INTER- AND INTRA-BLOCK ESTIMATION OF TREATMENT EFFECTS IN RANDOMIZED BLOCKS.

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ABSTRACT

Best linear unbiased estimation of treatment effects in randomized blocks is considered for experiments in which there are unequal numbers of observations on the treatments in each block. Treating block effects as fixed yields, of course, the same estimators as in any 2-way crossed classification fixed effects model without interaction; and for block effects taken as random a tractable expression for estimated treatment effects is developed. A particular application of that expression to balanced incomplete blocks, reconciliation of which is shown for the calculation-oriented expressions given for the inter-intra block estimator of treatment effects given in five standard texts.

1. INTRODUCTION

Randomized complete block experiments consisting of t treatments used in each of b blocks usually involve n observations on each treatment in each block. In many experiments $n = 1$. Special cases of block experiments are balanced incomplete blocks in which (with $n = 1$) only $k < t$ treatments are used in each block, such that each treatment is used in r blocks and each treatment pair occurs in λ blocks. In between these extremes of

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complete blocks and balanced incomplete blocks is the more general case where k_j treatments are used in block j, with $n_{ij} \geq 0$ observations on treatment i in that block and $n_{ij} > 0$ for k_j values of i. John (1971, p.228) considers this general case but only when treating block effects as fixed effects. This paper deals with the case of block effects being treated as random: estimators of treatment effects are derived, along with their variances, and the results are applied to balanced incomplete blocks to yield an expression for the inter- and intra-block estimator of a treatment effect that is simpler than that given by Kempthorne (1952), Federer (1955), Cochran and Cox (1957), Scheffé (1959) and John (1971).

We use a linear model that has equation

$$y_{ijp} = \tau_i + \beta_j + e_{ijp} \tag{1}$$

where y_{ijp} is the p'th observation on the i'th treatment in the j'th block, for $i = 1,\cdots,t$, $j = 1,\cdots,b$ and $p = 1,\cdots,n_{ij}$ with $n_{ij} \geq 0$, there being at least one $n_{ij} > 0$ for each value of i and for each value of j. In (1), τ_i is the effect due to treatment i, β_j is the effect due to block j and the e_{ijp}s are random errors having mean zero and variance σ_e^2, and all such terms are uncorrelated. The τ_is are considered as fixed effects and the point of interest is to estimate contrasts (linear combinations of differences $\tau_i - \tau_h$) among the τ_is, and to derive sampling variances of those estimators. Two interpretations for the β_js are available. One is that they are fixed effects, and contrasts among them can be estimated, just as with the treatment effects; and the other is that the β_js are random effects, which must be taken into account in estimating treatment contrasts.

Randomized complete blocks have the same number of observations on every treatment in every block, i.e., all $n_{ij} = n$. The best linear unbiased estimator (BLUE) of $\tau_i - \tau_h$ is then

$$\text{BLUE}(\tau_i - \tau_h) = \bar{y}_{i..} - \bar{y}_{h..} \tag{2}$$

whether the block effects are treated as fixed or random. This is well known. But with unbalanced data, i.e., not necessarily equal numbers of observations on the treatments either within a block or from block to block, the estimator of a treatment difference is not the simple difference between observed treatment means of balanced data that is evident in (2). Furthermore, with unbalanced data, the estimators of treatment differences when block effects are treated as fixed are not the same as when block effects are treated as random. It is these two sets of estimators which we specify in this paper.

In matrix notation, where $\underset{\sim}{y}$ and $\underset{\sim}{e}$ are vectors of elements y_{ijp} and e_{ijp} of (1) arrayed in lexicon order, and with $\underset{\sim}{\tau} = [\tau_1 \cdots \tau_t]'$ and $\underset{\sim}{\beta} = [\beta_1 \cdots \beta_b]'$, we write (1) as

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{\tau} + \underset{\sim}{Z}\underset{\sim}{\beta} + \underset{\sim}{e} \tag{3}$$

with $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ being incidence matrices corresponding to $\underset{\sim}{\tau}$ and $\underset{\sim}{\beta}$, respectively. Then $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ have the following forms.

$$\underset{\sim}{X} = \overset{i=t}{\underset{i=1}{(+)}} \underset{\sim}{1}_{n_i} \tag{4}$$

where (+) represents the operation of a direct sum of matrices and where $\underset{\sim}{1}_{n_i}$ is a summing vector (all elements unity) of order n_i. And

$$\underset{\sim}{Z} = \begin{bmatrix} \underset{\sim}{Z}_1 \\ \vdots \\ \underset{\sim}{Z}_i \\ \vdots \\ \underset{\sim}{Z}_t \end{bmatrix} \text{ with } \underset{\sim}{Z}_i = \overset{j=b}{\underset{j=1}{[+]}} \underset{\sim}{1}_{n_{ij}} \tag{5}$$

where [+] represents the same operation as (+) except that for any n_{ij} that is zero, we define $\tilde{1}_0$ in (5) to be a scalar zero that is to be treated as a matrix not of order 1 × 1 (as is often useful) but of order 0 × 1, i.e., no row and one column. The purpose of this is that it gives to $\tilde{Z}_i$ its correct structure, that for n_{ij} being non-zero, $\tilde{1}_{n_{ij}}$ in (5) occurs in column j of $\tilde{Z}_i$.

Example Suppose the numbers of observations in an experiment are as follows.

Numbers of observations: n_{ij}

i	j			$n_{i.}$
	1	2	3	
1	1	0	2	3
2	1	2	1	4
3	0	2	2	4

Then, using a dot to represent a matrix element of zero, $\tilde{X}$ and $\tilde{Z}$ are

$$\tilde{X} = \begin{bmatrix} 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \end{bmatrix} \quad \text{and} \quad \tilde{Z} = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \\ \hline 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \hline \cdot & 1 & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \end{bmatrix} . \tag{6}$$

The first and last sub-matrices of $\underset{\sim}{Z}$ in (6) illustrate the effect of defining $\underset{\sim}{1}_0$ as $0_{0\times 1}$; e.g., for $\underset{\sim}{Z}_1$ in (6) $n_{11} = 1$, $n_{12} = 0$ and $n_{13} = 2$ so that

$$\underset{\sim}{Z}_1 = \begin{bmatrix} \underset{\sim}{1}_1 & \cdot & \cdot \\ \cdot & \cdot & \underset{\sim}{1}_2 \end{bmatrix}$$

as is evident in the first three rows of $\underset{\sim}{Z}$.

2. BEST LINEAR UNBIASED ESTIMATION

When $E(\underset{\sim}{y}) = \underset{\sim}{W}\underset{\sim}{\theta}$ and $\text{var}(\underset{\sim}{y}) = \underset{\sim}{V}$ for $\underset{\sim}{V}$ positive definite, it is well known that the best linear unbiased estimator (BLUE) of $\underset{\sim}{\lambda}'\underset{\sim}{W}\underset{\sim}{\theta}$ is

$$\text{BLUE}(\underset{\sim}{\lambda}'\underset{\sim}{W}\underset{\sim}{\theta}) = \underset{\sim}{\lambda}'\underset{\sim}{W}\underset{\sim}{\theta}^o \text{ for } \underset{\sim}{\theta}^o = (\underset{\sim}{W}'\underset{\sim}{V}^{-1}\underset{\sim}{W})^-\underset{\sim}{W}'\underset{\sim}{V}^{-1}\underset{\sim}{y} , \tag{7}$$

where $\underset{\sim}{A}^-$, for any matrix $\underset{\sim}{A}$, is a generalized inverse of $\underset{\sim}{A}$ satisfying $\underset{\sim}{A}\underset{\sim}{A}^-\underset{\sim}{A} = \underset{\sim}{A}$. We use (7) to derive estimators of $\underset{\sim}{\tau}$ in (3).

3. BLOCKS TREATED AS FIXED EFFECTS

When, in (1), the block effects β_j are treated as fixed effects the equations corresponding to $\underset{\sim}{\theta}^o$ of (7), i.e., the normal equations, are

$$\begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} \\ \underset{\sim}{Z}'\underset{\sim}{X} & \underset{\sim}{Z}'\underset{\sim}{Z} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\tau}^o \\ \underset{\sim}{\beta}^o \end{bmatrix} = \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{y} \\ \underset{\sim}{Z}'\underset{\sim}{y} \end{bmatrix} . \tag{8}$$

The nature of $\underset{\sim}{X}$ and $\underset{\sim}{Z}$ in (4) and (5), illustrated in (6), leads to writing (8) as

$$\begin{bmatrix} D\{n_{i.}\} & \underset{\sim}{N} \\ \underset{\sim}{N}' & D\{n_{.j}\} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\tau}^o \\ \underset{\sim}{\beta}^o \end{bmatrix} = \begin{bmatrix} \{y_{i..}\} \\ \{y_{.j.}\} \end{bmatrix} , \tag{9}$$

where $\underset{\sim}{D}\{n_{i.}\}$ and $\underset{\sim}{D}\{n_{.j}\}$ are diagonal matrices having diagonal elements $n_{i.}$ and $n_{.j}$, respectively, $\underset{\sim}{N} = \{n_{ij}\}$ is the $t \times b$ matrix of n_{ij}s, $\{y_{i..}\}$

is the t × 1 vector of treatment totals $y_{i..}$ and $\{y_{.j.}\}$ is the b × 1 vector of block totals $y_{.j.}$ — in all cases, for i = 1,⋯,t and j = 1,⋯,b.

The nature of the sub-matrices in the (t + b)-order square matrix on the left-hand side of (9) is such that the sum of the rows through $\mathbf{D}\{n_{i.}\}$ equals the sum of the rows through $\mathbf{D}\{n_{.j}\}$. This leads to equations (9) having rank one less than their order. We therefore solve (9) using a generalized inverse of a partitioned matrix given, for example, as $\boldsymbol{\theta}^{**}$ in Searle (1982, Section 10.6):

$$\begin{bmatrix}\boldsymbol{\tau}^o \\ \boldsymbol{\beta}^o\end{bmatrix} = \left\{\begin{bmatrix}\mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}\{1/n_{.j}\}\end{bmatrix} + \begin{bmatrix}\mathbf{I} \\ -\mathbf{D}\{1/n_{.j}\}\mathbf{N}'\end{bmatrix} \mathbf{T}_*^-[\mathbf{I} \quad -\mathbf{N}\mathbf{D}\{1/n_{.j}\}]\right\} \begin{bmatrix}\{y_{i..}\} \\ \{y_{.j.}\}\end{bmatrix} \tag{10}$$

where

$$\mathbf{T}_* = \mathbf{D}\{n_{i.}\} - \mathbf{N}\mathbf{D}\{1/n_{.j}\}\mathbf{N}' \ . \tag{11}$$

With elements in row i of $\mathbf{T}_*$ summing to zero, i.e.,

$$n_{i.} - \textstyle\sum_j n_{ij}(1/n_{.j})\sum_i n_{ij} = n_{i.} - n_{i.}(n_{.j}/n_{.j}) = 0,$$

$\mathbf{T}_*$, of order t, has rank t - 1. Define $\mathbf{T}$ as the principle leading submatrix of order t - 1 in $\mathbf{T}_*$, i.e.,

$$\mathbf{T} = \{t_{ii'}\} \quad \text{for} \quad i,i' = 1,\cdots,t-1 \tag{12}$$

with, from (11),

$$t_{ii} = n_{i.} - \textstyle\sum_j n_{ij}^2/n_{.j} \quad \text{and} \quad t_{ii'} = -\sum_j n_{ij}n_{i'j} \quad \text{for} \quad i \neq i' \ . \tag{13}$$

Then for (10)

$$\mathbf{T}_*^- = \begin{bmatrix}\mathbf{T}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix} \tag{14}$$

and so (10) yields

$$\boldsymbol{\tau}^o = \begin{bmatrix}\mathbf{T}^{-1}\mathbf{u} \\ 0\end{bmatrix} \quad \text{for} \quad \mathbf{u} = \{y_{i..} - \textstyle\sum_j n_{ij}\bar{y}_{.j.}\} \quad \text{for} \quad i = 1,\cdots,t-1. \tag{15}$$

This is, of course, precisely the result implied in (69) and (70) of Searle (1981, Section 7.2).

Knowing, as we do, for the model (1) that $\tau_i - \tau_t$ for every $i = 1,\cdots,t-1$ is estimable we define

$$\underset{\sim}{\delta} = \{\tau_i - \tau_t\} \quad \text{for} \quad i = 1,2,\cdots,t-1 \ . \tag{16}$$

Then $\underset{\sim}{\delta}$ is estimable with BLUE

$$\text{BLUE}(\underset{\sim}{\delta}) = \hat{\underset{\sim}{\delta}} = \underset{\sim}{T}^{-1}\underset{\sim}{u} = \left[\underset{\sim}{D}\{n_{i.}\} - \Sigma_j \frac{n_{ij}n_{i'j}}{n_{.j}} \right]^{-1} \{y_{i..} - \Sigma_j n_{ij}\bar{y}_{.j.}\} \tag{17}$$

for $i,i' = 1,2,\cdots,t-1$.

Through considering the variance of each element of $\underset{\sim}{u}$ in (15), and the covariance of any pair of its elements, we find that the variance-covariance matrix of $\underset{\sim}{u}$ is $\text{var}(\underset{\sim}{u}) = \underset{\sim}{T}\sigma_e^2$, and

$$\text{var}(\hat{\underset{\sim}{\delta}}) = \underset{\sim}{T}^{-1}\sigma_e^2 \ . \tag{18}$$

Then, for any $\underset{\sim}{k}'$ of order $t - 1$

$$\text{BLUE}(\underset{\sim}{k}'\underset{\sim}{\delta}) = \underset{\sim}{k}'\hat{\underset{\sim}{\delta}} \text{ with variance } \underset{\sim}{k}'\underset{\sim}{T}^{-1}\underset{\sim}{k}\sigma_e^2 \ . \tag{19}$$

And σ_e^2 can be estimated unbiasedly by (with $N = n_{..}$)

$$\hat{\sigma}_e^2 = \frac{\Sigma_i\Sigma_j\Sigma_p y_{ijp}^2 - \Sigma_j n_{.j}\bar{y}_{.j.}^2 - \underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u}}{N - a - b + 1} \tag{20}$$

On assuming normality, hypotheses about $\underset{\sim}{k}'\underset{\sim}{\delta}$ can be tested using (19) and (20) in the usual t-statistic. And any hypothesis $H:\underset{\sim}{K}'\underset{\sim}{\delta} = \underset{\sim}{m}$, for $\underset{\sim}{K}'$ of full row rank less than t, can be tested using

$$F = (\underset{\sim}{K}'\hat{\underset{\sim}{\delta}} - \underset{\sim}{m})'(\underset{\sim}{K}'\underset{\sim}{T}^{-1}\underset{\sim}{K})^{-1}(\underset{\sim}{K}'\hat{\underset{\sim}{\delta}} - \underset{\sim}{m})/\hat{\sigma}_e^2 r_{\underset{\sim}{K}} \ , \tag{21}$$

an F-statistic on $r_{\underset{\sim}{K}}$ (rank of $\underset{\sim}{K}$) and $N - a - b + 1$ degrees of freedom.

4. BLOCKS TREATED AS RANDOM EFFECTS

4.1 The model

In treating blocks as random effects we take the β_js as random variables, all with zero mean, variance σ_β^2, and uncorrelated with each other and with the e_{ijp}s. This means that the model is (3) with

$$E(\boldsymbol{\beta}) = \mathbf{0} \ , \ \text{var}(\boldsymbol{\beta}) = \sigma_\beta^2 \mathbf{I}_b \text{ and } \text{var}(\boldsymbol{\beta}, \mathbf{e}') = \mathbf{0} \ , \tag{22}$$

and

$$\text{var}(\mathbf{y}) = \mathbf{V} = \text{var}(\mathbf{Z}\boldsymbol{\beta} + \mathbf{e}) = \sigma_\beta^2 \mathbf{Z}\mathbf{Z}' + \sigma_e^2 \mathbf{I} \ . \tag{23}$$

4.2 BLU estimation of treatment effects

Applying (7) to (3) for estimating $\boldsymbol{\tau}$ gives

$$\text{BLUE}(\boldsymbol{\tau}) = \hat{\boldsymbol{\tau}} = (\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1} \mathbf{X}'\mathbf{V}^{-1}\mathbf{y} \tag{24}$$

for $\mathbf{V}$ of (23). But, as indicated in Searle [1971, Section 10.8a, particularly equation (104)] $\hat{\boldsymbol{\tau}}$ of (24) is also obtainable from the equations

$$\begin{bmatrix} \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z} \\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z} + \rho\mathbf{I} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\tau}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} = \begin{bmatrix} \mathbf{X}'\mathbf{y} \\ \mathbf{Z}'\mathbf{y} \end{bmatrix} \tag{25}$$

for

$$\rho = \sigma_e^2/\sigma_\beta^2 \ . \tag{26}$$

Then, just as with (8) simplifying to (9), so does (25) become

$$\begin{bmatrix} \mathbf{D}\{n_{i.}\} & \mathbf{N} \\ \mathbf{N}' & \mathbf{D}\{n_{.j}+\rho\} \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{\tau}} \\ \tilde{\boldsymbol{\beta}} \end{bmatrix} = \begin{bmatrix} \{y_{i..}\} \\ \{y_{.j.}\} \end{bmatrix} \ . \tag{27}$$

The partitioned matrix on the left-hand side of (27) is non-singular. Analogous to the generalized inverse of a partitioned matrix used in (10), we can then use the regular inverse to solve (27) for $\hat{\boldsymbol{\tau}}$. But, since

(27) differs from (10) only in having $\underset{\sim}{D}\{n_{\cdot j} + \rho\}$ in place of $D\{n_{\cdot j}\}$, we immediately deduce that $\hat{\underset{\sim}{\tau}}$ is similar to $\hat{\underset{\sim}{\delta}}$ of (17) except that the order of $\hat{\underset{\sim}{\tau}}$ is t whereas that of $\hat{\underset{\sim}{\delta}}$ is t - 1; and $n_{\cdot j}$ in $\hat{\underset{\sim}{\delta}}$ is everywhere replaced by $n_{\cdot j} + \rho$. Thus from (17) and (27)

$$\hat{\underset{\sim}{\tau}} = \left[\underset{\sim}{D}\{n_{i\cdot}\} - \left\{\sum_j \frac{n_{ij}n_{i'j}}{n_{\cdot j} + \rho}\right\}\right]^{-1}\left\{y_{i\cdot\cdot} - \sum_j \frac{n_{ij}}{n_{\cdot j} + \rho}\, y_{\cdot j\cdot}\right\} \tag{28}$$

for i,i' = 1,2,⋯,t. This is, of course, the same as (17) except for (30) being of order t rather than t-1 as is (17), and with $n_{\cdot j}$ in (17) being replaced by $n_{\cdot j} + \rho$.

Using first principles to derive the variance of $w_i = y_{i\cdot\cdot} - \sum_j[n_{ij}y_{\cdot j\cdot}/(n_{\cdot j} + \rho)]$ and the covariance of w_i and $w_{i'}$, it will be found that

$$\text{var}(\underset{\sim}{w}) = \left[\underset{\sim}{D}\{n_{i\cdot}\} - \left\{\sum_{ij} \frac{n_{ij}n_{i'j}}{n_{\cdot j} + \rho}\right\}\right]\sigma_e^2 \tag{29}$$

so that from (28)

$$\text{var}(\hat{\underset{\sim}{\tau}}) = \left[\underset{\sim}{D}\{n_{i\cdot}\} - \left\{\sum_j \frac{n_{ij}n_{i'j}}{n_{\cdot j} + \rho}\right\}\right]^{-1}\sigma_e^2 \quad . \tag{30}$$

4.3 Generality

The generality of results (28) and (30) merits emphasis. Those results apply for any block designs with model equation (1) and the βs and es random. No matter what the pattern of treatments used in the blocks is, nor the pattern of numbers of observations available on those treatments, (28) and (30) apply. In all cases n_{ij} is the number of observations on treatment i in block j, with $n_{ij} = 0$ whenever treatment i is not used in block j. Thus (28) and (30) can be used whether there is a pattern to the n_{ij}s, such as with balanced incomplete blocks, or whether there is no pattern.

4.4 Special cases

It is instructive to see that (28) reduces to the well known results of certain special cases.

(i) $\sigma_\beta^2 = 0$. This means the model is $y_{ijp} = \tau_i + e_{ijp}$. Then ρ is infinite and (28) gives $\hat{\tau}_i = \bar{y}_{i..}$, as is to be expected.

(ii) *Balanced data, all* $n_{ij} = n$. The inverse matrix in (28) simplifies to

$$\left(bn\underset{\sim}{I}_t - \frac{bn^2}{tn+\rho}\underset{\sim}{J}_t\right)^{-1} = \frac{1}{bn}\left(\underset{\sim}{I} + n\underset{\sim}{J}/\rho\right) \quad , \tag{31}$$

where $\underset{\sim}{I}_t$ is an identity matrix of order t and $\underset{\sim}{J}_t$ is square of order t with every element unity. Thus (28) yields what one would expect:

$$\hat{\tau}_i = \frac{1}{bn}\left[y_{i..} + \frac{n}{\rho}y_{...} - \frac{\rho n}{tn+\rho}\left(y_{...} + \frac{nt}{\rho}y_{...}\right)\right] = \bar{y}_{i..}$$

with variance, from (30) and (31),

$$v(\hat{\tau}_i) = \frac{1}{bn}(1 + \rho n)\sigma_e^2 = (\sigma_e^2 + n\sigma_\beta^2)/bn = v(\bar{y}_{i..}) \quad ,$$

in the mixed model.

(iii) *The fixed effects model*, when $\sigma_\beta^2 \to \infty$. Then $\rho \to 0$, and (28) reduces back to (15).

(iv) *Balanced incomplete blocks*. These are a special case of block designs that impose particular patterns of values on the n_{ij} values. One broad class of such patterns is considered in Section 5.

4.5 Estimating variance components

With blocks treated as fixed effects, estimation of treatment contrasts, as in (17) and (19), does not involve σ_β^2. In contrast, the estimator (28) with the mixed model demands a value for $\rho = \sigma_e^2/\sigma_\beta^2$. Unless some a priori value of ρ is acceptable ρ must be estimated, and this introduces the need for estimating variance components from unbalanced data, with all the attendant difficulties, especially those relating to which method of estimation should be used. Available contenders include maximum likelihood and restricted maximum likelihood (both of which are available from the statistical computing package BMDP4V, see, for example, Searle and Grimes, 1980) and Henderson's Methods II and III. The maximum likelihood methods require iterative solution of non-linear equations but in this simple case of only one variance component other than σ_e^2, estimators obtained by Henderson's method III are available explicitly from Searle [1971, p. 466, equations (117) and (118)]. The first of these is identical to (20) and the second is

$$\hat{\sigma}_\beta^2 = \frac{\underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u} - \hat{\sigma}_e^2(t-1)}{\mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{Z} - \underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{X}\underset{\sim}{Z}]} = \frac{\underset{\sim}{u}'\underset{\sim}{T}^{-1}\underset{\sim}{u} - \hat{\sigma}_e^2(t-1)}{N - \sum_i\sum_j n_{ij}^2/n_{i.}} \quad . \tag{32}$$

Equality of the denominators in (32) is established by using the same matrix equivalents as are evident between (8) and (9):

$$\begin{aligned}\mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{Z} - \underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^-\underset{\sim}{X}'\underset{\sim}{Z}] &= \mathrm{tr}[D\{n_{.j}\}] - \mathrm{tr}[\underset{\sim}{N}'\underset{\sim}{D}\{1/n_{i.}\}N] \\ &= N - \mathrm{tr}[\underset{\sim}{D}\{1/n_{i.}\}\underset{\sim}{N}\underset{\sim}{N}' = N - \textstyle\sum_i\sum_j n_{ij}^2/n_{i.} \quad .\end{aligned}$$

The estimators (20) and (32) yield $\hat{\rho} = \hat{\sigma}_e^2/\hat{\sigma}_\beta^2$. Denote $\hat{\underset{\sim}{\tau}}$ of (28) by $\hat{\underset{\sim}{\tau}}(\rho)$, a function of ρ, and replace ρ therein by $\hat{\rho}$ to yield $\hat{\tau}(\hat{\rho})$. Then the Kackar and Harville (1981) results indicate that $\hat{\underset{\sim}{\tau}}(\hat{\rho})$ is an unbiased estimator of $\underset{\sim}{\tau}$. Similarly, replacing ρ and σ_e^2 in (30) by $\hat{\rho}$ and $\hat{\sigma}_e^2$ yields an estimate of the dispersion matrix of $\hat{\underset{\sim}{\tau}}(\hat{\rho})$. Kackar and Harville (1984) indicate how this estimate can be improved.

5. BALANCED INCOMPLETE BLOCKS

5.1 Specification

Balanced incomplete blocks are special cases of block designs with data that are unbalanced in a particularly interesting manner. Although in their most general form balanced incomplete block designs do not have to be confined to those wherein there is either only one or no observation on each treatment in each block, this is certainly the most common form of the design appearing in the literature and we too confine attention to that form. Nevertheless, (28) and (30), through their generality vis-á-vis unbalanced data, provide opportunity for considering balanced (and partially balanced) incomplete blocks of far broader form than considered here.

The usual characteristics of a balanced incomplete block design of t treatments in b blocks are that only $k < t$ treatments are used in each block, that each treatment is used in only $r < b$ blocks and that every pair of treatments occurs in precisely λ different blocks. Thus

$$tr = bk \quad \text{and} \quad \lambda(t-1) = r(k-1) \ . \tag{33}$$

Confining attention to the $n_{ij} = 0$ or 1 case gives

$$n_{i.} = r \quad \text{and} \quad n_{.j} = k \tag{34}$$

and also permits dropping the third subscript from y_{ijp} to denote the response from treatment i in block j as y_{ij} . Then with

$$\bar{y}_{i.} = \sum_{j=1}^{b} y_{ij}/r \quad \text{and} \quad \bar{y}_{.j} = \sum_{i=1}^{a} y_{ij}/k$$

and denoting by $\bar{y}_i^*$ the mean of the block means for the blocks containing treatment i, we have

$$\bar{y}_i^* = \frac{1}{r}\sum_{j=1}^{b} n_{ij}\bar{y}_{.j} = \frac{1}{rk}\sum_{j=1}^{b} n_{ij}y_{.j} = \frac{1}{rk}\, y_i^* \ , \tag{35}$$

so defining y_i^* .

4.2 Estimated treatment effects

The preceding notation immediately simplifies (28) to be

$$\underset{\sim}{\hat{\tau}} = \left(r\underset{\sim}{I}_t - \frac{1}{k+\rho} \left\{ \Sigma_j n_{ij} n_{i'j} \right\} \right)^{-1} \left\{ y_{i\cdot} - \frac{1}{k+\rho} \sum_{j=1} n_{ij} y_{\cdot j} \right\}$$

for $i, i' = 1, \cdots, t$. Then, by the nature of the balanced incomplete block design, $\Sigma_j n_{ij}^2 = r$ and $\Sigma_j n_{ij} n_{i'j} = \lambda$ for $i \neq i'$. Hence

$$\underset{\sim}{\hat{\tau}} = \left[r\underset{\sim}{I} - \frac{1}{k+\rho} \left(r\underset{\sim}{I} + \lambda \underset{\sim}{J} - \lambda \underset{\sim}{I} \right) \right]^{-1} \left\{ r\bar{y}_{i\cdot} - \frac{1}{k+\rho} rk\bar{y}_i^* \right\} \quad ; \tag{36}$$

and after some straightforward simplification this reduces to

$$\hat{\tau}_i = \frac{1}{\lambda t + r\rho} \left[r(k+\rho)\bar{y}_{i\cdot} - rk\bar{y}_i^* + \lambda t \bar{y}_{\cdot\cdot} \right] \tag{37}$$

for $i = 1, \cdots, t$. And similarly, from (30) and (36),

$$\text{var}(\underset{\sim}{\hat{\tau}}) = \frac{k+\rho}{\lambda t + r\rho} \left(\underset{\sim}{I} + \frac{\lambda}{r\rho} \underset{\sim}{J} \right) \sigma_e^2 \quad . \tag{38}$$

This gives the variance of $\hat{\tau}_i$, and the covariance of $\hat{\tau}_i$ and $\hat{\tau}_h$ for $i \neq h$ as

$$v(\hat{\tau}_i) = \frac{k+\rho}{\lambda t + r\rho} (1 + \lambda / r\rho) \sigma_e^2 = \frac{\lambda + r\rho}{r(\lambda t + r\rho)} (\sigma_e^2 + k\sigma_\beta^2) \tag{39}$$

and

$$\text{cov}(\hat{\tau}_i, \hat{\tau}_h) = \frac{(k+\rho)\lambda\sigma_e^2}{(\lambda t + r\rho) r\rho} = \frac{\lambda}{r(\lambda t + r\rho)} (\sigma_e^2 + k\sigma_\beta^2) \quad . \tag{40}$$

Also, from (37), a treatment difference is estimated as

$$\hat{\tau}_i - \hat{\tau}_h = \frac{1}{\lambda t + r\rho} \left[r(k+p)(\bar{y}_{i\cdot} - \bar{y}_{h\cdot}) - rk(\bar{y}_i^* - \bar{y}_h^*) \right] \tag{41}$$

with variance, from (39) and (40)

$$v(\hat{\tau}_i - \hat{\tau}_h) = \frac{2\rho}{\lambda t + r\rho} (\sigma^2_e + k\sigma^2_\beta) \quad . \tag{42}$$

5.3 Reconciliation with inter-intra block estimators

The treatment and treatment-difference estimators in (37) and (41) are precisely the same as those given in at least five well-known texts. Nevertheless, since text notations are not all the same and some lack the succinctness of (37) and (41), we indicate how those text results can be reconciled with (37) and (41).

(i) Kempthorne (1952, Section 26.4). The V_j and T_j used on page 533 are, respectively, our $y_{i.}$ and y_i^*; and with $W = 1/\sigma_e^2$ and $W' = 1/(\sigma_e^2 + k\sigma_\beta^2)$ on page 534, the ν on page 535 is

$$\nu = \frac{W \;-\; W'}{Wt(k-1) + W'(t-k)} = \frac{r}{(t-1)(\lambda t + r\rho)} \quad . \tag{43}$$

Then the y_j of his (11) is

$$\frac{V_j}{r} + \frac{\nu}{r}\left[(t-k)V_j - (t-1)T_j\right] = \hat{\tau}_i - \frac{\lambda t}{\lambda t + r\rho}\,\bar{y}_{..}$$

so that the difference between two of Kempthorne's y_j's is the difference between two $\hat{\tau}_i$s, as in (41).

(ii) Federer (1955, pages 416-418) Equation (XIII-14) shows an adjusted treatment mean as $\bar{x}_j' = \bar{x}_j + \mu W_j/r$ for $\bar{x}_j$ being $\bar{y}_{i.}$, μ being Kempthorne's ν and, from (XIII-7),

$$W_j = (t-k)y_{i.} - (t-1)y_i^* + (k-1)y_{..} \quad . \tag{44}$$

μ comes from the first part of (XIII-18) as

$$\mu = \frac{w - w'}{wt(k-1) + w'(t-k)}$$

with $w = 1/E_e$ from (XIII-16) and $E_e = \sigma_e^2$ from the table at the bottom of page 416. But w' is given as

$$w' = \frac{t(r-1)}{k(b-1)E_b - (t-k)E_e}$$

in (XIII-17). It is only when

$$E_b = \sigma_e^2 + \frac{kb-t}{b-1}\sigma_\beta^2$$

is used, also from the table at the bottom of page 416, that w' reduces to $1/(\sigma_e^2 + k\sigma_\beta^2)$, μ reduces to Kempthorne's ν of (43) and then $\bar{x}_j'$ becomes

$$\hat{\tau}_i - \frac{\lambda t}{\lambda t + r\rho}\bar{y}_i \quad .$$

(iii) Cochran and Cox (1957, p.444-445). Using $E_b = \sigma_e^2 + [k(r-1)/r]\sigma_\beta^2$ and $E_e = \sigma_e^2$ their μ in section 4 on page 445 is

$$\mu = \frac{r(E_b - E_e)}{rt(k-1)E_b + k(b-r-t+1)E_e} = \frac{r}{(t-1)(\lambda t + r\rho)} = \nu \text{ of } (43)$$

Then, with $T = y_{i.}$, $B_t = y_i^*$ and $G = y_{..}$, their W on page 444 is the same as (44) and their adjusted treatment mean from page 445 is

$$(T + \mu W)/r = \hat{\tau}_i - \frac{\lambda t}{\lambda t + r\rho}\bar{y}_{..} \quad .$$

(iv) Sheffé (1959, pages 160-178). His notation is so different from that of most other writers that it is necessary to display equivalent notations.

Scheffé	Here
p.161: I,J,r,k	t,b,r,k, respectively
p.162: K_{ij}	n_{ij}
p.164: g_i, h_j, G_i	$y_{i.}$, $y_{.j}$, $y_{i.} - r\bar{y}_i^*$, respectively
(5.2.10): T_i	$y_i^* = kr\bar{y}_i^*$
(5.2.17): $\delta = \frac{rk-r+\lambda}{rk} = \frac{(k-1)I}{k(I-1)}$	$\frac{\lambda t}{rk} = \frac{(k-1)t}{k(t-1)}$
(5.2.18): $\hat{\alpha}_i = G_i/r\delta$	$rk(\bar{y}_{i.} - \bar{y}_i^*)/\lambda t$
(5.2.34): $\alpha_i' = \frac{T_i - rJ^{-1}\Sigma_j h_j}{r - \lambda}$	$\frac{rk(\bar{y}_i^* - \bar{y}_{..})}{r - \lambda}$
(5.2.32b): $\sigma_f^2 = k^2\sigma_B^2 + k\sigma_e^2$	$k(\sigma_e^2 + k\sigma_\beta^2)$
p.172: $\psi = \Sigma c_i\alpha_i$ and $\psi' = \Sigma c_i\alpha_i'$, $\Sigma c_i = 0$.	
(5.2.41): $w = r\delta/\sigma_e^2$	$\lambda t/k\sigma_e^2$
$w' = (r-\lambda)/\sigma_f^2$	$(r-\lambda)/[k(\sigma_e^2 + k\sigma_\beta^2)]$
(5.2.42): $\psi^* = \frac{w\psi + w'\psi'}{w + w'}$	

ψ^* is described by Scheffé as being unbiased and having minimum variance. It therefore corresponds to a $\hat{\tau}_i$. Since ψ is a contrast of α_is it is also a contrast of $(\mu + \alpha_i)$-terms of Scheffé's model. To make ψ^* consistent with $\hat{\tau}_i$ we therefore consider

$$\hat{\psi}_i^* = \frac{w(\hat{\mu}+\hat{\alpha}_i) + w'(\hat{\mu}'+\hat{\alpha}_i')}{w + w'} \quad .$$

Scheffé gives $\hat{\alpha}_i$ on page 165, but nowhere shows a corresponding $\hat{\mu}$, the only intimation being the last line of page 164 in "correction term for the grand mean." From this we infer $\hat{\mu} = \bar{y}_{..}$; and from (5.2.33), $\hat{\mu}' = \bar{y}_{..}$. Then $\hat{\psi}_i^*$ simplifies to $\hat{\tau}_i$.

(v) John (1971, pages 223-240). This author defines

$$\psi^* = \frac{w_1\hat{\psi} + w_2\tilde{\psi}}{w_1 + w_2}$$

on page 236, with, from the first, and the fourth-to-last lines of page 237,

$$w_1 = \frac{\lambda t}{k\sigma_e^2} \ , \quad w_2 = \frac{r-\lambda}{\sigma_f^2} \ , \quad \text{where} \quad \sigma_f^2 = k(\sigma_e^2 + k\sigma_\beta^2) \quad .$$

Page 223 has $T_i = y_{i.}$ and $B_j = y_{.j}$, and page 234 has $T_i' = y_i^*$. Then from page 224

$$\hat{\psi} = (k/\lambda t)\sum c_i Q_i \text{ for } \sum c_i = 0 \quad \text{and} \quad kQ_i = kT_i - T'_i$$

and from page 236 $\tilde{\psi} = \sum c_i T_i'/(r-\lambda)$. With these substitutions

$$\psi^* = \frac{(k+\rho)r\Sigma c_i(\bar{y}_{i.} - \bar{y}_i^*) + r\rho\Sigma c_i\bar{y}_i^*}{\lambda t + r\rho} = \frac{(k+\rho)r\Sigma c_i\bar{y}_{i.} - rk\Sigma c_i\bar{y}_i^*}{\lambda t + r\rho} \ .$$

Thus if for some particular i and $h \neq i$ we take $c_i = 1$ and $c_h = -1$ and all other cs zero, then

$$\psi^* = \frac{1}{\lambda t + r\rho}\left[r(k+\rho)(\bar{y}_{i.} - \bar{y}_{h.}) - rk(\bar{y}_i^* - \bar{y}_h^*)\right]$$

which is $\hat{\tau}_i - \hat{\tau}_h$ of (41).

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