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AN IMPROVED ALGORITHM FOR FINDING OPTIMAL LOT SIZING POLICIES FOR FINITE PRODUCTION RATE ASSEMBLY SYSTEMS

by

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An Improved Algorithm for Finding Optimal Lot Sizing Policies for Finite Production Rate Assembly Systems

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Abstract

We show that an $O(n^3 \log n)$ algorithm can find optimal Power–of–Two Lot Size Policies for Finite Production Rate Assembly Systems. This improves an $O(n^5)$ algorithm proposed in Atkins, Queyranne and Sun's paper [1] (1992).

In their paper "Lot Sizing Policies for Finite Production Rate Assembly Systems" [1] (1992), Atkins, Queyranne and Sun provided an $O(n^5)$ algorithm to find optimal Power–of–Two Lot Size Policies for Finite Production Rate Assembly Systems. In this article we show that an $O(n^3 \log n)$ algorithm can solve the same problem. The organization of the paper is as follows. First, we rewrite the original relaxation problem (RP) in Atkins, Queyranne and Sun [1] (1992) to an equivalent problem (RP_1). Then, we present a mapping from this model to the model presented in Roundy [3] (1990). By using this mapping, we show an algorithm solving the original problem in $O(n^3 \log n)$. Finally, we give an example to illustrate the mapping procedure.

Refer to [1] (1992) for the notation, motivation, etc. We introduce the following equivalent formulation to the original relaxation problem of Atkins, Queyranne and Sun.

Lemma 1 (Equivalent Formulation)
Problem (RP)*:*

$$C^* = \min_Q f(Q) \triangleq \min_Q \sum_{i\in N}\left[\frac{K_i}{Q_i/\pi_0} + \sum_{j\in\langle i,1\rangle} H_{ij}\max_{\ell\in\langle i,j\rangle} Q_\ell\right]$$
$$\text{s.t.}\quad Q_i \geq 0 \qquad \forall\, i\in N$$

is equivalent to problem (RP_1)*:*

$$C_1^* = \min_q f_1(q) \triangleq \min_q \sum_{i\in N}\left[\frac{K_i}{q_{ii}/\pi_0} + \sum_{j\in\langle i,1\rangle} H_{ij}q_{ij}\right]$$
$$\text{s.t.}\quad q_{ij} \geq 0 \qquad \forall\, \langle i,j\rangle \in R, \tag{1a}$$
$$q_{ij} \leq q_{i,s(j)} \qquad \forall\, \langle i,s(j)\rangle \in R, \tag{1b}$$
$$q_{ij} \geq q_{s(i),j} \qquad \forall\, \langle s(i),j\rangle \in R, \tag{1c}$$

where R *is the set of all paths in* $G(N,A)$*.*

Proof. Suppose that $Q = (Q_1,\cdots,Q_n)$ is a feasible solution to (RP). Let

$$q_{ij} \triangleq \max_{\ell\in\langle i,j\rangle} Q_\ell, \qquad \forall\, \langle i,j\rangle \in R. \tag{2}$$

Then inequalities (1a), (1b) and (1c) hold, that is., $q = (q_{ij}|\langle i,j\rangle \in R)$ is also a feasible solution to (RP_1). Note also that $q_{ii} = Q_i, \forall\, i \in N$. Therefore, $f_1(q) = f(Q)$, and $C_1^* \leq C^*$.

Now suppose that q is a feasible solution to (RP_1). Let $Q_i = q_{ii}, \forall\, i\in N$. If $\ell \in \langle i,j\rangle \in R$, then by (1c) $q_{ij} \geq q_{s(i),j} \geq \cdots \geq q_{\ell j}$, and by (1b) $q_{\ell\ell} \leq q_{\ell,s(\ell)} \leq \cdots \leq q_{\ell j}$. Therefore, $Q_\ell = q_{\ell\ell} \leq q_{ij}$, $\forall\, \ell\in\langle i,j\rangle$, and $\max_{\ell\in\langle i,j\rangle} Q_\ell \leq q_{ij}$. Hence, $f(Q) \leq f_1(q)$, and $C^* \leq C_1^*$.

This implies $C^* = C_1^*$, i.e., problem (RP) is equivalent to problem (RP_1). □

The mapping is best described by defining three networks and by providing network-based reformulations of problem (RP_1). The three networks are defined as follows.

1. Network $G(N_1, A_1)$ corresponds directly to problem (RP_1).

$$
\begin{aligned}
N_1 &\triangleq \{\langle i,j\rangle \in R\}, \\
A_1 &\triangleq \{(\langle i,s(j)\rangle,\langle i,j\rangle)\,|\,\langle i,s(j)\rangle \in R\} \cup \{(\langle i,j\rangle,\langle s(i),j\rangle)\,|\,\langle s(i),j\rangle \in R\}
\end{aligned}
$$

For each node $\langle i,j\rangle$ in N_1, the setup cost and the holding cost are

$$
\begin{aligned}
K_{\langle i,j\rangle} &= \begin{cases} K_i\pi_0, & \text{if } i=j, \\ 0, & \text{if } i\neq j \end{cases} \\
H_{\langle i,j\rangle} &= H_{ij}.
\end{aligned}
$$

Note that problem (RP_1) can now be re-stated as

$$
\text{Problem } (RP_1^*): \quad \begin{cases} \min\limits_{q} & \sum\limits_{\langle i,j\rangle\in N_1} \left(\dfrac{K_{\langle i,j\rangle}}{q_{ij}} + H_{\langle i,j\rangle} q_{ij}\right) \\ \text{s.t.} & q_{ij} \geq q_{i'j'} \geq 0, \qquad\qquad \forall\, (\langle i,j\rangle,\langle i',j'\rangle) \in A_1. \end{cases}
$$

2. Let $D \triangleq \max\limits_{i_\ell\in L} |\langle i_\ell,1\rangle|$ be the length (the number of nodes in a route) of a longest leaf-route in $G(N,A)$, where L is the set of all leaves. Let $G(N',A')$ be the series system with

$$
\begin{aligned}
N' &\triangleq \{0,1,2,\cdots,D-1\} \\
A' &\triangleq \{(i,i-1)\,|\,i=1,2,\cdots,D-1\}.
\end{aligned}
$$

3. Let $G(N_2,A_2)$ be a graph defined by

$$
\begin{aligned}
N_2 &\triangleq \{\langle i,k\rangle\,|\,i\in N, k=0,1,2,\cdots,D-1\} \\
A_2 &\triangleq \{(\langle i,k\rangle,\langle s(i),k\rangle)\,|\,i\in N\backslash\{1\}, k=0,1,2,\cdots,D-1\} \\
&\quad \cup \{(\langle i,k\rangle,\langle i,k-1\rangle)\,|\,i\in N, k=1,2,\cdots,D-1\}
\end{aligned}
$$

The network $G(N_2,A_2)$ can be viewed as the cross product of $G(N',A')$ and of $G(N,A)$ (see Figures 1, 2 and 3). $G(N_2,A_2)$ has the structure that Roundy [3] requires. We embed $G(N_1,A_1)$ into $G(N_2,A_2)$ as follows.

$$
\text{node } \langle i,j\rangle \in N_1 \longmapsto \text{node } \langle i,k\rangle \in N_2, \qquad \text{with } k = D - |\langle j,1\rangle|.
$$

Costs for $G(N_2,A_2)$ are defined as follows.

$$
\begin{aligned}
K'_{\langle i,k\rangle} &= \begin{cases} K_{\langle i,j\rangle}, & \text{if } \langle i,j\rangle \in N_1 \longmapsto \langle i,k\rangle \in N_2, \qquad \text{with } k = D-|\langle j,1\rangle|, \\ 0, & \text{otherwise} \end{cases} \\
H'_{\langle i,k\rangle} &= \begin{cases} H_{\langle i,j\rangle}, & \text{if } \langle i,j\rangle \in N_1 \longmapsto \langle i,k\rangle \in N_2, \qquad \text{with } k = D-|\langle j,1\rangle|, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

We now define Problem (RP_2) as

$$\text{Problem } (RP_2): \quad \begin{cases} \min\limits_{q'} & \sum\limits_{\langle i,j\rangle \in N_2} \left(\dfrac{K'_{\langle i,j\rangle}}{q'_{ij}} + H'_{\langle i,j\rangle} q'_{ij} \right) \\ \text{s.t.} & q'_{ij} \geq q'_{i'j'} \geq 0, \qquad \forall\, (\langle i,j\rangle, \langle i',j'\rangle) \in A_2. \end{cases} \qquad 24;1H$$

The costs for $G(N_2, A_2)$ are obviously selected to make problems (RP_1^*) and (RP_2) equivalent. Let $S = \{\langle i,k\rangle \in N_2 : k \leq D - 1 - |\langle i,1\rangle|\}$. Note that nodes in S have no corresponding nodes in N_1. The setup costs and holding costs corresponding to these nodes are zero, and no arc in A_2 goes from a node in S to a node in $N_2 \backslash S$. Using these facts the equivalence between problems (RP_1^*) and (RP_2) is easily verified.

The algorithm for solving (RP) can be summarized as follows.

1. Construct $G(N_2, A_2)$ as described above

2. Use the algorithm suggested by Roundy [3] to solve Problem (RP_2) over network $G(N_2, A_2)$ in the time of $O(|N_2| D \log |N_2|)$. Note that $|N_2| \leq n^2$ and $D \leq n$, so $O(|N_2| D \log |N_2|) \leq O(n^3 \log n)$. Let the solution to Problem (RP_2) be $q'_{\langle i,k\rangle}$ for every node $\langle i,k\rangle \in N_2$.

3. In order to get the solution to the relaxation problem (RP) over the network $G(N_1, A_1)$, we use the inverse mapping from $G(N_2, A_2)$ to $G(N_1, A_1)$:

   $$\langle i,k\rangle \in N_2 \quad \longmapsto \quad \begin{cases} \emptyset, & \text{if} \quad |\langle i,1\rangle| \leq D - 1 - k \\ \langle i,j\rangle \in N_1, & \text{if} \quad \langle j,1\rangle \subseteq \langle i,1\rangle \in G(N,A) \text{ such that } |\langle j,1\rangle| = D - k \end{cases}$$

   The solution to problem (RP_1) over the network $G(N_1, A_1)$ is:

   $$q_{ij} = q_{\langle i,j\rangle} = q'_{\langle i,k\rangle}, \qquad \text{if } \langle i,k\rangle \in N_2 \longmapsto \langle i,j\rangle \in N_1.$$

   When carefully implemented, the run time for this step is $O(|N_2|) \leq O(n^2)$.

4. Let $Q_i \stackrel{\triangle}{=} q_{ii}$, $\forall\, i \in N$. We have the solution to the original problem (RP).

5. Using the optimal rounding method in Roundy [2] (1983) to derive, in $O(n \log n)$ time, an optimal power–of–two lot size policy for the finite production rate assembly systems with effectiveness at least 98%.

It is easy to see that the total time to solve the problem is bounded by the time to solve the relaxation problem over network $G(N_2, A_2)$ in step 2, which is $O(n^3 \log n)$.

The following example illustrates the mapping process.

Example. The following example of an assembly system $G(N, A)$ in Figure 1 with 7 facilities illustrates the embedding procedure. The length of the longest leaf–route, which corresponds to the series network $G(N', A')$ in Figure 2, is four, i.e., $D = 4$. Graph $G(N_2, A_2)$ in Figure 3 is the network corresponding to problem (RP_2). It is the Cartesian product of graph $G(N, A)$ and graph $G(N', A')$. The graph $G(N_1, A_1)$ in Figure 4 is imbedded in graph $G(N_2, A_2)$.

It is easy to verify the mapping from $G(N_1, A_1)$ to $G(N_2, A_2)$. The following examples illustrate the inverse mapping from $G(N_2, A_2)$ to $G(N_1, A_1)$:

If $\langle i, k\rangle = \langle 7, 3\rangle \in N_2$, then $i = 7$, $k = 3$ and $D - k - 1 = 4 - 3 - 1 = 0 < 4 = |\langle 7, 1\rangle| = |\langle i, 1\rangle|$, so there is a corresponding node in $G(N_2, A_2)$. Therefore, $|\langle j, 1\rangle| = D - k = 4 - 3 = 1$ and $j = 1$, i.e., $\langle 7, 3\rangle \in N_2 \longmapsto \langle 7, 1\rangle \in N_1$.

If $\langle i, k\rangle = \langle 7, 0\rangle \in N_2$, then $i = 7$, $k = 0$ and $D - k - 1 = 4 - 0 - 1 = 3 < 4 = |\langle 7, 1\rangle| = |\langle i, 1\rangle|$. Therefore, $|\langle j, 1\rangle| = D - k = 4 - 0 = 4$ and $j = 7$, i.e., $\langle 7, 0\rangle \in N_2 \longmapsto \langle 7, 7\rangle \in N_1$.

If $\langle i, k\rangle = \langle 6, 3\rangle \in N_2$, then $i = 6$, $k = 3$ and $D - k - 1 = 4 - 3 - 1 = 0 < 3 = |\langle 6, 1\rangle| = |\langle i, 1\rangle|$. Therefore, $|\langle j, 1\rangle| = D - k = 4 - 3 = 1$ and $j = 1$, i.e., $\langle 6, 3\rangle \in N_2 \longmapsto \langle 6, 1\rangle \in N_1$.

If $\langle i, k\rangle = \langle 6, 0\rangle \in N_2$, then $i = 6$, $k = 0$ and $D - k - 1 = 4 - 0 - 1 = 3 = |\langle 6, 1\rangle| = |\langle i, 1\rangle|$. Therefore, $\langle 6, 0\rangle \in N_2 \longmapsto \emptyset \in N_1$.

The following table summarizes the mapping between $G(N_1, A_1)$ and $G(N_2, A_2)$.

$\langle i,j\rangle \in N_1 \Longleftrightarrow \langle i,k\rangle \in N_2$	
$\langle 7,1\rangle$	$\langle 7,3\rangle$
$\langle 7,3\rangle$	$\langle 7,2\rangle$
$\langle 7,5\rangle$	$\langle 7,1\rangle$
$\langle 7,7\rangle$	$\langle 7,0\rangle$
$\langle 6,1\rangle$	$\langle 6,3\rangle$
$\langle 6,3\rangle$	$\langle 6,2\rangle$
$\langle 6,6\rangle$	$\langle 6,1\rangle$
$\langle 5,1\rangle$	$\langle 5,3\rangle$
$\langle 5,3\rangle$	$\langle 5,2\rangle$

$\langle i,j\rangle \in N_1 \Longleftrightarrow \langle i,k\rangle \in N_2$	
$\langle 5,5\rangle$	$\langle 5,1\rangle$
$\langle 4,1\rangle$	$\langle 4,3\rangle$
$\langle 4,2\rangle$	$\langle 4,2\rangle$
$\langle 4,4\rangle$	$\langle 4,1\rangle$
$\langle 3,1\rangle$	$\langle 3,3\rangle$
$\langle 3,3\rangle$	$\langle 3,2\rangle$
$\langle 2,1\rangle$	$\langle 2,3\rangle$
$\langle 2,2\rangle$	$\langle 2,2\rangle$
$\langle 1,1\rangle$	$\langle 1,3\rangle$

Figure 1: Graph $G(N, A)$

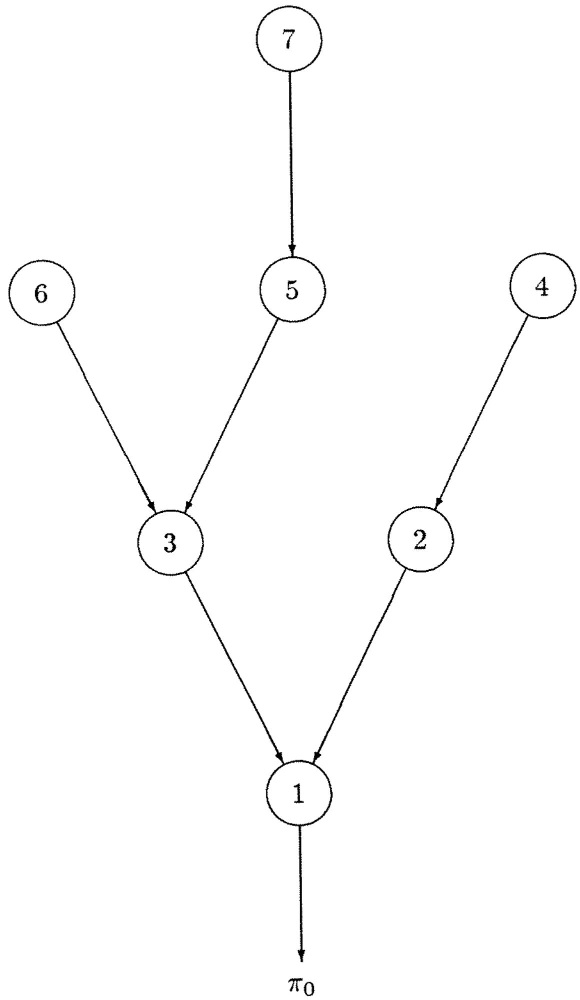


Figure 2: Graph $G(N', A')$

Figure 3: Graph $G(N_2, A_2) = G(N, A) \times G(N', A')$

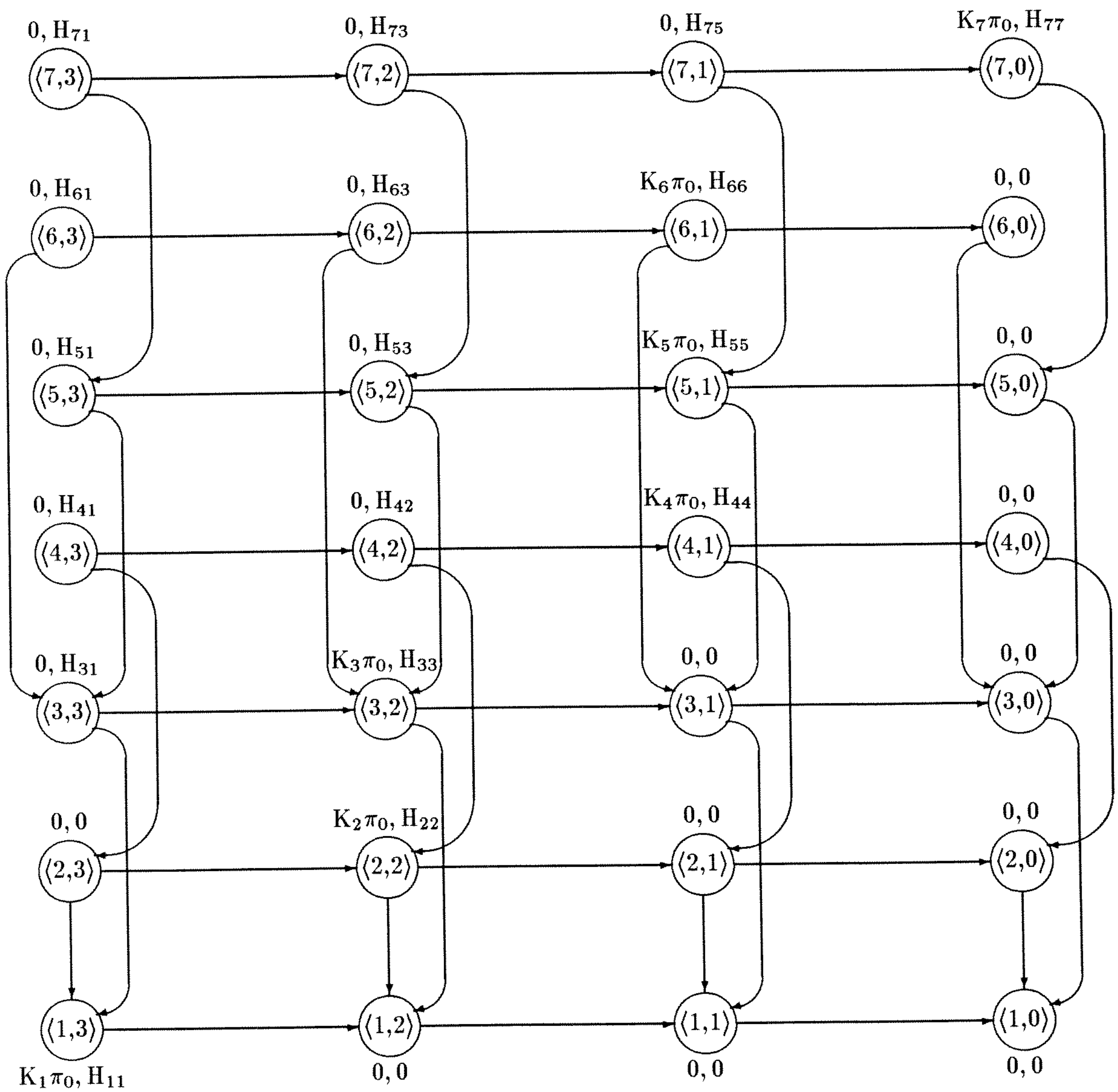
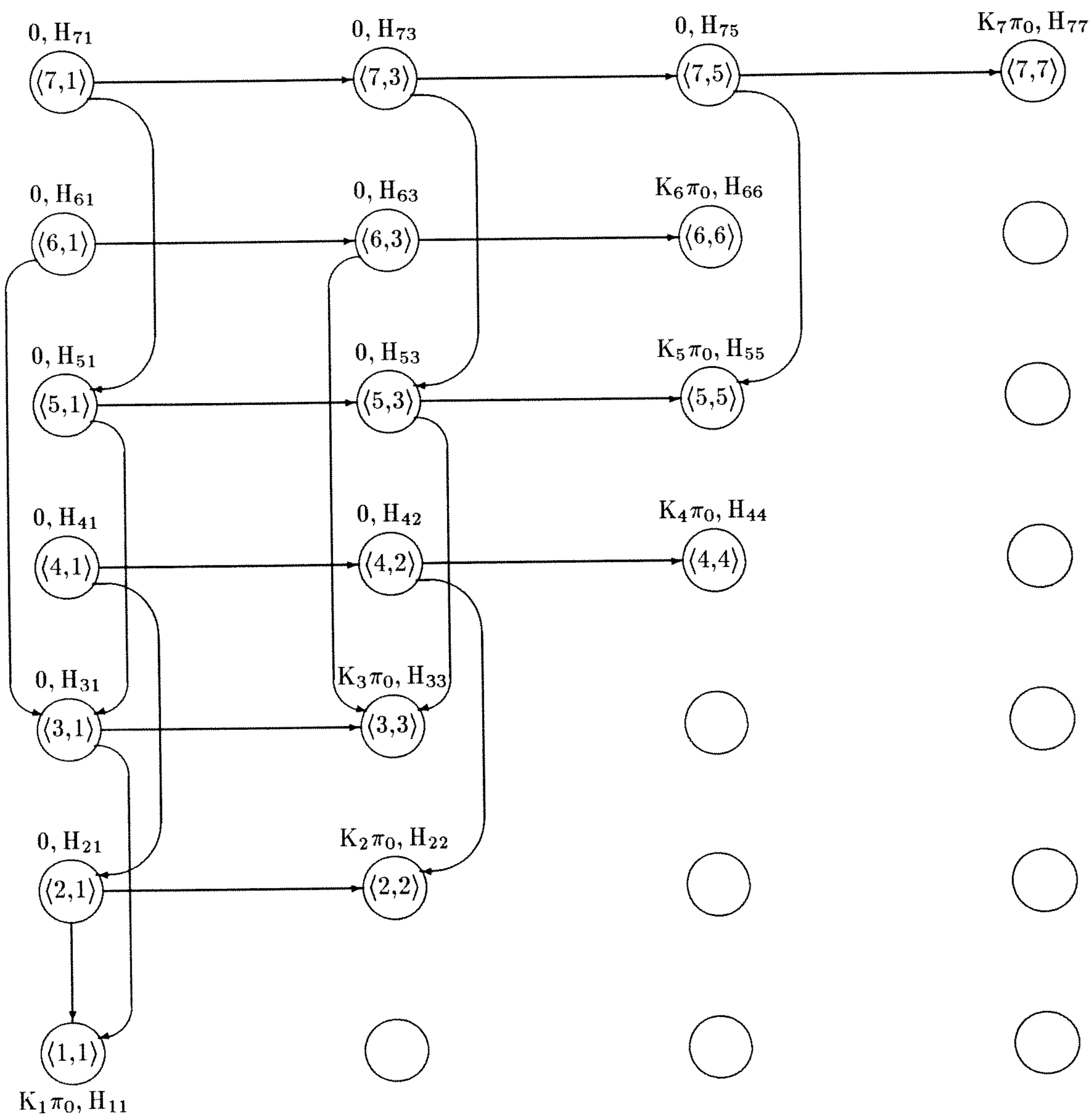


Figure 4: Graph $G(N_1, A_1)$, which is embedded in graph $G(N_2, A_2)$



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