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TECHNICAL REPORT NO. 1021

August 1992

RELIABILITY BOUNDS AND CRITICAL TIME OF THE BIRMBAUM-SAUNDERS DISTRIBUTION

by

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Summary and Conclusions-

The Birnbaum-Saunders distribution [3] has been shown to be the failure time distribution for fatigue failure caused by catastrophic crack size. In this paper, we obtain the point estimate for the critical time and show that, for any fixed scale parameter, the critical time is a monotonic decreasing function of the shape parameter. Then we demonstrate how to construct the reliability bounds and the confidence interval of the critical time from the confidence intervals of the parameters. Our method is not affected by censoring. Finally we illustrate the procedure by numerical examples.

1. Introduction

The Birnbaum-Saunders distribution arises as the failure time of a fatigue process where failure is defined to be the event that a crack length exceeds a treshold value. Although Birnbaum and Saunders [3] assumed that the crack length is a normal random variate under cyclic loading to arrive at the Birnbaum-Saunders distribution, it is later shown by Desmond [4] that a variety of distribution for crack size will still result in a Birnbaum-Saunders distribution. Moreover, as Desmond [5] has noted, even if the rate of crack growth in a common cycle is dependent on the crack size, the resulting failure distribution is still of the Birnbaum-Saunders type. These properties established the prominent role of the Birnbaum-Saunders distribution in modelling fatigue failure in particular, and wear-out failure in general.

In this paper, we address two issues concerning the confidence intervals related to the Birnbaum-Saunders distribution. The first one is the construction of reliability bounds (and hence the tolerance limits) while the second one is obtaining confidence intervals for the critical time of the failure rate. The first result will be useful to some design engineers who may need the lower reliability bound while the second one will be useful to reliability engineers in conducting burn-in process. The main idea used in obtaining these confidence intervals is similar to that of the equivariant confidence set discussed in Lehman [9]. Although the idea is the same, it should be noted that in the first case the reliability function is an explicit function; while in the second one, the critical time is an implicit function of the parameters.

We use the asymptotic distribution of the parameters given in Engelhardt *et. al.* [6] for statistical inference and use the result to obtain the confidence bounds of the reliability function and the critical time. To date, these bounds have yet been obtained at the best knowledge of the authors.

It should be noted that our approach is analytical implying that if the confidence intervals of the parameters were exact, the confidence bounds we obtained would also be exact. Moreover, our approach is applicable to censored data as long as the confidence intervals of the parameters can be established.

Notations

$\sim$	distributed as
MLE	Maximum likelihood estimator
α, β	The shape and scale parameters of the Birnbaum-Saunders Distribution
$g'(t;\cdot)$	first derivative of the function g w.r.t. t
ct	critical time of the failure rate
t^*	normalized critical time $= ct/\beta$.
Φ	cdf of standard s-normal distribution
$f(t;\cdot)$	pdf of the Birnbaum-Saunders Distribution with parameters $(\cdot)$
$R(t;\cdot)$	reliability function of the Birnbaum-Saunders Distribution with parameters $(\cdot)$
$\mathcal{N}(\cdot,\cdot')$	s-normal distribution with mean $(\cdot)$ and variance $(\cdot)'$
$\underline{(\cdot)}, \overline{(\cdot)}$	lower and upper confidence bounds of $(\cdot)$

$\alpha^*_{LB}(t), \alpha^*_{UB}(t)$ the α values that give the lower and upper bounds for $R(t)$ respectively.

$\gamma, \gamma_1, \gamma_2, \xi$ arbitary number whose sum is between 0 and 1.

Other standard notation is given in "information for Readers and Authors" at rear of each issue.

2. Failure Rate and its Critical Time

The pdf and the reliability function of the Birnbaum-Saunders distribution are given as follows.

$$f(t;\alpha,\beta) = \frac{1}{2\alpha\beta\sqrt{2\pi}}[(\frac{t}{\beta})^{-1/2} + (\frac{t}{\beta})^{-3/2}]\exp[-\frac{1}{2\alpha^2}(\frac{t}{\beta} - 2 + \frac{\beta}{t})] \quad (1)$$

$$R(t;\alpha,\beta) = \Phi[-\frac{1}{\alpha}((\frac{t}{\beta})^{1/2} - (\frac{t}{\beta})^{-1/2})] \quad (2)$$

$$h(t;\alpha,\beta) = \frac{f(t;\alpha,\beta)}{R(t;\alpha,\beta)} \qquad t \geq 0,\ \alpha > 0,\ \beta > 0 \quad (3)$$

where α is the shape parameter while β is the scale parameter.

From (1) we have

$$f'(t;\alpha,\beta) = -p(t;\alpha,\beta)f(t;\alpha,\beta)$$

where

$$p(t;\alpha,\beta) = \frac{1}{2t}(1 + \frac{2\beta}{t+\beta}) + \frac{1}{2\alpha^2\beta}[1 - (\frac{\beta}{t})^2]$$

It then follows that

$$\lim_{t\to\infty} h(t;\alpha,\beta) = \lim_{t\to\infty} -\frac{f'(t;\alpha,\beta)}{f(t;\alpha,\beta)} = \lim_{t\to\infty} p(t;\alpha,\beta) = \frac{1}{2\alpha^2\beta}$$

which exhibits behaviour similar to the inverse Gaussian distribution.

Also since

$$h'(t;\alpha,\beta) = \frac{R(t;\alpha,\beta)f'(t;\alpha,\beta) + f^2(t;\alpha,\beta)}{R^2(t;\alpha,\beta)} = h(t;\alpha,\beta)[-p(t;\alpha,\beta) + h(t;\alpha,\beta)]$$

the the solution to (if any)

$$h(t;\alpha,\beta) - p(t;\alpha,\beta) = 0 \tag{4}$$

is the critical time of the failure rate function.

We do not attempt to examine the condition of which the solution to (4) exists as, given the estimates of α and β, one can always plot the failure rate to see how it behaves. In figure 1, we plot the failure rate for various values of α for $\beta = 1$. It can be seen that for small value of α, the critical time is too high for its practical use. Also, the failure rate tends to be IFR as $\alpha \to 0$.

3. Parameters Inferences

Suppose that we have n random observations from the Birnbaum-Saunders distribution, say $\{t_1, \ldots, t_n\}$. The maximum likelihood estimator of β can be shown to be the positive root of the following equation.

$$\beta^2 - \beta[\frac{2n}{\sum_{i=1}^n t_i^{-1}} + \frac{n}{\sum_{i=1}^n (\beta + t_i)^{-1}}] + \frac{n}{\sum_{i=1}^n t_i^{-1}}[\frac{n}{\sum_{i=1}^n (\beta + t_i)^{-1}} + \frac{\sum_{i=1}^n t_i}{n}] = 0 \tag{5}$$

Using $\widehat{\beta}$ from above, the MLE of α can be obtained from

$$\widehat{\alpha} = [\frac{\sum_{i=1}^n t_i}{n\,\widehat{\beta}} + \frac{\widehat{\beta}\sum_{i=1}^n t_i^{-1}}{n} - 2]^{1/2} \tag{6}$$

One may deduce from the above estimation procedure that it is difficult, if not impossible, to come out with any exact sampling distribution for the parameters. Thus far, besides via extensive simulation which has been carried out by Engelhardt *et. al.* [6] , the only way is to use the asymptotic result for the maximum likelihood estimator.

It is well known that if (1) satisfies some regularity conditions (see for example Bickel and Doksum [2]), the asymptotic distribution for the MLE of α and β is normal with mean α and β; and the covariance matrix is given by the inverse of the expectation of the Fisher information. i.e. for a sample of size n we have

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \overset{a}{\sim} \mathcal{N}\left(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}, -\frac{1}{n}\left(\mathbf{E}\begin{pmatrix} \frac{\partial^2 ln(f)}{\partial\alpha^2} & \frac{\partial^2 ln(f)}{\partial\alpha\partial\beta} \\ \frac{\partial^2 ln(f)}{\partial\alpha\partial\beta} & \frac{\partial^2 ln(f)}{\partial\beta^2} \end{pmatrix}\right)^{-1} \right) \tag{7}$$

Engelhardt *et. al.* [6] computed the covariance matrix as

$$-\frac{1}{n}\left(\mathbf{E}\begin{pmatrix} \frac{\partial^2 ln(f)}{\partial\alpha^2} & \frac{\partial^2 ln(f)}{\partial\alpha\partial\beta} \\ \frac{\partial^2 ln(f)}{\partial\alpha\partial\beta} & \frac{\partial^2 ln(f)}{\partial\beta^2} \end{pmatrix}\right)^{-1} = \begin{pmatrix} \frac{\alpha^2}{2n} & 0 \\ 0 & \frac{\beta^2}{n[0.25+\alpha^{-2}+I(\alpha)]} \end{pmatrix} \tag{8}$$

where

$$I(\alpha) = 2\int_0^\infty \{[1+g(\alpha x)]^{-1} - 0.5\}^2 \, d\Phi(x) \tag{9}$$

$$g(y) = 1 + y^2/2 + y\sqrt{1+y^2/4} \tag{10}$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. It can be seen that $\hat{\alpha}$ and $\hat{\beta}$ are asymptotically independent of each other. Also $I(\alpha)$ can be evaluated numerically whenever $\hat{\alpha}$ is known. More importantly, the confidence intervals of $\hat{\alpha}$ and $\hat{\beta}$ can be computed from (8) using the standard normal table.

4. Derivation of the Reliability Bounds

In the case where the reliability function is monotone in its parameters, the confidence bounds can be obtained from some equivariant confidence set discussed in Lehman [9]. Similar approach has been adopted by Lawless [8] by finding the equivariant estimates. For the Birnbaum-Saunders distribution, however, $R(t)$ is not monotone w.r.t. α in (2). Consequently, there is no equivariant confidence set for $R(t)$. Nevertheless, we can still obtain a confidence bound for $R(t)$ as follows.

Suppose that the $(1-\gamma_1)$ confidence interval of α and the $(1-\gamma_2)$ confidence interval of β are given by

$$[\underline{\alpha},\ \overline{\alpha}] \quad \text{and} \quad [\underline{\beta},\ \overline{\beta}]$$

respectively. We have the following result.

Theorem 1 *The $(1-\gamma_1-\gamma_2+\gamma_1\gamma_2)$ level confidence bound for $R(t)$ in (2) is given by the interval*

$$[\inf_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\underline{\beta}),\ \sup_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\overline{\beta})] \quad \forall t > 0. \tag{11}$$

Proof: See Appendix. □

In practice, we can obtain the reliability bound of the Birnbaum-Saunders distribution as follows. We note from figure 2 that in (2), $R(t;\alpha,\beta)$ is monotone decreasing in α for all $t \leq \beta$ and monotone increasing in α for all $t \geq \beta$. This suggests the following :

(i) for $t \leq \beta$, $\inf_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\underline{\beta}) = R(t;\overline{\alpha},\underline{\beta})$ and $\sup_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\overline{\beta}) = R(t;\underline{\alpha},\overline{\beta})$;

(ii) for $t \geq \beta$, $\inf_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\underline{\beta}) = R(t;\underline{\alpha},\underline{\beta})$ and $\sup_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\overline{\beta}) = R(t;\overline{\alpha},\overline{\beta})$.

To determine $\alpha^*_{LB}(t)$ ($\alpha^*_{UB}(t)$) we only need to focus on the range between $\underline{\alpha}$ and $\overline{\alpha}$ in figure 2 and pick the one that gives the minimum (maximum) $R(t)$.

It should be noted that $\alpha^*_{LB}(t)$ is either equal to $\overline{\alpha}$ or $\underline{\alpha}$ and similarly for $\alpha^*_{UB}(t)$. This conforms to a "Bang-Bang" control strategy if we view the problem as a optimal control problem where one is looking for the control $\alpha^*_{LB}(t)$ to minimize $R(t)$ and $\alpha^*_{UB}(t)$ to maximize $R(t)$.

In addition, if we only want to evaluate the lower confidence bound using figure 2, we need only the one-sided confidence interval for β.

5. Confidence intervals of the Critical Time

By the invariance principle of MLE (see Zehna [11]), given $\hat{\alpha}$ and $\widehat{\beta}$ the MLE of the critical time is then the solution of (4) where α and β are replaced by $\hat{\alpha}$ and $\widehat{\beta}$ respectively.

To obtain the confidence interval of the critical time we observe that since β is the scale parameter there is no loss in generality to set it to one. This is equivalent to reparameterize the Birnbaum-Saunders random variable T to T/β. In doing so we can single out the effect of α on the critical time.

We solve (4) with $\beta = 1$, i.e.

$$h(t;\alpha,1) - p(t;\alpha,1) = 0$$

for different values of α ranging from 0.5 to 2.5. The resulting critical time, $t^*(\alpha)$, is depicted in figure 3. It can be seen that the critical time is a decreasing function of α. On the same plot we give the mean as a function of α (which is increasing) as it is practical to burn-in when the critical time does not exceed certain fraction of the mean life. From figure 3, one can see that for the purpose of burn-in, the evaluation of the critical time is useful only when α is large.

More importantly, as the critical time, $t^*(\alpha)$, is a monotonic decreasing function of α for all fixed β, we have

$$\alpha \in (\underline{\alpha},\ \overline{\alpha}) \quad \text{if and only if} \quad t^* \in (\underline{t^*}, \overline{t^*})$$

where $(\underline{\alpha},\ \overline{\alpha})$ is the $100(1-\gamma_1)\%$ confidence interval of α evaluated from (8) and the standard normal table. Using the concept of equivariant set in Lehman [9], we have

$$P[t^* \in (\underline{t^*}, \overline{t^*})] = P[\alpha \in (\underline{\alpha}, \overline{\alpha})] = 1 - \gamma_1. \tag{12}$$

meaning that $(\underline{t^*}, \overline{t^*})$ is the $100(1-\gamma_1)\%$ confidence interval of t^*. Now suppose we have the $100(1-\gamma_2)\%$ confidence of β as $(\underline{\beta}, \overline{\beta})$. As β is a scale parameter, it is easy to see that the $100(1-\gamma_1-\gamma_2+\gamma_1\gamma_2)\%$ confidence interval of the critical time is given by

$$(\underline{ct},\ \overline{ct}) = (\underline{\beta t^*},\ \overline{\beta t^*}) \tag{13}$$

using the fact that $\widehat{\alpha}$ and $\widehat{\beta}$ are asymptotically independent and (12).

6. Numerical Examples

Fatigue data

This is an example given in Birnbaum and Saunders [3] on the fatigue life (in cycle) of 6061-t6 aluminum coupons cut parallel to the direction of rolling. The data consist of $n = 101$ observations under a maximum stress of 31,000 psi with 18 cycle/s oscillation.

The MLE of the parameters are computed from (3) and (4) as

$$\widehat{\alpha} = 0.1704 \qquad \widehat{\beta} = 131.82.$$

It can be seen that $\widehat{\alpha}$ is small suggesting that the failure rate function is IFR. This is to be expected as it would be absurd to consider burn-in for fatigue. The failure rate for this data is depicted in figure 4 which confirms that it is indeed IFR.

For $\gamma_1 = \gamma_2 = 0.05$, the 95% confidence intervals of α and β can be obtained from (6), (7) and (8) as [0.1497, 0.1977] and [127.659, 136.262] respectively.

We obtain the 90.25% confidence bounds for $R(t; \alpha, \beta)$ by

(a) maximizing $R(t; \alpha, 136.262)$ subject to constraint that $\alpha \in [0.1497, 0.1977]$ iteratively for some fixed t; and

(b) minimizing $R(t; \alpha, 127.659)$ subject to constraint that $\alpha \in [0.1497, 0.1977]$ iteratively for some fixed t. .

(a) and (b) give the upper and lower confidence bounds respectively. The result is depicted in figure 5. It can be seen that these bounds are rather "tight" which suggests that our method is quite efficient in terms of the width of the bounds.

The $\alpha^*_{UB}(t)$ and $\alpha^*_{LB}(t)$ that give these bounds are plotted in figure 6. The same result is obtained using figure 2 given in the previous section.

It can be seen that $\alpha^*_{LB}(t)$ switches from $\overline{\alpha}$ to $\underline{\alpha}$ at $t = \underline{\beta} = 127.659$. Also $\alpha^*_{UB}(t)$ switches from $\underline{\alpha}$ to $\overline{\alpha}$ at $t = \overline{\beta} = 136.262$. As a result, there is a time region where $\alpha^*_{LB}(t) = \alpha^*_{UB}(t) = \underline{\alpha}$.

Repair time data

The data of this example consist of active repair time (in hour) for an airborne communication transceiver. This data set were used in [7]. It should also be noted that although the model used by Hsieh is inverse Gaussian distribution, the Birnbaum-Saunders distribution also provides a good fit with the Kolmogorov-Simirnov statistics = 0.0994.

Here, $n = 46$ and the MLE of the parameters are

$$\widehat{\alpha} = 1.2504 \qquad \widehat{\beta} = 2.0527$$

and the MLE for the critical time is obtained by solving (4) where α and β are replaced by $\widehat{\alpha}$ and $\widehat{\beta}$ as $\widehat{ct} = 0.5889$. For $\gamma_1 = \gamma_2 = 0.025$, the 97.5% confidence intervals of α and β are computed from (8) as [1.0137, 1.6314] and [1.6903, 2.6128] respectively.

We shall obtain the 95.0625% confidence interval of the critical time by using $\overline{\alpha}$ and $\underline{\beta}$ to compute $\underline{ct}$; and using $\underline{\alpha}$ and $\overline{\beta}$ to compute $\overline{ct}$ as in (13). The resulting interval is given by

$$(\underline{ct},\ \overline{ct}) = (0.2577,\ 1.2958).$$

The confidence interval of the critical time obtained by Hsieh [7] using jacknife-method is [0.3275, 0.9563]. Although our confidence interval is slightly wider our method guaranteed the desired coverage probability while the jacknife confidence interval only gives an approximation.

7. Concluding Remarks

1. In the region where $R(t)$ is monotone in both α and β, we have

$$\inf_{\alpha,\beta} P[\inf_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha} \\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) \leq R(t;\alpha,\beta) \leq \sup_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha} \\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta)]$$

$$= P[\underline{\alpha} \leq \alpha \leq \overline{\alpha},\ \ \underline{\beta} \leq \beta \leq \overline{\beta}] = 1 - \gamma_1 - \gamma_2 + \gamma_1\gamma_2.$$

This ensures the efficiency of the present procedure.

2. **Tolerance Limits**

Using the relationship between confidence bounds of reliability function and tolerance limits, the above procedure can be used to establish the ξ-content tolerance limits as follows (see Patel [10] for the definition).

Suppose that the $100(1-\gamma)\%$ upper and lower reliability bounds are obtained as in figure 7. Then the $100(1-\gamma)\%$ level ξ-content tolerance limits are given by the ordinates that correspond to ξ and $1-\xi$ on the lower and upper reliability bounds respectively.

3. **Censored Data**

It can be seen that our approach in constructing the reliability bounds is not affected by censoring as long as the confidence intervals of the parameters can be obtained.

Appendix: Proof of Theorem 1

By the definition of confidence bound in Bickel and Doksum [2], we only need to show that

$$P[\inf_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\overline{\alpha},\beta) \leq R(t;\alpha,\beta) \leq \sup_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\underline{\alpha},\beta)] \geq 1-\gamma_1-\gamma_2+\gamma_1\gamma_2$$

From basic probability, we know that if set $A \subseteq B$ then $P(A) \leq P(B)$. Now we have

$$\inf_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) \leq R(t;\alpha,\beta) \leq \sup_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) \quad \text{whenever} \quad \underline{\alpha}\leq\alpha\leq\overline{\alpha}, \;\; \underline{\beta}\leq\beta\leq\overline{\beta}.$$

So the latter is a subset of the former. Moreover the latter set is equivalent to

$$\widehat{\alpha} \in [\alpha \pm Z_{1-\alpha_1/2}\sqrt{\frac{\alpha^2}{n}}], \widehat{\beta} \in [\beta \pm Z_{1-\alpha_1/2}\sqrt{\frac{\beta^2}{n[0.25+\alpha^{-2}+I(\alpha)]}}]$$

This gives

$$\begin{aligned}
& P[\inf_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) \leq R(t;\alpha,\beta) \leq \sup_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta)] \\
\geq\; & P[\underline{\alpha}\leq\alpha\leq\overline{\alpha}, \;\; \underline{\beta}\leq\beta\leq\overline{\beta}] \\
=\; & P[\widehat{\alpha} \in [\alpha \pm Z_{1-\alpha_1/2}\sqrt{\frac{\alpha^2}{n}}], \widehat{\beta} \in [\beta \pm Z_{1-\alpha_1/2}\sqrt{\frac{\beta^2}{n[0.25+\alpha^{-2}+I(\alpha)]}}]\,] \\
=\; & 1-\gamma_1-\gamma_2+\gamma_1\gamma_2 \;\; \text{since } \widehat{\alpha} \text{ and } \widehat{\beta} \text{ are independent.}
\end{aligned}$$

From (2), it is readily seen that $R(t)$ is monotone increasing in β. Hence it follows that

$$\inf_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) = \inf_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\underline{\beta}) \quad \text{and} \quad \sup_{\substack{\underline{\alpha}\leq\alpha\leq\overline{\alpha}\\ \underline{\beta}\leq\beta\leq\overline{\beta}}} R(t;\alpha,\beta) = \sup_{\underline{\alpha}\leq\alpha\leq\overline{\alpha}} R(t;\alpha,\overline{\beta}). \tag{14}$$

□

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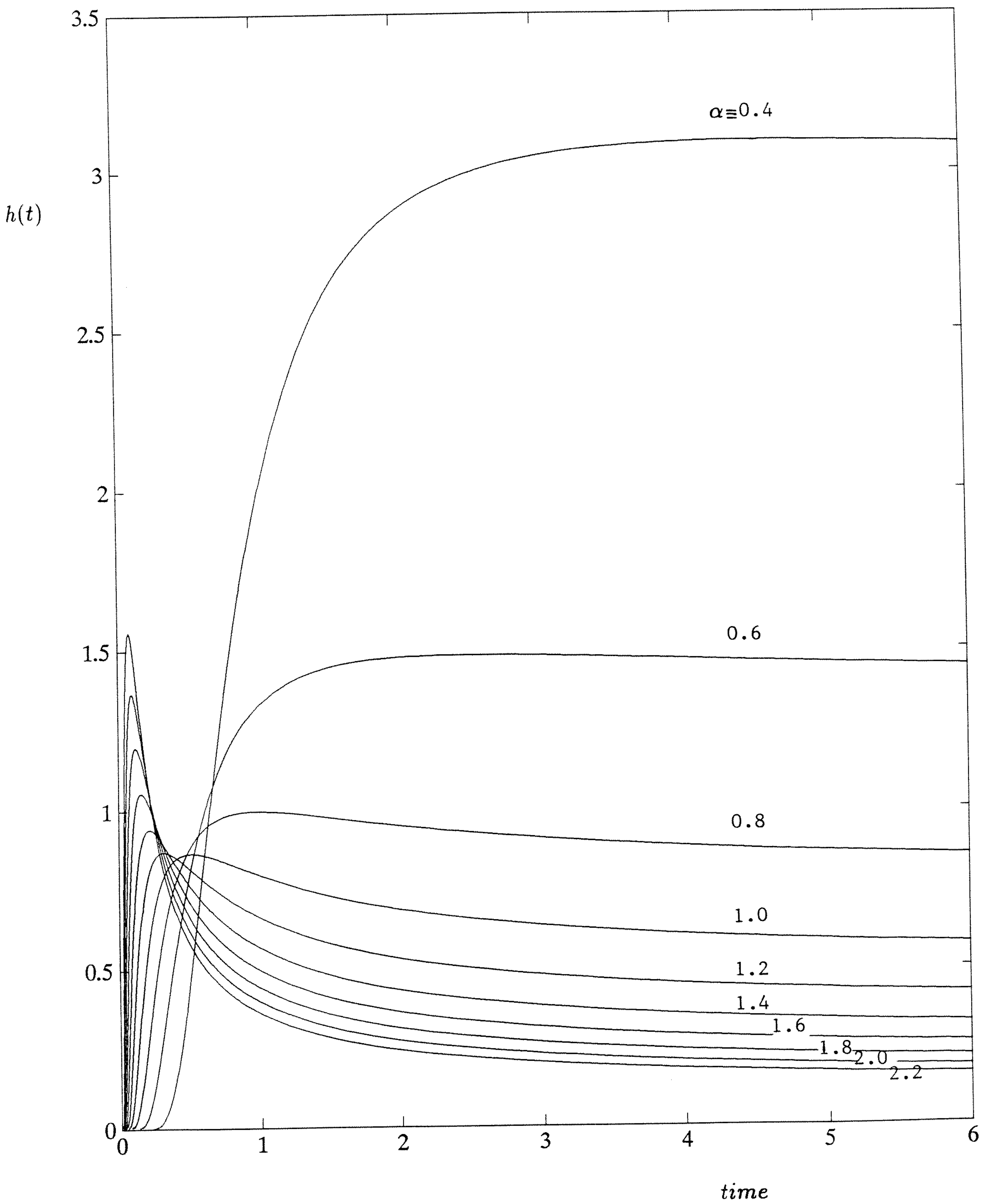

Figure 1: Failure rate of the Birnbaum-Saunders distribution for various α and $\beta = 1$.

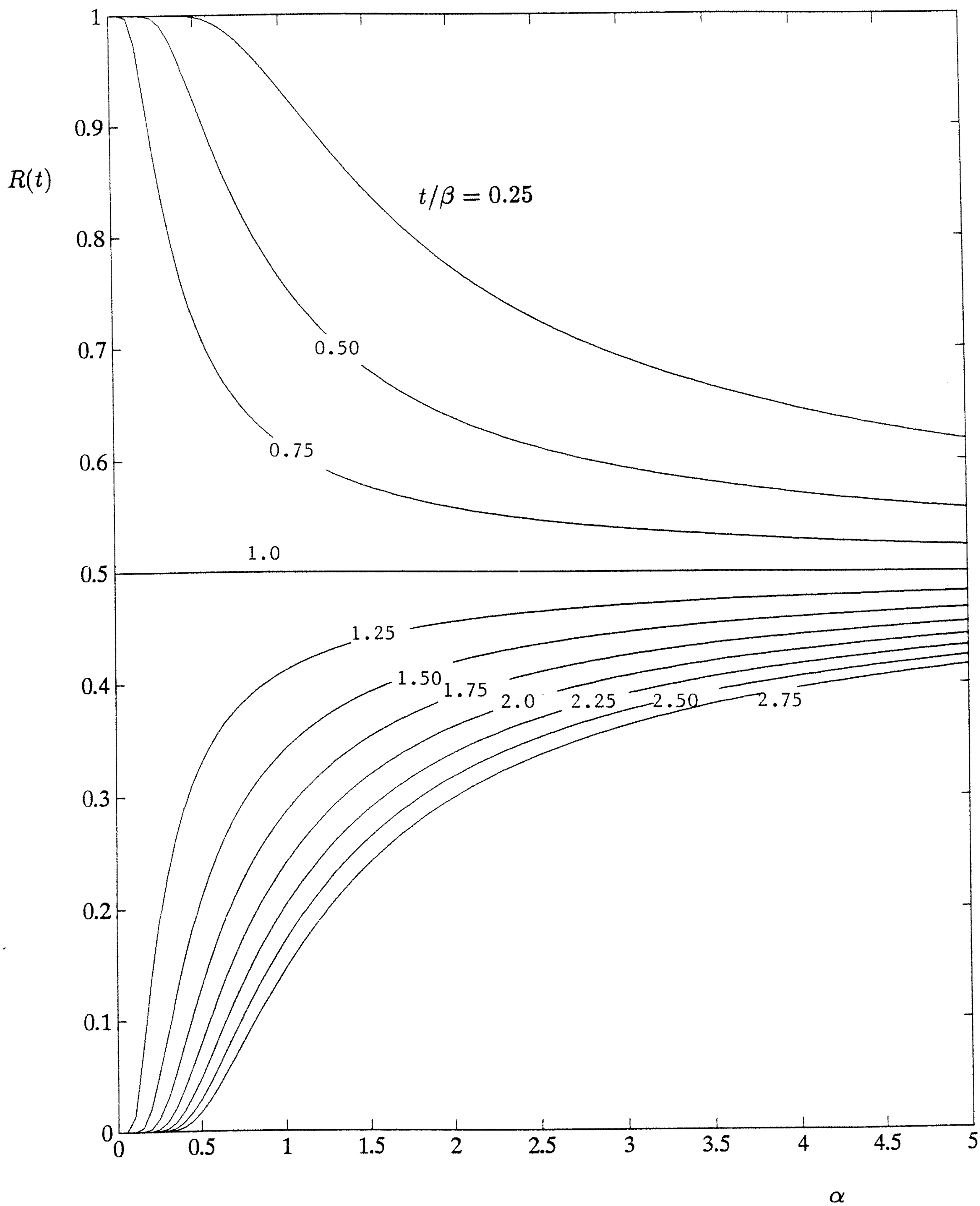

Figure 2: $R(t)$ as a function of α.

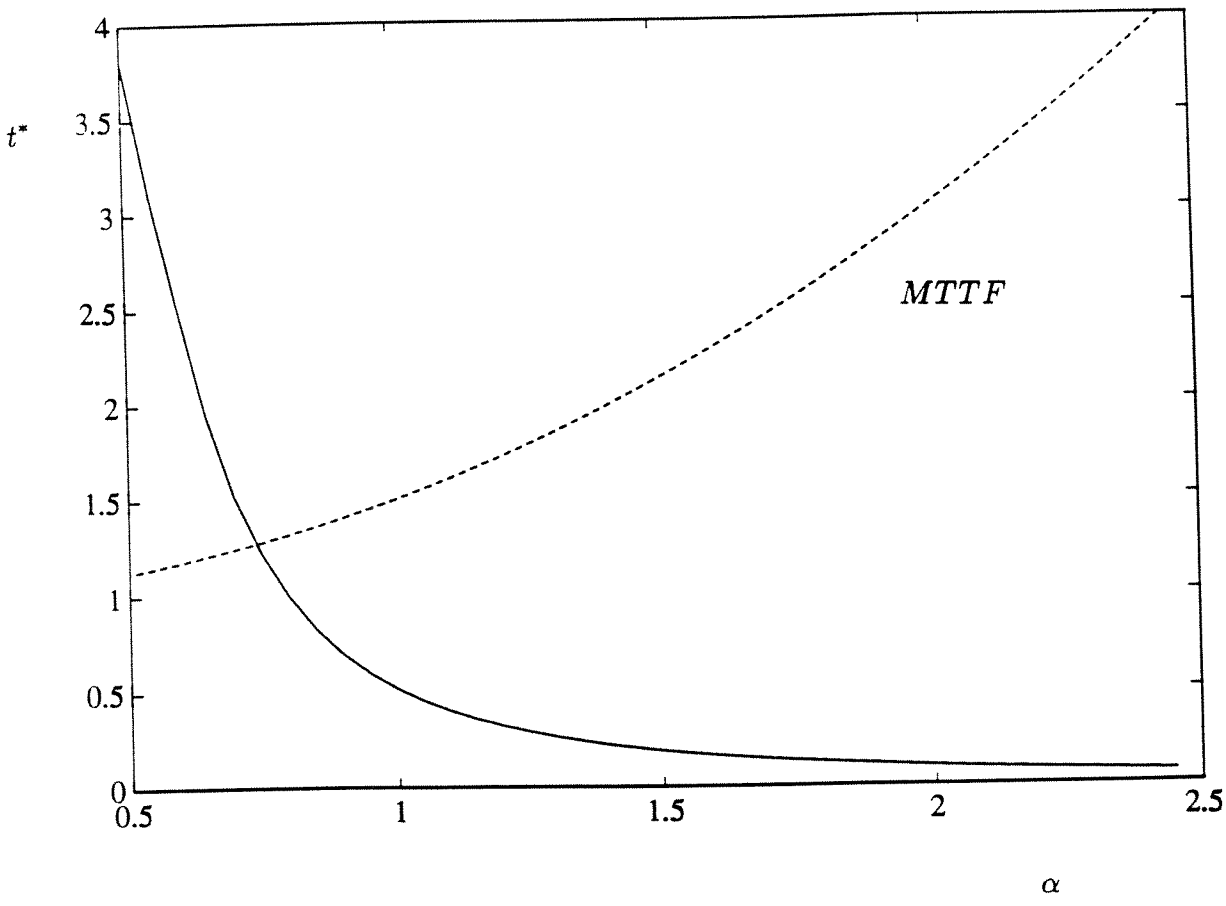

Figure 3 : Critical time as a function of α .

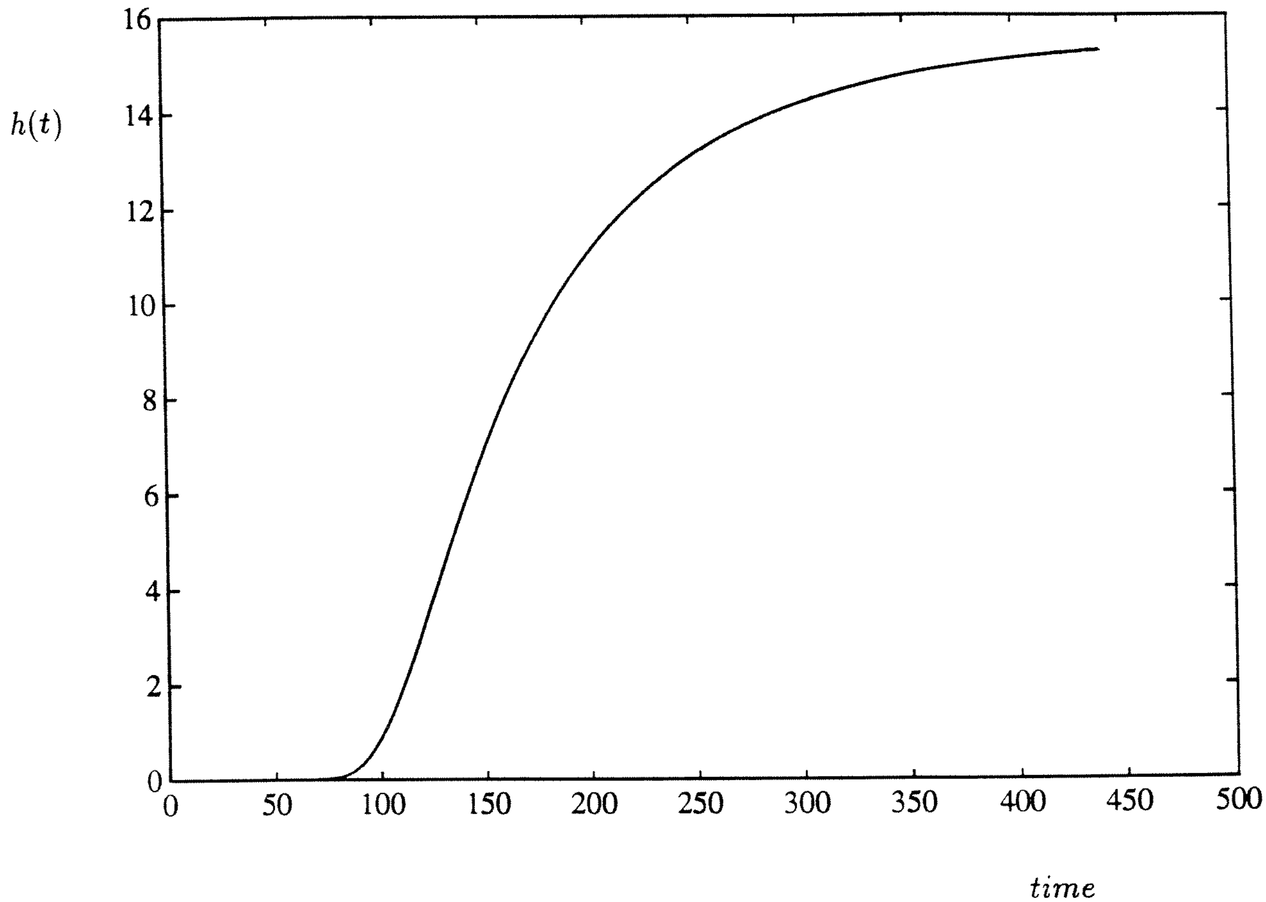

Figure 4 : Failure rate for fatigue data.

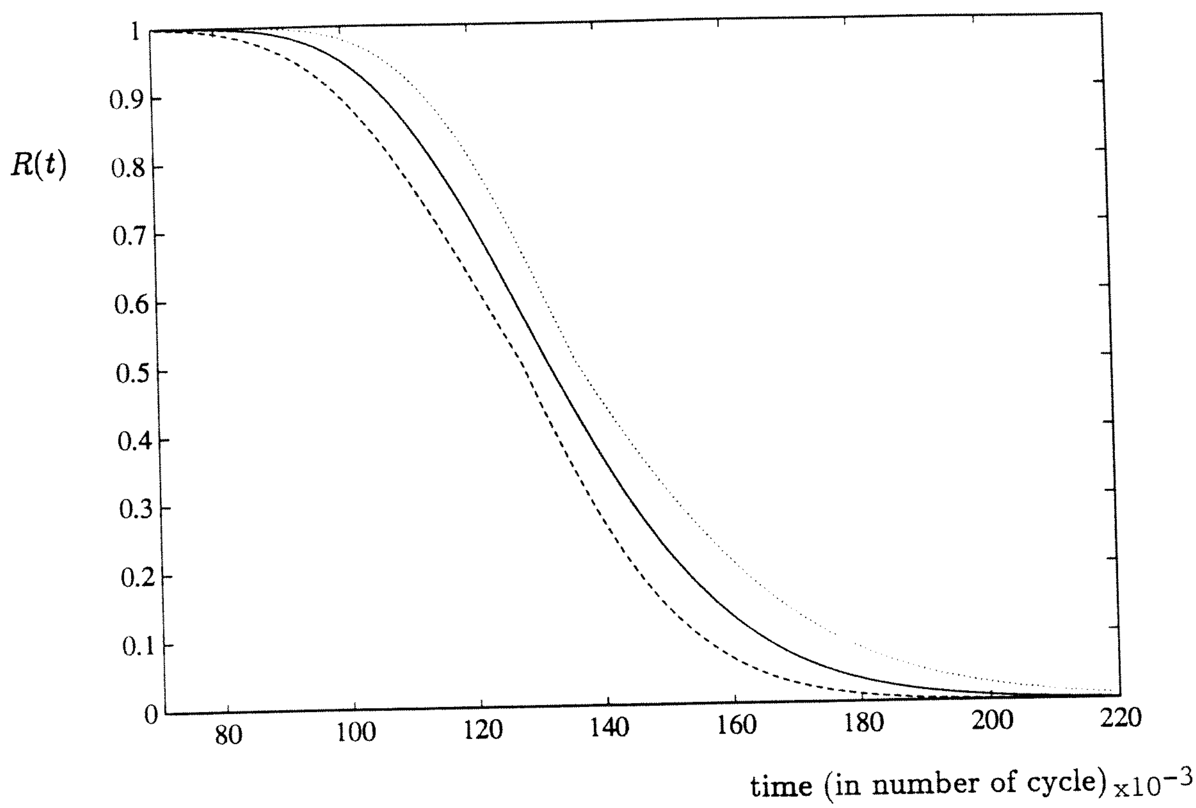

Figure 5 : Confidence bounds of $R(t)$ for example.

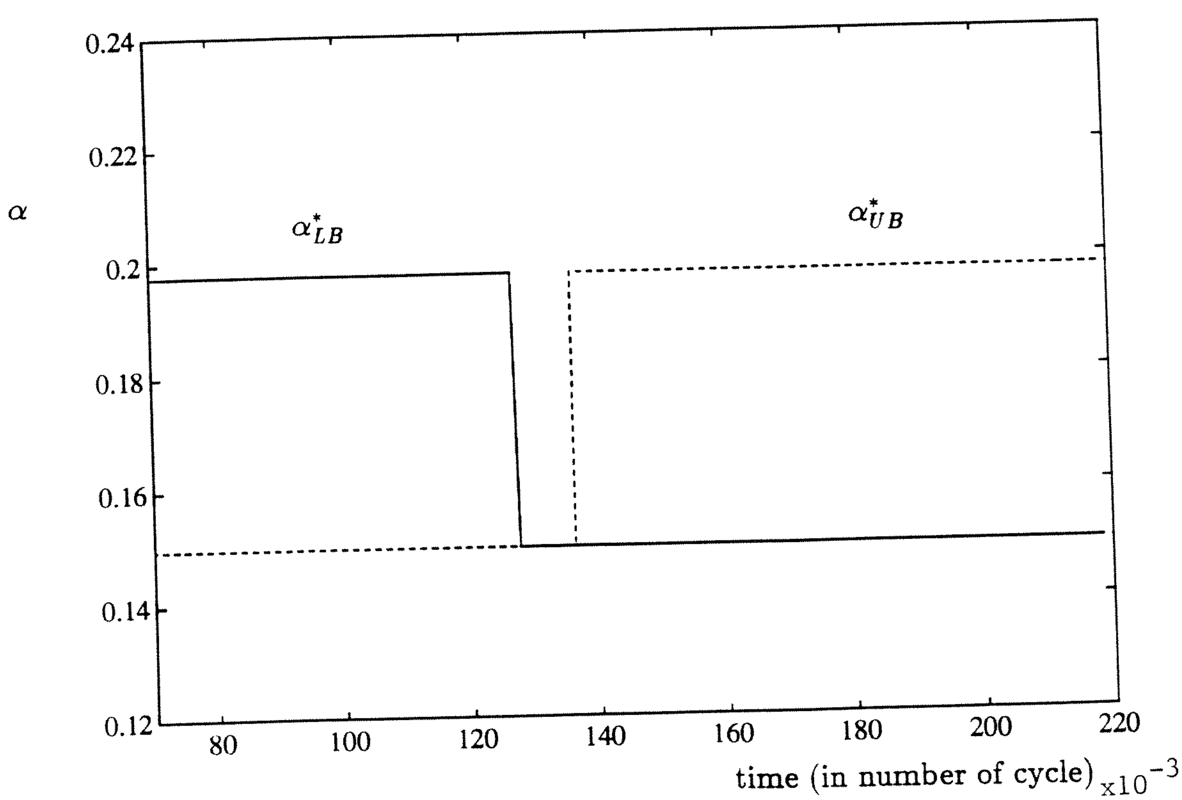

Figure6 : The Trajectories of α^*_{LB} and α^*_{UB}.

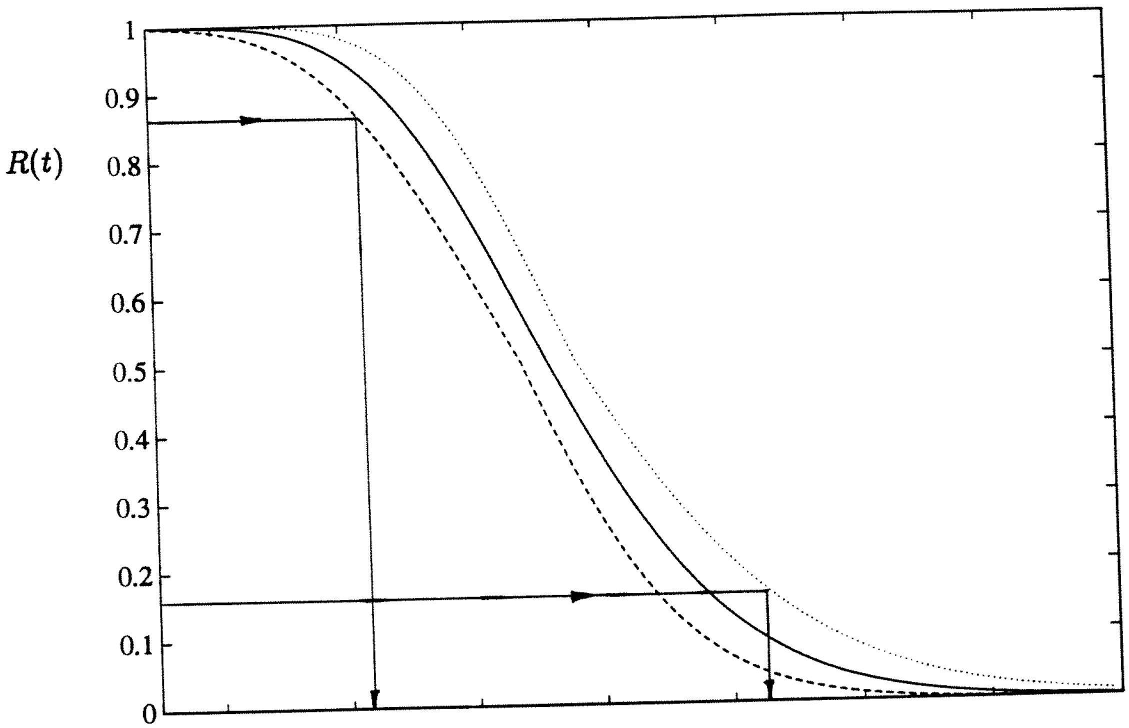

Figure 7 : A graphical illustration of obtaining the ξ-content tolerance limits from the lower and upper reliability bounds.