A NETWORK TRAFFIC MODEL WITH RANDOM TRANSMISSION RATE

KRISHANU MAULIK, SIDNEY RESNICK, AND HOLGER ROOTZÉN

Abstract. The infinite source Poisson network model assumes sources begin data transmissions at Poisson time points and continue for random lengths of time. The random transmission times have such heavy tails that the variance is infinite. Transmission rates have been assumed non-random and, usually constant. However, analysis of network data suggests that the transmission rate is also a random variable with a heavy tail. So we consider an infinite source Poisson model with sources transmitting for a random length of time at a random rate. Both the rates and lengths have infinite variance but finite mean and are assumed *asymptotically independent*, a concept made precise. We carefully discuss equivalent formulations of asymptotic independence and prove a limit theorem for the input process showing that the centered process under a suitable scaling converges to a totally skewed stable Lévy motion in the sense of finite dimensional distributions.

1. Introduction

Long range dependence, self-similarity and heavy tails are established concepts required for modeling broadband data networks. This is especially true when analyzing internet data, as described in, for example, [35]. The inadequacy of the finite variance model and short range dependence is well documented (cf. [8, 21, 36]).

Network traffic models generally contain many sources transmitting data. Transmissions are either modeled as superpositions of ON/OFF models ([11, 12, 13, 17, 30]) or by means of the infinite source Poisson model, sometimes called the M/G/∞ input model ([10, 13, 15, 14, 29, 23, 26]). In the first case, only mild assumptions are made about the tail of the ON/OFF periods such as existence of a finite mean. For the second model, the times between the starts of transmissions are modeled as iid exponentially distributed random variables. Thus, to account for the long range dependence and self-similar nature of the traffic, it becomes important to consider transmission times to be heavy tailed [37].

Most of the existing research assumes the rate of transmission to be constant and non-random. Konstantopoulos and Lin [16] replace the constant, non-random rate by a deterministic rate function which is regularly varying and are able to show that the input process at a large time scale is approximated by a stable Lévy motion. Their approximation is in the sense of convergence of finite dimensional distributions, which does not permit further weak convergence queueing results based on continuous mappings. Resnick and van den Berg [26] showed the convergence to hold on the space $D[0,\infty)$ of càdlàg functions with Skorohod's M_1 topology (cf. [27, 32, 33, 34]).

A recent empirical study on several internet traffic data sets by Guerin *et. al.* [10] shows that the infinite source Poisson model often gives an inadequate fit to data. This study suggests that the transmission rate is also a random variable with a heavy tail. There have been few studies of this aspect of the internet traffic data modeling. In a series of recent papers, Levy, Pipiras and

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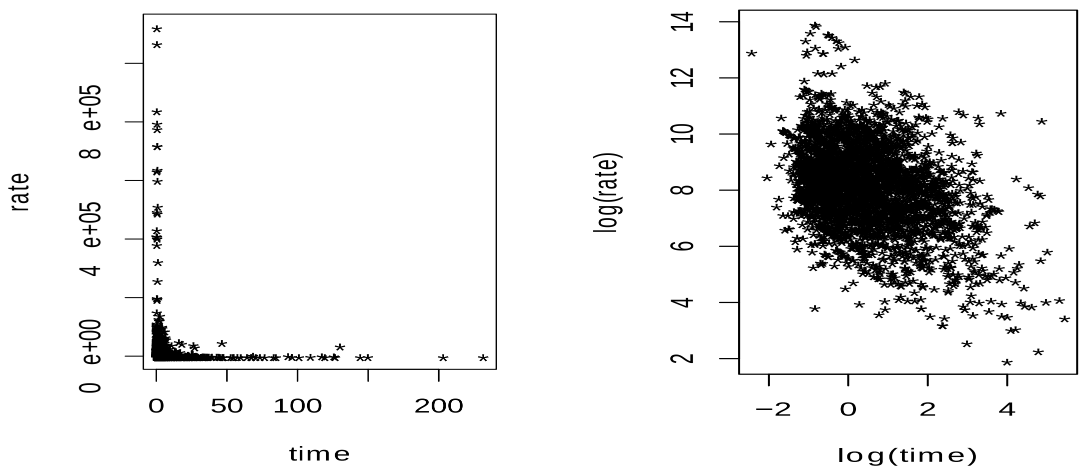

FIGURE 1. Plot of the time of transmission against the rate of the transmission of the BUburst data: *left*) in natural scale, *right*) in log-log scale

Taqqu [18, 19, 20] consider the case where the transmission rate is also random for a superposition of renewal reward processes. They show that the limiting behavior for large time scale and large number of superpositioned models can either be a stable Lévy process with stationary, *independent* increments or symmetric stable process with stationary, but *dependent* increments, depending on the relative rate of growth of the time scale and number of models. Their results parallel the results of Mikosch *et. al.* [29] for the infinite source Poisson model who also obtain two different limits depending on the growth rate time scale relative to the intensity of the Poisson process.

However, Taqqu *et. al.* [18, 19, 20] consider only the renewal-reward model and assume the transmission rate to be independent of the length of transmission. It is difficult to conclude from evidence in measured data that rate and the length of the transmission are always independent. There are cases where we may reasonably assume that the rate and the length of the transmission are at least asymptotically independent in a certain sense. As an example, we consider the BUburst dataset considered by Guerin *et. al.* [10]. This is data processed from the original 1995 Boston University data described in the report [5] and also catalogued at the Internet Traffic Archive (ITA) web site `www.acm.org/sigcomm/ITA/`. A plot of the transmission length against the transmission rate, (see Figure 1) shows that most of the data pairs hug the axes, which suggests the variables are at least asymptotically independent. However, if we plot the data in the log scale on both the axes, then a weak linear dependence is observable and the correlation coefficient between the two variables after log transform is approximately -0.379, which argues against an independence assumption. We consider the log transform to make the variables have finite second moment, so that correlation coefficient becomes meaningful.

The Hill estimates obtained for the transmission length, the transmission rate and the size of the transmitted file are 1.407, 1.138 and 1.157 respectively. These estimates are consistent with the observations made in Guerin *et. al.* [10]. The corresponding Hill plots are given in Figure 2. For each of the variables, the plots in the first column, named Hill plot, give plots of $\{(k, \hat{\alpha}_{k,n}) : 1 \leqslant k \leqslant n\}$, where $\hat{\alpha}_{k,n}$ is the Hill estimator of α based on k upper order statistics. The plots

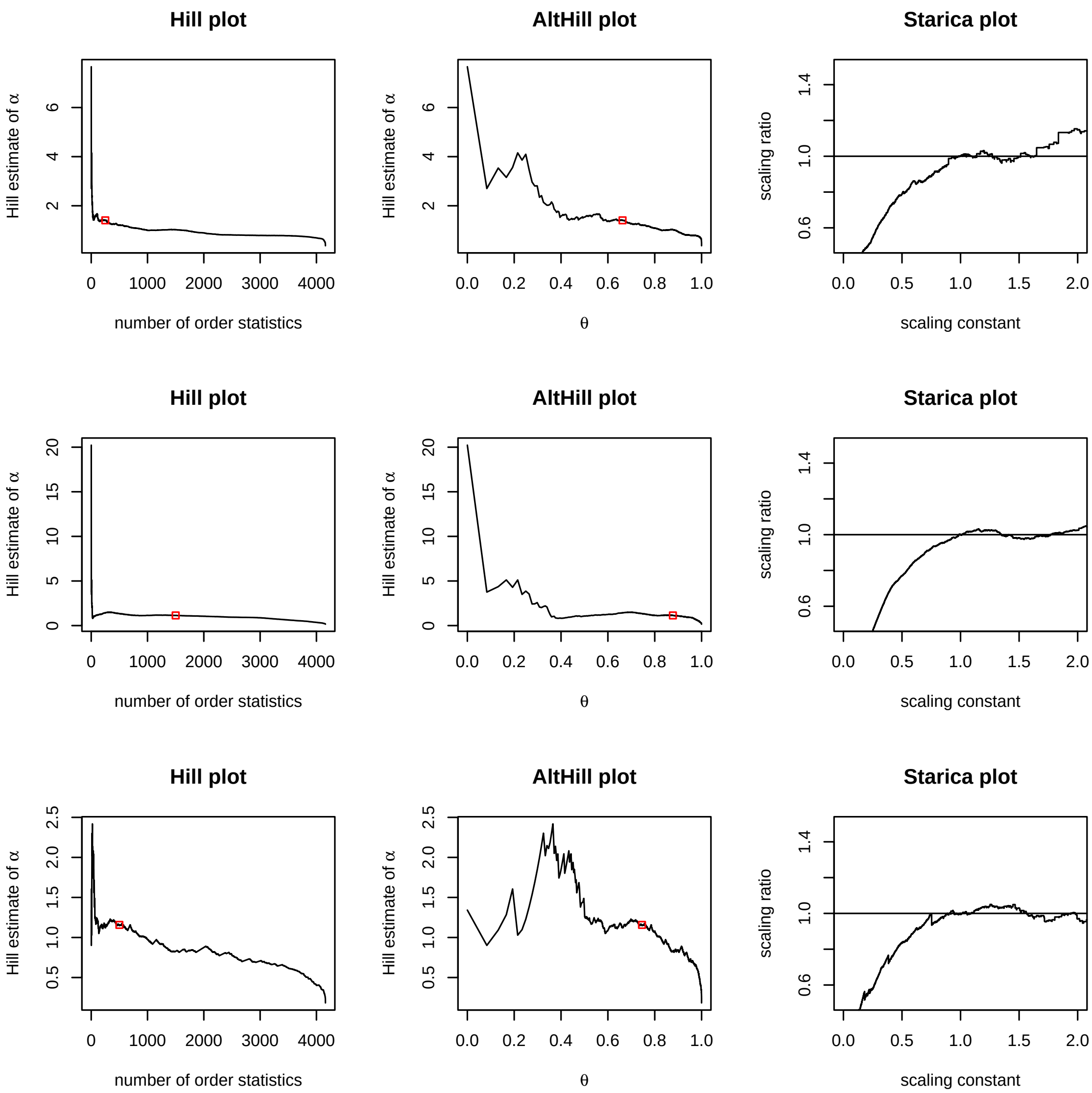

FIGURE 2. Hill plots of *top*) transmission length, *middle*) transmission rate, and *bottom*) transmitted file size

in the second column, named AltHill plot, give the Hill estimates in an alternative scale and plot $\{\left(\theta, \hat{\alpha}_{\lceil n^\theta \rceil, n}\right) : 0 \leqslant \theta \leqslant 1\}$ [25]. This plot blows up the original Hill plot on the left side and helps looking at that part more closely. The third plot, named Starica plot, is an exploratory device suggested by Stărică (cf. Section 7 of [28]) to decide on the number of upper order statistics to be used. It uses the fact that for a random variable X with Pareto tail of parameter α, we have

$$\lim_{T \to \infty} T \, \mathrm{P}\left[\frac{X}{T^{\frac{1}{\alpha}}} > r\right] = r^{-\alpha}.$$

For every k, we estimate the left hand side by

$$\hat{\nu}_{n,k}\big((r,\infty]\big) = \frac{1}{k}\sum_{i=1}^{n} \mathbf{I}\left[\frac{X_i}{(n/k)^{1/\hat{\alpha}_{n,k}}} > r\right].$$

We expect the ratio of $\hat{\nu}_{n,k}\big((r,\infty]\big)$ and $r^{-\hat{\alpha}_{n,k}}$, called the scaling ratio, to be approximately 1, at least for values of r in a neighborhood of 1, if we have made the correct choice of k. In the Starica plot, we plot the above scaling ratio against the scaling constant r, and choose k so that the graph hugs the horizontal line of height 1. The interesting point to be noted is the fact that the rate of the transmission has a much heavier tail than the length of transmission. This justifies the study of a model with a random rate with heavy tails. The tail of the size of the transmitted file, which is the product of the rate and the time of transmission, is comparable to the rate of the transmission, the heavier one between time and rate. This is in agreement with Theorem 3.2.

Since both the transmission length and the transmission rate have marginal distributions with heavy tails, it is further reasonable to assume that their bivariate distribution has a bivariate regularly varying tail, which is asymptotically independent (cf. [22]). The equivalent definitions of asymptotic independence are considered in (3.1) and (3.2). However, as described in Sections 3 and 4, the usual notion of asymptotic independence from extreme value theory (cf. [22, page 296]) is not sufficient for meaningful analysis. So we define an alternative and relevant definition of asymptotic independence in Section 3 and describe other equivalent formulations of the concept. Before that, Section 2 gives a quick review of the infinite source Poisson model, the case which we concentrate on. In Section 4, we give different examples and check the relevance of our hypotheses in those particular cases. In Section 5, we prove that in our setup, the input process of the infinite source Poisson model is approximated for large time scales by a positively skewed stable Lévy motion in the sense of convergence of finite dimensional distributions. Finally, in Section 6, we make some comments on the model with respect to the empirical findings and suggest some further avenues of research.

2. The infinite source Poisson model

We consider the M/G/∞ input model of incoming traffic to a communication network. Let $\{\Gamma_k, k \geq 1\}$ denote the points of a homogeneous Poisson process on $[0,\infty)$ with rate λ. Suppose at time Γ_k, a source starts a transmission, and continues to transmit for a period of length L_k, at a fixed rate R_k, both chosen at random. The total volume of traffic injected into the network between 0 and t is

$$A(t) = \sum_{k=1}^{\infty}((t-\Gamma_k)_+ \wedge L_k)R_k, \; t \geq 0. \tag{2.1}$$

We assume that (L_k, R_k) are iid with joint distribution function F and let F_L and F_R be the marginal distributions of L_k and R_k respectively. We make the following assumptions on the distribution of (L_k, R_k):

$$F(\mathbb{R}_+^2) = 1, \text{ where } \mathbb{R}_+^2 = (\mathbf{0},\boldsymbol{\infty}), \tag{2.2}$$

$$\overline{F}_L(x) = 1 - F_L(x) \in RV_{-\alpha_L}, \; \alpha_L \in (1,2), \tag{2.3}$$

$$\overline{F}_R(x) = 1 - F_R(x) \in RV_{-\alpha_R}, \; \alpha_R \in (1,2), \tag{2.4}$$

where RV_α stands for the class of regularly varying functions of index α, i. e., $G \in RV_\alpha$ if

$$\lim_{t\to\infty} \frac{G(xt)}{G(t)} = x^\alpha.$$

Note that we use the same symbols for the distribution function and probability measures interchangeably. The choice will be clear from the context.

Recall that we define the left-continuous inverse of a non-decreasing function Φ by

$$\Phi^{\leftarrow}(y) = \inf\{x : \Phi(x) \geqslant y\}.$$

We then define the quantile functions of L_1, R_1 and the product L_1R_1 respectively as follows:

$$b_L(T) = \inf\left\{x : \overline{F}_L(x) \leqslant \frac{1}{T}\right\} = \left(\frac{1}{\overline{F}_L}\right)^{\leftarrow}(T), \tag{2.5}$$

$$b_R(T) = \inf\left\{y : \overline{F}_R(y) \leqslant \frac{1}{T}\right\} = \left(\frac{1}{\overline{F}_R}\right)^{\leftarrow}(T), \tag{2.6}$$

$$b_P(T) = \inf\left\{z : \mathrm{P}[L_1R_1 > z] \leqslant \frac{1}{T}\right\}. \tag{2.7}$$

It is easy to see (cf. [6, 9, 3, 22]) that b_L and b_R are regularly varying functions of indices $1/\alpha_L$ and $1/\alpha_R$ respectively. Properties of the quantile function b_P will be given later.

3. Asymptotic Independence

For meaningful study of the quantile function b_P and the input process $\{A(t) : t \geqslant 0\}$, we need to make assumptions on the dependence structure of the distribution function F. We make assumptions which are somewhat weaker than independence, and which are a form of asymptotic independence. This is not the usual concept of asymptotic independence discussed in the context of extreme value theory, which requires that the distribution of the coordinatewise sample maxima, $(\vee_{i=1}^n L_i, \vee_{i=1}^n R_i)$, under suitable scaling, converges weakly to a product measure, and which is equivalent to the existence of regularly varying functions b_L and b_R, such that,

$$T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}, \frac{R_1}{b_R(T)}\right) \in \cdot\right] \xrightarrow{v} \nu(\cdot) \text{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \tag{3.1}$$

where ν is a measure satisfying $\nu((\mathbf{0},\boldsymbol{\infty}]) = 0$. The convergence above is vague convergence. This means that ν concentrates on the axes $\{0\}\times(0,\infty]$ and $(0,\infty]\times\{0\}$ (cf. [22, Chapter 5]). There is an equivalent formulation of the above concept where the variables are transformed so as to have the similar tails (cf. Section 4 of [7]), which states:

$$T\,\mathrm{P}\left[\frac{(b_L^{\leftarrow}(L), b_R^{\leftarrow}(R))}{T} \in \cdot\right] \xrightarrow{v} \tilde{\nu}(\cdot) \text{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \tag{3.2}$$

where $\tilde{\nu}$ satisfies $\tilde{\nu}((\mathbf{0},\boldsymbol{\infty}]) = 0$, and $\tilde{\nu}$ is also homogeneous of index -1. Thus if we define

$$\Phi(\theta) = \tilde{\nu}\left\{(s,t) : s \vee t > 1, \frac{t}{s} \leqslant \tan\theta\right\}, 0 \le \theta \le \frac{\pi}{2} \tag{3.3}$$

then the asymptotic independence is equivalent to the fact that Φ is supported on$\{0, \frac{\pi}{2}\}$. This traditional concept, however, does not fit the observed data, as we have seen in the Section 1. Also it fails to offer any useful result, as we shall illustrate through examples in the next section. So we need to strengthen the concept.

We have the following set of assumptions on the distribution function F. The assumptions differ depending on which one of the random variables, L and R has a heavier tail. We shall make two different cases accordingly. In the following we write ν_β for the measure on $(0,\infty]$ satisfying $\nu_\beta\big((x,\infty]\big) = x^{-\beta}$, $x > 0$, $\beta > 0$.

Case I: L has a heavier tail.

(IA) $\alpha_L < \alpha_R$.

(IB) $T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}, R_1\right) \in \cdot\right] \xrightarrow{v} \nu_{\alpha_L} \times G(\cdot)$ on $D := (0,\infty] \times [0,\infty]$,
where G is a probability measure with $G(\mathbb{R}_+) = 1$ and α_L-th moment finite, i. e.,

$$\int_0^\infty z^{\alpha_L}\, G(dz) < \infty. \tag{3.4}$$

(IC) $\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} \mathrm{E}\left[\left(\frac{L_1}{b_L(T)} R_1\right)^\delta \mathbf{1}_{\left[\frac{L_1}{b_L(T)} < \varepsilon\right]}\right] = 0$ for some $\delta > 0$.

Case II: R has a heavier tail.

(IIA) $\alpha_L > \alpha_R$.

(IIB) $T\,\mathrm{P}\left[\left(\frac{R_1}{b_R(T)}, L_1\right) \in \cdot\right] \xrightarrow{v} \nu_{\alpha_R} \times G(\cdot)$ on D,
where G is a probability measure with $G(\mathbb{R}_+) = 1$ and α_R-th moment finite.

(IIC) $\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} \mathrm{E}\left[\left(L_1 \frac{R_1}{b_R(T)}\right)^\delta \mathbf{1}_{\left[\frac{R_1}{b_R(T)} < \varepsilon\right]}\right] = 0$ for some $\delta > 0$.

In the data sets we have examined, both L and R have regularly varying tails and this motivates the form of the above assumptions. We consider the cone $D = (0,\infty] \times [0,\infty]$ instead of the more natural choice of $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ for the simple reason that we cannot have a desired characterization of (IB) or (IIB) by means of multivariate regular variation on the larger set without further assumptions, as shall become evident from Theorem 3.1, Lemma 3.2 and Remark 3.1. We first prove the multivariate regular variation condition on $(\mathbf{0},\boldsymbol{\infty})$ and then extend it to D. Under further moment condition, as in Lemma 3.2, we extend it to $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

The conditions (IB) and (IIB) are the required asymptotic independence conditions for the random variables L_1 and R_1. Neither (IB) nor (IIB) is symmetric in L and R. We also make certain assumptions about the truncated moment in (IC) and (IIC). The conditions (IB) and (IIB) of asymptotic independence are stronger than the usual concept of asymptotic independence discussed in the extreme value theory.

Lemma 3.1. *Assume the condition* (IB) *holds as well as (3.1). Then* ν *satisfies* $\nu\big((\mathbf{0},\boldsymbol{\infty}]\big) = 0$.

Proof. Fix $x > 0$. Let us define $\mathbf{x} = (x, x)$. Since $b_L(T) \to \infty$, we have, for all $K > 0$, $b_L(T)\,x > K$, for sufficiently large T. Hence we have, for all $K > 0$,

$$\nu\big((\mathbf{x},\boldsymbol{\infty}]\big) = \lim_{T\to\infty} T\,\mathrm{P}[L_1 > b_L(T)x, R_1 > b_R(T)x] \tag{3.5}$$

$$\leqslant \lim_{T\to\infty} T\,\mathrm{P}[L_1 > K, R_1 > b_R(T)x] \tag{3.6}$$

$$= \bar{G}(K)\nu_{\alpha_R}(x,\infty]. \tag{3.7}$$

Then letting $K \to \infty$, we get $\nu\big((\mathbf{x},\boldsymbol{\infty}]\big) = 0$, for all $x > 0$. Thus we have $\nu((\mathbf{0},\boldsymbol{\infty}]) = 0$. □

We now state and prove a condition which is equivalent to the asymptotic independence, that we defined.

Theorem 3.1. *Assume* $\mathbf{X} = (X_1, X_2)$ *is a random variable taking values in* $\mathbb{R}_+^2$, *i.e,* $\mathrm{P}[\mathbf{X} \in \mathbb{R}_+^2] = 1$. *Define* $b := \left(\frac{1}{1-F_{X_1}}\right)^{\leftarrow}$, *where* F_{X_1} *is the distribution function of* X_1. *Suppose*

$$T\,\mathrm{P}[X_1 > b(T)] \to 1. \tag{3.8}$$

Then the following are equivalent:

(i) $T\,\mathrm{P}\left[\left(\frac{X_1}{b(T)}, \frac{X_2}{X_1}\right) \in \cdot\right] \xrightarrow{v} (\nu_\alpha \times G)(\cdot)$ *on* D
for some $\alpha > 0$, *and* G *a probability measure satisfying* $G((0,\infty)) = 1$.

(ii) $T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right] \xrightarrow{v} \nu(\cdot)$ *on* D,
where $\nu(\{\mathbf{x} : x_1 > u\}) > 0$ *for all* $u > 0$.

In fact, ν *is homogeneous of order* $-\alpha$*; i. e.,* $\nu(u\cdot) = u^{-\alpha}\nu(\cdot)$ *on* D, *and is given by*

$$\nu = \begin{cases} (\nu_\alpha \times G) \circ \theta^{-1} & \text{on} \quad (0,\infty) \times [0,\infty) \\ 0 & \text{on} \quad D \setminus ((0,\infty) \times [0,\infty)) \end{cases}, \tag{3.9}$$

where $\theta(x,y) = (x, xy)$, *if* $(x,y) \in D \setminus \{(\infty, 0)\}$ *and* $\theta(\infty, 0)$ *is defined arbitrarily.*

Remark 3.1. In light of above theorem, we can rewrite the assumptions (IB) and (IIB) respectively as follows:

(IB′) $T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in \cdot\right] \xrightarrow{v} \nu(\cdot)$ on D,
where ν is a homogeneous Radon measure of order $-\alpha_L$.

(IIB′) $T\,\mathrm{P}\left[\frac{(R_1, L_1R_1)}{b_R(T)} \in \cdot\right] \xrightarrow{v} \nu(\cdot)$ on D,
where ν is a homogeneous Radon measure of order $-\alpha_R$.

This characterizes our asymptotic independence conditions in terms of standard multivariate regular variation on the cone D (cf. Chapter 5, [22]) and is in the spirit of the characterization of multivariate regular variation using a polar coordinate transformation (cf. [1]).

Remark 3.2. The function θ as defined above is Borel-measurable, irrespective of its value at $(\infty, 0)$. The result can be easily seen from the fact that the singleton subset $\{(\infty, 0)\}$ is a measurable subset of D.

Remark 3.3. Since $\mathrm{P}[\mathbf{X} \in (\mathbf{0}, \boldsymbol{\infty})] = 1$, we have $\frac{X_2}{X_1}$ is well-defined *almost everywhere.*

Remark 3.4. The condition (3.8) holds, for example, when X_1 has a regularly varying tail, as in our case.

Remark 3.5. Observe that the measure ν as defined above is Radon. To see this, note that the relatively compact sets in D are contained in $[a, \infty] \times [0, \infty]$. Now

$$\nu([a,\infty] \times [0,\infty]) = (\nu_\alpha \times G)([a,\infty] \times [0,\infty]) = a^{-\alpha} < \infty$$

and hence ν is Radon.

Proof of Theorem 3.1:

(i) $\Rightarrow$ **(ii):** Let $0 < s < \infty$ and $S \in \mathcal{B}([0,\infty])$. Define

$$V_{s,S} = \{\mathbf{x} \in D : s < x_1 < \infty, \frac{x_2}{x_1} \in S\}.$$

Now $V_{s,[t_1,t_2]}$ is relatively compact in D for all $0 \leqslant t_1 \leqslant t_2 \leqslant \infty$. Also if $G(\{t_1,t_2\}) = 0$, then we have,

$$T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in V_{s,[t_1,t_2]}\right] = T\,\mathrm{P}\left[\left(\frac{X_1}{b(T)}, \frac{X_2}{X_1}\right) \in (s,\infty)\times[t_1,t_2]\right]$$
$$\to (\nu_\alpha \times G)((s,\infty)\times[t_1,t_2]) = \nu\left(V_{s,[t_1,t_2]}\right),$$

where ν is as defined in (3.9). Now, fix $s_0 \in (0,\infty)$. Note $T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in V_{s_0,[0,\infty]}\right] \leqslant T < \infty$, and

$$\nu\left(V_{s_0,[0,\infty]}\right) = (\nu_\alpha \times G)((s_0,\infty)\times[0,\infty]) = s_0^{-\alpha} \in (0,\infty).$$

Hence $T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in V_{s_0,[0,\infty]}\right]$ is strictly positive and finite for all large T. So we can define probability measures $Q_T(\cdot)$ and $Q(\cdot)$ on $(s_0,\infty)\times[0,\infty]$ for all large T, by

$$Q_T(\cdot) = \frac{\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right]}{\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in V_{s_0,[0,\infty]}\right]} \text{ and } Q(\cdot) = \frac{\nu(\cdot)}{\nu\left(V_{s_0,[0,\infty]}\right)}.$$

Then

$$Q_T(V_{s,[t_1,t_2]}) \to Q(V_{s,[t_1,t_2]}) \quad \forall s \in (s_0,\infty), 0 \leqslant t_1 \leqslant t_2 \leqslant \infty \text{ with } G(\{t_1,t_2\}) = 0. \tag{3.10}$$

Let

$$\mathcal{P} = \{V_{s_1,[t_1,t_2]} \setminus V_{s_2,[t_1,t_2]} : s_0 < s_1 < s_2 < \infty, 0 \leqslant t_1 \leq t_2 \leqslant \infty\}.$$

Observe $B \in \mathcal{P}$ is a Q-continuity set iff $G(\{t_1,t_2\}) = 0$. So by (3.10), $Q_T(B) \to Q(B)$ for all Q-continuity sets $B \in \mathcal{P}$. Also, clearly for every $\mathbf{x}$ in $(s_0,\infty)\times[0,\infty]$ and positive ε, there is an A in $\mathcal{P}$, for which $\mathbf{x} \in A^\circ \subseteq A \subseteq B(\mathbf{x},\varepsilon)$, where A° is the interior of A and $B(\mathbf{x},\varepsilon)$ is the ball of radius ε around $\mathbf{x}$. Now $\mathcal{P}$ is a π-system. Then, by Theorem 2.3 of [2], we have

$$Q_T \Rightarrow Q \text{ on } (s_0,\infty)\times[0,\infty].$$

Thus $Q_T(B) \to Q(B)$ for all Borel sets B of $(s_0,\infty)\times[0,\infty]$ with boundary in $(s_0,\infty)\times[0,\infty]$ having zero Q-measure, for all $s_0 > 0$. Hence the same result holds with Q_T, Q replaced by $T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right]$, ν respectively.

Let K be relatively compact in D with $\nu(\partial_D K) = 0$. Then there exists $s_0 > 0$, such that $K \subset (s_0,\infty]\times[0,\infty]$. Define $B = K \cap \left((s_0,\infty)\times[0,\infty]\right)$. Then B is Borel in $(s_0,\infty)\times[0,\infty]$ and $\nu\left(\partial_{(s_0,\infty)\times[0,\infty]} B\right) = 0$. We have

$$\begin{aligned} T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n)} \in K\right] &= T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n)} \in B\right] && \text{since } \mathrm{P}[\mathbf{X} \in \mathbb{R}_+^2] = 1 \\ &\to \nu(B) = \nu(K) && \text{by definition of } \nu \text{ in (3.9).} \end{aligned}$$

Therefore

$$T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n)} \in \cdot\right] \xrightarrow{v} \nu(\cdot) \text{ on } D,$$

where ν is defined as in (3.9). Thus,

$$\nu(\{\mathbf{x} : x_1 > u\}) = (\nu_\alpha \times G)(\{\mathbf{x} : x_1 > u\}) = u^{-\alpha} > 0, \quad \forall u > 0.$$

(ii)⇒ (i): Define $U = \{\mathbf{x} \in D : x_1 > 1\}$.

Choose integer n_v such that $b(n_v) \leqslant v < b(n_v+1)$. Fix $0 < s \leqslant \infty$, $0 \leqslant t \leqslant \infty$, so that $\nu(\partial([s,\infty]\times[t,\infty])) = 0$. Then

$$\frac{n_v}{n_v+1}\cdot\frac{(n_v+1)\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n_v+1)}\in[s,\infty]\times[t,\infty]\right]}{n_v\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n_v)}\in U\right]} \leqslant \frac{\mathrm{P}\left[\frac{\mathbf{X}}{v}\in[s,\infty]\times[t,\infty]\right]}{\mathrm{P}\left[\frac{\mathbf{X}}{v}\in U\right]}$$
$$\leqslant \frac{n_v+1}{n_v}\cdot\frac{n_v\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n_v)}\in[s,\infty]\times[t,\infty]\right]}{(n_v+1)\,\mathrm{P}\left[\frac{\mathbf{X}}{b(n_v+1)}\in U\right]},$$

and taking the limit as $v\to\infty$, we find,

$$\frac{\mathrm{P}\left[\frac{\mathbf{X}}{v}\in[s,\infty]\times[t,\infty]\right]}{\mathrm{P}\left[\frac{\mathbf{X}}{v}\in U\right]} \to \frac{\nu\left([s,\infty]\times[t,\infty]\right)}{\nu(U)}, \quad 0<s\leqslant\infty, \quad 0\leqslant t\leqslant\infty.$$

Arguing as before, normalizing to probability measures and so on, we have

$$\frac{\mathrm{P}[\mathbf{X}\in v\cdot]}{\mathrm{P}[\mathbf{X}\in vU]} \xrightarrow{v} \frac{\nu(\cdot)}{\nu(U)} \text{ on } D.$$

Then, by the usual argument, (cf. [22]), $\frac{\nu(\cdot)}{\nu(U)}$ is homogeneous on D of order $-\alpha$, for some $\alpha>0$, and hence this is true for $\nu(\cdot)$; i. e., $\nu(s\cdot) = s^{-\alpha}\nu(\cdot)$ on D.

Now, by (3.8) and the fact that $\mathrm{P}[\mathbf{X}\in\mathbb{R}_+^2]=1$,

$$1 = \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{X_1}{b(T)}>1\right] = \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{X_1}{b(T)}>1, \frac{X_2}{X_1}\in(0,\infty)\right] = \nu(V_{1,(0,\infty)}).$$

Thus $G(\cdot) := \nu(V_{1,\cdot})$ is a probability measure on $[0,\infty]$ with $G(\mathbb{R}_+)=1$.

Also, for all $s>0$, $S\in\mathcal{B}([0,\infty])$,

$$\nu(V_{s,S}) = (\nu_\alpha\times G)\big((s,\infty]\times S\big) = (\nu_\alpha\times G)(\theta^{-1}V_{s,S}),$$

and thus ν has a form as defined in (3.9).

Again, $V_{s,[t_1,t_2]}$ is a ν-continuity set iff $(s,\infty]\times[t_1,t_2]$ is a $(\nu_\alpha\times G)$-continuity set iff $G(\{t_1,t_2\})=0$. Thus for all $s>0$, all $0\leqslant t_1\leqslant t_2\leqslant\infty$ with $G(\{t_1,t_2\})=0$, i. e., $V_{s,[t_1,t_2]}$ a ν-continuity set, we have, for all $s>0$, $S\in\mathcal{B}([0,\infty])$,

$$T\,\mathrm{P}\left[\frac{X_1}{b(T)}>s, \frac{X_2}{X_1}\in[t_1,t_2]\right] = T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in V_{s,[t_1,t_2]}\right] \to \nu(V_{s,[t_1,t_2]}) = (\nu_\alpha\times G)\big((s,\infty]\times[t_1,t_2]\big).$$

Then, arguing by normalizing to probability measures as before, we get,

$$T\,\mathrm{P}\left[\left(\frac{X_1}{b(T)},\frac{X_2}{X_1}\right)\in\cdot\right] \xrightarrow{v} (\nu_\alpha\times G)(\cdot) \text{ on } D.$$

□

Note that we have neither used the moment condition (3.4) on G, nor have we used the condition (IC) or (IIC). Assuming these and (3.1), stronger conclusions are possible.

Lemma 3.2. *Assume $\mathbf{X}=(X_1,X_2)$ is a random variable taking values in $\mathbb{R}_+^2$. Let $b := \left(\frac{1}{1-F_{X_1}}\right)^{\leftarrow}$. Suppose $T\,\mathrm{P}[X_1>b(T)]\to 1$. Further assume*

(i) $T\,\mathrm{P}\left[\left(\frac{X_1}{b(T)},\frac{X_2}{X_1}\right)\in\cdot\right] \xrightarrow{v} \nu_\alpha\times G$ *on* D,
for some $\alpha>0$, and G a probability measure satisfying $G((0,\infty))=1$ and having finite α-th moment,

(ii) $\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{X_2}{b(T)}\right)^{\delta}\mathbf{1}_{\left[\frac{X_1}{b(T)}\leqslant\varepsilon\right]}\right]=0$ *for some* $\delta>0$.

Then

$$T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in\cdot\right]\xrightarrow{v}\tilde{\nu}\ \text{on}\ [\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}, \tag{3.11}$$

where

$$\tilde{\nu}=\begin{cases}\nu & \text{on } D\\ 0 & \text{on } \{0\}\times(0,\infty]\end{cases}, \tag{3.12}$$

with ν *defined as in (3.9).*

Proof. First we observe that $\tilde{\nu}$, as defined in the lemma, is Radon on $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$, given the existence of α-th moment of G. To see this, note that a relatively compact set in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$ is contained in $[\mathbf{0},\boldsymbol{\infty}]\setminus[\mathbf{0},\mathbf{a}]$ for some $a>0$, where $\mathbf{a}=(a,a)$. So it is enough to check the finiteness of $\tilde{\nu}([\mathbf{0},\boldsymbol{\infty}]\setminus[\mathbf{0},\mathbf{a}])$. We consider the set $[\mathbf{0},\boldsymbol{\infty}]\setminus[\mathbf{0},\mathbf{a}]$ in two disjoint components, namely, $(a,\infty]\times[0,\infty]$ and $[0,a]\times(a,\infty]$. Now

$$\tilde{\nu}((a,\infty]\times[0,\infty])=(\nu_\alpha\times G)((a,\infty]\times[0,\infty])=a^{-\alpha}<\infty,$$

and

$$\begin{aligned}\tilde{\nu}([0,a]\times(a,\infty])&=\nu((0,a]\times(a,\infty])=(\nu_\alpha\times G)(\{\mathbf{x}:0<x_1\leqslant a,x_1x_2>a\})\\&=\int\limits_{(1,\infty)}\left(\left(\frac{a}{x_2}\right)^{-\alpha}-a^{-\alpha}\right)G(dx_2)=a^{-\alpha}\int\limits_{(1,\infty)}x_2^{\alpha}\,G(dx_2)-G((1,\infty)).\end{aligned}$$

Thus $\tilde{\nu}$ is Radon iff $\int_{(1,\infty)}x_2^{\alpha}\,G(dx_2)<\infty$ iff G has finite α-th moment, which has been assumed.

Now we consider the vague convergence. We have already seen in Theorem 3.1 that, (i) implies vague convergence in (3.11) on D.

Let K be relatively compact in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$ with $\tilde{\nu}(\partial K)=0$. Choose $\varepsilon_k\downarrow 0$ such that $K_{\varepsilon_k}:=K\cap([\varepsilon_k,\infty]\times[0,\infty])$ satisfy $\tilde{\nu}(\partial K_{\varepsilon_k})=0$. Then

$$\liminf_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K\right]\geqslant\lim_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K_{\varepsilon_k}\right]=\nu(K_{\varepsilon_k})=\tilde{\nu}(K_{\varepsilon_k}).$$

Letting $k\to\infty$, and using the definition of $\tilde{\nu}$,

$$\liminf_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K\right]\geqslant\tilde{\nu}(K\cap D)=\tilde{\nu}(K).$$

Since K is relatively compact in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$, there exists $s\in(0,\infty)$ such that $K\subseteq[\mathbf{0},\mathbf{s}]^c$, where $\mathbf{s}=(s,s)$. Therefore,

$$\begin{aligned}\limsup_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K\right]&\leqslant\lim_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K_{\varepsilon_k}\right]+\limsup_{T\to\infty}T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)}\in K\cap([0,\varepsilon_k)\times(s,\infty])\right]\\&\leqslant\tilde{\nu}(K_{\varepsilon_k})+\limsup_{T\to\infty}T\,\mathrm{P}\left[\frac{X_1}{b(T)}<\varepsilon_k,\frac{X_2}{b(T)}>s\right]\\&\leqslant\tilde{\nu}(K_{\varepsilon_k})+s^{-\delta}\limsup_{T\to\infty}\mathrm{E}\left[\left(\frac{X_2}{b(T)}\right)^{\delta}\mathbf{1}_{\left[\frac{X_1}{b(T)}<\varepsilon_k\right]}\right].\end{aligned}$$

Letting $k \to \infty$, by (ii),

$$\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in K\right] \leqslant \tilde{\nu}(K).$$

Hence,

$$T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right] \xrightarrow{v} \tilde{\nu}(\cdot) \text{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}.$$

□

In fact, it is easily seen from Theorem 3.1 and Lemma 3.2 that the converse also holds, which is summarized in the following corollary.

Corollary 3.1. *Assume* $\mathbf{X} = (X_1, X_2)$ *is a random variable taking values in* $\mathbb{R}_+^2$*. Let* $b = \left(\frac{1}{1-F_{X_1}}\right)^{\leftarrow}$*. Suppose* $T\,\mathrm{P}[X_1 > b(T)] \to 1$*. Assume the moment condition:*

$$\lim_{\varepsilon\downarrow 0}\limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{X_2}{b(T)}\right)^{\delta} \mathbf{1}_{\left[\frac{X_1}{b(T)}\leqslant\varepsilon\right]}\right] = 0 \textit{ for some } \delta > 0.$$

Then

$$T\,\mathrm{P}\left[\left(\frac{X_1}{b(T)}, \frac{X_2}{X_1}\right) \in \cdot\right] \xrightarrow{v} \nu_\alpha \times G(\cdot) \textit{ on } D = (0,\infty] \times [0,\infty]$$

for some $\alpha > 0$*, and some probability measure* G *on* $[0,\infty]$ *with* $G((0,\infty)) = 1$ *and finite* α*-th moment, with* $\nu_\alpha((x,\infty]) = x^{-\alpha}$
iff

$$T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right] \xrightarrow{v} \tilde{\nu}(\cdot) \textit{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$$

for some Radon measure $\tilde{\nu}$ *satisfying*

$$\tilde{\nu}(\{\mathbf{x} : x_1 > u\}) > 0 \textit{ for some } u > 0. \tag{3.13}$$

In fact, $\tilde{\nu}$ *is homogeneous of order* $-\alpha$ *and is given as in (3.9) and (3.12).*

The importance of the above lemma and corollary lies in the fact that we could extend the multivariate regular variation condition to the set $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$, which is the natural domain for studying the vague convergence of $\frac{\mathbf{X}}{b}$. Note that under the set of assumptions (IA)-(IC) or (IIA)-(IIC), the assumptions of the above corollary hold and we get $\alpha = \alpha_L$ or α_R respectively. Also for $X_1 = L_k$ or R_k, which have regularly varying tail, $T\,\mathrm{P}\left[\frac{\mathbf{X}}{b(T)} \in \cdot\right]$ converges to the measure ν_α with appropriate α, and hence the positivity condition (3.13) is satisfied. The specialization of Corollary 3.1 to case (I) is given next.

Corollary 3.2. *Assume* (L_1, R_1) *is a random variable taking values in* $\mathbb{R}_+^2$*. Suppose* L_1 *has a regularly varying tail of order* $-\alpha_L$*. Also assume the condition* (IC) *holds.*

Then the condition (IB) *holds iff*

$$T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in \cdot\right] \xrightarrow{v} \tilde{\nu}(\cdot) \textit{ on } [\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\} \tag{3.14}$$

for some Radon measure $\tilde{\nu}$ *satisfying* (3.13) *and which happens to be homogeneous of order* $-\alpha$*.*

Note (3.14) is a strengthening of (IB′) as it extends the vague convergence to the natural domain $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$. Thus, in presence of the condition (IC), (IB) is equivalent to the fact that (L_1, L_1R_1) is multivariate regularly varying. Similar results hold for case (II).

Next we consider another interesting equivalent statement of (IB).

Lemma 3.3. *Assume (L,R) takes values in $\mathbb{R}_+^2$, where L has a marginal with regularly varying tail of index $-\alpha_L$. Then*

$$T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)},R_1\right)\in\cdot\right]\xrightarrow{v}\nu_{\alpha_L}\times G(\cdot)\ \text{on } D \tag{3.15}$$

iff

$$\mathrm{P}[R_1\in\cdot|L_1>x]\Rightarrow G(\cdot)\ \text{as } x\to\infty, \tag{3.16}$$

where the second convergence is the usual weak convergence.

Proof. Observe that (3.15) holds iff for all $x>0$ and $y\geqslant 0$, we have

$$T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}>x,R_1\leqslant y\right]\to x^{-\alpha_L}G(y). \tag{3.17}$$

However,

$$T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}>x,R_1\leqslant y\right]=T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}>x\right]\mathrm{P}[R_1\leqslant y|L_1>b_L(T)x],$$

and, by the nature of the marginal distribution of L, we have,

$$T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}>x\right]\to x^{-\alpha_L}.$$

Thus (3.17) holds iff

$$\mathrm{P}[R_1\leqslant y|L_1>b_L(T)x]\to G(y)$$

as $T\to\infty$, for all $y\geqslant 0$. This is equivalent to (3.16), since $b_L(T)\to\infty$. □

Now let us consider sufficient conditions under which the conditions (IA)-(IC) or (IIA)-(IIC) hold. We check that the conditions are indeed generalizations of independence, i. e. they hold in particular, when F is a product measure. We first check the moment condition. We only consider the condition (IC). The case for the condition (IIC) is exactly similar.

Lemma 3.4. *Let L_1 and R_1 be independent random variables taking values in $(0,\infty)$ with respective marginal distributions F_L and F_R and quantile functions b_L and b_R. Let $\overline{F}_L\in RV_{-\alpha_L}$ and $\overline{F}_R\in RV_{-\alpha_R}$ with $\alpha_L<\alpha_R$. Then (IC) holds for $\delta\in(\alpha_L,\alpha_R)$.*

Proof. We have by independence and $\delta<\alpha_R$ and Karamata's theorem that as $T\to\infty$

$$\begin{aligned}
T\,\mathrm{E}\left[\left(\frac{L_1}{b_L(T)}R_1\right)^\delta\mathbf{1}_{\left[\frac{L_1}{b_L(T)}\leqslant\varepsilon\right]}\right]&=\mathrm{E}(R_1^\delta)\frac{T}{(b_L(T))^\delta}\,\mathrm{E}\left[L_1^\delta\mathbf{1}_{\left[\frac{L_1}{b_L(T)}\leqslant\varepsilon\right]}\right]\\
&=\mathrm{E}(R_1^\delta)\frac{T}{(b_L(T))^\delta}\int_{(0,b_L(T)\varepsilon]}\delta x^{\delta-1}\,\mathrm{P}[x<L_1\leqslant b_L(T)\varepsilon]\,dx\\
&=\mathrm{E}(R_1^\delta)\frac{T}{(b_L(T))^\delta}\left[\int_{(0,b_L(T)\varepsilon]}\delta x^{\delta-1}\overline{F}_L(x)\,dx-(b_L(T)\varepsilon)^\delta\,\overline{F}_L(b_L(T)\varepsilon)\right]\\
&\sim\mathrm{E}(R_1^\delta)\frac{T}{(b_L(T))^\delta}\left[(b_L(T)\varepsilon)^\delta\overline{F}_L(b_L(T)\varepsilon)\frac{\delta}{\delta-\alpha_L}-(b_L(T)\varepsilon)^\delta\,\overline{F}_L(b_L(T)\varepsilon)\right]\\
&=\frac{\alpha_L}{\delta-\alpha_L}\,\mathrm{E}(R_1^\delta)\varepsilon^\delta T\overline{F}_L(b_L(T)\varepsilon)\\
&\sim\frac{\alpha_L}{\delta-\alpha_L}\,\mathrm{E}(R_1^\delta)\varepsilon^{\delta-\alpha_L}\to 0
\end{aligned}$$

as $\epsilon \to 0$, since $\delta > \alpha_L$. Thus,

$$\lim_{\varepsilon \downarrow 0} \limsup_{T\to\infty} \mathrm{E}\left[\left(\frac{L_1}{b_L(T)} R_1\right)^{\delta} \mathbf{1}_{\left[\frac{L_1}{b_L(T)} < \varepsilon\right]}\right] = 0, \quad \forall \delta \in (\alpha_L, \alpha_R).$$

□

Next we consider the condition of asymptotic independence. This also holds in particular under independence. We again consider the case (IB) only.

Lemma 3.5. *Let L_1 and R_1 be independent random variables taking values in $(0,\infty)$ with respective marginal distributions F_L and F_R and quantile functions b_L and b_R. Let $\overline{F}_L \in RV_{-\alpha_L}$ and $\overline{F}_R \in RV_{-\alpha_R}$ with $\alpha_L < \alpha_R$. Then* (IB) *holds.*

Proof. For $S_1 \in \mathcal{B}((0,\infty])$ and $S_2 \in \mathcal{B}([0,\infty])$,

$$\begin{aligned} \lim_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}, R_1\right) \in S_1 \times S_2\right] &= \mathrm{P}[R_1 \in S_2] \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)} \in S_1\right] && \text{by independence} \\ &\to \mathrm{P}[R_1 \in S_2]\nu_{\alpha_L}(S_1) && \text{since } \overline{F}_L \in RV_{-\alpha_L}. \end{aligned}$$

So (IB) holds with $G = F_R$. Since $\mathrm{P}[R_1 \in (0,\infty)] = 1$, we have $G((0,\infty)) = 1$. Also $\alpha_L < \alpha_R$ implies G has finite α_L-th moment. □

Finally, empowered with all these tools we study the quantile function of the product, b_P. Again, we consider the case I only.

Theorem 3.2. *Suppose (L_1, R_1) is a random variable on $\mathbb{R}_+^2$, where L_1 has a regularly varying tail of index $-\alpha_L$. Let (L_1, R_1) satisfy the conditions* (IB) *and* (IC)*. Then*

$$T\,\mathrm{P}\left[\frac{L_1 R_1}{b_L(T)} > z\right] \sim z^{-\alpha_L} \int_0^\infty u^{\alpha_L}\, G(du),$$

and hence

$$b_P(T) \sim \left(\int_0^\infty u^{\alpha_L}\, G(du)\right)^{\frac{1}{\alpha_L}} b_L(T).$$

Proof. Let

$$A_\varepsilon = \{(x,y) : \varepsilon < x < \varepsilon^{-1}, xy > z\}.$$

Note A_ε is relatively compact in D and

$$\partial A_\varepsilon = \left(\{\varepsilon\} \times \left[\frac{z}{\varepsilon}, \infty\right]\right) \cup \left(\{\varepsilon^{-1}\} \times [z\varepsilon, \infty]\right) \cup \{(x,y) : \varepsilon < x < \varepsilon^{-1}, xy = z\}.$$

Choose a sequence $\varepsilon_k \downarrow 0$ such that $(\nu_{\alpha_L} \times G)(\partial A_{\varepsilon_k}) = 0$, for all k. Then

$$\liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)} R_1 > z\right] \geqslant \lim_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(n)}, R_1\right) \in A_{\varepsilon_k}\right] = (\nu_{\alpha_L} \times G)(A_{\varepsilon_k}) \text{ by (IB)}.$$

Taking the limit as $k \to \infty$,

$$\begin{aligned} \liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)} R_1 > z\right] &\geqslant (\nu_{\alpha_L} \times G)(\{(x,y) : xy > z, 0 < x < \infty\}) \\ &= (\nu_{\alpha_L} \times G)(\{(x,y) : xy > z, 0 < x \leq \infty\}), \end{aligned}$$

since $\nu_{\alpha_L}(\{\infty\}) = 0$. Also, since $\{(x,y) \in D : x \geq \varepsilon_k^{-1}\}$ is a $\nu_{\alpha_L} \times G$ continuity set, we have,

$$\begin{aligned}
\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > z\right] \leqslant\ & \lim_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}, R_1\right) \in A_{\varepsilon_k}\right] + \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)} \geqslant \varepsilon_k^{-1}\right] \\
& + \limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > z, \frac{L_1}{b_L(T)} \leqslant \varepsilon_k\right] \\
=\ & (\nu_{\alpha_L} \times G)(A_{\varepsilon_k}) + \nu_{\alpha_L}\left([\varepsilon_k^{-1}, \infty]\right) \\
& + \limsup_{T\to\infty} T\,\mathrm{P}\left[\left(\frac{L_1}{b_L(T)}R_1\right)\mathbf{1}_{\left[\frac{L_1}{b_L(T)} \leqslant \varepsilon_k\right]} > z\right] \\
\leqslant\ & (\nu_{\alpha_L} \times G)(A_{\varepsilon_k}) + \varepsilon_k^{\alpha_L} + z^{-\delta} \limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{L_1}{b_L(T)}R_1\right)^{\delta}\mathbf{1}_{\left[\frac{L_1}{b_L(T)} \leqslant \varepsilon_k\right]}\right].
\end{aligned}$$

Taking limits as $k \to \infty$, and using (IC) and the fact $\alpha_L > 0$

$$\begin{aligned}
\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > z\right] &\leqslant (\nu_{\alpha_L} \times G)(\{(x,y) : xy > z, 0 < x < \infty\}) \\
&= (\nu_{\alpha_L} \times G)(\{(x,y) \in D : xy > z\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\limsup_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > z\right] &\leqslant (\nu_{\alpha_L} \times G)(\{(x,y) \in D : xy > z\} \\
&= \int_0^\infty \nu_{\alpha_L}\left(\left(\frac{z}{u}, \infty\right]\right) G(du) = z^{-\alpha_L} \int_0^\infty u^{\alpha_L}\, G(du).
\end{aligned}$$

Therefore

$$\lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 \left(\int_0^\infty u^{\alpha_L}\, G(du)\right)^{-\frac{1}{\alpha_L}} > z\right] = z^{-\alpha_L},$$

and hence

$$b_P(T) \sim \left(\int_0^\infty u^{\alpha_L}\, G(du)\right)^{\frac{1}{\alpha_L}} b_L(T) \sim b_L\left(\int_0^\infty u^{\alpha_L}\, G(du) T\right).$$

□

We get a similar result for the case II by interchanging the roles of L_1 and R_1 and thus the quantile function of the product, b_P is a regularly varying function of index $\frac{1}{\alpha_P}$, where $\alpha_P := \alpha_L \wedge \alpha_R$, i. e., the product has a behavior similar to the factor random variable with the heavier tail.

4. Products, asymptotic independence, multivariate regular variation

In this section, we discuss some examples to illustrate the concept of asymptotic independence discussed in section 3 and show its difference from the usual concept used in extreme value theory.

The first example shows that the usual concept and (3.1) might hold, but the asymptotic independence, as defined by us, can fail.

Example 1. Let X and Y be random variables with regularly varying tails of indices $-\alpha_X$ and $-\alpha_Y$, with $1 < \alpha_X, \alpha_Y < 2$. Let b_X and b_Y be the corresponding quantile functions, defined as in (2.5). Let B be a Bernoulli random variable with probability of success 0.5, independent of X and Y. Then define

$$(L, R) = B(X, 0) + (1 - B)(0, Y).$$

Then we have

$$\begin{aligned} T\,\mathrm{P}\left[\left(\frac{L}{b_X(T)}, \frac{R}{b_Y(T)}\right) \in \cdot\right] &= \frac{1}{2} T\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, 0\right) \in \cdot\right] + \frac{1}{2} T\,\mathrm{P}\left[\left(0, \frac{Y}{b_Y(T)}\right) \in \cdot\right] \\ &\xrightarrow{v} \frac{1}{2}\nu_{\alpha_X} \times \varepsilon_0(\cdot) + \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_Y}(\cdot) \quad \text{on } [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \end{aligned}$$

where ε_0 is the Dirac measure at 0. Thus the limiting measure is concentrated on the axes; i. e., on the set $\big(\{0\} \times (0, \infty]\big) \cup \big((0, \infty] \times \{0\}\big)$.

But, observe that $LR \equiv 0$ and hence we do not get anything interesting about the product.

In the previous example, (3.1) holds, yet we observe that the product is degenerate at zero. In the next example, again (3.1) holds. The product LR is not degenerate, but still we cannot make any interesting conclusion about the tail behavior of the product, as the conditions (IB) and (IIB) fail.

Example 2. Let X, Y and B be as in the previous example. Define

$$(L, R) = B(X, \sqrt{X}) + (1 - B)(\sqrt{Y}, Y).$$

Suppose $\alpha_Y < \alpha_X$, so that Y has a heavier tail. Now observe that $\alpha_X < 2 < 2\alpha_Y$, since $\alpha_Y > 1$, and similarly also $\alpha_Y < 2 < 2\alpha_X$. So we have $\sqrt{b_X(T)} = o(b_Y(T))$ and $\sqrt{b_Y(T)} = o(b_X(T))$ as $T \to \infty$. Then

$$\begin{aligned} T\,\mathrm{P}\left[\left(\frac{L}{b_X(T)}, \frac{R}{b_Y(T)}\right) \in \cdot\right] &= \frac{T}{2}\,\mathrm{P}\left[\left(\frac{X}{b_X(T)}, \frac{\sqrt{X}}{b_Y(T)}\right) \in \cdot\right] + \frac{T}{2}\,\mathrm{P}\left[\left(\frac{\sqrt{Y}}{b_X(T)}, \frac{Y}{b_Y(T)}\right) \in \cdot\right] \\ &\xrightarrow{v} \frac{1}{2}\nu_{\alpha_X} \times \varepsilon_0(\cdot) + \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_Y}(\cdot) \quad \text{on } [\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}, \end{aligned}$$

and the limiting measure is concentrated on the axes.

However,

$$\begin{aligned} T\,\mathrm{P}\left[\frac{L}{b_X(T)} > x, R \leqslant y\right] &= \frac{T}{2}\,\mathrm{P}[X > b_X(T)x, \sqrt{X} \leqslant y] + \frac{T}{2}\,\mathrm{P}[\sqrt{Y} > b_X(T)x, Y \leqslant y] \\ &= \frac{T}{2}\,\mathrm{P}[b_X(T)x < X \leqslant y^2] + \frac{T}{2}\,\mathrm{P}[b_X(T)^2 x^2 < Y \leqslant y] \to 0, \end{aligned}$$

since $b_X(T) \to \infty$. Thus,

$$T\,\mathrm{P}\left[\left(\frac{L}{b_X(T)}, R\right) \in \cdot\right] \xrightarrow{v} 0 \qquad \text{on } D.$$

So, the condition (IB) fails. Similarly we can show that the limiting measure in the condition (IIB) is also identically zero.

Also $LR = BX^{3/2} + (1 - B)Y^{3/2}$. Then, since $\alpha_Y < \alpha_X$, we have,

$$\mathrm{P}[LR > x] \sim \frac{1}{2}\,\mathrm{P}[Y^{3/2} > x],$$

which is regularly varying of index $-\frac{2}{3}\alpha_Y$. Since $\mathrm{P}[L > \cdot] \in RV_{-\alpha_X}$, $\mathrm{P}[R > \cdot] \in RV_{-\alpha_Y}$, the tail behavior of LR cannot be concluded from the tail behavior of the factors even though (3.1) holds.

This example, as well as Example 1 reinforce the idea that the classical notion of asymptotic independence from extreme value theory contains little information about the tail behavior of the product.

For the next example, we need the following result.

Proposition 4.1. *Suppose (U, V) is multivariate regularly varying in the sense that there exists regularly varying functions b_U, b_V, such that*

$$T\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, \frac{V}{b_V(T)}\right) \in \cdot\right] \xrightarrow{v} \nu(\cdot) \not\equiv 0 \tag{4.1}$$

on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ and $\nu\Big(\big(\{\infty\} \times (0, \infty]\big) \cup \big((0, \infty] \times \{\infty\}\big)\Big) = 0$ and $\nu\big((\mathbf{0}, \boldsymbol{\infty}]\big) > 0$.

Then for some $\alpha_U > 0$, $\alpha_V > 0$,we have

$$\mathrm{P}[U > \cdot] \in RV_{-\alpha_U},$$
$$\mathrm{P}[V > \cdot] \in RV_{-\alpha_V}$$

and

$$\mathrm{P}[UV > \cdot] \in RV_{-\frac{\alpha_U \alpha_V}{\alpha_U + \alpha_V}}.$$

Proof. Let b_U and b_V be regularly varying with indices $1/\alpha_U$ and $1/\alpha_V$ respectively. Now, for any $u > 0$, such that $(u, \infty] \times [0, \infty]$ is a ν-continuity set, we have,

$$T\,\mathrm{P}\left[\frac{U}{b_U(T)} > u\right] = T\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, \frac{V}{b_V(T)}\right) \in (u, \infty] \times [0, \infty]\right] \to \nu\big((u, \infty] \times [0, \infty]\big) =: K_u,$$

where K_u is a positive and finite for some $u_0 > 0$, since $\nu\big((\mathbf{0}, \boldsymbol{\infty}]\big) > 0$.

Now, we have, for any $u > 0$,

$$\begin{aligned} T\,\mathrm{P}\left[\frac{U}{b_U(T)} > u\right] &= T\,\mathrm{P}\left[\frac{U}{\frac{u}{u_0} b_U(T)} > u_0\right] \\ &\sim \left(\frac{u}{u_0}\right)^{-\alpha} \left(\frac{u}{u_0}\right)^{\alpha} T\,\mathrm{P}\left[\frac{U}{b_U\Big(\big(\frac{u}{u_0}\big)^{\alpha} T\Big)} > u_0\right] \\ &\to (u_0)^{\alpha} K_{u_0} u^{-\alpha} \end{aligned}$$

and hence $\mathrm{P}[U > \cdot] \in RV_{-\alpha_U}$. Similarly, we can check that $\mathrm{P}[V > \cdot] \in RV_{-\alpha_V}$.

Define for $x > 0$ and any positive number K,

$$A_{K,x} := \{(u, v) : uv > x, u \leqslant K, v \leqslant K\}.$$

Then, for any $x > 0$, we have,

$$T\,\mathrm{P}\left[\frac{UV}{b_U(T) b_V(T)} > x\right] \geqslant T\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, \frac{V}{b_V(T)}\right) \in A_{K,x}\right].$$

Then, letting T got to ∞ first, and then letting K go to ∞ through a sequence so that $A_{K,x}$ is a ν-continuity set, we have

$$\limsup_{T \to \infty} T\,\mathrm{P}\left[\frac{UV}{b_U(T) b_V(T)} > x\right] \geqslant \nu\big(\{(u, v) : uv > x\}\big).$$

On the other hand, we have,

$$T\,\mathrm{P}\left[\frac{UV}{b_U(T)b_V(T)} > x\right] \leqslant T\,\mathrm{P}\left[\left(\frac{U}{b_U(T)}, \frac{V}{b_V(T)}\right) \in A_{K,x}\right] + T\,\mathrm{P}\left[\frac{U}{b_U(T)} > K\right] + T\,\mathrm{P}\left[\frac{U}{b_U(T)} > K\right]$$

Now, by regularly varying tails of U and V, the last two terms converge to $K^{-\alpha_U}$ and $K^{-\alpha_V}$ respectively, as $T \to \infty$. Then letting K go to ∞ through a sequence so that $A_{K,x}$ is a ν-continuity set, they go to zero. Hence, we have

$$\liminf_{T\to\infty} T\,\mathrm{P}\left[\frac{UV}{b_U(T)b_V(T)} > x\right] \leqslant \nu\big(\{(u,v) : uv > x\}\big).$$

Thus,

$$T\,\mathrm{P}\left[\frac{UV}{b_U(T)b_V(T)} > x\right] \to \nu\big(\{(u,v) : uv > x\}\big).$$

Then, since $b_U b_V$ is a regularly varying function of index $\frac{\alpha_U+\alpha_V}{\alpha_U\alpha_V}$, and $\nu\big(\{(u,v) : uv > x\}\big) > 0$ for some $x > 0$, we have, arguing as in the case of U,

$$\mathrm{P}[UV > \cdot] \in RV_{-\frac{\alpha_U\alpha_V}{\alpha_U+\alpha_V}}.$$

□

Now we consider an interesting example, where the vague limits in (3.1) are two different non-zero measures for two different choices of b_L and b_R on the sets $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$ and $(\mathbf{0},\boldsymbol{\infty}]$. In the first case the limiting measure is supported on the axes only. But in the second case there is a non-degenerate limit although with different sets of scaling. Further modification of the scaling shows that the conditions (IB) and (IIB) hold, but the conditions (IC) and (IIC) fail. We still fail to conclude anything meaningful about the tail behavior of the product using the tail behavior of the factors.

Example 3. Suppose we have independent vectors (U, V), (X, Y) which are independent of the Bernoulli random variable B with probability of success 0.5. We assume

(i) The random variables (X, Y) are independent with

$$\mathrm{P}[X > \cdot] \in RV_{-\alpha_1},\ \mathrm{P}[Y > \cdot] \in RV_{-\alpha_2}$$

with

$$1 < \alpha_2 < \alpha_1 < 2,$$

so that Y has the heavier tail.

(ii) The random variables (U, V) are dependent with multivariate regularly varying distribution in the sense that there exists regularly varying functions b_3 and b_4 of indices $1/\alpha_3$ and $1/\alpha_4$ respectively, such that

$$T\,\mathrm{P}\left[\left(\frac{U}{b_3(T)}, \frac{V}{b_4(T)}\right) \in \cdot\right] \xrightarrow{v} \nu_{U,V}(\cdot)$$

on $[\mathbf{0},\boldsymbol{\infty}] \setminus \{\mathbf{0}\}$, where $\nu_{U,V}\big((\mathbf{0},\boldsymbol{\infty}]\big) > 0$ but $\nu_{U,V}\big(\{\infty\} \times (0,\infty] \cup (0,\infty] \times \{\infty\}\big) = 0$ and $1 < \alpha_4 < \alpha_3 < 2$. Then by Proposition 4.1,

$$\mathrm{P}[U > \cdot] \in RV_{-\alpha_3} \text{ and } \mathrm{P}[V > \cdot] \in RV_{-\alpha_4},$$

and V has a heavier tail.

(iii) Assume further that

$$\alpha_1 < \alpha_3,\ \alpha_2 < \alpha_4. \tag{4.2}$$

We define

$$(L, R) = B(U, V) + (1 - B)(X, Y).$$

Then we have the following conclusions.

(1) We have

$$P[L > x] = \frac{1}{2}P[U > x] + \frac{1}{2}P[X > x] \in RV_{-\alpha_1},$$
$$P[R > x] = \frac{1}{2}P[V > x] + \frac{1}{2}P[Y > x] \in RV_{-\alpha_2}$$

so that R has the heavier tail.

(2) Define the measure $\nu_0 = \frac{1}{2}\varepsilon_0 \times \nu_{\alpha_2} + \frac{1}{2}\nu_{\alpha_1} \times \varepsilon_0$, and we have on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$

$$T\,\mathrm{P}\left[\left(\frac{L}{b_1(T)}, \frac{R}{b_2(T)}\right) \in \cdot\right] \xrightarrow{v} \nu_0,$$

where b_1 and b_2 are quantile functions of X and Y respectively. To see this, note

$$T\,\mathrm{P}\left[\left(\frac{L}{b_1(T)}, \frac{R}{b_2(T)}\right) \in \cdot\right] = \frac{T}{2}\,\mathrm{P}\left[\left(\frac{U}{b_1(T)}, \frac{V}{b_2(T)}\right) \in \cdot\right] + \frac{T}{2}\,\mathrm{P}\left[\left(\frac{X}{b_1(T)}, \frac{Y}{b_2(T)}\right) \in \cdot\right],$$

and the first term goes to zero since $b_i \in RV_{\frac{1}{\alpha_i}}, \quad i = 1, \dots, 4$, and (4.2) imply

$$b_3(T) = o(b_1(T)), \quad b_4(T) = o(b_2(T)),$$

and the vague limit of the second term is ν_0 on $[\mathbf{0}, \boldsymbol{\infty}] \setminus \{\mathbf{0}\}$.

(3) We obtain a different vague limit on the cone $(\mathbf{0}, \boldsymbol{\infty}]$. For $x > 0, y > 0$, we have, using assumption (ii), that

$$\begin{aligned} T\,\mathrm{P}\left[\left(\frac{L}{b_3(T)}, \frac{R}{b_4(T)}\right) \in (x, \infty] \times (y, \infty]\right] =& \frac{1}{2}T\,\mathrm{P}\left[\frac{U}{b_3(T)} > x \frac{V}{b_4(T)} > y\right] \\ &+ \frac{1}{2}T\,\mathrm{P}\left[\frac{X}{b_3(T)} > x\right]\mathrm{P}\left[\frac{Y}{b_4(T)} > y\right] \\ \to& \frac{1}{2}\nu_{U,V}\big((x, \infty] \times (y, \infty]\big) + 0 \end{aligned}$$

and thus the vague limit on $(\mathbf{0}, \boldsymbol{\infty}]$ is $\frac{1}{2}\nu_{U,V}$. To verify the limit of 0 for the second term, note $b_i^{\leftarrow} \in RV_{\alpha_i}$, $i = 1, \dots, 4$ and as $T \to \infty$

$$\begin{aligned} TP[X > b_3(T)x]P[Y > b_4(T)y] \sim& \frac{T}{b_1^{\leftarrow}(b_3(T)x)b_2^{\leftarrow}(b_4(T)y)} \\ \sim& \frac{Tx^{-\alpha_1}y^{-\alpha_2}}{b_1^{\leftarrow} \circ b_3(T) b_2^{\leftarrow} \circ b_4(T)} \end{aligned}$$

which as a function of T is regularly varying with index $1 - \frac{\alpha_1}{\alpha_3} - \frac{\alpha_2}{\alpha_4}$. The result follows if we show $\frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_4} > 1$. However

$$\frac{\alpha_1}{\alpha_3} + \frac{\alpha_2}{\alpha_4} > \frac{1}{2}(\alpha_1 + \alpha_2) > \frac{2}{2} = 1,$$

since $1 < \alpha_i < 2$, for $i = 1, \dots, 4$.

(4) We have that

$$\mathrm{P}[LR > \cdot] \in RV_{-\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4}}, \text{ and } \frac{1}{2} < \frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4} < 1.$$

Note

$$LR = B\,UV + (1-B)\,XY,$$

so that

$$\mathrm{P}[LR > \cdot] = \frac{1}{2}\,\mathrm{P}[UV > \cdot] + \frac{1}{2}\,\mathrm{P}[XY > \cdot].$$

From [4] or Lemma 3.4, Lemma 3.5 and Theorem 3.2, we have that $\mathrm{P}[XY > \cdot] \in RV_{-\alpha_2}$, and from Proposition 4.1, we have that $\mathrm{P}[XY > \cdot] \in RV_{-\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4}}$. But

$$\frac{\alpha_3\alpha_4}{\alpha_3+\alpha_4} < 1 < \alpha_2.$$

So we have

$$\mathrm{P}[LR > \cdot] \sim \frac{1}{2}\,\mathrm{P}[UV > \cdot],$$

which is surprisingly heavy – it has no first moment – considering the facts that

$$\mathrm{P}[L > \cdot] \in RV_{-\alpha_1} \text{ and } \mathrm{P}[R > \cdot] \in RV_{-\alpha_2},$$

and $1 < \alpha_1, \alpha_2 < 2$.

We conclude that the tail of the product is hidden from a condition like (3.1) or knowledge of the marginal distributions.

(5) We have that conditions (IB) and (IIB) hold but conditions (IC) and (IIC) fail.

For (IB) we have,

$$T\,\mathrm{P}\left[\frac{L}{b_1(T)} > x, R \leqslant y\right] = \frac{T}{2}\,\mathrm{P}\left[\frac{U}{b_1(T)} > x, V \leqslant y\right] + \frac{T}{2}\,\mathrm{P}\left[\frac{X}{b_1(T)} > x\right]\mathrm{P}[Y \leqslant y].$$

Then the second term converges to $\left(\frac{1}{2}\nu_{\alpha_1} \times F_Y\right)\left((x,\infty] \times [0,y]\right)$, where F_Y denotes the distribution function of Y. Also,

$$\frac{T}{2}\,\mathrm{P}\left[\frac{U}{b_1(T)} > x, V \leqslant y\right] \leqslant \frac{T}{2}\,\mathrm{P}\left[\frac{U}{b_3(T)} > \frac{b_1(T)}{b_3(T)}x\right] \to 0.$$

since $b_3(T) = o(b_1(T))$. Thus if we define $b_L(T) = 2^{-1/\alpha_1} b_1(T)$, then

$$T\,\mathrm{P}\left[\left(\frac{L}{b_L(T)}, R\right) \in \cdot\right] \xrightarrow{v} \left(\nu_{\alpha_1} \times F_Y\right)(\cdot)$$

on D.

Similarly, condition (IIB) holds with $b_R(T) = 2^{-1/\alpha_2} b_2(T)$.

For (IC), observe that,

$$T\,\mathrm{E}\left[\left(\frac{L}{b_1(T)}R\right)^{\delta}\mathbf{1}_{\left[\frac{L}{b_1(T)}<\varepsilon\right]}\right] = \frac{T}{2}\,\mathrm{E}\left[\left(\frac{X}{b_1(T)}\right)^{\delta}\mathbf{1}_{\left[\frac{X}{b_1(T)}<\varepsilon\right]}\right]\mathrm{E}\left[Y^{\delta}\right] + \frac{T}{2}\,\mathrm{E}\left[\left(\frac{U}{b_1(T)}V\right)^{\delta}\mathbf{1}_{\left[\frac{U}{b_1(T)}<\varepsilon\right]}\right] \tag{4.3}$$

So, if $\delta \geqslant \alpha_2$, then $\mathrm{E}\left(Y^\delta\right) = \infty$ and hence condition (IC) fails. Also a closer look at the proof of Lemma 3.4 will show

$$\lim_{\varepsilon \downarrow 0} \limsup_{T\to\infty} T\,\mathrm{E}\left[\left(\frac{X}{b_1(T)}\right)^\delta \mathbf{1}_{\left[\frac{X}{b_1(T)}<\varepsilon\right]}\right] = 0$$

iff $\delta > \alpha_1 > \alpha_2$. Thus, for no positive δ, the first term in the RHS of (4.3) can go to zero. Since $\frac{b_L(T)}{b_1(T)} = 2^{-1/\alpha_1}$, condition (IC) fails even if we scale by b_L.

For (IIC), observe that,

$$\begin{aligned} T\,\mathrm{E}\left[\left(\frac{R}{b_2(T)}L\right)^\delta \mathbf{1}_{\left[\frac{R}{b_2(T)}<\varepsilon\right]}\right] &= \frac{T}{2}\,\mathrm{E}\left[\left(\frac{Y}{b_2(T)}\right)^\delta \mathbf{1}_{\left[\frac{Y}{b_2(T)}<\varepsilon\right]}\right]\mathrm{E}(X^\delta) \\ &\quad + \frac{T}{2}\,\mathrm{E}\left[\left(\frac{V}{b_2(T)}U\right)^\delta \mathbf{1}_{\left[\frac{V}{b_2(T)}<\varepsilon\right]}\right] \end{aligned}$$

For any $\delta > 0$, we have,

$$\begin{aligned} T\,\mathrm{E}\left[\left(\frac{V}{b_2(T)}U\right)^\delta \mathbf{1}_{\left[\frac{V}{b_2(T)}<\varepsilon\right]}\right] &\geqslant \mathrm{E}\left[\left(\frac{V}{b_2(T)}U\right)^\delta \mathbf{1}_{\left[\frac{V}{b_2(T)}<\varepsilon, \frac{UV}{b_3(T)b_4(T)}>1\right]}\right] \\ &\geqslant \left(\frac{b_3(T)b_4(T)}{b_2(T)}\right)^\delta T\,\mathrm{P}\left[\frac{V}{b_2(T)} < \varepsilon, \frac{UV}{b_3(T)b_4(T)} > 1\right] \end{aligned}$$

Now,

$$\frac{b_3(T)b_4(T)}{b_2(T)} \in RV_{\frac{1}{\alpha_3}+\frac{1}{\alpha_4}-\frac{1}{\alpha_2}}.$$

Since $\frac{1}{\alpha_3} + \frac{1}{\alpha_4} > \frac{1}{2} + \frac{1}{2} = 1 > \frac{1}{\alpha_2}$, and $\delta > 0$, we have,

$$\left(\frac{b_3(T)b_4(T)}{b_2(T)}\right)^\delta \to \infty.$$

Also, we have

$$T\,\mathrm{P}\left[\frac{V}{b_2(T)} < \varepsilon, \frac{UV}{b_3(T)b_4(T)} > 1\right] = T\,\mathrm{P}\left[\frac{UV}{b_3(T)b_4(T)} > 1\right] - T\,\mathrm{P}\left[\frac{V}{b_2(T)} \geqslant \varepsilon, \frac{UV}{b_3(T)b_4(T)} > 1\right].$$

The second term is bounded by $T\,\mathrm{P}\left[\frac{V}{b_2(T)} \geqslant \varepsilon\right]$, which goes to zero, since $b_4(T) = o(b_2(T))$. The first term goes to $\nu\big(\{(u,v) : uv > 1\}\big)$, by Proposition 4.1. Thus

$$T\,\mathrm{E}\left[\left(\frac{V}{b_2(T)}U\right)^\delta \mathbf{1}_{\left[\frac{V}{b_2(T)}<\varepsilon\right]}\right] \to \infty.$$

Since $\frac{b_R(T)}{b_2(T)} = 2^{-\frac{1}{\alpha_2}}$, condition (IIC) also fails.

So we observe that the asymptotic independence condition, as we have defined alone is not enough to conclude about the tail behavior of the product and we need to have some truncated moment condition.

The examples in this section justify our conditions of asymptotic independence and truncated moments and show neither of the conditions can be dropped, if we expect any meaningful result.

5. Lévy Approximation

We now give a Lévy approximation to the cumulative input process when input rates are random. Observe from the Theorem 3.2 that the product L_1R_1 has a tail of index $\alpha_P := \alpha_L \wedge \alpha_R$, and hence has a finite mean. Let us call it $\mu_P := \mathrm{E}(L_1R_1)$. Then we have the following asymptotic behavior of the process $A(t)$, defined as in (2.1), measured at a large scale.

Theorem 5.1. *Under assumptions (2.2)–(2.4) and* (IA)–(IC) *or* (IIA)–(IIC), *we have*

$$X^{(T)}(\cdot) \stackrel{\text{fidi}}{\to} X_{\alpha_P}(\cdot),$$

where

$$X^{(T)}(t) = \frac{A(Tt) - \lambda T t \mu_P}{b_P(T)}$$

and X_α is a mean 0, skewness 1, α-stable Lévy motion with scale parameter $\left(\frac{\lambda}{C_\alpha}\right)^{\frac{1}{\alpha}}$ and

$$C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}.$$

We shall prove the theorem in two parts. First we prove the one-dimensional convergence and then we prove the finite dimensional convergence for any number of dimensions.

5.1. **One-dimensional convergence.** For the analysis, it helps to consider the Poisson point process,

$$M = \sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k, L_k, R_k)}$$

with mean measure $\lambda\, dt \times F$ on $(0,\infty)^3$.

The random variable $A(T)$ is a function of the random measure restricted to $\mathcal{R}^{(T)} = \{(x,y,z) \in (0,\infty)^3 : x < T\}$. It helps to split $\mathcal{R}^{(T)}$ into two disjoint sets

$$\mathcal{R}_1^{(T)} = \{(x,y,z) \in (0,\infty)^3 : x + y \leqslant T\},$$
$$\mathcal{R}_2^{(T)} = \{(x,y,z) \in (0,\infty)^3 : x < T < x + y\}.$$

The corresponding input processes are

$$A_1(T) = \sum_{k=1}^{\infty} R_k L_k \mathbf{1}_{\left[(\Gamma_k, L_k, R_k) \in \mathcal{R}_1^{(T)}\right]}, \tag{5.1}$$

$$A_2(T) = \sum_{k=1}^{\infty} R_k (T - \Gamma_k) \mathbf{1}_{\left[(\Gamma_k, L_k, R_k) \in \mathcal{R}_2^{(T)}\right]}. \tag{5.2}$$

with $A(T) = A_1(T) + A_2(T)$. Since $A_i(T)$, $i = 1,2$ are functions of $M|_{\mathcal{R}_i^{(T)}}$, $i = 1,2$ respectively, and $\mathcal{R}_1(T) \cap \mathcal{R}_2(T) = \emptyset$, we have $A_1(T)$ and $A_2(T)$ are independent.

Now,

$$\mathrm{E}\left(M\left(\mathcal{R}_1^{(T)}\right)\right) = \int_{x=0}^{T} \int_{y\in(0,T-x]} \int_{z\in(0,\infty)} \lambda\, dx\, F(dy, dz)$$

$$= \lambda \int_{x=0}^{T} F_L(T-x)\,dx = \lambda \int_{0}^{T} F_L(x)\,dx =: \lambda \widehat{F_L}(T),$$

and

$$\mathrm{E}\left(M\left(\mathcal{R}_2^{(T)}\right)\right) = \int_{x=0}^{T} \int_{y\in(T-x,\infty)} \int_{z\in(0,\infty)} \lambda\,dx\,F(dy,dz)$$
$$= \lambda \int_{x=0}^{T} \overline{F}_L(T-x)\,dx = \lambda \int_{x=0}^{T} \overline{F}_L(x)\,dx =: \lambda m_L(T) \to \lambda\mu_L, \text{ as } T\to\infty,$$

where $\mu_L = \mathrm{E}(L_1) < \infty$ as $\alpha_L > 1$. Since $\mathrm{E}\left(M\left(\mathcal{R}_i^{(T)}\right)\right) < \infty$, $i = 1, 2$, we have the representation

$$M|_{\mathcal{R}_1^{(T)}} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P(T)} \varepsilon_{\left(\tau_k^{(T)}, J_k^{(T)}, S_k^{(T)}\right)},$$

where $P(T) \sim \mathrm{POI}(\lambda\widehat{F}_L(T))$; i. e., a Poisson random variable with parameter $\lambda\widehat{F}_L(T)$, independent of the iid random vectors

$$\left(\tau_k^{(T)}, J_k^{(T)}, S_k^{(T)}\right) \sim \left.\frac{dx\,F(dy,dz)}{\widehat{F_L}(T)}\right|_{\mathcal{R}_1^{(T)}}, \tag{5.3}$$

where the above statement means the vector on the left has a distribution given on the right. Similarly

$$M|_{\mathcal{R}_2^{(T)}} \stackrel{\mathrm{d}}{=} \sum_{k=1}^{P'(T)} \varepsilon_{\left({\tau'_k}^{(T)}, {J'_k}^{(T)}, {S'_k}^{(T)}\right)},$$

where $P'(T) \sim \mathrm{POI}(\lambda m_L(T))$ independent of the iid random vectors

$$\left({\tau'_k}^{(T)}, {J'_k}^{(T)}, {S'_k}^{(T)}\right) \sim \left.\frac{dx\,F(dy,dz)}{m_L(T)}\right|_{\mathcal{R}_2^{(T)}}. \tag{5.4}$$

The key step in the entire analysis is to study the tail behavior of $J_1^{(T)} S_1^{(T)}$. We summarize this in the following lemma.

Lemma 5.1. *Under the assumptions of the model,*

$$T\,\mathrm{P}\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} > w\right] \to w^{-\alpha} \text{ as } T\to\infty,$$

where the convergence is uniform for $w \in [a,\infty)$, $\forall a > 0$.

Proof. We study the tail behavior in the cases (I) and (II) respectively.

First, we consider the case (I). Using (5.3) and the fact that $\frac{1}{T}\widehat{F}_L(T) = \frac{1}{T}\int_0^T F_L(u)\,du \sim F_L(T) \to 1$, we have,

$$T\,\mathrm{P}[J_1^{(T)} S_1^{(T)} > b_L(T)w] = \frac{T}{\widehat{F}_L(T)} \iint_{\substack{y\in(0,T)\\ yz>b_L(T)w}} (T-y)\,F(dy,dz)$$

$$\sim \iint\limits_{\substack{y\in(0,T)\\ yz>b_L(T)w}} \int\limits_y^T du\, F(dy,dz)$$

$$= \int\limits_0^T \iint\limits_{\substack{y\in(0,u]\\ yz>b_L(T)w}} F(dy,dz)\, du$$

$$= \frac{1}{T}\int\limits_0^T T\,\mathrm{P}\left[L_1 \leqslant u, \frac{L_1}{b_L(T)}R_1 > w\right] du \tag{5.5}$$

$$= \frac{b_L(T)}{T}\int\limits_0^{\frac{T}{b_L(T)}} T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right] du. \tag{5.6}$$

Now, $(0,u]\times(w,\infty)$ is bounded away from $\mathbf{0}$ and hence is relatively compact in $[\mathbf{0},\boldsymbol{\infty}]\setminus\{\mathbf{0}\}$ and has boundary with ν-measure zero. Hence by the assumptions (IB) and (IC) and using Corollary 3.1,

$$T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right] \to \nu\big((0,u]\times(w,\infty)\big) = (\nu_{\alpha_L}\times G)(A_{u,w}), \tag{5.7}$$

where $A_{u,w} = \{(y,z) : y \leqslant u, yz > w\}$ and

$$\lim_{u\to\infty}(\nu_{\alpha_L}\times G)(A_{u,w}) = (\nu_{\alpha_L}\times G)(\{(y,z) : yz > w\})$$

$$= w^{-\alpha_L}\int\limits_0^\infty u^{\alpha_L}\, G(du) =: c_w < \infty. \tag{5.8}$$

Also, by Theorem 3.2,

$$T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > w\right] \to c_w. \tag{5.9}$$

Fix $\varepsilon > 0$. Choose $M > 0$, by (5.8), such that $(\nu_{\alpha_L}\times G)(A_{M,w}) > c_w - \varepsilon$. Also observe, $\frac{T}{b_L(T)} \in RV_{1-\frac{1}{\alpha_L}}$ and since $\alpha_L > 1$, we have

$$\frac{T}{b_L(T)} \to \infty. \tag{5.10}$$

Then by (5.10), (5.7) and (5.9), choose T_0, such that $\forall T > T_0$, each of the following three inequalities hold

(i) $\frac{T}{b_L(T)} > M$,

(ii) $\forall u > M,\ T\,\mathrm{P}\left[\frac{(L_1,L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right] \geqslant T\,\mathrm{P}\left[\frac{(L_1,L_1R_1)}{b_L(T)} \in (0,M]\times(w,\infty)\right] > c_w - 2\varepsilon$,

(iii) $\forall u,\ T\,\mathrm{P}\left[\frac{(L_1,L_1R_1)}{b_L(T)} \in (0,u]\times(w,\infty)\right] \leqslant T\,\mathrm{P}\left[\frac{L_1}{b_L(T)}R_1 > w\right] < c_w + 2\varepsilon$.

Then, for all $T > T_0$,

$$c_w + 2\varepsilon \geqslant \frac{b_L(T)}{T} \int_0^{\frac{T}{b_L(T)}} T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u] \times (w,\infty)\right] du$$
$$\geqslant \frac{b_L(T)}{T} \int_M^{\frac{T}{b_L(T)}} T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u] \times (w,\infty)\right] du \geqslant (1 - \frac{Mb_L(T)}{T})(c_w - 2\varepsilon).$$

Taking limit as $T \to \infty$, and using (5.10) and the fact $\varepsilon > 0$ is arbitrary, we get

$$\frac{b_L(T)}{T} \int_0^{\frac{T}{b_L(T)}} T\,\mathrm{P}\left[\frac{(L_1, L_1R_1)}{b_L(T)} \in (0,u] \times (w,\infty)\right] du \to (\nu_{\alpha_L} \times G)(A_{\infty,w}) = w^{-\alpha_L} \int_0^\infty u^{\alpha_L}\, G(du).$$

Thus, by (5.6),

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_L(T)} > w\right] \to w^{-\alpha_L} \int_0^\infty u^{\alpha_L}\, G(du). \tag{5.11}$$

Now, we consider the case (II). As in (5.5), we have,

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_R(T)} > w\right] \sim \frac{1}{T}\int_0^T T\,\mathrm{P}\left[L_1 \leqslant u, L_1\frac{R_1}{b_R(T)} > w\right] du. \tag{5.12}$$

Since $\{(y,z) : y \leqslant u, yz > w\} =: A_{u,w}$ is bounded away from $[0,\infty] \times \{0\}$, it is relatively compact in $[0,\infty] \times (0,\infty]$. Also, if u is a continuity point of G, then $(G \times \nu_{\alpha_R})(\partial A_{u,w}) = 0$ and hence

$$T\,\mathrm{P}\left[L_1 \leqslant u, L_1\frac{R_1}{b_R(T)} > w\right] \to (G \times \nu_{\alpha_R})(A_{u,w}).$$

Then, arguing as in case (I), we get,

$$\frac{1}{T}\int_0^T T\,\mathrm{P}\left[L_1 \leqslant u, L_1\frac{R_1}{b_R(T)} > w\right] du \to c_w = w^{-\alpha_R} \int_0^\infty u^{\alpha_R}\, G(du).$$

Hence, by (5.12),

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_R(T)} > w\right] \to w^{-\alpha_R} \int_0^\infty u^{\alpha_R}\, G(du). \tag{5.13}$$

Combining (5.13) and (5.11), we have,

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b(T)} > w\right] \to w^{-\alpha_P} \int_0^\infty u^{\alpha_P}\, G(du),$$

where b is read with subscript L in the case (I) and R in the case (II). Hence

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{\tilde{b}(T)} > w\right] \to w^{-\alpha_P}, \tag{5.14}$$

where $\tilde{b}(T) := \left(\int_0^\infty u^{\alpha_P}\, G(du)\right)^{\frac{1}{\alpha_P}} b(T) \sim b_P(T)$ by Theorem 3.2. The LHS of (5.14) is monotone non-increasing and RHS is continuous in $(0,\infty)$. Hence, (cf. pg. 1 of [22]) pointwise convergence implies locally uniform convergence in $(0,\infty)$, and thus

$$T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > w\right] \to w^{-\alpha_P}. \tag{5.15}$$

Since LHS in (5.15) above is monotone non-increasing with a continuous pointwise limit on $(0,\infty)$ which has a finite limit at ∞, the convergence is uniform on $[a,\infty)$, $\forall a > 0$ (cf. pg. 1 of [22]). □

To complete the proof of Theorem 5.1 for one-dimensional convergence, we need to prove three more lemmas studying the moment conditions of $\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}$.

Lemma 5.2. *Under the model assumptions, we have*

$$\lim_{M\to\infty}\limsup_{T\to\infty} T\,\mathrm{E}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}>M\right]}\right) = 0. \tag{5.16}$$

Proof. First observe that, from (5.3),

$$\begin{aligned}
\mathrm{P}[J_1^{(T)}S_1^{(T)} > b_P(T)w] &= \frac{1}{\widehat{F}_L(T)}\int_0^T \iint_{\substack{y\in(0,u]\\ yz>b_P(T)w}} F(dy,dz)\,du \\
&= \frac{1}{\widehat{F}_L(T)}\int_0^T \mathrm{P}\left[L_1 \leqslant u, \frac{L_1R_1}{b_P(T)} > w\right] du. \qquad (5.17)
\end{aligned}$$

Now,

$$\begin{aligned}
T\,\mathrm{E}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}>M\right]}\right) &= MT\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > M\right] + \int_M^\infty T\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > x\right] dx \\
&= MT\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > M\right] + \frac{T}{\widehat{F}_L(T)}\int_M^\infty\int_0^T \mathrm{P}\left[L_1 \leqslant u, \frac{L_1R_1}{b_P(T)} > x\right] du\,dx \quad \text{by (5.17)} \\
&\leqslant MT\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > M\right] + \frac{T}{\widehat{F}_L(T)}\int_M^\infty T\,\mathrm{P}\left[L_1R_1 > b_P(T)x\right] dx \\
&= MT\,\mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > M\right] + \frac{T}{\widehat{F}_L(T)}\frac{T}{b_P(T)}\int_{Mb_P(T)}^\infty \mathrm{P}\left[L_1R_1 > x\right] dx.
\end{aligned}$$

Therefore, using Lemma 5.1 and Karamata's theorem,

$$\limsup_{T\to\infty} T\,\mathrm{E}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}\mathbf{1}_{\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}>M\right]}\right) \leqslant M^{1-\alpha_P} + 1\cdot\lim_{T\to\infty}\frac{T}{b_P(T)}\cdot\frac{Mb_P(T)\,\mathrm{P}[L_1R_1 > Mb_P(T)]}{\alpha_P - 1}$$

$$= M^{1-\alpha_P} + \frac{M}{\alpha_P - 1} \lim_{T\to\infty} T\,\mathrm{P}\left[\frac{L_1 R_1}{b_P(T)} > M\right]$$

$$= M^{1-\alpha_P} + \frac{M^{1-\alpha_P}}{\alpha_P - 1} = \frac{\alpha_P}{\alpha_P - 1} M^{1-\alpha_P}.$$

Thus, since $\alpha_P > 1$,

$$\lim_{M\to\infty} \limsup_{T\to\infty} T\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \mathbf{1}_{\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} > M\right]}\right) = 0.$$

□

Lemma 5.3. *Under the model assumptions, we have,*

$$\limsup_{T\to\infty} T\,\mathrm{Var}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \mathbf{1}_{\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \leqslant M\right]}\right) \leqslant \frac{\alpha_P}{2-\alpha_P} M^{2-\alpha_P} \quad \forall M > 0, \tag{5.18}$$

and hence

$$\lim_{\varepsilon\downarrow 0} \limsup_{T\to\infty} T\,\mathrm{Var}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \mathbf{1}_{\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \leqslant \varepsilon\right]}\right) = 0. \tag{5.19}$$

Proof. We know

$$T\,\mathrm{Var}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \mathbf{1}_{\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \leqslant M\right]}\right) \leqslant T\,\mathrm{E}\left(\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right)^2 \mathbf{1}_{\left[\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \leqslant M\right]}\right)$$

$$= \int_0^M 2tT\,\mathrm{P}\left[t < \frac{J_1^{(T)} S_1^{(T)}}{b_P(T)} \leqslant M\right] dt$$

$$= \frac{T}{\widehat{F}_L(T)} \int_0^M 2t \int_0^T \mathrm{P}\left[L_1 \leqslant u, t < \frac{L_1 R_1}{b_P(T)} \leqslant M\right] du\,dt \qquad \text{by (5.17)}$$

$$\sim \int_0^M 2t \int_0^T \mathrm{P}\left[L_1 \leqslant u, t < \frac{L_1 R_1}{b_P(T)} \leqslant M\right] du\,dt$$

$$\leqslant T \int_0^M 2t\ \mathrm{P}\left[t < \frac{L_1 R_1}{b_P(T)} \leqslant M\right] dt$$

$$= 2T \int_0^M t\ \mathrm{P}\left[\frac{L_1 R_1}{b_P(T)} > t\right] dt - TM^2\,\mathrm{P}\left[\frac{L_1 R_1}{b_P(T)} > M\right]$$

$$= \frac{2T}{(b_P(T))^2} \int_0^{b_P(T)M} t\ \mathrm{P}\left[L_1 R_1 > t\right] dt - TM^2\,\mathrm{P}\left[\frac{L_1 R_1}{b_P(T)} > M\right]$$

$$\sim \frac{2T}{2-\alpha_P} M^2 \,\mathrm{P}\left[\frac{L_1R_1}{b_P(T)} > M\right] - TM^2\,\mathrm{P}\left[\frac{L_1R_1}{b_P(T)} > M\right] \qquad \text{by Karamata's theorem}$$

$$= \frac{\alpha_P}{2-\alpha_P} TM^2\,\mathrm{P}\left[\frac{L_1R_1}{b_P(T)} > M\right] \to \frac{\alpha_P}{2-\alpha_P} M^{2-\alpha_P} \qquad \text{by Theorem 3.2.}$$

Therefore

$$\limsup_{T\to\infty} T\,\mathrm{Var}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} \mathbf{1}_{\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} \leqslant M\right]}\right) \leqslant \frac{\alpha_P}{2-\alpha_P} M^{2-\alpha_P} \quad \forall M > 0.$$

□

Lemma 5.4. *Under the model assumptions, we further have,*

$$\lim_{T\to\infty} \mathrm{E}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}\right) = 0. \tag{5.20}$$

Proof. We know

$$\mathrm{E}\left(\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)}\right) = \int_0^\infty \mathrm{P}\left[\frac{J_1^{(T)}S_1^{(T)}}{b_P(T)} > t\right] dt$$

$$= \frac{1}{\widehat{F}_L(T)} \int_0^\infty \int_0^T \mathrm{P}\left[L_1 \leqslant u, \frac{L_1R_1}{b_P(T)} > t\right] du\,dt \qquad \text{by (5.17)}$$

$$\leqslant \frac{T}{\widehat{F}_L(T)} \int_0^\infty \mathrm{P}\left[L_1R_1 > b_P(T)t\right] dt \tag{5.21}$$

$$\sim \frac{1}{b_P(T)} \int_0^\infty \mathrm{P}\left[L_1R_1 > t\right] dt = \frac{\mathrm{E}(L_1R_1)}{b_P(T)} \to 0 \qquad \text{since } \mathrm{E}(L_1R_1) < \infty.$$

□

Now, we are ready to prove Theorem 5.1 for the process A_1, as defined in (5.1), albeit with a different centering.

Theorem 5.2. *Under assumptions (2.2)–(2.4) and* (IA)–(IC) *or* (IIA)–(IIC)*, we have*

$$\frac{A_1(T) - P(T)\,\mathrm{E}\left(J_1^{(T)}S_1^{(T)}\right)}{b_P(T)} \Rightarrow X_{\alpha_P}(1), \tag{5.22}$$

where X_{α_P} is as defined in Theorem 5.1.

Proof. As in section 2 of [24], using (5.16), (5.18)–(5.20), we get

$$S_T \Rightarrow S_{\alpha_P} \text{ in } D([0,\infty)),$$

where

$$S_T(t) := \sum_{k=1}^{\lfloor Tt\rfloor} \left[\frac{J_k^{(T)}S_k^{(T)}}{b_P(T)} - \mathrm{E}\left(\frac{J_k^{(T)}S_k^{(T)}}{b_P(T)}\right)\right]$$

and S_{α_P} is an α_P-Levy motion with the skewness parameter 1, mean 0, and scaling parameter $C_{\alpha_P}^{-\frac{1}{\alpha_P}}$.

Since $P(T) \sim \mathrm{POI}(\lambda \widehat{F}_L(T))$ and $\lambda \widehat{F}_L(T) \sim \lambda T \to \infty$, by the central limit theorem,

$$\frac{P(T) - \lambda \widehat{F}_L(T)}{\sqrt{\lambda \widehat{F}_L(T)}} \Rightarrow \mathrm{N}(0,1) \text{ in } \mathbb{R}, \tag{5.23}$$

where $\mathrm{N}(0,1)$ is a standard normal random variable. Then, by Slutsky's theorem,

$$\frac{P(T) - \lambda \widehat{F}_L(T)}{\lambda \widehat{F}_L(T)} \xrightarrow{\mathrm{P}} 0,$$

and hence $P(T)/T \Rightarrow \lambda$ in $[0,\infty)$. By independence of S_T and $P(T)/T$, we have

$$\left(S_T, \frac{P(T)}{T}\right) \Rightarrow (S_{\alpha_P}, \lambda) \text{ in } D([0,\infty)) \times [0,\infty).$$

Hence, by [31],

$$S_T\left(\frac{P(T)}{T}\right) \Rightarrow S_{\alpha_P}(\lambda) \text{ in } \mathbb{R}.$$

Thus,

$$S_T\left(\frac{P(T)}{T}\right) = \frac{A_1(T)}{b_P(T)} - P(T)\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right) \Rightarrow S_{\alpha_P}(\lambda) \text{ in } \mathbb{R}.$$

Note, $S_{\alpha_P}(\lambda)$ is α_P-stable random variable with skewness parameter 1, mean 0 and scaling parameter $(\lambda/C_{\alpha_P})^{1/\alpha_P}$ and hence has same distribution as $X_{\alpha_P}(1)$, and the result is proved. □

Now, we consider A_2 and its negligibility in the following theorem.

Theorem 5.3. *If A_2 is defined as in (5.2), then*

$$\frac{A_2(T)}{b_P(T)} \xrightarrow{P} 0. \tag{5.24}$$

Proof. It is enough to show $\frac{A_2(T)}{b(T)} \xrightarrow{P} 0$, where $b(T)$ is $b_L(T)$ or $b_R(T)$ in cases (I) and (II) respectively, since $\frac{b_P(T)}{b(T)} \to$ constant, which is positive and finite.

Fix $\varepsilon > 0$, $\eta > 0$. Choose M such that $\mathrm{P}[P'(T) > M] < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}
\mathrm{P}\left[\frac{A_2(T)}{b(T)} > \eta\right] &\leqslant \mathrm{P}\left[\frac{A_2(T)}{b(T)} > \eta, P'(T) \leqslant M\right] + \mathrm{P}[P'(T) > M] \\
&\leqslant \mathrm{P}\left[\sum_{k=1}^{M} \left(T - {\tau'_k}^{(T)}\right) \frac{{S'_k}^{(T)}}{b(T)} > \eta\right] + \frac{\varepsilon}{2} \\
&\leqslant M\,\mathrm{P}\left[\left(T - {\tau'_1}^{(T)}\right) \frac{{S'_1}^{(T)}}{b(T)} > \frac{\eta}{M}\right] + \frac{\varepsilon}{2}.
\end{aligned}$$

Thus, to show $\frac{A_2(T)}{b(T)} \xrightarrow{P} 0$, it is enough to show $\mathrm{P}\left[\left(T-\tau_1'^{(T)}\right)\frac{S_1'^{(T)}}{b(T)} > \eta\right] \to 0\ \forall \eta > 0$, i. e., $\left(T-\tau_1'^{(T)}\right)\frac{S_1'^{(T)}}{b(T)} \xrightarrow{P} 0$. Now, by (5.4), we have,

$$\begin{aligned}\mathrm{P}\left[T-\tau_1'^{(T)} \leqslant s\right] &= \mathrm{P}\left[\tau_1'^{(T)} \geqslant T-s\right]\\ &= \frac{1}{m_L(T)}\int_{x=T-s}^{T}\int_{y>T-x}\int_{z\in(0,\infty)} F(dy,dz)\,dx\\ &= \frac{\int_{x=T-s}^{T}\overline{F}_L(T-x)\,dx}{m_L(T)} = \int_0^s \frac{\overline{F}_L(x)}{m_L(T)}\,dx.\end{aligned}$$

Thus, $T-\tau_1'^{(T)}$ has density $\frac{\overline{F}_L(\cdot)}{m_L(T)}$, supported on $(0,T)$, which converges pointwise to a density function $\frac{\overline{F}_L(\cdot)}{\mu_L}$, supported on $\mathbb{R}_+$. Hence, by Scheffé's theorem, $T-\tau_1'^{(T)}$ converges weakly to a positive random variable with density $\frac{\overline{F}_L(\cdot)}{\mu_L}$. So it is enough to show, by Slutsky's theorem, that

$$\frac{S_1'^{(T)}}{b(T)} \xrightarrow{P} 0. \tag{5.25}$$

Fix $\eta>0$. Now observe, by (5.4),

$$\begin{aligned}\mathrm{P}\left[S_1'^{(T)} > b(T)\eta\right] &= \frac{1}{m_L(T)}\int_{x<T}\int_{y>T-x}\int_{z>b(T)\eta} F(dy,dz)\,dx\\ &= \frac{1}{m_L(T)}\int_0^T \mathrm{P}\left[L_1>T-x, \frac{R_1}{b(T)}>\eta\right]dx\\ &= \frac{1}{m_L(T)}\int_0^T \mathrm{P}\left[L_1>x, \frac{R_1}{b(T)}>\eta\right]dx. \qquad (5.26)\end{aligned}$$

Now we show (5.25) by considering the cases (I) and (II) separately.

In case (I), we consider $b=b_L$. Now, by (5.26), with b replaced by b_L,

$$\begin{aligned}\mathrm{P}\left[S_1'^{(T)} > b_L(T)\eta\right] &= \frac{1}{m_L(T)}\int_0^T \mathrm{P}\left[L_1>x, \frac{R_1}{b_L(T)}>\eta\right]dx\\ &\leqslant \frac{1}{m_L(T)}T\,\mathrm{P}\left[\frac{R_1}{b_L(T)}>\eta\right] = \frac{1}{m_L(T)}T\,\mathrm{P}\left[\frac{R_1}{b_R(T)} > \frac{b_L(T)}{b_R(T)}\eta\right] \to 0 \quad \text{as } T\to 0,\end{aligned}$$

since $\frac{b_L(T)}{b_R(T)} \in RV_{\frac{1}{\alpha_L}-\frac{1}{\alpha_R}}$ and $\alpha_L<\alpha_R$ imply $\frac{b_L(T)}{b_R(T)} \to \infty$ and $m_L(T)\to\mu_L<\infty$. Therefore,

$$\frac{S_1'^{(T)}}{b_L(T)} \xrightarrow{P} 0.$$

In case (II), we consider $b = b_R$. Again, by (5.26), and with b replaced by b_R, we have, for all $T_0 > 0$,

$$
\begin{aligned}
\mathrm{P}\left[S_1'^{(T)} > b_R(T)\eta\right] &= \frac{1}{m_L(T)} \int_0^T \mathrm{P}\left[L_1 > x, \frac{R_1}{b_R(T)} > \eta\right] dx \\
&\leqslant \frac{T_0}{m_L(T)} \mathrm{P}[R_1 > b_R(T)\eta] + \frac{T}{m_L(T)} \mathrm{P}\left[L_1 > T_0, \frac{R_1}{b_R(T)} > \eta\right] \\
&\to \frac{1}{\mu_L} \bar{G}(T_0)\eta^{-\alpha_R} \text{ as } T \to \infty,
\end{aligned}
$$

by (IIB), for all continuity points T_0 of G. Then letting $T_0 \to \infty$ through continuity points of G, we conclude

$$
\mathrm{P}\left[\frac{S_1'^{(T)}}{b_R(T)} > \eta\right] \to 0,
$$

which proves the theorem. □

Then, combining (5.22) and (5.24) and using Slutsky's theorem, we get Theorem 5.1 for one-dimensional convergence with a random centering:

$$
\frac{A(T)}{b_P(T)} - P(T)\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right) \Rightarrow X_{\alpha_P}(1), \tag{5.27}
$$

Now, we prove Theorem 5.1 for one-dimensional convergence with correct centering.

Proof of Theorem 5.1: (for one-dimensional convergence)

Observe that we should replace the centering $P(T)\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right)$ by $\frac{\lambda T \mu_P}{b_P(T)}$ in (5.27) to get the required result. We shall show the difference of the above two expressions goes to 0 in probability. Now,

$$
\begin{aligned}
&P(T)\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right) - \frac{\lambda T \mu_P}{b_P(T)} \\
&= \frac{P(T) - \lambda \widehat{F}_L(T)}{\sqrt{\lambda \widehat{F}_L(T)}} \sqrt{\lambda \widehat{F}_L(T)}\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right) - \frac{\lambda}{b_P(T)}\left[T\mu_P - \widehat{F}_L(T)\,\mathrm{E}\left(J_1^{(T)} S_1^{(T)}\right)\right].
\end{aligned} \tag{5.28}
$$

Now, as in (5.21),

$$
\begin{aligned}
\sqrt{\lambda \widehat{F}_L(T)}\,\mathrm{E}\left(\frac{J_1^{(T)} S_1^{(T)}}{b_P(T)}\right) &\leqslant \frac{T\sqrt{\lambda}}{\sqrt{\widehat{F}_L(T)}} \int_0^\infty \mathrm{P}[L_1 R_1 > b_P(T)t]\,dt \\
&= \sqrt{\frac{T}{\widehat{F}_L(T)}} \frac{\sqrt{T\lambda}}{b_P(T)} \int_0^\infty \mathrm{P}[L_1 R_1 > t]\,dt \\
&\sim \frac{\sqrt{T\lambda}}{b_P(T)} \mathrm{E}(L_1 R_1) \to 0
\end{aligned}
$$

since $\frac{\sqrt{T}}{b_P(T)} \in RV_{\frac{1}{2}-\frac{1}{\alpha_P}}$ and $\alpha_P < 2$. Thus using (5.23), we get the first term in the RHS of (5.28) goes to 0 in probability. Thus, we only need to show the second term in the RHS of (5.28), which is just a number, goes to zero. Observe that, from (5.3),

$$T\mu_P - \widehat{F}_L(T)\,\mathrm{E}(J_1^{(T)}S_1^{(T)}) = \int_{x=0}^{T} \iint_{(y,z)\in\mathbb{R}_+^2} yz\,F(dy,dz)\,dx - \int_{x=0}^{T} \int_{y\leqslant T-x} \int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx$$

$$= \int_{x=0}^{T} \int_{y>T-x} \int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx = \int_{x=0}^{T} \int_{y>x} \int_{z\in(0,\infty)} yz\,F(dy,dz)\,dx \tag{5.29}$$

$$= \int_{x=0}^{T} \left[\int_{y>x} \int_{z\in(0,1]} yz\,F(dy,dz) + \int_{y>x} \int_{z>1} yz\,F(dy,dz) \right] dx$$

$$\leqslant \int_{x=0}^{T} \left[\int_{y>x} y\,F_L(dy) + \int_{u>x} u\,F_P(du) \right] dx, \tag{5.30}$$

where L_1R_1 has distribution F_P. Then, by (5.29) and (5.30), we get,

$$0 \leqslant \frac{\lambda}{b_P(T)} \left[T\mu_P - \widehat{F}_L(T)\,\mathrm{E}\left(J_1^{(T)}S_1^{(T)}\right) \right]$$

$$\leqslant \frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{y>x} y\,F_L(dy)\,dx + \frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{u>x} u\,F_P(du)\,dx. \tag{5.31}$$

Now, $\overline{F}_L \in RV_{-\alpha_L}$, and therefore, by Karamata's theorem, $\int_{y>x} y\,F_L(dy) \sim x\overline{F}_L(x)/(\alpha_L - 1) \in RV_{1-\alpha_L}$, so that, again by Karamata's theorem, we get,

$$\frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{y>x} y\,F_L(dy)\,dx \sim \frac{\lambda}{(2-\alpha_L)(\alpha_L-1)} \frac{T^2\overline{F}_L(T)}{b_P(T)} \in RV_{2-\alpha_L-\frac{1}{\alpha_P}}.$$

But

$$2 - \alpha_L - \frac{1}{\alpha_P} \leqslant 2 - \alpha_P - \frac{1}{\alpha_P} = -\frac{(\alpha_P-1)^2}{\alpha_P} < 0.$$

Hence

$$\frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{y>x} y\,F_L(dy)\,dx \to 0.$$

Similarly,

$$\frac{\lambda}{b_P(T)} \int_{x=0}^{T} \int_{u>x} u\,F_P(du)\,dx \sim \frac{\lambda}{(2-\alpha_P)(\alpha_P-1)} \frac{T^2\overline{F_P}(T)}{b_P(T)} \in RV_{2-\alpha_P-\frac{1}{\alpha_P}},$$

and hence goes to zero. Thus, by (5.31),

$$\frac{\lambda}{b_P(T)} \left[T\mu_P - \widehat{F}_L(T)\,\mathrm{E}\left(J_1^{(T)}S_1^{(T)}\right) \right] \to 0. \tag{5.32}$$

Thus, we have,

$$\frac{A(T) - \lambda T \mu_P}{b_P(T)} \Rightarrow X_{\alpha_P}(1) \text{ on } \mathbb{R}.$$

So,

$$\forall t \geqslant 0, \quad \frac{A(Tt) - \lambda Tt\mu_P}{b_P(T)} = \frac{b_P(Tt)}{b_P(T)} \frac{A(Tt) - \lambda Tt\mu_P}{b_P(Tt)} \Rightarrow t^{\frac{1}{\alpha_P}} X_{\alpha_P}(1) \stackrel{\mathrm{d}}{=} X_{\alpha_P}(t) \text{ in } \mathbb{R}.$$

This is the required one-dimensional convergence:

$$\frac{A(Tt) - \lambda Tt\mu_P}{b_P(T)} \Rightarrow X_{\alpha_P}(t) \text{ in } \mathbb{R} \quad \forall t \geqslant 0. \tag{5.33}$$

□

Finally, we consider the finite dimensional convergence, which will complete the proof of Theorem 5.1.

Proof of Theorem 5.1: (for finite dimensional convergence)

Let $0 < s < t$. Observe

$$A_1(Tt) - A_1(Ts) = \iiint\limits_{Ts < x+y \leqslant Tt} yz\, M(dx, dy, dz)$$

is independent of

$$A_1(Tu) = \iiint\limits_{x+y \leqslant Tu} yz\, M(dx, dy, dz) \quad \forall u \leqslant s,$$

since they are the functions of Poisson point process restricted to disjoint sets. Hence, $A_1(T\cdot)$ has independent increments. Also, let us define,

$$A_1(Tt) - A_1(Ts) = B_T(s,t) + C_T(s,t), \tag{5.34}$$

where

$$B_T(s,t) = \iiint\limits_{\substack{0 \leqslant x \leqslant Ts \\ Ts < x+y \leqslant Tt}} yz\, M(dx, dy, dz,) \text{ and } C_T(s,t) = \iiint\limits_{\substack{Ts < x \leqslant Tt \\ Ts < x+y \leqslant Tt}} yz\, M(dx, dy, dz).$$

Note that setting $N(A) = M(A + (Ts, 0, 0))$ gives

$$C_T(s,t) = \iiint\limits_{\substack{0 < u \leqslant T(t-s) \\ 0 < u+y \leqslant T(t-s)}} yz\, N(dx, dy, dz) \stackrel{\mathrm{d}}{=} \iiint\limits_{0 < x+y \leqslant T(t-s)} yz\, M(dx, dy, dz) = A_1(T(t-s)),$$

where the equality in distribution in the second last step follows from the fact, that by invariance of Lebesgue measure under translation, M and N have same mean measure and hence the same distribution. So, by (5.33), we get,

$$\frac{C_T(s,t) - \lambda T(t-s)\mu_P}{b_P(T)} \Rightarrow X_{\alpha_P}(t-s) \stackrel{\mathrm{d}}{=} X_{\alpha_P}(t) - X_{\alpha_P}(s). \tag{5.35}$$

Also,

$$\mathrm{E}\left(\frac{B_T(s,t)}{b_P(T)}\right) = \frac{\lambda}{b_P(T)} \iiint\limits_{\substack{0 \leqslant x \leqslant Ts \\ Ts < x+y \leqslant Tt}} yz\, F(dy, dz)\, dx \leqslant \frac{\lambda}{b_P(T)} \iiint\limits_{\substack{0 \leqslant x \leqslant Ts \\ x+y > Ts}} yz\, F(dy, dz)\, dx$$

$$= \frac{b_P(Ts)}{b_P(T)} \lambda \frac{Ts\mu_P - \widehat{F}_L(Ts) \,\mathrm{E}\left(J_1^{(Ts)} S_1^{(Ts)}\right)}{b_P(Ts)} \qquad \text{by (5.29)}$$
$$\to s^{-\alpha_P} \cdot 0 \qquad \text{by (5.32)},$$

which implies

$$\frac{B_T(s,t)}{b_P(T)} \stackrel{P}{\to} 0. \tag{5.36}$$

Set

$$X_1^{(T)}(t) = \frac{A_1(Tt) - \lambda Tt\mu_P}{b_P(T)}.$$

Then, by (5.34) – (5.36),

$$X_1^{(T)}(Tt) - X_1^{(T)}(Ts) = \frac{(A_1(Tt) - \lambda Tt\mu_P) - (A_1(Ts) - \lambda Ts\mu_P)}{b_P(T)} \Rightarrow X_{\alpha_P}(t) - X_{\alpha_P}(s) \text{ in } \mathbb{R}.$$

By independence increment property of $X_1^{(T)}$ and X_{α_P}, coordinatewise convergence of increments implies joint convergence of increments. Thus,

$$X_1^{(T)} \stackrel{\text{fidi}}{\to} X_{\alpha_P}. \tag{5.37}$$

Also, by (5.24),

$$\frac{A_2(Tt)}{b_P(T)} = \frac{A_2(Tt)}{b_P(Tt)} \cdot \frac{b_P(Tt)}{b_P(T)} \stackrel{P}{\to} 0.$$

Thus,

$$\frac{A_2(Tt_1), \cdots, A_2(Tt_k)}{b_P(T)} \stackrel{P}{\to} \mathbf{0}, \qquad \text{for all } 0 \leqslant t_1 < \cdots < t_k. \tag{5.38}$$

Now, $X^{(T)}(t) = \frac{A(Tt) - \lambda Tt\mu_P}{b_P(T)} = X_1^{(T)}(t) + \frac{A_2(Tt)}{b_P(T)}$ implies, by (5.37) and (5.38),

$$X^{(T)} \stackrel{\text{fidi}}{\to} X_{\alpha_P}.$$

6. Conclusion

Our result is crucially dependent on the modeling of the joint distribution of (L, R). If our model of asymptotic independence holds, then so does the classical asymptotic independence model. However, an estimate of the spectral measure, defined in (3.3), of the time and the rate of transmission (cf. Sections 4 and 5 of [7]), as given in Figure 3, does not seem to be supported on $\{0, \frac{\pi}{2}\}$, and so suggests a lack of asymptotic independence between the two random variables. This fact is reflected in the conclusion as well. Our result predicts that, if the model is true, then the input process measured at a large time scale should have independent increments. But this is not corroborated by the empirical findings reported in [10]. This observation suggests considering the case when the joint distribution of (L, R) is multivariate regularly varying in the sense of (4.1), and not asymptotically independent. We shall consider this in a later paper.

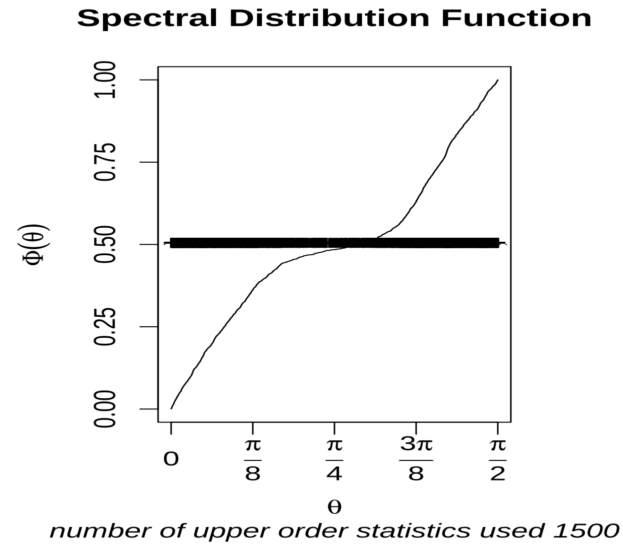

FIGURE 3. Spectral measure estimates of time and rate of transmission

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