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A NOTE ON THE TAIL DISTRIBUTION OF THE ACCUMULATED VALUE OF AN ADDITIVE FUNCTIONAL OF AN IRREDUCIBLE MARKOV CHAIN

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ABSTRACT

We show that the tail distribution of an accumulated value of an additive functional of an irreducible Markov chain is exponential if the state space is finite.

1. **Introduction.** Let $X_0, X_1, \ldots, X_\zeta$ be a Markov chain with finite state space E and random life-time ζ. We assume $P_i(0 < \zeta < \infty) = 1$ for each initial state i. We consider a non-zero non-negative function f on E and we put

$$S = \sum_{n=0}^{\zeta-1} f(X_n)$$

We give a Markov renewal equation for $P_i[S > t]$ and show that it is exponential in the limit.

2. Let $g(r)$ be the largest eigenvalue of the matrix $a_{ij}(r) = r^{f(i)}P_{ij}$. Since a_{ij} are continuous increasing functions of r, $g(1) < 1$ and $g(\infty) = \infty$, we must have $g(r) = 1$ for some $r > 1$. Denote this r by e^α. By the Perron-Frobenius theorem for non-negative irreducible matrices, there exists an eigenvector $\{x_i\}$ of $a_{ij}(e^\alpha)$ with strictly positive components which corresponds to the eigenvalue 1. Then $Q_{ij} = x_j a_{ij}(e^\alpha)x_i^{-1}$ is a stochastic matrix.

Notice that

$$P_i[S > t] = \underset{\sim}{1}_{[0,f(i))}(t) + \underset{\sim}{1}_{[f(i),\infty)}(t) \sum_{j\in E} P_{ij}P_j[S > t - f(i)]$$

which can be rewritten as the Markov renewal equation

$$R_i(t) = C_i(t) + \sum_{j\in E} \int_0^t A_{ij}(ds)R_j(t - s)$$

where

$$R_i(t) = \frac{e^{\alpha t} P_i[S>t]}{x_i} \qquad C_i(t) = \frac{e^{\alpha t} \underset{\sim}{1}_{[0,f(i))}(t)}{x_i}$$

and A_{ij} is the distribution function of a measure which charges the point $f(i)$ with mass Q_{ij}. Using standard results of Markov renewal theory, one gets, in the aperiodic case

$$\lim_{t\to\infty} R_i(t) = \frac{1}{\Sigma_j v_j f(j)} \int_0^\infty \sum_j v_j C_j(s)ds$$

where $\{v_j\}$ is the invariant distribution for the matrix Q_{ij}. Here $v_j = \pi_j x_j$ where $\{\pi_j\}$ is the left eigenvector of the matrix $a_{ij}(e^\alpha)$ corresponding to the eigenvalue 1, so that

$$\lim_{t\to\infty} P_i[S > t]e^{+\alpha t} = x_i \frac{\sum_j \pi_j(e^{\alpha f(j)} - 1)}{\alpha \sum_j \pi_j x_j f(j)}$$

Similarly in the case where the possible values of S are periodic with period δ,

$$\lim_{n\to\infty} P_i[S > n\delta]e^{\alpha n\delta} = c_i$$

where $\{c_i\}$ is an eigenvector of $a_{ij}(e^\alpha)$, but has no pleasant representation.

3. If the state space is countable, we can fix e^α as that r for which the convergence parameter of $a_{ij}(r)$ is 1. Then if $a_{ij}(e^\alpha)$ is 1-recurrent, and f is bounded, the same results will hold. Unfortunately, there seems to be no condition on P_{ij} which will guarantee $a_{ij}(e^\alpha)$ is 1-recurrent.

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