Partitioning Modified Hadamard Matrices and Construction of **m**-ary Balanced Incomplete Block Designs*

by

W. T. Federer and B. L. Raktoe**

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Abstract

A procedure is given for writing s^n factorials as a series of **m**-ary balanced block designs. The Hadamard matrix (X) for the 2^n factorial, composed of plus and minus ones, is transformed to a zero-one matrix as follows: (X + J)/2. A rearrangement of rows and columns allows this modified Hadamard matrix to be written as an array of blocked designs, some of which are balanced incomplete block designs. By a process of unionizing smaller block designs, larger balanced incomplete block designs are obtained. It is suggested that this procedure might be utilized to attack the problem of the existence of certain balanced incomplete block designs.

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1. Introduction

This is an investigation concerning the possibility of rearranging and modifying the contrast matrix $(X'X)$ in the 2^n factorial or any Hadamard matrix into a series of balanced incomplete block (BIB) designs. The modification made is to add a matrix of ones to the contrast matrix and utilize the $(X'X + J)/2$ matrix. Then, a study is made of unionizing sets of rows and columns in the resulting zero-one matrix to form BIB designs.

In section 2, the combinations in the 2^n and 3^n factorials are rearranged to form a series of balanced incomplete designs designated as $B(k,b,v = n, r,\lambda)$ designs or more simply as B_k designs for $k = 0,1,2,\ldots,n$. The extension to the s^n case is straightforward. In section 3, the zero-one matrix obtained from $(X'X + J)/2$ is rearranged in such a manner as to obtain BIB's or their duals (the roles of v and b and r and k interchanged) around the edges of the matrix. In section 4, subsets of the matrix $(X'X + J)/2$ for the 2^n are unionized to obtain BIB designs for $v = n + 1$ and $2n$ for n even. In the last section some additional possible studies are discussed.

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2. s^n Factorial as a Series of BIB's

By writing the combinations in a 2^n factorial in an appropriate manner, a series of balanced incomplete block designs for $v = n$ treatments is obtained. The procedure (see left hand side of Table 1) is to write all combinations at the zero level, then to write all combinations for $n - 1$ factors at the zero level and one of the n factors at the one level, then to write all combinations for $n - 2$ factors at the zero level and two factors at the one level,..., and lastly to write all n factors at the one level. Upon subdivision, the 2^n combinations for a continuing series of balanced incomplete block (BIB) designs may be written as indicated in the right hand portion of Table 1. This construction procedure leads to the following theorem:

Theorem 2.1. *The 2^n combinations in the 2^n factorial treatment design consisting of zero and one levels may be arranged in a series of binary zero-one balanced incomplete block designs with the parameters:*

$$v = n;\ k = 0,1,2,\ldots,n;\ b = \binom{n}{k};\ r = bk/n;\ \lambda = r(k - 1)/(n - 1).$$

The above procedure may be utilized to write the s^n factorial treatment designs as a series of m-ary balanced incomplete block designs. The procedure is illustrated for $s = 3$ in Table 2. The procedure and Theorem 1 may be generalized to cover the general case for the s^n factorial. The given m-ary design for a specified k is both pairwise and variance balanced.

For the $\prod_{i=1}^{n} k_i$ factorial, it is not possible to obtain variance-balanced incomplete blocks for n treatments in blocks of size k. It is possible in certain instances to obtain pairwise-balanced designs. For example, consider the following $2 \times 2 \times 3$ factorial:

0 0 0 1 0 0 0 1 0 0 0 1	
1 1 0 1 0 1 0 1 1 0 0 2	$rI - MN'/2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} - 1/2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 4 \end{pmatrix}$ and $\lambda = 1$.
1 1 1 1 0 2 0 1 2	
1 1 2	

3. The 2^n Hadamard Matrix and BIB Arrangements

The contrast matrix in a 2^n factorial is a Hadamard matrix. If the minus ones in the contrast matrix are replaced by zeros (i.e., $(X'X + J)/2$), if the parameters are rearranged to have the order of mean, main effects, two-factor interactions,..., n-factor interaction, and if the rows of the matrix are rearranged to have the same form for the 2^{nd} to the $n + 1^{st}$ columns as given in the left hand side of Table 1, then Tables 3 for 2^3, 4 for 2^4, 5 for 2^5, and 6 for 2^6 may be formed. The 2^n combinations are given in the 2^{nd} to the $n + 1^{st}$ columns and have been rearranged in the order given in Table 1. When this is accomplished and if we use the notation $B(k,b,v,r,\lambda)$ to designate a block design with the parameters k = block size, b = number of blocks, v = number of treatments, $r = bk/v$ = number of repetitions of each treatment, and $\lambda = r(k - 1)/(v - 1)$ = the number of times any arbitrary pair of treatments occurs together in the block, we may partition the zero-one matrix in Table 3 into four different balanced incomplete block designs or the duals (b and v and r and k interchanged) of these designs. The dual of a design in Table 3 is denoted with a prime on the B.

For the 2^4 factorial matrix in Table 4, the table can be partitioned into balanced incomplete block designs or their duals except for the part in rows 6 to 11 and columns 6 to 11. This part forms a split-plot design. Likewise, the 2^5 matrix in Table 5 forms a series of balanced incomplete block designs or their duals except for the four block designs in the center of the table involving rows 7 to 26 and columns 7 to 26. These four designs form four group divisible designs. We may note that a similar thing happens for the 2^6 factorial in Table 6.

From these enumerations we may obtain the following theorems.

Theorem 3.1. Let the parameters of the 2^n factorial be arranged in the order of the i^{th} factor interactions for $i = 0,1,2,\ldots,n$ and let the observations in the observation vector for the complete factorial be arranged as in Table 1 and Theorem 2.1, then the columns corresponding to the one factor interactions (main effects) of the matrix $(X'X + J)/2$ form the series of balanced incomplete block designs in Theorem 2.1.

Theorem 3.2. For the 2^n factorial as ordered in Theorems 2.1 and 3.1, the following represents the structure of $(X'X + J)/2$ after deleting the first column of ones:

factor								
1	2	3	4	5	6	...	n-1	n
B_0	rcb	B_0	rcb	B_0	rcb			B_0 for n odd rcb for n even
B_1	dual of B_{n-2}	complement of dual of B_{n-3}	dual of B_{n-4}	complement of dual of B_{n-5}	dual of B_{n-6}			
⋮								
B_{n-1}	dual of B_{n-2}	dual of B_{n-3}	dual of B_{n-4}	dual of B_{n-5}	dual of B_{n-6}		dual of B_1	
B_n	rcb	rcb	rcb	rcb	rcb		rcb	rcb

In the above theorem the complement of a design means replacing the zeros with ones and the ones with zeros in the original design. The symbol rcb means one replicate of a randomized complete block design for v treatments; this is also a balanced incomplete block design but since v varies the symbol B_n was retained for the rcb where v = n .

4. Unionizing BIB's

The binary balanced incomplete block designs described in the previous section are for v = n treatments. One item of interest is to determine if several of the designs in the previous section may be unionized, or put together, to form a balanced incomplete block design for v other than v = n. This can be done as is illustrated here with examples. Consider Table 4 and the columns for the main effects and the 4 factors interaction ABCD. Rows 6 to 15 for these parameters are given as follows:

row	Effect A	B	C	D	ABCD	k
6	1	1	0	0	1	3
7	1	0	1	0	1	3
8	1	0	0	1	1	3
9	0	1	1	0	1	3
10	0	1	0	1	1	3
11	0	0	1	1	1	3
12	1	1	1	0	0	3
13	1	1	0	1	0	3
14	1	0	1	1	0	3
15	0	1	1	1	0	3
r	6	6	6	6	6	30

This produces a balanced incomplete block design with $v = 5$, $r = 6$, $b = 10$, $k = 3$, $\lambda = 3$.

This example may be generalized for the 2^n factorial for n even to obtain a balanced incomplete block design with the parameters $v = n + 1$; $b = n(n-1)/2 + n(n-1)(n-2)/6$, $k = 3$, $r = n(n-1)/2$, and $\lambda = 2$. The design is constructed as follows:

B_2 B_3	$1_{n(n-1)/2}$ $0_{n(n-1)(n-2)/6}$
n main effects	n-factor interaction

Likewise, one could consider the following selection of rows and parameters to obtain a balanced incomplete block design.

	Effect								
rows	A	B	C	D	ABC	ABD	ACD	BCD	k
2	1	0	0	0	1	1	1	0	4
3	0	1	0	0	1	1	0	1	4
4	0	0	1	0	1	0	1	1	4
5	0	0	0	1	0	1	1	1	4
12	1	1	1	0	1	0	0	0	4
13	1	1	0	1	0	1	0	0	4
14	1	0	1	1	0	0	1	0	4
15	0	1	1	1	0	0	0	1	4
r	4	4	4	4	4	4	4	4	32

The parameters of the above design are $v = 8 = b$, $r = 4 = k$, $\lambda = 2$, and $NN' = 2(I + J)$. For the 2^n case for n even, the balanced incomplete block design for $v = 2n = b$, $r = n = k$, $\lambda = 2$, and $NN' = (n-2)I + 2J$ is obtained by unionizing B_1 and B_{n-1} designs as follows:

B_1	B_{n-1}
B_{n-1}	B_1
n main effects	n (n-1)-factor interactions

5. Discussion

The original purpose of this investigation was an attempt to obtain a solution for the existence of the balanced incomplete block design with parameters $v = 22$, $b = 33$, $r = 12$, $k = 8$, and $\lambda = 4$. One approach attempted was to unionize designs from the 2^6 factorial from the main effect columns (2^{nd} to $n + 1^{st}$) to obtain $b = 33$. For example, $B_0 \cup B_1 \cup B_1 \cup B_3$ produces $b = 1 + 6 + 6 + 20 = 33$ blocks, $r = 0 + 1 + 1 + 10 = 12$ replicates and $\lambda = 0 + 0 + 0 + 4 = 4$. This satisfies all requirements for 6 of the 22 entries but how does one obtain the remaining 16 treatments in order to have $v = 22$? This remains an unsolved problem. Also, if one proved that this procedure would not produce the desired BIB, this would not prove that the method could not work for some value of n other than 6 or for some other process of unionizing designs.

It should be noted that this method did produce two classes of BIB designs as described in section 4. Hence, the possibilities of the procedure to construct balanced incomplete block designs are not completely resolved.

Table 1. 2^n factorial arranged in a series of balanced incomplete block designs.

Combination	Parameters of b.i.b. design				
	k	b	v	r	λ
0 0 0 ··· 0 0	0	1	n	0	0
1 0 0 ··· 0 0 0 1 0 ··· 0 0 ⋮ 0 0 0 ··· 0 1	1	n	n	1	0
1 1 0 0 ··· 0 0 1 0 1 0 ··· 0 0 ⋮ 0 0 0 0 ··· 1 1	2	$\frac{n(n-1)}{2}$	n	(n-1)	1
1 1 1 0 ··· 0 0 1 1 0 1 ··· 0 0 ⋮ 0 0 0 ··· 1 1 1	3	$\frac{n(n-1)(n-2)}{2(3)}$	n	$\frac{(n-1)(n-2)}{2}$	n-2
1 1 1 1 ··· 0 0 1 1 1 0 ··· 0 0 ⋮ 0 0 ··· 1 1 1 1	4	$\frac{n(n-1)(n-2)(n-3)}{2(3)(4)}$	n	$\frac{(n-1)(n-2)(n-3)}{2(3)}$	$\frac{(n-2)(n-3)}{2}$
⋮					
1 1 1 1 ··· 1 0 1 1 1 1 ··· 0 1 ⋮ 0 1 1 1 ··· 1 1	n-1	n	n	n-1	n-2
1 1 1 1 ··· 1 1	n	1	n	1	1

Table 2. 3^n factorial arranged in a series of m-ary balanced block designs.

Combination	Parameters of balanced block design				
	k	b	v	r	λ
0 0 0 ··· 0	0	1	n	0	0
1 0 0 ··· 0 0 1 0 ··· 0 0 0 1 ··· 0 ⋮ 0 0 0 ··· 1	1	n	n	1	0
2 0 0 ··· 0 0 2 0 ··· 0 ⋮ 0 0 0 ··· 2 1 1 0 ··· 0 1 0 1 ··· 0 ⋮ 0 0 ··· 1 1	2	$n + \frac{n(n-1)}{2}$	n	n+1	1
1 1 1 ··· 0 1 1 0 ··· 0 ⋮ 0 ··· 1 1 1	3	$\frac{n(n-1)(n-2)}{2(3)}$	n	$\frac{(n-1)(n-2)}{2}$	n-2
1 2 0 ··· 0 1 0 2 ··· 0 ⋮ 2 1 0 ··· 0 ⋮ 0 0 ··· 2 1	3	n(n-1)	n	3(n-1)	4
⋮					
2 2 2 ··· 2	2n	1	n	2	4

Table 3. 2^3 factorial coefficient matrix X with -1's replaced by zeros, with parameter order as indicated, and with observations ordered as in Table 1.

2^3 factorial

M	A	B	C	AB	AC	BC	ABC
1	0	0	0	1	1	1	0
1	1	0	0	0	0	1	1
1	0	1	0	0	1	0	1
1	0	0	1	1	0	0	1
1	1	1	0	1	0	0	0
1	1	0	1	0	1	0	0
1	0	1	1	0	0	1	0
1	1	1	1	1	1	1	1

$B(k, b, v, r, \lambda)$

B(0, 1, 3, 0, 0)	B(3, 1, 3, 1, 1)	B (0, 1, 1, 0, 0)
B(1, 3, 3, 1, 0)	B(1, 3, 3, 1, 0)	B'(3, 1, 3, 1, 1)
B(2, 3, 3, 2, 1)	B(1, 3, 3, 1, 0)	B'(0, 1, 3, 0, 0)
B(3, 1, 3, 1, 1)	B(3, 1, 3, 1, 1)	B (1, 1, 1, 1, 0)

B_0	B_3	B_0
B_1	B_1	B'_3
B_2	B_1	B'_0
B_3	B_3	B_1

Table 4. 2^4 factorial coefficient matrix X with -1's replaced by zeros, with parameter order as indicated, and with observations ordered as in Table 1.

2^4 factorial

	M	A	B	C	D	AB	AC	AD	BC	BD	CD	ABC	ABD	ACD	BCD	ABCD
1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	1
2	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0
3	1	0	1	0	0	0	1	1	0	0	1	1	1	0	1	0
4	1	0	0	1	0	1	0	1	0	1	0	1	0	1	1	0
5	1	0	0	0	1	1	1	0	1	0	0	0	1	1	1	0
6	1	1	1	0	0	1	0	0	0	0	1	0	0	1	1	1
7	1	1	0	1	0	0	1	0	0	1	0	0	1	0	1	1
8	1	1	0	0	1	0	0	1	1	0	0	1	0	0	1	1
9	1	0	1	1	0	0	0	1	1	0	0	0	1	1	0	1
10	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1
11	1	0	0	1	1	1	0	0	0	0	1	1	1	0	0	1
12	1	1	1	1	0	1	1	0	1	0	0	1	0	0	0	0
13	1	1	1	0	1	1	0	1	0	1	0	0	1	0	0	0
14	1	1	0	1	1	0	1	1	0	0	1	0	0	1	0	0
15	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

B(0,1,4,0,0)	B(6,1,6,1,1)	B(0,1,4,0,0)	B(1,1,1,1,0)
B(1,4,4,1,0)	dual of B(2,6,4,3,1)	B(3,4,4,3,2)	dual of B(0,1,4,0,0)
B(2,6,4,3,1)	split-plot	B(2,6,4,3,1)	dual of B(6,1,6,1,1)
B(3,4,4,3,2)	dual of B(2,6,4,3,1)	B(1,4,4,1,0)	dual of B(0,1,4,0,0)
B(4,1,4,1,1)	B(6,1,6,1,1)	B(4,1,4,1,1)	B(1,1,1,1,0)

Table 5. 2^5 factorial coefficient matrix X with -1's replaced by zeros, with parameter order as indicated, and with observations ordered as in Table 1, and with the column for the mean omitted.

	A	B	C	D	E	AB	AC	AD	AE	BC	BD	BE	CD	CE	DE	ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE	ABCD	ABCE	ABDE	ACDE	BCDE	ABCDE
1	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0
2	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1
3	0	1	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	0	0	0	1	1	1	0	0	0	0	1	0	1
4	0	0	1	0	0	1	0	1	1	0	1	1	0	0	1	1	0	0	1	1	0	1	1	0	1	0	0	1	0	0	1
5	0	0	0	1	0	1	1	0	1	1	0	1	0	1	0	0	1	0	1	0	1	1	0	1	1	0	1	0	0	0	1
6	0	0	0	0	1	1	1	1	0	1	1	0	1	0	0	0	0	1	0	1	1	0	1	1	1	1	0	0	0	0	1
7	1	1	0	0	0	1	0	0	0	0	0	0	1	1	1	0	0	0	1	1	1	1	1	1	0	1	1	1	0	0	0
8	1	0	1	0	0	0	1	0	0	0	1	1	0	0	1	0	1	1	0	0	1	1	1	0	1	1	1	0	1	0	0
9	1	0	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	1	1	0	1	1	0	0
10	1	0	0	0	1	0	0	0	1	1	1	0	1	0	0	1	1	0	1	0	0	0	1	1	1	0	1	1	1	0	0
11	0	1	1	0	0	0	0	1	1	1	0	0	0	0	1	0	1	1	1	1	0	0	0	1	1	1	1	0	0	1	0
12	0	1	0	1	0	0	1	0	1	0	1	0	0	1	0	1	0	1	1	0	1	0	1	0	1	1	0	1	0	1	0
13	0	1	0	0	1	0	1	1	0	0	0	1	1	0	0	1	1	0	0	1	1	1	0	0	1	0	1	1	0	1	0
14	0	0	1	1	0	1	0	0	1	0	0	1	1	0	0	1	1	0	0	1	1	0	1	1	0	1	0	0	1	1	0
15	0	0	1	0	1	1	0	1	0	0	1	0	0	1	0	1	0	1	1	0	1	1	0	1	0	0	1	0	1	1	0
16	0	0	0	1	1	1	1	0	0	1	0	0	0	0	1	0	1	1	1	1	0	1	1	0	0	0	0	1	1	1	0
17	1	1	1	0	0	1	1	0	0	1	0	0	0	0	1	1	0	0	0	0	1	0	0	1	1	0	0	1	1	1	1
18	1	1	0	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1	1	1
19	1	1	0	0	1	1	0	0	1	0	0	1	1	0	0	0	0	1	1	0	0	1	0	0	1	1	0	0	1	1	1
20	1	0	1	1	0	0	1	1	0	0	0	1	1	0	0	0	0	1	1	0	0	0	1	1	0	0	1	1	0	1	1
21	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	1	0	0	1	0	1	0	1	0	1	0	1	0	1	1
22	1	0	0	1	1	0	0	1	1	1	0	0	0	0	1	1	0	0	0	0	1	1	1	0	0	1	1	0	0	1	1
23	0	1	1	1	0	0	0	0	1	1	1	0	1	0	0	0	0	1	0	1	1	1	0	0	0	0	1	1	1	0	1
24	0	1	1	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0	1	0	1	0	1	0	0	1	0	1	1	0	1
25	0	1	0	1	1	0	1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	1	0	1	1	0	1	0	1
26	0	0	1	1	1	1	0	0	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	1
27	1	1	1	1	0	1	1	1	0	1	1	0	1	0	0	1	1	0	1	0	0	1	0	0	0	1	0	0	0	0	0
28	1	1	1	0	1	1	1	0	1	1	0	1	0	1	0	1	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0
29	1	1	0	1	1	1	0	1	1	0	1	1	0	0	1	0	1	1	0	0	1	0	0	1	0	0	0	1	0	0	0
30	1	0	1	1	1	0	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0	1	0	0	0	1	0	0
31	0	1	1	1	1	0	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	0
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 5. (Continued)

2^5 factorial

B(0,1,5,0,0)	B(10,1,10,1,1)	B(0,1,10,0,0)	B(5,1,5,1,1)	B(0,1,1,0,1)
B_1 = B(1,5,5,1,0)	dual of B_3	dual of B_3	B_1 also dual of B_1	v = b = 5 k = r = 1 dual of B_5
B_2 = B(2,10,5,4,1)	NN' = 4 1 1 1 1 1 1 2 2 2 4 1 1 1 2 2 1 1 2 4 1 2 1 2 1 2 1 4 2 2 1 2 1 1 4 1 1 1 1 2 4 1 1 2 1 4 2 1 1 4 1 1 4 1 4 B(4,10,10,4,?)	B(6,10,10,6,?)	B_3	dual of B(0,1,10,0,0)
B_3 = B(3,10,5,6,3)	B(4,10,10,4,?)	B(4,10,10,4,?)	B_3	dual of B(10,1,10,1,1)
B_4 = B(4,5,5,4,3)	dual of B_3	dual of B_2	B_1 = dual of B_1	dual of B(0,1,5,0,0)
B_5 = B(5,1,5,1,1)	B(10,1,10,1,1)	B(10,1,10,1,1)	B(5,1,5,1,1)	B(1,1,1,1,1)

Table 6. 2^6 factorial coefficient matrix X with -1's replaced by zeros, with parameter order as indicated, with observations ordered as in Table 1, and with column for mean deleted

A	B	C	D	E	F	AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF	ABC	ABD	ABE	ABF	ACD	ACE	ACF	ADE	ADF	AEF	BCD	BCE	BCF	BDE	BDF	BEF	CDE	CDF	CEF	DEF
0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
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Table 6 (continued)

ABCD	ABCE	ABCF	ABDE	ABDF	ABEF	ACDE	ACDF	ACEF	ADEF	BCDE	BCDF	BCEF	BDEF	CDEF	ABCDE	ABCDF	ABCEF	ABDEF	ACDEF	BCDEF	ABCDEF
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1	0	1	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0	0
0	1	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0	0
0	0	0	1	1	1	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	0
0	0	0	0	0	0	1	1	1	1	0	0	0	0	1	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0	0	0	0	0	1	0
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 6. (Continued)

2^6 factorial

B(0,1,6,0,0) = B_0	B(15,1,15,1,1) = B^+	B(0,1,20,0,0) = B^-	B(15,1,15,1,1) = B^*	B(0,1,6,0,0) = B^X	B(1,1,1,1,1) = $B^{\div}$
B(1,6,6,1,0) = B_1	dual of B_4	dual of B_3	dual of B_2	dual of B_1	dual of B_0
B(2,15,6,5,1) = B_2	B(7,15,15,7,?)	B(12,15,20,9,?)	B(7,15,15,7,3) b.i.b.	B_2	dual of B^+
B(3,20,6,10,4) = B_3	B(9,20,15,12,?)	B(10,20,20,10,?)	B(9,20,15,12,?)	B_3	dual of B^-
B(4,15,6,10,6) = B_4	B(7,15,15,7,?)	B(4,15,20,3,?)	B(7,15,15,7,?)	B_4	dual of B^*
B(5,6,6,5,4) = B_5	dual of B_4	dual of B_3	dual of B_2	dual of B_1	dual of B^X
B(6,1,6,1,1) = B_6	B^+	Complement of B^-	B^*	Complement of B^X	$B^{\div}$