_L^3(y).$$

Model 2:

$$
\begin{aligned}
&\Pi^0_{\text{flag}^0 \otimes \text{flag}^0}(f_1^x \otimes g_1^y, f_2^x \otimes g_2^y, h^{x,y}) \\
:= &\sum_{I \times J \in \mathcal{I} \times \mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \langle B_I(f_1, f_2), \varphi_I^1 \rangle \langle \tilde{B}_J(g_1, g_2), \varphi_J^1 \rangle \langle h, \psi_I^2 \otimes \psi_J^2 \rangle \psi_I^3 \otimes \psi_J^3
\end{aligned}
$$

where

$$
\begin{aligned}
B_I(f_1, f_2)(x) &:= \sum_{K \in \mathcal{K}: |K| > |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \psi_K^1 \rangle \langle f_2, \psi_K^2 \rangle \varphi_K^3(x), \\
\tilde{B}_J(g_1, g_2)(y) &:= \sum_{L \in \mathcal{L}: |L| > |J|} \frac{1}{|L|^{\frac{1}{2}}} \langle g_1, \psi_L^1 \rangle \langle g_2, \psi_L^2 \rangle \varphi_L^3(y).
\end{aligned}
$$

In the rest of the chapter we will discuss in details how to reduce the boundedness of multiplier operators stated in Theorem 3.3.1 to the study of the discrete model operators specified in Theorem 4.1.1.

4.2 Littlewood-Paley Decomposition

4.2.1 Set Up

Let $\varphi \in \mathcal{S}(\mathbb{R})$ be a Schwartz function with $\text{supp}\widehat{\varphi} \subseteq [-2, 2]$ and $\widehat{\varphi}(\xi) = 1$ on $[-1, 1]$. Let

$$
\widehat{\psi}(\xi) = \widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)
$$

so that $\text{supp}\widehat{\psi} \subseteq [-2, -\frac{1}{2}] \cup [-\frac{1}{2}, 2]$. Now for every $k \in \mathbb{Z}$, define

$$
\widehat{\psi_k} := \widehat{\psi}(2^{-k}\xi)
$$

One important observation is that

$$
\sum_{k \in \mathbb{Z}} \widehat{\psi_k}(\xi) = 1
$$

We will adopt the notation non-lacunary for $(\widehat{\varphi_k})_k$ and lacunary for $(\widehat{\psi_k})_k$.

4.2.2 Special Symbols

We will first focus on a special case of the symbols and the general case will be studied as an extension afterwards. Suppose that

$$a((\xi_1,\eta_1),(\xi_2,\eta_2)) = a_1(\xi_1,\xi_2)a_2(\eta_1,\eta_2)$$

$$b((\xi_1,\eta_1),(\xi_2,\eta_2),(\xi_3,\eta_3)) = b_1(\xi_1,\xi_2,\xi_3)b_2(\eta_1,\eta_2,\eta_3)$$

where

$$a_1(\xi_1,\xi_2) = \sum_{k_1}\widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)$$
$$b_1(\xi_1,\xi_2,\xi_3) = \sum_{k_2}\widehat{\phi}_{k_2}(\xi_1)\widehat{\phi}_{k_2}(\xi_2)\widehat{\phi}_{k_2}(\xi_3)$$

At least one of the families $(\widehat{\phi}_{k_1}(\xi_1))_{k_1}$ and $(\widehat{\phi}_{k_1}(\xi_2))_{k_1}$ is lacunary and at least one of the families $(\widehat{\phi}_{k_2}(\xi_1))_{k_2}$, $(\widehat{\phi}_{k_2})(\xi_2))_{k_2}$ and $(\widehat{\phi}_{k_2}(\xi_3))_{k_2}$ is lacunary. Moroever,

$$a_2(\eta_1,\eta_2) = \sum_{j_1}\widehat{\phi}_{j_1}(\eta_1)\widehat{\phi}_{j_1}(\eta_2)$$

$$b_2(\eta_1,\eta_2,\eta_3) = \sum_{j_2}\widehat{\phi}_{j_2}(\eta_1)\widehat{\phi}_{j_2}(\eta_2)\widehat{\phi}_{j_2}(\eta_3)$$

where at least one of the families $(\widehat{\phi}_{j_1}(\eta_1))_{j_1}$ and $(\widehat{\phi}_{j_1}(\eta_2))_{j_1}$ is lacunary and at least one of the families $(\widehat{\phi}_{j_2}(\eta_1))_{j_2}$, $(\widehat{\phi}_{j_2})(\eta_2))_{j_2}$ and $(\widehat{\phi}_{j_2}(\eta_3))_{j_2}$ is lacunary.

Then

$$\begin{aligned} a_1(\xi_1,\xi_2)b_1(\xi_1,\xi_2,\xi_3) &= \sum_{k_1,k_2}\widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_2}(\xi_1)\widehat{\phi}_{k_2}(\xi_2)\widehat{\phi}_{k_2}(\xi_3) \\ &= \underbrace{\sum_{k_1\approx k_2}}_{I^1} + \underbrace{\sum_{k_1\ll k_2}}_{II^1} + \underbrace{\sum_{k_1\gg k_2}}_{III^1} \end{aligned}$$

Case I^1 gives rise to the symbol of paraproduct. More precisely,

$$I^1 = \sum_k \widehat{\tilde{\phi}}_k(\xi_1)\widehat{\tilde{\phi}}_k(\xi_2)\widehat{\phi}_k(\xi_3)$$

where $\widehat{\tilde{\phi}}_k(\xi_1) := \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_2}(\xi_1)$ and $\widehat{\tilde{\phi}}_k(\xi_2) := \widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_2}(\xi_2)$ when $k := k_1 \approx k_2$. The above expression can be completed as

$$I^1 = \sum_k \widehat{\tilde{\phi}}_k(\xi_1)\widehat{\tilde{\phi}}_k(\xi_2)\widehat{\phi}_k(\xi_3)\widehat{\phi}_k(\xi_1+\xi_2+\xi_3)$$

and at least two of the families $\widehat{\tilde{\phi}}_k(\xi_1)_k$, $(\widehat{\tilde{\phi}}_k(\xi_2))_k$, $(\widehat{\phi}_k(\xi_3))_k$, $(\widehat{\phi}_k(\xi_1+\xi_2+\xi_3))_k$ are lacunary.

Case II^1 and III^1 can be treated similarly. In Case II^1, the sum is non-degenerate when $(\phi_{k_2}(\xi_1))_{k_2}$ and $(\phi_{k_2}(\xi_2))_{k_2}$ are non-lacunary. In particular, one has

$$II^1 = \sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\varphi}_{k_2}(\xi_1)\widehat{\varphi}_{k_2}(\xi_2)\widehat{\psi}_{k_2}(\xi_3)$$

In the case when the symbols are assumed to take the special form, the above expression can be rewritten as

$$\sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\psi}_{k_2}(\xi_3),$$

which can be "completed" as

$$\sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_1}(\xi_1+\xi_2)\widehat{\varphi}_{k_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)\widehat{\psi}_{k_2}(\xi_1+\xi_2+\xi_3) \tag{4.2.1}$$

The exact same argument can be applied to $a_2(\eta_1,\eta_2)b_2(\eta_1,\eta_2,\eta_3)$ so that the symbol can be decomposed as

$$\underbrace{\sum_{j_1 \approx j_2}}_{I^2} + \underbrace{\sum_{j_1 \ll j_2}}_{II^2} + \underbrace{\sum_{j_1 \gg j_2}}_{III^2}$$

where

$$I^2 = \sum_j \widehat{\tilde{\phi}}_j(\eta_1)\widehat{\tilde{\phi}}_j(\xi_2)\widehat{\phi}_j(\eta_3)\widehat{\phi}_j(\eta_1+\eta_2+\eta_3)$$

with at least two of the families $(\widehat{\tilde{\phi}}_j(\eta_1))_j$, $(\widehat{\tilde{\phi}}_j(\eta_2))_j$, $(\widehat{\phi}_j(\eta_3))_j$ and $(\widehat{\phi}_j(\eta_1+\eta_2+\eta_3))_j$ are lacunary. Case II^2 and III^2 have similar expressions, where

$$II^2 = \sum_{j_1 \ll j_2} \widehat{\phi}_{j_1}(\eta_1)\widehat{\phi}_{j_1}(\eta_2)\widehat{\phi}_{j_1}(\eta_1+\eta_2)\widehat{\varphi}_{j_2}(\eta_1+\eta_2)\widehat{\psi}_{j_2}(\eta_3)\widehat{\psi}_{j_2}(\eta_1+\eta_2+\eta_3).$$

One can now combine the decompositions and analysis for a_1, a_2, b_1 and b_2 to study the original operator:

$$\begin{aligned}&T_{ab}(f_1 \otimes g_1, f_2 \otimes g_2, h)\\ =&T_{ab}^{I^1I^2} + T_{ab}^{I^1II^2} + T_{ab}^{I^1III^2} + T_{ab}^{II^1I^2} + T_{ab}^{II^1II^2} + T_{ab}^{II^1III^2} + T_{ab}^{III^1I^1} + T_{ab}^{III^2II^2} + T_{ab}^{III^1III^2}\end{aligned}$$

Because of the symmetry between frequency variables (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) and the symmetry between cases for frequency scales $k_1 \ll k_2$ and $k_1 \gg k_2$, $j_1 \ll j_2$ and $j_1 \gg j_2$, it suffices to consider the following operators and others can be proved using the same argument.

1. $T_{ab}^{I^1I^2}$ is a bi-parameter paraproduct;

2.

$$\begin{aligned}T_{ab}^{II^1I^2} =& \sum_{\substack{k_1 \ll k_2 \\ j \in \mathbb{Z}}} \int \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_1}(\xi+\xi_2)\widehat{\varphi}_{k_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)\widehat{\psi}_{k_2}(\xi_1+\xi_2+\xi_3)\\ &\qquad \widehat{\tilde{\phi}}_j(\eta_1)\widehat{\tilde{\phi}}_j(\eta_2)\widehat{\phi}_j(\eta_3)\widehat{\phi}_j(\eta_1+\eta_2+\eta_3)\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{g_1}(\eta_1)\widehat{g_2}(\eta_2)\widehat{h}(\xi_3,\eta_3)\\ &\qquad \cdot e^{2\pi i x(\xi_1+\xi_2+\xi_3)}e^{2\pi i y(\eta_1+\eta_2+\eta_3)}d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3\\ =& \sum_{\substack{k_1 \ll k_2 \\ j \in \mathbb{Z}}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \varphi_{k_2}\Big)(g_1 * \tilde{\phi}_j)(g_2 * \tilde{\phi}_j)(h * \psi_{k_2} \otimes \phi_j) * \psi_{k_2} \otimes \phi_j\end{aligned}$$

where at least two of the families $(\phi_{k_1})_{k_1}$ are lacunary and at least two of the families $(\phi_j)_j$ are lacunary.

3.

$$T_{ab}^{II^1 II^2} = \sum_{\substack{k_1 \ll k_2 \\ j_1 \ll j_2}} \int \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_1}(\xi + \xi_2)\widehat{\varphi}_{k_2}(\xi_1 + \xi_2)\widehat{\psi}_{k_2}(\xi_3)\widehat{\psi}_{k_2}(\xi_1 + \xi_2 + \xi_3)$$

$$\widehat{\phi}_{j_1}(\eta_1)\widehat{\phi}_{j_1}(\eta_2)\widehat{\phi}_{j_1}(\eta_1 + \eta_2)\widehat{\varphi}_{j_2}(\eta_1 + \eta_2)\widehat{\psi}_{j_2}(\eta_3)\widehat{\psi}_{j_2}(\eta_1 + \eta_2 + \eta_3)$$

$$\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{g_1}(\eta_1)\widehat{g_2}(\eta_2)\widehat{h}(\xi_3, \eta_3)$$

$$\cdot\, e^{2\pi i x(\xi_1+\xi_2+\xi_3)} e^{2\pi i y(\eta_1+\eta_2+\eta_3)} d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3$$

$$= \sum_{\substack{k_1 \ll k_2 \\ j_1 \ll j_2}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \varphi_{k_2}\Big)\Big(((g_1 * \phi_{j_1})(g_2 * \phi_{j_1}) * \phi_{j_1}) * \varphi_{j_2}\Big)$$

$$\cdot\, (h * \psi_{k_2} \otimes \psi_{j_2}) * \psi_{k_2} \otimes \psi_{j_2}$$

where at least two of the families $(\phi_{k_1})_{k_1}$ are lacunary and at least two of the families $(\phi_{j_1})_{j_1}$ are lacunary.

4.2.3 General Symbols

The extension from special symbols to general symbols can be treated as specified in Chapter 2.13 of [8]. With abuse of notations, we will proceed the discussion as in the previous section with recognition of the fact that bump functions do not necessarily equal to 1 on their supports, which prevents simple manipulation as before.

One notices that I^1 generates bi-parameter paraproduct as previously. In Case II^1, since $k_1 \ll k_2$, $\widehat{\varphi}_{k_2}(\xi_1)$ and $\widehat{\varphi}_{k_2}(\xi_2)$ behave like $\widehat{\varphi}_{k_2}(\xi_1 + \xi_2)$. One could obtain (4.2.1) as a result. To make the argument rigorous, one considers the Taylor expansions

$$\widehat{\varphi}_{k_2}(\xi_1) = \widehat{\varphi}_{k_2}(\xi_1 + \xi_2) + \sum_{l_1 > 0} \frac{\widehat{\varphi}_{k_2}^{(l_1)}(\xi_1 + \xi_2)}{l_1!}(-\xi_2)^{l_1}$$

$$\widehat{\varphi}_{k_2}(\xi_2) = \widehat{\varphi}_{k_2}(\xi_1+\xi_2) + \sum_{l_2>0} \frac{\widehat{\varphi}_{k_2}^{(l_2)}(\xi_1+\xi_2)}{l_2!}(-\xi_1)^{l_2}$$

There are some abuse of notations in the sense that $\widehat{\varphi}_{k_2}(\xi_1+\xi_2)$ in both equations do not represent for the same function - they correspond to $\widehat{\varphi}_{k_2}(\xi_1)$ and $\widehat{\varphi}_{k_2}(\xi_2)$ respectively, and share the common feature that they are non-lacunary bump functions. Let $\widetilde{\varphi}_{k_2}(\xi_1+\xi_2)$ denote the product of the two and one can rewrite II^1 as

$$\underbrace{\sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widetilde{\varphi}_{k_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)}_{II_0^1} +$$
$$\underbrace{\sum_{0<l_1+l_2\leq M}\sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\frac{\widehat{\varphi}_{k_2}^{(l_1)}(\xi_1+\xi_2)}{l_1!}\frac{\widehat{\varphi}_{k_2}^{(l_2)}(\xi_1+\xi_2)}{l_2!}(-\xi_1)^{l_2}(-\xi_2)^{l_1}\widehat{\psi}_{k_2}(\xi_3)}_{II_1^1} +$$
$$\underbrace{\sum_{l_1+l_2>M}\sum_{k_1 \ll k_2} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\frac{\widehat{\varphi}_{k_2}^{(l_1)}(\xi_1+\xi_2)}{l_1!}\frac{\widehat{\varphi}_{k_2}^{(l_2)}(\xi_1+\xi_2)}{l_2!}(-\xi_1)^{l_2}(-\xi_2)^{l_1}\widehat{\psi}_{k_2}(\xi_3)}_{II_{\text{rest}}^1}$$

where $M \gg |\alpha_1|$.

One observes that II_0^1 can be "completed" to obtain (4.2.1) as desired.

One can simplify II_1^1 as

$$\sum_{0<l_1+l_2\leq M}\sum_{\mu=100}^{\infty}\sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\frac{\widehat{\varphi}_{k_2}^{(l_1)}(\xi_1+\xi_2)}{l_1!}\frac{\widehat{\varphi}_{k_2}^{(l_2)}(\xi_1+\xi_2)}{l_2!}(-\xi_1)^{l_2}(-\xi_2)^{l_1}\widehat{\psi}_{k_2}(\xi_3)$$
$$= \sum_{0<l_1+l_2\leq M}\sum_{\mu=100}^{\infty}\sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)2^{-k_2l_1}\widehat{\varphi}_{k_2,l_1}(\xi_1+\xi_2)2^{-k_2l_2}\widehat{\varphi}_{k_2,l_2}(\xi_1+\xi_2)(-\xi_1)^{l_2}(-\xi_2)^{l_1}\widehat{\psi}_{k_2}(\xi_3)$$
$$\sim \sum_{0<l_1+l_2\leq M}\sum_{\mu=100}^{\infty}\sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)2^{-k_2l_1}\widehat{\varphi}_{k_2,l_1}(\xi_1+\xi_2)2^{-k_2l_2}\widehat{\varphi}_{k_2,l_2}(\xi_1+\xi_2)2^{k_1l_1}2^{k_1l_2}\widehat{\psi}_{k_2}(\xi_3)$$
$$= \sum_{0<l_1+l_2\leq M}\sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\varphi}_{k_2,l_1}(\xi_1+\xi_2)\widehat{\varphi}_{k_2,l_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)$$

$$= \sum_{0<l_1+l_2\leq M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \underbrace{\sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\tilde{\varphi}}_{k_2,l_1,l_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)}_{II^1_{1,\mu}}$$

where $\widehat{\tilde{\varphi}}_{k_2,l_1,l_2}(\xi_1+\xi_2)$ denotes an L^∞-normalized non-lacunary bump function supported at scale 2^{k_2}. One notices that $II^1_{1,\mu}$ has a form similar to (4.2.1) and can be rewritten as

$$\sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\phi}_{k_1}(\xi+\xi_2)\widehat{\tilde{\varphi}}_{k_2,l_1,l_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)\widehat{\psi}_{k_2}(\xi_1+\xi_2+\xi_3)$$

Meanwhile.

$$\begin{aligned} II^1_{\text{rest}} &= \sum_{l_1+l_2>M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} \sum_{k_2=k_1+\mu} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\tilde{\varphi}}_{k_2,l_1,l_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3) \\ &\leq \sum_{\mu=100}^{\infty} 2^{-\mu M} \underbrace{\sum_{k_2=k_1+\mu} \sum_{l_1+l_2>M} \widehat{\phi}_{k_1}(\xi_1)\widehat{\phi}_{k_1}(\xi_2)\widehat{\tilde{\varphi}}_{k_2,l_1,l_2}(\xi_1+\xi_2)\widehat{\psi}_{k_2}(\xi_3)}_{II^1_{\text{rest},\mu}} \end{aligned}$$

where $m^1_\mu := II^1_{\text{rest},\mu}$ is a Coifman-Meyer symbol satisfying

$$\left|\partial^{\alpha_1} m^1_\mu\right| \lesssim 2^{\mu|\alpha_1|} \frac{1}{|(\xi_1,\xi_2)|^{|\alpha_1|}}$$

for sufficiently many multi-indices α_1.

Same procedure can be applied to study $a_2(\eta_1,\eta_2)b_2(\eta_1\eta_2,\eta_3)$. One can now combine all the arguments above to decompose and study

$$T_{ab} = T_{ab}^{I^1I^2} + T_{ab}^{I^1II^2} + T_{ab}^{I^1III^2} + T_{ab}^{II^1I^2} + T_{ab}^{II^1II^2} + T_{ab}^{II^1III^2} + T_{ab}^{III^1I^1} + T_{ab}^{III^2II^2} + T_{ab}^{III^1III^2}$$

where each operator takes the form

$$\int_{\mathbb{R}^6} \text{symbol}\cdot\widehat{f_1}(\xi_1)\widehat{f_2}(\xi_2)\widehat{g_1}(\eta_1)\widehat{g_2}(\eta_2)\widehat{h}(\xi_3,\eta_3)e^{2\pi i x(\xi_1+\xi_2+\xi_3)}e^{2\pi i y(\eta_1+\eta_2+\eta_3)}d\xi_1 d\xi_2 d\xi_3 d\eta_1 d\eta_2 d\eta_3$$

with the symbol for each operator specified as follows.

1. $T_{ab}^{I^1 I^2}$ is a bi-parameter paraproduct as in the special case.

2. $T_{ab}^{II^1 I^2}$: $(II_0^1 + II_1^1 + II_{\text{rest}}^1) \otimes I^2$

   where the operator associated with each symbol can be written as

   (i)

$$T^{II_0^1 I^2} := \sum_{\substack{k_1 \ll k_2 \\ j \in \mathbb{Z}}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \varphi_{k_2} \Big)(g_1 * \tilde{\phi}_j)(g_2 * \tilde{\phi}_j)(h * \psi_{k_2} \otimes \phi_j) * \psi_{k_2} \otimes \phi_j$$

   (ii)

$$T^{II_1^1 I^2} := \sum_{0 < l_1 + l_2 \leq M} \sum_{\mu = 100}^{\infty} 2^{-\mu(l_1 + l_2)} T^{II_{1,\mu}^1 I^2}$$

   with

$$T^{II_{1,\mu}^1 I^2} := \sum_{\substack{k_2 = k_1 + \mu \\ j \in \mathbb{Z}}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \tilde{\varphi}_{k_2, l_1, l_2} \Big)(g_1 * \tilde{\phi}_j)(g_2 * \tilde{\phi}_j)(h * \psi_{k_2} \otimes \phi_j) * \psi_{k_2} \otimes \phi_j$$

   (iii)

$$T^{II_{\text{rest}}^1 I^2} := \sum_{\mu = 100}^{\infty} 2^{\mu M} T^{II_{\text{rest},\mu}^1 I^2}$$

   One notices that $II_{\text{rest},\mu}^1$ and I^2 are Coifman-Meyer symbols. $T^{II_{\text{rest},\mu}^1 I^2}$ is therefore a bi-parameter paraproduct and one can apply the Coifman-Meyer theorem on paraproducts to derive the bound of type $O(2^{|\alpha_1|\mu})$, which would suffice due to the decay factor $2^{-\mu M}$.

3. $T^{II^1 II^2}$: $(II_0^1 + II_1^1 + II_{\text{rest}}^1) \otimes (II_0^2 + II_1^2 + II_{\text{rest}}^2)$

   where the operator associated with each symbol can be written as

   (i)

$$T^{II_0^1 II_0^2} := \sum_{\substack{k_1 \ll k_2 \\ j_1 \ll j_2}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \varphi_{k_2} \Big)\Big(((g_1 * \phi_{j_1})(g_2 * \phi_{j_1}) * \phi_{j_1}) * \varphi_{j_2} \Big)$$
$$(h * \psi_{k_2} \otimes \psi_{j_2}) * \psi_{k_2} \otimes \psi_{j_2}$$

(ii)

$$T^{II_1^1 II_0^2} := \sum_{0<l_1+l_2\le M} \sum_{\mu=100}^{\infty} 2^{-\mu(l_1+l_2)} T^{II_{1,\mu}^1 II_0^2}$$

with

$$T^{II_{1,\mu}^1 II_0^2}$$

$$:= \sum_{\substack{k_2=k_1+\mu \\ j_1 \ll j_2}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \tilde{\varphi}_{k_2,l_1,l_2}\Big)\Big(((g_1 * \phi_{j_1})(g_2 * \phi_{j_1}) * \phi_{j_1}) * \varphi_{j_2}\Big)$$

$$(h * \psi_{k_2} \otimes \psi_{j_2}) * \psi_{k_2} \otimes \psi_{j_2}$$

(iii)

$$T^{II_{\text{rest}}^1 II_0^2} := \sum_{\mu=100}^{\infty} 2^{\mu M} T^{II_{\text{rest},\mu}^1 II_0^2}$$

where $T^{II_{\text{rest},\mu}^1 II_0^2}$ is a multiplier operator with the symbol

$$m_\mu^1 \otimes II_0^2$$

which generates a model similar as $T^{I^1 II_0^2}$ or, by symmetry, $T^{II_0^1 I^2}$.

(iv)

$$T^{II_1^1 II_1^2} := \sum_{\substack{0<l_1+l_2\le M \\ l_1'+l_2'\le M'}} \sum_{\mu,\mu'=100}^{\infty} 2^{-\mu(l_1+l_2)} 2^{\mu'(l_1'+l_2')} T^{II_{1,\mu}^1 II_{1,\mu'}^2}$$

with

$$T^{II_{1,\mu}^1 II_{1,\mu'}^2}$$

$$:= \sum_{\substack{k_2=k_1+\mu \\ j_2=j_1+\mu'}} \Big(((f_1 * \phi_{k_1})(f_2 * \phi_{k_1}) * \phi_{k_1}) * \tilde{\varphi}_{k_2,l_1,l_2}\Big)\Big(((g_1 * \phi_{j_1})(g_2 * \phi_{j_1}) * \phi_{j_1}) * \tilde{\varphi}_{j_2,l_1',l_2'}\Big)$$

$$(h * \psi_{k_2} \otimes \psi_{j_2}) * \psi_{k_2} \otimes \psi_{j_2}$$

(v)

$$T^{II_{\text{rest}}^1 II_1^2} := \sum_{\mu=100}^{\infty} 2^{\mu M} T^{II_{\text{rest},\mu}^1 II_1^2}$$

where $T^{II^1_{\text{rest},\mu}II^2_1}$ has the symbol

$$m^1_\mu \otimes II^2_1$$

which generates a model similar as $T^{I^1II^2_1}$ or $T^{II^1_1I^2}$.

(vi)

$$T^{II^1_{\text{rest}}II^2_{\text{rest}}} := \sum_{\mu,\mu'=100}^{\infty} 2^{\mu M}2^{\mu' M'} T^{II^1_{\text{rest},\mu}II^2_{\text{rest},\mu'}}$$

where $T^{II^1_{\text{rest},\mu}II^2_{\text{rest},\mu'}}$ is associated with the symbol

$$m^1_\mu \otimes m^2_{\mu'}$$

which generates a model similar as $T^{II^1_{\text{rest},\mu}I^2}$, $T^{I^1II^2_0}$ or $T^{II^1_0I^2}$.

4. $T^{III^1II^2}$, $T^{III^1I^2}$ and $T^{III^1III^2}$ can be studied by the exact same reasoning for $T^{II^1II^2}$, $T^{II^1I^2}$ and $T^{II^1II^2}$ by the symmetry between symbols II and III.

4.3 Discretization

With discretization procedure specified in Chapter 2.2 of [8], one can reduce the above operators into the following discrete model operators listed in Theorem 3.3.1:

Table 4.1: Operators vs. Discrete Model Operators

$T^{II^1_0I^2}$	$\longrightarrow$	$\Pi_{\text{flag}^0\otimes\text{paraproduct}}$
$T^{II^1_{1,\mu}I^2}$	$\longrightarrow$	$\Pi_{\text{flag}^\mu\otimes\text{paraproduct}}$
$T^{II^1_0II^2_0}$	$\longrightarrow$	$\Pi_{\text{flag}^0\otimes\text{falg}^0}$
$T^{II^1_0II^2_{1,\mu'}}$	$\longrightarrow$	$\Pi_{\text{flag}^0\otimes\text{flag}^{\mu'}}$
$T^{II^1_{1,\mu}II^2_{1,\mu'}}$	$\longrightarrow$	$\Pi_{\text{flag}^\mu\otimes\text{flag}^{\mu'}}$

CHAPTER 5

SIZES AND ENERGIES

Definition 5.0.1. We define sizes and energies as follows, where functions in non-lacunary and lacunary families are denoted by φ and ψ respectively.

(1)

$$\text{size}_{\mathcal{I}}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \sup_{I \in \mathcal{I}} \frac{|\langle f, \varphi_I \rangle|}{|I|^{\frac{1}{2}}}$$

(2)

$$\text{size}_{\mathcal{I}}((\langle f, \psi_I \rangle)_{I \in \mathcal{I}}) := \sup_{I_0 \in \mathcal{I}} \frac{1}{|I_0|} \left\| \Big(\sum_{\substack{I \in I_0 \\ I \in \mathcal{I}}} \frac{|\langle f, \psi_I \rangle|^2}{|I|} \chi_I \Big)^{\frac{1}{2}} \right\|_{1,\infty}$$

(3)

$$\text{energy}_{\mathcal{I}}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \sup_{n \in \mathbb{Z}} 2^n \sup_{\mathbb{D}} \sum_{I \in \mathbb{D}} |I|$$

where $\mathbb{D}$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying

$$\frac{|\langle f, \varphi_I \rangle|}{|I|^{\frac{1}{2}}} > 2^n$$

(4)

$$\text{energy}_{\mathcal{I}}((\langle f, \psi_I \rangle)_{I \in \mathcal{I}}) := \sup_{n \in \mathbb{Z}} 2^n \sup_{\mathbb{D}} \sum_{I \in \mathbb{D}} |I|$$

where $\mathbb{D}$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying

$$\frac{1}{|I|} \left\| \Big(\sum_{\substack{\tilde{I} \in I \\ \tilde{I} \in \mathcal{I}}} \frac{|\langle f, \psi_{\tilde{I}} \rangle|^2}{|\tilde{I}|} \chi_{\tilde{I}} \Big)^{\frac{1}{2}} \right\|_{1,\infty} > 2^n$$

(5)

$$\text{energy}^p_{\mathcal{I}}((\langle f, \varphi_I \rangle)_{I \in \mathcal{I}}) := \sum_{n \in \mathbb{Z}} 2^n \sup_{\mathbb{D}_n} \sum_{I \in \mathbb{D}_n} |I|$$

where $\mathbb{D}$ ranges over all collections of disjoint dyadic intervals in $\mathcal{I}$ satisfying

$$\frac{|\langle f, \varphi_I \rangle|}{|I|^{\frac{1}{2}}} > 2^n$$

5.1 Useful Facts about Sizes and Energies

The following propositions describe facts about size and energies which will be heavily employed later on. The proof of the propositions can be found in Chapter 2 of [8].

Proposition 5.1.1 (John-Nirenburg). *Let $\mathcal{I}$ be a finite collection of dyadic intervals. For any sequence $(a_I)_{I\in\mathcal{I}}$ and $r > 0$, define the BMO-norm for the sequence as*

$$\|(a_I)_I\|_{BMO(r)} := \sup_{I_0 \in \mathcal{I}} \frac{1}{|I_0|^{\frac{1}{r}}} \left\| \left(\sum_{I \subseteq I_0} \frac{|a_I|^2}{|I|} \chi_I(x) \right)^{\frac{1}{2}} \right\|_r$$

Then for any $0 < p < q < \infty$,

$$\|(a_I)_I\|_{BMO(p)} \simeq \|(a_I)_I\|_{BMO(q)}.$$

Proposition 5.1.2. *Suppose $f \in L^1(\mathbb{R})$. Then*

$$size_{\mathcal{I}}((\langle f, \varphi_I \rangle)_I), size_{\mathcal{I}}((\langle f, \psi_I \rangle)_I) \lesssim \sup_{I \in \mathcal{I}} \int_{\mathbb{R}} |f| \tilde{\chi}_I^M dx$$

for $M > 0$ and the implicit constant depends on M. $\tilde{\chi}_I$ is an L^∞-normalized bump function adapted to I.

Proposition 5.1.3. *Suppose $f \in L^1(\mathbb{R})$. Then*

$$energy_{\mathcal{I}}((\langle f, \varphi_I \rangle))_I, energy_{\mathcal{I}}((\langle f, \psi_I \rangle))_I \lesssim \|f\|_1.$$

Proposition 5.1.4. *Suppose that for any $I \in \mathcal{I}'$, $I \cap S_1 \neq \emptyset$ for some $S_1 \subset \mathbb{R}^2$ and for any $J \in \mathcal{J}'$, $J \cap S'_1 \neq \emptyset$ for some $S'_1 \subset \mathbb{R}^2$.Then*

$$size_{\mathcal{I}'}((\langle B_I^{\#_1}, \varphi_I\rangle)_{I\in\mathcal{I}'}) \lesssim \sup_{K\cap S_1\neq\emptyset} \frac{|\langle f_1, \phi_K^1\rangle|}{|K|^{\frac{1}{2}}} \sup_{K\cap S_1\neq\emptyset} \frac{|\langle f_2, \phi_K^2\rangle|}{|K|^{\frac{1}{2}}}$$

$$size_{\mathcal{J}'}((\langle \tilde{B}_J^{\#_2}, \varphi_J\rangle)_{J\in\mathcal{J}'}) \lesssim \sup_{L\cap S'_1\neq\emptyset} \frac{|\langle g_1, \phi_L^1\rangle|}{|L|^{\frac{1}{2}}} \sup_{L\cap S'_1\neq\emptyset} \frac{|\langle g_2, \phi_L^2\rangle|}{|L|^{\frac{1}{2}}}$$

Proposition 5.1.5. *Suppose for any $I \in \mathcal{I}'$, $I \cap S_1 \neq \emptyset$ and $\bigcup_{I\in\mathcal{I}'} I \subseteq S_2$, then for $0 \leq \theta_1, \theta_2, \theta_3 < 1$ and $\theta_1 + \theta_2 + \theta_3 = 1$, $1 \leq p < \infty$, $1 < p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$,*

(i)

$$energy_{\mathcal{I}'}((\langle B_I, \varphi_I\rangle)_{I\in\mathcal{I}'}) \lesssim size_{K\in\mathcal{K}:K\cap S_1\neq\emptyset}((\langle f_1, \varphi_K\rangle)_K)^{1-\theta_1} size_{K\in\mathcal{K}:K\cap S_1\neq\emptyset}((\langle f_2, \psi_K\rangle)_K)^{1-\theta_2} \cdot |F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3}$$

$$energy^p_{\mathcal{I}'}((\langle B_I, \varphi_I\rangle)_{I\in\mathcal{I}'}) \lesssim size_{K\in\mathcal{K}:K\cap S_1\neq\emptyset}((\langle f_1, \varphi_K\rangle)_K)^{1-\theta_1} size_{K\in\mathcal{K}:K\cap S_1\neq\emptyset}((\langle f_2, \psi_K\rangle)_K)^{1-\theta_2} \cdot |F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{1}{p'}}$$

$$energy^p_{\mathcal{J}'}((\langle \tilde{B}_J, \varphi_J\rangle)_{J\in\mathcal{J}'}) \lesssim size_{L\in\mathcal{L}:L\cap S'_1\neq\emptyset}((\langle g_1, \varphi_K\rangle)_L)^{1-\theta_1} size_{L\in\mathcal{L}:L\cap S'_1\neq\emptyset}((\langle g_2, \psi_L\rangle)_L)^{1-\theta_2} \cdot |G_1|^{\theta_1}|G_2|^{\theta_2}|S'_2|^{\theta_3-\frac{1}{p'}}$$

5.2 Proof of Propositions 5.1.4

Without loss of generality, we will prove the the first size estimate and the second follows the same argument. One recalls the definition of

$$\text{size}_{\mathcal{I}'}((\langle B_I^{\#_1}, \varphi_I\rangle)_{I\in\mathcal{I}'} = \frac{|\langle B_{I_0}^{\#_1}(f_1, f_2), \varphi_{I_0}^1\rangle|}{|I_0|^{\frac{1}{2}}}$$

for some $I_0 \in \mathcal{I}'$ with the property that $I_0 \cap S_1 \neq \emptyset$ by the assumption. Then

$$\begin{aligned}\frac{|\langle B^{\#_1}_{I_0}(f_1, f_2), \varphi^1_{I_0}\rangle|}{|I_0|^{\frac{1}{2}}} \leq & \frac{1}{|I_0|} \sum_{K:|K|\sim 2^{\#_1}|I_0|} \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \phi^1_K\rangle||\langle f_2, \phi^2_K\rangle||\langle \tilde{\chi}_{I_0}, \phi^3_K\rangle| \\ = & \frac{1}{|I_0|} \sum_{K:|K|\sim 2^{\#_1}|I_0|} \frac{|\langle f_1, \phi^1_K\rangle|}{|K|^{\frac{1}{2}}} \frac{|\langle f_2, \phi^2_K\rangle|}{|K|^{\frac{1}{2}}} |\langle \tilde{\chi}_{I_0}, |K|^{\frac{1}{2}} \phi^3_K\rangle|\end{aligned}$$

where $\tilde{\chi}_{I_0} := |I_0|^{\frac{1}{2}} \varphi^1_{I_0}$ is an L^∞-normalized bump function. In the Walsh case, $|K| > |I|$ implies that $K \supseteq I$ and $K \cap S_1 \neq \emptyset$. Therefore

$$\begin{aligned}\frac{|\langle B^{\#_1}_{I_0}(f_1, f_2), \varphi^1_{I_0}\rangle|}{|I_0|^{\frac{1}{2}}} \leq & \frac{1}{|I_0|} \sup_{K\cap S_1 \neq \emptyset} \frac{|\langle f_1, \phi^1_K\rangle|}{|K|^{\frac{1}{2}}} \sup_{K\cap S_1 \neq \emptyset} \frac{|\langle f_2, \phi^2_K\rangle|}{|K|^{\frac{1}{2}}} \sum_{K:|K|\sim 2^{\#_1}|I_0|} |\langle \tilde{\chi}_{I_0}, |K|^{\frac{1}{2}} \phi^3_K\rangle| \\ \lesssim & \frac{1}{|I_0|} \sup_{K\cap S_1 \neq \emptyset} \frac{|\langle f_1, \phi^1_K\rangle|}{|K|^{\frac{1}{2}}} \sup_{K\cap S_1 \neq \emptyset} \frac{|\langle f_2, \phi^2_K\rangle|}{|K|^{\frac{1}{2}}} \cdot |I_0|\end{aligned}$$

where the last inequality hold trivially if χ_{I_0} is a characteristic function of I_0 as in the Walsh case. In the general case when $\tilde{\chi}_{I_0}$ is a bump function adapted to I_0, the inequality follows from the fact that fix an interval I_0, $\{K : |K| \sim 2^{\#_1}|I|\}$ is a disjoint collection of intervals. This completes the proof of the proposition.

5.3 Proof of Proposition 5.1.5

The estimates described in Proposition 5.1.5 depends on the following one-dimensional stopping-time decomposition.

5.3.1 One-Dimensional Stopping-Time Decomposition - Maximal Intervals

Given finiteness of $\mathcal{K}$, there exists some $K_1 \in \mathbb{Z}$ such that $\frac{|\langle f_1, \varphi_K^1\rangle|}{|K|^{\frac{1}{2}}} \leq C_1 2^{K_1}\text{energy}_{\mathcal{K}}((\langle f_1, \varphi_K\rangle)_K)$. We can pick the largest interval $K_{\max}$ such that

$$\frac{|\langle f_1, \varphi_{K_{\max}}^1\rangle|}{|K_{\max}|^{\frac{1}{2}}} > C_1 2^{K_1-1}\text{energy}_{\mathcal{K}}((\langle f_1, \varphi_K\rangle)_K).$$

Then we define a tree

$$U := \{K \in \mathcal{K} : K \subseteq K_{\max}\},$$

and let $K_U := K_{\max}$, usually called as tree-top. Now we look at $\mathcal{K} \setminus U$ and repeat the above step to choose maximal intervals and collect their subintervals in their corresponding sets. Since $\mathcal{K}$ is finite, the process will eventually end. We then collect all U's in a set $\mathbb{U}_{K_1-1}$. Next we repeat the above algorithm to $\mathcal{K} \setminus \bigcup_{U \in \mathbb{U}_{K_1-1}} U$. We thus obtain a decomposition $\mathcal{K} = \bigcup_k \bigcup_{U \in \mathbb{U}_k} U$. If, otherwise, the sequence is formed in terms of bump functions in lacunary family, then the same procedure can be performed to

$$\frac{1}{|K|}\left\|\Big(\sum_{K' \subseteq K} \frac{|\langle f_2, \psi_{K'}\rangle|^2}{|K'|}\chi_{K'}\Big)^{\frac{1}{2}}\right\|_{1,\infty}$$

One simple observation is that the above procedure can be applied to general sequences indexed by dyadic intervals.

The next proposition summarizes the information from the stopping-time decomposition.

Proposition 5.3.1. *Suppose* $\mathcal{K} = \bigcup_k \bigcup_{\mathbb{U}_k} U$ *is a decomposition obtained from the stopping-time algorithm specified above, then for any* $k \in \mathbb{Z}$*, one has*

$$2^{k-1}\textit{energy}_{\mathcal{K}}((\langle f, \phi_K\rangle)_K) \leq \textit{size}_{\mathcal{K}}^k((\langle f, \phi_K\rangle)_K) \leq \min(2^k\textit{energy}_{\mathcal{K}}((\langle f, \phi_K\rangle)_K), \textit{size}_{\mathcal{K}}((\langle f, \phi_K\rangle)_K))$$

where

$$size^k_{\mathcal{K}}((\langle f, \phi_K\rangle)_K) := size_{\bigcup_{\mathbb{U}_k} U}((\langle f, \phi_K\rangle)_K)$$

In addition,

$$\sum_{U \in \mathbb{U}_k} |K_U| \lesssim 2^{-k}$$

The next lemma follows from the stopping-time decomposition and the proposition, whose proof is discussed in details in Chapter 2.9 of [8]. It plays an important role in proving Proposition 5.1.5.

Lemma 5.3.2. *Suppose $\mathcal{K}$ is a finite collection of dyadic intervals. Then*

$$\left|\sum_{K \in \mathcal{I}} \frac{1}{|K|}\langle f_1, \phi_K\rangle\langle f_2, \phi_K\rangle\langle f_3, \phi_K\rangle\right| \lesssim \prod_{i=1}^{3} size_{\mathcal{K}}((\langle f_i, \phi_K\rangle)_K)^{1-\theta_i} energy_{\mathcal{K}}((\langle f_i, \phi_K\rangle)_K)^{\theta_i}$$

We are now ready to prove the proposition.

Proof of Proposition 5.1.5. (i) By definition of energy, there exists $n_0 \in \mathbb{Z}$ and a disjoint collection of dyadic intervals $\mathbb{D}$ such that

$$\text{energy}_{\mathcal{I}'}((\langle B_I, \varphi_I\rangle)_{I \in \mathcal{I}'}) := 2^{n_0} \sum_{\substack{I \in \mathbb{D} \\ I \in \mathcal{I}'}} |I| \tag{5.3.1}$$

where

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} > 2^{n_0}$$

One recalls that in the Walsh case

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} := \frac{1}{|I|}\left|\sum_{\substack{K \in \mathcal{K} \\ K \supseteq I}} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1, \phi^1_K\rangle\langle f_2, \phi^2_K\rangle\langle \tilde{\chi}_I, \phi^3_K\rangle\right|$$

where $\tilde{\chi}_I := \frac{\varphi_I}{|I|^{\frac{1}{2}}}$ is an L^∞-normalized bump function. By the assumption that $I \cap S_1 \neq \emptyset$ for any $I \in \mathcal{I}'$, one can derive that $K \cap S_1 \neq \emptyset$ given $K \supseteq I$.

Therefore, one can rewrite

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} = \frac{1}{|I|}\Big|\sum_{\substack{K\in\mathcal{K}\\ K\supseteq I\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1, \phi_K^1\rangle\langle f_2, \phi_K^2\rangle\langle \varphi_I^\infty, \phi_K^3\rangle\Big|$$

Case I: ϕ_K^3 is lacunary. One notices that in this case the only degenerate case $\langle \psi_K^3, \varphi_I^1\rangle \neq 0$ is when $K \supseteq I$, which is usually called the biest trick:

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} = \frac{1}{|I|}\Big|\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1, \phi_K^1\rangle\langle f_2, \phi_K^2\rangle\langle \tilde{\chi}_I, \psi_K^3\rangle\Big|$$

One also notices that $\bigcup_{I\in\mathcal{I}'} I \subseteq S_2$, which gives the following localization in the Walsh case:

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} = \frac{1}{|I|}\Big|\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1, \phi_K^1\rangle\langle f_2, \phi_K^2\rangle\langle \tilde{\chi}_I, \psi_K^3\cdot\chi_{S_2}\rangle\Big|$$

Let $B_{S_2}^{S_1}(x) := \sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1, \phi_K^1\rangle\langle f_2, \phi_K^2\rangle\psi_K^3\cdot\chi_{S_2}(x)$ and one can rewrite

$$\text{energy}_{\mathcal{I}'}((\langle B_I, \varphi_I\rangle)_{I\in\mathcal{I}'}) = \text{energy}_{\mathcal{I}'}((\langle B_{S_2}^{S_1}, \varphi_I\rangle)_{I\in\mathcal{I}'}) \lesssim \sum_{\substack{I\in\mathbb{D}\\ I\in\mathcal{I}'}}|\langle B_{S_2}^{S_1}, \tilde{\chi}_I\rangle| \lesssim \|B_{S_2}^{S_1}\|_1$$

where the last inequality follows from the disjointness of I's.

Case II: ϕ_K^3 is non-lacunary. One can drop the condition $K \supseteq I$ in the sum and bound the above expression by

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} \leq \frac{1}{|I|}\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}|\langle f_1, \phi_K^1\rangle||\langle f_2, \phi_K^2\rangle|\langle |\tilde{\chi}_I|, |\varphi_K^3|\rangle$$

One can again apply the localization using the assumption that $\bigcup_{I\in\mathcal{I}'} I \subseteq S_2$ to deduce that

$$\frac{|\langle B_I, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} \leq \frac{1}{|I|}\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}|\langle f_1, \phi_K^1\rangle||\langle f_2, \phi_K^2\rangle|\langle |\tilde{\chi}_I|\chi_{S_2}, |\varphi_K^3|\rangle$$

Let $B^{S_1}_{S_2}(x) := \sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\phi^1_K\rangle||\langle f_2,\phi^2_K\rangle||\varphi^3_K|\chi_{S_2}(x)$,

$$\text{energy}_{\mathcal{I}'}((\langle B_I,\varphi_I\rangle)_{I\in\mathcal{I}'}) \leq \text{energy}_{\mathcal{I}'}((\langle B^{S_1}_{S_2},|\varphi_I|\rangle)_{I\in\mathcal{I}'}) \lesssim \sum_{\substack{I\in\mathbb{D}\\ I\in\mathcal{I}'}} |\langle B^{S_1}_{S_2},|\tilde{\chi}_I|\rangle| \lesssim \|B^{S_1}_{S_2}\|_1$$

Estimate of $\|B^{S_1}_{S_2}\|_1$. With the abuse of notations, $\|B^{S_1}_{S_2}\|_1$ represents for different functions in Case I and II. Nevertheless, they can be estimated by the same argument. One has that for some $\eta\in L^\infty$ with $\|\eta\|_{L^\infty}=1$,

$$\|B^{S_1}_{S_2}\|_1 \leq |\langle B^{S_1}_{S_2},\eta\rangle| \leq \sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\phi^1_K\rangle||\langle f_2,\phi^2_K\rangle||\langle \eta,\chi_{S_2}\cdot\phi^3_K\rangle|$$

where

$$\phi^3_K = \begin{cases} \psi^3_K & \text{in Case } I \\ |\varphi^3_K| & \text{in Case } II \end{cases}$$

One can now apply Lemma 5.3.2 and obtain

$$\|B^{S_1}_{S_2}\|_1 \lesssim$$
$$\text{size}_{K\cap S_1\neq\emptyset}((\langle f_1,\phi^1_K\rangle)_K)^{1-\theta_1}\text{size}_{K\cap S_1\neq\emptyset}((\langle f_2,\phi^2_K\rangle)_K)^{1-\theta_2}\text{size}_{\mathcal{K}}((\langle \eta\chi_{S_2},\phi^3_K\rangle)_K)^{1-\theta_3}$$
$$\text{energy}_{\mathcal{K}}((\langle f_1,\phi^1_K\rangle)_K)^{\theta_1}\text{energy}_{\mathcal{K}}((\langle f_2,\phi^2_K\rangle)_K)^{\theta_2}\text{energy}_{\mathcal{K}}((\langle \eta\chi_{S_2},\phi^3_K\rangle)_K)^{\theta_3}$$

for some $0\leq\theta_1,\theta_2,\theta_3<1$ with $\theta_1+\theta_2+\theta_3=1$. By applying Proposition 5.1.2 and using the fact that $\eta\cdot\chi_{S_2}\in L^\infty$ with $\|\eta\|_{L^\infty}=1$,

$$\text{size}_{K\in\mathcal{K}}((\langle \eta\chi_{S_2},\phi^3_K\rangle)_K) \lesssim 1$$

$$\text{energy}_{\mathcal{K}}((\langle \eta\chi_{S_2},\phi^3_K\rangle)_K) \lesssim \|\eta\chi_{S_2}\|_1 \leq |S_2|$$

One combines the above estimates with the energy estimates described in Proposition 5.3.1 to conclude that

$$\text{energy}_{\mathcal{I}'}((\langle B_I,\varphi_I\rangle)_{I\in\mathcal{I}'}) \lesssim$$

$$\text{size}_{K\cap S_1\neq\emptyset}((\langle f_1,\phi_K^1\rangle)_K)^{1-\theta_1}\text{size}_{K\cap S_1\neq\emptyset}((\langle f_2,\phi_K^2\rangle)_K)^{1-\theta_2}\cdot|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3}$$

for some $0\leq\theta_1,\theta_2,\theta_3<1$ with $\theta_1+\theta_2+\theta_3=1$.

(ii) One first observes that

$$2^n\sum_{\substack{I\in\mathbb{D}_n\\ I\in\mathcal{I}'}}|I|\leq(\sum_n 2^{np}\sum_{\substack{I\in\mathbb{D}_n\\ I\in\mathcal{I}'}}|I|)^{\frac{1}{p}}$$

for any collection of intervals $\mathbb{D}_n$ satisyfying

(a) $\mathbb{D}_n$ is a disjoint collection of intervals

(b) For any $I\in\mathbb{D}_n$,

$$\frac{|\langle B_I,\varphi_I\rangle|}{|I|^{\frac{1}{2}}}>2^n$$

One notices that for each n, there exists a disjoint collection of intervals, denoted by $\mathbb{D}_n^0$ satisfying conditions (a) and (b) specified above. One can rewrite

$$\text{energy}^p_{\mathcal{I}'}(((\langle B_I,\varphi_I\rangle)_{I\in\mathcal{I}'})=\Big(\sum_n 2^{np}\sum_{\substack{I\in\mathbb{D}_n^0\\ I\in\mathcal{I}'}}|I|\Big)^{\frac{1}{p}}\tag{5.3.2}$$

Fix $n\in\mathbb{Z}$, the localization argument in the proof of (i) can be applied here as well and one can rewrite

$$2^n<\frac{|\langle B_I,\varphi_I\rangle|}{|I|^{\frac{1}{2}}}\begin{cases}\overset{\text{Case}I}{=}\ \frac{|\langle B_{S_2}^{S_1},\varphi_I\rangle|}{|I|^{\frac{1}{2}}} & \text{where } B_{S_2}^{S_1}:=\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\psi_K^3\cdot\chi_{S_2}\\ \overset{\text{Case}II}{\leq}\ \frac{|\langle B_{S_2}^{S_1},|\varphi_I|\rangle|}{|I|^{\frac{1}{2}}} & \text{where } B_{S_2}^{S_1}:=\sum_{\substack{K\in\mathcal{K}\\ K\cap S_1\neq\emptyset}}\frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\phi_K^1\rangle||\langle f_2,\phi_K^2\rangle||\varphi_K^3|\chi_{S_2}\end{cases}$$

for any $I \in \mathbb{D}_n^0$. One also notices that for any $x \in I$,

$$M(B_{S_2}^{S_1})(x) \geq \begin{cases} \frac{|\langle B_{S_2}^{S_1}, \varphi_I\rangle|}{|I|^{\frac{1}{2}}} & \text{where } B_{S_2}^{S_1} := \sum_{\substack{K \in \mathcal{K} \\ K \cap S_1 \neq \emptyset}} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \psi_K^3 \cdot \chi_{S_2} \\ \frac{|\langle B_{S_2}^{S_1}, |\varphi_I|\rangle|}{|I|^{\frac{1}{2}}} & \text{where } B_{S_2}^{S_1} := \sum_{\substack{K \in \mathcal{K} \\ K \cap S_1 \neq \emptyset}} \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \phi_K^1 \rangle| |\langle f_2, \phi_K^2 \rangle| |\varphi_K^3| \chi_{S_2} \end{cases}$$

which implies that

$$I \subseteq \{M(B_{S_2}^{S_1})(x) > 2^n\}$$

for any $I \in \mathbb{I}'$ satisfying condition (b).

Then by the disjointness of $\mathbb{D}_n$, one can estimate the energy as follows

$$\text{energy}_{\mathcal{I}'}^p((\langle B_I, \varphi_I \rangle)_{I \in \mathcal{I}'} \leq (\sum_n 2^{np} |\{M(B_{S_2}^{S_1})(x) > 2^n\}|)^{\frac{1}{p}} \lesssim \|M(B_{S_2}^{S_1})\|_p$$

One can then apply the fact that the maximal operator $M : L^p \to L^p$ for $p > 1$ and derive

$$\|M(B_{S_2}^{S_1})\|_p \lesssim \|B_{S_2}^{S_1}\|_p$$

One can now apply dualization and find $\chi_S \in L^{p'}$ with $\|\chi_S\|_{L^{p'}} = 1$ such that

$$\|B_{S_2}^{S_1}\|_p \lesssim |\langle B_{S_2}^{S_1}, \chi_S \rangle|$$

The linear form can be estimated using a similar argument described in the proof of (i). In particular,

$$\begin{aligned} &|\langle B_{S_2}^{S_1}, \chi_S \rangle| \lesssim \\ &\quad \text{size}_{K \cap S_1 \neq \emptyset}((\langle f_1, \phi_K^1 \rangle)_K)^{1-\theta_1} \text{size}_{K \cap S_1 \neq \emptyset}((\langle f_2, \phi_K^2 \rangle)_K)^{1-\theta_2} \text{size}_{\mathcal{K}}((\langle \chi_S \cdot \chi_{S_2}, \phi_K^3 \rangle)_K)^{1-\theta_3} \\ &\quad \text{energy}_{\mathcal{K}}((\langle f_1, \phi_K^1 \rangle)_K)^{\theta_1} \text{energy}_{\mathcal{K}}((\langle f_2, \phi_K^2 \rangle)_K)^{\theta_2} \text{energy}_{\mathcal{K}}((\langle \chi_S \cdot \chi_{S_2}, \phi_K^3 \rangle)_K)^{\theta_3} \end{aligned} \tag{5.3.3}$$

for some $0 \leq \theta_1, \theta_2, \theta_3 < 1$ with $\theta_1 + \theta_2 + \theta_3 = 1$. Here ϕ_K^3 are defined differently in Case I and II. However, one applies the same straightforward estimates that

$$\text{size}_{K \in \mathcal{K}}((\langle \chi_S \cdot \chi_{S_2}, \phi_K^3 \rangle)_K) \lesssim 1$$

$$\text{energy}_{\mathcal{K}}((\langle \chi_S \cdot \chi_{S_2}, \phi_K^3 \rangle)_K) \lesssim \|\chi_S \chi_{S_2}\|_1 \leq |S|^{\kappa} |S_2|^{1-\kappa}$$

for any $0 \leq \kappa \leq 1$. By plugging in the above estimates into 5.3.3, one has

$$\begin{aligned} |\langle B_{S_2}^{S_1}, \chi_S \rangle| \lesssim\ & \text{size}_{K \in \mathcal{K}: K \cap S_1 \neq \emptyset}((\langle f_1, \phi_K^1 \rangle)_K)^{1-\theta_1} \text{size}_{K \in \mathcal{K}: K \cap S_1 \neq \emptyset}((\langle f_2, \phi_K^2 \rangle)_K)^{1-\theta_2} \\ \cdot\ & \text{energy}_{\mathcal{K}}((\langle f_1, \phi_K^1 \rangle)_K)^{\theta_1} \text{energy}_{\mathcal{K}}((\langle f_2, \phi_K^2 \rangle)_K)^{\theta_2} |S|^{\kappa\theta_3} |S_2|^{(1-\kappa)\theta_3} \end{aligned}$$

Let $\theta_3 \kappa = \frac{1}{p'}$, then $(1-\kappa)\theta_3 = \theta_3 - \frac{1}{p'}$. By applying the fact that $|S|^{\frac{1}{p'}} = 1$, one can conclude

$$\begin{aligned} \|B_{S_2}^{S_1}\|_p \lesssim\ & \text{size}_{K \in \mathcal{K}: K \cap S_1 \neq \emptyset}((\langle f_1, \phi_K^1 \rangle)_K)^{1-\theta_1} \text{size}_{K \in \mathcal{K}: K \cap S_1 \neq \emptyset}((\langle f_2, \phi_K^2 \rangle)_K)^{1-\theta_2} \\ \cdot\ & \text{energy}_{\mathcal{K}}((\langle f_1, \phi_K^1 \rangle)_K)^{\theta_1} \text{energy}_{\mathcal{K}}((\langle f_2, \phi_K^2 \rangle)_K)^{\theta_2} |S_2|^{\theta_3 - \frac{1}{p'}} \end{aligned}$$

which gives the desired estimate for the energies.

□

CHAPTER 6

PROOF OF THEOREM 3.3.1 - WALSH CASE

6.1 Estimates for $\Pi_{\mathbf{flag}^{\#1}\otimes\mathbf{flag}^{\#2}}$

One defines the exceptional set

$$\Omega := \Omega_1 \cup \Omega_2,$$

where

$$\begin{aligned}\Omega_1 :=& \bigcup_{n_1\in\mathbb{Z}}\{Mf_1 > C_1 2^{n_1}|F_1|\} \times \{Mg_1 > C_2 2^{-n_1}|G_1|\}\cup \\ & \bigcup_{m_1\in\mathbb{Z}}\{Mf_2 > C_1 2^{m_1}|F_2|\} \times \{Mg_2 > C_2 2^{-m_1}|G_2|\}\cup \\ \Omega_2 :=& \{SSh > C_3\|h\|_{L^s}\}\end{aligned}$$

and

$$\tilde{\Omega} := \{M\chi_\Omega > \frac{1}{100}\}.$$

6.1.1 Two-Dimensional Tensor-Type Stopping-Time Decomposition I - Level Sets

The first tensor-type stopping time decomposition relies on intersection with level sets which is required by the estimates for $\Pi_{\text{flag}^{\#1}\otimes\text{flag}^{\#2}}$. Another tensor-type stopping-time decomposition involves maximal intervals and is necessary for the discussion on $\Pi_{\text{flag}^{0}\otimes\text{flag}^{0}}$. We will focus on the first one in this section.

One-dimensional stopping time decompositions

One can perform a one-dimensional stopping-time decomposition on $\mathcal{I} := \{I : I \times J \in \mathcal{R}\}$. Let

$$\Omega^x_{N_1} := \{Mf_1 > C_1 2^{N_1}|F_1|\}$$

for some $N_1 \in \mathbb{Z}$ and

$$\mathcal{I}_{N_1} := \{I \in \mathcal{I} : |I \cap \Omega^x_{N_1}| > \frac{1}{10}|I|\}.$$

Define

$$\Omega^x_{N_1-1} := \{Mf_1 > C_1 2^{N_1-1}|F_1|\}$$

and

$$\mathcal{I}_{N_1-1} := \{I \in \mathcal{I} \setminus \mathcal{I}^{N_1} : |I \cap \Omega^x_{N_1-1}| > \frac{1}{10}|I|\}.$$

$$\vdots$$

The procedure generates the sets $(\Omega^x_{n_1})_{n_1}$ and $(\mathcal{I}_{n_1})_{n_1}$. Independently define

$$\Omega^x_{M_1} := \{Mf_2 > C_1 2^{M_1}|F_2|\}$$

for some $M_1 \in \mathbb{Z}$ and

$$\mathcal{I}_{M_1} := \{I \in \mathcal{I} : |I \cap \Omega^x_{M_1}| > \frac{1}{10}|I|\}.$$

Define

$$\Omega^x_{M_1-1} := \{Mf_2 > C_1 2^{M_1-1}|F_2|\}$$

and

$$\mathcal{I}_{M_1-1} := \{I \in \mathcal{I} \setminus \mathcal{I}^{M_1} : |I \cap \Omega^x_{M_1-1}| > \frac{1}{10}|I|\}.$$

$$\vdots$$

The procedure generates the sets $(\Omega^x_{m_1})_{m_1}$ and $(\mathcal{I}_{m_1})_{m_1}$. Now define $\mathcal{I}_{n_1,m_1} := \mathcal{I}_{n_1} \cap \mathcal{I}_{m_1}$ and the decomposition on $\mathcal{I} = \bigcup_{n_1,m_1} \mathcal{I}_{n_1,m_1}$.

Same decomposition procedure can be applied to $\mathcal{J} := \{J : I \times J \in \mathcal{R}\}$. Let

$$\Omega^y_{N_2} := \{Mg_1 > C_2 2^{N_2}|G_1|\}$$

for some $N_2 \in \mathbb{Z}$ and define

$$\mathcal{J}_{N_2} := \{J \in \mathcal{J} : |J \cap \Omega^y_{N_2}| > \frac{1}{10}|J|\}.$$

Iteratively define

$$\Omega^y_{N_2-1} := \{Mg_1 > C_2 2^{N_2-1}|G_1|\}$$

and

$$\mathcal{J}_{N_2-1} := \{J \in \mathcal{J} \setminus \mathcal{J}^{N_2} : |J \cap \Omega^y_{N_2-1}| > \frac{1}{10}|J|\}.$$

$$\vdots$$

The procedure produces the sets $(\Omega^y_{n_2})_{n_2}$ and $(\mathcal{J}_{n_2})_{n_2}$.

Independently define

$$\Omega^y_{M_2} := \{Mg_2 > C_2 2^{M_2}|G_2|\}$$

for some $M_2 \in \mathbb{Z}$ and define

$$\mathcal{J}_{M_2} := \{J \in \mathcal{J} : |J \cap \Omega^y_{M_2}| > \frac{1}{10}|J|\}.$$

Iteratively define

$$\Omega^y_{M_2-1} := \{Mg_2 > C_2 2^{M_2-1}|G_2|\}$$

and

$$\mathcal{J}_{M_2-1} := \{J \in \mathcal{J} \setminus \mathcal{J}^{M_2} : |J \cap \Omega^y_{M_2-1}| > \frac{1}{10}|J|\}.$$

$$\vdots$$

The procedure produces the sets $(\Omega^{y}_{m_2})_{m_2}$ and $(\mathcal{J}_{m_2})_{m_2}$. Therefore $\mathcal{J} = \bigcup_{n_2,m_2} \mathcal{J}_{n_2,m_2}$, where $\mathcal{J}_{n_2,m_2} := \mathcal{J}_{n_2} \cap \mathcal{J}_{m_2}$.

Tensor product of two one-dimensional stopping-time decompositions

If we assume that all dyadic rectangles satisfy $I \times J \cap \tilde{\Omega}^c \neq \emptyset$, then we have the following observations:

Observation 1. If $I \times J \in \mathcal{I}_{n_1,m_1} \times \mathcal{J}_{n_2,m_2}$, then $n_1, m_1, n_2, m_2 \in \mathbb{Z}$ satisfies $n_1 + n_2 < 0$ and $m_1 + m_2 < 0$. Equivalently, $\forall I \times J \cap \tilde{\Omega}^c \neq \emptyset$, $I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2}$, for some $n_2, m_2 \in \mathbb{Z}$ and $n, m > 0$.

Proof. Given $I \in \mathcal{I}_{n_1}$, one has $|I \cap \{Mf_1 > C_1 2^{n_1}|F_1|\}| > \frac{1}{10}|I|$; similarly, $J \in \mathcal{J}_{n_1'}$ implies that $|J \cap \{Mg_1 > C_2 2^{n_1'}|G_1|\}| > \frac{1}{10}|J|$. If $n_1 + n_1' \geq 0$, then $\{Mf_1 > C_1 2^{n_1}|F_1|\} \times \{Mg_1 > C_2 2^{n_1'}|G_1|\} \subseteq \Omega_1 \subseteq \Omega$. Then $|I \times J \cap \Omega| > \frac{1}{100}|I \times J|$, which implies that $I \times J \subseteq \tilde{\Omega}$ and contradicts with the assumption. Same reasoning applies to m_1 and m_2. □

6.1.2 Two-Dimensional General Level Sets Stopping-Time Decomposition

With the assumption that $R \cap \tilde{\Omega}^c \neq \emptyset$, one has that

$$|R \cap \Omega_0| \leq \frac{1}{100}|R|,$$

where

$$\Omega_0^2 := \{SSh > C_3\|h\|_s\}.$$

Then define

$$\Omega^2_{-1} := \{SSh > C_3 2^{-1}\|h\|_{L^s}\}$$

and

$$\mathcal{R}_{-1} := \{R \in \mathcal{R} : |R \cap \Omega^2_{-1}| > \frac{1}{100}|R|\}$$

Successively define

$$\Omega^2_{-1} := \{SSh > C_3 2^{-2}\|h\|_{L^s}\}$$

and

$$\mathcal{R}_{-2} := \{R \in \mathcal{R} \setminus \mathcal{R}_{-1} : |R \cap \Omega^2_{-2}| > \frac{1}{100}|R|\}$$

$$\vdots$$

This two-dimensional stopping-time decomposition generates the sets $(\Omega^2_{k_1})_{k_1}$ and $(\mathcal{R}_{k_1})_{k_1}$.

Independently one can apply the same algorithm involving $SS\chi_{E'}$ which generates $(\Omega^2_{k_2})_{k_2}$ and $(\mathcal{R}_{k_2})_{k_2}$.

6.1.3 Sparsity condition

One important property followed from the "tensor-type stopping-time decomposition - level sets" is the sparsity of dyadic intervals at different levels:

Proposition 6.1.1. *Suppose that $J_0 \in \mathcal{J}_{n_2-10}$. Then*

$$\sum_{\substack{J \in \mathcal{J}_{n_2} \\ J \cap J_0 \neq \emptyset}} |J| \leq \frac{1}{2}|J_0|$$

To prove the proposition, one would need the following claim about pointwise estimates for Mg_1 (and similarly for Mg_2) on $J \in \mathcal{J}_{n_2}$:

Claim 6.1.2. $J \in \mathcal{J}_{n_2} \Longrightarrow Mg_1 > 2^{-7} \cdot C_2 2^{n_2}|G_1|$ on J.

Proof on Claim $\Longrightarrow$ Proposition 6.1.1. We will first explain why the proposition follows from the claim and then prove the claim. One recalls that all the intervals are dyadic, which means $J \cap J_0 \neq \emptyset \Longrightarrow J \subseteq J_0$ or $J_0 \subseteq J$. If $J_0 \subseteq J$, then the claim implies that $J_0 \subseteq J \subseteq \{Mg_1 > C_2 2^{n_2-7}|G_1|\}$. But $J_0 \in \mathcal{J}_{n_2-10} \Longrightarrow \left|J_0 \cap \{Mg_1 > C_2 2^{n_2-7}\}\right| < \frac{1}{10}|J_0|$, which is a contradiction. If $J \subseteq J_0$, and suppose that

$$\sum_{\substack{J \in \mathcal{J}_{n_2} \\ J \subseteq J_0}} |J| > \frac{1}{2}|J_0|.$$

Then $J \in \mathcal{J}_{n_2} \Longrightarrow \left|J \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}\right| > \frac{1}{10}|J|$. Therefore

$$\sum_{\substack{J \in \mathcal{J}_{n_2} \\ J \subseteq J_0}} \left|J \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}\right| > \frac{1}{10} \sum_{\substack{J \in \mathcal{J}_{n_2} \\ J \subseteq J_0}} |J| > \frac{1}{20}|J_0|.$$

But by the disjointness of $(J)_{J \in \mathcal{J}_{n_2}}$,

$$\sum_{\substack{J \in \mathcal{J}_{n_2} \\ J \subseteq J_0}} \left|J \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}\right| \leq \left|J_0 \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}\right|.$$

Thus

$$\left|J_0 \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}\right| > \frac{1}{20}|J_0|,$$

Now the claim, with slight modifications, implies that $J_0 \subseteq \{Mg_1 > C_2 2^{n_2-8}|G_1|\}$. This is a contradiction as $J_0 \in \mathcal{J}_{n_2-10}$. □

We are now ready to prove the claim:

Proof of Claim. We prove the claim case by case:

Case (i): $\forall y \in \{Mg_1 > C_2 2^{n_2}|G_1|\}$, there exists $J_y \subseteq J$ such that $\text{ave}_{J_y}(g_1) > C_2 2^{n_2}|G_1|$;

Case (ii): There exists $y \in \{Mg_1 > C_2 2^{n_2}|G_1|\}$ and $J_y \nsubseteq J$ such that $\text{ave}_{J_y}(g_1) > C_2 2^{n_2}|G_1|$:

Case (iia): $\frac{1}{40}|J| \leq |J_y \cap J|$ and $|J_y| \leq |J|$;

Case (iib): $\frac{1}{40}|J| \leq |J_y \cap J|$ and $|J_y| > |J|$;

Case (iic): $|J_y \cap J| < \frac{1}{40}|J|$.

Proof of (i): In Case (i), once observes that $\{Mg_1 > C_2 2^{n_2}|G_1|\} \cap J$ can be rewritten as $\{M(g_1 \cdot \chi_J) > C_2 2^{n_2}|G_1|\} \cap J$. Thus

$$C_2 2^{n_2}|G_1||\{Mg_1 > C_2 2^{n_2}|G_1|\} \cap J| = C_2 2^{n_2}|G_1||\{M(g_1\chi_J) > C_2 2^{n_2}|G_1|\} \cap J| \leq \|g_1\chi_J\|_1$$

One recalls that $|\{Mg_1 > C_2 2^{n_2}|G_1|\} \cap J| > \frac{1}{10}|J|$, which implies that

$$C_2 2^{n_2}|G_1| \cdot \frac{1}{10}|J| \leq \|g_1\chi_J\|_1$$

equivalently,

$$\frac{\|g_1\chi_J\|_1}{|J|} \geq \frac{1}{10}C_2 2^{n_1}|G_1|.$$

Therefore $Mg_1 > 2^{-4}C_2 2^{n_2}|G_1|$.

Proof of (ii): We will prove that either (iia) or (iib) holds, then $Mg_1 > 2^{-7}C_2 2^{n_2}|G_1|$. If neither (iia) nor (iib) happens, then (iic) has to hold and in this case, $Mg_1 > 2^{-7}C_2 2^{n_2}|G_1|$.

If there exists $y \in \{Mg_1 > C_2 2^{n_2}|G_1|\}$ such that (iia) holds, then

$$\frac{\|g_1\chi_{J_y}\|_1}{|J_y|} \leq \frac{\|g_1\chi_{J_y \cup J}\|_1}{|J_y|} \leq \frac{\|g_1\chi_{J_y \cup J}\|_1}{|J_y \cap J|} \leq \frac{\|g_1\chi_{J_y \cup J}\|_1}{\frac{1}{40}|J|},$$

where the last inequality follows from $\frac{1}{40}|J| \leq |J_y \cap J|$. Moreover, $|J_y| \leq |J|$ and the implicit condition $y \in J_y \cup J \neq \emptyset$ implies that $|J_y \cup J| \leq 2|J|$. Thus

$$\frac{\|g_1\chi_{J_y \cup J}\|_1}{\frac{1}{20}|J|} \leq \frac{\|g_1\chi_{J_y \cup J}\|_1}{\frac{1}{40}\frac{1}{2}|J_y \cup J|},$$

which implies

$$\frac{\|g_1\chi_{J_y \cup J}\|_1}{|J_y \cup J|} > \frac{1}{80}C_2 2^{n_2}|G_1|$$

and as a result $Mg_1 > 2^{-7}C_2 2^{n_2}|G_1|$ on J.

If there exists $y \in \{Mg_1 > C_2 2^{n_2}|G_1|\}$ such that (iib) holds, then

$$\frac{\|g_1\chi_{J_y}\|_1}{|J_y|} \le \frac{\|g_1\chi_{J_y\cup J}\|_1}{|J_y|} = \frac{2\|g_1\chi_{J_y\cup J}\|_1}{2|J_y|} \le \frac{2\|g_1\chi_{J_y\cup J}\|_1}{|J_y \cup J|},$$

where the last inequality follows from $|J_y| > |J|$. As a consequence,

$$\frac{2\|g_1\chi_{J_y\cup J}\|_1}{|J_y \cup J|} > C_2 2^{n_2}|G_1|$$

and $Mg_1 > 2^{-1}C_2 2^{n_2}|G_1|$ on J.

If neither (i), (iia) nor (iib) happens, then on $\mathcal{S}_{(iic)} := \{y : Mg_1(y) > C_2 2^{n_2}|G_1| \text{ and } (i) \text{ does not hold}\}$, one has

$$\text{Given } \tilde{J} \nsubseteq J \text{ such that } y \in \tilde{J}, |\tilde{J} \cup J| \ge \frac{1}{40}|\tilde{J}| \Longrightarrow \text{ave}_{\tilde{J}}(g_1) \le C_2 2^{n_2}|G_1|$$

One direct geometric observation is that $|\mathcal{S}_{(iic)} \cap J| \le \frac{1}{20}|J|$. In particular, suppose $y \in \mathcal{S}_{(iic)}$, then any J_y with $\text{ave}_{J_y}(g_1) > C_2 2^{n_2}|G_1|$ has to contain the left endpoint or right endpoint of J, which we denote by J_{left} and J_{right}. If $J_{\text{left}} \in J_y$, then $|J_y \cap J| < \frac{1}{40}|J|$ implies the interval

$$|[J_{\text{left}}, y]| < \frac{1}{40}.$$

Same implication holds true for $y \in \mathcal{S}_{(iic)}$ with $J_{\text{right}} \in J_y$. Therefore, for any $y \in \mathcal{S}_{(iic)}$, $|[J_{\text{left}}, y]| < \frac{1}{40}$ or $|[y, J_{\text{right}}]| < \frac{1}{40}$, which can be concluded as

$$\left|\mathcal{S}_{(iic)} \cap J\right| < \frac{1}{20}|J|.$$

Since $\left|\{Mg_1 > C_2 2^{n_2}|G_1|\} \cap J\right| > \frac{1}{10}|J|$,

$$\left|(\{Mg_1 > C_2 2^{n_2}|G_1|\} \setminus \mathcal{S}_{(iic)}) \cap J\right| > \frac{1}{20}|J|,$$

in which case one can apply the argument for (i) with $\{Mg_1 > C_2 2^{n_2}|G_1|\}$ replaced by $\{Mg_1 > C_2 2^{n_2}|G_1|\} \setminus \mathcal{S}_{(iic)}$ to conclude that

$$Mg_1 > 2^{-5}C_2 2^{n_2}|G_1|.$$

This ends the proof for the claim. □

Proposition 6.1.3. *Given an arbitrary collection of dyadic rectangles $\mathcal{R}$. Let $\mathcal{R} \subseteq \bigcup_{n}\bigcup_{n_2} \mathcal{I}^{-n-n_2} \times \mathcal{J}_{n_2}$ denote the "tensor-type stopping-time decomposition I - level sets" for $\mathcal{R}$ with respect to Mf_1 and Mg_1. Then for any fixed n,*

$$\sum_{n_2\in\mathbb{Z}}\Big|\bigcup_{R\in\mathcal{I}_{-n-n_2}\times\mathcal{J}_{n_2}} R\Big| \lesssim \Big|\bigcup_{n_2\in\mathbb{Z}}\bigcup_{R\in\mathcal{I}_{-n-n_2}\times\mathcal{J}_{n_2}} R\Big|$$

Proof or Proposition 6.1.3. Proposition 6.1.1 gives a sparsity condition for intervals in y- direction, which is sufficient to generate sparsity for dyadic rectangles in $\mathbb{R}^2$. Proposition 6.1.3 thus follows. □

6.1.4 Hybrid of Stopping-Time Decompositions

Table 6.1: Stopping-Time for $\Pi_{\text{flag}^{\#_1}\otimes\text{flag}^{\#_2}}$

Tensor-type stopping-time decomposition I on $\mathcal{I}\times\mathcal{J}$	$\longrightarrow$	$I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}$ $(n_2,m_2\in\mathbb{Z}, n>0)$
$\Downarrow$		
General two-dimensional level sets decomposition on $\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}$	$\longrightarrow$	$I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\cap\mathcal{R}_{k_1,k_2}$ $(n_2,m_2\in\mathbb{Z}, n>0, k_1<0, k_2\leq K)$

With the stopping-time decompositions specified above, one can rewrite the linear form as

$$\Big|\sum_{\substack{n>0\\m>0\\k_1<0\\k_2\leq K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle$$
$$\cdot\langle h,\psi_I^2\otimes\psi_J^2\rangle\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle\Big|$$
$$=\sum_{\substack{n>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|I||J|$$

One recalls the two-dimensional level sets stopping-time decomposition that $I \times J \in \mathcal{R}_{k_1,k_2}$ if and only if

$$|I \cap (\Omega_{k_1})^c| \geq \frac{99}{100}|I \times J|$$

$$|I \cap (\Omega_{k_2})^c| \geq \frac{99}{100}|I \times J|$$

with $\Omega_{k_1} := \{SSh > C_3 2^{k_1}\|h\|_s\}$, and $\Omega_{k_2} := \{SS\chi_{E'} > C_3 2^{k_2}\}$.

One can therefore restrict $I \times J$ to its small smaller subset and rewrite the linear form as

$$\sum_{\substack{n>0\\k_1<0\\k_2\leq K}} \sum_{n_2\in\mathbb{Z}} \sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \frac{|\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}} \cdot \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}$$

$$\cdot |I \times J \cap (\Omega_{k_1})^c \cap (\Omega_{k_2})^c|$$

$$= \sum_{\substack{n>0\\m>0\\k_1>0\\k_2\geq -K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}} \sup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \sup_{J\in\mathcal{J}_{n_2,m_2}} \frac{|\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}} \cdot$$

$$\int \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \chi_I(x)\chi_J(y)dxdy$$

where the integrand can be estimated by Cauchy-Schwartz inequality:

$$|\Lambda| \lesssim \sum_{\substack{n>0\\m>0\\k_1>0\\k_2\geq -K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}} \sup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}} \frac{|\langle B_I^{l}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \sup_{J\in\mathcal{J}_{n_2,m_2}} \frac{|\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}} \cdot C_3 2^{k_1}\|h\|_{L^s} C_3 2^{k_2} \cdot$$

$$\Big|\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\Big|$$

To estimate $\sup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}$, One recalls the algorithm in the "two-dimensional tensor type stopping-time decomposition -level sets", which incorporates the following information:

$$I\in\mathcal{I}_{-n-n_2,-m-m_2}$$

implies that

$$|I\cap\{Mf_1<C_12^{-n-n_2}|F_1|\}|\geq\frac{9}{10}|I|$$
$$|I\cap\{Mf_2<C_12^{-m-m_2}|F_2|\}|\geq\frac{9}{10}|I|$$

which infers that

$$I\cap\{Mf_1<C_12^{-n-n_2}|F_1|\}\cap\{Mf_2<C_12^{-m-m_2}|F_2|\}\neq\emptyset$$

Then one can apply Proposition 5.1.4 with $S_1:=\{Mf_1<C_12^{-n-n_2}|F_1|\}\cap\{Mf_2<C_12^{-m-m_2}|F_2|\}$ together with the estimates

$$\sup_{K\cap S_1\neq\emptyset}\frac{\langle f_1,\phi_K^1\rangle}{|K|^{\frac{1}{2}}}\leq C_12^{-n-n_2}|F_1|$$
$$\sup_{K\cap S_1\neq\emptyset}\frac{\langle f_2,\phi_K^1\rangle}{|K|^{\frac{1}{2}}}\leq C_12^{-m-m_2}|F_2|$$

to derive that

$$\sup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\lesssim C_1^22^{-n-n_2}|F_1|2^{-m-m_2}|F_2|$$

Similarly, one can define $S_1':=\{Mg_1<C_22^{n_2}|G_1|\}\cap\{Mg_2<C_22^{m_2}|G_2|\}$ and apply Proposition 5.1.4 to conclude that

$$\sup_{J\in\mathcal{J}_{n_2,m_2}}\frac{|\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\lesssim C_2^22^{n_2}|G_1|2^{m_2}|G_2|$$

As a consequence, the linear form can be estimated by

$$C_1^2C_2^2C_3^2 \sum_{\substack{n>0\\ m>0\\ k_1<0\\ k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} 2^{-n-n_2}|F_1|2^{-m-m_2}|F_2|2^{n_2}|G_1|2^{m_2}|G_2| \cdot 2^{k_1}\|h\|_{L^s}2^{k_2} \cdot \Big| \bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\Big|$$

One can apply the sparsity condition, in particular, Proposition 6.1.3 repeatedly and obtain the following bound for the expression:

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \Big| \bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\Big| \lesssim \sum_{m_2\in\mathbb{Z}} \Big| \bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-m-m_2}\times\mathcal{J}_{m_2}}} R\Big| \lesssim \Big| \bigcup_{R\in\mathcal{R}_{k_1,k_2}} R\Big|$$

$$\Big| \bigcup_{R\in\mathcal{R}_{k_1,k_2}} R\Big| \lesssim \min(C_3^{-1}2^{-k_1}, C_3^{-1}2^{-k_2\gamma})$$

for any $\gamma > 1$. Lastly, with combination of the estimates above,

$$\begin{aligned} |\Lambda| \lesssim & C_1^2C_2^2C_3 \sum_{\substack{n>0\\ m>0\\ k_1<0\\ k_2\leq K}} 2^{-n}|F_1|2^{-m}|F_2||G_1|G_2| \cdot 2^{k_1}\|h\|_{L^s}2^{k_2}2^{-\frac{k_1}{2}}2^{-\frac{k_2\gamma}{2}} \\ \lesssim & C_1^2C_2^2C_3|F_1||F_2||G_1||G_2|\|h\|_{L^s} \end{aligned}$$

with appropriate choice of γ as previously.

Remark 6.1.4. One important observation is that thanks to Proposition 5.1.4, the sizes

$$\sup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}$$

and

$$\sup_{J\in\mathcal{J}_{n_2,m_2}} \frac{|\langle \tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}$$

can be estimated in the exactly same way as

$$\text{size}_{\mathcal{I}_{-n-n_2}}((f_1,\phi_I)_I)$$

and

$$\text{size}_{\mathcal{I}_{-n-n_2}}((f_2, \phi_I)_I)$$

respectively. Based on this observation, it is not difficult to verify that the discrete model $\Pi_{\text{flag}^{\#1}\otimes\text{paraproduct}}$ can be estimated by a essentially same argument as $\Pi_{\text{paraproduct}\otimes\text{paraproduct}}$ or $\Pi_{\text{flag}^{\#1}\otimes\text{flag}^{\#2}}$. In addition, $\Pi_{\text{flag}^0\otimes\text{flag}^{\#2}}$ can be studied similarly as $\Pi_{\text{flag}^0\otimes\text{paraproduct}}$.

6.2 Estimates for $\Pi^1_{\textbf{flag}^0\otimes\textbf{flag}^0}$

We define the exceptional set $\Omega := \Omega_1 \cup \Omega_2$, where

$$\begin{aligned}
\Omega_1 :=& \bigcup_{n_1\in\mathbb{Z}}\{Mf_1 > C_1 2^{n_1}|F_1|\} \times \{Mg_1 > C_2 2^{-n_1}|G_1|\}\cup \\
& \bigcup_{m_1\in\mathbb{Z}}\{Mf_2 > C_1 2^{m_1}|F_2|\} \times \{Mg_2 > C_2 2^{-m_1}|G_2|\}\cup \\
& \bigcup_{l_1\in\mathbb{Z}}\{MB > C_1 2^{l_1}\|B\|_1\} \times \{M\tilde{B} > C_2 2^{-l_1}\|\tilde{B}\|_1\} \\
\Omega_2 :=& \{SSh > C_3\|h\|_{L^s}\}
\end{aligned}$$

and

$$\tilde{\Omega} := \{M\chi_\Omega > \frac{1}{100}\}.$$

6.2.1 Two-Dimensional Tensor-Type Stopping-Time Decomposition II - Maximal Intervals

One-dimensional stopping-time decomposition

Given finiteness of $\mathcal{I}$, there exists some $L_1 \in \mathbb{Z}$ such that $\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \leq C_1 2^{L_1}\|B\|_1$. We can pick the largest interval $I_{\max}$ such that

$$\frac{|\langle B_{I_{\max}}(f_1,f_2),\varphi^1_{I_{\max}}\rangle|}{|I_{\max}|^{\frac{1}{2}}} \geq C_1 2^{L_1-1}\|B\|_1.$$

Then we define a tree

$$T := \{I \in \mathcal{I} : I \subseteq I_{\max}\},$$

and let $I_T := I_{\max}$, usually called as tree-top. Now we look at $\mathcal{I} \setminus T$ and repeat the above step to choose maximal intervals and collect their subintervals in their corresponding sets. Since $\mathcal{I}$ is finite, the process will eventually end. We then collect all T's in a set $\mathbb{T}_{L_1-1}$. Next we repeat the above algorithm to $\mathcal{I} \setminus \bigcup_{T\in\mathbb{T}_{L_1-1}} T$. We thus obtain a decomposition $\mathcal{I} = \bigcup_{l_1}\bigcup_{T\in\mathbb{T}_{l_1}} T$. One simple observation is that the above procedure can be applied to general sequences indexed by dyadic intervals. One can thus apply the same algorithm to $\mathcal{J} := \{J : I\times J \in \mathcal{R}\}$. We denote the decomposition as $\mathcal{J} = \bigcup_{l_2}\bigcup_{S\in\mathbb{S}_{l_2}} S$ with respect to the sequence $(\frac{|\langle B_J(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}})_{J\in\mathcal{J}}$, where S represents for a tree with the tree-top.

Tensor product of two one-dimensional stopping-time decompositions

Observation 2. If $I \times J \cap \tilde{\Omega}^c \neq \emptyset$ and $I \times J \in T \times S$ with $T \in \mathbb{T}_{l_1}$ and $S \in \mathbb{S}_{l_2}$, then $l_1, l_2 \in \mathbb{Z}$ satisfies $l_1 + l_2 < 0$. Equivalently, $I \times J \in T \times S$ with $T \in \mathbb{T}_{-l-l_2}$ and $S \in \mathbb{S}_{l_2}$

for some $l_2 \in \mathbb{Z}$, $l > 0$.

Proof. $I \in T$ with $T \in \mathbb{T}_{l_1}$ means that $I \subseteq I_T$ where $\frac{|\langle B_{I_T}(f_1,f_2),\varphi^1_{I_T}\rangle|}{|I_T|^{\frac{1}{2}}} > C_1 2^{l_1}\|B\|_1$. By the biest trick, $\frac{|\langle B_{I_T}(f_1,f_2),\varphi^1_{I_T}\rangle|}{|I_T|^{\frac{1}{2}}} = \frac{|\langle B(f_1,f_2),\varphi^1_{I_T}\rangle|}{|I_T|^{\frac{1}{2}}} \leq MB(x)$ for any $x \in I_T$. Thus $I_T \subseteq \{MB > C_1 2^{l_1}\|B\|_1\}$. By a similar reasoning, $J \in S$ with $S \in \mathbb{S}_{l_2}$ implies that $J \subseteq J_S \subseteq \{M\tilde{B} > C_2 2^{l_2}\|\tilde{B}\|_1\}$. If $l_1 + l_2 \geq 0$, then $\{MB > C_1 2^{l_1}\|B\|_1\} \times \{M\tilde{B} > C_2 2^{l_2}\|\tilde{B}\|_1\} \subseteq \Omega_1 \subseteq \Omega$. As a consequence, $I \times J \subseteq \Omega \subseteq \tilde{\Omega}$, which is a contradiction. □

6.2.2 Hybrid of Stopping-Time Decompositions

Table 6.2: Stopping-Time on $\mathcal{K}$ for $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$

One-dimensional stopping-time decomposition on $\mathcal{K}$	$\longrightarrow$	$K \in \mathcal{K}_{n_0}$ ($n_0 \in \mathbb{Z}$)

Table 6.3: Stopping-Time on $\mathcal{L}$ for $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$

One-dimensional stopping-time decomposition on $\mathcal{L}$	$\longrightarrow$	$L \in \mathcal{L}_{n_0'}$ ($n_0' \in \mathbb{Z}$)

Table 6.4: Stopping-Time on $\mathcal{R}$ for $\Pi^1_{\text{flag}^0 \otimes \text{flag}^0}$

Two-dimensional tensor-type stopping-time decomposition I on $\mathcal{I} \times \mathcal{J}$	$\longrightarrow$	$I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2}$ ($n_2, m_2 \in \mathbb{Z}, n, m > 0$)
$\Downarrow$		
General two-dimensional level sets decomposition on $\mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2}$	$\longrightarrow$	$I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2} \cap \mathcal{R}_{k_1,k_2}$ ($n_2, m_2 \in \mathbb{Z}, n, m > 0, k_1 < 0, k_2 \leq K$)
$\Downarrow$		
Two-dimensional tensor-type stopping-time decomposition II on $I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2} \cap \mathcal{R}_{k_1,k_2}$	$\longrightarrow$	$I \times J \in (\mathcal{I}_{-n-n_2,-m-m_2} \cap T) \times (\mathcal{J}_{n_2,m_2} \cap S) \cap \mathcal{R}_{k_1,k_2}$ with $T \in \mathbb{T}_{-l-l_2}$, $S \in \mathbb{S}_{l_2}$ ($n_2, m_2, l_2 \in \mathbb{Z}, n, m, l > 0, k_1 < 0, k_2 \leq K$)

With the stopping-time decomposition performed as described in the chart, one can estimate the linear form by

$$|\Lambda| \lesssim \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|\langle B_I(f_1,f_2),\varphi_I^1\rangle||\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|$$

$$\cdot |\langle h,\psi_I^2\otimes\psi_J^2\rangle||\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|$$

$$= \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}$$

$$\cdot \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|I||J|$$

The two-dimensional level sets stopping-time decomposition indicates that

$$I\times J\in\mathcal{R}_{k_1,k_2}$$

if and only if

$$|I\times J\cap(\Omega_{k_1}^2)^c| \geq \frac{99}{100}|I\times J|$$

$$|I\times J\cap(\Omega_{k_2}^2)^c| \geq \frac{99}{100}|I\times J|$$

where $\Omega_{k_1}^2 := \{SSh(x,y) > C_3 2^{k_1+1}\|h\|_s\}$ and $\Omega_{k_2}^2 := \{SS\chi_{E'}(x,y) > C_3 2^{k_2+1}\}$.

One can therefore restrict $I\times J$ to its smaller subset:

$$|\Lambda|$$

$$\lesssim \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\I\times J\in T\times S\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}$$

$$\cdot \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|I\times J\cap(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c|$$

$$= \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \sup_{I\in T} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \sup_{J\in S} \frac{|\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}} \cdot$$

$$\int_{(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \chi_I(x)\chi_J(y)dxdy \tag{6.2.1}$$

where the integral can be estimated using Cauchy-Schwartz inequality:

$$\int_{(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} \chi_I(x)\chi_J(y)dxdy$$

$$\leq \int_{(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c} \Big(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}} dxdy$$

$$\leq \int_{(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c} SSh(x,y)SS\chi_{E'}(x,y)\cdot\chi_{\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J}dxdy$$

$$\lesssim C_3^2 2^{k_1}\|h\|_s 2^{k_2}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big|$$

The last inequality follows from the point-wise bounds for SSh and $SS\chi_{E'}$ on $(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c$. By applying the above estimates into (6.2.1):

$$|\Lambda|$$

$$\lesssim C_1C_2C_3^2 \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} 2^{-l-l_2}\|B\|_1 2^{l_2}\|\tilde{B}\|_1\cdot 2^{k_1}\|h\|_{L^s}2^{k_2} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\ I\times J\in T\times S\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big|$$

$$=C_1C_2C_3^2 \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} 2^{-l}\|B\|_1\|\tilde{B}\|_1 \cdot 2^{k_1}\|h\|_{L^s}2^{k_2} \cdot \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \Big| \bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| \tag{6.2.2}$$

where by definition of the trees T, S and their corresponding tree tops I_T, J_S:

$$\Big| \bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| = \Big|\big(\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\cap S\\I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\big)\cap(I_T\times J_S)\Big|$$

$$\leq \Big|\big(\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\big)\cap(I_T\times J_S)\Big|$$

6.2.3 Sparsity Condition.

One can estimate the following term in two approaches:

$$\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\S\in\mathbb{S}_{l_2}}} \Big|\big(\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\big)\cap(I_T\times J_S)\Big|$$

The first approach relies on the sparsity condition which mimics the argument in the previous section. In particular, one observes that for any fixed l and l_2, $\{I_T\times J_S : T\in\mathbb{T}_{-l-l_2}, J\in\mathbb{S}_{l_2}\}$ is a disjoint collection of rectangles. One can therefore reduce the above expression to

$$\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \Big| \bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\Big|$$

which can be estimated by applying the sparsity condition (Proposition 6.1.3) twice:

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}\Bigg|\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}} R\Bigg| \lesssim \sum_{m_2\in\mathbb{Z}}\Bigg|\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-m-m_2}\times\mathcal{J}_{m_2}}} R\Bigg|$$
$$\lesssim \Bigg|\bigcup_{R\in\mathcal{R}_{k_1,k_2}} R\Bigg|$$
$$\lesssim \min(2^{-k_1}, 2^{-k_2\gamma})$$

for any $\gamma > 1$.

6.2.4 Fubini Argument

Alternatively, one can apply the"Fubini" argument and have

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ S\in\mathbb{S}_{l_2}}}\left|I_T\times J_S\right|$$

One now recalls that the stopping-time algorithms ensure that

$$I_T\in\mathcal{I}_{-n-n_2,-m-m_2}$$

and

$$J_S\in\mathcal{J}_{n_2,m_2}$$

Thus the above expression can be rewritten as

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|$$
$$=\underbrace{\sum_{\substack{n_2<0\\ m_2>0}}}_{I}+\underbrace{\sum_{\substack{n_2>0\\ m_2<0}}}_{II}+\underbrace{\sum_{\substack{n_2\le 0\\ m_2\le 0}}}_{III}+\underbrace{\sum_{\substack{n_2\ge 0\\ m_2\ge 0}}}_{IV}$$

One observes that Case I and Case II follow the same argument by symmetry and a similar observation holds for Case III and Case IV. Without loss of generality, we will focus on Case I and III.

Case I. One rewrites I as

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}(C_1 2^{-l-l_2}\|B\|_1)^{1+\epsilon}\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|\cdot(C_2 2^{l_2}\|\tilde{B}\|_1)^{1+\epsilon}\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|$$
$$\cdot 2^{l(1+\epsilon)}\|B\|_1^{-1-\epsilon}\|\tilde{B}\|_1^{-1-\epsilon}C_1^{-1-\epsilon}C_2^{-1-\epsilon}$$
$$\leq \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\underbrace{\sum_{l_2\in\mathbb{Z}}(C_1 2^{-l-l_2}\|B\|_1)^{1+\epsilon}\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|}_{a}\cdot\underbrace{\sum_{l_2\in\mathbb{Z}}(C_2 2^{l_2}\|\tilde{B}\|_1)^{1+\epsilon}\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|}_{b}$$
$$\cdot 2^{l(1+\epsilon)}\|B\|_1^{-1-\epsilon}\|\tilde{B}\|_1^{-1-\epsilon}C_1^{-1-\epsilon}C_2^{-1-\epsilon}$$

One observes that a and b take the form of L^p-energies, which allows one to apply the energy estimates stated in Proposition 5.1.5. In particular,

$$a \leq (\text{energy}^{1+\epsilon}_{\mathcal{I}_{-n-n_2,-m-m_2}}(((\langle B_I,\varphi_I^1\rangle)_I))^{1+\epsilon}$$

$$b \leq (\text{energy}^{1+\epsilon}_{\mathcal{J}_{n_2,m_2}}(((\langle \tilde{B}_J,\varphi_J^1\rangle)_I))^{1+\epsilon}$$

The stopping-time decomposition generates the set S_1, S_2, S'_1, and S'_2. More precisely, one recalls that $I_T\in\mathcal{I}_{-n-n_2,-m-m_2}$ translates into

$$|I_T\cap\{Mf_1\leq C_1 2^{-n-n_2}|F_1|\}|>\frac{9}{10}|I_T|$$

$$|I_T\cap\{Mf_2\leq C_1 2^{-m-m_2}|F_2|\}|>\frac{9}{10}|I_T|$$

which infers that

$$I_T \cap \{Mf_1 < C_1 2^{-n-n_2}|F_1|\} \cap \{Mf_2 < C_1 2^{-m-m_2}|F_2|\} \neq \emptyset$$

Another piece information carried by the stopping-time decomposition is that

$$|I_T \cap \{Mf_1 > C_1 2^{-n-n_2-1}|F_1|\}| > \frac{1}{10}|I|$$

$$|I_T \cap \{Mf_2 > C_1 2^{-m-m_2-1}|F_2|\}| > \frac{1}{10}|I|$$

which, by the point-wise estimate specified in Claim 6.1.2, yields

$$I_T \subseteq \{Mf_1 > C_1 2^{-n-n_2-10}|F_1|\}$$

$$I_T \subseteq \{Mf_2 > C_1 2^{-m-m_2-10}|F_2|\}$$

Then one can apply Proposition 5.1.5 with $S_1 := \{Mf_1 < C_1 2^{-n-n_2}|F_1|\} \cap \{Mf_2 < C_1 2^{-m-m_2}|F_2|\}$ and $S_2 := \{Mf_1 > C_1 2^{-n-n_2-10}|F_1|\} \cap \{Mf_2 > C_1 2^{-m-m_2-10}|F_2|\}$ to derive that

$$a \lesssim (\text{size}((\langle f_1, \varphi_K^1\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_1}\text{size}((\langle f_2, \psi_K^2\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_2}|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{\epsilon}{1+\epsilon}})^{1+\epsilon}$$

where by Proposition 5.1.2,

$$\text{size}((\langle f_1, \varphi_K^1\rangle)_{K\cap S_1\neq\emptyset}) \lesssim \sup_{K\cap S_1\neq\emptyset}\frac{|\langle f_1, \tilde{\chi}_K\rangle|}{|K|} \lesssim C_1 2^{-n-n_2}|F_1|$$

$$\text{size}((\langle f_2, \psi_K^2\rangle)_{K\cap S_1\neq\emptyset}) \lesssim \sup_{K\cap S_1\neq\emptyset}\frac{|\langle f_2, \tilde{\chi}_K'\rangle|}{|K|} \lesssim C_1 2^{-m-m_2}|F_2|$$

for some L^∞-normalized bump functions $\tilde{\chi}_K, \tilde{\chi}_K'$ adapted to K. As a consequence,

$$a \lesssim (C_1^{2-\theta_1-\theta_2}2^{(-n-n_2)(1-\theta_1)}2^{(-m-m_2)(1-\theta_2)}|F_1||F_2||S_2|^{\theta_3-\frac{\epsilon}{1+\epsilon}})^{1+\epsilon}$$

By letting $\theta_3 = \frac{\epsilon}{1+\epsilon}$, equivalently $\theta_1+\theta_2 = \frac{1}{1+\epsilon}$, one obtains

$$a \lesssim C_1^{1+2\epsilon}2^{(-n-n_2)(1-\theta_1)(1+\epsilon)}2^{(-m-m_2)(\epsilon+\theta_1(1+\epsilon))}|F_1|^{1+\epsilon}|F_2|^{1+\epsilon}$$

for some $0 < \theta_1 < 1$.

Let $S'_1 := \{Mg_1 < C_2 2^{n_2}|G_1|\} \cap \{Mg_2 < C_2 2^{m_2}|G_2|\}$ and $S'_2 := \{Mg_1 > C_2 2^{n_2-10}|G_1|\} \cap \{Mf_2 > C_2 2^{m_2-10}|G_2|\}$. An exactly same argument yields

$$b \lesssim C_2^{1+2\epsilon} 2^{n_2(1-\theta'_1)(1+\epsilon)} 2^{m_2(\epsilon+\theta'_1(1+\epsilon))} |G_1|^{1+\epsilon}|G_2|^{1+\epsilon}$$

for some $0 < \theta' < 1$.

Combining the estimates for a and b,

$$\begin{aligned} I \lesssim & C_1^{\epsilon} C_2^{\epsilon} 2^{-n(1-\theta_1)(1+\epsilon)} 2^{-m(\epsilon+\theta_1(1+\epsilon))} |F_1|^{1+\epsilon}|F_2|^{1+\epsilon}|G_1|^{1+\epsilon}|G_2|^{1+\epsilon} 2^{-l(1+\epsilon)} \|B\|_1^{-1-\epsilon} \|\tilde{B}\|_1^{-1-\epsilon} \\ & \cdot \sum_{\substack{n_2<0 \\ m_2<0}} 2^{n_2(\theta'_1-\theta_1)(1+\epsilon)} 2^{m_2(\theta'_1-\theta_1)(1+\epsilon)} \end{aligned}$$

By choosing $0 < \theta_1 < \theta'_1 < 1$, one can conclude that

$$I \lesssim C_!^{\epsilon} C_2^{\epsilon} 2^{-n(1-\theta_1)(1+\epsilon)} 2^{-m(\epsilon+\theta_1(1+\epsilon))} |F_1|^{1+\epsilon}|F_2|^{1+\epsilon}|G_1|^{1+\epsilon}|G_2|^{1+\epsilon} 2^{l(1+\epsilon)} \|B\|_1^{-1-\epsilon} \|\tilde{B}\|_1^{-1-\epsilon}.$$

Case III. Case III is more tricky and cannot be treated using the argument for Case I and II, there is no choice of $\theta_1, \theta_2, \theta'_1$ and θ'_2 with $\theta_1 + \theta_2 = \theta'_1 + \theta'_2$ and $\theta_1 < \theta'_1$, $\theta_2 < \theta'_2$. Similarly, Case IV cannot be resolved using the previous argument. We will invent an argument to treat those cases, which also provides a proof for Case I and II. But one will get a slightly weaker estimate than the previous argument in terms of the exponents for $|F_1|, |F_2|, |G_1|$ and $|G_2|$.

$$III \leq \underbrace{\sup_{\substack{n_2\in\mathbb{Z} \\ m_2\in\mathbb{Z}}} \sum_{l_2\in\mathbb{Z}} \Big(\sum_{\substack{T\in\mathbb{T}_{-l-l_2} \\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \Big)^{\frac{1}{2}} \Big(\sum_{\substack{S\in\mathbb{S}_{l_2} \\ J_S\in\mathcal{J}_{n_2,m_2}}} |J_S| \Big)^{\frac{1}{2}}}_{a} \cdot$$

$$\underbrace{\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\sup_{l_2\in\mathbb{Z}}\Big(\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|\Big)^{\frac{1}{2}}\Big(\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|\Big)^{\frac{1}{2}}}_{b}$$

To estimate a, one applies the Cauchy-Schwartz inequality and a Fubini-type argument as in Case I:

$$\begin{aligned}
&a\\
&\leq \sup_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\sum_{l_2\in\mathbb{Z}}(C_1 2^{-l-l_2}\|B\|_1)\Big(\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|\Big)^{\frac{1}{2}}\cdot(C_2 2^{l_2}\|\tilde{B}\|_1)\Big(\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|\Big)^{\frac{1}{2}}\cdot 2^l\|B\|_1^{-1}\|\tilde{B}\|_1^{-1}C_1^{-1}C_2^{-1}\\
&\leq \sup_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\underbrace{\Big[\sum_{l_2\in\mathbb{Z}}(C_1 2^{-l-l_2}\|B\|_1)^2\sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_T|\Big]^{\frac{1}{2}}}_{a^1}\cdot\underbrace{\Big[\sum_{l_2\in\mathbb{Z}}(C_2 2^{l_2}\|\tilde{B}\|_1)^2\sum_{\substack{S\in\mathbb{S}_{l_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}}|J_S|\Big]^{\frac{1}{2}}}_{a^2}\\
&\quad\cdot 2^l\|B\|_1^{-1}\|\tilde{B}\|_1^{-1}C_1^{-1}C_2^{-1}
\end{aligned}$$

One recalls that Proposition 5.1.5 can be applied with $p=2$, $S_1 := \{Mf_1 < C_1 2^{-n-n_2}|F_1|\}\cap\{Mf_2 < C_1 2^{-m-m_2}|F_2|\}$ and $S_2 := \{Mf_1 > C_1 2^{-n-n_2-10}|F_1|\}\cap\{Mf_2 > C_1 2^{-m-m_2-10}|F_2|\}$ to estimate

$$\begin{aligned}
a^1 \lesssim&\,\text{size}((\langle f_1,\varphi_K^1\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_1}\text{size}((\langle f_2,\psi_K^2\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_2}|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{1}{2}}\\
\lesssim&(C_1 2^{-n-n_2}|F_1|)^{1-\theta_1}(C_1 2^{-m-m_2}|F_2|)^{1-\theta_2}|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{1}{2}}
\end{aligned}$$

where the last inequality follows from the definition of S_1 and S_2 and the reasoning specified in Case I. Let $\theta_3=\frac{1}{2}$ and equivalently $\theta_1+\theta_2=\frac{1}{2}$, one can simplify the expression as

$$a^1 \lesssim C_1^{\frac{3}{2}}2^{(-n-n_2)(1-\theta_1)}2^{(-m-m_2)(\frac{1}{2}+\theta_1)}|F_1||F_2|$$

By similar reasoning, one can obtain

$$a^2 \lesssim C_2^{\frac{3}{2}} 2^{n_2(1-\theta_1')} 2^{m_2(\frac{1}{2}+\theta_1')} |G_1||G_2|$$

for any $0 < \theta_1' < 1$. One can now combine the estimates for a^1 and a^2 to derive:

$$a \lesssim C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} \sup_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} 2^{(-n-n_2)(1-\theta_1)} 2^{(-m-m_2)(\frac{1}{2}+\theta_1)} |F_1||F_2| 2^{n_2(1-\theta_1')} 2^{m_2(\frac{1}{2}+\theta_1')} |G_1||G_2| \cdot 2^l \|B\|_1^{-1} \|\tilde{B}\|_1^{-1}$$

With the choice $\theta_1 = \theta_1'$, one has

$$a \lesssim C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} 2^{-n(1-\theta_1)} 2^{-m(\frac{1}{2}+\theta_1)} |F_1||F_2||G_1||G_2| \cdot 2^l \|B\|_1^{-1} \|\tilde{B}\|_1^{-1}$$

One can apply a similar Fubini-type argument to estimate b. One first notices that

$$b = \sum_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} \Big(\sum_{\substack{T \in \mathbb{T}_{-l-\tilde{l}_2} \\ I_T \in \mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \Big)^{\frac{1}{2}} \Big(\sum_{\substack{S \in \mathbb{S}_{\tilde{l}_2} \\ J_S \in \mathcal{J}_{n_2,m_2}}} |J_S| \Big)^{\frac{1}{2}}$$

for some $\tilde{l}_2 \in \mathbb{Z}$. One can now rewrite for any $\epsilon > 0$, $0 < \mu < 1$,

$$\begin{aligned}
& b \\
= & \sum_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} (C_1 2^{-n-n_2}|F_1|)^{\frac{1}{2}\mu(1+\epsilon)} (C_1 2^{-m-m_2}|F_2|)^{\frac{1}{2}(1-\mu)(1+\epsilon)} \Big(\sum_{\substack{T \in \mathbb{T}_{-l-\tilde{l}_2} \\ I_T \in \mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \Big)^{\frac{1}{2}} \\
& (C_2 2^{n_2}|G_1|)^{\frac{1}{2}\mu(1+\epsilon)} (C_2 2^{m_2}|G_2|)^{\frac{1}{2}(1-\mu)(1+\epsilon)} \Big(\sum_{\substack{S \in \mathbb{S}_{\tilde{l}_2} \\ J_S \in \mathcal{J}_{n_2,m_2}}} |J_S| \Big)^{\frac{1}{2}} \\
& \cdot C_1^{-\frac{1}{2}(1+\epsilon)} C_2^{-\frac{1}{2}(1+\epsilon)} 2^{n \cdot \frac{1}{2}\mu(1+\epsilon)} 2^{m \cdot \frac{1}{2}(1-\mu)(1+\epsilon)} |F_1|^{-\frac{1}{2}\mu(1+\epsilon)} |F_2|^{-\frac{1}{2}(1-\mu)(1+\epsilon)} |G_1|^{-\frac{1}{2}\mu(1+\epsilon)} |G_2|^{-\frac{1}{2}(1-\mu)(1+\epsilon)} \\
\leq & \underbrace{\Big[\sum_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} (C_1 2^{-n-n_2}|F_1|)^{\mu(1+\epsilon)} (C_1 2^{-m-m_2}|F_2|)^{(1-\mu)(1+\epsilon)} \sum_{\substack{T \in \mathbb{T}_{-l-\tilde{l}_2} \\ I_T \in \mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \Big]^{\frac{1}{2}}}_{b^1} \\
& \underbrace{\Big[\sum_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} (C_2 2^{n_2}|G_1|)^{\mu(1+\epsilon)} (C_2 2^{m_2}|G_2|)^{(1-\mu)(1+\epsilon)} \sum_{\substack{S \in \mathbb{S}_{\tilde{l}_2} \\ J_S \in \mathcal{J}_{n_2,m_2}}} |J_S| \Big]^{\frac{1}{2}}}_{b^2}
\end{aligned}$$

$$\cdot C_1^{-\frac{1}{2}(1+\epsilon)} C_2^{-\frac{1}{2}(1+\epsilon)} 2^{n\cdot\frac{1}{2}\mu(1+\epsilon)} 2^{m\cdot\frac{1}{2}(1-\mu)(1+\epsilon)} |F_1|^{-\frac{1}{2}\mu(1+\epsilon)} |F_2|^{-\frac{1}{2}(1-\mu)(1+\epsilon)} |G_1|^{-\frac{1}{2}\mu(1+\epsilon)} |G_2|^{-\frac{1}{2}(1-\mu)(1+\epsilon)}$$

To estimate b^1, one observes that for any $I_T \in \mathcal{I}_{-n-n_2,-m-m_2}$, one has by the pointwise estimate stated in Claim 6.1.2 that

$$I_T \subseteq \{Mf_1 > C_1 2^{-n-n_2-10}|F_1|\} \cap \{Mf_2 > C_1 2^{-m-m_2-10}|F_2|\}$$

Moreover, $\{I_T : T \in \mathbb{T}_{-l-l_2}\}$ is a disjoint collection of intervals for any fixed $-l-l_2$. As a result,

$$\begin{aligned}
b^1 \lesssim & \Big[\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} (C_1 2^{-n-n_2}|F_1|)^{\mu(1+\epsilon)} (C_1 2^{-m-m_2}|F_2|)^{(1-\mu)(1+\epsilon)} \cdot \\
& \qquad \Big|\{Mf_1 > C_1 2^{-n-n_2}|F_1|\} \cap \{Mf_2 > C_1 2^{-m-m_2}|F_2|\}\Big|\Big]^{\frac{1}{2}} \\
\leq & \Big[\int (Mf_1(x))^{\mu(1+\epsilon)} (Mf_2(x))^{(1-\mu)(1+\epsilon)} dx\Big]^{\frac{1}{2}} \\
\leq & \Big[\Big(\int (Mf_1(x))^{\mu(1+\epsilon)\frac{1}{\mu}} dx\Big)^{\mu} \Big(\int (Mf_2(x))^{(1-\mu)(1+\epsilon)\frac{1}{1-\mu}} dx\Big)^{1-\mu}\Big]^{\frac{1}{2}}
\end{aligned}$$

where the last step follows from the Holder inequality. One can now use the mapping property for the Hardy-Littlewood maximal operator $M : L^p \to L^p$ for any $p > 1$:

$$\Big(\int (Mf_1(x))^{1+\epsilon} dx\Big)^{\mu} \lesssim \|f_1\|_{1+\epsilon}^{(1+\epsilon)\mu} = |F_1|^{\mu}$$

By the same reasoning, one has

$$\Big(\int (Mf_2(x))^{1+\epsilon} dx\Big)^{1-\mu} \lesssim \|f_2\|_{1+\epsilon}^{(1+\epsilon)(1-\mu)} = |F_2|^{1-\mu}$$

One thus has

$$b^1 \lesssim |F_1|^{\frac{\mu}{2}} |F_2|^{\frac{1-\mu}{2}}$$

By symmetric argument with $-n-n_2$ and $-m-m_2$ replaced by n_2 and m_2 correspondingly, one obtains

$$b^2 \lesssim |G_1|^{\frac{\mu}{2}} |G_2|^{\frac{1-\mu}{2}}$$

Combination of the estimates for b^1 and b^2 yields

$$b \lesssim C_1^{-\frac{1}{2}(1+\epsilon)} C_2^{-\frac{1}{2}(1+\epsilon)} |F_1|^{-\frac{\mu}{2}\epsilon} |F_2|^{-\frac{1-\mu}{2}\epsilon} |G_1|^{-\frac{\mu}{2}\epsilon} |G_2|^{-\frac{1-\mu}{2}\epsilon} 2^{n\cdot\frac{1}{2}\mu(1+\epsilon)} 2^{m\cdot\frac{1}{2}(1-\mu)(1+\epsilon)}$$

By applying the results for both a and b, one concludes that

$$III \lesssim C_1^{-\frac{1}{2}\epsilon} C_2^{-\frac{1}{2}\epsilon} 2^{-n(1-\theta_1-\frac{1}{2}\mu(1+\epsilon))} 2^{-m(\frac{1}{2}+\theta_1-\frac{1}{2}(1-\mu)(1+\epsilon))} |F_1|^{1-\frac{\mu}{2}\epsilon} |F_2|^{1-\frac{1-\mu}{2}\epsilon} |G_1|^{1-\frac{\mu}{2}\epsilon} |G_2|^{1-\frac{1-\mu}{2}\epsilon} \cdot 2^l \|B\|_1^{-1} \|\tilde{B}\|_1^{-1}$$

To sum up, one has

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ S\in\mathbb{S}_{l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}} \left|I_T \times J_S\right| \lesssim C_1^{-\frac{1}{2}\epsilon} C_2^{-\frac{1}{2}\epsilon} 2^{-n(1-\theta_1-\frac{1}{2}\mu(1+\epsilon))} 2^{-m(\frac{1}{2}+\theta_1-\frac{1}{2}(1-\mu)(1+\epsilon))} |F_1|^{1-\frac{\mu}{2}\epsilon} |F_2|^{1-\frac{1-\mu}{2}\epsilon} |G_1|^{1-\frac{\mu}{2}\epsilon} |G_2|^{1-\frac{1-\mu}{2}\epsilon} \cdot 2^l \|B\|_1^{-1} \|\tilde{B}\|_1^{-1}$$

for any $0 < \theta_1, \mu < 1$, $\epsilon > 0$. One can now combine the estimates obtained in two different ways:

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}} \sum_{\substack{T\in\mathbb{T}_{-l-l_2}\\ S\in\mathbb{S}_{l_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}\\ J_S\in\mathcal{J}_{n_2,m_2}}} \left|\Big(\bigcup_{R\in\mathcal{R}_{k_2}} R\Big) \cap (I_T \times J_S)\right|$$
$$\lesssim C_1^{-\frac{1}{2}\epsilon(1-\lambda)} C_2^{-\frac{1}{2}\epsilon(1-\lambda)} C_3^{-\gamma\lambda} 2^{-k_2\gamma\lambda} 2^{-n(1-\theta_1-\frac{1}{2}\mu(1+\epsilon))(1-\lambda)} 2^{-m(\frac{1}{2}+\theta_1-\frac{1}{2}(1-\mu)(1+\epsilon))(1-\lambda)}$$
$$(|F_1|^{1-\frac{\mu}{2}\epsilon} |F_2|^{1-\frac{1-\mu}{2}\epsilon} |G_1|^{1-\frac{\mu}{2}\epsilon} |G_2|^{1-\frac{1-\mu}{2}\epsilon} \cdot 2^l \|B\|_1^{-1} \|\tilde{B}\|_1^{-1})^{(1-\lambda)}$$

for some $0 \leq \lambda \leq 1$. By applying the above estimates to (6.2.2), one has

$$|\Lambda| \lesssim C_1^{1-\frac{1}{2}\epsilon(1-\lambda)} C_2^{1-\frac{1}{2}\epsilon(1-\lambda)} C_3^{2-\gamma\lambda} \sum_{\substack{n>0\\ m>0\\ l>0\\ k_1<0\\ k_2\leq K}} 2^{-l\lambda} 2^{k_1(1-\frac{\lambda}{2})} 2^{k_2(1-\frac{\lambda\gamma}{2})} 2^{-n(1-\theta_1-\frac{1}{2}\mu(1+\epsilon))(1-\lambda)} 2^{-m(\frac{1}{2}+\theta_1-\frac{1}{2}(1-\mu)(1+\epsilon))(1-\lambda)}$$

$$\cdot \left(|F_1|^{1-\frac{\mu}{2}\epsilon}|F_2|^{1-\frac{1-\mu}{2}\epsilon}|G_1|^{1-\frac{\mu}{2}\epsilon}|G_2|^{1-\frac{1-\mu}{2}\epsilon}\right)^{1-\lambda}\|B\|_1^{\lambda}\|\tilde{B}\|_1^{\lambda}\|h\|_{L^s}$$

One notices that as long as $0 < \epsilon < 2$,there exists $0 < \theta_1 < 1$ such that

$$1 - \theta_1 - \frac{1}{2}\mu(1+\epsilon) > 0$$

$$\frac{1}{2} + \theta_1 - \frac{1}{2}(1-\mu)(1+\epsilon) > 0$$

As a result, the geometric series involving 2^{-n} and 2^{-m} are convergent. Also, one observes that for any $0 < \lambda < 1$, the series involving 2^{-l} and 2^{k_1} are convergent as well. One can separate the cases when $k_2 > 0$ and $k_2 \leq 0$ to select $\gamma > 1$ to make the series about 2^{k_2} convergent. Therefore, one can estimate the above expression as

$$|\Lambda| \lesssim C_1C_2C_3^2\|h\|_{L^s} \cdot \left(|F_1|^{1-\frac{\mu}{2}\epsilon}|F_2|^{1-\frac{1-\mu}{2}\epsilon}|G_1|^{1-\frac{\mu}{2}\epsilon}|G_2|^{1-\frac{1-\mu}{2}\epsilon}\right)^{1-\lambda}\|B\|_1^{\lambda}\|\tilde{B}\|_1^{\lambda}$$

where

$$\|B\|_1 \lesssim |F_1|^{\rho}|F_2|^{1-\rho}$$

$$\|\tilde{B}\|_1 \lesssim |G_1|^{\rho'}|G_2|^{1-\rho'}$$

One thus obtains

$$|\Lambda| \lesssim C_1C_2C_3^2|F_1|^{(1-\frac{\mu}{2}\epsilon)(1-\lambda)+\rho\lambda}|F_2|^{(1-\frac{1-\mu}{2}\epsilon)(1-\lambda)+(1-\rho)\lambda}|G_1|^{(1-\frac{\mu}{2}\epsilon)(1-\lambda)+\rho'\lambda}|G_2|^{(1-\frac{1-\mu}{2}\epsilon)(1-\lambda)+(1-\rho')\lambda} \tag{6.2.3}$$

With the choice $\rho = \rho' = \frac{1}{2}$, one can simplify the above expression:

$$|\Lambda| \lesssim C_1C_2C_3^2|F_1|^{(1-\frac{\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda}|F_2|^{(1-\frac{1-\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda}|G_1|^{(1-\frac{\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda}|G_2|^{(1-\frac{1-\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda}$$

By choosing $0 < \epsilon, \mu, \lambda < 1$ such that $(1-\frac{\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda = \frac{1}{p}$ and $(1-\frac{1-\mu}{2}\epsilon)(1-\lambda)+\frac{1}{2}\lambda = \frac{1}{q}$, one completes the proof.

6.3 Estimates for $\Pi_{\text{flag}^0 \otimes \text{paraproduct}}$

Proof. We assume that $f_i \leq \chi_{E_i}$, $g_j \leq \chi_{F_j}$, for $1 \leq i, j \leq 2$. We define the exceptional set

$$\Omega := \Omega_1 \cup \Omega_2,$$

where

$$\begin{aligned}
\Omega_1 := & \bigcup_{n_1 \in \mathbb{Z}} \{Mf_1 > C_1 2^{n_1}|F_1|\} \times \{Mg_1 > C_2 2^{-n_1}|G_1|\} \cup \\
& \bigcup_{m_1 \in \mathbb{Z}} \{Mf_2 > C_1 2^{m_1}|F_2|\} \times \{Mg_2 > C_2 2^{-m_1}|G_2|\} \cup \\
& \bigcup_{l_1 \in \mathbb{Z}} \{MB > C_1 2^{l_1}\|B\|_1\} \times \{Mg_1 > C_2 2^{-l_1}|G_1|\} \cup \\
& \bigcup_{l_2 \in \mathbb{Z}} \{MB > C_1 2^{l_2}\|B\|_1\} \times \{Mg_2 > C_2 2^{-l_2}|G_2|\}; \\
\Omega_2 := & \{SSh > C_3 \|h\|_{L^s}\}
\end{aligned}$$

and

$$\tilde{\Omega} := \{M\chi_{\Omega} > \frac{1}{100}\}.$$

6.3.1 Hybrid of Stopping-Time Decompositions

Table 6.5: Stopping-Time on $\mathcal{K}$ for $\Pi^1_{\text{flag}^0 \otimes \text{paraproduct}}$

One-dimensional stopping-time decomposition on $\mathcal{K}$	$\longrightarrow$	$K \in \mathcal{K}_{n_0}$ $(n_0 \in \mathbb{Z})$

Observation 3. If $I \times J \in T \times \mathcal{J}_{n_2,m_2}$ with $T \in \mathbb{T}_{l_1}$, then $l_1, m_1, m_2 \in \mathbb{Z}$ satisfies

Table 6.6: Stopping-Time on $\mathcal{R}$ for $\Pi^1_{\text{flag}^0 \otimes \text{paraproduct}}$

Two-dimensional tensor-type stopping-time decomposition I on $\mathcal{I} \times \mathcal{J}$	$\longrightarrow$	$I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2}$ $(n_2, m_2 \in \mathbb{Z}, n > 0)$
$\Downarrow$		
Two-dimensional general level sets decomposition on $\mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2}$	$\longrightarrow$	$I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \times \mathcal{J}_{n_2,m_2} \cap \mathcal{R}_{k_1,k_2}$ $(n_2, m_2 \in \mathbb{Z}, n > 0, k_1 < 0, k_2 \leq K)$
$\Downarrow$		
One-dimensional stopping-time decomposition - maximal intervals on $I \in \mathcal{I}_{-n-n_2,-m-m_2} \cap \{I : I \times J \in \mathcal{R}_{k_1,k_2}\}$	$\longrightarrow$	$I \times J \in (\mathcal{I}_{-n-n_2,-m-m_2} \cap T) \times \mathcal{J}_{n_2,m_2} \cap \mathcal{R}_{k_1,k_2}$ with $T \in \mathbb{T}_{-l-n_2}$ $(n_2, m_2 \in \mathbb{Z}, n, l > 0, k_1 < 0, k_2 \leq K)$

$l_1 + m_1 < 0$ and $l_1 + m_2 < 0$. Equivalently, $\forall I \times J \cap \tilde{\Omega}^c \neq \emptyset$, $I \times J \in T \times \mathcal{J}_{n_2,m_2}$, with $T \in \mathbb{T}_{-l-n_2}$ and $T \in \mathbb{T}_{-l'-m_2}$ for some $n_2, m_2 \in \mathbb{Z}$, $l, l' > 0$ and $-l - n_2 = -l' - m_2$.

Proof. $I \in T$ with $T \in \mathbb{T}_{l_1}$ means that $\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} > C_1 2^{l_1} \|B\|_1$. By the biest trick, $\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} = \frac{|\langle B(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \leq MB(x)$ for any $x \in I$. Thus $I \subseteq \{MB > C_1 2^{l_1}\|B\|_1\}$ Meanwhile $J \in \mathcal{J}_{n_2}$ implies that $|J \cap \{Mg_1 > C_2 2^{n_2}|G_1|\}| > \frac{1}{10}|J|$. If $l_1 + n_2 \geq 0$, then $\{MB > C_1 2^{l_1}\|B\|_1\} \times \{Mg_1 > C_2 2^{n_2}|G_1|\} \subseteq \Omega_1 \subseteq \Omega$. As a consequence, $|I \times J \cap \Omega| > \frac{1}{10}|I \times J|$ and $I \times J \subseteq \tilde{\Omega}$, which is a contradiction. Same reasoning applies to l_1 and m_2. □

6.3.2 Application of Stopping-Time Decompositions

We combine the stopping-time decompositions of dyadic rectangles and rewrite the linear form as

$$|\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\sum_{T\in\mathbb{T}_{-l-n_2}}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{1}{|I|^{\frac{1}{2}}|J|}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle g_1,\varphi_J^1\rangle\langle g_2,\psi_J^2\rangle$$

$$\langle h,\psi_I^2\otimes\varphi_J^2\rangle\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle\Big|$$

$$=\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\sum_{T\in\mathbb{T}_{-l-n_2}}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\frac{|\langle g_2,\psi_J^2\rangle|}{|J|^{\frac{1}{2}}}\cdot$$

$$\cdot\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|I||J|$$

One notices that for any $I\times J\in\mathcal{R}_{k_1,k_2}$,

$$|I\times J\cap(\Omega_{k_1}^2)^c|\geq\frac{99}{100}|I\times J|$$

$$|I\times J\cap(\Omega_{k_2}^2)^c|\geq\frac{99}{100}|I\times J|$$

where $\Omega_{k_1}^2:=\{SMh(x,y)>C_3 2^{k_1+1}\|h\|_s\}$ and $\Omega_{k_2}^2:=\{SS\chi_{E'}(x,y)>C_3 2^{k_2+1}\}$.

As a result, one can restrict $I\times J$ in the sum to its smaller subset:

$$|\Lambda|$$

$$\lesssim\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\sum_{T\in\mathbb{T}_{-l-n_2}}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\frac{|\langle g_2,\psi_J^2\rangle|}{|J|^{\frac{1}{2}}}\cdot$$

$$\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}|I\times J\cap(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c|$$

where the inner-most sum can be rewritten as

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\frac{|\langle g_2,\psi_J^2\rangle|}{|J|^{\frac{1}{2}}}\cdot$$

$$\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy$$

By applying Cauchy-Schwartz inequality twice with respect to the sums over I and J, one obtains

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c} \sup_{\substack{I\in T\\ I\in\mathcal{I}_{-n-n_2,-m-m_2}}} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\chi_I(x) \sum_{\substack{J\in\mathcal{J}_{n_2,m_2}\\ J\in\{J:I\times J\in\mathcal{R}_{k_1,k_2}\}}} \frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\frac{|\langle g_2,\psi_J^2\rangle|}{|J|^{\frac{1}{2}}}\chi_J(y)\cdot$$

$$\Big(\sum_{\substack{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\\ I\in\{I:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}\Big(\sum_{\substack{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\\ I\in\{I:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}dxdy$$

$$\leq \int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c} \sup_{\substack{I\in T\\ I\in\mathcal{I}_{-n-n_2,-m-m_2}}} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\chi_I(x) \sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\Big(\sum_{J\in\mathcal{J}_{m_2}}\frac{|\langle g_2,\psi_J^2\rangle|^2}{|J|}\chi_J(y)\Big)^{\frac{1}{2}}\cdot$$

$$\sup_{\substack{J\in\mathcal{J}_{n_2,m_2}\\ J\in\{J:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{1}{|J|^{\frac{1}{2}}}\Big(\sum_{\substack{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\\ I\in\{I:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|^2}{|I||J|}\chi_I(x)\Big)^{\frac{1}{2}}\chi_J(y)$$

$$\Big(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle \chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}dxdy$$

One can then use the Holder inequality to obtain:

$$\sup_{I\in T}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\Big[\int\chi_{\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I}(x)\sum_{J\in\mathcal{J}_{n_2,m_2}}\frac{|\langle g_2,\psi_J^2\rangle|^2}{|J|}\chi_J(y)dxdy\Big]^{\frac{1}{2}}$$

$$\cdot\Big[\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\Big(\sup_{\substack{J\in\mathcal{J}_{n_2,m_2}\\ J\in\{J:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{1}{|J|^{\frac{1}{2}}}\sum_{\substack{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\\ I\in\{I:I\times J\in\mathcal{R}_{k_1,k_2}\}}}\frac{|\langle h,\psi_I^2\otimes\varphi_J^2\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^2$$

$$\left(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\right)^{\frac{1}{2}\cdot 2} dxdy\Big]^{\frac{1}{2}}$$

$$\leq \sup_{I\in T}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot\left[\int\chi_{\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I}(x)\sum_{J\in\mathcal{J}_{n_2,m_2}}\frac{|\langle g_2,\psi_J^2\rangle|^2}{|J|}\chi_J(y)dxdy\right]^{\frac{1}{2}}$$

$$\left[\int_{\substack{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c\cap\\ \bigcup_{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}} I\times J\cap\\ \bigcup_{I\times J\in\mathcal{R}_{k_1,k_2}} I\times J}}(SMh(x,y))^2(SS\chi_{E'}(x,y))^2dxdy\right]^{\frac{1}{2}}$$

$$\lesssim \sup_{I\in T}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot\left[\int\chi_{\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I}(x)\sum_{J\in\mathcal{J}_{n_2,m_2}}\frac{|\langle g_2,\psi_J^2\rangle|^2}{|J|}\chi_J(y)dxdy\right]^{\frac{1}{2}}$$

$$\cdot C_3^2 2^{k_1}\|h\|_s 2^{k_2}\left|\bigcup_{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}} I\times J\cap\bigcup_{I\times J\in\mathcal{R}_{k_1,k_2}} I\times J\right|^{\frac{1}{2}}$$

where the second inequality follows from the definition of the hybrid maximal and square functions. The last step follows from the two-dimensional general level sets stopping-time decomposition. In particular, for any $(x,y)\in(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c$,

$$SMh(x,y)\lesssim C_3 2^{k_1}\|h\|_s$$

$$SS\chi_{E'}(x,y)\lesssim C_3 2^{k_2}$$

One can also simplify the other integral term

$$\left[\int\chi_{\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I}(x)\sum_{J\in\mathcal{J}_{n_2,m_2}}\frac{|\langle g_2,\psi_J^2\rangle|^2}{|J|}\chi_J(y)dxdy\right]^{\frac{1}{2}}$$

$$=\left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I\right|^{\frac{1}{2}}\left(\sum_{J\in\mathcal{J}_{n_2,m_2}}|\langle g_2,\psi_J^2\rangle|^2\right)^{\frac{1}{2}}$$

By the John-Nirenburg inequality described in Proposition 5.1.1, one has

$$\left(\sum_{J\in\mathcal{J}_{n_2,m_2}}|\langle g_2,\psi_J^2\rangle|^2\right)^{\frac{1}{2}}\lesssim\sup_{J_0\in\mathcal{J}_{n_2,m_2}}\frac{1}{|J_0|}\left\|\sum_{J\subseteq J_0}\frac{\langle g_2,\psi_J^2\rangle}{|J|}\chi_J\right\|_{1,\infty}\cdot\sup_{J_0\in\mathcal{J}_{n_2,m_2}}|J_0|^{\frac{1}{2}}$$

Proposition 5.1.2 provides estimates for the $L^{1,\infty}$-size:

$$\sup_{J_0\in\mathcal{J}_{n_2,m_2}}\frac{1}{|J_0|}\left\|\sum_{J\subseteq J_0}\frac{\langle g_2,\psi_J^2\rangle}{|J|}\chi_J\right\|_{1,\infty}\sup_{J_0\in\mathcal{J}_{n_2,m_2}}|J_0|$$

$$\lesssim \sup_{J_0 \in \mathcal{J}_{n_2,m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \sup_{J_0 \in \mathcal{J}_{n_2,m_2}} |J_0|^{\frac{1}{2}}$$

$$\leq \sup_{J_0 \in \mathcal{J}_{n_2,m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \Big| \bigcup_{J_0 \in \mathcal{J}_{n_2,m_2}} J_0 \Big|^{\frac{1}{2}}$$

where $\tilde{\chi}_{J_0}$ denotes an L^∞-normalized bump function adapted to J_0. One therefore has the following estimate:

$$\Big[\int \chi_{\bigcup_{I \in \mathcal{I}_{-n-n_2,-m-m_2} \cap T} I}(x) \sum_{J \in \mathcal{J}_{n_2,m_2}} \frac{|\langle g_2, \psi_J^2 \rangle|^2}{|J|} \chi_J(y) dx dy \Big]^{\frac{1}{2}}$$

$$\lesssim \sup_{J_0 \in \mathcal{J}_{n_2,m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \Big| \bigcup_{I \in \mathcal{I}_{-n-n_2,-m-m_2} \cap T} I \Big|^{\frac{1}{2}} \Big| \bigcup_{J_0 \in \mathcal{J}_{n_2,m_2}} J_0 \Big|^{\frac{1}{2}}$$

By applying the above result to the estimate for the linear form, one has

$$|\Lambda| \lesssim \sum_{\substack{n>0 \\ m>0 \\ l>0 \\ k_1<0 \\ k_2 \leq K}} \sum_{\substack{n_2 \in \mathbb{Z} \\ m_2 \in \mathbb{Z}}} \sum_{T \in \mathbb{T}_{-l-n_2}} \sup_{I \in T} \frac{|\langle B_I(f_1, f_2), \varphi_I^1 \rangle|}{|I|^{\frac{1}{2}}} \sup_{J \in \mathcal{J}_{n_2}} \frac{|\langle g_1, \varphi_J^1 \rangle|}{|J|^{\frac{1}{2}}} \sup_{J_0 \in \mathcal{J}_{n_2,m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \cdot$$
$$\Big| \bigcup_{I \in \mathcal{I}_{-n-n_2,-m-m_2} \cap T} I \Big|^{\frac{1}{2}} \Big| \bigcup_{J_0 \in \mathcal{J}_{n_2,m_2}} J_0 \Big|^{\frac{1}{2}} C_3^2 2^{k_1} \|h\|_s 2^{k_2} \Big| \bigcup_{I \times J \in \mathcal{I}_{-n-n_2,-m-m_2} \cap T \times \mathcal{J}_{n_2,m_2}} I \times J \cap \bigcup_{I \times J \in \mathcal{R}_{k_1,k_2}} I \times J \Big|^{\frac{1}{2}} \tag{6.3.1}$$

where, by recalling the definition of $T \in \mathbb{T}_{-l-n_2}$, $\mathcal{I}_{-n-n_2,-m-m_2}$, and $\mathcal{J}_{n_2,m_2}$ in the tensor-product type stopping-time decomposition,

$$\sup_{I \in T} \frac{|\langle B_I(f_1, f_2), \varphi_I^1 \rangle|}{|I|^{\frac{1}{2}}} \lesssim C_1 2^{-l-n_2} \|B\|_1$$

$$\sup_{J \in \mathcal{J}_{n_2}} \frac{|\langle g_1, \varphi_J^1 \rangle|}{|J|^{\frac{1}{2}}} \lesssim C_2 2^{n_2} |G_1|$$

$$\sup_{J_0 \in \mathcal{J}_{n_2,m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \leq \sup_{J_0 \in \mathcal{J}_{m_2}} \frac{1}{|J_0|} \int |g_2(y)\tilde{\chi}_{J_0}(y)| dy \lesssim C_2 2^{m_2} |G_2|$$

Meanwhile, since $g_j \leq \chi_{G_j}$ for $j = 1, 2$,

$$\sup_{J \in \mathcal{J}} \frac{|\langle g_1, \varphi_J^1 \rangle|}{|J|^{\frac{1}{2}}} \leq 1,$$

$$\sup_{J\in\mathcal{J}}\frac{|\langle g_2,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\leq 1.$$

One way to integrate the above estimates is

$$\sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\leq \min(1,C_2 2^{n_2}|G_1|),$$

$$\sup_{J\in\mathcal{J}_{m_2}}\frac{|\langle g_2,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\leq \min(1,C_2 2^{m_2}|G_2|).$$

By plugging in the above estimates into expression (6.3.1), one obtains

$$|\Lambda| \lesssim C_1C_3^2\sum_{\substack{n>0\\m>0\\l>0\\k_1>0\\k_2\geq -K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\sum_{T\in\mathbb{T}_{-l-n_2}}2^{-l-n_2}\|B\|_1\cdot\min(1,C_2 2^{n_2}|G_1|)\cdot\min(1,C_2 2^{m_2}|G_2|)2^{k_1}\|h\|_s 2^{k_2}\cdot$$
$$\Big|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I\Big|^{\frac{1}{2}}\Big|\bigcup_{J_0\in\mathcal{J}_{n_2,m_2}}J_0\Big|^{\frac{1}{2}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}I\times J\Big|^{\frac{1}{2}}$$

Now one can apply the Cauchy-Schwartz inequality to obtain

$$C_1C_3^2\sum_{\substack{n>0\\m>0\\l>0\\k_1>0\\k_2\geq -K}}\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}2^{-l-n_2}\|B\|_1\cdot\min(1,C_2 2^{n_2}|G_1|)\cdot\min(1,C_2 2^{m_2}|G_2|)2^{k_1}\|h\|_s 2^{k_2}\cdot$$
$$\Big(\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I\Big|\Big|\bigcup_{J_0\in\mathcal{J}_{n_2,m_2}}J_0\Big|\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}I\times J\Big|\Big)^{\frac{1}{2}}$$
$$=C_1C_3^2\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}}\|B\|_1\|h\|_{L^s}2^{k_1}2^{k_2}\cdot$$
$$\sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}}}\#\cdot\Big(\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T}I\Big|\Big|\bigcup_{J_0\in\mathcal{J}_{n_2,m_2}}J_0\Big|\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}I\times J\Big|\Big)^{\frac{1}{2}}$$

(6.3.2)

where

$$\begin{aligned}
\# &:= 2^{-l-n_2}\min(1, C_2 2^{n_2}|G_1|)\min(1, C_2 2^{m_2}|G_2|) \\
&\leq \min\left(2^{-l-n_2}(C_2 2^{n_2} G_1)^{1-\theta}(C_2 2^{m_2}|G_2|)^{\theta}, C_2^2 2^{-l-n_2} 2^{n_2}|G_1| 2^{m_2}|G_2|\right) \\
&\leq \min(C_2|G_1|^{1-\theta}|G_2|^{\theta}, C_2^2 2^{-l-n_2} 2^{n_2} 2^{m_2}|G_1||G_2|).
\end{aligned}$$

where the last step uses the fact that $l > 0$ and $l + n_2 - m_2 > 0$ by the definition of our exceptional set. Now 6.3.2 can be estimated by

$$\begin{aligned}
&C_1 C_3{}^2 \sum_{\substack{n>0\\ m>0\\ l>0\\ k_1<0\\ k_2\leq K}} \|B\|_1 \|h\|_{L^s} 2^{k_1} 2^{k_2} \cdot \\
&\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \left[\# \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\right| \left|\bigcup_{J_0\in\mathcal{J}_{n_2,m_2}} J_0\right|\right]^{\frac{1}{2}} \cdot \left[\# \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\right|\right]^{\frac{1}{2}} \\
\leq & C_1 C_3{}^2 \sum_{\substack{n>0\\ m>0\\ l>0\\ k_1<0\\ k_2\leq K}} \|B\|_1 \|h\|_{L^s} 2^{k_1} 2^{k_2} \cdot \\
&\left[\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \# \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\right| \left|\bigcup_{J_0\in\mathcal{J}_{n_2,m_2}} J_0\right|\right]^{\frac{1}{2}} \cdot \left[\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \# \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\right|\right]^{\frac{1}{2}}
\end{aligned} \tag{6.3.3}$$

where the last inequality follows from the Cauchy-Schwartz inequality.

6.3.3 Sparsity Condition

One can apply the sparsity condition to capture the contribution from the stopping-time decompositions $\mathcal{R} = \bigcup_{k_1,k_2} \bigcup_{R\in\mathcal{R}_{k_1,k_2}} R$. In particular, one first uses the fact that $\{I : I \in T, T \in \mathbb{T}_{-l-n_2}\}$ is a disjoint collection of intervals for any fixed $-l-n_2$ to simplify

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| = \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big|$$

One can now apply Proposition 6.1.3 to deduce that

$$\begin{aligned}\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| &\lesssim \sum_{m_2\in\mathbb{Z}}\Big|\bigcup_{n_2\in\mathbb{Z}}\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| \\ &\leq \sum_{m_2\in\mathbb{Z}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-m-m_2}\times\mathcal{J}_{m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| \\ &\lesssim \Big|\bigcup_{m_2\in\mathbb{Z}}\bigcup_{\substack{I\times J\in\mathcal{I}_{-m-m_2}\times\mathcal{J}_{m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big| \\ &\leq \Big|\bigcup_{I\times J\in\mathcal{R}_{k_1,k_2}} I\times J\Big| \\ &\lesssim \min(2^{-k_1}, 2^{-k_2\gamma})\end{aligned}$$

for any $\gamma > 1$.

Fubini Argument. One also observes that a trivial bound can be obtained for

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\#\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big|$$

by forgetting the information from the stopping-time decompositions with respect to $SM(h)$ and $SS(\chi_{E'})$. In particular, the above expression can be estimated by

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\#\sum_{T\in\mathbb{T}_{-l-n_2}}\Big|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\Big|\Big|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\Big|.$$

which already appear in the expression (6.3.2).

One can apply the bound for # to estimate

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \# \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\right|\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right|$$

$$\leq C_2^2 \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} 2^{-l-n_2}2^{n_2}|G_1|2^{m_2}|G_2| \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\right|\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right|$$

$$\leq \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} C_1 2^{-l-n_2}\|B\|_1 \sum_{T\in\mathbb{T}_{-l-n_2}} \left|\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T} I\right| \cdot C_2 2^{n_2}|G_1|C_2 2^{m_2}|G_2|\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right| \cdot \|B\|_1^{-1}C_1^{-1}$$

$$\leq \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} C_1 2^{-l-n_2}\|B\|_1 \sum_{\substack{T\in\mathbb{T}_{-l-n_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \cdot C_2 2^{n_2}|G_1|C_2 2^{m_2}|G_2|\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right| \cdot \|B\|_1^{-1}C_1^{-1} \qquad (6.3.4)$$

For any fixed n_2, m_2, one can apply Proposition 5.1.5 with $S_1 := \{Mf_1 \leq C_1 2^{-n-n_2}|F_1|\} \cap \{Mf_2 \leq C_1 2^{-m-m_2}|F_2|\}$ and $\theta_3 = 0$:

$$C_1 2^{-l-n_2}\|B\|_1 \sum_{\substack{T\in\mathbb{T}_{-l-n_2}\\ I_T\in\mathcal{I}_{-n-n_2,-m-m_2}}} |I_T| \lesssim \min(1, C_1 2^{-n-n_2}|F_1|)^{1-\theta_1} \min(1, C_1 2^{-m-m_2}|F_2|)^{\theta_1}|F_1|^{\theta_1}|F_2|^{1-\theta_1}$$

$$\leq C_1^{\alpha_1(1-\theta_1)+\alpha_2\theta_1} 2^{(-n-n_2)\alpha_1(1-\theta_1)}2^{(-m-m_2)\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}$$

By applying the above energy estimate into (6.3.4):

$$C_1^{\alpha_1(1-\theta_1)+\alpha_2\theta_1-1} \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} 2^{(-n-n_2)\alpha_1(1-\theta_1)}2^{(-m-m_2)\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}$$

$$\cdot C_2 2^{n_2}|G_1|C_2 2^{m_2}|G_2|\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right| \cdot \|B\|_1^{-1}$$

$$=C_1^{\alpha_1(1-\theta_1)+\alpha_2\theta_1-1} \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} 2^{(-n-n_2)\alpha_1(1-\theta_1)}2^{(-m-m_2)\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}$$

$$\cdot (C_2 2^{n_2}|G_1|)^{(1-\theta_1')(1+\epsilon)}(C_2 2^{m_2}|G_2|)^{\theta_1'(1+\epsilon)}\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right| \cdot$$

$$\|B\|_1^{-1}(C_2 2^{n_2}|G_1|)^{1-(1-\theta_1')(1+\epsilon)}(C_2 2^{m_2}|G_2|)^{1-\theta_1'(1+\epsilon)}$$

$$\leq C_1 C_2^2 \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} 2^{-n\alpha_1(1-\theta_1)}2^{-m\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}$$

$$\cdot (C_2 2^{n_2}|G_1|)^{(1-\theta_1')(1+\epsilon)}(C_2 2^{m_2}|G_2|)^{\theta_1'(1+\epsilon)}\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right|\cdot$$

$$\|B\|_1^{-1}|G_1|^{1-(1-\theta_1')(1+\epsilon)}|G_2|^{1-\theta_1'(1+\epsilon)}2^{n_2(1-(1-\theta_1')(1+\epsilon)-\alpha_1(1-\theta_1))}2^{m_2(1-\theta_1'(1+\epsilon)-\alpha_2\theta_1)}$$

If

$$1-(1-\theta_1')(1+\epsilon)=\alpha_1(1-\theta_1)$$
$$1-\theta_1'(1+\epsilon)=\alpha_2\theta_1 \tag{6.3.5}$$

then the above expression can be estimated by

$$C_1C_2^2\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}(2^{n_2}|G_1|)^{(1-\theta_1')(1+\epsilon)}(2^{m_2}|G_2|)^{\theta_1'(1+\epsilon)}\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right|\cdot$$
$$\|B\|_1^{-1}2^{-n\alpha_1(1-\theta_1)}2^{-m\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}|G_1|^{1-(1-\theta_1')(1+\epsilon)}|G_2|^{1-\theta_1'(1+\epsilon)} \tag{6.3.6}$$

where by applying the point-wise estimate specified in Claim 6.1.2:

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}(C_2 2^{n_2}|G_1|)^{(1-\theta_1')(1+\epsilon)}(C_2 2^{m_2}|G_2|)^{\theta_1'(1+\epsilon)}\left|\bigcup_{J\in\mathcal{J}_{n_2,m_2}} J\right|$$
$$\leq \sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}(C_2 2^{n_2}|G_1|)^{(1-\theta_1')(1+\epsilon)}(C_2 2^{m_2}|G_2|)^{\theta_1'(1+\epsilon)}\,|\{Mg_1 > C_2 2^{n_2}|G_1|\}\cap\{Mg_2 > C_2 2^{m_2}|G_2|\}|$$
$$\lesssim \int (Mg_1(y))^{(1-\theta_1')(1+\epsilon)}(Mg_2(y))^{\theta_1'(1+\epsilon)}dy$$
$$\leq \left[\int (Mg_1(y))^{(1-\theta_1')(1+\epsilon)\frac{1}{1-\theta_1'}}dy\right]^{1-\theta_1'}\left[\int (Mg_2(y))^{(\theta_1')(1+\epsilon)\frac{1}{\theta_1'}}dy\right]^{\theta_1'}$$

The last inequality is the application of Holder inequality. Now one can apply the mapping property for Hardy-Littlewood maximal operator to derive the following bound

$$\|g_1\|_{1+\epsilon}^{(1+\epsilon)(1-\theta_1')}\|g_2\|_{1+\epsilon}^{(1+\epsilon)\theta_1'} = |G_1|^{1-\theta_1'}|G_2|^{\theta_1'}$$

As a result, (6.3.6) can be estimated by

$$C_1C_2^2\|B\|_1^{-1}2^{-n\alpha_1(1-\theta_1)}2^{-m\alpha_2\theta_1}|F_1|^{\alpha_1(1-\theta_1)+\theta_1}|F_2|^{\alpha_2\theta_1+(1-\theta_1)}|G_1|^{1-(1-\theta_1')\epsilon}|G_2|^{1-\theta_1'\epsilon}$$

Assuming the following equations hold:

$$1-(1-\theta_1')\epsilon = \alpha_1(1-\theta_1)+\theta_1$$

$$1-\theta_1'\epsilon = \alpha_2\theta_1+(1-\theta_1)$$

one would derive equations (6.3.5) if $\theta_1 = 1-\theta_1'$. As a result, one has

$$C_1C_2^2\|B\|_1^{-1}2^{-n(1-\theta_1(1+\epsilon))}2^{-m(1-(1-\theta_1)(1+\epsilon))}|F_1|^{1-\theta_1\epsilon}|F_2|^{1-(1-\theta_1)\epsilon}|G_1|^{1-\theta_1\epsilon}|G_2|^{1-(1-\theta_1)\epsilon}$$

Combination of Sparsity Condition and Fubini Argument. Now, one can combine both estimates by interpolating between them:

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\#\sum_{T\in\mathbb{T}_{-l-n_2}}\left|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T\times\mathcal{J}_{n_2,m_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\right|$$
$$\lesssim\left(C_2|G_1|^{1-\theta}|G_2|^{\theta}\min(C_3^{-1}2^{-k_1},C_3^{-\gamma}2^{-k_2\gamma})\right)^{\lambda}$$
$$\cdot\left(C_1C_2^2\|B\|_1^{-1}2^{-n(1-\theta_1(1+\epsilon))}2^{-m(1-(1-\theta_1)(1+\epsilon))}|F_1|^{1-\theta_1\epsilon}|F_2|^{1-(1-\theta_1)\epsilon}|G_1|^{1-\theta_1\epsilon}|G_2|^{1-(1-\theta_1)\epsilon}\right)^{1-\lambda}$$

One can now plug in the above estimates into the expression (6.3.2) and obtain:

$$|\Lambda|\lesssim C_1^2C_2^2C_3^2\sum_{\substack{n>0\\ m>0\\ l>0\\ k_1<0\\ k_2\le K}}\|B\|_1\|h\|_{L^s}2^{k_1}2^{k_2}\left(|G_1|^{1-\theta}|G_2|^{\theta}2^{-\frac{1}{2}k_1}2^{-\frac{1}{2}k_2\gamma}\right)^{\frac{\lambda}{2}}$$
$$\cdot\left(\|B\|_1^{-1}2^{-n(1-\theta_1(1+\epsilon))}2^{-m(1-(1-\theta_1)(1+\epsilon))}|F_1|^{1-\theta_1\epsilon}|F_2|^{1-(1-\theta_1)\epsilon}|G_1|^{1-\theta_1\epsilon}|G_2|^{1-(1-\theta_1)\epsilon}\right)^{\frac{1-\lambda}{2}+\frac{1}{2}}$$

$$=C_1^2C_2^2C_3^2\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\le K}}\|h\|_{L^s}2^{k_1(1-\frac{\lambda}{4})}2^{k_2(1-\frac{\lambda}{4}\gamma)}2^{-n(1-\theta_1(1+\epsilon))(1-\frac{\lambda}{2})}2^{-m(1-(1-\theta_1)(1+\epsilon))(1-\frac{\lambda}{2})}$$

$$\cdot|F_1|^{(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|F_2|^{(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_1|^{(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_2|^{(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}\|B\|_1^{\frac{\lambda}{2}}$$

where one can use the trivial estimate for $\|B\|_1$:

$$\|B\|_1\lesssim|F_1|^{\rho}|F_2|^{1-\rho},$$

for any $0<\rho<1$. One can then simplify the expression further as

$$C_1^2C_2^2C_3^2\sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\le K}}\|h\|_{L^s}2^{k_1(1-\frac{\lambda}{4})}2^{k_2(1-\frac{\lambda}{4}\gamma)}2^{-n(1-\theta_1(1+\epsilon))(1-\frac{\lambda}{2})}2^{-m(1-(1-\theta_1)(1+\epsilon))(1-\frac{\lambda}{2})}$$

$$\cdot|F_1|^{(1-\rho)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|F_2|^{\rho\frac{\lambda}{2}+(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_1|^{(1-\theta)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_2|^{\theta\frac{\lambda}{2}+(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}$$

One first observes that for $0<\lambda<1,\epsilon>0$, $0<\theta_1<1$ such that

$$\begin{aligned}&1-\theta_1(1+\epsilon)>0\\&1-(1-\theta_1)(1+\epsilon)>0\end{aligned}\tag{6.3.7}$$

the series involving 2^{k_1}, 2^{-n} and 2^{-m} are convergent. Also, for $k_2>0$, as long as $0<\lambda<1$ and $\gamma>1$ sufficiently large, the series converges. For $k_2<0$, if $0<\lambda<1$ and $\gamma>1$ close to 1, one has a convergent series as well. Thus

$$|\Lambda|$$
$$\lesssim C_1^2C_2^2C_3^2|F_1|^{(1-\rho)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|F_2|^{\rho\frac{\lambda}{2}+(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_1|^{(1-\theta)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}|G_2|^{\theta\frac{\lambda}{2}+(1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})}$$

One can choose $0<\rho=\theta<1$, $0<\lambda<1$ close to 0, and $\epsilon>0$ close to 0 such that (6.3.7) hold and

$$(1-\rho)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})=(1-\theta)\frac{\lambda}{2}+(1-\theta_1\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2})=\frac{1}{p}$$

$$\rho\frac{\lambda}{2} + (1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2}) = \theta\frac{\lambda}{2} + (1-(1-\theta_1)\epsilon)(\frac{1}{2}+\frac{1-\lambda}{2}) = \frac{1}{q}$$

As a consequence,

$$\Lambda \lesssim C_1^2 C_2^2 C_3^2 |F_1|^{\frac{1}{p}} |F_2|^{\frac{1}{q}} |G_1|^{\frac{1}{p}} |G_2|^{\frac{1}{q}} \|h\|_{L^s(\mathbb{R}^2)}.$$

6.4 Estimates for $\Pi^0_{\mathbf{flag}^0\otimes\mathbf{flag}^0}$ and $\Pi^0_{\mathbf{flag}^0\otimes\mathbf{flag}^{\#_2}}$

One defines the exceptional set in a similar fashion as in the previous section with a modification on B:

$$\begin{aligned}\Omega_1 :=& \bigcup_{n_1\in\mathbb{Z}}\{Mf_1 > C_1 2^{n_1}|F_1|\} \times \{Mg_1 > C_2 2^{-n_1}|G_1|\} \cup \\ & \bigcup_{m_1\in\mathbb{Z}}\{Mf_2 > C_1 2^{m_1}|F_2|\} \times \{Mg_2 > C_2 2^{-m_1}|G_2|\} \cup \\ & \bigcup_{l_1\in\mathbb{Z}}\{MB^+ > C_1 2^{l_1}\|B^+\|_1\} \times \{M\tilde{B}^+ > C_2 2^{-l_1}\|\tilde{B}^+\|_1\} \\ \Omega_2 :=& \{SSh > C_3\|h\|_{L^s}\}\end{aligned}$$

where

$$\begin{aligned}B^+ &:= \sum_K \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\psi_K^1\rangle||\langle f_2,\psi_K^2\rangle||\varphi_K^3| \\ \tilde{B}^+ &:= \sum_L \frac{1}{|L|^{\frac{1}{2}}}|\langle g_1,\psi_L^1\rangle||\langle g_2,\psi_L^2\rangle||\varphi_L^3|\end{aligned}$$

Then define

$$\begin{aligned}\Omega &:= \Omega_1 \cup \Omega_2 \\ \tilde{\Omega} &:= \{M\chi_\Omega > \frac{1}{100}\}.\end{aligned}$$

6.4.1 Two-dimensional stopping-time decompositions II - maximal intervals

One can perform a same stopping-time decomposition as in the previous section with $\|B\|_1$ replaced by $\|B^+\|_1$. In particular,

$$\mathcal{I} = \bigcup_{l_1} \bigcup_{T^+ \in \mathbb{T}^+_{l_1}} T^+,$$

where T^+ represents for the tree with the tree top I_{T^+} satisfying

$$\frac{|\langle B_{I_{T^+}}, \varphi^1_{I_{T^+}} \rangle|}{|I_{T^+}|^{\frac{1}{2}}} > C_1 2^{l_1} \|\tilde{B^+}\|_1$$

In addition,

$$\mathcal{J} = \bigcup_{l_2} \bigcup_{S^+ \in \mathbb{S}^+_{l_2}} S^+,$$

where S^+ represents for the tree with the tree top J_S satisfying

$$\frac{|\langle \tilde{B}_{J_{S^+}}, \varphi^1_{J_{S^+}} \rangle|}{|J_{S^+}|^{\frac{1}{2}}} > C_2 2^{l_2} \|\tilde{B^+}\|_1$$

Observation 4. If $I \times J \cap \tilde{\Omega}^c \neq \emptyset$ and $I \times J \in T^+ \times S^+$ with $T^+ \in \mathbb{T}^+_{l_1}$ and $S^+ \in \mathbb{S}^+_{l_2}$, then $l_1, l_2 \in \mathbb{Z}$ satisfies $l_1 + l_2 < 0$. Equivalently, $I \times J \in T^+ \times S^+$ with $T^+ \in \mathbb{T}^+_{-l-l_2}$ and $S^+ \in \mathbb{S}^+_{l_2}$ for some $l_2 \in \mathbb{Z}$, $l > 0$.

Proof of Proposition. One has

$$\begin{aligned}
\frac{|\langle B_{I_{T^+}}(f_1, f_2), \varphi^1_{I_{T^+}} \rangle|}{|I_{T^+}|^{\frac{1}{2}}} =& \frac{1}{|I_{T^+}|^{\frac{1}{2}}} \Big| \sum_{|K| > |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \psi^1_K \rangle \langle f_2, \psi^2_K \rangle \langle \varphi^3_K, \varphi^1_{I_{T^+}} \rangle \Big| \\
\leq& \frac{1}{|I_{T^+}|^{\frac{1}{2}}} \sum_{K} \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \psi^1_K \rangle| |\langle f_2, \psi^2_K \rangle| \langle |\varphi^3_K|, |\varphi^1_{I_{T^+}}| \rangle \\
=& \frac{|\langle B^+(f_1, f_2), |\varphi^1_{I_{T^+}}| \rangle|}{|I_{T^+}|^{\frac{1}{2}}} \leq MB^+(x)
\end{aligned}$$

for any $x \in I_T$, where $B^+ := \sum_K \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \psi_K^1 \rangle| |\langle f_2, \psi_K^2 \rangle| |\varphi_K^3|$. As a consequence,

$$I_T \subseteq \{MB^+ > C_1 2^{l_1} \|B^+\|_1\}.$$

Similarly, from the stopping-time decomposition on $\mathcal{J}$, one can deduce that

$$J_S \subseteq \{M\tilde{B}^+ > C_2 2^{l_2} \|\tilde{B}^+\|_1\}.$$

If $l_1 + l_2 \geq 0$, then $\{MB^+ > C_1 2^{l_1} \|B^+\|_1\} \times \{M\tilde{B}^+ > C_2 2^{l_2} \|\tilde{B}^+\|_1\} \subseteq \Omega_1 \subseteq \Omega$, which implies that $I \times J \subseteq \Omega \subseteq \tilde{\Omega}$ and contradicts the assumption. □

6.4.2 Hybrid of Stopping-Time Decompositions

Table 6.7: Stopping-Time on $\mathcal{K}$ for $\Pi^0_{\text{flag}^0 \otimes \text{flag}^0}$

One-dimensional stopping-time decomposition on $\mathcal{K}$	$\longrightarrow$	$K \in \mathcal{K}_{n_0}$ $(n_0 \in \mathbb{Z})$

Table 6.8: Stopping-Time on $\mathcal{L}$ for $\Pi^0_{\text{flag}^0 \otimes \text{flag}^0}$

One-dimensional stopping-time decomposition on $\mathcal{L}$	$\longrightarrow$	$L \in \mathcal{L}_{n_0'}$ $(n_0' \in \mathbb{Z})$

One can apply the essentially same argument for Π with $\|B\|_1$ replaced by $\|B^+\|_1$. For the sake completeness, we provides outlines of the proof:

$$|\Lambda| \lesssim \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\S^+\in\mathbb{S}^+_{l_2}}} \sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\I\times J\in\mathcal{R}_{k_1,k_2}}} \frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}} |\langle B_I(f_1,f_2),\varphi_I^1\rangle| |\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|$$

$$\cdot |\langle h, \psi_I^2 \otimes \psi_J^2 \rangle| |\langle \chi_{E'}, \psi_I^3 \otimes \psi_J^3 \rangle|$$

$$\lesssim \sum_{\substack{n>0\\m>0\\l>0\\k_1<0\\k_2\leq K}} \sum_{\substack{n_2\in\mathbb{Z}\\m_2\in\mathbb{Z}\\l_2\in\mathbb{Z}}} \sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\S^+\in\mathbb{S}^+_{l_2}}} \sup_{I\in T^+} \frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \sup_{J\in S^+} \frac{|\langle \tilde{B}_J(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}.$$

Table 6.9: Stopping-Time on $\mathcal{R}$ for $\Pi^0_{\text{flag}^0\otimes\text{flag}^0}$

Two-dimensional tensor-type stopping-time decomposition I on $\mathcal{I}\times\mathcal{J}$	$\longrightarrow$	$I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}$ $(n_2,m_2\in\mathbb{Z},n>0)$
$\Downarrow$		
Two-dimensional general level sets decomposition on $\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}$	$\longrightarrow$	$I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\cap\mathcal{R}_{k_1,k_2}$ $(n_2,m_2\in\mathbb{Z},n>0,k_1<0,k_2\leq K)$
$\Downarrow$		
Two-dimensional tensor-type stopping-time decomposition II on $I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}\cap\mathcal{R}_{k_1,k_2}$	$\longrightarrow$	$I\times J\in(\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+)\times(\mathcal{J}_{n_2,m_2}\cap S^+)$ $\cap\quad\mathcal{R}_{k_1,k_2}$ with $T^+\in\mathbb{T}^+_{-l-l_2}$, $S^+\in\mathbb{S}^+_{l_2}$ $(n_2,m_2,l_2\in\mathbb{Z},n,l>0,k_1<0,k_2\leq K,)$

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi^2_I\otimes\psi^2_J\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi^3_I\otimes\psi^3_J\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy$$

where $\Omega^2_{k_1}:=\{SSh(x,y)>C_3 2^{k_1+1}\|h\|_s\}$ and $\Omega^2_{k_2}:=\{SS\chi_{E'}(x,y)>C_3 2^{k_2+1}\}$.

Based on the tensor-type stopping-time decomposition,

$$\sup_{I\in T^+}\frac{|\langle B_I(f_1,f_2),\varphi^1_I\rangle|}{|I|^{\frac{1}{2}}}\lesssim C_1 2^{-l-l_2}\|B^+\|_1$$

$$\sup_{J\in S^+}\frac{|\langle\tilde{B}_J(g_1,g_2),\varphi^1_J\rangle|}{|J|^{\frac{1}{2}}}\lesssim C_2 2^{l_2}\|\tilde{B^+}\|_1$$

Meanwhile, the integral can be estimated in the same manner as in the previous section:

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi^2_I\otimes\psi^2_J\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi^3_I\otimes\psi^3_J\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy$$

$$\leq\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\Big(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi^2_I\otimes\psi^2_J\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}dxdy$$

$$\lesssim C_3^2 2^{k_1}\|h\|_s 2^{k_2}\Big|\bigcup_{\substack{I\times J\in\mathcal{I}_{-n-n_2,-m-m_2}\cap T^+\times\mathcal{J}_{n_2,m_2}\cap S^+\\ I\times J\in\mathcal{R}_{k_1,k_2}}} I\times J\Big|$$

where the last inequality follows from the point-wise estimates on the set $(\Omega_{k_1}^2)^c\cap(\Omega_{k_2}^2)^c$.

6.4.3 Sparsity Condition.

One can apply the sparsity condition as in the previous section to derive that

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}\sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\ S^+\in\mathbb{S}^+_{l_2}}}\Big|\Big(\bigcup_{\substack{R\in\mathcal{R}_{k_1,k_2}\\ R\in\mathcal{I}_{-n-n_2,-m-m_2}\times\mathcal{J}_{n_2,m_2}}}R\Big)\cap(I_{T^+}\times J_{S^+})\Big|\lesssim\min(C_3^{-1}2^{-k_1},C_3^{-\gamma}2^{-k_2\gamma})$$

6.4.4 Fubini Argument.

$$\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}\\ l_2\in\mathbb{Z}}}\sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\ S^+\in\mathbb{S}^+_{l_2}}}\big|I_{T^+}\times J_{S^+}\big|$$

$$\leq\underbrace{\sup_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\sum_{l_2\in\mathbb{Z}}\Big(\sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\ I_{T^+}\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_{T^+}|\Big)^{\frac{1}{2}}\Big(\sum_{\substack{S^+\in\mathbb{S}^+_{l_2}\\ J_{S^+}\in\mathcal{J}_{n_2,m_2}}}|J_{S^+}|\Big)^{\frac{1}{2}}}_{a}\cdot$$

$$\underbrace{\sum_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}}\sup_{l_2\in\mathbb{Z}}\Big(\sum_{\substack{T^+\in\mathbb{T}^+{}_{-l-l_2}\\ I_{T^+}\in\mathcal{I}_{-n-n_2,-m-m_2}}}|I_{T^+}|\Big)^{\frac{1}{2}}\Big(\sum_{\substack{S^+\in\mathbb{S}^+_{l_2}\\ J_{S^+}\in\mathcal{J}_{n_2,m_2}}}|J_{S^+}|\Big)^{\frac{1}{2}}}_{b}$$

One first notices that b can be estimated by the exactly same argument as in the previous section as the only fact about T^+ and S^+ is that $\{I_{T^+} : T^+ \in \mathbb{T}^+_{-l-l_2}\}$, $\{J_{S^+} : S^+ \in \mathbb{S}^+_{l_2}\}$ form disjoint collections of intervals. As a result, the following estimate holds for b:

$$b \lesssim |F_1|^{-\frac{\mu}{2}\epsilon}|F_2|^{-\frac{1-\mu}{2}\epsilon}|G_1|^{-\frac{\mu}{2}\epsilon}|G_2|^{-\frac{1-\mu}{2}\epsilon}2^{n\cdot\frac{1}{2}\mu(1+\epsilon)}2^{m\cdot\frac{1}{2}(1-\mu)(1+\epsilon)}$$

with $0 < \mu < 1$, $\epsilon > 0$.

The estimate for a requires slight modification:

$$\begin{aligned}
a \leq & \sup_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \sum_{l_2\in\mathbb{Z}} (C_1 2^{-l-l_2}\|B^+\|_1)\Big(\sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\ I_{T^+}\in\mathcal{I}_{-n-n_2,-m-m_2}}} |I_{T^+}|\Big)^{\frac{1}{2}} \cdot (C_2 2^{l_2}\|\tilde{B}^+\|_1)\Big(\sum_{\substack{S^+\in\mathbb{S}^+_{l_2}\\ J_{S^+}\in\mathcal{J}_{n_2,m_2}}} |J_{S^+}|\Big)^{\frac{1}{2}} \\
& \cdot 2^l\|B^+\|_1^{-1}\|\tilde{B}^+\|_1^{-1}C_1^{-1}C_2^{-1} \\
\leq & \sup_{\substack{n_2\in\mathbb{Z}\\ m_2\in\mathbb{Z}}} \underbrace{\Big[\sum_{l_2\in\mathbb{Z}} (C_1 2^{-l-l_2}\|B^+\|_1)^2 \sum_{\substack{T^+\in\mathbb{T}^+_{-l-l_2}\\ I_{T^+}\in\mathcal{I}_{-n-n_2,-m-m_2}}} |I_{T^+}|\Big]^{\frac{1}{2}}}_{a^1} \cdot \underbrace{\Big[\sum_{l_2\in\mathbb{Z}} (C_2 2^{l_2}\|\tilde{B}^+\|_1)^2 \sum_{\substack{S^+\in\mathbb{S}^+_{l_2}\\ J_{S^+}\in\mathcal{J}_{n_2,m_2}}} |J_{S^+}|\Big]^{\frac{1}{2}}}_{a^2} \\
& \cdot 2^l\|B^+\|_1^{-1}\|\tilde{B}^+\|_1^{-1}C_1^{-1}C_2^{-1}
\end{aligned}$$

One notices that Proposition 5.1.5 is applicable with $p = 2$, $S_1 := \{Mf_1 < C_1 2^{-n-n_2}|F_1|\} \cap \{Mf_2 < C_1 2^{-m-m_2}|F_2|\}$ and $S_2 := \{Mf_1 > C_1 2^{-n-n_2-10}|F_1|\} \cap \{Mf_2 > C_1 2^{-m-m_2-10}|F_2|\}$:

$$\begin{aligned}
a^1 \lesssim & \text{size}((\langle f_1, \psi^1_K\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_1}\text{size}((\langle f_2, \psi^2_K\rangle)_{K\cap S_1\neq\emptyset})^{1-\theta_2}|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{1}{2}} \\
\lesssim & (C_1 2^{-n-n_2}|F_1|)^{1-\theta_1}(C_2 2^{-m-m_2}|F_2|)^{1-\theta_2}|F_1|^{\theta_1}|F_2|^{\theta_2}|S_2|^{\theta_3-\frac{1}{2}} \\
\lesssim & C_1^{\frac{3}{2}}(2^{-n-n_2})^{1-\theta_1}(2^{-m-m_2})^{\frac{1}{2}+\theta_1}|F_1||F_2|
\end{aligned}$$

where the last inequality follows by letting $\theta_3 = \frac{1}{2}$, $0 < \theta_1 < 1$. By a similar reasoning, one can derive

$$a^2 \lesssim C_2^{\frac{3}{2}} 2^{n_2(1-\theta_1')} 2^{m_2(\frac{1}{2}+\theta_1')} |G_1||G_2|$$

for any $0 < \theta_1' < 1$. With the choice $\theta_1 = \theta_1'$, one obtains

$$a \lesssim C_1^{\frac{1}{2}} C_2^{\frac{1}{2}} 2^{-n(1-\theta_1)} 2^{-m(\frac{1}{2}+\theta_1)} |F_1||F_2||G_1||G_2| \cdot 2^l \|B^+\|_1^{-1} \|\tilde{B}^+\|_1^{-1}$$

One observes that the estimates for the current model match with the ones for $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ except that $\|B\|_1$ and $\|\tilde{B}\|_1$ are replaced by $\|B^+\|_1$ and $\|\tilde{B}^+\|_1$. One can therefore apply the estimates for the linear form of $\Pi_{\text{flag}^0 \otimes \text{flag}^0}$ with appropriate modifications:

$$|\Lambda| \lesssim C_1 C_2 C_3^2 \|h\|_{L^s} \cdot (|F_1|^{1-\frac{\mu}{2}\epsilon} |F_2|^{1-\frac{1-\mu}{2}\epsilon} |G_1|^{1-\frac{\mu}{2}\epsilon} |G_2|^{1-\frac{1-\mu}{2}\epsilon})^{1-\lambda} \|B^+\|_1^{\lambda} \|\tilde{B}^+\|_1^{\lambda}$$

where

$$\|B^+\|_1 \lesssim |F_1|^{\rho} |F_2|^{1-\rho}$$

$$\|\tilde{B}^+\|_1 \lesssim |G_1|^{\rho'} |G_2|^{1-\rho'}$$

The above estimates for $\|B^+\|_1$, $\|\tilde{B}^+\|_1$ agree with the estimates for $\|B\|_1$, $\|\tilde{B}\|_1$, which allow one to conclude that

$$|\Lambda| \lesssim |F_1|^{\frac{1}{p}} |F_2|^{\frac{1}{q}} |G_1|^{\frac{1}{p}} |G_2|^{\frac{1}{q}}$$

with proper choice of $\mu, \epsilon, \lambda, \rho, \rho'$ which agrees with the choice in expression (6.2.3).

CHAPTER 7

PROOF OF THEOREM 3.3.2 - WALSH CASE

7.1 Estimates for $\Pi_{\mathbf{flag}^{\#_1}\otimes\mathbf{flag}^{\#_2}}$

One defines the exceptional set

$$\Omega := \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 := \bigcup_{n_1\in\mathbb{Z}} \{Mf_1 > C_1 2^{n_1}|F_1|^{\frac{1}{p}}\} \times \{Mg_1 > C_2 2^{-n_1}|G_1|^{\frac{1}{p}}\}$$

$$\Omega_2 := \{SSh > C_3\|h\|_{L^s}\}$$

and

$$\tilde{\Omega} := \{M\chi_\Omega > \frac{1}{100}\}.$$

7.1.1 Hybrid of Stopping-Time Decompositions.

Table 7.1: Stopping-Time on $\mathcal{R}$ for $\Pi_{\text{flag}^{\#_1}\otimes\text{flag}^{\#_2}}$

Tensor-type stopping-time decomposition I on $\mathcal{I}'\times\mathcal{J}'$	$\longrightarrow$	$I\times J \in \mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}$ $(n_2\in\mathbb{Z}, n>0)$
$\Downarrow$		
General wo-dimensional level sets decomposition on $\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}$	$\longrightarrow$	$I\times J \in \mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\cap\mathcal{R}_{k_1,k_2}$ $(n_2\in\mathbb{Z}, n>0, k_1<0, k_2\leq K)$

where

$$\mathcal{I}'_{-n-n_2} := \{I\in\mathcal{I}\setminus\mathcal{I}'_{-n-n_2+1} : \left|I\cap\Omega'^{x}_{-n-n_2}\right| > \frac{1}{10}|I|\}$$

$$\mathcal{J}'_{n_2} := \{J \in \mathcal{J} \setminus \mathcal{J}'_{n_2+1} : \left|I \cap \Omega'^{y}_{n_2}\right| > \frac{1}{10}|J|\}$$

with

$$\Omega'^{x}_{-n-n_2} := \{Mf_1 > C_1 2^{-n-n_2}|F_1|^{\frac{1}{p}}\}$$

$$\Omega'^{y}_{n_2} := \{Mg_1 > C_2 2^{n_2}|G_1|^{\frac{1}{p}}\}$$

The stopping-time decompositions can now be applied to the linear form:

$$\Big|\sum_{\substack{n>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle\Big|$$

$$\lesssim \sum_{\substack{n>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot$$
$$\int_{(\Omega_{k_1})^c\cap(\Omega_{k_2})^c}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy$$
$$\lesssim \sum_{\substack{n>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\frac{|\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot$$
$$\int_{(\Omega_{k_1})^c\cap(\Omega_{k_2})^c}SSh(x,y))SS\chi_{E'}(x,y)dxdy$$

$$\lesssim \sum_{\substack{n>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sup_{I\in\mathcal{I}'_{-n-n_2}}\frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\cdot\sup_{J\in\mathcal{J}'_{n_2}}\frac{|\langle\tilde{B}_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot C_3 2^{k_1}\|h\|_{L^s}2^{k_2}\cdot$$
$$\sum_{\substack{I\times J\in\mathcal{I}'_{-n-n_2}\times\mathcal{J}'_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\Big|\big(\bigcup_{R\in\mathcal{R}_{k_1,k_2}}R\big)\cap\big(\bigcup_{I\in\mathcal{I}_{-n-n_2,-m-m_2}}I\times\bigcup_{J\in\mathcal{J}_{n_2,m_2}}J\big)\Big|\cdot \tag{7.1.1}$$

where for the second inequality one has again used the fact that

$$|I\times J\cap(\Omega_{k_1})^c| \geq \frac{99}{100}|I\times J|$$

$$|I \times J \cap (\Omega_{k_2})^c| \geq \frac{99}{100}|I \times J|$$

with $\Omega_{k_1} := \{SSh > C_3 2^{k_1}\|h\|_s\}$, and $\Omega_{k_2} := \{SS\chi_{E'} > C_3 2^{k_2}\}$.

To estimate $\sup_{I \in \mathcal{I}'_{-n-n_2}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}$, one can now apply Proposition 5.1.5 with $S_1 := \{Mf_1 \leq C_1 2^{-n-n_2}|F_1|^{\frac{1}{p}}\}$:

$$\sup_{I \in \mathcal{I}'_{-n-n_2}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \lesssim \sup_{K \cap S_1 \neq \emptyset} \frac{|\langle f_1,\varphi_K^1\rangle|}{|K|^{\frac{1}{2}}} \sup_{K \cap S_1 \neq \emptyset} \frac{|\langle f_2,\phi_K^2\rangle|}{|K|^{\frac{1}{2}}}$$

where by the definition of S_1,

$$\sup_{K \cap S_1 \neq \emptyset} \frac{|\langle f_1,\varphi_K^1\rangle|}{|K|^{\frac{1}{2}}} \lesssim 2^{-n-n_2}|F_1|^{\frac{1}{p}}$$

and by the fact that $f_2 \in L^\infty$,

$$\sup_{K \cap S_1 \neq \emptyset} \frac{|\langle f_2,\phi_K^2\rangle|}{|K|^{\frac{1}{2}}} \lesssim 1$$

As a result,

$$\sup_{I \in \mathcal{I}'_{-n-n_2}} \frac{|\langle B_I^{\#_1}(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}} \lesssim 2^{-n-n_2}|F_1|^{\frac{1}{p}}$$

By a similar reasoning,

$$\sup_{J \in \mathcal{J}'_{n_2}} \frac{|\langle B_J^{\#_2}(g_1,g_2),\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}} \lesssim 2^{n_2}|G_1|^{\frac{1}{p}}$$

When combining the above estimates into (7.1.1):

$$\begin{aligned}
|\Lambda| \lesssim & C_1C_2C_3^2 \sum_{\substack{n>0\\k_1<0\\k_2\leq K}} \sum_{n_2 \in \mathbb{Z}} 2^{-n-n_2}|F_1|^{\frac{1}{p}} 2^{n_2}|G_1|^{\frac{1}{p}} 2^{k_1}\|h\|_{L^s} 2^{k_2} \cdot \Big|\big(\bigcup_{R \in \mathcal{R}_{k_1,k_2}} R\big) \cap \big(\bigcup_{I \in \mathcal{I}_{-n-n_2,-m-m_2}} I \times \bigcup_{J \in \mathcal{J}_{n_2,m_2}} J\big)\Big| \\
= & C_1C_2C_3^2 \sum_{\substack{n>0\\k_1<0\\k_2\leq K}} 2^{-n}|F_1|^{\frac{1}{p}}|G_1|^{\frac{1}{p}} C_3 2^{k_1}\|h\|_{L^s} 2^{k_2} \cdot \sum_{n_2 \in \mathbb{Z}} \Big|\big(\bigcup_{R \in \mathcal{R}_{k_1,k_2}} R\big) \cap \big(\bigcup_{I \in \mathcal{I}_{-n-n_2,-m-m_2}} I \times \bigcup_{J \in \mathcal{J}_{n_2,m_2}} J\big)\Big| \\
\lesssim & C_1C_2C_3^2 \sum_{\substack{n>0\\k_1<0\\k_2\leq K}} 2^{-n}|F_1|^{\frac{1}{p}}|G_1|^{\frac{1}{p}} C_3 2^{k_1}\|h\|_{L^s} 2^{k_2} \cdot 2^{-\frac{k_1}{2}} 2^{-\frac{k_2\gamma}{2}}
\end{aligned}$$

where the last inequality follows from the sparsity condition. With proper choice of $\gamma > 1$, one obtains the desired estimate.

7.2 Estimates for $\Pi^1_{\mathbf{flag}^0\otimes\mathbf{flag}^0}$ and $\Pi^0_{\mathbf{flag}^0\otimes\mathbf{flag}^0}$

One first defines

$$\Omega := \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 := \bigcup_{l_2\in\mathbb{Z}}\{MB > C_1 2^{-l_2}\|B\|_p\} \times \{M\tilde{B} > C_2 2^{l_2}\|\tilde{B}\|^{\frac{1}{p}}\}$$

$$\Omega_2 := \{SSh > C_3\|h\|_{L^s}\}$$

and

$$\tilde{\Omega} := \{M\chi_\Omega > \frac{1}{100}\}.$$

7.2.1 Hybrid of Stopping-Time Decompositions.

Table 7.2: Stopping-Time on $\mathcal{R}$ for $\Pi_{\text{flag}^0\otimes\text{flag}^0}$

Tensor-type stopping-time decomposition I on $\mathcal{I}\times\mathcal{J}$	$\longrightarrow$	$I\times J \in \mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}$ $(l_2\in\mathbb{Z}, l>0)$
$\Downarrow$		
General wo-dimensional level sets decomposition on $\mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}$	$\longrightarrow$	$I\times J \in \mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}\cap\mathcal{R}_{k_1,k_2}$ $(l_2\in\mathbb{Z}, l>0, k_1<0, k_2\le K)$

where

$$\mathcal{I}''_{-l-l_2} := \{I \in \mathcal{I} \setminus \mathcal{I}''_{-l-l_2+1} : \left|I \cap \Omega''^{x}_{-l-l_2}\right| > \frac{1}{10}|I|\}$$

$$\mathcal{J}''_{l_2} := \{J \in \mathcal{J} \setminus \mathcal{J}''_{l_2+1} : \left|I \cap \Omega''^{y}_{l_2}\right| > \frac{1}{10}|J|\}$$

with

$$\Omega''^{x}_{-l-l_2} := \{MB > C_1 2^{-l-l_2}\|B\|_p\}$$

$$\Omega''^{y}_{l_2} := \{M\tilde{B} > C_2 2^{l_2}\|\tilde{B}\|_p\}$$

With the tensor-type stopping-time decomposition and the general 2-dimensional level sets stopping-time decomposition, the linear form can be rewritten as

$$\Big|\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\sum_{l_2\in\mathbb{Z}}\sum_{\substack{I\times J\in\mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{1}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle\tilde{B}_J(g_1,g_2)\varphi_J^1\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle\Big|$$

$$\lesssim\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\sum_{l_2\in\mathbb{Z}}\sup_{I\in\mathcal{I}''_{-l-l_2}}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\sup_{J\in\mathcal{J}''_{l_2}}\frac{|\langle\tilde{B}_J(g_1,g_2)\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\cdot$$

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\sum_{\substack{I\times J\in\mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)dxdy \tag{7.2.1}$$

where the integrand can be estimated using Cauchy-Schwartz inequality as before:

$$\int_{(\Omega^2_{k_1})^c\cap(\Omega^2_{k_2})^c}\Big(\sum_{\substack{I\times J\in\mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}$$

$$\Big(\sum_{\substack{I\times J\in\mathcal{I}''_{-l-l_2}\times\mathcal{J}''_{l_2}\\ I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|^2}{|I||J|}\chi_I(x)\chi_J(y)\Big)^{\frac{1}{2}}dxdy$$

$$\leq \int_{(\Omega_{k_1}^2)^c \cap (\Omega_{k_2}^2)^c} SSh(x,y)SS\chi_{E'}(x,y)\chi_{\bigcup_{\substack{I\times J \in \mathcal{I}''_{-l-l_2}\times \mathcal{J}''_{l_2} \\ I\times J\in \mathcal{R}_{k_1,k_2}}} I\times J}(x,y)dxdy$$

$$\lesssim C_3^2 2^{k_1}\|h\|_s 2^{k_2}\Bigg|\bigcup_{\substack{I\times J \in \mathcal{I}''_{-l-l_2}\times \mathcal{J}''_{l_2} \\ I\times J\in \mathcal{R}_{k_1,k_2}}} I\times J\Bigg|$$

Now one combines the above estimate into expression (7.2.1) to generate the following bound:

$$|\Lambda| \lesssim C_1C_2C_3^2 \sum_{\substack{l>0 \\ k_1<0 \\ k_2\leq K}} 2^{-l}\|B\|_p\|\tilde{B}\|_p C_3 2^{k_1}\|h\|_{L^s(\mathbb{R}^2)}2^{k_2}\cdot \sum_{l_2\in\mathbb{Z}}\Bigg|\bigcup_{\substack{I\times J \in \mathcal{I}''_{-l-l_2}\times \mathcal{J}''_{l_2} \\ I\times J\in \mathcal{R}_{k_1,k_2}}} I\times J\Bigg|$$

The sparsity condition can be applied to estimate

$$\sum_{l_2\in\mathbb{Z}}\Bigg|\bigcup_{\substack{I\times J \in \mathcal{I}''_{-l-l_2}\times \mathcal{J}''_{l_2} \\ I\times J\in \mathcal{R}_{k_1,k_2}}} I\times J\Bigg| \lesssim \Bigg|\bigcup_{I\times J\in \mathcal{R}_{k_1,k_2}} I\times J\Bigg| \lesssim \min(C_3^{-1}2^{-k_1}, C_3^{-\gamma}2^{-k_2\gamma})$$

for any $\gamma > 1$. As a consequence,

$$|\Lambda| \lesssim C_1C_2C_3^2 \sum_{\substack{l>0 \\ k_1<0 \\ k_2\leq K}} 2^{-l}\|B\|_p\|\tilde{B}\|_p C_3 2^{k_1}\|h\|_{L^s(\mathbb{R}^2)}2^{k_2(1-\gamma)} \lesssim \|B\|_p\|\tilde{B}\|_p$$

with appropriate choice of $\gamma > 1$. The theorem would follow if one obtains

$$\|B\|_p\| \lesssim |F_1|^{\frac{1}{p}}$$

$$\|\tilde{B}\|_p\| \lesssim |G_1|^{\frac{1}{p}}$$

which is exactly the case that will be clarified next.

Estimate of $\|B\|_p$. Without loss of generality, we focuses on the estimate for $\|B\|_p$, which would apply to $\|\tilde{B}\|_p$ as well. One notices that

$$\|B\|_p \leq |\langle B, \chi_S\rangle|$$

for some $\chi_S \in L^{p'}$ with $\|\chi_S\|_{p'} = 1$. One then applies the size-energy estimates for the linear form

$$
\begin{aligned}
|\langle B, \chi_S\rangle| \leq & \sum_K \frac{1}{|K|^{\frac{1}{2}}} |\langle f_1, \phi_K^1\rangle||\langle f_2, \phi_K^2\rangle||\langle \chi_S, \phi_K^3\rangle| \\
& \lesssim \text{size}((\langle f_1, \varphi_K^1\rangle)_{K\in\mathcal{K}})^{1-\theta_1} \text{size}((\langle f_2, \psi_K^2\rangle)_{K\in\mathcal{K}})^{1-\theta_2} \text{size}((\langle \chi_S, \psi_K^3\rangle)_{K\in\mathcal{K}})^{1-\theta_3} \\
& \quad \text{energy}((\langle f_1, \phi_K^1\rangle)_{K\in\mathcal{K}})^{\theta_1} \text{energy}((\langle f_2, \phi_K^2\rangle)_{K\in\mathcal{K}})^{\theta_2} \text{energy}((\langle \chi_S, \phi_K^3\rangle)_{K\in\mathcal{K}})^{\theta_3}
\end{aligned}
\tag{7.2.2}
$$

where one uses the trivial estimates

$$
\text{size}((\langle f_1, \varphi_K^1\rangle)_{K\in\mathcal{K}}), \text{size}((\langle f_2, \psi_K^2\rangle)_{K\in\mathcal{K}}).\text{size}((\langle \chi_S, \psi_K^3\rangle)_{K\in\mathcal{K}}) \leq 1
$$

since $|f_1| \leq \chi_{F_1}$, $|f_2| \leq \chi_{F_2}$.

Moreover,

$$
\text{energy}((\langle f_1, \varphi_K^1\rangle)_{K\in\mathcal{K}}) \lesssim \|f_1\|_1 = |F_1|
$$

and

$$
\text{energy}((\langle \chi_S, \psi_K^3\rangle)_{K\in\mathcal{K}}) \lesssim \|\chi_S\|_1 = |S|
$$

By taking $\theta_2 = 0$, $\theta_3 = \frac{1}{p'}$ and applying the above estimates to (7.2.2), one has

$$
\|B\|_p \leq |\langle B, \chi_S\rangle| \lesssim |F_1|^{\frac{1}{p}} |S|^{\frac{1}{p'}} = |F_1|^{\frac{1}{p}}.
$$

as desired.

7.3 Estimates for $\Pi_{\text{flag}^0 \otimes \text{paraproduct}}$

One first defines

$$
\Omega := \Omega_1 \cup \Omega_2,
$$

where

$$\Omega_1 := \bigcup_{n_2\in\mathbb{Z}} \{MB > C_1 2^{-n_2}\|B\|_p\} \times \{Mg_1 > C_2 2^{n_2}|G_1|^{\frac{1}{p}}\}$$
$$\Omega_2 := \{SSh > C_3\|h\|_{L^s}\}$$

and

$$\tilde{\Omega} := \{M\chi_{\Omega} > \frac{1}{100}\}.$$

7.3.1 Hybrid of Stopping-Time Decompositions.

Table 7.3: Stopping-Time on $\mathcal{R}$ for $\Pi_{\text{flag}^0 \otimes \text{paraproduct}}$

Tensor-type stopping-time decomposition I on $\mathcal{I} \times \mathcal{J}$	$\longrightarrow$	$I \times J \in \mathcal{I}''_{-l-n_2} \times \mathcal{J}'_{n_2}$ ($n_2 \in \mathbb{Z}, l > 0$)
$\Downarrow$		
General wo-dimensional level sets decomposition on $\mathcal{I}''_{-l-n_2} \times \mathcal{J}'_{n_2}$	$\longrightarrow$	$I \times J \in \mathcal{I}''_{-l-n_2} \times \mathcal{J}'_{n_2} \cap \mathcal{R}_{k_1,k_2}$ ($n_2 \in \mathbb{Z}, l > 0, k_1 < 0, k_2 \leq K$)

where

$$\mathcal{I}''_{-l-n_2} := \{I \in \mathcal{I} \setminus \mathcal{I}''_{-l-n_2+1} : \left|I \cap \Omega''^{x}_{-l-n_2}\right| > \frac{1}{10}|I|\}$$
$$\mathcal{J}'_{n_2} := \{J \in \mathcal{J} \setminus \mathcal{J}'_{n_2+1} : \left|I \cap \Omega'^{y}_{n_2}\right| > \frac{1}{10}|J|\}$$

with

$$\Omega''^{x}_{-l-n_2} := \{MB > C_1 2^{-l-n_2}\|B\|_p\}$$
$$\Omega'^{y}_{n_2} := \{Mg_1 > C_2 2^{n_2}|G_1|^{\frac{1}{p}}\}$$

With the tensor-type stopping-time decomposition and the general 2-dimensional level sets stopping-time decomposition, the linear form can be rewritten as

$$\Big|\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{T\in\mathbb{T}_{-l-n_2}}\sum_{\substack{I\times J\in T\times\mathcal{J}_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{1}{|I|^{\frac{1}{2}}|J|}\langle B_I(f_1,f_2),\varphi_I^1\rangle\langle g_1,\varphi_J^1\rangle\langle g_2,\varphi_J^2\rangle\langle h,\psi_I^2\otimes\psi_J^2\rangle\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle\Big|$$

$$\lesssim\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{T\in\mathbb{T}_{-l-n_2}}\sup_{I\in T}\frac{|\langle B_I(f_1,f_2),\varphi_I^1\rangle|}{|I|^{\frac{1}{2}}}\sup_{J\in\mathcal{J}_{n_2}}\frac{|\langle g_1,\varphi_J^1\rangle|}{|J|^{\frac{1}{2}}}\frac{|\langle g_2,\varphi_J^2\rangle|}{|J|^{\frac{1}{2}}}\cdot$$

$$\int\sum_{\substack{I\times J\in T\times\mathcal{J}_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy$$

$$\lesssim C_1C_2\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\sum_{n_2\in\mathbb{Z}}\sum_{T\in\mathbb{T}_{-l-n_2}}2^{-l-n_2}\|B\|_p2^{n_2}|G_1|^{\frac{1}{p}}\cdot 1\cdot 1$$

$$\cdot\int\sum_{\substack{I\times J\in T\times\mathcal{J}_{n_2}\\I\times J\in\mathcal{R}_{k_1,k_2}}}\frac{|\langle h,\psi_I^2\otimes\psi_J^2\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\frac{|\langle\chi_{E'},\psi_I^3\otimes\psi_J^3\rangle|}{|I|^{\frac{1}{2}}|J|^{\frac{1}{2}}}\chi_I(x)\chi_J(y)dxdy, \tag{7.3.1}$$

where the integral can be estimated as

$$C_3^2 2^{k_1}\|h\|_{L^s(\mathbb{R}^2)}2^{k_2}\cdot\Big|\Big(\bigcup_{R\in\mathcal{R}_{k_1,k_2}}R\Big)\cap\Big(I_T\cap\bigcup_{I\in\mathcal{I}_{-n-n_2}}I\times\bigcup_{J\in\mathcal{J}_{n_2}}J\Big)\Big|$$

Now one combines the above estimate into expression (7.3.1):

$$C_1C_2C_3^2\sum_{\substack{l>0\\k_1<0\\k_2\leq K}}\|B\|_p|G_1|^{\frac{1}{p}}C_32^{k_1}\|h\|_{L^s(\mathbb{R}^2)}2^{k_2}\cdot$$

$$\sum_{n_2\in\mathbb{Z}}\sum_{T\in\mathbb{T}_{-l-n_2}}2^{-l-n_2}2^{n_2}\cdot\Big|\Big(\bigcup_{R\in\mathcal{R}_{k_1,k_2}}R\Big)\cap\Big(I_T\cap\bigcup_{I\in\mathcal{I}_{-n-n_2}}I\times\bigcup_{J\in\mathcal{J}_{n_2}}J\Big)\Big|$$

One can again use the sparsity condition as before and estimate

$$\sum_{n_2\in\mathbb{Z}}\sum_{T\in\mathbb{T}_{-l-n_2}}2^{-l-n_2}2^{n_2}\Big|\Big(\bigcup_{R\in\mathcal{R}_{k_1,k_2}}R\Big)\cap\Big(I_T\times\bigcup_{J\in\mathcal{J}_{n_2}}J\Big)\Big|$$

$$\lesssim 2^{-l}\Big|\bigcup_{R\in\mathcal{R}_{k_2}} R\Big|$$

$$\lesssim 2^{-l}C_3^{\gamma}2^{-k_2\gamma}$$

for any $\gamma > 1$.

One can apply the above estimates to derive the following bound for the linear form

$$|\Lambda| \lesssim C_1C_2C_3^2 \sum_{\substack{l>0\\k_1<0\\k_2\leq K}} 2^{-l}2^{k_1}2^{k_2(1-\gamma}\|h\|_{L^s(\mathbb{R}^2)}\|B\|_p|G_1|^{\frac{1}{p}}$$

where one can again apply the estimate $\|B\|_p \lesssim |F_1|^{\frac{1}{p}}$. Thus

$$|\Lambda| \lesssim C_1C_2C_3^2 \sum_{\substack{l>0\\k_1<0\\k_2\leq K}} 2^{-l}2^{k_1(1-\frac{\alpha}{2})}2^{k_2(1-\frac{\alpha\gamma}{2})}\|h\|_{L^s(\mathbb{R}^2)}|F_1|^{\frac{1}{p}}|G_1|^{\frac{1}{p}}.$$

As before, one can separate the case when $0 \leq k_2 \leq K$ and $k_2 < 0$, where in the former one lets $\gamma > 1$ to be sufficiently large and the latter $\gamma > 1$ close to 1. As a consequence,

$$|\Lambda| \lesssim C_1C_2C_3^2|F_1|^{\frac{1}{p}}|G_1|^{\frac{1}{p}}\|h\|_{L^s(\mathbb{R}^2)}$$

as desired.

CHAPTER 8

GENERALIZATION TO FOURIER CASE

The general case can be treated as follows:

$$\Lambda(f_1 \otimes g_1, f_2 \otimes g_2, h, \chi_{E'}) := \sum_{\tau_1,\tau_2 \in \mathbb{N}} 2^{-100(\tau_1+\tau_2)} \sum_{I \times J \in \mathcal{I} \times \mathcal{J}} \frac{1}{|I|^{\frac{1}{2}}|J|} \langle B_I(f_1, f_2), \phi_I^1 \rangle \langle g_1, \phi_J^1 \rangle \langle g_2, \phi_J^2 \rangle \cdot \langle h, \phi_I^2 \otimes \phi_J^2 \rangle \langle \chi_{E'}, \phi_I^{3,\tau_1} \otimes \phi_J^{3,\tau_2} \rangle$$

Now one modifies the previous argument by re-defining the exceptional set and replacing C_1, C_2 by $C_1 2^{\tau_1}$, $C_2 2^{\tau_2}$ respectively. One then focuses on the inner sum and apply the argument before. One can see from the previous argument that C_1 and C_2 only appear in polynomial powers, which would translate to $O(2^{20\tau_1})$ and $O(2^{20\tau_2})$. This would not be an issue because of the fast decay factor $2^{-100(\tau_1+\tau_2)}$. The only nontrivial problem in this general case is that one can no longer easily localize B as before. In particular, one would like to obtain same estimates for the size and energy terms in Proposition 5.1.4 and Proposition 5.1.5.

Although the localization cannot be implemented as before, there are still some information one can extract if I intersects some set nontrivially and if $|K| > |I|$. In particular, one considers

$$\frac{|\langle B_I, \varphi_I^1 \rangle|}{|I|^{\frac{1}{2}}} = \frac{1}{|I|} \left| \sum_{K:|K| \geq |I|} \frac{1}{|K|^{\frac{1}{2}}} \langle f_1, \phi_K^1 \rangle \langle f_2, \phi_K^2 \rangle \langle \tilde{\chi}_I^1, \phi_K^3 \rangle \right|$$

where $\tilde{\chi}_I^1$ denotes the L^∞ smooth bump function adapted to I. Using the lemma about the decomposition of smooth bump functions, one can rewrite the above expression as

$$\frac{1}{|I|}\Big|\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4} \sum_{K:|K|>|I|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\langle \tilde{\chi}_I^{1,\tau_3},\phi_K^{3,\tau_4}\rangle\Big|$$

where $\tilde{\chi}_I^{1,\tau_3}$ is an L^∞-normalized bump function adapted to I with the additional property that $\mathrm{supp}(\tilde{\chi}_I^{1,\tau_3}) \subseteq 2^{\tau_3}I$, and ϕ_K^{3,τ_4} is an L^2-normalized bump function with $\mathrm{supp}(\phi_K^{3,\tau_3}) \subseteq 2^{\tau_3}K$. If $\int \phi_K^3 = 0$, then the functions ϕ_K^{3,τ_3} can be chosen such that $\int \phi_K^{3,\tau_3} = 0$. With the property of being compactly supported, one has that if

$$\langle \tilde{\chi}_I^{1,\tau_3},\phi_K^{3,\tau_4}\rangle \neq 0,$$

then

$$2^{\tau_3}I \cap 2^{\tau_4}K \neq \emptyset.$$

One also recalls that $I \cap S_1 \neq \emptyset$ and $|I| \leq |K|$, it follows that

$$\frac{\mathrm{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}.$$

8.1 Size

One recalls the definition of

$$\mathrm{size}_{\mathcal{I}_{S_1}}((\langle B_I^{\#_1},\varphi_I\rangle)_{I\in\mathcal{I}_{S_1}} = \frac{|\langle B_{I_0}^{\#_1}(f_1,f_2),\varphi_{I_0}^1\rangle|}{|I_0|^{\frac{1}{2}}}$$

for some $I_0 \in \mathcal{I}_{S_1}$ where $I \cap S_1 \neq \emptyset$ for any $I \in \mathcal{I}_{S_1}$. Then

$$\begin{aligned}&\frac{|\langle B_{I_0}^{\#_1}(f_1,f_2),\varphi_{I_0}^1\rangle|}{|I_0|^{\frac{1}{2}}}\\ =&\frac{1}{|I_0|}\Big|\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4} \sum_{K:|K|\sim 2^{\#_1}|I_0|} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\langle \tilde{\chi}_{I_0}^{1,\tau_3},\phi_K^{3,\tau_4}\rangle\Big|\end{aligned}$$

where $\tilde{\chi}_{I_0}^{1,\tau_3}$ and ϕ_K^{3,τ_4} are defined similarly as before. Since $|K| \sim 2^{\#_1}|I_0|$ implies that $|K| > |I_0|$, one can apply the geometric interpretation and obtain

$$\sum_{K:|K|\sim 2^{\#_1}|I_0|} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1, \phi_K^1\rangle||\langle f_2, \phi_K^2\rangle||\langle \tilde{\chi}_{I_0}^{1,\tau_3}, \phi_K^{3,\tau_4}\rangle|$$

$$= \sum_{K:|K|\sim 2^{\#_1}|I_0|} \frac{|\langle f_1, \phi_K^1\rangle|}{|K|^{\frac{1}{2}}}\frac{|\langle f_2, \phi_K^2\rangle|}{|K|^{\frac{1}{2}}}|\langle \tilde{\chi}_{I_0}, |K|^{\frac{1}{2}}\phi_K^3\rangle|$$

$$\leq \sup_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_1, \phi_K^1\rangle|}{|K|^{\frac{1}{2}}} \sup_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_2, \phi_K^2\rangle|}{|K|^{\frac{1}{2}}}$$

$$\cdot \sum_{K:|K|\sim 2^{\#_1}|I_0|} |\langle \tilde{\chi}_{I_0}, |K|^{\frac{1}{2}}\phi_K^3\rangle|$$

One notices that

$$\sup_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_1, \phi_K^1\rangle|}{|K|^{\frac{1}{2}}} \lesssim 2^{\tau_3+\tau_4} \sup_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_1, \phi^1_{2^{\tau_3+\tau_4}K}\rangle|}{|2^{\tau_3+\tau_4}K|^{\frac{1}{2}}} \leq 2^{\tau_3+\tau_4} \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \phi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}}$$

Similarly,

$$\sup_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_2, \phi_K^2\rangle|}{|K|^{\frac{1}{2}}} \lesssim 2^{\tau_3+\tau_4} \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \phi_{K'}^2\rangle|}{|K'|^{\frac{1}{2}}}$$

As a result,

$$\frac{|\langle B_{I_0}^{\#_1}(f_1, f_2), \varphi_{I_0}^1\rangle|}{|I_0|^{\frac{1}{2}}}$$

$$\lesssim \frac{1}{|I_0|}\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4}2^{2(\tau_3+\tau_4)} \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \phi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}} \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \phi_{K'}^2\rangle|}{|K'|^{\frac{1}{2}}}|I_0|$$

$$\lesssim \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \phi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}} \sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \phi_{K'}^2\rangle|}{|K'|^{\frac{1}{2}}}$$

which is exactly the same estimate for the corresponding term in Proposition 5.1.4.

8.2 Energy

8.2.1 Case I: ϕ^3_{K,τ_3} is lacunary.

One recalls that $I_T \in T$ with $T \in \mathbb{T}_{l_1}$ are the maximal intervals which satisfy

$$\begin{aligned} 2^{l_1}\|B\|_1 <& \frac{|\langle B_{I_T}, \tilde{\chi}^1_{I_T}\rangle|}{|I_T|} \\ =& \frac{1}{|I_T|}\Big|\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4} \sum_{\substack{K:|K|\geq|I_T| \\ \frac{\operatorname{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}}} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi^1_K\rangle\langle f_2,\phi^2_K\rangle\langle\tilde{\chi}^{1,\tau_3}_{I_T},\phi^{3,\tau_4}_K\rangle\Big| \end{aligned}$$

With the biest trick, the above expression can be rewritten as

$$2^{l_1}\|B\|_1 < \frac{1}{|I_T|}\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4}\Big|\sum_{K:\frac{\operatorname{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi^1_K\rangle\langle f_2,\phi^2_K\rangle\langle\tilde{\chi}^{1,\tau_3}_{I_T},\psi^{3,\tau_4}_K\rangle\Big|$$

Let us denote $B^{\tau_3,\tau_4}_{S_1} := \sum_{K:\frac{\operatorname{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi^1_K\rangle\langle f_2,\phi^2_K\rangle\psi^{3,\tau_4}_K(x)$. Then

$$\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4}\frac{|\langle B^{\tau_3,\tau_4}_{S_1},\tilde{\chi}^{1,\tau_3}_{I_T}\rangle|}{|I_T|} \gtrsim 2^{l_1}\|B\|_1$$

The expression on the left-hand side can be majorized by

$$\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4}M(B^{\tau_3,\tau_4}_{S_1})(x)$$

for any $x \in I_T$. By the pigeonhole principle, there must exist a pair $(\tau_3,\tau_4) \in \mathbb{N}^2$ such that

$$2^{100\tau_3}2^{-100\tau_4}M(B^{\tau_3,\tau_4}_{S_1})(x) \gtrsim 2^{l_1}\|B\|_1$$

With some careful treatment of implicit constants in the inequalities, one has that for any $x \in I_T$,

$$M(B^{\tau_3,\tau_4}_{S_1})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B\|_1$$

for some pair $(\tau_3, \tau_4) \in \mathbb{N}^2$. Equivalently,

$$I_T \subseteq \bigcup_{\tau_3,\tau_4\in\mathbb{N}} \{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B\|_1\}$$

By the disjointness of $(I_T)_{T\in\mathbb{T}_{l_1}}$,

$$\sum_{T\in\mathbb{T}_{l_1}} |I_T| \leq \sum_{\tau_3,\tau_4\in\mathbb{N}} \left|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B\|_1\}\right|.$$

8.2.2 Case II: ϕ^3_{K,τ_3} is non-lacunary.

In this case, the stopping-time decomposition is performed in a slightly different way, with $\|B\|_1$ replaced by $\|B^+\|_1$. More precisely, $I_T \in T$ with $T \in \mathbb{T}_{l_1}$ if and only if I_T is a maximal interval such that

$$2^{l_1}\|B^+\|_1 < \frac{|\langle B_{I_T}, \tilde{\chi}^1_{I_T}\rangle|}{|I_T|}$$

where the right hand side of the inequality can be estimated by

$$\frac{1}{|I_T|}\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{100\tau_3}2^{-100\tau_4} \sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1, \psi^1_K\rangle||\langle f_2, \psi^2_K\rangle|\langle|\tilde{\chi}^{1,\tau_3}_{I_T}|, |\varphi^{3,\tau_4}_K|\rangle$$

Let us denote $B_{S_1}^{\tau_3,\tau_4} := \displaystyle\sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1, \psi^1_K\rangle||\langle f_2, \psi^2_K\rangle||\varphi^{3,\tau_4}_K(x)|$. Then by the same reasoning applied in Case I, one has

$$\sum_{T\in\mathbb{T}_{l_1}} |I_T| \leq \sum_{\tau_3,\tau_4\in\mathbb{N}} \left|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^+\|_1\}\right|.$$

8.2.3 $L^{1,\infty}$-energy.

With a little abuse of notations, we will summarize both Case I and II as follows:

$$B_{S_1}^{\tau_3,\tau_4} := \begin{cases} \displaystyle\sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}\langle f_1,\phi_K^1\rangle\langle f_2,\phi_K^2\rangle\psi_K^{3,\tau_4}(x) & \text{in Case } I \\ \displaystyle\sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}} \frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\psi_K^1\rangle||\langle f_2,\psi_K^2\rangle||\varphi_K^{3,\tau_4}(x)| & \text{in Case } II \end{cases}$$

and

$$B^{(+)} := \begin{cases} B & \text{in Case } I \\ B^{+} & \text{in Case } II \end{cases}$$

One first recalls the definition of $L^{1,\infty}$-energy that for some $l_1 \in \mathbb{Z}$,

$$\begin{aligned} \text{energy}_{\mathcal{I}_{S_1}}^{1,\infty}((\langle B_I,\varphi_I^1\rangle))_I =& 2^{l_1}\sum_{T\in\mathbb{T}_{l_1}}|I_T| \\ \leq& \sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{l_1}\|B^{(+)}\|_1\left|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^{(+)}\|\|_1\}\right|\|B^{(+)}\|_1^{-1} \end{aligned}$$

The last inequality follows from the discussion in the previous subsections. One can then focus on the summand and rewrite it as

$$\underbrace{2^{l_1}2^{80\tau_3}2^{80\tau_4}\|B^{(+)}\|_1\left|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^{(+)}\|_1\}\right|^{\alpha}}_{(*)}\cdot 2^{-80\tau_3}2^{-80\tau_4},$$

where

$$(*) \leq \|M(B_{S_1}^{\tau_3,\tau_4})\|_{L^{1,\infty}} \lesssim \|B_{S_1}^{\tau_3,\tau_4}\|_1$$

The estimate for $\|B_{S_1}^{\tau_3,\tau_4}\|_1$ will be specified in the later subsection.

8.2.4 L^p-energy for $p > 1$.

$$\text{energy}_{\mathcal{I}_{S_1}}^{p}((\langle B_I,\varphi_I^1\rangle))_I$$

$$
\begin{aligned}
=&\Big(\sum_{l_1\in\mathbb{Z}} 2^{l_1 p}\sum_{T\in\mathbb{T}_{l_1}}|I_T|\Big)^{\frac{1}{p}}\\
\leq&\Big(\sum_{l_1\in\mathbb{Z}}\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{l_1 p}\|B^{(+)}\|_1^p\big|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^{(+)}\|\|_1\}\big|\Big)^{\frac{1}{p}}\\
&\qquad\cdot\|B^{(+)}\|_1^{-1}\\
=&\Big(\sum_{\tau_3,\tau_4\in\mathbb{N}} 2^{-80\tau_3 p}2^{-80\tau_4 p}\underbrace{\sum_{l_1\in\mathbb{Z}}(2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^{(+)}\|_1)^p\big|\{M(B_{S_1}^{\tau_3,\tau_4})(x) > 2^{80\tau_3}2^{80\tau_4}2^{l_1}\|B^{(+)}\|\|_1\}\big|}_{(*)}\Big)^{\frac{1}{p}}\\
&\qquad\cdot\|B^{(+)}\|_1^{-1}
\end{aligned}
\tag{8.2.1}
$$

where

$$(*)\leq\|M(B_{S_1}^{\tau_3,\tau_4})\|_p^p\lesssim\|B_{S_1}^{\tau_3,\tau_4}\|_p^p$$

8.2.5 Estimate for $\|B_{S_1}^{\tau_3,\tau_4}\|_p$ for $1\leq p<\infty$.

One again uses the duality of the operator norm. For some $\chi_S\in L^{\frac{1}{p'}}$, one has

$$
\begin{aligned}
&|\langle B_{S_1}^{\tau_3,\tau_4},\chi_S\rangle|\\
\lesssim&\begin{cases}
\displaystyle\sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}}\frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\varphi_K^1\rangle||\langle f_2,\psi_K^2\rangle||\langle\chi_S,\psi_K^{3,\tau_4}\rangle| & \text{in Case } I\\
\displaystyle\sum_{K:\frac{\text{dist}(K,S_1)}{|K|}\lesssim 2^{\tau_3+\tau_4}}\frac{1}{|K|^{\frac{1}{2}}}|\langle f_1,\psi_K^1\rangle||\langle f_2,\psi_K^2\rangle|\langle\chi_S,|\varphi_K^{3,\tau_4}|\rangle & \text{in Case } II
\end{cases}
\end{aligned}
$$

where both expression can be estimated using size-energy estimates stated in Lemma 5.3.2. In particular,

$$|\langle B_{S_1}^{\tau_3,\tau_4},\chi_S\rangle|$$

$$\lesssim \left(\text{size}((\langle f_1, \varphi_K^1\rangle)_{K: \frac{\text{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}}\right)^{1-\theta_1} \left(\text{size}((\langle f_2, \psi_K^2\rangle)_{K: \frac{\text{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}}\right)^{1-\theta_2}$$
$$\left(\text{size}((\langle \chi_S, \psi_K^3\rangle)_K)\right)^{1-\theta_3} \left(\text{energy}((\langle f_1, \varphi_K^1\rangle)_K)\right)^{\theta_1} \left(\text{energy}((\langle f_2, \psi_K^2\rangle)_K)\right)^{\theta_2} \left(\text{energy}((\langle \chi_S, \psi_K^3\rangle)_K)\right)^{\theta_3}$$

The condition on the collection of intervals K generates the following estimates:

$$\text{size}((\langle f_1, \varphi_K^1\rangle)_{K: \frac{\text{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}}) \lesssim 2^{\tau_3+\tau_4} \sup_{K: \frac{\text{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}} \frac{|\langle f_1, \varphi^1_{2^{\tau_3+\tau_4}K}\rangle|}{|2^{\tau_3+\tau_4}K|^{\frac{1}{2}}} \leq 2^{\tau_3+\tau_4} \sup_{K'\cap S_1 \neq \emptyset} \frac{|\langle f_1, \varphi^1_{K'}\rangle|}{|K'|^{\frac{1}{2}}}$$

Similar reasoning gives

$$\text{size}((\langle f_2, \psi_K^2\rangle)_{K: \frac{\text{dist}(K,S_1)}{|K|} \lesssim 2^{\tau_3+\tau_4}}) \lesssim 2^{\tau_3+\tau_4} \sup_{K'\cap S_1 \neq \emptyset} \frac{|\langle f_2, \tilde{\chi}_{K'}\rangle|}{|K'|}$$

where $\tilde{\chi}_{K'}$ denotes an L^∞-normalized bump function adapted to K'.

Also, using the fact that X_S is a characteristic function, one has

$$\text{size}((\langle \chi_S, \psi_K^3\rangle)_K) \leq 1$$

Moreover,

$$\text{energy}((\langle f_1, \varphi_K^1\rangle)_K) \lesssim |F_1|$$
$$\text{energy}((\langle f_2, \psi_K^2\rangle)_K) \lesssim |F_2|$$
$$\text{energy}((\langle \chi_S, \psi_K^3\rangle)_K) \lesssim |S|$$

Combining all the size-energy estimates,

$$|\langle B^{\tau_3,\tau_4}_{-n-n_2,-m-m_2}, \chi_S\rangle|$$
$$\lesssim 2^{(2-\theta_1-\theta_2)(\tau_3+\tau_4)}\Big(\sup_{K'\cap S_1 \neq \emptyset} \frac{|\langle f_1, \varphi^1_{K'}\rangle|}{|K'|^{\frac{1}{2}}}\Big)^{1-\theta_1}\Big(\sup_{K'\cap S_1 \neq \emptyset} \frac{|\langle f_2, \tilde{\chi}_{K'}\rangle|}{|K'|}\Big)^{1-\theta_2} \cdot 1^{1-\theta_3} \cdot |F_1|^{\theta_1}|F_2|^{\theta_2}|S|^{\theta_3}$$

where $0 \leq \theta_1, \theta_2, \theta_3 < 1$ and $\theta_1 + \theta_2 + \theta_3 = 1$. By choosing $\theta_3 = \frac{1}{p'}$, which implies that $\theta_1 + \theta_2 = \frac{1}{p}$, we can conclude that

$$\|B_{S_1}^{\tau_3,\tau_4}\|_p \lesssim 2^{(\tau_3+\tau_4)(1+\frac{1}{p'})}\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \varphi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}}\Big)^{1-\theta_1}\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \tilde{\chi}_{K'}\rangle|}{|K'|}\Big)^{\theta_1+\frac{1}{p'}} \cdot |F_1|^{\theta_1}|F_2|^{\frac{1}{p}-\theta_1}.$$

By applying the above estimate to (8.2.1), one has

$$\left[\sum_{\tau_3,\tau_4} 2^{-80\tau_3 p}2^{-80\tau_4 p}\left(2^{(\tau_3+\tau_4)(1+\frac{1}{p'})}\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \varphi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}}\Big)^{1-\theta_1}\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \tilde{\chi}_{K'}\rangle|}{|K'|}\Big)^{\theta_1+\frac{1}{p'}} \cdot |F_1|^{\theta_1}|F_2|^{\frac{1}{p}-\theta_1}\right)^p\right]^{\frac{1}{p}}$$

$$\|B^{(+)}\|_1^{-1}$$

$$\lesssim\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_1, \varphi_{K'}^1\rangle|}{|K'|^{\frac{1}{2}}}\Big)^{1-\theta_1}\Big(\sup_{K'\cap S_1\neq\emptyset} \frac{|\langle f_2, \tilde{\chi}_{K'}\rangle|}{|K'|}\Big)^{\theta_1+\frac{1}{p'}} \cdot |F_1|^{\theta_1}|F_2|^{\frac{1}{p}-\theta_1}\|B^{(+)}\|_1^{-1}$$

which is exactly the same estimate for the corresponding terms in Proposition 5.1.5.

□

BIBLIOGRAPHY

[1] Kato, T. and Ponce, G. *Commutator estimates and the Euler and Navier-Stokes equations*,Comm. Pure App. Math. ,41, 891-907, [1988]. 2

[2] Coifman, R. R. and Meyer, Y. *Operateurs multilineaires*, Hermann, Paris, [1991]. 2

[3] Kenig, C. *On the local and global well-posedness theory for the KP-I equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21, 827-838 [2004] 3

[4] Germain, P., Masmoudi, N. and Shatah, J. *Global Solutions for the Gravity Water Waves Equation in Dimension 3*, [2009] 2

[5] Miyachi, A. and Tomita, N. *Estimates for trilinear flag paraproducts on L^∞ and Hardy spaces*, Math. Z. 282, 577 - 613, [2016] 2

[6] Muscalu, C. and Schlag, W. *Classical and Multilinear Harmonic Analysis*, [2013]. 10, 19, 24, 26, 30

[7] Muscalu, C. *Paraproducts with flag singularities I. A case study*, Rev. Mat. Iberoamericana, vol. 23 705-742, [2007]. 2

[8] Muscalu, C., Pipher, J., Tao, T. and Thiele, C. *Bi-parameter paraproducts*, Acta Math., vol. 193, 269-296, [2004]. 3