K-THEORY OF WEIGHT VARIETIES AND DIVIDED DIFFERENCE OPERATORS IN EQUIVARIANT KK-THEORY

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

Ho Hon Leung

May 2011

K-THEORY OF WEIGHT VARIETIES AND DIVIDED DIFFERENCE OPERATORS IN EQUIVARIANT KK-THEORY

Ho Hon Leung, Ph.D.

Cornell University 2011

This thesis consists of two chapters. In the first chapter, we compute the K-theory of weight varieties by using techniques in Hamiltonian geometry. In the second chapter, we construct a set of divided difference operators in equivariant KK-theory.

Let T be a compact torus and (M, ω) a Hamiltonian T-space. In Chapter 1, we give a new proof of the K-theoretic analogue of the Kirwan surjectivity theorem in symplectic geometry (see [HL1]) by using the equivariant version of the Kirwan map introduced in [G2]. We compute the kernel of this equivariant Kirwan map. As an application, we find the presentation of the K-theory of weight varieties, which are the symplectic quotients of complete flag varieties G/T, as the quotient ring of the T-equivariant K-theory of flag varieties by the kernel of the Kirwan map, where G is a compact, connected and simply-connected Lie group.

Demazure [D1], [D2], [D3] defined a set of isobaric divided difference operators on the representation ring $R(T)$. It can be seen as a decomposition of the classical Weyl character formula. In [HLS], Harada, Landweber and Sjamaar defined an analogous set of divided difference operators on the equivariant K-theory. In Chapter 2, we explicitly define these operators in the setting of equivariant KK-theory first defined by Kasparov [K1], [K2]. It is a generalization of the results in [D3] and [HLS]. Due to the elegance and generality of equivariant KK-theory, some interesting applications of the result will also be given.

BIOGRAPHICAL SKETCH

Ho Hon Leung was born in Hong Kong on 24th September, 1984. He attended Queen's College in Hong Kong, where he started to love both Mathematics and Physics, from 1996 to 2003. He then earned a Bachelor of Science in Mathematics at Imperial College London in England in 2006. After that, he came to Cornell University to pursue his Ph.D. in the field of Mathematics.

To my parents and my brother.

ACKNOWLEDGEMENTS

I would like to thank Reyer Sjamaar for his amazing supervision over the last five years. I owe a great debt to him for introducing me to the topics in and around equivariant symplectic geometry and the related algebraic topology. In particular, I first heard about Kasparov theory from him in September 2009. It has drawn almost all of my attention and interest ever since. He is always willing to listen to my latest half-baked idea and help me get around the problems in my research projects. Not only has he taught me to be a good researcher, but also he has shown me how to be a wise and an organized person. He is always there giving me sounding advice whenever I have any difficulties in either my research or my daily life. Needless to say, he is a perfect role model for me as an advisor, an expositor and a researcher.

I am also deeply grateful to Tara Holm for teaching me the topics in symplectic geometry since Spring 2008. That was the starting point for my first research project. Many thanks go to Allen Knuston for his great patience and generosity in answering my questions. I am also grateful to Tomoo Matsumura for both mathematical and non-mathematical discussions. I am thankful also to all the people in our symplectic reading group, Tim, Alex, Fatima, Milena, Shisen, and some others. Many thanks also go to Ken Brown, Bill Thurston, James West and Allen Hatcher, Cao Xiaodong and Cai Mingzhong for their advice, both mathematical and otherwise. Many thanks go to Maria Terrell for her teaching advice. Also, I am grateful to all students of the lively Math 1106 class in Spring 2010, you made me realize that I can be a good teacher, thank you. Thanks also go to all staff members in the Department who make it such a wonderful place to stay in.

I owe special debts to the Summer School on Operator Algebras and Noncommutative Geometry in the University of Victoria, Canada in 2010. Over there I was very fortunate to have attended the presentations given by Heath Emerson and Ralf Meyer. They had significantly improved my understanding of the motivations behind the abstract nonsense by Kasparov. Also thanks Ralf for his great patience in answering my emails.

My stay in Cornell would not be the same without my friends, Simon, ,Chor, Eugenia, Song and many others, thanks all. Thanks also go to all graduate students in the Department for a lot of relaxing and enjoyable discussions and many other activities. Last but not least, I am also deeply grateful to my parents, my brother, uncle and grandmother for their truthful and unceasing support. This thesis is dedicated to all of them.

TABLE OF CONTENTS

CHAPTER 1

K-THEORY OF WEIGHT VARIETIES

1.1 Background

1.1.1 Symplectic Geometry

A *symplectic manifold* is a pair (M, ω) consisting of a smooth manifold M and a *symplectic form* ω which is a 2-form that is *closed*, i.e. $d\omega = 0$ and *nondegenerate*, i.e. for all $p \in M$, there does not exist non-zero $X \in \mathrm{T}M$ such that $\omega(X, Y) = 0$ for all $Y \in \mathrm{T}M$.

Remark 1 Note that ω is skew-symmetric, that is, $\omega(X, Y) = -\omega(Y, X)$ for all $X, Y \in \mathrm{T}M$. Recall that in odd dimensions antisymmetric matrices are not invertible. Since ω is a non-dengerate 2-form, the skew-symmetric condition implies that all symplectic manifolds (M, ω) have even dimensions.

The symplectic form ω on M allows us to associate to each function $H \in C^\infty(M)$ a vector field X_H, called its *Hamiltonian vector field*

$$dH = \iota_{X_H}\omega$$

Note that X_H is unique by the non-dengeneracy condition on ω.

Conversely, given a vector field X on M, if $X = X_H$ for some functions $H \in C^\infty(M)$, then X is called a *Hamiltonian vector field* and H is called its *Hamiltonian function.* The Hamiltonian function H is unique only up to an additive constant.

Let G be a compact connected Lie group acting smoothly on M. This action is called *symplectic* if it preserves the symplectic form ω, that is

$$g^*\omega = \omega$$

for all $g \in G$. The G-action on M is called *Hamiltonian* if it is symplectic and each ξ_M, $\xi \in \mathfrak{g}$, is a Hamiltonian vector field. In this case, there is a map $\phi\colon M \longrightarrow \mathfrak{g}^*$, called a *moment map*, satisfying the following properties:

(i) ϕ is equivariant with respect to the G-action on M and the coadjoint action of G on $\mathfrak{g}^*$, that is,

$$\phi(g.p) = \mathrm{Ad}^*(g)(\phi(p))$$

for all $p \in M$ and $g \in G$.

(ii) For each $\xi \in \mathfrak{g}^*$, the function $\phi^\xi \in C^\infty(M)$ defined by $\phi^\xi(p) = \langle \phi(p), \xi \rangle$ is a Hamiltonian function for the vector field ξ_M:

$$d\phi^\xi = \iota_{\xi_M}\omega$$

A compact symplectic manifold (M, ω) on which the G-action is Hamiltonian is called a *compact Hamiltonian G-space.*

In this Chapter, we will only deal with a compact torus action, so we will use the T-action on M as our notation instead, where T is a compact torus. Let T' be a subtorus in T, $\phi|_{T'}\colon M \to \mathfrak{t}'^*$ is the restriction of the T-action to the T'-action. We call $\phi|_{T'}$ the *component* of the moment map corresponding to T' in T.

1.1.2 Representation ring and Equivariant K-theory

Let G be a compact Lie group, the *representation ring* of G, $R(G)$, consists of all formal differences of isomorphism classes of finite dimensional complex-linear representations of G. Addition in $R(G)$ is given by the direct sum of representations. Multiplication in $R(G)$ is given by the tensor products of representations over $\mathbb{C}$. Alternatively, $R(G)$ can be defined as the free abelian group generated by all irreducible characters. For example, let T be a maximal torus in G, let $\mathscr{X}(T) = Hom(T, U(1))$ be the character group of T. Then $R(T) = \mathbb{Z}[\mathscr{X}(T)]$. Note that $\mathscr{X}(T)$ is a discrete group. The multiplication is defined by $(\sum \lambda_g g)(\sum \mu_h h) = \sum_{g,h} \lambda_g \mu_h gh$ for $g \in G, h \in H, \lambda_g, \mu_h \in \mathbb{Z}$. In fact, the character group of a torus of rank n is isomorphic to $\mathbb{Z}^n$. Thus $R(T) \cong \mathbb{Z}[a_1, a_1^{-1}, ..., a_n, a_n^{-1}]$ which is a ring of Laurent polynomials with coefficients in $\mathbb{Z}$.

The *G-equivariant K-theory* of a compact G-space M, $K_G^0(M)$, is the Grothendieck ring of virtual G-equivariant complex bundles over M. In particular, if M is a point, then

$$K_G^0(pt) \cong R(G)$$

In this case a G-vector bundle is just a (finite-dimensional) G-module. If G is trivial, then we use the notation $K^0(M)$ instead.

Given a continuous map $M \to N$ where M, N are compact G-spaces, we can *pullback* a G-vector bundle on N to the corresponding G-vector bundle on M. This operation is well-behaved with respect to the isomorphism classes of vector bundles. We obtain a map $f^*\colon K_G^0(N) \to K_G^0(M)$. So, K_G^0 is a functor from compact G-spaces to commutative rings. Note that $K_G^0(M)$ is naturally endowed with a $R(G)$-module structure because any G-space X has a natural map onto a point (so that we have the map $R(G) \to K_G^0(M)$). For further properties about equivariant K-theory, see [S].

The main theme of Chapter 2 is to compute the K-theory of certain compact manifolds, *Weight Varieties*, by using techniques in Hamiltonian geometry.

Alternatively, equivariant K-theory can be defined by using equivariant KK-theory of C*-algebras, see Section 2.2.

1.2 Introduction

For M a compact Hamiltonian T-space, where T is a compact torus, we have a moment map $\phi\colon M \to \mathfrak{t}^*$. For any regular value μ of ϕ, $\phi^{-1}(\mu)$ is a submanifold of M and has a locally free T-action by the invariance of ϕ. The *symplectic reduction* of M at μ is defined as $M//T(\mu) := \phi^{-1}(\mu)/T$. The parameter μ is surpressed when $\mu = 0$. Kirwan [K] proved that the natural map, which is now called the *Kirwan map*,

$$\kappa\colon H_T^*(M;\mathbb{Q}) \to H_T^*(\phi^{-1}(0);\mathbb{Q}) \cong H^*(M//T;\mathbb{Q})$$

induced from the inclusion $\phi^{-1}(0) \subset M$ is a surjection when $0 \in \mathfrak{t}^*$ is a regular value of ϕ. This result was done in the context of rational Borel equivariant cohomology. In the context of complex K-theory, a theorem of Harada and Landweber [HL1] showed that

$$\kappa\colon K_T^*(M) \to K_T^*(\phi^{-1}(0))$$

is a surjection. This result was done over $\mathbb{Z}$.

In Section 1.3, we give another proof of this theorem by using *equivariant*

Kirwan map, which was first introduced by Goldin [G2] in the context of rational cohomology. It can also be seen as an equivariant version of the Kirwan map.

Theorem 2 *Let T be a compact torus and M be a compact Hamiltonian T-space with moment map $\phi\colon M \to \mathfrak{t}^*$. Let S be a circle in T, and $\phi|_S := M \to \mathbb{R}$ be the corresponding component of the moment map. For a regular value $0 \in \mathfrak{t}^*$ of $\phi|_S$, the equivariant Kirwan map*

$$\kappa_S\colon K_T^*(M) \to K_T^*(\phi|_S^{-1}(0))$$

is a surjection.

As an immediate corollary of a result in [HL1], we also find the kernel of this equivariant Kirwan map.

In Section 1.4, for the special case $G = SU(n)$, we find an explicit formula for the K-theory of weight varieties, the symplectic reduction of flag varieties $SU(n)/T$. The main result is Theorem 12. The results in this section are the K-theoretic analogues of [G1].

1.3 Equivariant Kirwan map in K-theory

We fix the notations about Morse theory. Let $f\colon M \to \mathbb{R}$ be a Morse function on a compact Riemannian manifold M. Consider its negative gradient flow on M, let $\{C_i\}$ be the connected components of the critical set of f. Define the stratum S_i to be the set of points of M which flow down to C_i by their paths of steepest descent. There is an ordering on I: $i \leq j$ if $f(C_i) \leq f(C_j)$. Hence we obtain a smooth stratification of $M = \cup S_i$. For all $i, j \in I$, denote

$$M_i^+ = \bigcup_{j \le i} S_j, \quad M_i^- = \bigcup_{j < i} S_j$$

As we are working in the equivariant category, we require that the Morse function and the Riemannian metric to be T-invariant.

In the following, we will consider the norm square of the moment map. In general, it is not a Morse function due to the possible presence of singularities of the critical sets but the norm square of the moment map still yields a smooth stratifications and the results of the Morse-Bott theory still holds in this general setting (Such functions are now called the Morse-Kirwan functions). For the descriptions and properties of these functions, see [K]. Kirwan proved that the Morse-Kirwan functions are equivariantly perfect in the context of rational cohomology. For more results in this direction, see [K] and [L]. In the context of equivariant K-theory, the following result is shown in [HL1]:

Lemma 3 (Harada and Landweber) *Let T be a compact torus and (M, ω) be a compact Hamiltonian T-space with moment map $\phi\colon M \to \mathfrak{t}^*$. Let $f = ||\phi||^2$ be the norm square of the moment map. Let $\{C_i\}$ be the connected components of the critical sets of f and the stratum S_i be the set of points of M which flow down to C_i by their paths of steepest descent. The inclusion $C_i \to S_i$ of a critical set into its stratum induces an isomorphism $K_T^*(S_i) \cong K_T^*(C_i)$.*

For a smooth stratification $M = \cup S_i$ defined by a Morse-Kirwan function f, i.e. the strata S_i are locally closed submanifolds of M and each of them satisfies the closure property $\overline{S}_i \subseteq M_i^+$. We have a T-normal bundle N_i to S_i in M. By excision, we have

$$K_T^*(M_i^+, M_i^-) \cong K_T^*(N_i, N_i \backslash S_i)$$

If N_i is complex, by Thom Isomorphism we have

$$K_T^*(N_i, N_i \backslash S_i) \cong K_T^{*-d(i)}(S_i)$$

where the degree $d(i)$ of the stratum is the rank of its normal bundle N_i. Since the collection of the sets M_i^+ gives a filtration of M, we obtain a filtration of $K_T^*(M)$ and a spectral sequence

$$E_1 = \bigoplus_{i \in I} K_T^*(M_i^+, M_i^-) = \bigoplus_{i \in I} K_T^{-d(i)}(S_i), \quad E_\infty = \mathrm{Gr} K_T^*(M)$$

which converges to the associated graded algebra of the equivariant K-theory of M. By Lemma 3, the spectral sequence becomes

$$E_1 = \bigoplus_{i \in I} K_T^{*-d(i)}(C_i), \quad E_\infty = \mathrm{Gr} K_T^*(M)$$

Definition 4 *The function f is called* equivariantly perfect *for equivariant K-theory if the above spectral sequence for equivariant K-theory collapses at the E_1 page, or equivalently speaking, we have the following short exact sequences:*

$$0 \longrightarrow K_T^{*-d(i)}(C_i) \longrightarrow K_T^*(M_i^+) \longrightarrow K_T^*(M_i^-) \longrightarrow 0$$

for each $i \in I$.

In [HL1], Harada and Landweber showed the following theorem. (Indeed, they showed it for compact Lie group G. But in our paper, we only need to consider the abelian case.)

Theorem 5 (Harada and Landweber) *Let T be a compact torus and (M, ω) be a compact Hamiltonian T-space with the moment map $\phi \colon M \to \mathfrak{t}^*$. The norm square of the moment map $f = ||\phi||^2$ is an equivariantly perfect Morse-Kirwan* *function for equivariant K-theory. By the Bott-periodicity in complex equivariant K-theory, we can rewrite the short exact sequences as:*

$$0 \longrightarrow K_T^*(C_i) \longrightarrow K_T^*(M_i^+) \longrightarrow K_T^*(M_i^-) \longrightarrow 0$$

Let $\phi|_S \colon M \to \mathbb{R}$ be the component of the moment map ϕ corresponding to a circle S in T. Equivalently we are considering a compact Hamiltonian S-manifold with the moment map $\phi|_S$. By Theorem 5 above, the norm square of the moment map $||\phi|_S||^2$ is an equivariantly perfect Morse-Kirwan function for equivariant K-theory. We can give our proof of Theorem 2 now.

Proof of Theorem 2. Our proof is essentially the K-theoretic analogue of Theorem 1.2 in [G2]. For the Morse-Kirwan function $f = ||\phi|_S||^2$, denote $C_0 = f^{-1}(0) = \phi|_S^{-1}(0)$.

First, we need to show that $K_T^*(M_i^+) \to K_T^*(\phi|_S^{-1}(0))$ is surjective for all $i \in I$. We will show it by induction.

Notice that $K_T^*(M_0^+) \cong K_T^*(C_0) = K_T^*(\phi|_S^{-1}(0))$ by Theorem 5. Assume the inductive hypothesis that $K_T^*(M_i^+) \to K_T^*(C_0)$ is surjective for $0 \leq i \leq k-1$. By the equivariant homotopy equivalence, we have

$$K_T^*(M_k^-) \cong K_T^*(M_{k-1}^+)$$

Hence, we now have the surjection of

$$K_T^*(M_k^-) \cong K_T^*(M_{k-1}^+) \to K_T^*(C_0) \tag{1.1}$$

By Theorem 5, we know that $K_T^*(M_i^+) \to K_T^*(M_i^-)$ is a surjection for each i. Using it and equation (2.12), $K_T^*(M_k^+) \to K_T^*(C_0)$ is a surjection and hence our induction works.

Notice that $K_T^*(M) = K_T^*(\varinjlim M_i^+) = \varprojlim K_T^*(M_i^+)$, these equalities hold because we have the surjections $K_T^*(M_i^+) \to K_T^*(M_i^-)$ for all i. Hence we have the surjection result for $\kappa_S\colon K_T^*(M) \to K_T^*(C_0) = K_T^*(\phi|_S^{-1}(0))$, as desired. ■

Corollary 6 *Let T be a compact torus and M be a compact Hamiltonian T-space with moment map $\phi\colon M \to \mathfrak{t}^*$. Suppose that T acts freely on the zero level set of the moment map. Then*

$$\kappa\colon K_T^*(M) \to K^*(M//T)$$

is a surjection.

Proof. Choose a splitting of $T = S_1 \times S_2 \times ... \times S_{\mathrm{dim}T}$ where each S_i is quotiented out one at a time. Since T acts freely on the zero level set of the moment map, by Theorem 2, we have

$$\kappa_{S_1}\colon K_T^*(M) \to K_T^*(\phi|_{S_1}^{-1}(0)) \cong K_{T/S_1}^*(M//S_1)$$

is a surjection. By reduction in stages, we have

$$K_T^*(M) \to K_{T/S_1}^*(M//S_1) \to K_{T/(S_1\times S_2)}^*(M//(S_1\times S_2)) \to ... \to K_{T/T}^*(M//T) = K^*(M//T)$$

as desired. ■

We compute the kernel of our equivariant Kirwan map, which can be seen as a K-theoretic analogue of [G2].

Theorem 7 *Let T be a compact torus and M be a compact Hamiltonian T-space with moment map $\phi\colon M \to \mathfrak{t}^*$. Let T' be a subtorus in T. Let $\phi|_{T'}$ be the corresponding moment map for the Hamiltonian T'-action on M. For 0 a regular value of $\phi|_{T'}$, the kernel of the equivariant Kirwan map*

$$\kappa_{T'}\colon K_T^*(M) \to K_T^*(\phi|_{T'}^{-1}(0))$$

is the ideal $\langle K_T^{\mathfrak{t}'} \rangle$ *generated by* $K_T^{\mathfrak{t}'} = \cup_{\xi \in \mathfrak{t}'} K_T^{\xi}$ *where*

$K_T^{\xi} = \{\alpha \in K_T^*(M) \mid \alpha|_C = 0$ *for all connected components* C *of* M^T *with* $\langle \phi(C), \xi \rangle \leq 0\}$

Proof. Choose a splitting of $T' = S \times S \times ... \times S$. For each S in T', let $\phi|_S$ be the corresponding component of the moment map ϕ. By Theorem 3.1 in [HL2], the kernel of the equivariant Kirwan map κ_S is generated by K_T^{ξ} and $K_T^{-\xi}$ for a choice of generator $\xi \in \mathfrak{s}$. By successive application of this result to one-dimensional subtori of T', we get our result as desired. ■

1.4 K-theory of weight variety

1.4.1 Weight varieties

If $G = SU(n)$, we can naturally identify the set of Hermitian matrices H with $\mathfrak{g}^*$ by the trace map, i.e. $tr\colon (H) \to \mathfrak{g}^*$ defined by $A \mapsto i.tr(A)$. So $\lambda \in \mathfrak{t}^*$ is just a real diagonal matrix with entries $\lambda_1, \lambda_2, ..., \lambda_n$ in the diagonal. Through this identification, $M = \mathcal{O}_\lambda$ is an adjoint orbit of G through λ. The moment map corresponding to the T-action on $\mathcal{O}_\lambda$ takes a matrix to its diagonal entries, call it $\mu \in \mathfrak{t}^*$. Hence, $\mathcal{O}_\lambda//T(\mu)$, $\mu \in \mathfrak{t}^*$ is the symplectic quotient by the action of diagonal matrices at $\mu \in \mathfrak{t}^*$. The symplectic quotient consists of all Hermitian matrices with spectrum $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ and diagonal entries $\mu = (\mu_1, \mu_2, ..., \mu_n)$. We call this symplectic quotient $\mathcal{O}_\lambda//T(\mu)$ a *weight variety.*

If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ has the property that all entries have distinct values, then $\mathcal{O}_\lambda$ is a generic coadjoint orbit of $SU(n)$. It is symplectomorphic to a complete flag variety in $\mathbb{C}^n$. In this section, we mainly deal with the generic case unless otherwise stated. For more about the properties of weight varieties, see [Kn]. For the Weyl element action of any $\gamma \in W$ on $\lambda \in \mathfrak{t}^*$, we are going to use the notation $\lambda_\gamma = (\lambda_{\gamma^{-1}(1)}, ..., \lambda_{\gamma^{-1}(n)})$ for our notational convenience in our proof.

1.4.2 Divided difference operators and double Grothendieck polynomials

Let f be a polynomial in n variables, call them $x_1, x_2, ..., x_n$ (and possibly some other variables), the *divided difference operator* ∂_i is defined as

$$\partial_i f(..., x_i, x_{i+1}, ...) = \frac{f(..., x_i, x_{i+1}, ...) - f(...x_{i+1}, x_i, ...)}{x_i - x_{i+1}}$$

The *isobaric divided difference operator* is

$$\pi_i(f) = \partial_i(x_i f) = \frac{x_i f(..., x_i, x_{i+1}, ...) - x_{i+1} f(..., x_{i+1}, x_i, ...)}{x_i - x_{i+1}}$$

The *top Grothendieck polynomial* is

$$G_{id}(x, y) = \prod_{i<j}(1 - \frac{y_j}{x_i})$$

Note that the isobaric divided difference operator acts on G_{id} naturally by $\pi_i(G_{id})$. And $\pi_i(P.Q) = \pi_i(P)Q$ provided that Q is a symmetric polynomial in $x_1, x_2, ...x_n$. So this operator preserves the ideal generated by all differences of elementary symmetric polynomials $e_i(x_1, ..., x_n) - e_i(y_1, ..., y_n)$ for all $i = 1, ..., n$, denote this ideal by I. That is, the operator π_i acts on the ring R defined by

$$R = \frac{\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}, y_1^{\pm 1}, ..., y_n^{\pm 1}]}{I}$$

For any element $\omega \in S_n$, ω has reduced word expression $\omega = s_{i_1} s_{i_2} ... s_{i_l}$ (where each s_{i_j} is a transposition between i_j, i_{j+1}). We can define the corresponding operator:

$$\pi_{s_{i_1}s_{i_2}...s_{i_l}} = \pi_{s_{i_1}}...\pi_{s_{i_l}}$$

which is independent of the choice of the reduced word expression.

For any $\mu \in S_n$, the *double Grothendieck polynomial* G_μ is:

$$\pi_{\mu^{-1}} G_{id} = G_\mu$$

Define the permuted double Grothendieck polynomials G_ω^γ by

$$G_\omega^\gamma(x, y) = G_{\gamma^{-1}\omega}(x, y_\gamma) = \pi_{\omega^{-1}\gamma} G_{id}(x, y_\gamma)$$

where y_γ means the permutation of the $y_1, ..., y_n$ variables by γ.

Example 8 For $G = SU(3), W = S_3$, we have

$$G_{id} = (1 - \frac{y_2}{x_1})(1 - \frac{y_3}{x_1})(1 - \frac{y_3}{x_2})$$

$$\begin{aligned} G_{(23)}^{(12)} &= \pi_{(23)(12)} G_{id}(x, y_{(12)}) \\ &= \pi_{(23)(12)} \left(1 - \frac{y_3}{x_1}\right)\left(1 - \frac{y_1}{x_1}\right)\left(1 - \frac{y_3}{x_2}\right) \\ &= \pi_{(23)} \left(\frac{x_1 \left(1 - \frac{y_3}{x_1}\right)\left(1 - \frac{y_1}{x_1}\right)\left(1 - \frac{y_3}{x_2}\right) - x_2 \left(1 - \frac{y_3}{x_2}\right)\left(1 - \frac{y_1}{x_2}\right)\left(1 - \frac{y_3}{x_2}\right)}{x_1 - x_2} \right) \\ &= \pi_{(23)} \left(1 - \frac{y_3}{x_1}\right)\left(1 - \frac{y_3}{x_2}\right) \\ &= \left(1 - \frac{y_3}{x_1}\right) \end{aligned}$$

1.4.3 T-equivariant K-theory of flag varieties

We have the following formula for $K_T^*(SU(n)/T)$ (see [F]):

$$K_T^*(SU(n)/T) \cong R(T) \otimes_{R(G)} R(T) \cong R(T) \otimes_{\mathbb{Z}} R(T)/J$$

where $R(G) \cong R(T)^W$ and $R(T)$ are the character rings of G, T where $G = SU(n)$ respectively. $J \subset R(T) \otimes_{\mathbb{Z}} R(T)$ is the ideal generated by $a \otimes 1 - 1 \otimes a$ for all elements $a \in R(T)^W$. $R(T)^W$ is the Weyl group invariant of $R(T)$.

$R(T)$ can be written as a polynomial ring:

$$R(T) = K_T^*(pt) \cong \mathbb{Z}[a_1^{\pm 1}, ..., a_{n-1}^{\pm 1}]$$

In the equation $K_T^*(X) = R(T) \otimes_{\mathbb{Z}} R(T)/J$, denote the first copy of $R(T)$ by $\mathbb{Z}[y_1^{\pm 1}, ..., y_{n-1}^{\pm 1}]$ and the second copy of $R(T)$ by $\mathbb{Z}[x_1^{\pm 1}, ..., x_{n-1}^{\pm 1}]$. Then the ideal J is generated by $e_i(y_1, ..., y_{n-1}) - e_i(x_1, ..., x_{n-1}), i = 1, ..., n-1$, where e_i is the i-th symmetric polynomial in the corresponding variables. Equivalently,

$$K_T^*(Fl(\mathbb{C}^n)) \cong \frac{\mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}, x_1, ..., x_n]}{(J, (\prod_{i=1}^n y_i) - 1)} \tag{1.2}$$

Notice that $x_i^{-1}, i = 1, ..., n$ can be generated by some elements in the ideal J, where J is the ideal generated by $e_i(y_1, ..., y_n) - e_i(x_1, ..., x_n)$, for all $i = 1, ..., n$.

Let $G^{\mathbb{C}}$ be the complexification of a compact Lie group G, $B \subset G^{\mathbb{C}}$ be a Borel subgroup. In our case, $G = SU(n), G^{\mathbb{C}} = SL(n, \mathbb{C})$. Then $G/T \approx G^{\mathbb{C}}/B$. $G^{\mathbb{C}}/B$ consists of even-real-dimensional Schubert cells, C_ω indexed by the elements in the Weyl Group W. That is,

$$C_\omega = B_{-}\omega B/B, \omega \in W$$

The closures of these cells are called *Schubert varieties*:

$$X_\omega = \overline{B_{-}\omega B}/B, \omega \in W$$

For each Schubert variety $X_\omega, \omega \in W$, denote the T-equivariant structure sheaf on $X_\omega \subset G^{\mathbb{C}}/B$ by $[\mathcal{O}_{X_\omega}]$. It extends to the whole of $G^{\mathbb{C}}/B$ by defining it to be zero in the complement of X_ω. Since $[\mathcal{O}_{X_\omega}]$ is a T-equivariant coherent sheaf on $G^{\mathbb{C}}/B$, it determines a class in $K_0(T, G^{\mathbb{C}}/B)$, the Grothendieck group constructed from the semigroup whose elements are the isomorphism classes of T-equivariant locally free sheaves. The elements $[\mathcal{O}_{X_\omega}]_{\omega \in W}$ form a $R(T)$-basis for the $R(T)$-module $K_0(T, G^{\mathbb{C}}/B)$. Since there is a canonical isomorphism between $K_0(T, G^{\mathbb{C}}/B)$ and $K_T(G^{\mathbb{C}}/B) = K_T(G/T)$ (see [KK]), by abuse of notation we also denote $[\mathcal{O}_{X_\omega}]_{\omega \in W}$ as a linear basis in $K_T^*(G/T)$ over $R(T)$.

On the other hand, the double Grothendieck polynomials $G_\omega, \omega \in W$, as Laurent polynomials in variables $x_i, y_i, i = 1, 2, ..., n$ form a basis of $K_{T\times B}(pt) \cong R(T) \otimes_{\mathbb{Z}} R(T)$ over $K_T(pt) \cong R(T)$. By the equivariant homotopy principle,

$$K_{T\times B}(pt) = K_{T\times B}(M_{n\times n})$$

where $M_{n\times n}$ denote the set of all $n \times n$ matrices over $\mathbb{C}$. By a theorem of [KM], we are able to identify the classes generated by matrix Schubert varieties in $K_{T\times B}(M_{n\times n})$ (matrix Schubert varieties form a cell decomposition of $M_{n\times n}/B$) with the double Grothendieck polynomials in $K_{T\times B}(pt)$. The open embedding $\iota\colon GL(n, \mathbb{C}) \to M_{n\times n}$ induces a map in equivariant K-theory:

$$\iota^*\colon K_{T\times B}(M_{n\times n}) \to K_{T\times B}(GL(n, \mathbb{C})) = K_T(GL(n, \mathbb{C})/B) = K_T(SU(n)/T)$$

Under this map, the classes generated by the matrix Schubert varieties in $K_{T\times B}(M_{n\times n})$ are mapped to the classes, $[\mathcal{O}_{X_\omega}] \in K_T(SU(n)/T)$, of the corresponding Schubert varieties in $SU(n)/T$. By identifications of the double Grothendieck polynomials in $K_{T\times B}(pt)$ and the classes generated by the matrix Schubert varieties in $K_{T\times B}(M_{n\times n})$, the map ι^* sends the double Grothendieck polynomials to the T-equivariant structure sheaves $[\mathcal{O}_{X_\omega}]_{\omega \in W}$, as a $R(T)$-basis in $K_T(G/T) \cong R(T) \otimes_{R(G)} R(T)$. For more results about the geometry and combinatorics of double Grothendieck polynomials and matrix Schubert varieties, see [KM].

By abuse of notations, from now on, we will take the double Grothendieck polynomials $G_\omega(x,y), \omega \in W$ as a basis in $K_T^*(SU(n)/T)$ over $R(T)$. Under our notations, notice that the top double Grothendieck polynomial $G_{id}(x,y)$ corresponds to the T-equivariant structure sheaf $[\mathcal{O}_{X_{\omega_0}}]$, where $\omega_0 \in W$ is the permutation with the longest length, i.e. $\omega_0 = s_n s_{n-1} ... s_3 s_2 s_1$.

For more about K-theory and T-equivariant K-theory of flag varieties, for example, see [F] and [KK].

1.4.4 Restriction of T-equivariant K-theory of flag varieties to the fixed-point sets

Recall that the flag variety is compact, by [HL2], we have the Kirwan injectivity map, i.e. the map

$$\iota^* \colon K_T^*(Fl(\mathbb{C}^n)) \to K_T^*(Fl(\mathbb{C}^n)^T)$$

induced by the inclusion ι from $Fl(\mathbb{C}^n)^T$ to $Fl(\mathbb{C})$ is injective. We compute the restriction explicitly here. Notice that $Fl(\mathbb{C}^n)^T$ is indexed by the elements in the Weyl group $W = S_n$. The T-action on $\mathbb{C}^n$ splits into a sum of 1-dimensional vector spaces, call them $l_1, ..., l_n$. The fixed points of T-action are the flags which can be written as:

$$p_\omega = \langle l_{\omega(1)} \rangle \subset \langle l_{\omega(1)}, l_{\omega(2)} \rangle \subset \langle l_{\omega(1)}, l_{\omega(2)}, l_{\omega(3)} \rangle \subset ... \subset \langle l_{\omega(1)}, ..., l_{\omega(n)} \rangle = \mathbb{C}^n$$

where $\omega \in W$ and call

$$p_{id} = \langle l_1 \rangle \subset \langle l_1, l_2 \rangle \subset \langle l_1, l_2, l_3 \rangle \subset ... \subset \langle l_1, ..., l_n \rangle = \mathbb{C}^n$$

the base flag of $\mathbb{C}^n$. The description of the restriction map is as follow:

Theorem 9 *Let p_ω be a fixed point in $Fl(\mathbb{C}^n)^T$ as above. The inclusion $\iota_\omega\colon p_\omega \to Fl(\mathbb{C}^n)$ induces a restriction*

$$\iota_\omega^*\colon K_T^*(Fl(\mathbb{C}^n)) \to K_T^*(p_\omega) = R(T) = \mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}]$$

such that $\iota_\omega^*\colon y_i^{\pm 1} \to y_i^{\pm 1}, \iota_\omega^*\colon x_i \to y_{\omega(i)}, i = 1, ..., n$. Also, the inclusion map $\iota\colon Fl(\mathbb{C}^n)^T \to Fl(\mathbb{C}^n)$ induces a map

$$\iota^*\colon K_T^*(Fl(\mathbb{C}^n)) \to K_T^*(Fl(\mathbb{C}^n)^T) = \oplus_{p_\omega, \omega \in W} \mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}]$$

whose further restriction to each component in the direct sum is just the map ι_ω^*.

Proof. Consider $K_T^*(Fl(\mathbb{C}^n))$ as a module over $K_T^*(pt) = \mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}]$, the map

$$K_T^*(Fl(\mathbb{C}^n)) \to K_T^*(p)$$

induced by mapping any point p into $Fl(\mathbb{C}^n)$ is a surjective $R(T)$-module homomorphism and $K_T^*(Fl(\mathbb{C}^n))$ has a linear basis over $K_T^*(p) = R(T) = \mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}]$. Hence we must have $\iota_\omega^*\colon y_i^{\pm 1} \to y_i^{\pm 1}, i = 1, ..., n$, for all $\omega \in W$. To find the image of x_i under ι_ω^*, first, notice that in $K_T^*(pt)$, $y_i = [pt \times \mathbb{C}_i]$. $\mathbb{C}_i$ corresponds to the action of $T = S^1 \times ... \times S^1$ on the i-th copy of $\mathbb{C}^n = \mathbb{C} \times ... \times \mathbb{C}$ with weight 1 and acting trivally on all the other copies of $\mathbb{C}$. More generally, $y_{\omega(i)} = [pt \times \mathbb{C}_{\omega(i)}]$. In $K_T^*(p_\omega)$, $y_{\omega(i)} = [p_\omega \times \mathbb{C}_{\omega(i)}]$, where $p_\omega \times \mathbb{C}_{\omega(i)}$ is the T-line bundle over the point p_ω. By the Hodgkin's result (see [Ho]), $K_T^*(G/T) = R(T) \otimes_{R(G)} K_G^*(G/T) (\cong R(T) \otimes_{R(G)} R(T))$. Following our use of notations in 1.4.3, x_i comes from the second copy of $R(T)$ (which is isomorphic to $K_G^*(G/T)$ under our identification). Hence, each x_i is the class represented by the G-line bundle $G \times_T \mathbb{C}_i$ over G/T. T acts on $G \times \mathbb{C}_i$ diagonally and $G \times_T \mathbb{C}_i$ is the orbit space of the T-action. In particular, x_i is a T-line bundle over G/T by restriction of G-action to T-action. So, $\iota_\omega^*(x_i)$ is just the pullback T-line bundle of the map $\iota_\omega\colon p_\omega \to Fl(\mathbb{C}^n)$. For $i = 1$, we have $\iota_\omega^*(x_1) = [p_\omega \times \mathbb{C}_{\omega(1)}] = y_{\omega(1)}$. Similarly, $\iota_\omega^*(x_i) = y_{\omega(i)}$ for $i = 2, ..., n$. And hence the result. ■

1.4.5 Relations between double Grothendieck polynomials and the Bruhat Ordering

Recall our definition of the permuted double Grothendieck polynomials G_ω^γ in Section 1.4.2:

$$G_\omega^\gamma(x,y) = G_{\gamma^{-1}\omega}(x,y_\gamma) = \pi_{\omega^{-1}\gamma}G_{id}(x,y_\gamma)$$

where y_γ indicates the permutation of the $y_1,...,y_n$ variables by γ. For $\gamma \in W$, define the permuted Bruhat ordering by

$$v \leq_\gamma \omega \Leftrightarrow \gamma^{-1}v \leq \gamma^{-1}\omega$$

Notice that the permuted Bruhat ordering is related to the Schubert varieties in the following way: Each of the T-fixed points of a Schubert variety X_ω sits in one Schubert cell C_v (the interior of a Schubert variety) for $v \leq \omega$. So the T-fixed point set can be identified as:

$$(X_\omega)^T = \{v \mid v \leq \omega\}$$

For a fixed $\gamma \in W$, we can define the permuted Schubert varieties by

$$X_\omega^\gamma = \overline{\gamma B_- \gamma^{-1}\omega B}/B$$

for any $\omega \in W$. Then the T-fixed point set of X_ω^γ are

$$(X_\omega^\gamma)^T = \{v \mid v \leq_\gamma \omega\}$$

Notice that $\{X_\omega^\gamma\}_{\omega\in W}$ also form a cell decomposition of $G^{\mathbb{C}}/B \approx G/T$.

Define the support of the permuted double Grothendieck polynomials by

$$\mathrm{Supp}(G_\omega^\gamma) = \{z \in W \mid G_\omega^\gamma|_z \neq 0\}$$

Here we consider G^{γ}_{ω} as an element in $K^*_T(Fl(\mathbb{C}^n))$ (see Section 1.4.3). So $G^{\gamma}_{\omega}|_z$ is the image of G^{γ}_{ω} under the restriction of the Kirwan injective map at the point $z \in W$. That is,

$$\iota^*|_z \colon K^*_T(Fl(\mathbb{C}^n)) \to K^*_T(p_z)$$

Notice that the restriction rule follows Theorem 9. That is,

$$G^{\gamma}_{\omega}(x,y)|_z = G^{\gamma}_{\omega}(x_1, x_2, ..., x_n, y_1, ..., y_n)|_z = G_{\omega}(y_{z(1)}, y_{z(2)}, ..., y_{z(n)}, y_1, ..., y_n)$$

Example 10 Using the same notations as in the example in 1.4.2, $G^{(12)}_{(23)} = (1 - \frac{y_3}{x_1}) \in K^*_T(Fl(\mathbb{C}^3))$. There are six fixed points for each element in S_3,

$$G^{(12)}_{(23)}|_{(23)} \neq 0, G^{(12)}_{(23)}|_{(123)} \neq 0, G^{(12)}_{(23)}|_{(13)} = 0$$

$$G^{(12)}_{(23)}|_{(132)} = 0, G^{(12)}_{(23)}|_{(12)} \neq 0, G^{(12)}_{(23)}|_{id} \neq 0$$

So the support of a permuted double Grothendieck polynomial contains $id, (12), (23), (123)$. On the other hand,

$$\begin{aligned}(X^{(12)}_{(23)})^T &= \{v \in S_3 \mid (12)v \leq (12)(23) = (123)\} \\ &= \{v \in S_3 \mid (12)v \leq id, (12), (23) \quad or \quad (123)\} \\ &= \{v \in S_3 \mid v \leq (12), id, (123) \quad or \quad (23)\}\end{aligned}$$

which is the same as $\mathrm{Supp}(G^{(12)}_{(23)})$.

Now we show a fundamental relation between the permuted double Grothendieck polynomials and the permuted Bruhat Orderings:

Theorem 11 *The support of a permuted double Grothendieck polynomial G^{γ}_{ω} is $\{v \mid v \leq_{\gamma} \omega\}$*

Proof. We need to show $\text{Supp}(G_\omega) = (X_\omega)^T$ first. We do it by induction on the length of $v \in W$, $l(v)$, which stands for the minimum number of transpositions in all the possible choices of word expressions of v.

For $\omega = id$, G_{id} is just the top Grothendieck polynomial. It is non-zero only at the identity and zero at all the other elements. Assume the inductive hypothesis that $\text{Supp}(G_\omega) = (X_\omega)^T$ for all $l(\omega) \leq l-1$. Consider $v \in W, l(v) = l$, write $v = s_{i_1}s_{i_2}...s_{i_l}$ where each s_{i_j} is a transposition of elements i_j, i_j+1, let $\omega = vs_{i_l} = s_{i_1}...s_{i_{l-1}}$, so $l(\omega) = l-1$ and

$$\begin{aligned} G_v|_z &= \pi_{v^{-1}}G|_z = \pi_{i_l}\pi_{i_{l-1}}...\pi_{i_1}G|_z = \pi_{i_l}G_\omega|_z \\ &= \frac{x_{i_l}G_\omega(x,y) - x_{i_l+1}G_\omega(x_{s_{i_l}},y)}{x_{i_l} - x_{i_l+1}}|_z \\ &= \frac{y_{z(i_l)}G_\omega(y_z,y) - y_{z(i_l+1)}G_\omega(y_{zs_{i_l}},y)}{y_{z(i_l)} - y_{z(i_l+1)}} \end{aligned} \tag{1.3}$$

First, to prove that $\text{Supp}(G_v) \subset (X_v)^T$, suppose that $z \notin (X_v)^T$, then $z \notin (X_\omega)^T$ since $\omega \leq v$. Since $l(\omega) = l-1$, we have $z \notin \text{Supp}(G_\omega)$. That is $G_\omega(y_z, y) = 0$. Hence,

$$G_v|_z = \frac{-y_{z(i_l+1)}G_\omega(y_{zs_{i_l}},y)}{y_{z(i_l)} - y_{z(i_l+1)}}$$

We claim that it is zero. If it were not zero, then $G_\omega(y_{zs_{i_l}}, y) = G_\omega(x,y)|_{zs_{i_l}} \neq 0$. Equivalently, $zs_{i_l} \in \text{Supp}(G_\omega) = (X_\omega)^T$. If $z < zs_{i_l}$, then $z \in (X_\omega)^T$ which contradicts $z \notin \text{Supp}(G_\omega)$ shown before. If $z > zs_{i_l}$, then s_{i_l} increases the length of zs_{i_l}. Then $zs_{i_l} \in (X_\omega)^T$ implies that $z \in (X_v)^T$ which contradicts $z \notin (X_v)^T$. So the claim is proved. i.e. $z \notin (X_v)^T \Rightarrow G_v|_z = 0 \Leftrightarrow z \notin \text{Supp}(G_v)$.

Second, we need to prove that $(X_v)^T \subset \text{Supp}(G_v)$. Suppose that $z \notin \text{Supp}(G_v)$, i.e. $G_v|_z = 0$. Assume that $z \in (X_v)^T$. From (1.3),

$$y_{z(i_l)}G_\omega(y_z,y) = y_{z(i_l+1)}G_\omega(y_{zs_{i_l}},y) \tag{1.4}$$

Now there are two cases, $z = v$ and $z \neq v$. We consider these two cases separately.

If $z = v$, then $z \not\leq w$ (since $l(\omega) = l - 1$ and $l(z) = l(v) = l$)$\Leftrightarrow z \notin (X_\omega)^T = \text{Supp}(G_\omega) \Leftrightarrow G_\omega|_z = 0 \Leftrightarrow G_\omega(y_z, y) = 0 \Leftrightarrow G_\omega(y_{zs_{i_l}}, y) = 0$. The last equality is by (1.4). So we now have $G_\omega(x,y)|_{zs_{i_l}} = 0 \Leftrightarrow zs_{i_l} \notin \text{Supp}(G_\omega) = (X_\omega)^T$. Since $zs_{i_l} = vs_{i_l} = \omega \in (X_\omega)^T$, it's a contradiction.

If $z \neq v$, then $l(z) < l(v)$, then $l(z) \leq l-1$. Let $t \in W$ with $l(t) = l-1$ such that $z \leq t$. Although t may not be the same as ω but $t = v's_{i_j}$ for some $j \in 1, ..., l$ (v' is another word expression for v) By our inductive hypothesis, $\text{Supp}(G_t) = (X_t)^T$, so

$$z \in \text{Supp}(G_t) \Leftrightarrow G_t(y_z, y) = G_t(x,y)|_z \neq 0 \tag{1.5}$$

But $zs_{i_j} \not\leq t$ implies that $zs_{i_j} \notin (X_t)^T = \text{Supp}(G_t)$. By (1.4), (but now we have ω replaced by t), $G_t(y_{zs_{i_j}}, y) = 0$. By (1.3) and (1.5), we have $G_v|_z \neq 0$ contradicting our initial assumption that $z \notin \text{Supp}(G_v)$.

Hence, we have $z \notin \text{Supp}(G_v) \Rightarrow z \notin (X_v)^T$. The induction step is done.

Then we need to show that the statement holds for the permuted double Grothendieck polynomials, i.e. $\text{Supp}(G^\gamma_\omega) = (X^\gamma_\omega)^T$. By definition, $G^\gamma_\omega(x,y) = G_{\gamma^{-1}\omega}(x, y_\gamma)$, so,

$$\text{Supp}G_{\gamma^{-1}\omega}(x,y) = (X_{\gamma^{-1}\omega})^T = \{v \in W \mid v \leq \gamma^{-1}\omega\}$$

By permuting the y's variables by γ, we obtain

$$\begin{aligned}
\text{Supp}(G^\gamma_\omega) &= \text{Supp}G_{\gamma^{-1}\omega}(x, y_\gamma) \\
&= \{\gamma v \in W \mid v \leq \gamma^{-1}\omega\} \\
&= \{v \in W \mid \gamma^{-1}v \leq \gamma^{-1}\omega\} \\
&= \{(X^\gamma_\omega)^T\}
\end{aligned}$$

■

1.4.6 Main theorem

In this subsection, we prove the following result:

Theorem 12 *Let $\mathcal{O}_\lambda$ be a generic coadjoint orbit of $SU(n)$. Then*

$$K^*(\mathcal{O}_\lambda//T(\mu)) \cong \frac{\mathbb{Z}[x_1, ..., x_n, y_1^{\pm 1}]}{(I, ((\prod_{i=1}^n y_i) - 1), \pi_v G(x, y_r))}$$

for all $v, r \in S_n$ such that $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{r(i)}$ for some $k = 1, ..., n-1$. I is the difference between $e_i(x_1, ..., x_n) - e_i(y_1, ..., y_n)$ for all $i = 1, ..., n$, where e_i is the i-th elementary symmetric polynomial.

It is a K-theoretic analogue of the main result in [G1].

To make the symplectic picture more explicit, we denote $M = \mathcal{O}_\lambda \approx SU(n)/T$ to be the generic coadjoint orbit. So we have $K_T^*(M) = K_T^*(\mathcal{O}_\lambda) = K_T^*(Fl(\mathbb{C}^n))$. For $\lambda \in \mathfrak{t}^*, \lambda = (\lambda_1, ..., \lambda_n)$, assume that $\lambda_1 > \lambda_2 > ... > \lambda_n$, and $\lambda_1 + ... + \lambda_n = 0$. Since $M = \mathcal{O}_\lambda$ is compact, M^T has only a finite number of points. The kernel of the Kirwan map κ is generated by a finite number of components, see Theorem 7 and [HL2]. More specifically, let $M_\xi^\mu \subset M, \xi \in \mathfrak{t}$ be the set of points where the image under the moment map ϕ lies to one side of the hyperplane $\xi^\perp$ through $\mu = (\mu_1, ..., \mu_n) \in \mathfrak{t}^*$, i.e.

$$M_\xi^\mu = \{m \in M \mid \langle \xi, \phi(m) \rangle \leq \langle \xi, \mu \rangle\}$$

Then the kernel of κ is generated by

$$K_\xi = \{\alpha \in K_T^*(M) \mid \mathrm{Supp}(\alpha) \subset M_\xi^\mu\}$$

That is,

$$\ker(\kappa) = \sum_{\xi \in \mathfrak{t}} K_\xi$$

Now, we are going to compute the kernel explicitly. Our proof is similar to the results in [G1]. In [G1], Goldin proved a very similar result in rational cohomology by using the permuted double Schubert polynomials as a linear basis of $H_T^*(M)$ over $H_T^*(pt)$. In K-theory, the permuted double Grothendieck polynomials are used as a linear basis of $K_T^*(M)$ over $K_T^*(pt) \cong R(T)$. The following lemma will be used in our proof of Theorem 12:

Lemma 13 *Let $\mathcal{O}_\lambda$ be a generic coadjoint orbit of $SU(n)$ through $\lambda \in \mathfrak{t}^*$. Let $\alpha \in K_T^*(\mathcal{O}_\lambda)$ be a class with Supp$(\alpha) \subset (\mathcal{O}_\lambda)^\mu_\xi$. Then there exists some $\gamma \in W$ such that if α is decomposed in the $R(T)$-basis $\{G^\gamma_\omega\}_{\omega \in W}$ as*

$$\alpha = \sum_{\omega \in W} a^\gamma_\omega G^\gamma_\omega$$

where $a^\gamma_\omega \in R(T)$, then $a^\gamma_\omega \neq 0$ implies Supp$(G^\gamma_\omega) \subset (\mathcal{O}_\lambda)^\mu_\xi$. Indeed, γ can be chosen such that ξ attains its minimum at $\phi(\lambda_\gamma)$, where $\lambda_\gamma = (\lambda_{\gamma^{-1}(1)}, ..., \lambda_{\gamma^{-1}(n)}) \in \mathfrak{t}^*$.

Proof. The proof is essentially the same as Theorem 3.1 in [G1]. ■

Proof of Theorem 12. : Let e_i be the coordinate functions on $\mathfrak{t}^*$. That is, for $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathfrak{t}^*$, $e_i(\lambda) = \lambda_i$. For $\gamma \in S_n$, define $\eta^\gamma_k$by

$$\eta^\gamma_k = \sum_{i=k+1}^{n} e_{\gamma(i)}$$

We compute $M^\mu_{\eta^\gamma_k}$ explicitly:

$$\begin{aligned} M^\mu_{\eta^\gamma_k} &= \{m \in M \mid \langle \eta^\gamma_k, \phi(m) \rangle \leq \langle \eta^\gamma_k, \mu \rangle\} \\ &= \{m \in M \mid \eta^\gamma_k(\phi(m)) \leq \eta^\gamma_k(\mu)\} \\ &= \{m \in M \mid \eta^\gamma_k(\phi(m)) \leq \textstyle\sum_{i=k+1}^{n} \mu_{\gamma(i)}\} \end{aligned}$$

For any $\omega \in W$,

$$\begin{aligned}\eta_k^\gamma(\lambda_\omega) &= \sum_{i=k+1}^{n} e_{\gamma(i)}(\lambda_\omega) = \sum_{i=k+1}^{n} e_{\gamma(i)}(\lambda_{\omega^{-1}(1)}, ..., \lambda_{\omega^{-1}(n)}) \\ &= \sum_{i=k+1}^{n} \lambda_{\omega^{-1}\gamma(i)}\end{aligned}$$

Notice that η_k^γ attains minimum at λ_γ (due to our assumption that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$) and respects the permuted Bruhat ordering, i.e.

$$\eta_k^\gamma(\lambda_v) \leq \eta_k^\gamma(\lambda_\omega)$$

if $v \leq_\gamma \omega$. By restriction to the domain $\mathrm{Supp}(G_\omega^\gamma) = (X_\omega^\gamma)^T = \{v \in W \mid v \leq_\gamma w\} = \{v \in W \mid \gamma^{-1}v \leq \gamma^{-1}\omega\}$, η_k^γ attains its maximum at λ_ω and minimum at λ_γ. If $\eta_k^\gamma(\lambda_\omega) = \sum_{i=k+1}^n \lambda_{\omega^{-1}\gamma(i)} < \sum_{i=k+1}^n \mu_{\gamma(i)}$, then for $v \in (X_\omega^\gamma)^T$,

$$\eta_k^\gamma(\lambda_v) = \sum_{i=k+1}^{n} \lambda_{v^{-1}\gamma(i)} \leq \sum_{i=k+1}^{n} \lambda_{\omega^{-1}\gamma(i)} < \sum_{i=k+1}^{n} \mu_{\gamma(i)}$$

and hence

$$\mathrm{Supp}(G_\omega^\gamma) = (X_\omega^\gamma)^T = \{v \in W \mid \gamma^{-1}v \leq \gamma^{-1}\omega\} \subset M^\mu_{\eta_k^\gamma}$$

Since $G_\omega^\gamma(x,y) = \pi_{\omega^{-1}\gamma}G(x,y_\gamma)$, we have $\pi_v G(x,y_\gamma) \in \ker(\kappa)$ if $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\gamma(i)}$.

For the other direction, we need to show that the classes $\pi_v G(x,y_\gamma)$ with $v, \gamma \in W$ having the property that $\sum_{i=k+1}^n \lambda_{v(i)} < \sum_{i=k+1}^n \mu_{\gamma(i)}$ for some $k \in \{1, ..., n-1\}$ actually generate the whole kernel. Let $\alpha \in K_T^*(M)$ be a class in $\ker(\kappa)$, so $\mathrm{Supp}(\alpha) \subset M_\xi^\mu$ for some $\xi \in \mathfrak{t}$. We take $\gamma \in W$ such that $\xi(\lambda_\gamma)$ attains its minimum. Decompose α over the $R(T)$-basis $\{G_\omega^\gamma\}_{\omega\in W}$,

$$\alpha = \sum_{\omega\in W} a_\omega^\gamma G_\omega^\gamma$$

where $a^\gamma_\omega \in R(T)$. By Lemma 13, we need to show that $\mathrm{Supp}(G^\gamma_\omega) \subset M^\mu_{\eta^\gamma_k}$ for some k. Since η^γ_k is preserved by the permuted Bruhat ordering and attains its maximum at λ_ω in the domain $\mathrm{Supp}(G^\gamma_\omega)$, we just need to show that

$$\eta^\gamma_k(\lambda_\omega) < \eta^\gamma_k(\mu) \tag{1.6}$$

for some k. It is actually purely computational: Suppose (1.6) does not hold for all k. We have

$$\begin{array}{rcl} \lambda_{\omega^{-1}\gamma(n)} & \geq & \mu_{\gamma(n)} \\ & \vdots & \\ \lambda_{\omega^{-1}\gamma(2)} + ... + \lambda_{\omega^{-1}\gamma(n)} & \geq & \mu_{\gamma(2)} + ... + \mu_{\gamma(n)} \end{array}$$

For $\xi = \sum_{i=1}^n b_i e_i, b_1, ..., b_n \in \mathbb{R}$ (recall that ξ attains its minmum at λ_γ by our choice of $\gamma \in W$), we have $\xi(\lambda_\gamma) \leq \xi(\lambda_{s_i\gamma})$ where s_i is a transposition of i and $i+1$. And hence

$$b_i\lambda_{\gamma^{-1}(i)} + b_{i+1}\lambda_{\gamma^{-1}(i+1)} \leq b_i\lambda_{\gamma^{-1}(i+1)} + b_{i+1}\lambda_{\gamma^{-1}(i)}$$

By our assumption that $\lambda_i > \lambda_{i+1}$, we get $b_{\gamma(i)} \leq b_{\gamma(i+1)}$. And hence $b_{\gamma(1)} \leq b_{\gamma(2)} \leq ... \leq b_{\gamma(n)}$. Then,

$$\begin{array}{rcl} (b_{\gamma(n)} - b_{\gamma(n-1)})\lambda_{\omega^{-1}\gamma(n)} & \geq & (b_{\gamma(n)} - b_{\gamma(n-1)})\mu_{\gamma(n)} \\ (b_{\gamma(n-1)} - b_{\gamma(n-2)})(\lambda_{\omega^{-1}\gamma(n-1)} + \lambda_{\omega^{-1}\gamma(n)}) & \geq & (b_{\gamma(n-1)} - b_{\gamma(n-2)})(\mu_{\gamma(n-1)} + \mu_{\gamma(n)}) \\ & \vdots & \\ (b_{\gamma(2)} - b_{\gamma(1)})(\lambda_{\omega^{-1}\gamma(2)} + ... + \lambda_{\omega^{-1}\gamma(n)}) & \geq & (b_{\gamma(2)} - b_{\gamma(1)})(\mu_{\gamma(2)} + ... + \mu_{\gamma(n)}) \end{array}$$

Using $\sum_{i=1}^{n}\lambda_i = 0 = \sum_{i=1}^{n}\mu_i$ and summing up all the above inequalities to get

$$\begin{aligned} \sum_{i=1}^{n} b_{\gamma(i)}\lambda_{\omega^{-1}\gamma(i)} &\geq \sum_{i=1}^{n} b_i\mu_i \\ \Leftrightarrow \sum_{i=1}^{n} b_i\lambda_{\omega^{-1}(i)} &\geq \sum_{i=1}^{n} b_i\mu_i \\ \Leftrightarrow \xi(\lambda_\omega) &\geq \xi(\mu) \end{aligned}$$

the last inequality contradicts $\mathrm{Supp}(\alpha) \subset M_\xi^\mu$ since λ_ω has the property that $\omega \in \mathrm{Supp}(\alpha)$. So (1.6) is true.

So the kernel $\ker(\kappa)$ is generated by the set $\pi_v G(x, y_\gamma)$ for $v, \gamma \in W$ satisfying $\sum_{i=k+1}^{n}\lambda_{v(i)} < \sum_{i=k+1}^{n}\mu_{\gamma(i)}$ for some $k = 1, ..., n-1$. By (1.2) and the surjectivity of the Kirwan map κ,

$$\kappa\colon K_T^*(SU(n)/T) = K_T^*(\mathcal{O}_\lambda) \to K_T^*(\phi^{-1}(\mu)) \cong K^*(\mathcal{O}_\lambda//T(\mu))$$

It implies that

$$K^*(\mathcal{O}_\lambda//T(\mu)) = K_T^*(\mathcal{O}_\lambda)/\ker(\kappa)$$

With $\ker(\kappa)$ explicitly computed and by (1.2), Theorem 12 is proved. ∎

1.5 K-theory of symplectic reduction of generic coadjoint orbits

The goal of this section is to generalize the results in 1.4 to the K-theory of symplectic reduction of generic coadjoint orbits.

For a compact, connected and simply connected Lie group G, we consider the coadjoint orbit $\mathcal{O}_\lambda$ of G through a point $\lambda \in \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of Lie algebra of the maximal torus $T \subset G$. $\mathcal{O}_\lambda$ is diffeomorphic to the flag variety G/T. $\mathcal{O}_\lambda$ is a symplectic manifold with a symplectic form ω known as the Kirillow-Kostant-Souriau form. The torus T acts on $\mathcal{O}_\lambda$ by left multiplication on the coset gT. The T-action on $\mathcal{O}_\lambda$ is Hamiltonian. Hence, there is a moment map

$$\phi\colon \mathcal{O}_\lambda \to \mathfrak{t}^*$$

The image of the moment map ϕ is the convex hull of $W.\lambda$, a Weyl group orbit of λ. We assume that λ sits in the fundamental chamber in $\mathfrak{t}^*$. For a regular value $\mu \in \phi(\mathcal{O}_\lambda)$, we have the symplectic reduction at μ:

$$\phi^{-1}(\mu)/T = \mathcal{O}_\lambda//T(\mu)$$

By Corollary 6, we have the Kirwan surjective map:

$$\kappa\colon K_T^*(\mathcal{O}_\lambda) \to K_T^*(\phi^{-1}(\mu))$$

For the T-equivariant K-theory of $\mathcal{O}_\lambda \cong G/T$, we have the following formula for $K_T^*(G/T)$, see [KK]:

$$K_T^*(G/T) \cong R(T) \otimes_{R(G)} R(T)$$

The inclusion i_T from $(G/T)^T$ to G/T induces a map

$$i_T^*\colon K_T(G/T) \to K_T((G/T)^T) \cong F(W, R(T))$$

where $F(W, R(T))$ is the set of functions from the Weyl group W to $R(T)$. It is shown in [KK] that i_T^* is injective and the image $i_T^*(K_T(G/T))$ is isomorphic to a $R(T)$-subalgebra in $F(W, R(T))$, in which a $R(T)$-basis $\{\phi_\omega\}_{\omega\in W}$ exists. By pulling this $R(T)$-basis back through i_T^*, we obtain a $R(T)$-basis of $K_T(G/T)$, denote each element in this basis by $x_\omega = (i_T^*)^{-1}(\phi_\omega)$ for all $\omega \in W$. For the details of the proof and the construction of the basis $\{\phi_\omega\}_{\omega\in W}$ in $F(W, R(T))$, see [KK].

Define the support of of any class $\alpha \in K_T^*(\mathcal{O}_\lambda) = K_T^*(G/T)$ by

$$\mathrm{Supp}(\alpha) = \{v\lambda \colon i_T^*(\alpha)(v) \neq 0\}$$

In particular, it is shown in [KK] that

$$\mathrm{Supp}(x_\omega) = \{v\lambda \colon \omega \leq v\}$$

where the elements $\omega, v \in W$ are ordered by the Bruhat order. Fix $\gamma \in W$ and for all $\omega \in W$, define ϕ_ω^γ by

$$\phi_\omega^\gamma := \gamma.\phi_{\gamma^{-1}\omega}$$

where the action of $\gamma \in W$ on $\phi_{\gamma^{-1}\omega}$ is defined by

$$\gamma.\phi_{\gamma^{-1}\omega}(v) = \phi_{\gamma^{-1}\omega}(\gamma^{-1}v)$$

Define

$$x_\omega^\gamma := (i_T^*)^{-1}(\phi_\omega^\gamma)$$

It is quite obvious that

$$\mathrm{Supp}(x_\omega^\gamma) = \{v\lambda \colon \gamma^{-1}\omega \leq \gamma^{-1}v\}$$

and $\{x_\omega^\gamma\}_{\omega \in W}$ form a $R(T)$-basis of $K_T(G/T) = K_T(\mathcal{O}_\lambda)$.

For $\xi \in \mathfrak{t}$, define $f_\xi(x)$ on $\mathcal{O}_\lambda = G/T$ by

$$f_\xi(x) := \langle \xi, \phi(x) \rangle$$

It is well-known that f_ξ is a Morse-Bott function.

Let $\lambda_1, ..., \lambda_l \in \mathfrak{t}^*$ be the fundamental weights associated to the positive Weyl chamber of $\mathfrak{t}^*$. Denote the Weyl chamber explicitly by

$$C = \{a_1\lambda_1 + a_2\lambda_2 + ... + a_l\lambda_l \mid a_i > 0, i = 1, 2, , , l\}$$

Denote the closure by $\overline{C}$. We have the following lemma on the behaviour of the Morse-Bott function f_ξ in terms of the fixed-point set $W.\lambda = \{\omega\lambda \mid \omega \in W\}$ in $\mathfrak{t}^*$, see [GM].

Lemma 14 (Goldin and Mare) *Let $\gamma \in W$ and $\xi \in \gamma C$. If $\gamma^{-1}v \leq \gamma^{-1}\omega$, then $f_\xi(v\lambda) \leq f_\xi(\omega\lambda)$.*

Lemma 15 *Suppose that $x \in K_T^*(\mathcal{O}_\lambda)$ has the property that*

$$\phi(Supp(x)) \subset \{y \in \mathfrak{t}^* \mid \langle \xi, \mu \rangle \leq \langle \xi, y \rangle\}$$

When x is decomposed in the basis $R(T)$-basis $\{x_\omega^\gamma\}_{\omega \in W}$ as

$$x = \sum_{\omega \in W} a_\omega^\gamma x_\omega^\gamma$$

where $a_\omega^\gamma \in K_T^*(pt) \cong R(T)$, such that if $a_\omega^\gamma \neq 0$ then

$$\phi(Supp(x_\omega^\gamma)) \subset \{y \in \mathfrak{t}^* \mid \langle \xi, \mu \rangle \leq \langle \xi, y \rangle\}$$

Proof. Suppose $\xi \in \gamma C$, we look at the decomposition of x in the $R(T)$-basis $\{x_\omega^\gamma\}_{\omega \in W}$. Let

$$W' = \{\omega \in W \mid \langle \xi, \mu \rangle \leq \langle \xi, \omega\lambda \rangle\}$$

Then write

$$x = \sum_{\omega \in W} a_\omega^\gamma x_\omega^\gamma = \sum_{\omega \in W'} a_\omega^\gamma x_\omega^\gamma + a_{v_1}^\gamma x_{v_1}^\gamma + ... + a_{v_n}^\gamma x_{v_n}^\gamma$$

For all v_i, $i = 1, 2, ..., n$,

$$\langle \xi, v_i \rangle < \langle \xi, \mu \rangle$$

and

$$a_{v_i}^\gamma \neq 0$$

We can rearrange v_i such that v_1 has the property that there exists no $j > 1$ such that $\gamma^{-1}v_j < \gamma^{-1}v_1$. Since $\langle\xi, v_1\lambda\rangle < \langle\xi, \mu\rangle \leq \langle\xi, \omega\lambda\rangle$ for $\omega \in W'$ and by Lemma 14, we know that $v_1\lambda \notin \mathrm{Supp}(x_\omega^\gamma)$ for $\omega \in W'$. Hence, we have

$$i_T^*(x_\omega^\gamma)(v_1) = 0$$

for $\omega \in W'$. Similarly,

$$i_T^*(x_{v_j}^\gamma)(v_1) = 0$$

since $\gamma^{-1}v_j \nleq \gamma^{-1}v_1$. Hence,

$$i_T^*(\sum_{\omega \in W'} a_\omega^\gamma x_\omega^\gamma + a_{v_1}^\gamma x_{v_1}^\gamma + ... + a_{v_n}^\gamma x_{v_n}^\gamma)(v_1) = a_{v_1}^\gamma \neq 0$$

So it means that $i_T^*(x)(v_1) \neq 0$. That is, $v_1\lambda \in \mathrm{Supp}(x)$. But $\langle\xi, v_1\lambda\rangle < \langle\xi, \mu\rangle$. Contradiction. ■

Now we can state our main theorem:

Theorem 16 *Let $\mathcal{O}_\lambda \cong G/T$ be a generic coadjoint orbit of a compact, connected, simply-connected Lie group G. $K_T^*(\phi^{-1}(\mu))$ is isomorphic to the quotient of $K_T^*(G/T)$ by the ideal generated by*

$$\{x_v^\gamma \mid \textit{there exists } j \textit{ such that } \langle\lambda_j, \gamma^{-1}\mu\rangle \leq \langle\lambda_j, \gamma^{-1}v\lambda\rangle\}$$

Proof. Suppose $v, \gamma \in W$ have the property that

$$\langle\lambda_j, \gamma^{-1}\mu\rangle \leq \langle\lambda_j, \gamma^{-1}v\lambda\rangle$$

for some $1 \leq j \leq l$. Let $\xi = \gamma\lambda_j \in \gamma C$, if $\omega\lambda \in \mathrm{Supp}(x_v^\gamma)$, then $\gamma^{-1}v \leq \gamma^{-1}\omega$. By lemma 14, we have

$$\langle\xi, \mu\rangle \leq \langle\xi, v\lambda\rangle \leq \langle\xi, \omega\lambda\rangle$$

Hence, $x_v^\gamma \in \ker(\kappa)$, where κ is the Kirwan map

$$\kappa\colon K_T^*(\mathcal{O}_\lambda) \to K_T^*(\phi^{-1}(\mu))$$

For another direction of the proof, let us consider a class $x \in K_T^*(\mathcal{O}_\lambda)$ sitting in $\ker(\kappa)$. Equivalently, x has the property

$$\mathrm{Supp}(x) \subset \{y \in \mathfrak{t}^* \mid \langle \xi, \mu \rangle \leq \langle \xi, y \rangle\}$$

for some $\xi \in \mathfrak{t}^*$. Suppose $\gamma \in W$ has the property that $\xi \in \gamma C$, x can be decomposed over the $R(T)$-basis $\{x_\omega^\gamma\}_{\omega \in W}$:

$$x = \sum_{\omega \in W} a_\omega^\gamma x_\omega^\gamma$$

By lemma 15, if $a_\omega^\gamma \neq 0$, then

$$\langle \xi, \mu \rangle \leq \langle \xi, \omega\lambda \rangle$$

We can write $\xi \in \mathfrak{t}^*$ as

$$\xi = \gamma \sum_{j=1}^{l} a_j \lambda_j$$

where $a_j \geq 0$ for all j. These two equations imply that we must have

$$\langle \gamma\lambda_j, \mu \rangle \leq \langle \gamma\lambda_j, \omega\lambda \rangle$$

for some $j \in \{1, 2, ..., l\}$. It means that any class $x \in \ker(\kappa) \subset K_T^*(\mathcal{O}_\lambda)$ is generated by some classes x_ω^γ described in theorem 16. ■

Remark 17 This result is very similar to [GM], where the rational cohomology $H^*(\mathcal{O}_\lambda//T(\mu))$ is computed. Our result is slightly different since our T-equivariant K-theory $K_T^*(\mathcal{O}_\lambda)$ is over $\mathbb{Z}$, instead of $\mathbb{Q}$. Hence, due to the possible presence of torsion elements, $K_T^*(\phi^{-1}(\mu))$ may not be isomorphic to $K^*(\mathcal{O}_\lambda//T(\mu))$. This isomorphism holds when $G = SU(n)$, or at the *very regular value* μ of the moment map for any flag variety G/T where G is a compact connected Lie group, see [Sj].

CHAPTER 2

DIVIDED DIFFERENCE OPERATORS ON KASPAROV'S EQUIVARIANT KK-THEORY

2.1 Introduction

Let G be a compact connected Lie group, T be a maximal torus of G and X be a compact G-space. In [A], Atiyah showed that $K_G^*(X)$ is a direct summand of $K_T^*(X)$. The restriction map from the G-equivariant K-ring $K_G^*(X)$ to the T-equivariant K-ring $K_T^*(X)$ has a natural left inverse. This pushforward homomorphism is defined by means of the Dolbeault operator associated with an invariant complex structure on the homogeneous space G/T. In [HLS], Harada, Landweber and Sjamaar showed that the action of the Weyl group W on $K_T^*(X)$ extends to an action of a Hecke ring $\mathscr{D}$ generated by *divided difference operators*, which was first introduced in the context of Schubert calculus by Demazure [D3]. The ring $\mathscr{D}$ contains an *augmentation left ideal* $I(\mathscr{D})$ and they showed that $K_G^*(X)$ is isomorphic to the subring of $K_T^*(X)$ annihilated by $I(\mathscr{D})$.

This chapter can be seen as a natural generalization of these results from equivariant K-theory to equivariant KK-theory introduced by Kasparov [K1], [K2]. First, we extend the action of the ring $\mathscr{D}$ to the Kasparov's T-equivariant KK-group $KK_T(A, B)$ where A and B are G-C^*-algebras. Next, we show that $KK_G(A, B)$ is isomorphic to $KK_T(A, B)$ annihilated by $I(\mathscr{D})$. The key results of this paper rely on theorems due to Wasserman [W]. Since it is unpublished, I will prove Wasserman's Theorems in Section 2.6 and 2.7.

2.2 The definition and properties of KK-theory

Kasparov's KK-theory is a bivariant functor that assigns an abelian group $KK(A, B)$ to the C*-algebras A and B. The abelian group $KK(A, B)$ is contravariant in A and covariant in B. If G is a group acting on A and B in a reasonably nice way, then we also have the equivariant KK-theory group $KK_G(A, B)$. As in the case of K-theory, KK-theory has an even and an odd part, we will only deal with the even part in this thesis.

The construction of KK-theory was motivated by index theory, and in particular by a desire to find generalizations and more elegant proofs of the Atiyah-Singer Index Theorem. The definition of KK-theory is fairly technical. This section may serve as a rapid introduction to the basic properties of KK-theory. More information in KK-theory can be found in Kasparov's original papers [K1], [K2], see also [B] and [JT].

Definition 18 A C*-*algebra* is a complex Banach space $(A, ||.||)$ equipped with an associative bilinear product $(a, b) \mapsto ab$ and an anti-linear map $a \mapsto a^*$ of order 2, such that for all $a, b \in A$, we have the following properties:

$$\begin{aligned} (ab)^* &= b^* a^* \\ ||ab|| &\leq ||a||||b|| \\ ||aa^*|| &= ||a||^2 \end{aligned}$$

A $*$-*homomorphism* of C*-algebras is a homomorphism of algebras that intertwines the star operations. These homomorphisms are automatically bounded.

It follows from the definition of C*-algebra that $||a^*|| = ||a||$ for all a in a

C*-algebra.

Example 19 Let X be a locally compact Hausdorff space. A complex-valued function f on X is said to *vanish at infinity* if for all $\epsilon > 0$ there is a compact subset $C \subset X$ such that for all $x \in X - C$, we have $|f(x)| < \epsilon$. The vector space of continuous functions on X vanishing at infinity is denoted by $C_0(X)$. The norm on this space is the supremum norm. The multiplication of two functions is defined by point-wise multiplication. The anti-linear map is defined by $f^*(x) := \overline{f(x)}$. Then $C_0(X)$ is a commutative C*-algebra. Note that if X is compact, then all functions on X vanish at infinity. In this case, we use the notation $C(X)$ to stand for the set of all continuous functions on X.

In fact, every commutative C*-algebra is isomorphic to the C*-algebra of continuous functions that vanish at infinity on a locally compact Hausdorff space, by Gelfand-Naimark Theorem.

In this thesis, all C*-algebras are assumed to be *separable*. This assumption is necessary for the definition of *Kasparov product* in KK-theory. A commutative C*-algebra $C_0(X)$ is separable if X is metrisable. Because we usually work with smooth manifolds, this assumption is not an important restriction.

Remark 20 Let A, B be C*-algebras, we can form the algebraic tensor product $A \otimes B$ with the *-map defined by $(a \otimes b)^* = a^* \otimes b^*$. It is easy to show that at least one norm can be defined on $A \otimes B$. In general, there may be more than one way to define a C*-norm on $A \otimes B$. The minimal C*-norm on $A \otimes B$ is called the *spatial norm.* And by abuse of notations, we denote $A \otimes B$ the C*-completion of the algebraic tensor product of A and B under the spatial norm and call it *spatial* *tensor product.* All tensor products of C*-algebras in this thesis are taken to be the spatial tensor products. A is called a *nuclear* C*-*algebra* if $A \otimes B$ admits only one norm for any C*-algebra B. The set of nuclear C*-algebras forms an important class of C*-algebras and has been studied extensively by C*-algebraists. For an introductory course on this topic, see [Mu]. We will not make use of any technical aspect of this theory in this thesis. But it is worth pointing out an important theorem by Takesaki that every abelian C*-algebra is nuclear, see Theorem 6.4.15 in [Mu].

Definition 21 Let A be a C*-algebra. A *Hilbert A-module* is a complex vector space E, equipped with the structure of a right A-module, and with an 'A-valued inner product' $\langle -, - \rangle \colon E \times E \to A$ which is additive in both entries and has the following properties for all $e, f \in E$, $a \in A$:

$$\begin{aligned} \langle e, fa \rangle &= \langle e, f \rangle a \\ \langle e, f \rangle &= \langle f, e \rangle^* \\ \langle e, e \rangle &\geq 0 \end{aligned}$$

and E is complete in the norm $||.||$ defined by $||e||^2 = ||\langle e, e \rangle||_A$.

A homomorphism of Hilbert A-modules is a A-module map that preserves the A-valued inner products. An isomorphism is a bijective homomorphism.

If $A = \mathbb{C}$, then a Hilbert $\mathbb{C}$-module is nothing more than a Hilbert space. So Hilbert modules over C*-algebras can be seen as a generalization of Hilbert spaces. The motivating example of Hilbert A-modules that is used in this thesis is the following.

Example 22 Let X be a locally compact Hausdorff space, and let E be a complex vector bundle over X, with a Hermitian structure $\langle -, - \rangle$. Let $\Gamma_0(E)$ be the space of continuous sections s of E such that the function $x \mapsto \langle s(x), s(x) \rangle$ vanishes at infinity. Then $\Gamma_0(E)$ is a Hilbert $C_0(X)$-module, whose module structure is given by pointwise multiplication and with the $C_0(X)$-valued inner product

$$\langle s, t \rangle(x) := \langle s(x), t(x) \rangle_E$$

for all $s, t \in \Gamma_0(X)$ and $x \in X$.

As an analogue to the tensor product of two Hilbert spaces, we can form a tensor product in a similar way as follows.

Let E be a Hilbert B-module and F a Hilbert C-module. The algebraic tensor product $E \otimes_{\mathbb{C}} F$ is a right module over the algebraic tensor product $B \otimes_{\mathbb{C}} C$ such that $(e \otimes_{\mathbb{C}} f) b \otimes_{\mathbb{C}} c = eb \otimes_{\mathbb{C}} fc$ for $e \in E, f \in F, b \in B, c \in C$. By considering $B \otimes_{\mathbb{C}} C$ as a dense *-subalgebra of the spatial tensor product $B \otimes C$, we can define a $B \otimes C$-valued 'inner product' on $E \otimes_{\mathbb{C}} F$ as the map $\langle -, - \rangle \colon E \otimes_{\mathbb{C}} F \times E \otimes_{\mathbb{C}} F \to B \otimes C$ by

$$\langle e \otimes_{\mathbb{C}} f, e_1 \otimes_{\mathbb{C}} f_1 \rangle = \langle e, e_1 \rangle \otimes \langle f, f_1 \rangle$$

Then $E \otimes_{\mathbb{C}} F$ is almost a pre-Hilbert $B \otimes C$-module, the difference being that it is only a right module over the dense *-subalgebra $B \otimes_{\mathbb{C}} C$ of $B \otimes C$, not over $B \otimes C$ itself. Then we consider the $B \otimes_{\mathbb{C}} C$-submodule $N = \{x \in E \otimes_{\mathbb{C}} F | \langle x, x \rangle = 0\}$. Take the completion of $E \otimes_{\mathbb{C}} F/N$ in the norm $||\langle -, - \rangle||^{\frac{1}{2}}$. It is a right $B \otimes_{\mathbb{C}} C$-module and we have the inequality $||zb|| \leq ||z|| ||b||$ for all $z \in E \otimes_{\mathbb{C}} F/N$ and $b \in B \otimes_{\mathbb{C}} C$. Therefore we can extend the right $B \otimes_{\mathbb{C}} C$-module structure by continuity in two steps to obtain a right $B \otimes C$-module structure. We call such a construction *external tensor product* of E and F, which turns a product of Hilbert B-module and Hilbert C-module into a Hilbert $B \otimes C$-module. By abuse of notations, we denote $E \otimes F$ the external tensor product of E and F. It is not to be confused with the *internal tensor product* of E and F that will be defined and used extensively a while later.

As an analogue to the algebras of bounded operators on a Hilbert space, we have the following generalization to Hilbert C*-modules.

Definition 23 Let A be a C*-algebra, and let E be a Hilbert A-module. The *algebra* $\mathrm{B}(E)$ *of adjointable operators* on E consists of the $\mathbb{C}$-linear A-module map $T\colon E \to E$ such that there is another $\mathbb{C}$-linear A-module map T^* that satisfies

$$\langle Ta, b\rangle = \langle a, T^*b\rangle$$

for all $a, b \in E$.

By definition, it is plain to show that all adjointable operators are bounded with respect to the norm $||.||_E$. A simple argument by Riesz Representation Theorem shows that every bounded linear operator on a Hilbert space is adjointable. But in general, it is not true that every $\mathbb{C}$-linear A-module map is adjointable for a Hilbert A-module when $A \neq \mathbb{C}$, see [Sk].

$\mathrm{B}(E)$ is a C*-algebra in the operator norm, with the anti-linear map defined by $T \mapsto T^*$.

Next, we will define the set of compact operators on Hilbert A-modules as an analogue to the set of compact operators on Hilbert spaces.

Definition 24 The subalgebra $\mathrm{F}(E) \subset \mathrm{B}(E)$ of finite rank operators on E is algebraically generated by operators of the form

$$\theta_{e_1,e_2}\colon e_3 \mapsto e_1\langle e_2, e_3\rangle$$

for $e_1, e_2, e_3 \in E$. The C*-*algebra* $\mathrm{K}(E)$ *of compact operators* on E is the norm closure of $\mathrm{F}(E)$ in $\mathrm{B}(E)$.

Note that, by the following computation:

$$\begin{aligned}
\langle \theta_{e_1,e_2}(x), y \rangle &= \langle e_1 \langle e_2, x \rangle, y \rangle \\
&= \langle y, e_1 \langle e_2, x \rangle \rangle^* \\
&= (\langle y, e_1 \rangle \langle e_2, x \rangle)^* \\
&= \langle e_2, x \rangle^* \langle y, e_1 \rangle^* \\
&= \langle x, e_2 \rangle \langle e_1, y \rangle \\
&= \langle x, e_2 \langle e_1, y \rangle \rangle \\
&= \langle x, \theta_{e_2,e_1}(y) \rangle
\end{aligned}$$

We have $\theta_{e_1,e_2} = \theta^*_{e_2,e_1} \in \mathrm{F}(E)$.

The basic building blocks of KK-theory are the Kasparov bimodules.

Definition 25 Let A, B be C*-algebras. A *Kasparov* (A, B)*-module* is a triple (E, ϕ, F) such that

(i) E is a countably generated Hilbert B-module.

(ii) $\phi\colon A \to \mathrm{B}(E)$ is $*$-homomorphism.

(iii) $F \in \mathrm{B}(E)$ is an adjointable operator such that for all $a \in A$, $[F, \phi(a)] \in \mathrm{K}(E), (F - F^*)\phi(a) \in \mathrm{K}(E)$ and $(F^2 - 1)\phi(a) \in \mathrm{K}(E)$.

To define *equivariant* KK-theory, we need to use $\mathbb{Z}_2$-graded Kasparov modules which are equipped with suitable actions by a group G. We always assume that G is a locally compact Hausdorff group that is second countable.

Definition 26 A $\mathbb{Z}_2$*-graded Hilbert* A*-module* is a Hilbert A-module E with a decomposition $E_0 \oplus E_1$ such that $ea \in E_k$ for all $a \in A$ and $e \in E_k$ where $k = 0, 1$.

Note that a $\mathbb{Z}_2$-grading on a Hilbert A-module E naturally induces $\mathbb{Z}_2$-gradings on the C*-algebras $\mathrm{B}(E)$ and $\mathrm{F}(E)$.

Definition 27 A C*-algebra A is a G*-C*-algebra* if G acts on A by $*$-automorphism and the map $g \mapsto g.a$ is a continuous map. If A is a G-C*-algebra, then a G*-Hilbert* A*-module* is a Hilbert A-module equipped with a continuous left action of G by bounded, invertible $\mathbb{C}$-linear operators such that

(i) For all $e_1, e_2 \in E$ and $g \in G$, one has $\langle g.e_1, g.e_2 \rangle = g.\langle e_1, e_2 \rangle$.

(ii) For all $a \in A$, $g \in G$, $e \in E$, one has $g.(ea) = (g.e)(g.a)$.

The G-C-alebras we will use are all of the from $C_0(X)$, where X is a G-space.

A $\mathbb{Z}_2$-graded G-Hilbert A-module is a G-Hilbert A-module with a $\mathbb{Z}_2$-grading and the G-action respects the grading. An operator $F \in \mathrm{B}(E)$ has *degree 1* if F reverses the grading on $E = E_0 \oplus E_1$, that is, F sends elements in E_0 (the even part) to elements in E_1 (the odd part), and sends elements in E_1 to elements in E_0.

Definition 28 Let A, B be G-C*-algebras. A $\mathbb{Z}_2$*-graded equivariant Kasparov* (A, B)*-module* is a Kasparov (A, B)-module (E, ϕ, F) with the following additional properties:

(i) E is a $\mathbb{Z}_2$-graded G-Hilbert B-module

(ii) $\phi\colon A \to \mathrm{B}(E)$ is a G-equivariant $*$-homomorphism which respects the $\mathbb{Z}_2$-gradings, where G acts on $\mathrm{B}(E)$ by conjugation.

(iii) $F \in \mathrm{B}(E)$ has degree 1 and has the properties that the map $g \mapsto gFg^{-1}$ from G to $\mathrm{B}(E)$ is norm-continuous. And $(gFg^{-1} - F)\phi(a)$ is compact, that is, $(gFg^{-1} - F)\phi(a) \in \mathrm{K}(E)$.

Remark 29 By Prop. 20.2.4. in [B], when G is compact, $F \in \mathrm{B}(E)$ can be assumed to be G-invariant. Then in Definition 28 (iii) above, $(gFg^{-1} - F)\phi(a) = 0 \in \mathrm{K}(E)$. We will make use of this proposition in Section 2.6.

Define $\mathbb{E}_G(A, B)$ to be the set of all $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules. We have the following operations on $\mathbb{E}_G(A, B)$.

(i) *Direct Sum*: Let $(E_1, \phi_1, F_1), (E_2, \phi_2, F_2) \in \mathbb{E}_G(A, B)$. We can then form the G-Hilbert B-module $E_1 \oplus E_2$. Given $F_1, F_2 \in \mathrm{B}(E)$, we can define an element $F_1 \oplus F_2 \in \mathrm{B}(E_1 \oplus E_2)$ by

$$F_1 \oplus F_2(e_1, e_2) = (F_1e_1, F_2e_2)$$

for $e_1 \in E_1$ and $e_2 \in E_2$. It is easy to see that $F_1 \oplus F_2 \in \mathrm{K}(E_1 \oplus E_2)$ if and only if $F_1 \in \mathrm{K}(E_1)$ and $F_2 \in \mathrm{K}(E_2)$. Similarly, define $\phi_1 \oplus \phi_2\colon A \to \mathrm{B}(E_1 \oplus E_2)$ by

$$\phi_1 \oplus \phi_2(a) = \phi_1(a) \oplus \phi_2(a)$$

Then $(E_1 \oplus E_2, \phi_1 \oplus \phi_2, F_1 \oplus F_2) \in \mathbb{E}_G(A, B)$.

(ii) *Pullback*: Let $(E, \phi, F) \in \mathbb{E}_G(A, B)$ and let $\psi\colon C \to A$ be a G-equivariant $*$-homomorphism. Then $(E, \phi \circ \psi, F) \in \mathbb{E}_G(C, B)$ which is also denoted by $\psi^*(E, \phi, F)$.

(iii) *Pushout*: Let $(E, \phi, F) \in \mathbb{E}_G(A, B)$ and $\psi\colon B \to C$ be a G-equivariant $*$-homomorphism. We can form the G-Hilbert C-module $E \otimes_\psi C$ as the *internal tensor product* of two Hilbert modules. It is defined as follows: First we form the algebraic tensor product $E \otimes_B C$ which is a right C-module in the obvious way: $(x \otimes_B y)c = x \otimes_B yc$. We can define a map $\langle -, - \rangle\colon E \otimes_B C \times E \otimes_B C \to C$ to be the map which is linear in the first variable and conjugate linear in the second, and satisfies

$$\langle x_1 \otimes_B x_2, y_1 \otimes_B y_2 \rangle = \langle x_2, \psi(\langle x_1, y_1 \rangle) y_2 \rangle$$

for $x_1, y_1 \in E$, $x_2, y_2 \in C$. This is legitimate since

$$\langle \psi(b) x_2, \psi(\langle x_1, y_1 \rangle) y_2 \rangle = \langle x_2, \psi(\langle x_1 b, y_1 \rangle) y_2 \rangle$$

$$\langle x_2, \psi(\langle x_1, y_1 \rangle) \psi(b) y_2 \rangle = \langle x_2, \psi(\langle x_1, y_1 b \rangle) y_2 \rangle$$

for all $b \in B$. Set $N = \{z \in E \otimes_B C | \langle z, z \rangle = 0\}$. Then N is an C-submodule and we can consider the quotient $E \otimes_B C/N$ and the quotient map $q\colon E \otimes_B C \to E \otimes_B C/N$. Then $E \otimes_B C/N$ is a right C-module by $q(x)c = q(xc), x \in E \otimes_B C, c \in C$. And we can define the C-valued inner product on $E \otimes_B C/N$ by $\langle q(x), q(y) \rangle = \langle x, y \rangle, x, y \in E \otimes_B C$. The completion with respect to this pre-norm is denoted by $E \otimes_\psi C$. The G-action on $E \otimes_\psi C$ is defined by $g.(e \otimes_\psi c) = (ge \otimes_\psi gc), g \in G, e \in E, c \in C$. $E \otimes_\psi C$ is called the internal tensor product of E and C. Then the pushout $\psi_*(E, \phi, F)$ is defined by $(E \otimes_\psi C, \phi \otimes id, F \otimes id)$, which is an element in $\mathbb{E}_G(A, C)$.

The equivariant KK-theory $KK_G(A, B)$ is the set $\mathbb{E}_G(A, B)$ modulo certain *unitary equivalence* and *homotopy relation* as defined as follows.

Definition 30 Two $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules (E_0, ϕ_0, F_0), (E_1, ϕ_1, F_1) are said to be *unitarily equivalent* if there is a G-equivariant isomorphism of Hilbert B-modules $E_0 \cong E_1$ that respects the gradings, and intertwines F_0 and F_1, and $\phi_0(a)$ and $\phi(a)$ for all $a \in A$.

Definition 31 Two $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules (E_0, ϕ_0, F_0), (E_1, ϕ_1, F_1) are said to be *homotopic* if there exists a $\mathbb{Z}_2$-graded equivariant Kasparov $(A, C([0, 1], B))$-module (E, ϕ, F) with the following property. For $j = 0, 1$, let $ev_j \colon C([0, 1], B) \to B$ be the evaluation map at j. Then $(ev_j)_*(E, \phi, F) = (E \otimes_{ev_j} B, \phi \otimes id, F \otimes id)$ is unitarily equivalent to (E_j, ϕ_j, F_j).

Remark 32 A special case of homotopy of $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules is *operator homotopy.* Two $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules (E, ϕ, F) and (E, ϕ, F') are said to be *operator homotopic* if there is a norm-continuous map $t \mapsto F_t$ from $[0, 1]$ to $\mathrm{B}(E)$ such that for all t, $(E, \phi, F_t) \in \mathbb{E}_G(A, B)$ and $F_0 = F$ and $F_1 = F$. If two $\mathbb{Z}_2$-graded Kasparov (A, B)-modules are operator homotopic, then they are homotopic. The two homotopy relations are equivalent when the C*-algebra A of $\mathbb{E}_G(A, B)$ is separable, see Section 2.1 in [JT].

Definition 33 The *equivariant KK-theory of A and B* is the abelian group $KK_G(A, B)$ of $\mathbb{Z}_2$-graded equivariant Kasparov (A, B)-modules modulo homotopy, with addition induced by the direct sum. The inverse is given by

$$-(E_0 \oplus E_1, \phi, F) = (E_1 \oplus E_0, \phi, -F)$$

We call an element $(E, \phi, F) \in \mathbb{E}_G(A, B)$ *degenerate* when $[F, \phi(a)] = (F^2 - 1)\phi(a) = (F^* - F)\phi(a) = 0$ for all $a \in A$. The class of degenerate elements is denoted by $\mathbb{D}_G(A, B)$. It is not too difficult to show that every element in $\mathbb{D}_G(A, B)$ is homotopic to 0, see Lemma 2.1.20 in [JT].

Example 34 Let (M, ω) be a symplectic manifold. There is a natural almost complex structure associated with the symplectic form ω of M. Let $A = C_0(M)$

and $B = \mathbb{C}$. Let $D' = \bar{\partial} + \bar{\partial}^*$ be the Dolbeault operator acting on smooth forms with compact support. Let $\mathbb{H}$ be the Hilbert space of L^2-forms of bidegree $(0,^*)$ on M, that is, $\mathbb{H} = L^2(\wedge^{0,*}(M))$. $\mathbb{H}$ is a Hilbert space graded by decomposing the forms into even and odd forms. Then D' is an essentially self-adjoint operator (see [HR]) of degree 1. Note that D' is an unbounded operator. Let f be the real-valued function defined by $f(x) = x/\sqrt{1+x^2}$. By functional calculus, define $F = f(D')$. F is now a bouned operator acting on the smooth forms with compact support. Extend such an action to $\mathbb{H}$ by continuity. By abuse of notation, this operator is denoted by F. Let m be the function multiplication of $C_0(M)$ on $\mathbb{H}$. Then $[\mathbb{H}, m, F] \in KK(C_0(M), \mathbb{C})$. It is also called the *Dolbeault element* of M, denoted by $[\bar{\partial}_M]$.

Remark 35 The Dolbeault element serves as an important motivating example for KK-theory. An element similar to it can also be defined in equivariant KK-theory. It will be introduced in the next section, in which its properties will be exploited to give results that are important to our main theorems.

$KK_G(A, B)$ is a *homotopy invariant bifunctor*. It is *contravariant in the first variable*: If $\psi \colon D \to A$, then we have the map $\psi^* \colon KK_G(A, B) \to KK_G(D, A)$ given by the pullback construction. It is *covariant in the second variable*: If $\xi \colon B \to C$, then we have the map $\xi_* \colon KK_G(A, B) \to KK_G(A, C)$ given by the pushforward construction.

If the group G is trivial, we omit it from the notation and write it as $KK(A, B)$.

In general, the equivariant K-homology of a G-C*-algebra A is defined as

$$K^0_G(A) \colon = KK_G(A, \mathbb{C})$$

In particular, if M is a locally compact Hausdorff space on which G acts properly, then we can define the *equivariant K-homology of M* as:

$$K_0^G(M) :\, = KK_G(C_0(M), \mathbb{C})$$

On the other hand,

$$KK_G(\mathbb{C}, B) \cong K_0^G(B)$$

where $K_0^G(B)$ is the K-theory of G-C*-algebras B, see Proposition 17.5.5 and Theorem 18.5.3 in [B]. For the properties of K-theory of C*-algebras, see also [B]. We will not use the general theory of $K_0^G(B)$ here but only the following particular case: If M is a compact G-space, we have

$$KK_G(\mathbb{C}, C(M)) \cong K_G^0(M)$$

where $K_G^0(M)$ is just the equivariant K-theory of M. A special case comes out of it automatically: If M is a point, then

$$KK_G(\mathbb{C}, \mathbb{C}) \cong R(G)$$

where $R(G)$ is the representation ring of G.

The introduction to KK-theory would be incomplete without mentioning the *Kasparov Product*, which is the most important feature in KK-theory. The most general form of it is the map:

$$KK_G(A_1, B_1 \otimes C) \times KK_G(C \otimes A_2, B_2) \xrightarrow{\otimes_C} KK_G(A_1 \otimes A_2, B_1 \otimes B_2)$$

It is a bilinear map. We will use the following notation for the Kasparov product:

$$(x, y) \mapsto x \otimes_C y$$

Its definition is highly sophisticated so we will not define it here. A complete discussion of this product can be found in [B], or [JT]. We will only use some special cases of the Kasparov product:

(i) When $B_1 = A_2 = \mathbb{C}$, the Kasparov product becomes

$$KK_G(A_1, C) \times KK_G(C, B_2) \xrightarrow{\otimes_C} KK_G(A_1, B_2), (x, y) \mapsto x \otimes_C y$$

(ii) When $C = \mathbb{C}$, the Kasparov product becomes

$$KK_G(A_1, B_1) \times KK_G(A_2, B_2) \xrightarrow{\otimes_{\mathbb{C}}} KK_G(A_1 \otimes A_2, B_1 \otimes B_2), (x, y) \mapsto x \otimes_{\mathbb{C}} y$$

We also note the following two properties of Kasparov product, which will be used frequently in the upcoming sections:

(i) The Kasparov product is *associative*. That is, if $x \in KK_G(A, D)$, $y \in KK_G(D, E)$, $z \in KK_G(E, B)$, then

$$(x \otimes_D y) \otimes_E z = x \otimes_D (y \otimes_E z)$$

(ii) $KK_G(A, B)$ is endowed with a $R(G)$-module structure by the Kasparov product:

$$KK_G(\mathbb{C}, \mathbb{C}) \times KK_G(A, B) \xrightarrow{\otimes_{\mathbb{C}}} KK_G(A, B)$$

2.3 Main results

Let G be a compact Lie group and T be its maximal torus. Let $i \colon T \to G$ be the inclusion from T to G. Then every G-C*-algebra A can be naturally considered as an T-C*-algebra via i, that is, $t.x = i(t)x$ where $t \in T$ and $x \in A$. Hence we have a map naturally induced from i,

$$i^* \colon KK_G(A, B) \longrightarrow KK_T(A, B)$$

for all G-C*-algebras A and B. This map is also called the *restriction* map and we will also make use of a more descriptive notation as follows:

$$res_T^G\colon KK_G(A,B) \longrightarrow KK_T(A,B)$$

The goal of Sections 2.3.1 to 2.3.4 is to show that there is a left inverse $i_!\colon KK_T(A,B) \to KK_G(A,B)$ of $i^*\colon KK_G(A,B) \to KK_T(A,B)$. That is,

$$i_! \circ i^* = 1\colon KK_G(A,B) \to KK_G(A,B)$$

where $i^*\colon KK_G(A,B) \to KK_T(A,B)$ is induced by the inclusion $i\colon T \to G$. Then we will prove our main Theorem 54 in 2.3.5 which describes the subgroup $i^*(KK_G(A,B))$ in terms of the *divided difference operators*.

2.3.1 Construction of $[i^*] \in KK_G(\mathbb{C}, C(G/T))$

If A is an G-C*-algebra, define $Ind_T^G(A)$ to be the G-C*-algebra of all continuous functions $f\colon G \to A$ such that $f(gt) = t^{-1}f(g)$ for all $g \in G$, $t \in E$ and $||f||$ vanishes at infinity. The G-action on $Ind_T^G(A)$ is by left translation. Then there is a fairly natural way to define the *induction* map

$$ind_T^G\colon KK_T(A,B) \longrightarrow KK_G(Ind_T^G(A), Ind_T^G(B))$$

for all T-C*-algebras A and B. Its definition and properties will be explained in details in Section 2.6.

If B is an G-C*-algebra, denote $Res_T^G(B)$ to be the T-C*-algebra by restricting the G-action to T-action. It can be shown that for all G-C*-algebras A, $Ind_T^G(Res_T^G(A))$ is *equivariantly isomorphic* to $A \otimes C(G/T)$, see Section 2.6.

We will construct an element $[i^*] \in KK_G(\mathbb{C}, C(G/T))$ corresponding to

$$i^*\colon KK_G(A,B) \to KK_T(A,B)$$

Define

$$[i^*] = [C(G/T), id_{\mathbb{C}}, 0] \in KK_G(\mathbb{C}, C(G/T))$$

where $id_{\mathbb{C}}$ stands for the scalar multiplication and $C(G/T)$ is naturally viewed as a G-Hilbert $C(G/T)$-module. We need the following result by Wasserman [W].

Theorem 36 (Wasserman) *Let G be a compact group, and T be its closed subgroup. If A and B are G-C*-algebras, then $KK_T(A,B) \cong KK_G(A, B \otimes C(G/T))$. Precisely speaking, if $x \in KK_T(A,B)$, then there is an isomorphism $x \mapsto j^*(ind_T^G(x))$ where j^* is the map induced by the inclusion $j\colon A \cong A \otimes 1 \longrightarrow A \otimes C(G/T) \cong Ind_T^G(A)$. And the inverse is given by $y \mapsto ev_*(res_T^G(y))$ for $y \in KK_G(A, B \otimes C(G/T))$ where $ev\colon B \otimes C(G/T) \to B$ is the evaluation at identity, i.e. $b \otimes f \mapsto bf(1)$.*

For a proof of it, see Section 2.6. Let θ be the isomorphism $ev_* \circ res_T^G\colon KK_G(A, B \otimes C(G/T)) \to KK_T(A,B)$.

Lemma 37 *For any element $x \in KK_G(A,B)$,*

$$\theta(x \otimes_{\mathbb{C}} [i^*]) = i^*(x) \in KK_T(A,B)$$

Proof. It can be done by routine checking. Let $x = [E, \phi, F] \in KK_G(A,B)$, then

$$x \otimes_{\mathbb{C}} [i^*] = [E \otimes C(G/T), \phi \otimes id, F \otimes id]$$

where $E \otimes C(G/T)$ is the same as the external tensor product of two G-Hilbert modules and hence is a G-Hilbert $B \otimes C(G/T)$-module.

$$\theta(x \otimes_{\mathbb{C}} [i^*]) = ev_* \circ res_T^G(x \otimes_{\mathbb{C}} [i^*]) = [(E \otimes C(G/T)) \otimes_{ev} B, \phi \otimes id_{\mathbb{C}} \otimes id_B, F \otimes id \otimes id_B]$$

where $(E \otimes C(G/T)) \otimes_{ev} B$ is a T-Hilbert B-module. It is clear that $(E \otimes C(G/T)) \otimes_{ev} B$ is isomorphic to E as a T-Hilbert B-module. Let f be the isomorphism from $(E \otimes C(G/T)) \otimes_{ev} B$ to E. Then it is straightforward to check that

$$f \circ (\phi \otimes id \otimes id_B)(a) = \phi(a) \circ f$$

and

$$f \circ (F \otimes id \otimes id_B) = F \circ f$$

for any $a \in A$, ϕ is viewed as a T-equivariant map and F is viewed as a T-Hilbert B-module map by restricting the G-action to T-action. Hence, $\theta(x \otimes_{\mathbb{C}} [i^*])$ and $i^*(x)$ are unitarily equivalent in $\mathbb{E}_T(A, B)$ and our result follows. ■

2.3.2 Construction of $[i_!] \in KK_G(C(G/T), \mathbb{C})$

G/T is equipped with a G-equivariant complex structure corresponding to a choice of positive root system relative to (G/T). Then we can construct an equivariant Dolbeault element $KK_G(C(G/T), \mathbb{C})$ in almost the same way as in Example 34: The G-action on $C(G/T)$ is defined by

$$g.f(x) = f(g^{-1}x)$$

for any $g \in G$, $x \in G/T$ and $f \in C(G/T)$. The G-action on any smooth $(0,*)$-form is defined by

$$g.s(x) = g(s(g^{-1}x))$$

where $g \in G$, $x \in G/T$ and s is a smooth section of vector bundle $\Omega^{(0,*)}$ of complex differential forms of degree $(0,*)$ over M. This action extends to an action on $L^2(M, \Omega^{(0,*)})$ by continuity. Then let $\partial + \bar{\partial}^*$ be the G-equivariant Dolbeault operator acting on smooth forms on G/T. From here, we simply use the same technique as in Example 34 to construct an (equivariant) Dolbeault element $[\bar{\partial}_{G/T}]$ in $KK_G(C(G/T), \mathbb{C})$. Define $[i_!]$ to be $[\bar{\partial}_{G/T}]$.

Remark 38 If $A = \mathbb{C}$, $B = C(M)$, where M is a compact G-space, then $KK_G(\mathbb{C}, C(M)) \cong K_G(M)$ and $i_!$ is the *holomorphic induction* from $K_T(X)$ to $K_G(X)$ by Atiyah , see [A].

2.3.3 Kasparov product $[i^*] \otimes_{C(G/T)} [i_!]$

Following the definition of Kasparov product, we can get the following:

$$[i^*] \otimes_{C(G/T)} [i_!] = [C(G/T) \otimes_m L^2(G/T, S), i, 1 \otimes D]$$

where $C(G/T) \otimes_m L^2(G/T, S)$, as an internal tensor product of two Hilbert modules, is viewed as a G-Hilbert space. G acts on it by

$$g.(f \otimes_m h) = (g.f) \otimes_m (g.h)$$

where $g \in G$, $f \in C(G/T)$ and $h \in C^\infty(G/T, S)$. We can extend this action to an action on $C(G/T) \otimes_m L^2(G/T, S)$ by continuity. i is the scalar multiplication on $C(G/T) \otimes_m L^2(G/T, S)$.

In general, the Kasparov product is hard to compute. But in our particular case, Kasparov [K2] showed the following result:

Theorem 39 *Let G be a compact group and M be a compact G-manifold. Let $[E] \in K^0_G(M)$ be an element in the equivariant K-theory of M and let $[\bar{\partial}_M] \in KK_G(C(M), \mathbb{C}) \cong K^G_0(M)$ be the equivariant Dolbeault element. Then*

$$[E] \otimes_{C(M)} [D] = G\text{-}index((\bar{\partial}_M)_E)$$

where $(\bar{\partial}_M)_E$ *is the Dolbeault operator with coefficient in* E*.*

Remark 40 If D is, say, an order-zero elliptic operator and E is a complex vector bundle over a compact manifold M. In general it is permissible that D acts on sections of bundles like the Dolbeault operator. But for the sake of notational simplification we pretend that D acts on functions. We should think of D as a bounded operator, by some basic functional calculus, on $L^2(M)$. Then we can construct D_E as an operator

$$D_E\colon L^2(M,E) \longrightarrow L^2(M,E)$$

acting on sections of E. In general we define D_E by using the local triviality of E together with a partition of unity argument. Thus we choose a partition of unity $\{f_1, ..., f_k\}$ for M such that each f_i is supported within an open set U_i over which the bundle E is trivializable. Choosing trivializations and hence isomorphisms $L^2(U_i, E|_{U_i}) \cong L^2(U_i) \otimes \mathbb{C}^k$ where k is the dimension of the bundle, we define operators $(f_i^{1/2} D f_i^{1/2})_E$ on $L^2(U_i, E|_{U_i})$ by pulling back the operators $f_i^{1/2} D f_i^{1/2} \otimes 1$ on $L^2(U_i) \otimes \mathbb{C}^k$ via these isomorphisms. Finally we define D_E to be the operator

$$D_E = \sum_{i=1}^{k} (f_i^{1/2} D f_i^{1/2})_E$$

on $L^2(M,E)$. The operator we obtain in this way depends on the choice of partition of unity. However, whatever the choices D_E is a Fredholm operator and its index does not depend on the choices. In this way we obtain an index $\mathrm{ind}(D_E) \in \mathbb{Z}$ for every $[E] \in K^0(M)$. In the equivariant case where G is compact, D_E is then a G-equivariant Fredholm operator for $[E] \in K_G^0(M)$. The kernel and cokernel are now (finite-dimensional) G-vector spaces and hence we get the G-index $G-index(D_E) \in R(G)$.

Topologically, the element $[i^*] \in KK_G(\mathbb{C}, C(G/T)) \cong K^0_G(C(G/T))$ corresponds to the trivial G-bundle E_0 over G/T. The homogeneous pseudo-differential operator D_{E_0} has G-index $1_G \in R(G)$ by a result of Bott, see [Bo]. By Theorem 39, we have the following result:

Theorem 41 $[i^*] \otimes_{C(G/T)} [i_!] = 1 \in KK_G(\mathbb{C}, \mathbb{C})$

2.3.4 Push-pull operators

Recall the notation from Section 2.3.1 that $\theta\colon KK_G(A, B \otimes C(G/T)) \to KK_T(A, B)$ denote the isomorphism by Wasserman's Theorem. Then let $\theta^{-1}\colon KK_T(A, B) \to KK_G(A, B \otimes C(G/T))$ be the inverse of θ. Define $i_!\colon KK_T(A, B) \to KK_G(A, B)$ by

$$i_!(y) = \theta^{-1}(y) \otimes_{C(G/T)} [i_!]$$

for $y \in KK_T(A, B)$.

Lemma 42 *$i_! \circ i^* = 1$ as an action on $KK_G(A, B)$.*

Proof. By Lemma 37 and by associativity of Kasparov product,

$$\begin{aligned} i_!(i^*(x)) &= i_!(\theta(x \otimes_{\mathbb{C}} [i^*])) \\ &= (x \otimes_{\mathbb{C}} [i^*]) \otimes_{C(G/T)} [i_!] \\ &= x \otimes_{\mathbb{C}} ([i^*] \otimes_{C(G/T)} [i_!]) \\ &= x \otimes_{\mathbb{C}} 1 \\ &= x \end{aligned}$$

for all $x \in KK_G(A, B)$ as desired. $\blacksquare$

Define $\sigma\colon KK_T(A, B) \longrightarrow KK_T(A, B)$ by

$$\sigma = i^* \circ i_!$$

Some properties of σ can be stated immediately.

Lemma 43 *$\sigma^2 = \sigma$ and $\sigma(i^*(x)) = i^*(x)$ for any $x \in KK_G(A, B)$.*

Proof. By Section 2.3.3 and associativity of Kasparov product,

$$([i_!] \otimes [i^*]) \otimes ([i_!] \otimes [i^*]) = [i_!] \otimes ([i^*] \otimes [i_!]) \otimes [i^*] = [i_!] \otimes [i^*]$$

Now it is obvious that $\sigma^2 = \sigma$ and $\sigma(i^*(x)) = i^*(x)$ for any $x \in KK_G(A, B)$. $\blacksquare$

Remark 44 If $A = \mathbb{C}$ and $B = C(SU(n)/T)$, then $KK_T(\mathbb{C}, C(SU(n)/T)) \cong K_T(SU(n)/T)$. Then σ is simply the divided difference operator ∂_{ω_0} where ω_0 is the longest element in S_n, the symmetric group of n letters, see Section 1.4.2. See Section 2.3.5 for further explanations.

In particular, if $A = \mathbb{C}$, $B = \mathbb{C}$, then $KK_T(\mathbb{C}, \mathbb{C}) \cong R(T)$ and $KK_G(\mathbb{C}, \mathbb{C}) \cong R(G)$. σ is the *top Demazure's operator* ∂_{ω_0} acting on $R(T)$, where ω_0 is the longest element in the Weyl Group W. More generally, Demazure [D3] defined a set of operators δ_ω for every Weyl element ω, see Section 2.3.5 for a very brief introduction.

We do not introduce the definiton of the top Demazure's operator at this point. For the properties of this operator, see 2.3.5. But we just want to point out that the most important property of ∂_{ω_0} is its relation to the Weyl character formula. Let $\mathscr{R}$ be the root system of (G,T) and W be the Weyl Group. Let $\mathscr{X}(T) = \mathrm{Hom}(T, U(1))$ be the character group of T. We denote by e^λ the element of $R(T)$ defined by a character $\lambda \in \mathscr{X}(T)$. We fix a basis of the root system and let

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathscr{R}^+} \alpha$$

be the half-sum of all positive roots. The the Weyl character formula can be interpreted as the following formula:

$$\mathrm{ch}(u) = \frac{\mathrm{A}(u)}{\mathrm{d}} \tag{2.1}$$

for all $u \in R(T)$. $\mathrm{A}(u)$ is the following alternating sums of elements in $R(T)$:

$$\mathrm{A}(u) = \sum_{\omega \in W} (-1)^{l(\omega)} e^{-\rho} \omega(e^\rho u)$$

where $l(w)$ is the length of the Weyl element ω as explained in Section 1.4. d is defined as follows:

$$\mathrm{d} = \prod_{\alpha \in \mathscr{R}^+} (1 - e^{-\alpha})$$

In [D3], Demazure showed the following formula:

$$\partial_{\omega_0}(u) = \frac{\mathrm{A}(u)}{\mathrm{d}} \tag{2.2}$$

for all $u \in R(T)$. Recall that the classical proof of the Weyl character formula was done by using theory of compact Lie group and its Lie algebra, for example, see [BD]. But in [AB], it was shown that the Weyl character formula can also be interpreted as a computation of the character of an induced representation by an analytic Lefschetz fixed-point formula. In terms of our definition of $\sigma = i^* \circ i_!$ where $i^* \colon KK_G(\mathbb{C}, \mathbb{C}) \longrightarrow KK_T(\mathbb{C}, \mathbb{C})$ and $i_! \colon KK_T(\mathbb{C}, \mathbb{C}) \longrightarrow KK_G(\mathbb{C}, \mathbb{C})$ in this special case, this interpretation is equivalent to the following result:

$$\sigma(u) = \frac{\mathrm{A}(u)}{\mathrm{d}} \tag{2.3}$$

in which we have used the identification $KK_T(\mathbb{C},\mathbb{C}) \cong R(T)$. By (2.2), we have

$$\sigma(u) = \partial_{\omega_0}(u) \tag{2.4}$$

In the other words, the operator $\sigma\colon KK_T(A,B) \longrightarrow KK_T(A,B)$ can be interpreted as generalizations of both the Weyl character formula and the top Demazure's operator to Kasparov's KK-theory.

We call a compact Lie group G a *Hodgkin group* if it is connected and has a torsion-free fundamental group. In [Ho], Hodgkin proved the following result in equivariant K-theory:

$$K_T^*(M) \cong R(T) \otimes_{R(G)} K_G^*(M)$$

where G is a Hodgkin group, T is a maximal torus of G and M is any G-space which is locally contractible and of finite covering dimension. Note that it is an isomorphism of $R(T)$-modules. The following generalization of Hodgkin's result to KK-theory was due to A. Wasserman [W]. See Section 2.7 for a proof of it.

Theorem 45 (Wasserman) *Let G be a Hodgkin group and T be a maximal torus in G. For all G-C*-algebras A and B,*

$$KK_T(A,B) \cong KK_G(A,B) \otimes_{R(G)} R(T)$$

They are isomorphic as $R(T)$-modules. The map $KK_G(A,B) \otimes_{R(G)} R(T) \to KK_T(A,B)$ is given by $x \otimes a \mapsto a.i^*(x)$ where $i\colon T \to G$ is the inclusion map.

The next result is crucial for the constructions of divided difference operators in Section 2.3.5.

Theorem 46 *Assume that G is a Hodgkin group. Identify the $R(T)$-modules $KK_T(A,B)$ and $KK_G(A,B)\otimes_{R(G)} R(T)$ via Theorem 45, then $\sigma = 1\otimes\partial_{\omega_0}$, where 1 denotes the identity operator of $KK_G(A,B)$.*

Proof. By the Wasserman's Isomorphism $\theta\colon KK_G(A, B\otimes C(G/T)) \to KK_T(A,B)$ and Theorem 45, we can identify $KK_G(A,B)\otimes_{R(G)} R(T)$ with $KK_G(A, B\otimes C(G/T))$. But $R(T)$ is isomorphic to $KK_G(\mathbb{C}, C(G/T))$. Hence we can consider $KK_G(A,B)\otimes_{R(G)} KK_G(\mathbb{C}, C(G/T))$ instead. Note that the relation $(xb)\otimes c = x\otimes(bc) \in KK_G(A,B)\otimes_{R(G)} KK_G(\mathbb{C}, C(G/T))$ where $x\in KK_G(A,B)$, $b\in R(G)$ and $c\in KK_G(\mathbb{C}, C(G/T))$ is equivalent to (after making identifications of $R(G)\cong KK_G(\mathbb{C},\mathbb{C})$) the associativity of the Kasparov product $(x\otimes_{\mathbb{C}} b)\otimes_{\mathbb{C}} c = x\otimes_{\mathbb{C}}(b\otimes_{\mathbb{C}} c)$. Then this theorem is almost trivial. For any $x\otimes a\in KK_G(A,B)\otimes_{R(G)} R(T)$, the operator $1\otimes\partial_{\omega_0}$ acts on $KK_G(A,B)\otimes_{R(G)} KK_G(\mathbb{C}, C(G/T))$ by

$$\begin{aligned} 1\otimes\partial_{\omega_0}(x\otimes a) &= x\otimes\partial_{\omega_0} a \\ &= x\otimes(a\otimes_{C(G/T)}[i_!]\otimes_{\mathbb{C}}[i^*]) \end{aligned}$$

In terms of Kasparov product, $x\otimes_{\mathbb{C}}(a\otimes_{C(G/T)}[i_!]\otimes_{\mathbb{C}}[i^*]) = (x\otimes_{\mathbb{C}} a)\otimes_{C(G/T)}[i_!]\otimes_{\mathbb{C}}[i^*]$. But then $(x\otimes_{\mathbb{C}} a)\otimes_{C(G/T)}[i_!]\otimes_{\mathbb{C}}[i^*]$ is essentially the same as $\sigma(a.i^*(x))$. ■

The next result is analogous to a result by Snaith [Sn].

Lemma 47 *Let $\tilde{T}$ be a torus and $s\colon \tilde{T}\to T$ a covering homomorphism. Then the map $s^*\colon KK_T(A,B)\to KK_{\tilde{T}}(A,B)$ is injective for all T-C*-algebras A and B.*

Proof. Let $t\colon C\to\tilde{T}$ be the kernel of s. Let $\mathbb{E}_T$ be

$$\mathbb{E}_T = \prod_{\lambda\in\mathscr{X}(C)} \mathbb{E}_T(A,B)$$

where $\mathscr{X}(C)$ is the character group of C. We write an object of $\mathbb{E}_T$ as an $\mathscr{X}(C)$-tuple $([E_\lambda, \phi_\lambda, F_\lambda])_{\lambda \in \mathscr{X}(C)}$, where each $[E_\lambda, \phi_\lambda, F_\lambda]$ is an element in $\mathbb{E}_T(A, B)$. The restriction homomorphism $s^*\colon \mathscr{X}(\tilde{T}) \to \mathscr{X}(C)$ is surjective, see [Sn]. We choose a set-theoretic left inverse τ. Let $\mathbb{E}_{\tilde{T}} = \mathbb{E}_{\tilde{T}}(A, B)$ and $[E, \phi, F] \in \mathbb{E}_{\tilde{T}}$. Since C acts trivially on T-C*-algebra B, the C-invariant subspace E^C of E is a well-defined T-Hilbert B-module. For all objects $[E, \phi, F]$ in $\mathbb{E}_{\tilde{T}}$, define $\nu\colon \mathbb{E}_{\tilde{T}} \to \mathbb{E}_T$ by

$$\nu([E, \phi, F]) = [Hom(V_{\tau(\lambda)}, E)^C, \tilde{\phi}_\lambda, \tilde{F}_\lambda]_{\lambda \in \mathscr{X}(C)}$$

where $Hom(V_{\tau(\lambda)}, E)$ is the set of all $\tilde{T}$-maps from $V_{\tau(\lambda)}$ to E. It is a $\tilde{T}$-Hilbert B-module with the B-module structure defined by

$$fb(v) = f(v)b$$

for all $b \in B$ and $v \in V_{\tau(\lambda)}$. Then $Hom(V_{\tau(\lambda)}, E)^C$ is a T-Hilbert B-module. $\tilde{\phi}_\lambda\colon A^C \to \mathrm{B}(Hom(V_{\tau(\lambda)}, E)^C)$ where A^C is a T-C*-algebra by taking C-invariant of the $\tilde{T}$-action on A, is defined by

$$(\tilde{\phi}_\lambda(a)f)(v) = \phi(a)(f(v))$$

for all $f \in Hom(V_{\tau(\lambda)}, E)$, $v \in V_{\tau(\lambda)}$ and $\lambda \in \mathscr{C}(C)$. It is easy to check that $\tilde{\phi}_\lambda$ is a T-$*$-homomorphism. Similarly, $\tilde{F}_\lambda \in \mathrm{B}(Hom(V_{\tau(\lambda)}, E)^C)$ is defined by

$$(\tilde{F}_\lambda(f))(v) = F(f(v))$$

for all $f \in Hom(V_{\tau(\lambda)}, E)^C$ and $v \in V_{\tau(\lambda)}$. Again, it is routine to check that $\tilde{F}_\lambda$ is a T-Hilbert B-module map.

For all objects $[E_\lambda, \phi_\lambda, F_\lambda]_{\lambda \in \mathscr{X}(C)}$ in $\mathbb{E}_T$, define $\mu\colon \mathbb{E}_T \to \mathbb{E}_{\tilde{T}}$ by

$$\mu([E_\lambda, \phi_\lambda, F_\lambda]_{\lambda \in \mathscr{X}(C)}) = \bigoplus_{\lambda \in \mathscr{X}(C)} [V_{\tau(\lambda)} \otimes s^* E_\lambda, id \otimes s^* \phi_\lambda, id \otimes s^* F_\lambda]$$

where s^*E_λ is regarded as a $\tilde{T}$-Hilbert B-module through s. Likewise, $s^*\phi_\lambda$ and s^*F_λ are regarded as $\tilde{T}$-$*$-homomorphism and $\tilde{T}$-Hilbert B-module map via s respectively. $V_{\tau(\lambda)} \otimes s^*E_\lambda$ is the external tensor product of $V_{\tau(\lambda)}$ (as a $\tilde{T}$-Hilbert space) and s^*E_λ. Hence it is an $\tilde{T}$-Hilbert B-module itself after identifying $\mathbb{C} \otimes B$ with B as $\tilde{T}$-C*-algebras.

Then, for all $[E_\lambda, \phi_\lambda, F_\lambda]_{\lambda \in \mathscr{X}(C)}$ in $\mathbb{E}_T$,

$$\nu(\mu([E_\lambda, \phi_\lambda, F_\lambda]_\lambda)) = [Hom(V_{\tau(\psi)}, \bigoplus_\lambda V_{\tau(\lambda)} \otimes s^*E_\lambda)^C, \bigoplus_\lambda (\widetilde{id \otimes s^*\phi_\lambda})_\psi, \bigoplus_\lambda (\widetilde{id \otimes s^*F_\lambda})_\psi]_{\psi \in \mathscr{X}(C)}$$

And

$$\begin{aligned} Hom(V_{\tau(\psi)}, \bigoplus_\lambda V_{\tau(\lambda)} \otimes s^*E_\lambda)^C &= \bigoplus_\lambda Hom(V_{\tau(\psi)}, V_{\tau(\lambda)} \otimes s^*E_\lambda)^C \\ &= \bigoplus_\lambda Hom(V_{\tau(\psi)}, V_{\tau(\lambda)})^C \otimes (s^*E_\lambda)^C \\ &= E_\psi \end{aligned}$$

From here it is easily verified that

$$(\widetilde{id \otimes s^*\phi_\lambda})_\psi = \phi_\psi$$
$$(\widetilde{id \otimes s^*F_\lambda})_\psi = F_\psi$$

if $\lambda = \psi$. And $(\widetilde{id \otimes s^*\phi_\lambda})_\psi = 0$, $(\widetilde{id \otimes s^*F_\lambda})_\psi = 0$ otherwise. and Hence,

$$\nu\mu([E_\lambda, \phi_\lambda, F_\lambda]_\lambda) = [E_\lambda, \phi_\lambda, F_\lambda]_\lambda$$

For all objects $[E, \phi, F]$ in $\mathbb{E}_{\tilde{T}}$,

$$\mu(\nu([E, \phi, F])) = \bigoplus_\lambda [V_{\tau(\lambda)} \otimes s^*(Hom(V_{\tau(\lambda)}, E)^C), id \otimes s^*\tilde{\phi}_\lambda, id \otimes s^*\tilde{F}_\lambda]$$

We have

$$\bigoplus_\lambda V_{\tau(\lambda)} \otimes s^*(Hom(V_{\tau(\lambda)}, E)^C) \cong E$$

by virtue of Chapter III (6.4) in [BD]. From here it is easily verified that

$$\bigoplus_{\lambda} id \otimes s^*\tilde{\phi}_\lambda \cong \phi$$
$$\bigoplus_{\lambda} id \otimes s^*\tilde{F}_\lambda \cong F$$

Hence, we have

$$\mu\nu([E, \phi, F]) = [E, \phi, F]$$

We conclude that the categories $\mathbb{E}_{\tilde{T}}$ and $\mathbb{E}_T$ are equivalent.

If two elements in $x, y \in \mathbb{E}_{\tilde{T}}(A, B)$ are homotopic, i.e. they represent the same class in $KK_{\tilde{T}}(A, B)$, then there exists an element $a \in \mathbb{E}_{\tilde{T}}(A, B[0, 1])$ such that $(ev_0)_*(a) = x$ and $(ev_1)_*(a) = y$, where $ev_j \colon B([0, 1]) \to B$ is the evaluation at j, $j = 0, 1$. We consider the element $\nu(a) = (a_\lambda)_{\lambda\in\mathscr{X}(C)} \in \prod_\lambda \mathbb{E}_T(A, B([0, 1]))$. Then $(ev_0)_*((a_\lambda)_{\lambda\in\mathscr{X}(C)})$ and $(ev_1)_*((a_\lambda)_{\lambda\in\mathscr{X}(C)})$ are homotopic in $\prod_\lambda \mathbb{E}_T(A, B)$. A couple of definition-tracing arguments show that $\mu((ev_0)_*((a_\lambda)_\lambda)) = x$ and $\mu((ev_1)_*((a_\lambda)_\lambda)) = y$ in $\mathbb{E}_{\tilde{T}}(A, B)$. It means that there is a well-defined injective map from $KK_{\tilde{T}}(A, B)$ to $\oplus_\lambda KK_T(A, B)$. A very similar argument starting from two homotopic elements in $\prod_\lambda \mathbb{E}_T(A, B)$ shows the reverse inclusion and hence we obtain

$$\bigoplus_{\lambda\in\mathscr{X}(C)} KK_T(A, B) \cong KK_{\tilde{T}}(A, B)$$

The isomorphism $\oplus_\lambda KK_T(A, B) \to KK_{\tilde{T}}(A, B)$ is defined by

$$[E_\lambda, \phi_\lambda, F_\lambda]_{\lambda\in\mathscr{X}(C)} \mapsto \sum_{\lambda\in\mathscr{X}(C)} [V_{\tau(\lambda)}] \otimes_{\mathbb{C}} s^*([E_\lambda, \phi_\lambda, F_\lambda])$$

where $[V_{\tau(\lambda)}] \in R(\tilde{T}) \cong KK_{\tilde{T}}(\mathbb{C}, \mathbb{C})$ and $\otimes_{\mathbb{C}}$ is the Kasparov product over $\mathbb{C}$. In particular, setting $A = \mathbb{C}$ and $B = \mathbb{C}$ gives

$$\bigoplus_{\lambda\in\mathscr{X}(C)} R(T) \cong R(\tilde{T})$$

and hence

$$\bigoplus_{\lambda \in \mathscr{C}(C)} KK_T(A,B) \cong R(\tilde{T}) \otimes_{R(T)} KK_T(A,B)$$

Hence, we have

$$KK_{\tilde{T}}(A,B) \cong R(\tilde{T}) \otimes_{R(T)} KK_T(A,B)$$

which proves the lemma. ■

2.3.5 Main Theorem

In this section, we will show our main theorems, Theorem 52 and Theorem 54.

Let $\mathscr{R}$ be the root system of (G,T) and W be the Weyl group. We fix a basis of $\mathscr{R}$. Let α be a root, G_α be the centralizer in G of $\ker \alpha$ and $i_\alpha \colon T \to G_\alpha$ be the inclusion. Motivated by the definition of $i_!$, we want to define a 'pushforward' map $i_{\alpha,!} \colon KK_T(A,B) \to KK_{G_\alpha}(A,B)$ for every root α. First, we choose a complex structure on G_α/T. We do this by identifying G_α/T with the complex homogeneous space $(G_\alpha)_{\mathbb{C}}/B$ where B_α is the Borel subgroup of $(G_\alpha)_{\mathbb{C}}$ generated by $T_{\mathbb{C}}$ and the root space $\mathfrak{g}_{\mathbb{C}}^{-\alpha}$. Then $[i_{\alpha,!}]$ is defined in the same way as $[i_!]$ in Section 2.3.2. Moreover, the map $i_{\alpha,!} \colon KK_T(A,B) \to KK_{G_\alpha}(A,B)$ is also defined in the same way as $i_!$, see 2.3.4.

Define $\sigma_\alpha \colon KK_T(A,B) \longrightarrow KK_T(A,B)$ by

$$\sigma_\alpha = i_\alpha^* \circ i_{\alpha,!}$$

for every root α.

By Lemma 43 for $G = G_\alpha$, σ_α has the properties that $\sigma_\alpha^2 = \sigma_\alpha$ and $\sigma_\alpha(i_\alpha^*(x)) = i_\alpha^*(x)$ for $x \in KK_{G_\alpha}(A,B)$.

Definition 48 σ_α as defined above is called the *divided difference operator corresponding to the root* α. The set $\{\sigma_\alpha | \alpha \in \mathscr{R}\}$ is called the set of divided difference operators which act on $KK_T(A, B)$.

Under the same assumptions as in Theorem 46 we have $\sigma_\alpha = 1 \otimes \delta_\alpha$ for all roots α.

Remark 49 As stated before, the power of equivariant KK-theory comes from the fact that it generalizes both equivariant K-theory and equivariant K-homology. On the K-theory side, when $A = \mathbb{C}$ and $B = C(M)$ where M is a compact G-space, our set of divided difference operators specializes to a set of divided difference operators in T-equivariant K-theory of M, $K_T(M)$, which was first defined in [HLS]. On the other hand, if $B = \mathbb{C}$, then it simply means that we have now abstractly defined a set of divided difference operators in $K_T^0(A)$, which is clearly a new result.

The isobaric divided difference operators were introduced by Demazure [D3] on $R(T)$. The precise definitions were as follows. Let $s_\alpha \in W$ be the reflection element in the root α. Let $\mathscr{X}(T)$ be the character group of T and $\lambda \in \mathscr{X}(T)$, the element $e^\lambda - e^{-\alpha} e^{s_\alpha(\lambda)}$ is uniquely divisible by $1 - e^\alpha$, then a $\mathbb{Z}$-linear endomorphism δ_α of $R(T)$ is defined by

$$\delta_\alpha(u) = \frac{u - e^{-\alpha} s_\alpha(u)}{1 - e^{-\alpha}} \tag{2.5}$$

for all $u \in R(T)$. It has the following important property:

$$\delta_\alpha^2 = \delta_\alpha$$

and

$$\delta_\alpha(1) = 1$$

Alternatively, in a series of earlier papers [D1], [D2], Demazure defined the operators

$$\delta'_\alpha(u) = \frac{u - s_\alpha(u)}{1 - e^{-\alpha}} \tag{2.6}$$

It is easy to see that

$$(\delta'_\alpha)^2 = \delta'_\alpha$$

and

$$\delta'_\alpha(1) = 0$$

In the literature, δ_α are usually called *isobaric divided difference operators*. For any $\omega \in W$ and any reduced expression $\omega = s_{\beta_1} s_{\beta_2} ... s_{\beta_l}$ in terms of simple reflections, the composition $\delta_{\beta_1}\delta_{\beta_2}...\delta_{\beta_l}$ takes the same value ∂_ω. Similarly, the composition $\delta'_{\beta_1}\delta'_{\beta_2}...\delta'_{\beta_l}$ takes the same value $\partial'_\omega = e^{-\rho}\partial_\omega e^{-\rho}$, see [D3]. For the longest element ω_0, we call ∂_{ω_0} the top Demazure's operator.

Remark 50 When $A = \mathbb{C}$, $B = C(SU(n)/T)$, the set of divided difference operators σ_α is the same as ∂_i we used in Section 1.4.2, where i stands for the reflection element $s_i = (i, i+1) \in S_n$. S_n is the Weyl Group in this case.

$$KK_T(\mathbb{C}, C(SU(n)/T)) \cong K^0_T(SU(n)/T) \cong R(T) \otimes_{R(SU(n))} R(T)$$

Then by Theorem 46, σ_α acts as $1 \otimes \delta_\alpha$ on $R(T) \otimes_{R(SU(n))} R(T)$. By the identification of $R(T) \otimes_{R(SU(n))} R(T)$ with $\frac{\mathbb{Z}[y_1^{\pm 1}, ..., y_n^{\pm 1}, x_1, ..., x_n]}{(J, (\prod_{i=1}^n y_i) - 1)}$ as we have done in equation (1.2) in Section 1.4.3, it is now clear that $1 \otimes \delta_\alpha$ acts as the divided difference operator ∂_i that we defined in Section 1.4.2.

Let $\mathscr{E} = End_{R(G)}(R(T))$ be the $R(G)$-algebra of $R(G)$-linear endomorphisms of $R(T)$. Let $\mathscr{D}$ be the subalgebra of $\mathscr{E}$ generated by δ_α and the elements of $R(T)$

(as multiplication operators). By definition of ∂_ω, ∂'_ω, we have $\partial_\omega, \partial'_\omega \in \mathscr{D}$ for all ω. As a ring $\mathscr{D}$ is isomorphic to the Hecke algebra over $\mathbb{Z}$ of the extended affine Weyl group $\mathscr{X}(T) \rtimes W$, see [KL]. In [HLS] $\mathscr{D}$ is called the Hecke algebra.

The *augmentation left ideal* of $\mathscr{D}$ is the annihilator of the identity element $1 \in R(T)$, that is

$$I(\mathscr{D}) = \{\Delta \in \mathscr{D} | \Delta(1) = 0\}$$

By (2.5), $\mathscr{D}$ contains the group ring $\mathbb{Z}[W]$ when $\mathbb{Z}[W]$ is viewed as an algebra of endomorphisms of $R(T)$. Hence $I(\mathscr{D})$ naturally contains the augmentation ideal $I(W)$ of $\mathbb{Z}[W]$. Since $\partial'_\omega(1) = 0$ for $\omega \neq 1$, $I(\mathscr{D})$ contains all ∂'_ω when $\omega \neq 1$.

Some properties of $\mathscr{D}$ and $I(\mathscr{D})$ are noted as follows.

Theorem 51 (Harada, Landweber, Sjamaar) *(i) $(\partial_\omega)_{\omega \in W}$ is a basis of the left $R(T)$-module $\mathscr{D}$.*

(ii) $(\partial'_\omega)_{\omega \in W}$ is a basis of the left $R(T)$-module $\mathscr{D}$.

(iii) $(\partial_\omega)_{\omega \neq 1}$ is a basis of the left $R(T)$-module $I(\mathscr{D})$.

Let M be a left $\mathscr{D}$-module. We say an element of M is *$\mathscr{D}$-invariant* if it is annihilated by all operators in the augmentation left ideal $I(\mathscr{D})$. We denote $M^{I(\mathscr{D})}$ the group of invariants. By Theorem 51,

$$M^{I(\mathscr{D})} = \{m \in M | \partial'_\omega(m) = 0, \text{for all } \omega \neq 1\}$$

Since $I(\mathscr{D})$ contains the augmentation left ideal $I(W)$ of $\mathbb{Z}[W]$, we have

$$M^{I(\mathscr{D})} \subseteq M^W \tag{2.7}$$

where M^W contains elements that are invariant under the Weyl group action.

We now show that $KK_T(A,B)$ is equipped with a left $\mathscr{D}$-module structure in Theorem 52. Then, by (2.7), we have the following

$$KK_T(A,B)^{I(\mathscr{D})} \subseteq KK_T(A,B)^W \tag{2.8}$$

We will discuss (2.8) in Section 2.5.

Theorem 52 *The operators σ_α for $\alpha \in \mathscr{R}$, together with the natural $R(T)$-module structure generate a unique $\mathscr{D}$-module structure on $KK_T(A,B)$.*

Proof. The proof is very similar to Prop. 4.5 in [HLS] and is essentially an application of Theorem 45, Theorem 46 and Lemma 47. First, assume that G is a Hodgkin group. Idenitfy $KK_T(A,B)$ with $KK_G(A,B) \otimes_{R(G)} R(T)$ through the isomorphism of Theorem 45. Let

$$\mathscr{E}(A,B) = KK_G(A,B) \otimes \mathscr{E}$$

Then the map $\mathscr{D} \to \mathscr{E}(A,B)$ defined by $\Delta \mapsto 1 \otimes \Delta$, where 1 is the identity map of $KK_G(A,B)$, is a well-defined algebra homomorphism. Since $\sigma_\alpha = 1 \otimes \delta_\alpha$, σ_α generates an well-defined action of $\mathscr{D}$ on $KK_T(A,B)$.

If G is not a Hodgkin group, we choose a covering $s\colon \tilde{G} \to G$ such that $\tilde{G}$ is a Hodgkin group. By Lemma 47 the pullpack

$$s^*\colon KK_T(A,B) \to KK_{\tilde{T}}(A,B)$$

is injective, where $\tilde{T}$ is the maximal torus $s^{-1}(T)$ of $\tilde{G}$. Let $\tilde{\sigma}_\alpha = \tilde{i}_\alpha^* \circ \tilde{i}_{\alpha,!}$ be the operator on $KK_{\tilde{T}}(A,B)$ corresponding to α, where $\tilde{i}_\alpha\colon \tilde{T} \to \tilde{G}_\alpha$ is the inclusion. By the naturality properties of i_α^* and $i_{\alpha,!}$

$$s^*\sigma_\alpha \quad = \quad \tilde{\sigma}_\alpha s^* \tag{2.9}$$

By Lemma 2.4 [HLS], s induces an injective algebra homomorphism

$$\overline{s}\colon \mathscr{D} \to \tilde{\mathscr{D}}$$

We already know that $\tilde{\sigma}_\alpha$ generate a well-defined $\tilde{\mathscr{D}}$-action on $KK_{\tilde{T}}(A,B)$. This $\tilde{\mathscr{D}}$-module structure on $KK_{\tilde{T}}(A,B)$ is unique due to Theorem 46. The restriction of the $\tilde{\mathscr{D}}$-action to the subalgebra $\mathscr{D}$ preserves the submodule $KK_T(A,B)$ and by (2.9), the elements σ_α act in the required fashion. It is clear that the $\mathscr{D}$-module structure on $KK_T(A,B)$ so defined is unique. ■

By Theorem 52, it is now clear that if $A = B = \mathbb{C}$, our set of divided difference operators σ_α that acts on $KK_T(A,B) = KK_T(\mathbb{C},\mathbb{C}) \cong R(T)$ is the same as the set of Demazure's operators δ_α.

If G is a Hodgkin group, let $\mathscr{U} = \mathscr{D}$-Mod and $\mathscr{B} = R(G)$-Mod be the categories of left modules over the rings $\mathscr{D}$ and $R(G)$ respectively. Before stating our next theorem, we invoke the following result shown in [HLS].

Theorem 53 (Harada, Landweber, Sjamaar) *If G is a Hodgkin group, then the functor $\mathscr{G}: \mathscr{B} \to \mathscr{U}$ defined by*

$$B \mapsto B \otimes_{R(G)} R(T)$$

is an equivalence with inverse $\mathscr{F}: \mathscr{U} \to \mathscr{B}$ given by

$$A \mapsto Hom_{\mathscr{D}}(R(T), A)$$

Moreover, $\mathscr{F}$ is naturally isomorphic to the functor $\mathscr{J}: \mathscr{U} \to \mathscr{B}$ given by

$$A \mapsto A^{I(\mathscr{D})}$$

The following result describes $KK_G(A,B)$ as a direct summand of $KK_T(A,B)$. More precisely, $KK_G(A,B)$ is isomorphic to $KK_T(A,B)$ annihilated by 'divided difference operators'.

Theorem 54 *For all G-C*-algebras A and B, the map i^* is an isomorphism from $KK_G(A,B)$ onto $KK_T(A,B)^{I(\mathscr{D})}$ where i is the inclusion $T \to G$.*

Proof. First assume that G is a Hodgkin group, consider the $\mathscr{D}$-module $A = KK_T(A,B)$ and the $R(G)$-module $B = KK_G(A,B)$. By Theorem 45,

$$\mathscr{G}(B) = A$$

Hence, by Theorem 53,

$$B \cong \mathscr{F}(A) \cong \mathscr{J}(A) = A^{I(\mathscr{D})}$$

If G is not a Hodgkin group, we use the same trick as in the proof of Theorem 52 to get our desired result. ■

2.4 Some applications of Theorem 54

If $A = \mathbb{C}$ and $B = C(M)$ where M is a compact G-space. Theorem 54 specializses to equivariant K-theory:

$$K_G(M) \cong K_T(M)^{I(\mathscr{D})}$$

which is one of the main results in [HLS].

On the other hand, if $B = \mathbb{C}$, then Theorem 54 gives the corresponding result in equivariant K-homology, that is

Corollary 55 *If A is a G-C*-algebra, then*

$$K^0_G(A) \cong K^0_T(A)^{I(\mathscr{D})}$$

In particular, if $A = C(M)$ where M is a compact G-manifold, then we have

Corollary 56 *Let M be a compact G-manifold, then*

$$K_0^G(M) \cong K_0^T(M)^{I(\mathscr{D})}$$

2.5 The difference between $KK_T(A,B)^{I(\mathscr{D})}$ and $KK_T(A,B)^W$

Note that if $A = B = \mathbb{C}$, then the equivariant KK-group $KK_G(\mathbb{C},\mathbb{C})$ is isomorphic to $R(G)$. And Theorem 54 gives the following result:

$$R(G) \cong R(T)^{I(\mathscr{D})}$$

But $R(G)$ is also isomorphic to the Weyl invariant of $R(T)$, $R(T)^W$. It means that in the case of character ring of T, $R(T)^W = R(T)^{I(\mathscr{D})}$. One may wonder whether this result generalizes to the equivariant KK-group for any G-C*-algebras A and B. But the following example clearly shows that it is far from being true even for equivariant K-theory, let alone equivariant KK-theory.

Example 57 It was first given by Mcleod [M]. Let $M = SU(2) \times \mathbb{R}P^2$ be a G-space with $G = SU(2)$ acting freely on the $SU(2)$ factor and trivally on the second factor $\mathbb{R}P^2$. We have the following:

$$K_G(M) = K_{SU(2)}(SU(2) \times \mathbb{R}P^2) \cong K(\mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2$$

while

$$K_T(M) = K_{U(1)}(SU(2) \times \mathbb{R}P^2) \cong K(S^2 \times \mathbb{R}P^2) \cong (\mathbb{Z} \oplus \mathbb{Z}H) \otimes (\mathbb{Z} \oplus \mathbb{Z}_2)$$

where H is the Hopf bundle. The Weyl group is isomorphic to S_2 which acts on the Hopf bundle by $H \mapsto H^{-1} = 2 - H$. Thus,

$$K_T(M)^W = K_{U(1)}(SU(2) \times \mathbb{R}P^2)^{S_2} = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$$

For a generalization of this example, see [HLS].

Mcleod gave a criterion for $K_G(M)$ to be isomorphic to $K_T(M)^W$ as follows.

Theorem 58 (Mcleod) *If $K_T(M)$ is a free module over $R(T)$, then*

$$K_G(M) \cong K_T(M)^W$$

However, the previous example showed that the free module requirement is very restrictive.

If M is a compact Hamiltonian G-manifold, then the restriction map $K_T(M) \longrightarrow K_T(M^T)$ induced by $M^T \longrightarrow M$ is injective by Theorem 2.5 in [HL2]. Based on this result, it was shown in [HLS] that

$$K_G(M) \cong K_T(M)^W \tag{2.10}$$

In [K2], Kasparov constructed a map $\tau\colon KK_G(C(M), \mathbb{C}) \longrightarrow KK_G(\mathbb{C}, C(M))$ for any even-dimensional compact G-manifolds M with G-equivariant spin^c-structure and used it to show that it is an isomorphism in G-equivariant KK-theory:

$$KK_G(C(M), \mathbb{C}) \cong KK_G(\mathbb{C}, C(M)) \tag{2.11}$$

It is called the *Poincare duality* in equivariant KK-theory. The generalization of this result to other topological spaces M is one of the most important themes in KK-theory.

For a compact Hamiltonian G-manifold M with a G-equivariant symplectic form ω, there is an G-equivariant almost complex structure naturally associated with ω. It is canonical in the sense that it is unique up to homotopy. We obtain a G-equivariant spin^c-structure on M by this equivariant almost complex structure.

Thus, we can combine Kasparov's result (2.11) with (2.10) to give the following corollary.

Corollary 59 *If M is a compact Hamiltonian G-manifold, then*

$$K_0^G(M) \cong K_0^T(M)^W$$

where $K_0^G(M)$ is the G-equivariant K-homology of M.

Finally, we state some criteria for $KK_G(A,B)$ to be isomorphic to $KK_T(A,B)^W$ in this section. Recall that $\mathrm{d} = \prod_{\alpha \in \mathscr{R}^+}(1-e^{-\alpha}) \in R(T)$ is the Weyl denominator in (2.1).

Lemma 60 *Assume that the Weyl denominator $d = \prod_{\alpha \in \mathscr{R}^+}(1-e^{-\alpha}) \in R(T)$ is not a zero divisor in the $R(T)$-module $KK_T(A,B)$, then the map i^* is an isomorphism from $KK_G(A,B)$ to $KK_T(A,B)^W$ where i is the inclusion $T \to G$.*

Proof. It follows immediately from Lemma 3.5 in [HLS]. ■

The following corollary is immediate by Lemma 60. It is a generalization of Theorem 58.

Corollary 61 *If $KK_T(A,B)$ is a free module over $R(T)$, then*

$$KK_G(A,B) \cong KK_T(A,B)^W$$

2.6 Proof of Theorem 36

Theorem 36 is a version of Frobenius Reciprocity in equivariant KK-theory. As promised in section 2.1 a proof will be provided here. We will only prove it for the case that G is a compact group and A, B are G-C*-algebras.

Recall from 2.3 that if A, B are G-C*-algebras, the we have the restriction map:

$$res_T^G \colon KK_G(A, B) \to KK_T(A, B)$$

which is defined by sending $x = [E, \phi, F] \in KK_G(A, B)$ to $x|_T = [E|_T, \phi|_T, F|_T] \in KK_T(A, B)$ where $E|_T$ is regarded as an T-Hilbert B-module. ϕ is regarded as an T-$*$ homomorphism and F is regarded as an T-bounded operator in $\mathrm{B}(E|_T)$. To avoid notational confusion, we will also use the notations $Res_T^G E$, $Res_T^G F$, $Res_T^G \phi$ for $E|_T$, $F|_T$, $\phi|_T$ respectively.

On the other hand, if M is an T-C*-algebra, then $Ind_T^G(M)$ is the G-C*-algebra of all continuous functions $f \colon G \to M$ such that $f(gh) = h^{-1}f(g), \forall g \in G, h \in T$ and such that $\| f \|$ vanishes at infinity. Since we are dealing with the case that G/T is compact, the C*-norm of each element in $Ind_T^G(M)$ is just the maximum norm. The G-action on $Ind_T^G(M)$ is left translation.

If A is an G-C*-algebra, then $Ind_T^G(Res_T^G(A))$ is *equivariantly isomorphic* to $A \otimes C(G/T)$. We denote the isomorphism from $Ind_T^G(Res_T^G(A))$ to $A \otimes C(G/T)$ by Φ. More explicitly, if $F_A \in Ind_T^G(Res_T^G(A))$, then $\Phi(F_A)([g]) = gF_A(g)$. The inverse map $\Phi^{-1} \colon A \otimes C(G/T) \to Ind_T^G(Res_T^G(A))$ is defined as follows: for $a \otimes f \in A \otimes C(G/T)$, $\Phi^{-1}(a \otimes f)(g) = f(g)g^{-1}a$.

We are going to describe an *induction* map from the T-equivariant KK-theory to the G-equivariant KK-theory for any G-C*-algebras A, B.

Let E is an T-Hilbert B-module, define $\tilde{E} := Ind_T^G E$ by

$$Ind_T^G E = \{f_E \colon G \to E \mid f(gt) = t^{-1}f(g)\}$$

It has an $Ind_T^G B$-valued inner product defined by

$$\langle f_E, f'_E\rangle(g) := \langle f_E(g), f'_E(g)\rangle$$

for any $f_E, f'_E \in Ind_T^G(E)$ and $g \in G$.

Lemma 62 *$\tilde{E}$ is an G-Hilbert $Ind_T^G B$-module.*

Proof. For $f_B \in Ind_T^G(B)$ and $f_E \in Ind_T^G(E)$, we have

$$\begin{aligned}(f_E f_B)(gt) &= f_E(gt)f_B(gt) \\ &= (t^{-1}f_E(g))(t^{-1}f_B(g)) \\ &= t^{-1}(f_E(g)f_B(g)) \\ &= t^{-1}(f_E f_B)(g)\end{aligned}$$

Hence $f_E f_B \in Ind_T^G(E)$. Moreover,

$$\begin{aligned}\langle f_E, f'_E\rangle(gt) &= \langle f_E(gt), f'_E(gt)\rangle \\ &= \langle t^{-1}f_E(g), t^{-1}f'_E(g)\rangle \\ &= t^{-1}\langle f_E(g), f'_E(g)\rangle \\ &= t^{-1}(\langle f_E, f'_E\rangle(g))\end{aligned}$$

Hence, $\langle f_E, f'_E\rangle \in Ind_T^G(B)$. It is easy to check that $\langle f_E, f'_E f_B\rangle = \langle f_E, f'_E\rangle f_B$ and other properties of Hilbert $Ind_T^G B$-module are easily verified. The G-action on $Ind_T^G(E)$ is left translation for all $f_E \in Ind_T^G(E)$. Then

$$\begin{aligned}g\langle f_E, f'_E\rangle(x) &= \langle f_E, f'_E\rangle(g^{-1}x) \\ &= \langle f_E(g^{-1}x), f'_E(g^{-1}x)\rangle \\ &= \langle gf_E(x), gf'_E(x)\rangle\end{aligned}$$

Similarly, other properties of G-Hilbert module structure are easily verified. ■

If $\phi\colon A \to \mathrm{B}(E)$ an T-$*$-homomorphism, define $\tilde{\phi} := Ind_T^G\phi\colon Ind_T^G A \to \mathrm{B}(Ind_T^G E)$ by

$$\tilde{\phi}(f_A)(f_E)(g)\colon = \phi(f_A(g))(f_E(g))$$

for all $g \in G, f_A \in Ind_T^G A, f_E \in Ind_T^G E$.

Lemma 63 *$\tilde{\phi}$ is a well-defined G-$*$-homomorphism.*

Proof. First of all, we need to check that it is well-defined:

$$\begin{aligned}
\tilde{\phi}(f_A)(f_E)(gt) &= \phi(f_A(gt))(f_E(gt)) \\
&= \phi(t^{-1}f_A(g))(t^{-1}f_E(g)) \\
&= (t^{-1}\phi(f_A(g))t)(t^{-1}f_E(g)) \\
&= t^{-1}\phi(f_A(g))(f_E(g)) \\
&= t^{-1}\tilde{\phi}(f_A)(f_E)(g)
\end{aligned}$$

So $\tilde{\phi}(f_A)(f_E) \in Ind_T^G(E)$. And

$$\begin{aligned}
\| \tilde{\phi}(f_A)(f_E)(g) \|^2 &= \| \phi(f_A(g))(f_E(g)) \|^2 \\
&\leq \| \phi(f_A(g)) \|^2 \| f_E(g) \|^2 \\
&\leq \| \tilde{\phi}(f_A) \|^2 \| f_E \|^2
\end{aligned}$$

Hence, $\tilde{\phi}(f_A) \in \mathrm{B}(Ind_T^G(E))$. It is straightforward to see that $\tilde{\phi}(f_A)^*$ exists and

$\tilde{\phi}(f_A)^* \in \mathrm{B}(Ind_T^G(E))$. It is readily checked that $\tilde{\phi}$ is an G-$*$-homomorphism:

$$\begin{aligned}
(g\tilde{\phi}(f_A)g^{-1})(f_E)(x) &= g\tilde{\phi}(f_A)(g^{-1}f_E)(x) \\
&= g\phi(f_A(x))(g^{-1}f_E(x)) \\
&= g\phi(f_A(x))(f_E(gx)) \\
&= g\phi(gf_A(gx))(f_E(gx)) \\
&= g\tilde{\phi}(gf_A)(f_E)(gx) \\
&= \tilde{\phi}(gf_A)(f_E)(x)
\end{aligned}$$

Hence, $(g\tilde{\phi}(f_A)g^{-1})(f_E) = \tilde{\phi}(gf_A)(f_E)$. ■

Let $F \in \mathrm{B}(E)$ where E is an T-Hilbert B-module. We construct $\tilde{F} \in \mathrm{B}(Ind_T^G(E))$ as follows:

$$\tilde{F}(f_E)(g) := F(f_E(g))$$

Lemma 64 *$\tilde{F}$ is a well-defined operator on Hilbert $Ind_T^G B$-module map E. $\tilde{F}$ is G-invariant.*

Proof.

$$\begin{aligned}
\tilde{F}(f_E)(gt) &= F(f_E(gt)) = F(t^{-1}f_E(g)) \\
&= t^{-1}F(f_E(g))t = t^{-1}.F(f_E(g)) \\
&= t^{-1}.\tilde{F}(f_E)(g)
\end{aligned}$$

So, $\tilde{F}(f_E) \in Ind_T^G E$.

$$\begin{aligned}
\tilde{F}(f_E f_B)(g) &= F(f_E f_B(g)) = F(f_E(g)f_B(g)) \\
&= F(f_E(g))f_B(g) = \tilde{F}(f_E)(f_B)(g)
\end{aligned}$$

i.e. $\tilde{F}(f_E f_B) = \tilde{F}(f_E) f_B$. Hence, $\tilde{F}$ is an $Ind_T^G B$-module map.

$$\begin{aligned}
\| \tilde{F}(f_E) \|_{Ind_T^G E} &= \sup \| \tilde{F}(f_E)(g) \| = \sup \| F(f_E(g)) \| \\
&\leq \sup \| F \| \| f_E(g) \| \\
&= \| F \| \sup \| f_E(g) \| \\
&= \| F \| \| f_E \|
\end{aligned}$$

So, $\tilde{F} \in \mathrm{B}(Ind_T^G E)$. Define $\tilde{F^*}(f_E)(g) := F^*(f_E(g))$.

$$\begin{aligned}
\langle \tilde{F}(f_E), f'_E \rangle (g) &= \langle \tilde{F}(f_E)(g), f'_E(g) \rangle \\
&= \langle F(f_E(g)), f'_E(g) \rangle \\
&= \langle f_E(g), F^*(f'_E(g)) \rangle \\
&= \langle f_E, \tilde{F^*}(f'_E) \rangle (g)
\end{aligned}$$

So, $\tilde{F^*} = \tilde{F}^*$. $\tilde{F}$ is also G-continuous. i.e. $g \mapsto g.\tilde{F}$ is continuous in norm topology.

$$\begin{aligned}
g.\tilde{F}(f_E)(x) &= g\tilde{F}g^{-1}(f_E)(x) \\
&= \tilde{F}(g^{-1} f_E)(g^{-1}x) \\
&= F(g^{-1} f_E(g^{-1}x)) \\
&= F(f_E(x)) \\
&= \tilde{F}(f_E)(x)
\end{aligned}$$

So, $\tilde{F}$ is indeed G-invariant. ■

The *induction homomorphism*

$$ind_T^G \colon KK_T(A, B) \to KK_G(Ind_T^G(A), Ind_T^G(B))$$

is defined by sending $x = [E, \phi, F] \in KK_T(A, B)$ to $ind_T^G(x) = [\tilde{E}, \tilde{\phi}, \tilde{F}] \in KK_G(Ind_T^G A, Ind_T^G B)$. It is clear that it is well-defined.

We give a proof of Theorem 36 now.

Proof of Theorem 36: Let $x = [E, \phi, F] \in KK_T(A, B)$ and $i^*(ind_T^G(x)) = [\tilde{E}, \tilde{\phi} \circ i, \tilde{F}]$ where $\tilde{\phi} \circ i\colon A \to \mathrm{B}(\tilde{E})$. For $a \in A$, define $K_a \in Ind_T^G(A)$ by $K_a(g) = g^{-1}a$. Note that the G-action on K_a gives $g.K_a = K_{ga}$. Under the isomorphism between $A \otimes C(G/T)$ and $Ind_T^G(A)$, we can identify $a \otimes 1 \in A \otimes C(G/T)$ with $K_a \in Ind_T^G(A)$ for each $a \in A$.

$$\begin{aligned}(\tilde{\phi} \circ i)(a)(f_E)(g) &= \tilde{\phi}(K_a)(f_E)(g) \\ &= \phi(K_a(g))(f_E(g)) \\ &= \phi(g^{-1}a)(f_E(g))\end{aligned}$$

And $res_T^G \circ i^* \circ ind_T^G(x) = [\tilde{E} \mid_T, (\tilde{\phi} \circ i) \mid_T, \tilde{F} \mid_T]$

For a G-$*$-homomorphism $f\colon B \to D$, the pushforward $f_*\colon KK_G(A, B) \to KK_G(A, D)$ is, by definition, $[M, \xi, N] \mapsto [M \otimes_f D, \xi \otimes id_D, N \otimes id_D]$ where $M \otimes_f D$ is the internal tensor product of G-Hilbert B-module with D, viewed as a Hilbert D-module. For $x = [E, \phi, F] \in KK_T(A, B)$, we have

$$ev_* \circ res_T^G \circ i^* \circ ind_T^G(x) = [res_T^G(\tilde{E}) \otimes_{ev} B, (res_T^G(\tilde{\phi} \circ i^*)) \otimes id_B, res_T^G(\tilde{F}) \otimes id_B]$$

which is an element in $KK_T(A, B)$. $res_T^G(\tilde{E})$ is a T-Hilbert $B \otimes C(G/T)$-module, $res_T^G(\tilde{E}) \otimes_{ev} B$ is then a T-Hilbert B-module, where $ev\colon B \otimes C(G/T) \to B$ is the evaluation at identity. For $f_E, f'_E \in res_T^G(\tilde{E})$, $b_1, b_2 \in B$,

$$\begin{aligned}\langle f_E \otimes b_1, f'_E \otimes b_2 \rangle_{res_T^G(\tilde{E}) \otimes_{ev} B} &= b_1^* ev(\langle f_E, f'_E \rangle) b_2 \\ &= b_1^* \langle f_E, f'_E \rangle (1) b_2 \\ &= b_1^* \langle f_E(1), f'_E(1) \rangle b_2 \\ &= \langle f_E(1) b_1, f'_E(1) b_2 \rangle\end{aligned}$$

Our goal is to show that $x = ev_* \circ res_T^G \circ i^* \circ \iota_T^G(x) \in KK_T(A, B)$.

Claim: $\tilde{E} \otimes_{ev} B$ is isomorphic to E as T-Hilbert B-modules, i.e. $res_T^G(\tilde{E}) \otimes_{ev} B \cong E$.

Proof of claim: Define $Q\colon res_T^G(\tilde{E}) \otimes_{ev} B \to E$ by $f_E \otimes b \mapsto f_E(1)b$.

$$\begin{aligned} Q((f_E \otimes b)b_1) &= Q(f_E \otimes bb_1) = f_E(1)bb_1 = (f_E(1)b)b_1 \\ &= Q(f_E \otimes b)b_1 \end{aligned}$$

$$\begin{aligned} Q(t(f_E \otimes b)) &= Q(tf_E \otimes tb) = (tf_E(1))(t(b)) = t(f_E(1)b) \\ &= tQ(f_E \otimes b) \end{aligned}$$

Hence, Q is a T-Hilbert B-module map. Since G is compact, for each $x \in E$, we can choose a constant function $f_x\colon G \to E$ in $\tilde{E}$ defined by $f_x(g) = x$ for all $g \in G$. Then $Q(f_x \otimes b) = xb$ for all $b \in B$. So Q is surjective. Notice that

$$\langle f'_E \otimes b_1, f''_E \otimes b_2 \rangle = \langle f'_E(1)b_1, f''_E(1)b_2 \rangle$$

$$\langle Q(f'_E \otimes b_1), Q(f''_E \otimes b_2) \rangle = \langle f'_E(1)b_1, f''_E(1)b_2 \rangle$$

So, Q is isometric. Hence Q is an isomorphism between $\tilde{E} \otimes_{ev} B$ and E as T-Hilbert B-modules.

Claim: For any $a \in A$, $b \in B$, the following diagram is commutative:

$$\begin{array}{ccc} res_T^G(\tilde{E}) \otimes_{ev} B & \xrightarrow{(res_T^G(\tilde{\phi} \circ i) \otimes id_B)(a \otimes b)} & res_T^G(\tilde{E}) \otimes_{ev} B \\ \Big\downarrow Q & & \Big\downarrow Q \\ E & \xrightarrow{\phi(a)} & E \end{array}$$

Proof of claim: For any $f_E \otimes b \in res_T^G(\tilde{E}) \otimes_{ev} B$,

$$\begin{aligned} Q((res_T^G(\tilde{\phi} \circ i) \otimes id_B)(a \otimes b)(f_E \otimes b)) &= Q(res_T^G(\tilde{\phi} \circ i)(a)(f_E) \otimes id_B(b)) \\ &= (res_T^G(\tilde{\phi} \circ i)(a))(f_E)(1)b \\ &= \phi(K_a(1))(f_E(1))b \\ &= \phi(a)(f_E(1))b \end{aligned}$$

And

$$\phi(a)(Q(f_E \otimes b)) = \phi(a)(f_E(1)b) = \phi(a)(f_E(1))b$$

So the claim is proved.

Claim: The following diagram is commutative:

$$\begin{array}{ccc} res_T^G(\tilde{E}) \otimes_{ev} B & \xrightarrow{res_T^G(\tilde{F})\otimes id_B} & res_T^G(\tilde{E}) \otimes_{ev} B \\ \big\downarrow Q & & \big\downarrow Q \\ E & \xrightarrow{F} & E \end{array}$$

Proof of claim: For any $f_E \otimes b \in res_T^G(\tilde{E}) \otimes_{ev} B$,

$$\begin{aligned} Q((res_T^G\tilde{F}) \otimes id_B)(f_E \otimes b) &= Q(res_T^G\tilde{F}(f_E) \otimes id_B(b)) \\ &= \tilde{F}(f_E)(1)b \\ &= F(f_E(1))b \end{aligned}$$

And

$$F(Q(f_E \otimes b)) = F(f_E(1)b) = F(f_E(1))b$$

The claim is proved. We have shown that $x = ev_* \circ res_T^G \circ i^* \circ ind_T^G(x) \in KK_T(A, B)$.

On the other hand, take any $y = [V, \psi, W] \in KK_G(A, B \otimes C(G/T))$. By Prop.20.2.4 in [B], we can assume that W is G-invariant. V is a G-Hilbert $B \otimes C(G/T)$-module. $Ind_T^G(res_T^G(V) \otimes_{ev} B)$ is a G-Hilbert $B \otimes C(G/T)$-module.

Claim: V is isomorphic to $Ind_T^G(res_T^G(V) \otimes_{ev} B)$ as G-Hilbert $B \otimes C(G/T)$-module.

Proof of claim: Define $\Phi\colon V \to Ind_T^G(res_T^G(V) \otimes_{ev} B)$ by $\Phi(ef_B)(g) = g^{-1}e \otimes$

$f_B(g)$ for any $e \in V, f_B \in Ind_T^G(B) \cong B \otimes C(G/T), g \in G$. Then

$$\begin{aligned}
\| \Phi(ef_B) \|^2 &= \max \| \Phi(ef_B)(g) \|^2 \\
&= \max \| g^{-1}e \otimes f_B(g) \|^2 \\
&= \max \| \langle g^{-1}e \otimes f_B(g), g^{-1}e \otimes f_B(g) \rangle \| \\
&= \max \| f_B(g)^* \langle g^{-1}e, g^{-1}e \rangle (1) f_B(g) \| \\
&= \max \| f_B(g)^* g^{-1} \langle e, e \rangle (1) f_B(g) \|
\end{aligned}$$

$$\begin{aligned}
\| ef_B \|^2 &= \max \| ef_B(g) \|^2 \\
&= \max \| f_B(g)^* \langle e, e \rangle (g) f_B(g) \| \\
&= \max \| f_B(g)^* g^{-1} \langle e, e \rangle (1) f_B(g) \|
\end{aligned}$$

So, Φ preserves the norm.

$$\begin{aligned}
\Phi(ef_B f'_B)(g) &= g^{-1}e \otimes f_B(g) f'_B(g) = (g^{-1}e \otimes f_B(g)) f'_B(g) = \Phi(ef_B)(g) f'_B(g) \\
&= (\Phi(ef_B) f'_B)(g)
\end{aligned}$$

$$\begin{aligned}
g\Phi(ef_B)(g_1) &= \Phi(ef_B)(g^{-1}g_1) \\
&= (g^{-1}g_1)^{-1}e \otimes f_B(g^{-1}g_1) \\
&= g_1^{-1}ge \otimes gf_B(g_1) \\
&= \Phi((ge)(gf_B))(g_1) \\
&= \Phi(g(ef_B))(g_1)
\end{aligned}$$

So, Φ is a G-Hilbert $B \otimes C(G/T)$-module map. And it is clear that Φ is surjective so it defines an isomorphism between $Ind_T^G(res_T^G(V) \otimes_{ev} B)$ and V as G-Hilbert $B \otimes C(G/T)$ modules.

Claim: For any $a \in A$, the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{\psi(a)} & V \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ Ind_T^G(res_T^G V \otimes_{ev} B) & \xrightarrow{Ind_T^G(res_T^G \psi \otimes Id_B) \circ i(a)} & Ind_T^G(res_T^G V \otimes_{ev} B) \end{array}$$

Proof of claim: For any $e \in V$, $f_B \in Ind_T^G(B) \cong B \otimes C(G/T)$, $g \in G$,

$$\begin{aligned} \Phi(\psi(a)(ef_B))(g) &= \Phi((\psi(a)(e))f_B)(g) \\ &= g^{-1}(\psi(a)(e)) \otimes f_B(g) \\ &= \psi(g^{-1}a)(g^{-1}e) \otimes f_B(g) \end{aligned}$$

The last equality is due to:

$$\psi(g^{-1}a)(g^{-1}e) = g^{-1}\psi(a)gg^{-1}e = g^{-1}\psi(a)(e)$$

On the other hand,

$$\begin{aligned} (Ind_T^G(res_T^G\psi \otimes Id_B) \circ i(a))(\Phi(ef_B))(g) &= (Ind_T^G(res_T^G\psi \otimes Id_B))(K_a)(\Phi(ef_B))(g) \\ &= (res_T^G\psi \otimes Id_B)(K_a(g))(\Phi(ef_B)(g)) \\ &= (res_T^G\psi \otimes Id_B)(g^{-1}a)(g^{-1}e \otimes f_B(g)) \\ &= \psi(g^{-1}a)(g^{-1}e) \otimes f_B(g) \end{aligned}$$

It proves the claim.

Claim: The following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{W} & V \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ Ind_T^G(res_T^G V \otimes_{ev} B) & \xrightarrow{Ind_T^G(res_T^G W \otimes Id_B)} & Ind_T^G(res_T^G V \otimes_{ev} B) \end{array}$$

Proof of claim: For any $v \in V$, $f_B \in Ind_T^G(B)$, $g \in G$,

$$\begin{aligned} Ind_T^G(res_T^G W \otimes Id_B)(\Phi(vf_B))(g) &= (res_T^G W \otimes Id_B)(\Phi(vf_B)(g)) \\ &= (res_T^G W \otimes Id_B)(g^{-1}v \otimes f_B(g)) \\ &= W(g^{-1}v) \otimes f_B(g) \end{aligned}$$

$$\begin{aligned} \Phi \circ W(vf_B)(g) &= \Phi(W(vf_B))(g) = \Phi(W(v)f_B)(g) \\ &= g^{-1}(W(v)) \otimes f_B(g) \end{aligned}$$

Since W is G-invariant, then

$$g^{-1}(W(v)) = g^{-1}(W(gg^{-1}v)) = g^{-1}.W(g^{-1}v) = W(g^{-1}v)$$

The last equality is by G-invariance of W. Hence we have shown that $y = i^* \circ ind_T^G \circ ev_* \circ res_T^G(y) \in KK_G(A, B \otimes C(G/T))$. It concludes our proof of the theorem.

2.7 Proof of Theorem 45

In this section, we give a sketch proof of Theorem 45:

Proof. The basic idea is similar to the one proved by Rosenberg and Schochet in Theorem 3.7 (i) of [RS] for the case of K-theory of C*-algebras. Therefore we content ourselves here with a sketch of proof. A particular case of a theorem in [K2] showed that there is a Poincare duality

$$\delta \colon KK_G(C(G/T), \mathbb{C}) \to KK_G(\mathbb{C}, C(G/T))$$

which is an isomorphism. And more generally, we have an isomorphism

$$\delta_{C(G/T)} \colon KK_G(C(G/T), C(G/T)) \to KK_G(\mathbb{C}, C(G/T) \otimes C(G/T))$$

By a theorem of Mcleod [M],

$$KK_G(\mathbb{C}, C(G/T) \otimes C(G/T)) \cong K_G^*(G/T \times G/T) \cong K_T^*(G/T) \cong R(T) \otimes_{R(G)} R(T)$$

Steinberg's theorem [St] provides a free basis $\{e_\omega\}_{\omega \in W}$ for $R(T)$ as a $R(G)$-module, where $W \cong N(T)/T$ is the Weyl group of (G, T). Then there exist an unique set of elements $\{b_\omega\}_{\omega \in W}$ of $R(T) \cong KK_G(\mathbb{C}, C(G/T))$ such that

$$\delta_{C(G/T)}(1_{C(G/T)}) = \sum_{\omega \in W} b_\omega \otimes_{\mathbb{C}} e_\omega$$

Note that $\otimes_{\mathbb{C}}$ is the Kasparov product. For $\omega \in W$, let

$$a_\omega = \delta^{-1}(b_\omega)$$

Then we have, for $1_{C(G/T)} \in KK_G(C(G/T), C(G/T))$,

$$\begin{aligned} 1_{C(G/T)} &= \delta^{-1}_{C(G/T)}(\delta_{C(G/T)}(1_{C(G/T)})) \\ &= \delta^{-1}_{C(G/T)}(\sum_{\omega \in W} b_\omega \otimes_{\mathbb{C}} e_\omega) \\ &= \sum_{\omega \in W} \delta^{-1}(b_\omega) \otimes_{\mathbb{C}} e_\omega \\ &= \sum_{\omega \in W} a_\omega \otimes_{\mathbb{C}} e_\omega \end{aligned}$$

The third equality is done by the associativity of Kasparov product. Then we have the following calculation for any $v \in W$:

$$\begin{aligned} e_v &= e_v \otimes_{C(G/T)} 1_{C(G/T)} = e_v \otimes_{C(G/T)} (\sum_{\omega \in W} a_\omega \otimes_{\mathbb{C}} e_\omega) \\ &= \sum_{\omega \in W} (e_v \otimes_{C(G/T)} a_\omega) \otimes_{\mathbb{C}} e_\omega \end{aligned}$$

which means that if $v = \omega$, $e_v \otimes_{C(G/T)} a_\omega = 1_{R(G)}$. And $e_v \otimes_{C(G/T)} a_\omega = 0$ otherwise. For any element $y \in KK_T(A, B) \cong KK_G(A, B \otimes C(G/T))$ (the isomorphism is by

Theorem 36),

$$\begin{aligned} y &= y \otimes_{C(G/T)} 1_{C(G/T)} \\ &= y \otimes_{C(G/T)} (\sum_{\omega \in W} a_\omega \otimes_{\mathbb{C}} e_\omega) \\ &= \sum_{\omega \in W} (y \otimes_{C(G/T)} a_\omega) \otimes_{\mathbb{C}} e_\omega \end{aligned} \tag{2.12}$$

Note that $y \otimes_{C(G/T)} a_\omega \in KK_G(A, B)$. If

$$y = \sum_{\omega \in W} x_\omega \otimes_{\mathbb{C}} e_\omega$$

for some $x_\omega \in KK_G(A, B)$. Then

$$\begin{aligned} y \otimes_{C(G/T)} a_u &= (\sum_{\omega \in W} x_\omega \otimes_{\mathbb{C}} e_\omega) \otimes_{C(G/T)} a_u \\ &= \sum_{\omega \in W} x_\omega \otimes_{\mathbb{C}} (e_\omega \otimes_{C(G/T)} a_u) \\ &= \sum_{\omega \in W} x_\omega \otimes_{\mathbb{C}} \delta_{uw} \\ &= x_u \end{aligned}$$

Hence, equation (2.12) is an unique expression for $y \in KK_T(A, B)$. It means that $KK_T(A, B)$ and $R(T) \otimes_{R(G)} KK_G(A, B)$ are isomorphic as $R(G)$-module. It is clear that they are also isomorphic as $R(T)$-module. ■

BIBLIOGRAPHY

[A] M. Atiyah, *Bott periodicity and the index of elliptic operators*, Quart. J. Math. Oxford Ser. (2) **19** (1968), 113-140.

[AB] M. Atiyah and R. Bot, *A Lefschetz fixed point formula for elliptic complexes: II. Applications.*, Ann. of Math., **88** (1968) 451-491.

[B] B. Blackadar, *K-Theory for Operator Algebras*, Second Edition, Cambridge University Press, 1998. MR **99g**:46104

[Bo] R. Bott, *The index theorem for homogeneous differential operators*, Differential and Combinatorial Topology, A Symposium in Honor of Marston Morse (Princeton, NJ) (S. Cairns, ed.), Princeton Mathematical Series, vol.27, Princeton University Press, 1965, pp. 167-186.

[BD] T. Brocker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985

[D1] M. Demazure, *Invariants symétriques entiers es groupes de Weyl et torsion*, Invent. Math. **21** (1973), 287-301.

[D2] M. Demazure, *Désingularisation des variétés de Schubert géréralisées*, Ann. Sci. École Norm. Sup. (4) 7 (1974), 53-88.

[D3] M. Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. (2) **98** (1974), no. 3, 163-172.

[F] W. Fulton, *A pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. Journal **76** (1994) no.3. 711-729

[G1] R.F. Goldin, *The cohomology ring of weight varieties and polygon spaces*, Advances in Mathematics **160** (2001) no. 2. 175-204

[G2] R.F. Goldin, *An effective algorithm for the cohomology ring of symplectic reductions*, Geom. Func. Anal. **12** (2002) 567-583

[GM] R.F. Goldin and A.-L. Mare, *Cohomology of symplectic reductions of generic coadjoint orbits*, Proc. of Amer. Math. Soc. **132** (2004) no.10. 3069-3074

[HL1] M. Harada and G.D. Landweber, *Surjectivity for Hamiltonian G-spaces in K-theory*, Trans. Amer. Math. Soc. **259** (2007) no.12. 6001-6025

[HL2] M. Harada and G.D. Landweber, *The K-theory of abelian symplectic quotients*, Math. Res. Lett. **15** (2008) no.1. 57-72

[HLS] M. Harada, G. Landweber and R. Sjamaar, *Divided differences and the Weyl character formula in equivariant K-theory*, Math. Res. Lett. **17** (2010), no. 3, 507-527.

[H] A. Hatcher, *Vector bundles and K-theory*, http://www.math.cornell.edu/ Hatcher/ (2003) Version 2.0.

[HR] N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[Ho] L. Hodgkin, *The equivariant Künneth theorem in K-theory*, Lecture notes in Mathematics, vol. 496, Springer-Verlag, Berlin, 1975, pp.1-101.

[JT] K. K. Jensen and K. Thomsen, *Elements of KK-theory*, Birkhäuser, Boston, 1991.

[J] P. Julg, *Induction holomorphe pour le produit croisé d'une* C^**-algèbre par un groupe de Lie compact*, C.R. Acad. Sci. Paris, Sér. I **294** (1982), 193-196.

[K1] G. G. Kasparov, *Topological invariants of elliptic operators, I: K-homology*, Izv. Akad. Nauk SSSR Ser. Mat. **39**:4 (1975), 796-838. In Russian; translated in Math. USSR Izv. **9**:4 (1975), 751-792.

[K2] G. G. Kasparov, *Equivariant KK-theory and the Novikov conjecture*, Invent. Math. **91**:1 (1988), 147-201.

[KL] D. Kazhdan and G. Lusztig, *Proof of Deligne-Langlands conjecture for Hecke algebras*, Invent. Math. **87** (1987), no.1, 153-215.

[K] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*, volume 31 of Mathematical Notes, Princeton University Press Princeton, N.J. (1984)

[KK] B. Kostant and S. Kumar, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geo. **32** (1987) no.2. 549-603

[KM] A. Knutson and E. Miller, *Grobner Geometry of Schubert polynomials*, Ann. of Math. (2) **161** (2005) no.3. 1245-1318

[Kn] A. Knutson, *Weight Varieties* MIT Ph.D. thesis (1996)

[L] E. Lerman, *Gradient flow of the norm squared of a moment map*, Enseign. Math. (2) **51** (2005), 117-127

[M] J. McLeod, *The Künneth formula in equivariant K-theory*, Algebraic Topology (Waterloo, 1978) (P. Hoffman and V. Snaith, eds.), Lecture Notes in Mathematics, vol. 741, Springer-Verlag, Berlin, 1979, pp. 316-333.

[Mu] G. J. Murphy, C^**-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.

[RS] J. Rosenberg and C. Schochet, *The Künneth theorem and the universal coefficient theorem in equivariant K-theory and KK-theory*, Mem. Amer. Math. Soc. **62** (1986), no. 348, vi+95.

[S] G. Segal, *Equivariant K-theory*, Inst. Hausts Etudes Sci. Publ.Math. **34** (1968) 129-151

[Sj] R. Sjamaar, *Radicial subgroups and Weight Varieties*, preprint, 2009

[Sk] G. Skandalis, *Kasparov's bivariant K-theory and applications*, Exposition. Math. 9 (1991) no.3. 193-250

[Sn] V. Snaith, *On the Künneth formula spectral sequence in equivariant K-theory*, Proc. Cambridge Philos. Soc. **72** (1972), 167-177.

[St] R. Steinberg, *On a theorem of Pittie*, Topology **14** (1975), 173-177.

[TW] S. Tolman and J. Weitsman, *The cohomology rings of symplectic quotients*, Comm. Anal. Geom. **11** (2003) no.4. 751-773

[V] A. Valette, *Introduction to the Baum-Connes Conjecture*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2002, From notes taken by Indira Chatterji, With an appendix by Guido Mislin. MR **2003f**:58047

[W] A. Wasserman, *Equivariant K-theory II: the Hodgkin spectral sequence in Kasparov theory*, Preprint, 1983.