LEAST SQUARES ESTIMATES OF EFFECTS AND SUMS OF SQUARES FOR A TIED DOUBLE CHANGE-OVER DESIGN

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A group of designs is described on pages 454 to 456 of Federer's Experimental Design. The purpose of this paper is to give the least squares estimates of effects and to verify the sums of squares for the second design given on page 454. This design is called a "tied double change-over design" to distinguish it from the double change-over designs. The linear model for this design is:

$$X_{ijh} = \mu + \gamma_i + \beta_j + \delta_h + \rho_p + \epsilon_{ijh}, \qquad (1)$$

where $i = 1,2,\ldots,6$ and the two groups are not separated; $j = 1,2,3,\ldots,7$; $h = A,B,C$; $p = A,B,C$ and is determined by treatment preceding a given treatment; μ = general mean effect, γ_i = effect of ith store or column, β_j = effect of jth period or row, δ_h = direct effect of the hth treatment, ρ_p = residual effect of pth treatment, and ϵ_{ijh} is a random effect associated with the ijhth observation.

The design and observations are:

Store Number

Period	Group 1			Group 2			Period Totals
	1	2	3	4	5	6	
1	X_{11A}	X_{21B}	X_{31C}	X_{41A}	X_{51B}	X_{61C}	$X_{.1.}$
2	X_{12B}	X_{22C}	X_{32A}	X_{42C}	X_{52A}	X_{62B}	$X_{.2.}$
3	X_{13C}	X_{23A}	X_{33B}	X_{43B}	X_{53C}	X_{63A}	$X_{.3.}$
4	X_{14A}	X_{24B}	X_{34C}	X_{44A}	X_{54B}	X_{64C}	$X_{.4.}$
5	X_{15C}	X_{25A}	X_{35B}	X_{45B}	X_{55C}	X_{65A}	$X_{.5.}$
6	X_{16B}	X_{26C}	X_{36A}	X_{46C}	X_{56A}	X_{66B}	$X_{.6.}$
7	X_{17A}	X_{27B}	X_{37C}	X_{47A}	X_{57B}	X_{67C}	$X_{.7.}$
Store totals	$X_{1..}$	$X_{2..}$	$X_{3..}$	$X_{4..}$	$X_{5..}$	$X_{6..}$	$X_{...}$

The treatment totals are $X_{..A}$, $X_{..B}$, and $X_{..C}$ and are obtained by summing all observations which received the hth treatment.

The residual sum of squares is:

$$Res = \sum_{ijh}\sum\sum (Y_{ijh} - \mu - \gamma_i - \beta_j - \delta_h - \rho_p)^2 ; \qquad (2)$$

using the restrictions that

$$\Sigma\hat{\gamma}_i = 0, \qquad (3)$$

$$\Sigma\hat{\beta}_j = 0, \qquad (4)$$

$$\Sigma\hat{\delta}_h = 0, \qquad (5)$$

and

$$\Sigma\hat{\rho}_p = 0, \qquad (6)$$

the resulting normal equations are:

<u>For the mean</u>

$$42\,\hat{\mu} = X_{...} \qquad (7)$$

<u>For the columns or stores</u>

$$7(\hat{\mu} + \hat{\gamma}_1) + \hat{\delta}_A = X_{1..} \qquad (8)$$

$$7(\hat{\mu} + \hat{\gamma}_2) + \hat{\delta}_B = X_{2..} \qquad (9)$$

$$7(\hat{\mu} + \hat{\gamma}_3) + \hat{\delta}_C = X_{3...} \qquad (10)$$

$$7(\hat{\mu} + \hat{\gamma}_4) + \hat{\delta}_A = X_{4..} \qquad (11)$$

$$7(\hat{\mu} + \hat{\gamma}_5) + \hat{\delta}_B = X_{5..} \qquad (12)$$

$$7(\hat{\mu} + \hat{\gamma}_6) + \hat{\delta}_C = X_{6..} \qquad (13)$$

<u>For the rows or periods</u>

$$6(\hat{\mu} + \hat{\beta}_1) = X_{.1.} \qquad (14)$$

$$6(\hat{\mu} + \hat{\beta}_2) = X_{.2.} \qquad (15)$$

$$6(\hat{\mu} + \hat{\beta}_3) = X_{.3.} \qquad (16)$$

$$6(\hat{\mu} + \hat{\beta}_4) = X_{.4.} \qquad (17)$$

$$6(\hat{\mu} + \hat{\beta}_5) = X_{.5.} \qquad (18)$$

$$6(\hat{\mu} + \hat{\beta}_6) = X_{.6.} \qquad (19)$$

$$6(\hat{\mu} + \hat{\beta}_7) = X_{.7.} \qquad (20)$$

<u>For direct effects</u>

$$14(\hat{\mu} + \hat{\delta}_A) - 6\,\hat{\rho}_A + \hat{\gamma}_1 + \hat{\gamma}_4 = X_{..A} \qquad (21)$$

$$14(\hat{\mu} + \hat{\delta}_B) - 6\,\hat{\rho}_B + \hat{\gamma}_2 + \hat{\gamma}_5 = X_{..B} \qquad (22)$$

$$14(\hat{\mu} + \hat{\delta}_C) - 6\,\hat{\rho}_C + \hat{\gamma}_3 + \hat{\gamma}_6 = X_{..C} \qquad (23)$$

<u>For residual effects</u>

$$12(\hat{\mu} + \hat{\rho}_A) - 6\,\hat{\delta}_A - 2\,\hat{\beta}_1 = a \qquad (24)$$

$$12(\hat{\mu} + \hat{\rho}_B) - 6\,\hat{\delta}_B - 2\,\hat{\beta}_1 = b \qquad (25)$$

$$12(\hat{\mu} + \hat{\rho}_C) - 6\,\hat{\delta}_C - 2\,\hat{\beta}_1 = c \qquad (26)$$

where

$$a = X_{12B} + X_{15C} + X_{24B} + X_{26C} + X_{33B} + X_{37C} + X_{42C} + X_{45B} + X_{53C} + X_{57B} + X_{64C} + X_{66B}\,, \qquad (27)$$

$$b = X_{13C} + X_{17A} + X_{22C} + X_{25A} + X_{34C} + X_{36A} + X_{44A} + X_{46C} + X_{52A} + X_{55C} + X_{63A} + X_{67C}\,, \qquad (28)$$

and

$$c = X_{14A} + X_{16B} + X_{23A} + X_{27B} + X_{32A} + X_{35B} + X_{43B} + X_{47A} + X_{54B} + X_{56A} + X_{62B} + X_{65A}\,. \qquad (29)$$

Substitution of $\hat{\delta}_A$ from formula (8), (11) and (21), β_1 from formula (14) and $\hat{\mu}$ from formula (7) in formula (24) results in the following estimate for $\hat{\rho}_A$:

$$\hat{\rho}_A = \frac{8}{75(48)}\left\{48A + 21X_{..A} - 3X_{1..} - 3X_{4..} + 16X_{.1.} - 22X_{...}\right\}$$

$$= \frac{1}{450}\left\{16(3a + X_{.1.} - X_{...} = a^*) + 3(7X_{..A} - X_{1..} - X_{4..} - 2X_{...} = X_A^*)\right\}$$

$$= \frac{1}{450}\left\{16a^* + 3X_A^*\right\} \qquad (30)$$

Likewise

$$\hat{\rho}_B = \frac{16}{(150)(48)}\left\{48b + 21X_{..B} - 3X_{2..} - 3X_{5..} + 16X_{.1.} - 22X_{...}\right\}$$

$$= \frac{1}{450}\left\{16(3b + X_{.1.} - X_{...} = b^*) + 3(7X_{..B} - X_{2..} - X_{5..} - 2X_{...} = X_B^*)\right\}$$

$$= \frac{1}{450}\left\{16b^* + 3X_B^*\right\}; \qquad (31)$$

$$\hat{\rho}_C = \frac{1}{450}\left\{48c + 21X_{..C} - 3X_{3..} - 3X_{6..} + 16X_{.1.} - 22X_{...}\right\}$$

$$= \frac{1}{450}\left\{16(3c + X_{.1.} - X_{...} = c^*) + 3(7X_{..C} - X_{3..} - X_{6..} - 2X_{...} = X_C^*)\right\}$$

$$= \frac{1}{450}\left\{16c^* + 3X_C^*\right\} . \qquad (32)$$

The quantities with an asterisk (a^*, b^*; c^*, X_A^*, X_B^*; X_C^*) are defined within the parentheses in the above equations.

The direct effects are estimated to be:

$$\hat{\delta}_A = \frac{1}{450}\left\{6X_A^* + 7a^*\right\}; \qquad (33)$$

$$\hat{\delta}_B = \frac{1}{450}\left\{6X_B^* + 7b^*\right\}; \qquad (34)$$

$$\hat{\delta}_C = \frac{1}{450}\left\{6X_C^* + 7c^*\right\}. \qquad (35)$$