

COHOMOLOGY OF CONTACT TORIC  
MANIFOLDS AND HARD LEFSCHETZ PROPERTY  
OF HAMILTONIAN GKM MANIFOLDS

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This thesis consists of two parts. Each part solves a topological problem in equivariant symplectic geometry using combinatorial methods.

The classification problem of symplectic toric manifolds was solved by Delzant [Del88] and their cohomology rings were later computed in various ways by Davis-Januskiewicz [DJ91] and Tolman-Weitsman [TW03]. Contact toric manifolds are the analogue of symplectic toric manifolds. The classification problem was solved by Lerman [Ler03a]. In the first part of the thesis, we compute the cohomology ring of contact toric manifolds utilizing techniques in [DJ91].

The equivariant cohomology of Hamiltonian GKM manifolds can be computed as the graph cohomology of their GKM graphs. The combinatorial properties of GKM graphs were studied by Guillemin and Zara [GZ99], [GZ01]. In the second part of this thesis, we study the graph cohomology for a more general class of graphs and prove many properties about it. The most interesting one of these is a weak version of the Hard Lefschetz Property (HLP). It was known that HLP always holds for Kähler manifolds, but generally may fail for symplectic manifolds. Hamiltonian GKM manifolds are symplectic but generally not Kähler. Our result provides a new class of manifolds for which a weak version of HLP holds.

## BIOGRAPHICAL SKETCH

Shisen Luo was born on September 3rd, 1986 in a beautiful small village in southern China. He showed his love for mathematics from an early age. Having mastered multiplying single digit numbers at four, he thought math was all about multiplying bigger and bigger numbers. He dreamt one day he would be able to multiply two six-digit numbers and he thought that would be really cool. It quickly turned out math had a lot more than that and he was really fascinated.

He attended University of Science and Technology of China in 2004 and earned a Bachelor of Science degree in mathematics in 2008. After that, he went to Cornell University to pursue a PhD degree in mathematics. In 2013, he completed his thesis under the supervision of his advisor, Tara Holm.

This work is dedicated to my parents and my grandma, for their truthful love  
and unceasing support

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CHAPTER 1  
BACKGROUND

## 1.1 Symplectic manifolds and Hamiltonian actions

A *symplectic manifold* is a pair  $(M, \omega)$  where  $M$  is a smooth manifold and  $\omega$  is a two-form that is

- *closed*, i.e.  $d\omega = 0$ , and
- *non-degenerate*, i.e. for all  $p \in M$  and any non-zero vector  $X \in T_pM$ , there exists  $Y \in T_pM$ , such that  $\omega_p(X, Y) \neq 0$ .

The two-form  $\omega$  is called a *symplectic form* on  $M$ . We say  $M$  admits a symplectic structure if there exists a symplectic form on it.

**Remark 1.1.1.** One equivalent description of the non-degeneracy condition on  $\omega$  is that  $M$  is of even dimension, say  $2n$ , and the top power of  $\omega$ , namely  $\omega^n$ , is a volume form on  $M$ . As a consequence,  $M$  has to be orientable. Moreover, when  $M$  is compact,  $\omega^k$  is a non-trivial element in  $H^{2k}(M)$  for  $0 \leq k \leq n$ . Because of this fact, the only sphere that admits symplectic structure is  $S^2$ .

For each function  $H \in C^\infty(M)$ , we may associate to it a vector field  $X_H$  on  $M$  by

$$dH = \iota_{X_H}\omega.$$

Note that  $X_H$  is completely determined by the above formula because of the non-degeneracy of  $\omega$ . The vector field  $X_H$  is called the *Hamiltonian vector field* of  $H$ .

Conversely, given a smooth vector field  $X$  on  $M$ , if there exists a function  $H$  such that  $X = X_H$ , then we call  $X$  a *Hamiltonian vector field* and  $H$  is called the *Hamiltonian function* of  $X$ . Note that the Hamiltonian function does not necessarily exist for any smooth vector field on  $M$ . If one does exist, it is only unique up to the addition of a constant.

Let  $G$  be a compact connected Lie group acting smoothly on  $M$ . We call the action *symplectic* if it preserves the symplectic form  $\omega$ , that is

$$g^*\omega = \omega$$

for all  $g \in G$ .

Denote by the Lie algebra of  $G$  by  $\mathfrak{g}$  and its dual by  $\mathfrak{g}^*$ . The action is called a *Hamiltonian action* if there exists a map

$$\phi : M \rightarrow \mathfrak{g}^*$$

such that:

(a) For each  $X \in \mathfrak{g}$ , let

- $\phi^X : M \rightarrow \mathbb{R}$ ,  $\phi^X := \langle \phi(p), X \rangle$ , be the component of  $\phi$  along  $X$ ,
- $X^\#$  be the vector field on  $M$  generated by the one-parameter subgroup  $\{\exp(tX) \mid t \in \mathbb{R}\} \subseteq G$ .

Then

$$d\phi^X = \iota_{X\#}\omega,$$

i.e.,  $\phi^X$  is a Hamiltonian function for the vector field  $X^\#$ .

(b)  $\phi$  is *equivariant* with respect to the given  $G$ -action on  $M$  and the coadjoint action of  $G$  on  $\mathfrak{g}^*$ :

$$\phi(g.p) = \text{Ad}_g^*(\phi(p)) \quad \text{for all } p \in M, g \in G.$$

The vector  $(M, \omega, G, \phi)$  is then called a *Hamiltonian  $G$ -space* and  $\phi$  is called a *moment map*. Note that a Hamiltonian action is necessarily symplectic.

Throughout this thesis, we will be concerned with torus actions. In this case, the coadjoint action  $G \curvearrowright \mathfrak{g}^*$  is trivial. So the above condition (b) is simplified to “ $\phi$  is  $G$ -invariant”. Moreover, the dual of the Lie algebra  $\mathfrak{g}^*$  of the torus can be identified with the Euclidean space  $\mathbb{R}^m$ , where  $m$  is the dimension of the torus.

The action  $G \curvearrowright M$  is called *effective* if each group element  $g \neq e$  moves at least one  $p \in M$ , that is,

$$\bigcap_{p \in M} G_p = \{e\},$$

where  $G_p = \{g \in G \mid g.p = p\}$  is the *stabilizer* of  $p$ .

The following proposition is a well known fact.

**Proposition 1.1.2.** *If an  $m$ -dimensional compact torus  $\mathbb{T}^m$  acts effectively on a symplectic manifold  $(M, \omega)$  of dimension  $2n$  in a Hamiltonian fashion, then  $m \leq n$ .*

When  $m = n$  in the above proposition, we call  $M$  a *toric manifold* and the action *toric*.

The moment map of a torus action has the following important property, usually referred to as the *Convexity Theorem*.

**Theorem 1.1.3** (Atiyah [Ati82], Guillemin-Sternberg [GS82]). *Let  $(M, \omega, \mathbb{T}^m, \phi)$  be a compact connected Hamiltonian manifold. Then*

- (a) *the non-empty level sets of  $\phi$  are connected;*
- (b) *the image of  $\phi$  is the convex hull of the image of the fixed points of the action.*

The image  $\phi(M)$  of the moment map is hence called the *moment polytope*.

Generally, the moment polytope does not determine the Hamiltonian action, but it does when the action is toric.

**Definition 1.1.4.** A *Delzant polytope*  $\Delta$  in  $\mathbb{R}^n$  is a convex polytope satisfying

- it is *simple*, i.e., there are  $n$  edges meeting at each vertex;
- it is *rational*, i.e., the edges meeting at the vertex  $p$  are rational in the sense that each edge is of the form  $p + tu_i, t \geq 0$ , where  $u_i \in \mathbb{Z}^n$ ;
- it is *smooth*, i.e., for each vertex, the corresponding  $u_1, \dots, u_n$  can be chosen to be a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^n$ .

The smoothness condition above is equivalent to that the  $n$  primitive inward-pointing normal vectors to the  $n$  facets containing  $p$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^n$ . The following theorem is usually referred to as *Delzant Theorem*.

**Theorem 1.1.5** (Delzant [Del88]). *Symplectic toric manifolds are classified by Delzant polytopes. More specifically, there is the following one-to-one correspondence*

$$\begin{aligned} \{\text{isomorphism classes of toric manifolds}\} &\xrightarrow{1-1} \{\text{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \phi) &\mapsto \phi(M). \end{aligned}$$

The inverse map involves a construction called *symplectic reduction*. We will give explicit description of the inverse map in Chapter 2.

Theorem 1.1.5 is an important bridge that connects symplectic geometry with combinatorics. Another one is the GKM theorem which we will briefly discuss in Section 1.4.

A Delzant polytope is determined by its vertex set, which is the image of the fixed point set. So the Delzant Theorem says we may recover all information about the symplectic toric manifold  $(M, \omega, \mathbb{T}^n, \phi)$  from the fixed point set data  $(M^{\mathbb{T}^n}, \phi)$ . This is a recurring theme in symplectic geometry.

## 1.2 Equivariant cohomology

Let  $G$  be a finite dimensional Lie group. Denote by  $EG$  a contractible space upon which  $G$  acts freely. Such spaces always exist. The quotient space  $EG/G$  is the

classifying space of  $G$  and is denoted by  $BG$ .

**Example 1.2.1.** Identify  $S^{2k+1}$  with  $\{(z_1, z_2, \dots, z_{k+1}) \in \mathbb{C}^{k+1} \mid \sum_{i=1}^{k+1} |z_i|^2 = 1\}$ . The unit circle  $S^1$  acts on  $S^{2k+1}$  by complex multiplication on each component. We have a natural  $S^1$ -equivariant inclusion  $S^{2k-1} \hookrightarrow S^{2k+1}$  induced by

$$\begin{aligned} \mathbb{C}^k &\longrightarrow \mathbb{C}^{k+1} \\ (z_1, \dots, z_k) &\mapsto (z_1, \dots, z_k, 0) \end{aligned}$$

Define  $S^\infty$  to be the direct limit of  $S^1 \hookrightarrow S^3 \hookrightarrow S^5 \hookrightarrow \dots$ . We may take  $S^\infty$  as  $ES^1$ . The quotient  $BS^1 = ES^1/S^1$  is  $\mathbb{C}P^\infty$ .

When  $G = \mathbb{T}^m = (S^1)^m$ , we may take  $EG = (S^\infty)^m$  with  $\mathbb{T}^m$  acting diagonally and then  $BG = (\mathbb{C}P^\infty)^m$ , the  $m$ -fold Cartesian product of infinite dimensional complex projective spaces.

Suppose  $M$  is a  $G$ -space and let  $G$  act on  $EG \times M$  diagonally. The quotient space, denoted by  $EG \times_G M$ , is called the *Borel construction* of  $G \curvearrowright M$ . The *equivariant cohomology* of  $G \curvearrowright M$  is defined to be the singular cohomology of  $EG \times_G M$ , denoted by  $H_G^*(M)$ . That is,

$$H_G^*(M; R) := H^*(EG \times_G M; R),$$

where  $R$  is a ring, the default is  $\mathbb{C}$  in this thesis.

The Borel construction is often infinite dimensional, so the equivariant cohomology groups may be non-zero in infinitely many degrees. A simple and important example is when  $G = \mathbb{T}^m$  and  $M$  is just a single point with a trivial  $G$  action.

Then

$$H_{\mathbb{T}^m}^*(pt; R) = H^*(B\mathbb{T}^m; R) = H^*((\mathbb{C}P^\infty)^m; R) \simeq R[x_1, x_2, \dots, x_m],$$

where  $x_i \in H^2(B\mathbb{T}^m; R)$  for  $1 \leq i \leq m$ .

For every  $\mathbb{T}^m$  space  $M$ , there is a projection map  $\pi : E\mathbb{T}^m \times_{\mathbb{T}^m} M \rightarrow B\mathbb{T}^m$  making  $E\mathbb{T}^m \times_{\mathbb{T}^m} M$  a fiber bundle over  $B\mathbb{T}^m$  with fiber  $M$ . The map  $\pi$  induces a map in equivariant cohomology

$$\pi^* : H_{\mathbb{T}^m}^*(pt; R) \rightarrow H_{\mathbb{T}^m}^*(M; R),$$

which gives  $H_{\mathbb{T}^m}^*(M; R)$  an  $H_{\mathbb{T}^m}^*(pt; R) \simeq R[x_1, x_2, \dots, x_m]$ -module structure.

Equivariant cohomology of Hamiltonian manifolds has many nice properties. Among them are the following two beautiful theorems. The first one is due to Kirwan [Kir84] and commonly referred to as *Kirwan's injectivity theorem*.

**Theorem 1.2.2** (Kirwan). *Let a torus  $\mathbb{T}$  act on a compact symplectic manifold  $M$  in a Hamiltonian fashion. Denote the fixed point set by  $M^{\mathbb{T}}$  and let  $r : M^{\mathbb{T}} \hookrightarrow M$  be the natural inclusion. Then the induced map in equivariant cohomology*

$$r^* : H_{\mathbb{T}}^*(M; \mathbb{C}) \rightarrow H_{\mathbb{T}}^*(M^{\mathbb{T}}; \mathbb{C})$$

*is an injection.*

The second one is due to Kirwan [Kir84] and Ginzburg [Gin87] and is usually called *equivariant formality* of Hamiltonian manifolds.



**Theorem 1.2.3.** *Let a torus  $\mathbb{T}$  act on a compact symplectic manifold  $M$  in a Hamiltonian fashion. We have the following isomorphism*

$$H_{\mathbb{T}}^*(M; \mathbb{C}) \simeq H^*(B\mathbb{T}; \mathbb{C}) \otimes_{\mathbb{C}} H^*(M; \mathbb{C})$$

*as modules over  $H^*(B\mathbb{T}; \mathbb{C})$ .*

### 1.3 Contact toric manifolds

Contact manifolds are the odd-dimensional relatives of symplectic manifolds. This section mostly follows [Ler03a].

Assume  $M$  is a manifold of dimension  $2n - 1$ . Let  $\alpha$  be a 1-form on  $M$  that is nowhere zero and let  $\xi \subset TM$  be the field of hyperplanes defined by  $\xi = \ker \alpha$ . If  $\alpha \wedge (d\alpha)^{n-1}$  is a volume form on  $M$ , we call  $\xi$  is a (co-oriented) contact structure on  $M$  and  $\alpha$  is called a contact form on  $M$ . We call  $(M, \xi, \alpha)$  a (co-oriented) contact manifold.

The condition that  $\alpha \wedge (d\alpha)^{n-1}$  being a volume form on  $M$  is equivalent to  $d\alpha$  being nondegenerate on  $\xi$ . So  $\alpha$  gives an orientation on  $\xi$ , that is, the orientation which makes  $(d\alpha)^{n-1}$  a positive volume form.

For any positive function  $f$  on  $M$ , we have  $\ker(f\alpha) = \ker \alpha = \xi$  and we view  $(M, \xi, \alpha)$  and  $(M, \xi, f\alpha)$  as the same contact manifolds.

In this thesis, whenever we say “contact manifold”, we always mean compact

connected co-oriented contact manifold.

A *contactomorphism* is diffeomorphism  $g$  of a contact manifold  $(M, \xi, \alpha)$  which preserves the co-oriented contact structure, i.e.,

$$g_*\xi = \xi$$

and  $g_*$  preserves orientation of  $\xi$ , or equivalently

$$g^*\alpha = f\alpha$$

for some positive function  $f$  on  $M$ .

A group action  $G \curvearrowright (M, \xi, \alpha)$  is called a *contact action* if each element  $G$  induces a contactomorphism. We may assume the contact form  $\alpha$  is  $G$ -invariant.

Assume  $G \curvearrowright (M, \xi, \alpha)$  is a contact action preserving  $\alpha$ . The cotangent bundle  $T^*M$  has a natural symplectic structure. The  $G$  action on  $M$  is lifted to an action on  $T^*M$  and this action is Hamiltonian. Define a subbundle  $\xi_+^\circ \subset T^*M$  by

$$\xi_+^\circ = \{(p, b\alpha_p) \in T^*M \mid p \in M, b > 0\},$$

where  $\alpha_p$  is the value of  $\alpha$  at  $p$ .

The canonical symplectic form is again symplectic when restricting to  $\xi_+^\circ$ . Moreover, because  $G$  preserves  $\alpha$ , the  $G$  action on  $T^*M$  restricts to an action on  $\xi_+^\circ$ . This makes  $\xi_+^\circ$  a (non-compact) Hamiltonian  $G$ -manifold. It is diffeomorphic to  $M \times \mathbb{R}$ . We call  $\xi_+^\circ$  the *symplectization* of  $M$ .

The moment map  $\Phi : T^*M \rightarrow \mathfrak{g}^*$  for the  $G \curvearrowright T^*M$  is defined by

$$\langle \Phi(p, \eta), X \rangle = \langle \eta, X^\#(p) \rangle,$$

where  $X^\#$  be the vector field on  $M$  generated by the one-parameter subgroup  $\{\exp(tX) \mid t \in \mathbb{R}\} \subseteq G$ .

If we restrict  $\Phi$  to  $\xi_+^\circ$ , we get  $\Psi = \Phi|_{\xi_+^\circ} : \xi_+^\circ \rightarrow \mathfrak{g}^*$  and this is called the *contact moment map* for  $G \curvearrowright (M, \xi, \alpha)$ . We let

$$\Psi_\alpha = \Psi \circ \alpha,$$

and call it the  $\alpha$ -*moment map*. In other words,  $\Psi_\alpha : M \rightarrow \mathfrak{g}^*$  is defined by

$$\langle \Psi_\alpha(p), X \rangle = \alpha_p(X^\#(p))$$

for all  $p \in M$  and  $X$  in the Lie algebra  $\mathfrak{g}$  of  $G$ .

Note that contact moment map is determined by the contact structure, while  $\alpha$ -moment map also depends on the contact form  $\alpha$ .

Analogous to the symplectic case, a contact torus action  $\mathbb{T}^m \curvearrowright (M, \xi, \alpha)$  is called toric if it is effective and  $m = n = \frac{\dim M + 1}{2}$ . We call  $M$  a contact toric manifold. The full classification of contact toric manifold is given by Lerman [Ler03a]. The most interesting case is when  $\dim M \geq 5$  and the torus action is not free. In this case, they are fully classified by the *good cones* which can be viewed as an analogue to the Delzant polytopes. Precise definition of good cones will be given in Chapter 2. The main theorem of Chapter 2, Theorem 2.5.43, computes the

cohomology ring of a contact toric manifold in terms of the combinatorial data of the associated good cone.

## 1.4 Hamiltonian GKM manifolds

Unless stated otherwise, the manifolds in this section are always assumed to be compact and connected.

Every Hamiltonian manifold  $(M, \omega, \mathbb{T}, \phi)$  has a  $\mathbb{T}$ -invariant almost complex structure and the space of such structures is contractible, so the weights of the isotropy representation of  $\mathbb{T}$  on  $T_p M$  at each fixed point  $p$  is well-defined.

The Hamiltonian manifold  $(M, \omega, \mathbb{T}, \phi)$  is called *GKM* if

- $M^{\mathbb{T}}$  consists of isolated fixed points  $\{p_1, p_2, \dots, p_m\}$ ;
- For every  $p \in M^{\mathbb{T}}$ , the weights

$$\alpha_{i,p} \in \mathfrak{t}^*, \quad i = 1, \dots, d = \frac{\dim M}{2},$$

of the isotropy representation of  $\mathbb{T}$  on  $T_p M$  are pairwise linearly independent.

Denote by  $M_1$  the set of points in  $M$  whose isotropy group is  $\mathbb{T}$  or a codimension-1 (not necessarily connected) torus. Then the GKM condition im-

plies that  $M_1 \setminus M^\mathbb{T}$  can be written as the disjoint union of connected components

$$\bigsqcup_i X_i,$$

where each connected component  $X_i$  is homeomorphic to  $S^2 \setminus \{\text{north pole, south pole}\}$  and the closure of  $X_i$  is homeomorphic to  $S^2$ .

Given  $(M, \omega, \mathbb{T}^n, \phi)$  a Hamiltonian GKM manifold, define its GKM graph  $\Gamma = (V, E)$  by

- $V = M^\mathbb{T}$ , and
- Any two vertices  $p_1$  and  $p_2$  are incident if and only if there exists a codimension-1 subtorus  $K$  of  $\mathbb{T}$  such that  $p_1$  and  $p_2$  are in the same connected component of  $M^K$ . Such a connected component is one of the  $S^2$ 's described above.

Note that the moment map  $\phi : M \rightarrow \mathfrak{t}^* \simeq \mathbb{R}^n$  induced a map from  $V$  to  $\mathbb{R}^n$  by restriction. This map is also denoted by  $\phi$ . When we talk about a GKM graph, we always think of it as a triple  $(V, E, \phi : V \rightarrow \mathbb{R}^n)$ . For later use, we define a map

$$\alpha : E \rightarrow \mathbb{C}[x_1, \dots, x_n]_1,$$

where  $\mathbb{C}[x_1, \dots, x_n]_1$  stands for the set of non-zero homogeneous linear polynomials in  $x_1, \dots, x_n$ , by  $\alpha(e_{ij}) = (\phi(v_i) - \phi(v_j)) \cdot (x_1, x_2, \dots, x_n)$ . In other words, if  $\phi(v_i) = (p_1, p_2, \dots, p_n)$  and  $\phi(v_j) = (q_1, q_2, \dots, q_n)$ , then

$$\alpha(e_{ij}) = (p_1 - q_1)x_1 + (p_2 - q_2)x_2 + \dots + (p_n - q_n)x_n.$$

The map  $\alpha$  is only well-defined up to sign, but this is sufficient for our purpose. The second one of the GKM conditions can now be translated into “ $\alpha(e_{ij})$  and  $\alpha(e_{ik})$  are linearly independent for  $j \neq k$ ”.

The equivariant cohomology of Hamiltonian GKM manifolds can be easily computed from the combinatorial data of the GKM graph by the following theorem of Goresky, Kottwitz and MacPherson [GKM97], which we will refer to as the *GKM Theorem* from now on.

**Theorem 1.4.1** ([GKM97]). *Assume  $(M, \omega, \mathbb{T}^n, \phi)$  is a Hamiltonian GKM manifold with GKM graph  $\Gamma = (V, E)$  and  $\alpha : E \rightarrow \mathbb{C}[x_1, \dots, x_n]_1$  is induced by  $\phi$  as defined above. Then we have the following isomorphism*

$$H_{\mathbb{T}^n}^*(M; \mathbb{C}) \simeq \left\{ (f_1, f_2, \dots, f_{|V|}) \in \bigoplus_{i=1}^{|V|} \mathbb{C}[x_1, \dots, x_n] \mid \alpha(e_{ij}) \mid (f_i - f_j) \text{ for all } e_{ij} \in E \right\}$$

as  $H^*(B\mathbb{T}^n; \mathbb{C}) \simeq \mathbb{C}[x_1, \dots, x_n]$ -algebras.

The right-hand side of the isomorphism can be defined for any  $(\Gamma, \alpha)$  pair. We call it the *(equivariant) graph cohomology* of  $(\Gamma, \alpha)$  and denote it by  $H_{\mathbb{T}^n}^*(\Gamma, \alpha)$ . The GKM Theorem may be restated in this language as that the equivariant cohomology of the manifold is isomorphic to the *(equivariant) graph cohomology* of its GKM graph.

The original version of GKM theorem applies in more general circumstances. But this version here is sufficient for us and is the most convenient to use in the context of this thesis.

Combining the GKM Theorem with the equivariant formality of Hamiltonian manifolds, Theorem 1.2.3, we can compute the ordinary cohomology of Hamiltonian GKM manifolds.

**Corollary 1.4.2.** *With the same setup as Theorem 1.4.1, let  $I \subseteq H_{\mathbb{T}^n}^*(\Gamma, \alpha)$  be the ideal generated by*

$$\{(x_i, x_i, \dots, x_i) \mid 1 \leq i \leq n\}.$$

*Then  $H^*(M; \mathbb{C}) \simeq H_{\mathbb{T}^n}^*(\Gamma, \alpha)/I$  as vector spaces, and the isomorphism preserves gradings.*

In Chapter 3, we will use the above corollary and some combinatorial techniques developed there to prove the following result, which will be restated as Theorem 3.5.1.

**Theorem 1.4.3.** *If  $M$  is a  $2d$ -dimensional compact connected Hamiltonian GKM manifold whose moment map is in general position (we will make this precise in Chapter 3), we have  $\beta_2(M) = \beta_{2d-2}(M) \leq \frac{m-2}{d-1}$ , where  $\beta_i(M)$  is the  $i$ -th Betti number of  $M$  and  $m$  is the sum of all Betti numbers.*

In low dimensions, this theorem implies a weak version of the Hard Lefschetz Property.

**Corollary 1.4.4.** *If  $M$  is an 8 or 10-dimensional compact connected Hamiltonian GKM manifold whose moment map is in general position, then  $M$  has nondecreasing even Betti numbers up to half dimension:  $\beta_0(M) \leq \beta_2(M) \leq \beta_4(M)$ .*

CHAPTER 2  
COHOMOLOGY OF CONTACT TORIC MANIFOLDS

## 2.1 Introduction

A *contact toric manifold* of dimension  $(2n - 1)$  is a compact connected  $(2n - 1)$ -dimensional contact manifold equipped with an effective Hamiltonian  $\mathbb{T}^n$ -action. Motivated by the work of Banyaga and Molino [BM91],[BM96],[Ban99], and of Boyer and Galicki [BG00], Lerman [Ler03a] gave a classification of contact toric manifolds. According to Lerman's theorem, when  $n \geq 3$  and the torus action is not free, contact toric manifolds are classified by their *moment cones*, which by definition are the union of *{origin}* and the moment map image of their symplectizations. These moment cones are all "good cones".

**Definition 2.1.1.** A *good cone* in  $\mathbb{R}^n$  is a rational polyhedral cone given by

$$(2.1.2) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\},$$

where  $F_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle = 0\}$  is a facet of  $C$  and  $v_i \in \mathbb{Z}^n$  is the inward-pointing primitive normal vector to  $F_i$ . In addition, this cone must satisfy:

- (i) For  $0 < l < n$ , each codimension  $l$  face  $F$  of  $C$  is contained in exactly  $l$  facets:

$$F = F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_l};$$

and

- (ii) The  $\mathbb{Z}$ -module generated by  $v_{i_1}, \dots, v_{i_l}$  is a direct summand of  $\mathbb{Z}^n$  of rank  $l$ .



**Definition 2.1.3.** A contact toric manifold  $M$  is called a *good contact toric manifold* if  $\dim M > 3$  and the moment cone of  $M$  is a strictly convex good cone.

**Remark 2.1.4.** When the moment cone is not strictly convex, the construction of a contact toric manifold with the given moment cone can be found in the proof of Proposition 4.7 in [LS02]. It is topologically  $\mathbb{T}^k \times S^{k+2l-1}$ .

We introduced the equivariant cohomology of a point with a trivial torus action in Section 1.2. It is particularly important for this chapter of the thesis. Let  $\mathbb{T}^m$  be connected  $m$ -dimensional torus acting trivially on a point. The Borel construction of the point  $E\mathbb{T}^m \times_{\mathbb{T}^m} pt = E\mathbb{T}^m / \mathbb{T}^m$  is the classifying space of  $\mathbb{T}^m$ , denoted by  $B\mathbb{T}^m$ . If we use  $(S^\infty)^m$  as the model for  $E\mathbb{T}^m$ , then  $B\mathbb{T}^m$  is the product of  $m$  copies of  $\mathbb{C}P^\infty$ . The equivariant cohomology of a point is thus a polynomial ring with  $m$  variables, i.e,

$$(2.1.5) \quad H_{\mathbb{T}^m}^*(pt; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m],$$

where  $x_i \in H_{\mathbb{T}^m}^2(pt; \mathbb{Z})$ .

The variable  $x_i$  is the first Chern class of the fiber bundle

$$(2.1.6) \quad E\mathbb{T}^m \times_{\mathbb{T}^m} \mathbb{C} \rightarrow E\mathbb{T}^m \times_{\mathbb{T}^m} pt,$$

where in the total space,  $\mathbb{T}^m$  acts on  $\mathbb{C}$  with weight  $-(0, 0, \dots, 1, \dots, 0)$ , where the 1 is in the  $i^{th}$  position.

The generators for the various cohomology rings in this chapter will always be the images of these  $x_i$ 's under certain maps.

It is not hard to see that equivariant cohomology is an equivariant homeomorphism invariant. In this chapter, we are concerned with equivariant and ordinary cohomology rings. Since both of these are equivariant homeomorphism invariants, two  $\mathbb{T}^n$ -spaces that are equivariantly homeomorphic are considered as identical spaces. Because of this, we may exploit an idea of Davis and Januskiewicz in [DJ91], where they define a topological counterpart for toric manifolds and compute the corresponding cohomology rings.

In Section 2.2, we imitate the construction given in [DJ91] to define several topological spaces with torus actions, using combinatorial methods. These spaces will be shown in later sections to be equivariantly homeomorphic to toric symplectic cones, good contact toric manifolds and symplectic toric manifolds respectively.

In Section 2.3, exploiting the techniques in [DJ91] with some modification, we compute the equivariant cohomology of a good contact toric manifold. The main theorem is Theorem 2.3.33.

In Section 2.4, we collect some facts about symplectic toric manifolds and set up some notation for Section 2.5.

In Section 2.5, we compute the ordinary cohomology of a good contact toric manifold. The work in this section is not carried out in full generality. We must first assume  $C_0$ , the cone minus the origin, is contained in the upper half space

$$(2.1.7) \quad U\mathbb{R}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}.$$

We will show that this assumption will not cause any loss of generality.

The intersection of  $C$  with the hyperplane

$$(2.1.8) \quad H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\}$$

is a simple rational convex polytope. We denote it by  $P$ . Assume  $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$  is the primitive inward-pointing normal vector to  $F_i$ , the  $i^{\text{th}}$  facet of the cone. We need to impose the following criterion on  $C$ .

**Smoothness Criterion:** The polytope  $P$  is a Delzant polytope in  $H$ , and the vector  $(v_{i1}, v_{i2}, \dots, v_{i,n-1})$  is primitive in  $\mathbb{Z}^{n-1}$  for all  $i$ .

This hypothesis is where the generality is lost. We call this a *smoothness criterion*, since a Delzant polytope is by definition a smooth simple rational convex polytope. We make this hypothesis so we do not need to deal with orbi-bundles over orbifolds.

Under this smoothness criterion, we show that  $M$ , a good contact toric manifold, is a principal  $S^1$ -bundle over a symplectic toric manifold  $N$ . Then using the Gysin sequence of the  $S^1$ -bundle, we compute the ordinary cohomology group of  $M$  and also show how to take the product of any two even degree cohomology classes, and the product of one even degree cohomology class and one odd degree cohomology class. This is Theorem 2.5.43. Then by relating the Euler class of the  $S^1$ -bundle to the symplectic form on  $N$  and using the Hard Lefschetz Theorem, we show that the odd degree cohomology of  $M$  vanishes in degree lower than half, and hence show the product of any two odd degree cohomology classes of

$M$  is zero for dimension reasons. This is Theorem 2.5.48.

Finally in the Appendix, we give several equivalent descriptions of the generators of the cohomology ring of a symplectic toric manifold.

Some of the geometric computations we describe in this chapter have been computed in a more algebraic fashion by other authors. The virtue of our computation is that it is more explicit and geometric. Moreover, the geometry allows us to use the Hard Lefschetz Theorem on symplectic toric manifold to deduce Theorem 5.15. The consequent vanishing of certain Betti numbers is much more obscure in the existing algebraic description of the cohomology ring. The integral equivariant cohomology ring of a general smooth toric variety, which includes the symplectization of good contact toric manifolds, was identified with the Stanley-Reisner ring earlier by Franz [Fra06, Sec. 3] with a different proof from our proof of Theorem 2.3.33. In Theorem 1.2 of the same paper of Franz, he got an expression of the integral ordinary cohomology of a smooth toric variety in terms of Tor modules. The ordinary cohomology ring of the quotient of a moment-angle complex, which includes the case we consider in Section 2.5, was expressed also using Tor modules in [BP02, Thm.7.37]. To relate the description in Theorem 2.5.43 in this chapter to the results of [Fra06] and [BP02], an algebraic argument is presented in [SLM12].

**A remark on notations:** If a group  $G$  acts on two spaces  $X$  and  $Y$ , we use  $X \times_G Y$  and  $(X \times Y)/G$  interchangeably to denote the quotient space of  $X \times Y$  under the diagonal  $G$  action. And we will use  $[x, y]$  to denote the element of

$X \times_G Y$  that is the equivalence class of  $(x, y) \in X \times Y$ , so  $[gx, gy] = [x, y]$ , for  $g \in G$ .

## 2.2 Basic definitions, constructions and examples

A  $2n$  dimensional *symplectic toric manifold* is a compact connected symplectic manifold equipped with an effective Hamiltonian action of an  $n$ -torus  $\mathbb{T}^n$ . Delzant showed in [Del88] that these geometric objects are classified by their moment image, which is a simple rational smooth polytope, called a *Delzant polytope* in the symplectic literature. More information about symplectic toric manifolds may be found in Chapter 28 of [dS01].

The analogous result for contact toric manifolds was given by Lerman in [Ler03a], where he showed that a large class of contact toric manifolds are classified by *good cones*, defined in Definition 2.1.1.

The full classification theorem of contact toric manifolds is Theorem 2.18 in [Ler03a]. The terminology *good contact toric manifold*, which is defined in Definition 2.1.3, used in this chapter belongs to the case (4) in Lerman's classification theorem. It is actually a proper subcase, since here we also require the moment cone to be strictly convex. All other cases are topologically simple.

A *toric symplectic cone* is the symplectization of a contact toric manifold. Toric symplectic cones and contact toric manifolds are in one-to-one correspondence. There is more information about symplectic cones in [Ler03b] and [AM10]. We

will review the method to obtain toric symplectic cones and contact toric manifolds from strictly convex good cones in Section 2.3.

The Delzant polytope, which is the moment image of symplectic toric manifold, is in fact the orbit space of the  $\mathbb{T}^n$  action, and the moment map is just the point-to-orbit map. Using the ideas of [DJ91], with simple combinatorial methods we can construct manifolds that are  $\mathbb{T}^n$ -equivariantly homeomorphic to symplectic toric manifolds. The constructions we describe below are a variation and generalization of those in [DJ91].

We let  $P$  (or  $C_0$ ) be a simple convex polytope (or a strictly convex good cone minus the origin) in  $\mathbb{R}^n$ . The set of facets of  $P$  (or  $C_0$ ) is denoted  $\mathcal{F}$ , and we write  $F_i$  for the  $i^{\text{th}}$  facet. A *characteristic map* is a map

$$\lambda : \mathcal{F} \rightarrow \mathbb{Z}^l$$

where  $\mathbb{Z}^l$  is the integral lattice in  $\mathbb{R}^l$ . Here  $l$  and  $n$  are not necessarily equal. Denote  $\lambda(F_i)$  by  $\lambda_i$ .

Every finite subset  $U$  of  $\mathbb{Z}^l$  determines a subgroup of  $\mathbb{T}^l$  generated by

$$\{(e^{iu_1\theta}, e^{iu_2\theta}, \dots, e^{iu_l\theta}) \in \mathbb{T}^l : \theta \in \mathbb{R}, (u_1, \dots, u_l) \in U\}$$

It is in fact a closed subgroup. For every point  $p$  in  $P$  (or  $C_0$ ), denote by  $S_p^\lambda$  the subgroup of  $\mathbb{T}^l$  determined by vectors

$$\{\lambda_i : p \in F_i\}.$$

Define an equivalence relation  $\Delta$  on

$T^l \times P$  by:

$$(2.2.1) \quad (g, p)\Delta(h, q) \Leftrightarrow p = q, \text{ and } g^{-1}h \in S_p^\lambda.$$

We say  $\Delta$  is the equivalence relation determined by  $\lambda$ . Let  $\Delta_p$  denote  $S_p^\lambda$ , and  $P^\lambda$  denote the quotient space  $(\mathbb{T}^l \times P)/\Delta$ . With the quotient topology and the  $\mathbb{T}^l$  action on the first coordinate by multiplication,  $P^\lambda$  is in the category of  $\mathbb{T}^l$ -topological spaces.

We now give three fundamental examples which will be heavily used throughout the chapter.

**Example 2.2.2.** Let

$$(2.2.3) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

be a strictly convex good cone in  $\mathbb{R}^n$ . The vector  $v_i$  is the inward-pointing primitive normal vector to  $F_i$ . We assume the equations are *minimal*; that is, removing any one of the equations will give a different set. We will always assume that the equations used to define convex polytopes or cones are minimal. We let  $C_0 = C \setminus \{\text{origin}\}$ . We set  $l = n$  and define a characteristic map  $\lambda$  by  $\lambda(F_i) = v_i$ . We denote by  $\Delta$  the equivalence relation on  $\mathbb{T}^n \times C_0$  determined by  $\lambda$ . Then  $C_0^\lambda = (\mathbb{T}^n \times C_0)/\Delta$ , with  $\mathbb{T}^n$  acting on the first coordinate by multiplication, is  $\mathbb{T}^n$ -equivariantly homeomorphic to a toric symplectic cone. This claim will be proved in Section 2.3.

**Example 2.2.4.** For  $C$  as in the previous example, it is a cone over a simple convex polytope  $P$  (such  $P$  obviously exists, we will not specify it as the particular choice

of it does not matter). Notice that the facets of  $P$  are in one-to-one correspondence with those of  $C_0$ . We let  $\tilde{F}_i$  denote the facet of  $P$  contained in  $F_i$ . We may define a characteristic map as in Example 2.2.2 by sending  $\tilde{F}_i$  to  $v_i$ . By abuse of notation, we still call this map  $\lambda$  and denote by  $\Delta$  the equivalence relation on  $(\mathbb{T}^n \times P)$  determined by  $\lambda$ . We emphasize that  $\lambda_i = \lambda(\tilde{F}_i)$  is the normal vector to  $F_i$ , not to  $\tilde{F}_i$  in  $P$ . Then  $P^\lambda = (\mathbb{T}^n \times P)/\Delta$ , with  $\mathbb{T}^n$  acting on the first coordinate by multiplication, is  $\mathbb{T}^n$ -equivariantly homeomorphic to the contact toric manifold  $M$  associated to  $C$ . The space  $C_0^\lambda$  in the previous example is the symplectization of  $M = P^\lambda$ . These claims will be proved in Section 2.3.

The last example is related to the polytope  $P$  itself, without reference to the cone  $C$ .

**Example 2.2.5.** Let  $P$  be a Delzant polytope, i.e. a convex simple rational smooth polytope, in  $\mathbb{R}^k$  given by

$$(2.2.6) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

The equations are again assumed to be minimal, as in Example 2.2.2. We denote the  $i^{\text{th}}$  facet by  $\tilde{F}_i$ . Then  $\tilde{v}_i$  is the inward-pointing primitive normal vector to the facet  $\tilde{F}_i$ . We set  $l = k$  and define a characteristic map  $\tilde{\lambda}$  by setting  $\tilde{\lambda}(\tilde{F}_i) = \tilde{v}_i$ . Denote by  $\tilde{\Delta}$  the equivalence relation on  $\mathbb{T}^k \times P$  determined by  $\tilde{\lambda}$ . Then  $P^{\tilde{\lambda}} = (\mathbb{T}^k \times P)/\tilde{\Delta}$ , with  $\mathbb{T}^k$  acting on the first coordinate by multiplication, is  $\mathbb{T}^k$ -equivariantly homeomorphic to the symplectic toric manifold associated to  $P$ . This will be restated in Section 2.4.



Notice the subtle difference between  $P^\lambda$  and  $P^{\tilde{\lambda}}$ . Their relation is crucial in this chapter.

## 2.3 Equivariant cohomology of good contact toric manifolds

In this section, we imitate the computation of the equivariant cohomology of a symplectic toric manifold given in [DJ91] to compute the equivariant cohomology of a good contact toric manifold.

**Remark 2.3.1.** In their paper [DJ91], Davis and Januszkiewicz defined a class of manifolds equipped with torus action which they called *toric manifolds*, now called *quasitoric manifolds*, and computed their equivariant and ordinary cohomology rings. The cohomology ring of symplectic toric manifolds is a particular example of the cohomology ring of a quasitoric manifold. Quasitoric manifolds are a strictly larger class than symplectic toric manifolds. More information may be found in [GP]. Constructions in [DJ91] were later generalized to orbifolds in [PS].

We begin with a brief review of the construction of a toric symplectic cone from a strictly convex good cone given in Lemma 6.4 in [Ler03a]. Let the cone be as in Example 2.2.2:

$$(2.3.2) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

Define a map

$$(2.3.3) \quad \pi_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^n$$

by sending the  $i^{\text{th}}$  standard basis vector  $e_i$  of  $\mathbb{Z}^m$  to  $v_i$ . This induces a map

$$(2.3.4) \quad \pi_{\mathbb{R}} : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

by tensoring with  $\mathbb{R}$ . We then get a map

$$(2.3.5) \quad \pi_{\mathbb{T}} : \mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m \rightarrow \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$$

The subscripts  $\mathbb{Z}, \mathbb{R}, \mathbb{T}$  may be omitted when it will not cause confusion.

Let  $K = \ker(\pi_{\mathbb{T}})$ . Then we have a short exact sequence of groups

$$(2.3.6) \quad 1 \longrightarrow K \xrightarrow{i} \mathbb{T}^m \xrightarrow{\pi_{\mathbb{T}}} \mathbb{T}^n \longrightarrow 1.$$

This induces maps between Lie algebras

$$(2.3.7) \quad 0 \longrightarrow \mathfrak{k} \xrightarrow{i^*} \mathfrak{t}^m \xrightarrow{\pi^*} \mathfrak{t}^n \longrightarrow 0,$$

with dual maps between the duals of the Lie algebras

$$(2.3.8) \quad 0 \longleftarrow \mathfrak{k}^* \xleftarrow{i^*} (\mathfrak{t}^m)^* \xleftarrow{\pi^*} (\mathfrak{t}^n)^* \longleftarrow 0.$$

Let  $u : \mathbb{C}^m \rightarrow (\mathfrak{t}^m)^*$  be defined by

$$(2.3.9) \quad u(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2).$$

Lerman showed in [Ler03a] that

$$(2.3.10) \quad S = ((i^* \circ u)^{-1}(0) \setminus \{0\}) / K$$

is the toric symplectic cone associated to  $C$ .

The standard action of  $\mathbb{T}^m$  on  $\mathbb{C}^m$  restricts to a  $\mathbb{T}^m$ -action on  $(i^* \circ u)^{-1}(0) \setminus \{0\}$ , and hence a  $K$ -action on  $(i^* \circ u)^{-1}(0) \setminus \{0\}$ . It induces a  $\mathbb{T}^n \cong \mathbb{T}^m/K$  action on  $S = ((i^* \circ u)^{-1}(0) \setminus \{0\})/K$ .

In other words, the action of  $t_n \in \mathbb{T}^n$  on  $S$  is induced by the standard action of any element in  $\pi^{-1}(t_n)$  on  $(i^* \circ u)^{-1}(0) \setminus \{0\}$ . This action is Hamiltonian with moment map

$$(2.3.11) \quad \nu : S \rightarrow (\mathfrak{t}^n)^*$$

$$(2.3.12) \quad [z_1, \dots, z_m] \mapsto (\pi^*)^{-1}(u(z_1, \dots, z_m)),$$

using that  $\pi^*$  is injective. The image of  $\nu$  is  $C_0 = C \setminus \{0\}$  according to [Ler03a]. It is not hard to see from the construction that  $C_0$  is the orbit space of the  $\mathbb{T}^n$  action on  $S$ .

Now we are ready to prove the claim made in Example 2.2.2 as follows.

**Proposition 2.3.13.** *With the same notation as in Example 2.2.2,  $C_0^\lambda$  is  $\mathbb{T}^n$ -equivariantly homeomorphic to  $S$ , the symplectic toric cone associated with  $C$ .*

*Proof.* If we restrict the domain of  $u$  defined in (2.3.9) to

$$(2.3.14) \quad (\mathbb{R}^m)^+ = \{(z_1, \dots, z_m) \in \mathbb{C}^m : z_i \in \mathbb{R}_{\geq 0}, \forall i\},$$

it is a homeomorphism onto its image. Call this restriction map  $u_0$ .

Now define a map

$$(2.3.15) \quad \gamma_0 : \mathbb{T}^n \times C_0 \rightarrow S$$

by:

$$(t_n, p) \mapsto t_n \cdot [u_0^{-1}(\pi^*(p))]$$

We note that  $\pi^*(p) \in (\mathbb{R}^m)^+$  because of the defining equations for  $C$ . Thus,  $u_0^{-1}(\pi^*(p))$  is well-defined, and it is straightforward to check that

$$u_0^{-1}(\pi^*(p)) \in (i^* \circ u)^{-1}(0) \setminus \{0\}.$$

We let  $[u_0^{-1}(\pi^*(p))] \in S$  be the equivalence class of  $u_0^{-1}(\pi^*(p))$ . Finally,  $t_n \in \mathbb{T}^n$  acts on  $S$  as noted earlier.

This map  $\gamma_0$  is a continuous. It is also surjective because  $C_0$  is the orbit space of the  $\mathbb{T}^n$ -action on  $S$ .

According to Lemma 2.3.17, which will be stated and proved right after this proposition,  $\gamma_0$  induces a bijective map

$$(2.3.16) \quad \gamma : (\mathbb{T}^n \times C_0)/\Delta \rightarrow S$$

The inverse of this map is also continuous since  $\gamma$  is an open map. Finally,  $\gamma$  is obviously  $\mathbb{T}^n$ -equivariant.  $\square$

**Lemma 2.3.17.** *The stabilizer of  $[u_0^{-1}(\pi^*(p))] \in S$  is  $\Delta_p \leq \mathbb{T}^n$ .*

*Proof.* For simplicity and without loss of generality, we may assume that the facets that contain  $p$  are exactly  $F_1, \dots, F_j$ . Thus, the coordinates of  $\pi^*(p)$  that are zero are exactly the first  $j$  coordinates. Consequently, the stabilizer of  $[u_0^{-1}(\pi^*(p))]$  must be

$$\{\pi_T(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_j}, 1, 1, \dots, 1) : \theta_1, \dots, \theta_j \in \mathbb{R}\}.$$

This is precisely  $\Delta_p$ . □

A few more lines will prove the claims made in Example 2.2.4.

**Proposition 2.3.18.** *Let  $M$  denote the good contact toric manifold associated to  $C$ . With the same notation as in Example 2.2.4,  $P^\lambda$  is  $\mathbb{T}^n$ -equivariantly homeomorphic to  $M$ .*

*Proof.* By definition, the symplectic cone  $S$  is the symplectization of  $M$ . Topologically,  $S = M \times \mathbb{R}$ . The map  $\nu$  defined in (2.3.11) is proportional on the second coordinate.

For each  $q \in M$ , there is a unique  $x = x(q) \in \mathbb{R}$ , such that

$$\nu(q, x(q)) \in P.$$

Notice that  $P^\lambda = (\mathbb{T}^n \times P)/\Delta$  is a subset of  $(\mathbb{T}^n \times C_0)/\Delta$ , so  $(\mathbb{T}^n \times P)/\Delta$  is  $\mathbb{T}^n$ -equivariantly homeomorphic to its image under  $\gamma$  as defined in (2.3.16). This is precisely the pre-image of  $P$  under  $\nu$ , which is

$$\{(q, x(q)) \in S = M \times \mathbb{R} : q \in M\}.$$

This is clearly  $\mathbb{T}^n$ -equivariantly homeomorphic to  $M$ . □

We now compute the  $\mathbb{T}^n$ -equivariant cohomology of  $M$ , or equivalently, of  $P^\lambda$ . Recall that  $C$  is a cone over  $P$ . We continue to let  $\lambda$  denote the characteristic map defined in Example 2.2.4. Suppose  $v_i = (v_{i1}, v_{i2}, \dots, v_{in})$ . Let  $\tilde{\mathcal{F}}$  be the set of facets of  $P$ , and  $m = |\tilde{\mathcal{F}}|$ , the number of facets. Define a characteristic map

$$(2.3.19) \quad \mu : \tilde{\mathcal{F}} \rightarrow \mathbb{Z}^m$$

by sending  $\tilde{F}_i$  to  $e_i$ , the  $i^{\text{th}}$  standard basis vector of  $\mathbb{Z}^m$ . Denote by  $\Omega$  the equivalence relation on  $\mathbb{T}^m \times P$  determined by  $\mu$ . This space  $P^\mu = (\mathbb{T}^m \times P)/\Omega$  was first defined in [DJ91].

**Lemma 2.3.20.**  *$P^\mu$  is a fiber bundle over  $P^\lambda$  with fiber  $K$ , where  $K$  was defined in (2.3.6).*

*Proof.* The group  $K$  acts naturally on  $P^\mu = (\mathbb{T}^m \times P)/\Omega$  by multiplication on the first coordinate. Since  $C$  is a good cone,  $K \cap \Omega_p = 1$  for every  $p \in P$ . Thus the  $K$  action on  $P^\mu$  is free. The orbit space is exactly  $P^\lambda = (\mathbb{T}^n \times P)/\Delta$ . The projection from the total space to the orbit space is given by the natural map

$$\begin{aligned} \pi' : P^\mu = (\mathbb{T}^m \times P)/\Omega &\rightarrow P^\lambda = (\mathbb{T}^n \times P)/\Delta \\ [t_m, p] &\mapsto [\pi_T(t_m), p]. \end{aligned}$$

□

The torus  $K$  may be disconnected. Denote by  $K_0$  the connected component of  $K$  containing the identity.

**Lemma 2.3.21.** *There exists a group homomorphism  $r : \mathbb{T}^m \rightarrow K_0$ , satisfying  $r \circ i|_{K_0} = id_{K_0}$ , where  $id_{K_0}$  denotes the identity map on  $K_0$  and  $i$  is the inclusion of  $K$  into  $\mathbb{T}^m$  as defined in (2.3.6).*

*Proof.* Consider the maps

$$0 \longrightarrow \ker(\pi_{\mathbb{Z}}) \xrightarrow{j} \mathbb{Z}^m \xrightarrow{\pi_{\mathbb{Z}}} \mathbb{Z}^n,$$

where  $\pi_{\mathbb{Z}}$  is defined in (2.3.3) and  $j$  is the inclusion map. As a subgroup of the free abelian group  $\mathbb{Z}^m$ ,  $\ker(\pi_{\mathbb{Z}})$  is also a free abelian group. So with suitably chosen basis, the map  $j$  looks like an inclusion

$$b_1\mathbb{Z} \oplus b_2\mathbb{Z} \oplus \cdots \oplus b_k\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}^{m-k}.$$

But notice that for every  $x \in \mathbb{Z}^m$  and every  $t \in \mathbb{Z} \setminus \{0\}$ ,

$$x \in \ker(\pi_{\mathbb{Z}}) \Leftrightarrow tx \in \ker(\pi_{\mathbb{Z}}).$$

So all of the  $b_i$  must be 1. This means  $\ker(\pi_{\mathbb{Z}})$  is a direct summand of  $\mathbb{Z}^m$ , so there is a group homomorphism

$$(2.3.22) \quad r_0 : \mathbb{Z}^m \rightarrow \ker(\pi_{\mathbb{Z}}),$$

such that

$$r_0 \circ j = id_{\ker(\pi_{\mathbb{Z}})}.$$

This  $r_0$  induces a map

$$(2.3.23) \quad r : \mathbb{T}^m \rightarrow K_0$$

that satisfies the requirement of the lemma. □

Using Lemma 2.3.21, we can define an action of  $\mathbb{T}^m$  on  $EK_0 \times E\mathbb{T}^n$  by first sending  $t_m \in \mathbb{T}^m$  to  $(r(t_m), \pi(t_m)) \in K_0 \times \mathbb{T}^n$ , then using diagonal action of this element on  $EK_0 \times E\mathbb{T}^n$ .

**Lemma 2.3.24.** *The space  $(EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega)$  is a fiber bundle over  $E\mathbb{T}^n \times_{\mathbb{T}^n} ((\mathbb{T}^n \times P)/\Delta)$  with fiber  $EK_0$ . Thus these two spaces are homotopy equivalent.*

*Proof.* Define the projection map by

$$[x, y, [t, p]] \mapsto [y, [\pi(t), p]],$$

where  $x \in EK_0$ ,  $y \in E\mathbb{T}^n$ ,  $t \in \mathbb{T}^m$ ,  $p \in P$ . The square brackets  $[ ]$  are used whenever there is an equivalence relation involved.

It is easily shown that the map is well-defined. As for the fiber, pick any point  $[y_0, [t_n, p]]$  in the base space, with  $y_0 \in E\mathbb{T}^n$ ,  $t_n \in \mathbb{T}^n$ ,  $p \in P$ . Assume  $[x, y, [t, p]]$  is in the pre-image, then

$$[y, [\pi(t), p]] = [y_0, [t_n, p]].$$

So there exists  $s \in \mathbb{T}^n$  such that

$$sy = y_0, \quad \text{and}$$

$$[s \cdot \pi(t), p] = [t_n, p].$$

Since  $\pi_{\mathbb{T}}$  is surjective, so there is an  $s' \in \mathbb{T}^m$  such that  $\pi(s') = s$ . Then

$$[x, y, [t, p]] = [r(s')x, \pi(s')y, [s't, p]] = [r(s')x, y_0, [s't, p]], \quad \text{and}$$

$$\pi'([s't], p) = [\pi(s')\pi(t), p] = [s \cdot \pi(t), p] = [t_n, p],$$

where  $\pi'$  is defined in Lemma 2.3.20.

This means when considering the fiber over  $[y_0, [t_n, p]]$ , we only need to use representatives of the form  $[x, y_0, [t, p]]$ , where  $y_0$  is fixed, and  $\pi'[t, p] = [t_n, p]$ .

By Lemma 2.3.20, we know that the set of points  $[t, p] \in P^\lambda$  that satisfy  $\pi'[t, p] = [t_n, p]$  is homeomorphic to  $K$ . The elements in  $\mathbb{T}^m$  that fix  $y_0$  are  $K$ . So the fiber



over  $[y, [t_n, p]]$  is homeomorphic to  $(EK_0 \times K)/K$ , which is homeomorphic to  $EK_0$  via the map

$$[x, k] \mapsto r(k)^{-1}x,$$

noticing that  $K$  acts on  $EK_0$  by first mapping  $k \in K$  to  $r(k) \in K_0$ , then applying  $r(k)$  to  $EK_0$ .  $\square$

Notice that the cohomology of  $E\mathbb{T}^n \times_{\mathbb{T}^n} ((\mathbb{T}^n \times P)/\Delta)$  is exactly what we want to compute:  $H_{\mathbb{T}^n}^*(M; \mathbb{Z})$ .

Davis and Januskiewicz computed  $H_{\mathbb{T}^m}^*(P^\mu; \mathbb{Z}) = H^*(E\mathbb{T}^m \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega); \mathbb{Z})$  in [DJ91]. More details can be found in [DJ91, 434-436]. So the remaining task is to compare the two spaces

$$(EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^n} ((\mathbb{T}^n \times P)/\Omega)$$

and

$$E\mathbb{T}^m \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega).$$

**Lemma 2.3.25.** *The two spaces*

$$(2.3.26) \quad (EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^n} ((\mathbb{T}^n \times P)/\Omega)$$

and

$$(2.3.27) \quad E\mathbb{T}^m \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega)$$

are homotopy equivalent.

*Proof.* Let  $W = EK_0 \times E\mathbb{T}^n \times E\mathbb{T}^m$ . An element  $t_m \in \mathbb{T}^m$  acts on  $W$  by the diagonal action of  $(r(t_m), \pi(t_m), t_m)$ .

The projection

$$(2.3.28) \quad p_1 : W \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega \rightarrow E\mathbb{T}^m \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega$$

is a fiber bundle with fiber  $EK_0 \times E\mathbb{T}^n$ , which is contractible, so

$$(2.3.29) \quad W \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega \stackrel{h.e.}{\sim} E\mathbb{T}^m \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega,$$

where  $\stackrel{h.e.}{\sim}$  stands for ‘homotopy equivalent’.

Suppose  $t_m \in \mathbb{T}^m$ . If  $t_m \notin K$ , then  $\pi(t_m) \neq 1$ , so  $t_m$  acts on  $EK_0 \times E\mathbb{T}^n$  with no fixed points. If  $t_m \notin \Omega_p$  for any  $p \in P$ , then  $t_m$  acts on  $(\mathbb{T}^m \times P)/\Omega$  with no fixed points. Since  $K \cap \Omega_p = 1$  for all  $p \in P$ , the diagonal action of  $\mathbb{T}^m$  on  $(EK_0 \times E\mathbb{T}^n) \times (\mathbb{T}^m \times P)/\Omega$  is free.

So the projection

$$(2.3.30) \quad p_2 : W \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega \rightarrow (EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega$$

is a fiber bundle with fiber  $E\mathbb{T}^m$ . So

$$(2.3.31) \quad W \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega \stackrel{h.e.}{\sim} (EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^m} (\mathbb{T}^m \times P)/\Omega.$$

Combining (2.3.29) and (2.3.31) completes the proof.  $\square$

**Definition 2.3.32.** Define  $\mathcal{I}$  to be the ideal of  $\mathbb{Z}[x_1, x_2, \dots, x_m]$  or  $\mathbb{Q}[x_1, x_2, \dots, x_m]$  generated by monomials

$$\left\{ x_{i_1} x_{i_2} \cdots x_{i_l} : \bigcap_{j=1}^l \widetilde{F}_{i_j} = \emptyset \right\}.$$

The coefficients used will be clear in context.

**Theorem 2.3.33.** *If  $M$  is the contact toric manifold associated with the strictly convex good cone*

$$C = \bigcap_{i=1}^m \{x \in (t^n)^* = \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

then

$$H_{\mathbb{T}^n}^*(M; \mathbb{Z}) \simeq \mathbb{Z}[x_1, \dots, x_m]/\mathcal{I},$$

where  $x_i \in H_{\mathbb{T}^n}^2(M; \mathbb{Z})$ .

*Proof.* According to Lemma 2.3.24,

$$H_{\mathbb{T}^n}^*(M; \mathbb{Z}) = H^*(E\mathbb{T}^n \times_{\mathbb{T}^n} P^\lambda, \mathbb{Z}) = H^*((EK_0 \times E\mathbb{T}^n) \times_{\mathbb{T}^m} P^\mu; \mathbb{Z})$$

According to Lemma 2.3.25, they are equal to

$$H_{\mathbb{T}^m}^*(P^\mu; \mathbb{Z}).$$

Then Theorem 2.3.33 follows from Theorem 4.8 in [DJ91]. □

**Corollary 2.3.34.** *For a good contact toric manifold  $M$ , we have  $H^1(M; \mathbb{Z}) = 0$ .*

*Proof.* Consider the spectral sequence of the fiber bundle

$$(2.3.35) \quad M \hookrightarrow E\mathbb{T}^n \times_{\mathbb{T}^n} M \rightarrow B\mathbb{T}^n$$

If  $E_2^{0,1} = H^1(M; \mathbb{Z})$  has torsion, it will be in the kernel of

$$(2.3.36) \quad d_2 : E_2^{0,1} \rightarrow E_2^{2,0}$$

since  $E_2^{2,0} = H^2(B\mathbb{T}^n; \mathbb{Z}) = \mathbb{Z}^n$  is free. So the torsion part will live to  $E_\infty^{0,1}$ . Thus  $H^1(E\mathbb{T}^n \times_{\mathbb{T}^n} M; \mathbb{Z})$  will have torsion and contradicts Theorem 2.3.33, from which it easily follows that  $H^1(E\mathbb{T}^n \times_{\mathbb{T}^n} M; \mathbb{Z}) = 0$ . So  $H^1(M; \mathbb{Z})$  is torsion-free.

Theorem 1.1 in [Ler04] tells us that  $\pi_1(M)$  is a finite group. So  $H_1(M; \mathbb{Z})$  is finite, and  $H^1(M; \mathbb{Z})$  has no free part.

Thus, we may conclude that  $H^1(M; \mathbb{Z}) = 0$ . □

## 2.4 Cohomology and equivariant cohomology of symplectic toric manifolds

In this section, we recall some old construction and facts about symplectic manifolds. This will help to set up notations for Section 2.5. In Section 2.5, we will take  $k = n - 1$ .

First we will briefly recall the classical Delzant construction of a symplectic toric manifold. Let the convex Delzant polytope be as in (2.2.6):

$$(2.4.1) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

Then define a map

$$(2.4.2) \quad \pi_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^k$$

by sending the  $i^{\text{th}}$  standard basis vector  $e_i$  of  $\mathbb{Z}^m$  to  $\tilde{v}_i$ . This induces a map

$$(2.4.3) \quad \pi_{\mathbb{R}} : \mathbb{R}^m \rightarrow \mathbb{R}^k$$

by tensoring with  $\mathbb{R}$ . We then get a map

$$(2.4.4) \quad \pi_{\mathbb{T}} : \mathbb{T}^m = \mathbb{R}^m / \mathbb{Z}^m \rightarrow \mathbb{T}^k = \mathbb{R}^k / \mathbb{Z}^k$$

The subscripts  $\mathbb{Z}, \mathbb{R}, \mathbb{T}$  will be omitted sometimes when it will not cause confusion.

Letting  $K = \ker(\pi_{\mathbb{T}})$ , we have a short exact sequence of groups

$$(2.4.5) \quad 1 \longrightarrow K \xrightarrow{i} \mathbb{T}^m \xrightarrow{\pi_{\mathbb{T}}} \mathbb{T}^k \longrightarrow 1.$$

This induces maps on Lie algebras

$$(2.4.6) \quad 0 \longrightarrow \mathfrak{k} \xrightarrow{i_*} \mathfrak{t}^m \xrightarrow{\pi_*} \mathfrak{t}^k \longrightarrow 0,$$

and maps on the duals of Lie algebras

$$(2.4.7) \quad 0 \longleftarrow \mathfrak{k}^* \xleftarrow{i^*} (\mathfrak{t}^m)^* \xleftarrow{\pi^*} (\mathfrak{t}^k)^* \longleftarrow 0.$$

Let  $u : \mathbb{C}^m \rightarrow (\mathfrak{t}^m)^*$  be defined by

$$(2.4.8) \quad u(z_1, \dots, z_m) = (|z_1|^2, \dots, |z_m|^2)$$

Then

$$(2.4.9) \quad N = ((i^* \circ u)^{-1}(i^*(-\eta)))/K$$

is the symplectic toric manifold associated to the polytope  $P$ , where  $\eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m = (\mathfrak{t}^m)^*$ .

The standard action of  $\mathbb{T}^m$  on  $\mathbb{C}^m$  restricts to a  $\mathbb{T}^m$ -action on  $(i^* \circ u)^{-1}(i^*(-\eta))$ , and hence a  $K$ -action on  $(i^* \circ u)^{-1}(i^*(-\eta))$ . It induces a  $\mathbb{T}^k \cong \mathbb{T}^m/K$  action on  $N = ((i^* \circ u)^{-1}(i^*(-\eta)))/K$ .

In other words, the action of  $t_k \in \mathbb{T}^k$  on  $N$  is induced by the standard action of any element in  $\pi^{-1}(t_k)$  on  $(i^* \circ u)^{-1}(i^*(-\eta))$ . This action is Hamiltonian with moment map

$$(2.4.10) \quad \nu : N \rightarrow (\mathfrak{t}^k)^*$$

$$(2.4.11) \quad [z_1, \dots, z_m] \mapsto (\pi^*)^{-1}(u(z_1, \dots, z_m) + \eta),$$

using that  $\pi^*$  is injective. The image of  $\nu$  is  $P$ . It is easy to check from the construction that  $P$  is the orbit space of the  $\mathbb{T}^k$ -action on  $N$ .

Following the same line as Proposition 2.3.18, we have the following.

**Proposition 2.4.12.** *With the same notation as in Example 2.2.5,  $P^{\tilde{\lambda}}$  is  $\mathbb{T}^k$ -equivariantly homeomorphic to  $N$ , the symplectic manifold associated to  $P$ .*

The following diagram was used in [DJ91] to compute  $H^*(N; \mathbb{Z}) = H^*((\mathbb{T}^k \times P)/\tilde{\Delta}; \mathbb{Z})$ .

$$(2.4.13) \quad N \xrightarrow{f_1} E\mathbb{T}^k \times_{\mathbb{T}^k} N \xleftarrow{f_2} E\mathbb{T}^m \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega) \xrightarrow{f_3} E\mathbb{T}^m \times_{\mathbb{T}^m} pt .$$

In this diagram,  $E\mathbb{T}^m$  is taken to be  $EK \times E\mathbb{T}^k$ . Since  $K$  is connected, there is a

map  $r : \mathbb{T}^m \rightarrow K$ , such that  $r|_K$  is identity map. A group element  $t_m \in \mathbb{T}^m$  acts on  $E\mathbb{T}^m = EK \times E\mathbb{T}^k$  via the diagonal action of  $(r(t_m), \pi(t_m))$ .

The map  $f_1$  is inclusion of fiber,  $f_3$  is the projection onto the first factor. The map  $f_2$  is given by  $[(x, y), [t_m, p]] \mapsto [y, [\pi(t_m), p]]$ , where  $x \in EK, y \in E\mathbb{T}^k, t_m \in \mathbb{T}^m, p \in P$ .

**Theorem 2.4.14.** [DJ91] *Taking cohomology of diagram (2.4.13) allows us to compute the equivariant cohomology of the symplectic toric manifold*

$$(2.4.15) \quad H_{\mathbb{T}^k}^*(N; \mathbb{Z}) = H^*(E\mathbb{T}^k \times_{\mathbb{T}^k} (\mathbb{T}^k \times P)/\tilde{\Delta}; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m]/\mathcal{I},$$

where  $x_i \in H_{\mathbb{T}^k}^2(N; \mathbb{Z}), 1 \leq i \leq m$ , are images of generators of  $H^*(E\mathbb{T}^m \times_{\mathbb{T}^m} pt, \mathbb{Z})$ . And the ideal  $\mathcal{I}$  was defined in Definition 2.3.32.

**Definition 2.4.16.** Suppose  $\tilde{v}_j = (v_{j1}, v_{j2}, \dots, v_{jk})$ , for  $1 \leq j \leq m$ . Let

$$(2.4.17) \quad J_i = \sum_{j=1}^m v_{ji} x_j,$$

for  $1 \leq i \leq k$ . We define

$$(2.4.18) \quad \mathcal{J} = \langle J_1, J_2, \dots, J_k \rangle$$

to be the ideal in  $\mathbb{Q}[x_1, x_2, \dots, x_m]$  generated by the linear terms  $\{J_1, J_2, \dots, J_k\}$ .

**Theorem 2.4.19.** [DJ91] *Taking cohomology of diagram (2.4.13) allows us to compute the singular cohomology of symplectic toric manifold.*

$$(2.4.20) \quad H^*(N; \mathbb{Z}) = \mathbb{Z}[x_1, x_2, \dots, x_m]/\langle \mathcal{I}, \mathcal{J} \rangle,$$

where  $x_i \in H^2(N; \mathbb{Z}),$  for  $1 \leq i \leq m$ .

If we denote by  $\mathbb{T}^r$  the subgroup of  $\mathbb{T}^k = (S^1)^k$  that is the last  $r$  copies of  $S^1$ , then the  $\mathbb{T}^k$  action on  $N$  restricts to a  $\mathbb{T}^r$  action on  $N$ . We may compute the  $\mathbb{T}^r$ -equivariant cohomology of  $N$  as an easy corollary of the Theorem 2.4.14 and Theorem 2.4.19.

**Corollary 2.4.21.**  $H_{\mathbb{T}^r}^*(N; \mathbb{Z}) \simeq \mathbb{Z}[x_1, x_2, \dots, x_m] / \langle \mathcal{I}, J_1, J_2, \dots, J_{k-r} \rangle$

*Proof.* The spectral sequence of the singular cohomology with  $\mathbb{Z}$  coefficients of the fiber bundle

$$N \hookrightarrow E\mathbb{T}^r \times_{\mathbb{T}^r} N \rightarrow B\mathbb{T}^r$$

degenerates at the  $E^2$  term, since both  $N$  and  $B\mathbb{T}^r$  have cohomology only in even degrees. Thus  $E\mathbb{T}^r \times_{\mathbb{T}^r} N$  also has cohomology only in even degrees.

Denote by  $\mathbb{T}^{k-r}$  the subgroup of  $\mathbb{T}^k$  that is the product of the first  $k - r$  copies of  $S^1$  in  $\mathbb{T}^k = (S^1)^k$ .

We complete the proof by applying the argument in Theorem 4.14 in [DJ91] to the fiber bundle

$$E\mathbb{T}^r \times_{\mathbb{T}^r} N \hookrightarrow E\mathbb{T}^{k-r} \times_{\mathbb{T}^{k-r}} (E\mathbb{T}^r \times_{\mathbb{T}^r} N) \rightarrow B\mathbb{T}^{k-r}$$

and also noticing that  $E\mathbb{T}^{k-r} \times_{\mathbb{T}^{k-r}} (E\mathbb{T}^r \times_{\mathbb{T}^r} N) = E\mathbb{T}^k \times_{\mathbb{T}^k} N$ . □



## 2.5 The singular cohomology of good contact toric manifolds

Given a strictly convex good cone

$$(2.5.1) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\},$$

there is a good contact toric manifold  $M$  associated to it. As we proved in Proposition 2.3.18,  $M$  is  $\mathbb{T}^n$ -equivariantly homeomorphic to  $P^\lambda$ , which was defined in Example 2.2.4.

To make computations easier, we first move  $C_0 = C \setminus \{\vec{0}\}$  into the upper half space  $U\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$  using a transformation in  $SL(n; \mathbb{Z})$ , where  $SL(n; \mathbb{Z})$  is naturally included into  $SL(n; \mathbb{R})$  as a subgroup. We will show in Proposition 2.5.4 that we can always do this. This will not change the homeomorphism type of the good contact toric manifolds associated to the cone.

**Remark 2.5.2.** The good contact toric manifolds associated to two cones that differ by a transformation in  $SL(n; \mathbb{Z})$  are contactomorphic. This fact was not stated in [Ler03a], but it easily follows from the classification theorem there.

**Lemma 2.5.3.** *The following conditions are equivalent:*

1.  $C_0 \subset U\mathbb{R}^n$ , where  $U\mathbb{R}^n$  stands for the ‘upper half space’:  $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$ .
2.  $(0, 0, \dots, 0, 1) \in (C^\vee)^\circ$ , where  $C^\vee = \{y \in \mathbb{R}^n : \langle y, x \rangle \geq 0, \forall x \in C\}$ , and  $(C^\vee)^\circ$  is its interior  $\{y \in \mathbb{R}^n : \langle y, x \rangle > 0, \forall x \in C_0\}$ .

3.  $(0, 0, \dots, 0, 1)$  can be expressed as a linear combination of  $\{v_i, 1 \leq i \leq m\}$  with positive coefficients.

*Proof.* Denote  $(0, 0, \dots, 0, 1)$  by  $\vec{a}$ .

(1) $\Rightarrow$ (2): For any  $x \in C_0$ ,  $\langle \vec{a}, x \rangle = x_n > 0$ .

(2) $\Rightarrow$ (3): The dual cone  $C^\vee$  is spanned by rays along  $v_1, v_2, \dots, v_m$ . So any vector in the interior of it can be expressed as linear combination of  $\{v_i, 1 \leq i \leq m\}$  with positive coefficients.

(3) $\Rightarrow$ (1): For any  $x \in C$ , we have  $\langle x, v_i \rangle \geq 0$ , for  $1 \leq i \leq m$ , with all equalities if and only if  $x = \vec{0}$ . Suppose

$$\vec{a} = \sum_{i=1}^m k_i v_i, \text{ with } k_i > 0 \forall i.$$

Then

$$x_n = \langle x, \vec{a} \rangle = \sum_{i=1}^m k_i \langle x, v_i \rangle \geq 0.$$

The equality holds if and only if  $x = \vec{0}$ . So for any  $x \in C_0$ ,  $x_n > 0$ , i.e.,  $C_0 \subset U\mathbb{R}^n$ .  $\square$

**Proposition 2.5.4.** *There is an element  $B$  in  $SL(n; \mathbb{Z})$  so that  $B(C_0)$  is in  $U\mathbb{R}^n$ .*

*Proof.* Let  $v = \sum_{i=1}^m v_i$  and  $u$  be the primitive vector in the direction of  $v$ . Suppose

$$u = \frac{1}{k}v, \text{ for } k \in \mathbb{Z}_{>0}.$$

Since  $u$  is primitive, there exists  $D \in SL(n, \mathbb{Z})$ , such that,  $D(u) = (0, 0, \dots, 0, 1)$ .

Define a transformation  $B$  of  $\mathbb{R}^n$  by:

$$(2.5.5) \quad x \mapsto Bx = D^T x,$$

where the matrix for  $D^T$  is the transpose of the matrix for  $D$ . Since  $\det D^T = \det D = 1$ , we still have  $B \in SL(n; \mathbb{Z})$ .

Under this transformation  $B$ , the cone  $C = \bigcup_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$  is taken to

$$(2.5.6) \quad B(C) = \bigcup_{i=1}^m \{x \in \mathbb{R}^n : \langle x, D(v_i) \rangle \geq 0\}.$$

So the normal vectors to the facets of the new cone  $B(C)$  are  $\{D(v_i) : 1 \leq i \leq m\}$ . Moreover,

$$(2.5.7) \quad (0, 0, \dots, 0, 1) = D(u) = \sum_{i=1}^m \frac{1}{k} D(v_i).$$

Finally, using Lemma 2.5.3, we see that the new cone  $B(C_0)$  is in  $U\mathbb{R}^n$ .  $\square$

So without loss of generality, we may now assume that  $C_0 \subset U\mathbb{R}^n$ . By intersecting  $C$  with the hyperplane:

$$(2.5.8) \quad H = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n = 1\},$$

we get a polytope, which we will denote by  $P$ .

The intersection of facet  $F_i$  with  $H$  is given by

$$\tilde{F}_i := \left\{ (x_1, \dots, x_{n-1}, 1) \in \mathbb{R}^n : \sum_{j=1}^{n-1} x_j v_{ij} + v_{in} \geq 0 \right\}.$$

Let  $\mathcal{F} = \{\tilde{F}_i : 1 \leq i \leq m\}$ .

If we identify  $H$  with  $\mathbb{R}^{n-1}$  via

$$(x_1, \dots, x_{n-1}, 1) \mapsto (x_1, \dots, x_{n-1}),$$

then  $P$  can be thought of as a polytope in  $\mathbb{R}^{n-1}$  given by

$$(2.5.9) \quad \{x \in \mathbb{R}^{n-1} : \langle x, \tilde{v}_i \rangle \geq -v_{in}\},$$

where  $\tilde{v}_i = (v_{i1}, \dots, v_{i,n-1})$  is normal to  $\tilde{F}_i$  in  $H = \mathbb{R}^{n-1}$ .

To compute the singular cohomology ring of  $M$  with integer coefficients, we need to add the following assumption.

**Smoothness Criterion:** The polytope  $P$  is a Delzant polytope in  $\mathbb{R}^{n-1}$  and  $\tilde{v}_i$  is a primitive vector in  $\mathbb{Z}^{n-1}$ .

**Remark 2.5.10.** This assumption allows us to stay in the smooth category in the following discussion. In general,  $P$  is just a simple rational convex polytope, not necessarily smooth, so the results in this section do not hold in full generality. However, if we only care about rational cohomology, the argument and theorems in this section will still be valid in general. In that case, we need to use toric orbifolds instead of manifolds, and  $S^1$  orbi-bundles instead of “honest”  $S^1$  bundles.

Recall that there is a symplectic toric manifold  $N$  associated with  $P$  which is equal to  $P^{\tilde{\Delta}}$ , where  $P^{\tilde{\Delta}}$  was defined in Example 2.2.5, where we let  $k = n - 1$  and  $\eta_i = -v_{in}$ . The symbol  $\tilde{\Delta}$  will still be used here just as in Example 2.2.5.

**Proposition 2.5.11.** *With the notation as above, the good contact toric manifold  $M$  is a principal  $S^1$  bundle over  $N$ . The projection map*

$$d : M = (\mathbb{T}^n \times P)/\Delta \rightarrow N = (\mathbb{T}^{n-1} \times P)/\tilde{\Delta}$$

is given by

$$[t, p] \mapsto [\tilde{t}, p],$$

where  $t = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}) \in \mathbb{T}^n$ , and  $\tilde{t} = (e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{n-1}}) \in \mathbb{T}^{n-1}$  is just  $t$  with the last coordinate dropped. The notation  $[t, p]$  and  $[\tilde{t}, p]$  refers to the equivalence classes of  $(t, p)$  and  $(\tilde{t}, p)$  respectively.

*Proof.* Let

$$(2.5.12) \quad S^1 = \{(1, 1, \dots, 1, e^{i\theta_n}) \in \mathbb{T}^n : \theta_n \in \mathbb{R}\}$$

act on  $M = (\mathbb{T}^n \times P)/\Delta$  by multiplication on  $\mathbb{T}^n$ . Because of the smoothness criterion we imposed,  $S^1 \cap \Delta_p = 1$ , for any  $p \in P$ , and so this action is free.

The quotient space is

$$(2.5.13) \quad (\mathbb{T}^n \times P)/\langle \Delta, S^1 \rangle = (\mathbb{T}^n \times P)/\langle \tilde{\Delta}, S^1 \rangle = (\mathbb{T}^n/S^1 \times P)/\tilde{\Delta} = (\mathbb{T}^{n-1} \times P)/\tilde{\Delta}.$$

Therefore  $N$  is the orbit space, and from the construction, we can see the map from the space  $M$  to  $N$  just drops the last coordinate of  $t$ .  $\square$

From this perspective, then, we get the Gysin sequence of the  $S^1$ -bundle,



Define the odd part of  $H^*(M; \mathbb{Z})$  as

$$(2.5.18) \quad H^{odd}(M; \mathbb{Z}) =: \{\alpha \in H^k(M; \mathbb{Z}) : k \text{ is an odd integer}\}.$$

**Remark 2.5.19.** It is easy to see that  $H^{even}(M; \mathbb{Z})$  is a subring of  $H^*(M; \mathbb{Z})$ , while  $H^{odd}(M; \mathbb{Z})$  is a module over  $H^{even}(M; \mathbb{Z})$ .

Put together the exact sequences (2.5.15) of different degrees, we get the following exact sequence:

$$(2.5.20) \quad 0 \longrightarrow H^{odd}(M; \mathbb{Z}) \xrightarrow{d^*} H^*(N; \mathbb{Z}) \xrightarrow{\cup e} H^*(N; \mathbb{Z}) \xrightarrow{d^*} H^{even}(M; \mathbb{Z}) \longrightarrow 0.$$

From this exact sequence, we can easily prove the following.

**Proposition 2.5.21.** Denote by  $\rho$  the map

$$\cup e : H^*(N; \mathbb{Z}) \rightarrow H^*(N; \mathbb{Z}).$$

Then

$$(2.5.22) \quad H^{even}(M; \mathbb{Z}) \simeq \text{coker } \rho = H^*(N; \mathbb{Z}) / \langle e \rangle,$$

where  $\langle e \rangle$  stands for the ideal of  $H^*(N; \mathbb{Z})$  generated by the Euler class  $e$ . This is a ring isomorphism, and it preserves the degree of the cohomology classes since it is induced by the ring map  $d^*$ .

Moreover,

$$(2.5.23) \quad H^{odd}(M; \mathbb{Z}) \simeq \ker \rho = \text{Ann}(e),$$

where  $\text{Ann}(e)$  denotes the annihilator of  $e$  in  $H^*(N; \mathbb{Z})$ . This isomorphism maps a cohomology class in  $H^{2k+1}(M; \mathbb{Z})$  to a cohomology class in  $H^{2k}(N; \mathbb{Z})$ , lowering the degree by 1. It is an  $H^{\text{even}}(M; \mathbb{Z})$ -module isomorphism in the following sense.

The odd cohomology  $H^{\text{odd}}(M; \mathbb{Z})$  is an  $H^{\text{even}}(M; \mathbb{Z})$ -module. Moreover,  $\ker \rho = \text{Ann}(e)$  is a  $\text{coker } \rho = H^*(N; \mathbb{Z})/\langle e \rangle$ -module in the natural way, so it is also an  $H^{\text{even}}(M; \mathbb{Z})$ -module via the identification of  $\text{coker } \rho$  and  $H^{\text{even}}(M; \mathbb{Z})$  given in (2.5.22). The isomorphism in (2.5.23) respects these module structures.

*Proof.* The isomorphism (2.5.22) follows from the exactness of (2.5.20) at the second  $H^*(N; \mathbb{Z})$  and  $H^{\text{even}}(M; \mathbb{Z})$  and also the fact that  $d^* : H^*(N; \mathbb{Z}) \rightarrow H^{\text{even}}(M; \mathbb{Z})$  is a ring homomorphism.

As for (2.5.23), first notice that it follows from (2.5.14) that the map

$$(2.5.24) \quad d_* : H^{\text{odd}}(M; \mathbb{Z}) \rightarrow H^*(N; \mathbb{Z})$$

is a  $H^*(N; \mathbb{Z})$ -module homomorphism. Exactness of (2.5.20) at  $H^{\text{odd}}(M; \mathbb{Z})$  and the first  $H^{2k}(N; \mathbb{Z})$  shows that

$$(2.5.25) \quad d_* : H^{\text{odd}}(M; \mathbb{Z}) \rightarrow \ker \rho$$

is an  $H^*(N; \mathbb{Z})$ -module isomorphism, hence an  $H^*(N; \mathbb{Z})/\langle e \rangle$ -module isomorphism. □

So to compute the cohomology of  $M$ , we only need to describe the Euler class  $e$  explicitly as a polynomial in  $x_1, x_2, \dots, x_m$ , where  $x_i$ 's are generators of  $H^*(N; \mathbb{Z})$ , as in Theorem 2.4.19.



**Definition 2.5.26.** Define  $\tilde{\pi}_{\mathbb{Z}} : \mathbb{Z}^m \rightarrow \mathbb{Z}^{n-1}$  by mapping the  $i^{\text{th}}$  standard basis vector  $e_i$  of  $\mathbb{Z}^m$  to  $\tilde{v}_i$ . This induces a map

$$(2.5.27) \quad \tilde{\pi} : \mathbb{T}^m \rightarrow \mathbb{T}^{n-1}.$$

in the same way we defined  $\pi_T$  in (2.4.4). Let  $\tilde{K} = \ker \tilde{\pi}$ . Since  $P$  is Delzant and  $\tilde{v}_i$  is primitive,  $\tilde{K}$  is connected, and there exists a splitting  $\tilde{r} : \mathbb{T}^m \rightarrow \tilde{K}$  such that  $\tilde{r}|_{\tilde{K}} = id_{\tilde{K}}$ .

We will use the following diagram to compute the Euler class  $e$ .

$$\begin{array}{ccccc}
 M & L_1 = E\mathbb{T}^{n-1} \times_{\mathbb{T}^{n-1}} M & L_2 & L_3 = E\mathbb{T}^m \times_{\mathbb{T}^m} S^1 & \\
 d \downarrow & & d_1 \downarrow & & d_2 \downarrow & d_3 \downarrow \\
 N & \xrightarrow{f_1} E\mathbb{T}^{n-1} \times_{\mathbb{T}^{n-1}} N & \xleftarrow{f_2} (E\tilde{K} \times E\mathbb{T}^{n-1}) \times_{\mathbb{T}^m} P^\mu & \xrightarrow{f_3} E\mathbb{T}^m \times_{\mathbb{T}^m} pt & 
 \end{array}$$

The base spaces and the maps are virtually the same as that of (2.4.13). An element  $t_m \in \mathbb{T}^m$  acts on  $E\tilde{K} \times E\mathbb{T}^{n-1}$  via the diagonal action of  $(\tilde{r}(t_m), \tilde{\pi}(t_m))$ . We now turn to the top line. On  $L_1$ ,  $\mathbb{T}^{n-1} \curvearrowright M = (\mathbb{T}^n \times P)/\Delta$  by multiplication of  $\mathbb{T}^{n-1}$  on the first  $n-1$  coordinates of  $\mathbb{T}^n$ . On  $L_3$ ,  $\mathbb{T}^m \curvearrowright S^1$  with weight  $-(v_{1n}, v_{2n}, \dots, v_{mn})$ . We will use  $\vec{v}^l$  to denote the vector  $(v_{1n}, v_{2n}, \dots, v_{mn})$ . Finally,  $L_2$  is defined as the pull-back  $f_2^*(L_1)$ .

**Lemma 2.5.28.** *As principal- $S^1$ -bundles over  $N$ ,*

$$(2.5.29) \quad f_1^*(L_1) = M.$$

As principal  $S^1$ -bundles over  $(E\tilde{K} \times E\mathbb{T}^{n-1}) \times_{\mathbb{T}^m} P^\mu$ ,

$$(2.5.30) \quad L_2 = f_2^* L_1 = f_3^* L_3.$$

*Proof.* The map  $f_1$  is an inclusion, so the pull back of  $L_1$  by  $f_1$  is just by restriction. Then the equation (2.5.29) follows from the definition.

To show (2.5.30), it's enough if we can construct an  $S^1$ -equivariant map from  $f_3^* L_3$  to  $f_2^* L_1$ , which lifts the identity map of the base space  $(E\tilde{K} \times E\mathbb{T}^{n-1}) \times_{\mathbb{T}^m} P^\mu$ .

Let  $q = [x, y, [e^{i\vec{\theta}}, p]]$  be a point in the base space  $(E\tilde{K} \times E\mathbb{T}^{n-1}) \times_{\mathbb{T}^m} P^\mu$ , where  $x \in E\tilde{K}$ ,  $y \in E\mathbb{T}^{n-1}$ ,  $\vec{\theta} = (\theta_1, \dots, \theta_m)$ ,  $e^{i\vec{\theta}} = (e^{i\theta_1}, \dots, e^{i\theta_m}) \in \mathbb{T}^m$ , and  $p \in P$ .

The fiber of  $f_3^* L_3$  over the point  $q$  is, by the definition of pull-back, the fiber of  $L_3$  over the point  $f_3(q) = [(x, y), pt]$ , namely

$$(2.5.31) \quad f_3^* L_3|_q = \{[(x, y), e^{i\beta}] : \beta \in \mathbb{R}\}.$$

The fiber of  $f_2^* L_1$  over the point  $q$  is, by the definition of pull-back, the fiber of  $L_1$  over the point  $f_2(q) = [y, [\tilde{\pi}(e^{i\vec{\theta}}), p]]$ , namely

$$(2.5.32) \quad f_2^* L_1|_q = \{[y, [(\tilde{\pi}(e^{i\vec{\theta}}), e^{i\alpha}), p]] : \alpha \in \mathbb{R}\}.$$

Define a map from  $f_3^* L_3|_q$  to  $f_2^* L_1|_q$  by sending

$$(2.5.33) \quad [x, y, e^{i\beta}] \mapsto [y, [(\tilde{\pi}(e^{i\vec{\theta}}), e^{i\beta + i\langle v^t, \vec{\theta} \rangle}), p]].$$

We call this map  $s_q$ . To show  $s_q$  is well-defined, we need to show the definition is independent of the choice of representative of  $q$ .

First, assume  $e^{i\vec{\phi}} \in \mathbb{T}^m$ , and so

$$(2.5.34) \quad [x, y, [e^{i\vec{\theta}}, p]] = [\tilde{r}(e^{i\vec{\phi}})x, \tilde{\pi}(e^{i\vec{\phi}})y, [e^{i(\vec{\theta}+\vec{\phi})}, p]].$$

Using this representative of  $q$ , then

$$\begin{aligned} s_q([x, y, e^{i\beta}]) &= s_q([\tilde{r}(e^{i\vec{\phi}})x, \tilde{\pi}(e^{i\vec{\phi}})y, e^{i(\beta+\langle -\vec{v}^l, \vec{\phi} \rangle)}]) \\ &= [\tilde{\pi}(e^{i\vec{\phi}})y, [(\tilde{\pi}(e^{i(\vec{\theta}+\vec{\phi})}), e^{i\beta-i\langle \vec{v}^l, \vec{\phi} \rangle+i\langle \vec{v}^l, (\vec{\theta}+\vec{\phi}) \rangle}), p]] \\ &= [\tilde{\pi}(e^{i\vec{\phi}})y, [(\tilde{\pi}(e^{i\vec{\phi}})\tilde{\pi}(e^{i\vec{\theta}}), e^{i\beta+i\langle \vec{v}^l, \vec{\theta} \rangle}), p]]. \end{aligned}$$

This equals to the RHS of (3.4.19) by the definition of  $L_1$  as given before Lemma 2.5.28.

Second, choose a different representative by letting  $e^{i\vec{\delta}} \in \Omega_p$ , and then

$$(2.5.35) \quad [x, y, [e^{i\vec{\theta}}, p]] = [x, y, [e^{i\vec{\delta}}e^{i\vec{\theta}}, p]].$$

Using this representative of  $q$ , we have:

$$\begin{aligned} s_q([x, y, e^{i\beta}]) &= [y, [(\tilde{\pi}(e^{i\vec{\delta}}e^{i\vec{\theta}}), e^{i\beta+i\langle \vec{v}^l, \vec{\delta}+\vec{\theta} \rangle}), p]] \\ &= [y, [(\tilde{\pi}(e^{i\vec{\delta}}) \cdot \tilde{\pi}(e^{i\vec{\theta}}), e^{i\langle \vec{v}^l, \vec{\delta} \rangle} \cdot e^{i\beta+i\langle \vec{v}^l, \vec{\theta} \rangle}), p]]. \end{aligned}$$

This also equals to the RHS of (3.4.19) since

$$(2.5.36) \quad (\tilde{\pi}(e^{i\vec{\delta}}), e^{i\langle \vec{v}^l, \vec{\delta} \rangle}) \in \Delta_p,$$

which is true by definition of  $\Delta$ ,  $\tilde{\Delta}$  and  $\vec{v}^l$ .

So the map  $s_q$  defined by (3.4.19) is well-defined. It's obviously  $S^1$ -equivariant. □

Since the Euler class of  $L_3$  is  $\sum_{i=1}^m v_{in}x_i$ , using Lemma 2.5.28 and the naturality of Euler classes, we conclude

$$(2.5.37) \quad e = \sum_{i=1}^m v_{in}x_i.$$

**Definition 2.5.38.** We define linear forms

$$(2.5.39) \quad J_k = \sum_{i=1}^m v_{ik}x_i, 1 \leq k \leq n,$$

and the ideals

$$(2.5.40) \quad \mathcal{J} = \langle J_1, \dots, J_{n-1}, J_n \rangle$$

and

$$(2.5.41) \quad \tilde{\mathcal{J}} = \langle J_1, \dots, J_{n-1} \rangle,$$

where  $\langle S \rangle$  denotes the ideal in  $\mathbb{Z}[x_1, \dots, x_m]$  generated by the elements of  $S$ .

**Remark 2.5.42.** Notice that by definition we have  $e = J_n$ .

Combining Theorem 2.4.19, Proposition 2.5.21 and (2.5.37), we have proved the following:

**Theorem 2.5.43.** *Assume*

$$(2.5.44) \quad C = \bigcap_{i=1}^m \{x \in \mathbb{R}^n : \langle x, v_i \rangle \geq 0\}$$

*is a strictly convex good cone and  $M$  is the good contact toric manifold associated with it.*

*Further assume that  $C_0 = C \setminus \{\vec{0}\} \subset U\mathbb{R}^n$  and the smoothness criterion for  $C$  holds. Let*

$$(2.5.45) \quad \rho : \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle \mathcal{I}, \tilde{\mathcal{J}} \rangle} \rightarrow \frac{\mathbb{Z}[x_1, \dots, x_m]}{\langle \mathcal{I}, \tilde{\mathcal{J}} \rangle}$$

be multiplication by  $J_n$ . Then:

$$(2.5.46) \quad H^{\text{even}}(M; \mathbb{Z}) \simeq \text{coker } \rho \simeq \mathbb{Z}[x_1, \dots, x_m] / \langle \mathcal{I}, \mathcal{J} \rangle$$

as rings, where  $x_i$  represents a cohomology class of degree two.

Moreover,

$$(2.5.47) \quad H^{\text{odd}}(M; \mathbb{Z}) \simeq \ker \rho$$

as  $(H^{\text{even}}(M; \mathbb{Z}) \simeq \text{coker } \rho)$ -modules. A homogeneous polynomial of degree  $k$  represents a cohomology class of degree  $2k + 1$  under this isomorphism.

The next theorem states that half of the Betti numbers of  $M$  vanish.

**Theorem 2.5.48.** *Under the same assumption for  $M$  as in Theorem 2.5.43, we have*

$$(2.5.49) \quad H^{2k+1}(M; \mathbb{Z}) = 0$$

for  $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$ , and

$$(2.5.50) \quad H^{2k}(M; \mathbb{Q}) = 0$$

for  $\{k \in \mathbb{N} : n \leq 2k \leq 2n - 2\}$ .

*Proof.* According to Theorem 6.3 in [Gui94a], the cohomology class of the Kähler form on  $N$  is given by

$$(2.5.51) \quad [\omega] = 2\pi \sum_{i=1}^m v_{in} D_i,$$

where  $D_i$  denotes the cohomology class in  $H^2(N; \mathbb{R})$  that is the Poincare dual to  $(\mathbb{T}^{n-1} \times \tilde{F}_i)/\tilde{\Delta}$ . According to Proposition 2.6.8 in the Appendix,  $D_i = -x_i$ . So

$$(2.5.52) \quad [\omega] = -2\pi e.$$

Using the Hard Lefschetz Theorem on  $N$ , it follows easily from Theorem 2.5.43 that

$$(2.5.53) \quad H^{2k+1}(M; \mathbb{Q}) = 0$$

for  $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$ , and

$$(2.5.54) \quad H^{2k}(M; \mathbb{Q}) = 0$$

for  $\{k \in \mathbb{N} : n \leq 2k \leq 2n - 2\}$ . Furthermore, as the cohomology ring of a symplectic toric manifold, the ring  $\mathbb{Z}[x_1, x_2, \dots, x_m]/\{\mathcal{L}, \tilde{\mathcal{J}}\}$  in Theorem 2.5.43 is torsion-free (see [Ful93]). Therefore as an ideal of it,  $\ker \rho$  is also torsion-free. Hence

$$(2.5.55) \quad H^{2k+1}(M; \mathbb{Z}) = 0$$

for  $\{k \in \mathbb{N} : 1 \leq 2k + 1 \leq n - 1\}$ . □

**Corollary 2.5.56.** *Under the same assumption for  $M$  as in Theorem 2.5.43, the product of two odd-degree cohomology classes of  $M$  is zero.*

*Proof.* This is because of dimension considerations. □

**Remark 2.5.57.** Theoretically, Theorem 2.5.43 already tells us all the information of  $H^*(M; \mathbb{Z})$  as an additive group. Perhaps there should be a combinatorial proof for Theorem 2.5.48, but the author could not find one.

**Remark 2.5.58.** Theorem 2.5.43 tells us what the elements of  $H^*(M; \mathbb{Z})$  are, (3.4.23) tells us how to multiply two even-degree cohomology classes, (3.4.24) tells us how to multiply an even-degree cohomology class and an odd-degree cohomology class. Corollary 2.5.56 tells us the product of two odd-degree cohomology classes must be zero. So the ring structure of  $H^*(M; \mathbb{Z})$  is now completely determined.

## 2.6 Appendix: Several equivalent descriptions of the generators of cohomology rings of symplectic toric manifolds

There is a wide variety of descriptions of the generators of the cohomology ring of a symplectic toric manifold in the literature. In this appendix we list some of them and discuss their relations.

Let  $P$  be as in (2.2.6),

$$(2.6.1) \quad P = \bigcap_{i=1}^m \{x \in \mathbb{R}^k : \langle x, \tilde{v}_i \rangle \geq \eta_i\}.$$

We will use  $N$  to denote the symplectic toric manifold associated to  $P$ . We will keep our notation from Example 2.2.5, so  $N = P^{\tilde{\Delta}} = (\mathbb{T}^k \times P)/\tilde{\Delta}$ .

The cohomology in this section are assumed to be integral cohomology.

The generators in Theorem 2.4.19 of this chapter, which we denoted by  $x_i$ , is the image of the generators of  $H^*(B\mathbb{T}^m)$  under the composition:

$$(2.6.2) \quad H^*(B\mathbb{T}^m) \rightarrow H^*(E\mathbb{T}^m \times_{\mathbb{T}^m} ((\mathbb{T}^m \times P)/\Omega)) \xrightarrow{\sim} H^*(E\mathbb{T}^k \times_{\mathbb{T}^k} ((\mathbb{T}^k \times P)/\tilde{\Delta})) \rightarrow H^*(N)$$

We will now give an easier description for these generators, starting by defining a  $S^1$ -bundle over  $N = (\mathbb{T}^k \times P)/\tilde{\Delta}$ .

For any fixed  $j \in \{1, 2, \dots, k\}$ , define a characteristic map from the set of facets of  $P$  to  $\mathbb{Z}^{k+1}$ :

$$\begin{aligned} \sigma_j : \mathcal{F} &\rightarrow \mathbb{Z}^{k+1} \\ \tilde{F}_j &\mapsto (v_{j1}, v_{j2}, \dots, v_{jk}, 1); \\ \tilde{F}_i &\mapsto (v_{i1}, v_{i2}, \dots, v_{ik}, 0) \text{ for } i \neq j. \end{aligned}$$

Then  $P^{\sigma_j}$  is a principal  $S^1$ -bundle over  $P^{\tilde{\lambda}}$  for similar reasons as in Proposition 2.5.11.

For the same reason as (2.5.30) and (2.5.37), we have the following proposition.

**Proposition 2.6.3.** *Each generator  $x_j$  in Theorem 2.4.19 is the Euler class, or equivalently, the first Chern class, of the principal  $S^1$ -bundle  $P^{\sigma_j}$  over  $N = P^{\tilde{\lambda}}$ .*

Recall from (2.4.9),  $N$  is the quotient of a set  $(i^* \circ u)^{-1}(i^*(-\eta))$ , which we will denote by  $Z$ , by a torus  $K$ . Then  $Z$  is a principal- $K$  bundle over  $N$ .

In the book [GS99], the cohomology of a symplectic toric manifold is computed in a very different way (Theorem 9.8.6 of the book). The generators for the



cohomology ring are  $c_1, c_2, \dots, c_m$ , where  $(c_1, c_2, \dots, c_m)$  are the Chern classes of the bundle  $Z \rightarrow N$ . These  $c_i$ 's can also be described in the following way, as illustrated in both Section 2.2 of [Gui94b] and Section 9.8 of [GS99].

The torus  $\mathbb{T}^m$  acts on  $\mathbb{C}^m$  in the standard way and  $\mathbb{C}^m$  splits

$$(2.6.4) \quad \mathbb{C}^m = \mathbb{C}_1 \oplus \mathbb{C}_2 \oplus \cdots \oplus \mathbb{C}_m.$$

The torus  $K$ , as a subgroup of  $\mathbb{T}^m$ , also acts on  $\mathbb{C}^m$  and preserves this splitting. Then

$$(2.6.5) \quad (Z \times \mathbb{C}_i)/K \rightarrow Z/K$$

is a complex line bundle over  $N = Z/K$ , where the action of  $K$  on  $\mathbb{C}_i$  is by first including  $K$  into  $\mathbb{T}^m$ , then acting with weight  $-(0, \dots, 0, 1, 0, \dots, 0)$ , where the only 1 is on the  $i^{\text{th}}$  position. The  $K$  action on  $Z \times \mathbb{C}_i$  is the diagonal action. Then the first Chern class of this line bundle is  $c_i$ .

**Proposition 2.6.6.** *The generator  $c_i$  is exactly the generator  $x_i$  used in Theorem 2.4.19.*

*Proof.* Assume without loss of generality that  $i = 1$ . According to Proposition 2.6.3, it suffices to show  $P^{\sigma_1}$  and  $(Z \times S^1)/K$  are isomorphic principal- $S^1$  bundles over  $P^{\tilde{\lambda}} = Z/K$ , where the action of  $K$  on  $S^1$  is by first including  $K$  into  $\mathbb{T}^m$ , then acting with weight  $-(1, 0, 0, \dots, 0)$ . The base spaces are identified by Proposition 2.4.12.

Fixing a splitting  $\alpha : \mathbb{T}^k \rightarrow \mathbb{T}^m$ , such that

$$(2.6.7) \quad \pi \circ \alpha = id_{\mathbb{T}^k}.$$

Define

$$f : P^{\sigma_1} \rightarrow (Z \times S^1)/K$$

$$[t_k, e^{i\theta}, p] \mapsto [\alpha(t_k) \cdot u_0^{-1}(\pi^*(p) - \eta), \alpha(t_k) \cdot e^{i\theta}],$$

where  $t_k \in \mathbb{T}^k$ ,  $\theta \in \mathbb{R}$ ,  $p \in P$ , and  $u_0 : (\mathbb{R}_{\geq 0})^m \rightarrow (\mathbb{R}_{\geq 0})^m$  is given by  $(z_1, \dots, z_m) \mapsto (z_1^2, \dots, z_m^2)$ . The verification that this map is well-defined is routine. It is easy to see this map lifts the identity map of the base space, is  $S^1$ -equivariant and non-trivial on each fiber. So  $P^{\sigma_1}$  and  $(Z \times S^1)/K$  are isomorphic, hence  $c_1 = x_1$ .  $\square$

The set  $(\mathbb{T}^k \times \tilde{F}_i)/\tilde{\Delta}$  is a submanifold of  $N$ . It is the pre-image of  $\tilde{F}_i$  under the moment map  $\nu$  as defined in (2.4.10). Its Poincare dual, by definition, is a cohomology class in  $H^2(N; \mathbb{R})$ . We denote it by  $D_i$ .

**Proposition 2.6.8.** *The class  $D_i$  equals  $-c_i = -x_i$ .*

*Proof.* Without loss of generality, assume  $i = 1$ . It is obvious that  $-c_1$  is the first Chern class of the complex line bundle

$$(2.6.9) \quad (Z \times \mathbb{C})/K \rightarrow Z/K,$$

where  $K$ , as a subgroup of  $\mathbb{T}^m$ , acts on  $\mathbb{C}$  with weight  $(1, 0, 0, \dots, 0)$ . According to Proposition 12.8 in [BT82], it suffices to find a transversal section of this line bundle, such that the zero locus of the section is exactly  $\nu^{-1}(\tilde{F}_1)$ . Define a section

$$s : Z/K \rightarrow (Z \times \mathbb{C})/K$$

$$[z_1, z_2, \dots, z_m] \mapsto [(z_1, z_2, \dots, z_m), z_1],$$

noticing that  $Z$  is a subset of  $\mathbb{C}^m$ . It is easy to see this map is well-defined, and it is straightforward to show that

$$(2.6.10) \quad z_1 = 0 \Leftrightarrow \nu([z_1, \dots, z_m]) \in \tilde{F}_1.$$

Finally, to see it is transversal to the zero section, simply notice it is holomorphic and obviously not tangent to the zero section along the zero locus.  $\square$

In [TW03], as a corollary of a more general theorem, there is yet another way of computing the cohomology ring of a symplectic toric manifold. To describe the generators there, we draw a diagram first:

$$(2.6.11) \quad (EK \times E\mathbb{T}^k) \times_{\mathbb{T}^m} \mathbb{C}^m \xleftarrow{g_1} EK \times_K \mathbb{C}^m \xleftarrow{g_2} EK \times_K Z \xrightarrow{g_3} Z/K .$$

In the diagram,  $g_1$  and  $g_2$  are just inclusion maps,  $g_3$  is a fiber bundle with fiber  $EK$ . The group  $\mathbb{T}^m$  acts on  $EK \times E\mathbb{T}^k$  as explained in Section 2.4. Now

$$(2.6.12) \quad H^*((EK \times E\mathbb{T}^k) \times_{\mathbb{T}^m} \mathbb{C}^m) = \mathbb{Z}[y_1, \dots, y_m],$$

where  $y_i$  is the Chern class of the principal  $S^1$ -bundle

$$(2.6.13) \quad (EK \times E\mathbb{T}^k) \times_{\mathbb{T}^m} (\mathbb{C}^m \times S^1) \rightarrow (EK \times E\mathbb{T}^k) \times_{\mathbb{T}^m} \mathbb{C}^m,$$

where  $\mathbb{T}^m$  acts on  $S^1$  with weight  $-(0, 0, \dots, 0, 1, 0, \dots, 0)$ , where the only 1 is on the  $i^{th}$  position. In Theorem 7 of [TW03], the generators of the cohomology ring of a symplectic toric manifold are the images of these  $y_i$ 's under the composed map

$$(2.6.14) \quad H^*((EK \times E\mathbb{T}^k) \times_{\mathbb{T}^m} \mathbb{C}^m) \xrightarrow{g_1^*} H^*(EK \times_K \mathbb{C}^m) \xrightarrow{g_2^*} H^*(EK \times_K Z) \xrightarrow{(g_3^*)^{-1}} H^*(Z/K) .$$

It follows easily from the naturality of Chern class that  $g_2^* g_1^*(y_i) = g_3^*(c_i)$ , whence we may conclude our final proposition.

**Proposition 2.6.15.** *The generators for the cohomology ring of a symplectic toric manifold in Theorem 7 in [TW03] are the same as the ones used in Theorem 2.4.19 in this chapter.*

CHAPTER 3  
 THE HARD LEFSCHETZ PROPERTY OF HAMILTONIAN GKM  
 MANIFOLDS

### 3.1 Introduction

As we described in Section 1.4, Goresky, Kottwitz and MacPherson [GKM97] provided a means to turn the computation of equivariant cohomology into a combinatorial one. Assume  $(M, \omega, \mathbb{T}^n, \phi)$  is Hamiltonian GKM manifold with GKM graph  $\Gamma = (V, E)$  and a function  $\alpha : V \rightarrow \mathbb{C}[x_1, \dots, x_n]_1$  induced by  $\phi$ , then there is an isomorphism between the equivariant cohomology of  $M$  and the (equivariant) graph cohomology of  $(\Gamma, \alpha)$

$$(3.1.1) \quad H_{\mathbb{T}^n}^*(M; \mathbb{C}) \cong \left\{ (f_1, f_2, \dots, f_{|V|}) \in \bigoplus_{i=1}^{|V|} \mathbb{C}[x_1, x_2, \dots, x_n] \mid \alpha(e_{ij}) \mid f_i - f_j, \forall e_{ij} \in E \right\}.$$

The right-hand side of the isomorphism is the definition of the (equivariant) graph cohomology of  $(\Gamma, \alpha)$ , denoted by  $H_{\mathbb{T}^n}^*(\Gamma, \alpha)$ , or by  $H_{\mathbb{T}^n}^*(\Gamma)$  when the context is clear. Cohomology rings are assumed to be with  $\mathbb{C}$ -coefficients throughout the chapter. The ring  $H_{\mathbb{T}^n}^*(\Gamma)$  is naturally graded by assigning degree 1 to each variable  $x_i$ . The isomorphism (3.1.1) divides the degrees in half. From this isomorphism we can see that  $H_{\mathbb{T}^n}^*(M; \mathbb{C})$  only has elements of even degree.

Inspired by [GKM97], Gullemin and Zara [GZ99] studied the combinatorial properties of  $H_{\mathbb{T}^n}^*(\Gamma)$ , and showed that many familiar theorems in geometry have

natural counterparts in graph theory. In this part of the thesis, we take the opposite approach and study the combinatorics of GKM graphs to deduce properties of GKM manifolds.

For any  $n \geq 2$ , we may restrict the torus action  $\mathbb{T}^n \curvearrowright M$  to a smaller torus action  $\mathbb{T}^2 \curvearrowright M$  that is still GKM. So we will only consider the case  $n = 2$  in the following.

**Definition 3.1.2.** We say  $\phi : V \rightarrow \mathbb{R}^2$  is in *general position* if no three points in  $V$  are colinear. In particular, this implies  $\phi$  is injective. Assume  $\phi(v_i) = (z_{i1}, z_{i2})$ , we have a map  $\alpha : E \rightarrow \mathbb{C}[x, y]_1$  induced by  $\phi$  defined by

$$\alpha(e_{ij}) = (z_{j1} - z_{i1})x + (z_{j2} - z_{i2})y,$$

for any  $e_{ij} \in E$ . As we are dealing with an undirected graph,  $\alpha$  is only well-defined up to sign, but this ambiguity does not cause problem with the divisibility condition in (3.1.1).

The original GKM condition requires that  $\alpha(e_{ij})$  and  $\alpha(e_{ik})$  to be linearly independent for  $j \neq k$ . This is a weaker assumption than  $\phi$  being in general position. However in this part of the thesis, we always assume  $\phi$  to be in general position. This point is emphasized again in the following.

We assume the following setup for the remainder of the chapter.

**Setup 3.1.3.** Let  $\Gamma = (V, E)$  be a simple graph, and  $\phi : V \rightarrow \mathbb{R}^2$  a map in general position. For simplicity, we will always use  $m$  to stand for  $|V|$ . Let  $\alpha : E \rightarrow \mathbb{C}[x, y]_1$

be induced by  $\phi$ , and define

$$H_{\mathbb{T}}^*(\Gamma) = \left\{ (f_1, f_2, \dots, f_m) \in \bigoplus_{i=1}^m \mathbb{C}[x, y] \mid \alpha(e_{ij}) \mid f_i - f_j, \forall e_{ij} \in E \right\}.$$

We show in Section 3.2 that  $H_{\mathbb{T}}^*(\Gamma)$  is always a free module over  $\mathbb{C}[x, y]$  of dimension  $m$ . For any homogeneous basis of  $H_{\mathbb{T}}^*(\Gamma)$  as a module over  $\mathbb{C}[x, y]$ ,

$$\gamma_1, \gamma_2, \dots, \gamma_m,$$

we set  $c_i(\Gamma) = |\{j \in \mathbb{N} \mid \text{degree}(\gamma_j) = i\}|$  to be the number of degree  $i$  elements in the basis. Although the basis is not unique, the number of basis elements in a fixed degree is independent of the choice of the basis, hence  $c_i(\Gamma)$  is well-defined.

**Definition 3.1.4.** We call  $c_i(\Gamma)$  the  $i$ -th *characteristic number* of  $\Gamma$ .

These  $c_i$ 's will serve as the combinatorial counterpart for the geometric Betti numbers of the manifolds in this chapter. They are different from the combinatorial Betti numbers in [GZ99], whose definition will be given as follows.

Choose a *generic direction*  $\xi \in \mathbb{R}^2$ , namely one that ensures  $\phi(v_i) \cdot \xi \neq \phi(v_j) \cdot \xi$  for  $i \neq j$ . For any  $v_i \in V$ , define the index of  $v_i$  to be

$$\sigma(v_i) = \left| \{v_j \in V \mid e_{ij} \in E, \phi(v_j) \cdot \xi < \phi(v_i) \cdot \xi\} \right|.$$

The  $k$ -th (*combinatorial*) *Betti number* of  $\Gamma$  is defined as the number of vertices of index  $k$  and denoted by  $\beta_k(\Gamma)$ . Guillemin and Zara showed in [GZ99] that with a fixed *axial function*, although the indices do depend on the choice of  $\xi$ , the combinatorial Betti numbers do not. We do not need axial function in this thesis, hence

will not give the definition here. Interested readers are referred to [GZ99] for more details.

When  $\Gamma$  is the GKM graph of a manifold, then these three notions, characteristic numbers, combinatorial Betti numbers and geometric Betti numbers, all agree. Guillemin and Zara introduced the  $\beta_i$ 's as the combinatorial counterpart of geometric Betti numbers [GZ99]. They do exhibit some properties that resemble the geometric ones, including Poincare duality. But there are certain shortfalls. First of all, they are well-defined only for a very restrictive class of graphs: regular graphs with axial functions. This makes inductive arguments difficult. Secondly, for a connected regular graph equipped with an axial function, one would expect  $\beta_0$  to be 1. This is not the case, as the following simple example shows.

**Example 3.1.5.** Consider a cycle on five vertices, mapped into  $\mathbb{R}^2$  as in the following figure. The axial function on  $\Gamma$  is induced by the given embedding. The arrow

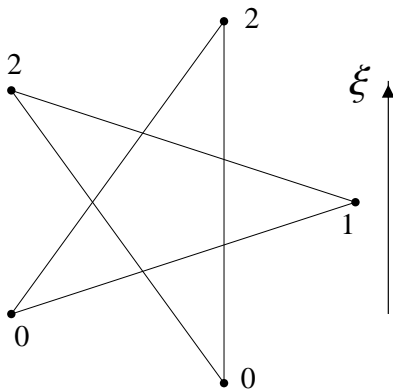


Figure 3.1: A regular graph with axial function



points in the  $\xi$  direction, and the number beside a vertex indicates the index of the that vertex. We can see although Poincare duality still holds, the graph has the undesirable property that  $\beta_0 = 2$ .

In contrast, the characteristic numbers are defined in a much more general setting, and  $c_0$  is equal to the number of connected components of the graph. In Section 3.3, we study the basic properties of characteristic numbers, and give comparisons with the geometric Betti numbers. In the remaining two sections, we study the properties of graphs that are in fact the GKM graphs of Hamiltonian GKM manifolds. In Section 3.4, using our tools of characteristic numbers and the fact that top Betti number of a compact oriented manifold is 1, we deduce some connectivity properties of (actual) GKM graphs, under the assumption that the moment map is in general position.

Results in Section 3.5 is the goal for developing all these tools. The section is largely motivated by a question posed by Yael Karshon.

**Question 3.1.6.** *[JHK<sup>+</sup>05, Problem 4.2] Suppose that a symplectic manifold  $(M, \omega)$  admits a Hamiltonian  $S^1$  action with isolated fixed points. Does  $(M, \omega)$  satisfy the hard Lefschetz property?*

Susan Tolman pointed out an easier version of the question, which we have simplified to our setting.

**Question 3.1.7.** *[JHK<sup>+</sup>05, Special case of Problem 4.3] Suppose  $M$  is a Hamiltonian*

GKM manifold of dimension  $2d$ , do the Betti numbers of  $M$  satisfy

$$\beta_0 \leq \beta_2 \leq \cdots \leq \beta_{2\lfloor \frac{d}{2} \rfloor},$$

where  $2\lfloor \frac{d}{2} \rfloor$  is the largest even integer no greater than  $d$ ?

Using the tools and results we develop in Section 3.3 and 3.4, in Section 3.5 we give an upper bound for  $\beta_2$  of Hamiltonian GKM manifold whose moment map is in general position. This is the content of Theorem 3.5.1. In dimensions 8 and 10, this implies that the even Betti numbers of these manifolds must be non-decreasing up to half the dimension. So our work provides a class of examples for which there is a positive answer to Tolman's question. We hope this framework can be developed to give further results in higher dimensions.

## 3.2 Graph cohomology is a free module in dimension two

The main theorem in this section, Theorem 3.2.1, holds in a much more general setting than we need in the rest of the chapter. In particular, it does not require the existence of the map  $\phi$  and there is no constraint on the map  $\alpha$ . The proof uses some basic techniques from algebraic geometry. The corresponding statement in higher dimension does not hold.

**Theorem 3.2.1.** *Given  $\Gamma = (V, E)$  a simple graph and  $\alpha : E \rightarrow \mathbb{C}[x, y]_1$ , then  $H_{\mathbb{T}}^*(\Gamma) \cong (\mathbb{C}[x, y])^m$  as modules over  $\mathbb{C}[x, y]$ , where  $m = |V|$ .*

*Proof.* Let  $M = H_{\mathbb{T}}^*(\Gamma)$ . As a graded module over  $\mathbb{C}[x, y]$ ,  $M$  defines a quasicoherent sheaf over  $\text{Proj } \mathbb{C}[x, y] = \mathbb{C}P^1$ . We denote this sheaf by  $\mathcal{F}$ . As a module over  $\mathbb{C}[x, y]$ ,  $M$  also defines a quasicoherent sheaf over  $\text{Spec } \mathbb{C}[x, y] = \mathbb{C}^2$ . We denote this sheaf by  $\mathcal{G}$ . The restriction of  $\mathcal{G}$  to  $\mathbb{C}^2 \setminus \{0\}$  will be denoted by  $\mathcal{H}$ , which is a quasicoherent sheaf over  $\mathbb{C}^2 \setminus \{0\}$ .

**Lemma 3.2.2.** *The sheaf  $\mathcal{F}$  is locally free.*

*Proof of Lemma 3.2.2.*  $\mathbb{C}P^1$  is covered by  $D(x) = \text{Spec } \mathbb{C}[\frac{y}{x}]$  and  $D(y) = \text{Spec } \mathbb{C}[\frac{x}{y}]$ . Now that  $\mathcal{F}(D(x)) = (M_x)_0$ , the degree 0 part of the localized module  $M_x$ , it follows from the definition of  $M$  that  $\mathcal{F}(D(x))$  is a torsion free  $\mathbb{C}[\frac{y}{x}]$ -module, hence a free  $\mathbb{C}[\frac{y}{x}]$ -module. Similarly  $\mathcal{F}(D(y))$  is a free  $\mathbb{C}[\frac{x}{y}]$ -module.  $\square$

**Lemma 3.2.3.** *Let  $\pi : \mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{C}P^1$  be the natural projection. Then  $\pi^* \mathcal{F} = \mathcal{H}$ .*

*Proof of Lemma 3.2.3.* This can be proved by looking at the distinguished open subsets of both spaces. The space  $\mathbb{C}^2 \setminus \{0\}$  is covered by  $\text{Spec } \mathbb{C}[x, \frac{1}{x}, y]$  and  $\text{Spec } \mathbb{C}[x, y, \frac{1}{y}]$ , while  $\mathbb{C}P^1$  is covered by  $\text{Spec } \mathbb{C}[\frac{y}{x}]$  and  $\text{Spec } \mathbb{C}[\frac{x}{y}]$ .

We have the following natural isomorphism

$$(M_x)_0 \otimes_{\mathbb{C}[\frac{y}{x}]} \mathbb{C}[x, \frac{1}{x}, y] = (M_x)_0 \otimes_{\mathbb{C}[\frac{y}{x}]} (\mathbb{C}[\frac{y}{x}] \otimes_{\mathbb{C}} \mathbb{C}[x, \frac{1}{x}]) = (M_x)_0 \otimes_{\mathbb{C}} \mathbb{C}[x, \frac{1}{x}] = M_x.$$

Similarly

$$(M_y)_0 \otimes_{\mathbb{C}[\frac{x}{y}]} \mathbb{C}[x, y, \frac{1}{y}] = M_y.$$

These exactly say  $\pi^* \mathcal{F} = \mathcal{H}$ .  $\square$

As an immediate consequence,  $\mathcal{H}$  is also locally free. By a theorem of Grothendieck [Gro57],  $\mathcal{F}$  splits as a direct sum of line bundles (the corresponding statement fails for locally free sheaves over  $\mathbb{C}P^2$ , this is the key reason why graph cohomology is special in dimension two). So  $\mathcal{H}$  is also a direct sum of line bundles, but the Picard group of  $\mathbb{C}^2 \setminus \{0\}$  is trivial, so  $\mathcal{H}$  must be a trivial vector bundle. In particular,  $\mathcal{H}(\mathbb{C}^2 \setminus \{0\})$  must be a free module over  $\mathcal{O}_{\mathbb{C}^2 \setminus \{0\}}(\mathbb{C}^2 \setminus \{0\}) = \mathbb{C}[x, y]$ .

Now  $\mathcal{G}(\mathbb{C}^2 \setminus \{0\}) = \mathcal{H}(\mathbb{C}^2 \setminus \{0\})$  and  $\mathcal{G}(\mathbb{C}^2) = M$ , so the following lemma will enable us to conclude that  $M$  is a free  $\mathbb{C}[x, y]$ -module.

**Lemma 3.2.4.** *The restriction map  $r : \mathcal{G}(\mathbb{C}^2) \rightarrow \mathcal{G}(\mathbb{C}^2 \setminus \{0\})$  is an isomorphism.*

*Proof of Lemma 3.2.4.* We can think of  $\mathcal{G}(\mathbb{C}^2 \setminus \{0\})$  as  $\bigcap_{a \neq 0 \text{ or } b \neq 0} M_{ax+by}$ . The intersection makes sense since  $M$  is torsion-free by definition and thus  $M_{ax+by}$  can be thought of as a subset of  $M_{(0)} = \mathbb{C}(x, y)^m$ , the localization of  $M$  at the zero prime ideal.

Assume  $(f_1, \dots, f_m) \in \mathcal{G}(\mathbb{C}^2 \setminus \{0\}) = \bigcap_{a \neq 0 \text{ or } b \neq 0} M_{ax+by} \subseteq \mathbb{C}(x, y)^m$ . Then we immediately see that  $f_i \in \mathbb{C}[x, y]$  for all  $i$ . Since the graph is finite, we can pick a non-zero linear polynomial  $px + qy$ , such that it is not a multiple of any  $\alpha(e_{ij})$ . Then  $(f_1, \dots, f_m) \in M_{px+qy}$  says that there exists  $n$ , such that

$$\alpha(e_{ij}) \Big| (px + qy)^n (f_i - f_j)$$

for all  $e_{ij} \in E$ . So

$$\alpha(e_{ij}) \Big| (f_i - f_j),$$

which says exactly  $(f_1, \dots, f_m) \in M$ . □

Now we know  $M$  is a free  $\mathbb{C}[x, y]$ -module. To see its dimension, let  $f = \prod_{e_{ij} \in E} \alpha(e_{ij})$ . The dimension of  $M$  as module over  $\mathbb{C}[x, y]$  equals the dimension of  $M_f$  as a module over  $\mathbb{C}[x, y]_f$ . But  $M_f$  is generated by  $(1, 0, \dots, 0), (0, 1, \dots, 0), \dots, (0, 0, \dots, 1)$  as a  $\mathbb{C}[x, y]_f$ -module. So  $\dim M = m$ . This completes the proof of the theorem. □

The similar result does not hold in higher dimensions in general, even if we assume  $\Gamma$  is a regular graph and  $\alpha$  is induced from a map  $\phi : V \rightarrow \mathbb{R}^n$ . An easy counterexample is the following.

**Example 3.2.5.** Consider  $\Gamma = (V, E)$  given by  $V = \{v_1, v_2, v_3, v_4\}$  and  $E = \{e_{12}, e_{23}, e_{34}, e_{14}\}$ . Note that  $\Gamma$  is a regular graph of degree 2. Define  $\phi : V \rightarrow \mathbb{R}^3$  by

$$\phi(v_1) = (0, 0, 0), \phi(v_2) = (1, 0, 0), \phi(v_3) = (1, 1, 0), \phi(v_4) = (1, 1, 1).$$

Then  $\phi$  induces  $\alpha : E \rightarrow \mathbb{C}[x, y, z]_1$  given by

$$\alpha(e_{12}) = x, \alpha(e_{23}) = y, \alpha(e_{34}) = z, \alpha(e_{14}) = x + y + z.$$

Then  $H_{\mathbb{T}^3}^*(\Gamma, \alpha)$  as a module over  $\mathbb{C}[x, y, z]$  is generated by

$$(1, 1, 1, 1), (0, x, x + y, x + y + z), (0, xy, 0, 0), (0, 0, yz, 0), \text{ and } (0, xz, xz, 0).$$

This is not free as a  $\mathbb{C}[x, y, z]$ -module, since the generators are related by

$$y(0, xz, xz, 0) = z(0, xy, 0, 0) + x(0, 0, yz, 0).$$

This naturally leads to the following open question.

**Question 3.2.6.** *Given a regular graph  $\Gamma = (V, E)$  together with an axial function  $\alpha : E \rightarrow \mathbb{C}[x_1, \dots, x_n]_1$  in the sense of definition 1 in 2.1 in [GZ99], is the graph cohomology necessarily a free module over  $\mathbb{C}[x_1, \dots, x_n]$ ? If not, what are the combinatorial criteria on  $(\Gamma, \alpha)$  that guarantee  $H_{\mathbb{T}^n}^*(\Gamma, \alpha)$  to be free?*

### 3.3 Properties of characteristic numbers

In this section we study some basic properties of characteristic numbers, defined in Definition 3.1.4. These numbers do not satisfy Poincare duality in general, but we will prove a weaker version of it. We will discuss how the “top characteristic number” behaves for regular graphs. Then we discuss their relationship to the combinatorial Betti numbers. Next, we will compute the characteristic numbers for complete graphs, which are (graph theoretically) GKM graphs for complex projective spaces. Finally, we prove an analogue of Künneth formula.

**Notation 3.3.1.** Continuing from Setup 3.1.3, the vertices of  $\Gamma$  are labeled as  $v_1, v_2, \dots, v_m$ . The edge connecting  $v_i$  and  $v_j$  will be denoted by  $e_{ij}$ . We do not distinguish between  $e_{ij}$  and  $e_{ji}$ , but most of the time we will use the smaller number as the first index. We will use  $\lambda(v_i)$  to denote the degree of  $v_i$ , i.e., the number of edges incident to  $v_i$ . We use  $\lambda_\Gamma(v_i)$  to denote the degree of  $v_i$  when we want to emphasize  $v_i$  as a vertex of  $\Gamma$ .

The characteristic number  $c_i$  is not affected by composing  $\phi : V \rightarrow \mathbb{R}^2$  with a

linear automorphism of  $\mathbb{R}^2$ , so without loss of generality, we may assume  $\phi(v_i)$  and  $\phi(v_j)$  have different second component for any  $i \neq j$ . Also we notice the definition of graph cohomology remains unchanged if we scale  $\alpha$ , so if we assume  $\phi(v_i) = (p_i, q_i)$ ,  $\phi(v_j) = (p_j, q_j)$  and let  $a_{ij} = -\frac{p_j - p_i}{q_j - q_i}$ , we may redefine  $\alpha$  as

$$\alpha(e_{ij}) = y - a_{ij}x.$$

For any positive integers  $p \leq q$ , we will use  $\mathbf{b}_p^q$  to denote the  $p$ -th standard basis vector in  $\mathbb{C}^q$ , i.e., the  $p$ -th entry of  $\mathbf{b}_p^q$  is 1 and that is the only non-zero entry.

For  $1 \leq i < j \leq m$ , we let

$$\begin{aligned} \mathbf{v}_{ij}^0 &= \mathbf{b}_i^m - \mathbf{b}_j^m = (0, \dots, 1, 0, \dots, -1, 0, \dots, 0) \in \mathbb{C}^m; \\ \mathbf{v}_{ij}^1 &= \mathbf{b}_i^{2m} - \mathbf{b}_j^{2m} + a_{ij}\mathbf{b}_{i+m}^{2m} - a_{ij}\mathbf{b}_{j+m}^{2m} \\ &= (0, \dots, 1, 0, \dots, -1, 0, \dots, 0, 0, \dots, a_{ij}, 0, \dots, -a_{ij}, 0, \dots, 0) \\ &= (\mathbf{v}_{ij}^0, a_{ij}\mathbf{v}_{ij}^0) \in \mathbb{C}^{2m}; \\ \mathbf{v}_{ij}^2 &= \mathbf{b}_i^{3m} - \mathbf{b}_j^{3m} + a_{ij}\mathbf{b}_{i+m}^{3m} - a_{ij}\mathbf{b}_{j+m}^{3m} + a_{ij}^2\mathbf{b}_{i+2m}^{3m} - a_{ij}^2\mathbf{b}_{j+2m}^{3m} \\ &= (\mathbf{v}_{ij}^1, a_{ij}^2\mathbf{v}_{ij}^0) \in \mathbb{C}^{3m}; \end{aligned}$$

and so forth, where we use  $(\mathbf{u}, \mathbf{v})$  to denote concatenation of two vectors  $\mathbf{u}$  and  $\mathbf{v}$ .

In general,

$$\begin{aligned} \mathbf{v}_{ij}^k &= \mathbf{b}_i^{(k+1)m} - \mathbf{b}_j^{(k+1)m} + a_{ij}\mathbf{b}_{i+m}^{(k+1)m} - a_{ij}\mathbf{b}_{j+m}^{(k+1)m} + \dots + a_{ij}^k\mathbf{b}_{i+km}^{(k+1)m} - a_{ij}^k\mathbf{b}_{j+km}^{(k+1)m} \\ &= (\mathbf{v}_{ij}^{k-1}, a_{ij}^k\mathbf{v}_{ij}^0) \in \mathbb{C}^{(k+1)m}. \end{aligned}$$

Denote by  $M_k(\Gamma)$  the matrix of size  $|E| \times (k+1)m$ , whose rows are indexed by  $E$  and the row corresponding to  $e_{ij}$  is  $\mathbf{v}_{ij}^k$ . Let  $r_k(\Gamma) = \text{rank}M_k(\Gamma)$ . This is

the dimension of the vector space spanned by the vectors  $\{\mathbf{v}_{ij}^k | e_{ij} \in E\}$ . We set  $r_{-1}(\Gamma) = 0$ . Let  $s_k(\Gamma) = |E| - r_k(\Gamma)$ . This is the dimension of the vector space of linear relations among the vectors  $\{\mathbf{v}_{ij}^k | e_{ij} \in E\}$ . We set  $s_{-1}(\Gamma) = |E|$ .

Throughout the rest of the chapter, we will use the following notation as introduced in Setup 3.1.3 and just above:

$$\begin{aligned} & \Gamma, V, E, m = |V|; \\ & v_1, v_2, \dots, v_m, \lambda(v_i), \lambda_\Gamma(v_i), e_{ij}, \text{ for } 1 \leq i, j \leq m; \\ & \phi, \alpha, a_{ij}, \mathbf{v}_{ij}^k, \text{ for } 1 \leq i, j \leq m \text{ and } k \geq 0; \\ & M_k(\cdot), c_k(\cdot) \text{ for } k \geq 0; r_k(\cdot), s_k(\cdot), \text{ for } k \geq -1, \end{aligned}$$

where  $\cdot$  in the last row can denote any graph in the sense of Setup 3.1.3.

**Remark 3.3.2.** If we make  $\Gamma$  into a directed graph by letting each edge in  $E$  leave the vertex of smaller subscript and enters the vertex of larger subscript, then  $M_0(\Gamma)$  is the transpose of the *incidence matrix* of this directed graph. The matrix  $M_1(\Gamma)$  is closely related to the *rigidity matrix* of the graph  $\Gamma$  and the map  $\phi$ , which is an important concept in rigidity theory. More information about the rigidity matrix and rigidity theory can be found in [GSS93]. The work in this chapter is closely related to some work in rigidity theory by the author [Luo12a, Luo12b]. We do not include those work in this thesis. But the connection between these two subjects is fascinating and it would very interesting to see more exploitation of it.

**Example 3.3.3.** Figure 3.2 shows a graph on four vertices. We have marked the coordinates of the image of  $\phi$  and of the image of  $\alpha$ . The edge set is



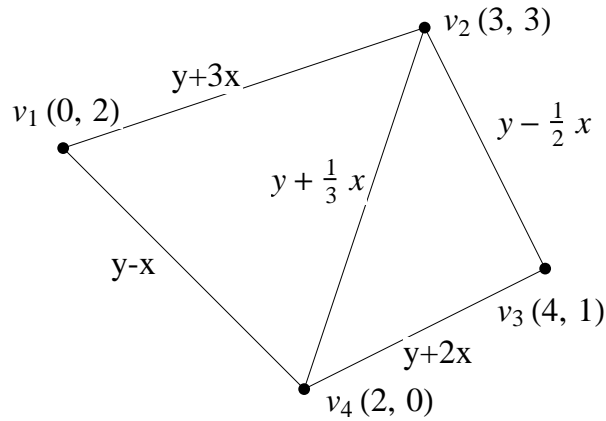


Figure 3.2: An example of  $\Gamma, \phi$  and  $\alpha$

$E = \{e_{12}, e_{14}, e_{23}, e_{24}, e_{34}\}$ . So  $M_0(\Gamma)$  is a  $5 \times 4$  matrix with rows indexed by  $E$  and columns indexed by  $V$ . In this example, we have

$$M_0(\Gamma) = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left( \begin{array}{cccc} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array} \right) \end{array}.$$

We can compute  $M_k(\Gamma)$  by recursively adding columns. As labeled in Figure 3.2,

we can see  $a_{12} = -3, a_{14} = 1, a_{23} = \frac{1}{2}, a_{24} = -\frac{1}{3}$  and  $a_{34} = -2$ , so  $M_2(\Gamma)$  is

$$M_2(\Gamma) = \begin{matrix} & v_1^0 & v_2^0 & v_3^0 & v_4^0 & v_1^1 & v_2^1 & v_3^1 & v_4^1 & v_1^2 & v_2^2 & v_3^2 & v_4^2 \\ \begin{matrix} e_{12} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{matrix} & \left( \begin{array}{cccccccccccc} 1 & -1 & 0 & 0 & a_{12} & -a_{12} & 0 & 0 & a_{12}^2 & -a_{12}^2 & 0 & 0 \\ 1 & 0 & 0 & -1 & a_{14} & 0 & 0 & -a_{14} & a_{14}^2 & 0 & 0 & 0 & -a_{14}^2 \\ 0 & 1 & -1 & 0 & 0 & a_{23} & -a_{23} & 0 & 0 & a_{23}^2 & -a_{23}^2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & a_{24} & 0 & -a_{24} & 0 & a_{24}^2 & 0 & 0 & -a_{24}^2 \\ 0 & 0 & 1 & -1 & 0 & 0 & a_{34} & -a_{34} & 0 & 0 & a_{34}^2 & -a_{34}^2 & 0 \end{array} \right) \end{matrix}.$$

A direct computation shows that  $r_0(\Gamma) = 3$  and  $r_k(\Gamma) = 5$  for all  $k \geq 1$ . It follows that  $s_0(\Gamma) = 2$  and  $s_k(\Gamma) = 0$  for all  $k \geq 1$ .

**Proposition 3.3.4.** *The characteristic numbers may be computed as follows:*

$$(3.3.5) \quad c_0(\Gamma) = \pi_0(\Gamma), \text{ the number of connected components of } \Gamma,$$

$$(3.3.6) \quad = m - r_0(\Gamma) = m - |E| + s_0(\Gamma);$$

$$(3.3.7) \quad c_k(\Gamma) = 2r_{k-1}(\Gamma) - r_k(\Gamma) - r_{k-2}(\Gamma)$$

$$(3.3.8) \quad = s_k(\Gamma) + s_{k-2}(\Gamma) - 2s_{k-1}(\Gamma), \forall k \geq 1.$$

*Proof.* Let  $f = (f_1, f_2, \dots, f_m) \in \mathbb{C}^m$ . Then it follows from the definition of graph cohomology that  $f$  is a class in  $H_{\mathbb{T}}^0(\Gamma)$  if and only if  $f_i = f_j \in \mathbb{C}$  whenever  $v_i$  and  $v_j$  are in the same connected component of  $\Gamma$ . So  $c_0$  equals to the number of connected components of  $\Gamma$ , which we denote by  $\pi_0(\Gamma)$ . This proves (3.3.5).

For any  $k \geq 0$ , assume

$$f = \left( \sum_{n=0}^k z_{1n} x^{k-n} y^n, \sum_{n=0}^k z_{2n} x^{k-n} y^n, \dots, \sum_{n=0}^k z_{mn} x^{k-n} y^n \right) \in H_{\mathbb{T}}^k(\Gamma),$$

where  $z_{st} \in \mathbb{C}$  for  $1 \leq s \leq m, 0 \leq t \leq k$ . Then by the definition of graph cohomology, this is equivalent to

$$(y - a_{ij}x) \left| \left( \sum_{n=0}^k z_{in} x^{k-n} y^n \right) - \left( \sum_{n=0}^k z_{jn} x^{k-n} y^n \right) \right|,$$

for each  $e_{ij}$  in  $E$ . This in turn means when we substitute  $y$  with  $a_{ij}x$  in

$$\left( \sum_{n=0}^k z_{in} x^{k-n} y^n \right) - \left( \sum_{n=0}^k z_{jn} x^{k-n} y^n \right),$$

we should get the zero polynomial:

$$\begin{aligned} 0 &= \left( \sum_{n=0}^k a_{ij}^n z_{in} x^k \right) - \left( \sum_{n=0}^k a_{ij}^n z_{jn} x^k \right) \\ &= \left( \sum_{n=0}^k a_{ij}^n (z_{in} - z_{jn}) \right) x^k \end{aligned}$$

So  $\sum_{n=0}^k a_{ij}^n (z_{in} - z_{jn}) = 0$ , and this expression can be rewritten as

$$\mathbf{v}_{ij}^k \cdot (z_{10}, z_{20}, \dots, z_{m0}, z_{11}, z_{21}, \dots, z_{m1}, z_{12}, \dots, z_{1k}, z_{2k}, \dots, z_{mk}) = 0.$$

Therefore

$$\dim H_{\mathbb{T}}^k(\Gamma) = (k+1)m - r_k(\Gamma).$$

By definition of  $c_i$ 's, we have

$$\dim H_{\mathbb{T}}^k(\Gamma) = \sum_{n=0}^k (k+1-n)c_n,$$

and so

$$(3.3.9) \quad \sum_{n=0}^k (k+1-n)c_n = (k+1)m - r_k(\Gamma).$$

In particular, when  $k = 0$ , this immediately simplifies to

$$c_0 = m - r_0,$$

proving (3.3.6).

The formula (3.3.7) for  $c_k$  for  $k \geq 1$  can be proved inductively. For the base case  $k = 1$ , we have by equation (3.3.9),

$$\begin{aligned} c_1 &= 2m - r_1 - 2c_0 \\ &= 2m - r_1 - 2(m - r_0) \\ &= 2r_0 - r_1 \\ &= 2r_0 - r_1 - r_{-1} \\ &= s_1 + s_{-1} - 2s_0. \end{aligned}$$

Now for  $k > 1$ , we assume the formula (3.3.7) holds for smaller  $k$ . Then by equation (3.3.9), we have

$$\begin{aligned} c_k &= (k+1)m - r_k - \sum_{n=0}^{k-1} (k+1-n)c_n \\ &= (k+1)m - r_k - \sum_{n=1}^{k-1} (k+1-n)(2r_{n-1} - r_{n-2} - r_n) - (k+1)(m - r_0). \end{aligned}$$

We easily simplify this to conclude that

$$c_k = 2r_{k-1} - r_k - r_{k-2} = s_k + s_{k-2} - 2s_{k-1},$$

as desired. □

**Example 3.3.10.** We continue with Example 3.3.3. Applying Proposition 3.3.4 to this example gives  $c_0(\Gamma) = 1$ ,  $c_1(\Gamma) = 1$  and  $c_2(\Gamma) = 2$ .

**Notation 3.3.11.** For any graph  $\Gamma = (V, E)$  and  $v \in V$ , we write  $E_v$  for the set of edges incident to  $v$  and write  $\Gamma - v$  for the subgraph of  $\Gamma$  obtained by deleting the vertex  $v$ . We obviously have  $\Gamma - v = (V \setminus \{v\}, E \setminus E_v)$ . Similarly, for any subset  $U \subseteq V$ , we write  $E_U$  for the set of edges incident to some vertex in  $U$  and write  $\Gamma - U$  for  $(V \setminus U, E \setminus E_U)$ .

The following lemma will be used repeatedly throughout the chapter. We call it the *Deletion Lemma*. This concerns how the  $s_k$ 's, hence the characteristic numbers, behave when we delete a vertex from a graph.

**Lemma 3.3.12 (Deletion Lemma).** *Assume a vertex  $v_t \in V$  is of degree less than or equal to  $k + 1$ , i.e.,  $\lambda(v_t) \leq k + 1$ . Then  $s_k(\Gamma) = s_k(\Gamma - v_t)$ .*

*Proof.* For simplicity, we write  $\lambda(v_t) = d$ . Without loss of generality, we may assume  $t = 1$  and the  $d$  edges incident to  $v_t$  are  $e_{12}, e_{13}, \dots, \text{and } e_{1,d+1}$ . Now assume there is a linear relation among  $\{\mathbf{v}_{ij}^k \mid e_{ij} \in E\}$ :

$$(3.3.13) \quad \sum_{e_{ij} \in E} u_{ij} \mathbf{v}_{ij}^k = \mathbf{0}.$$

We now consider the sub matrix of  $M_k(\Gamma)$  defined by the first  $d$  rows, and columns

$1, m + 1, \dots, km + 1$ :

$$N = \begin{pmatrix} 1 & a_{12} & a_{12}^2 & \cdots & a_{12}^k \\ 1 & a_{13} & a_{13}^2 & \cdots & a_{13}^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{1,d+1} & a_{1,d+1}^2 & \cdots & a_{1,d+1}^k \end{pmatrix} = \begin{pmatrix} \mathbf{x}_{12}^k \\ \mathbf{x}_{13}^k \\ \vdots \\ \mathbf{x}_{1,d+1}^k \end{pmatrix},$$

where  $\mathbf{x}_{1,i}^k = (1, a_{1i}, \dots, a_{1i}^k)$  is the vector consisting of the  $1, m + 1, \dots, (km + 1)$ th entries of  $\mathbf{v}_{ij}^k$ . This is a Vandermonde matrix. The assumption of  $\phi$  in general position guarantees that  $a_{1i} \neq a_{1j}$  for  $i \neq j$ , so the matrix  $N$  has full row rank. For the columns  $1, m + 1, \dots, (km + 1)$  of  $M_k(\Gamma)$ , the only nonzero entries are in the first  $d$  rows, so it follows from (3.3.13) that

$$\sum_{j=2}^{d+1} u_{1j} \mathbf{x}_{1j}^k = \mathbf{0}.$$

We conclude that  $u_{1j} = 0$  for all  $2 \leq j \leq d + 1$ . Therefore (3.3.13) reduces to

$$\sum_{e_{ij} \in E \setminus E_{v_t}} u_{ij} \mathbf{v}_{ij}^k = \mathbf{0}.$$

This means (3.3.13) is in fact a linear relation among  $\{\mathbf{v}_{ij}^k | e_{ij} \in E \setminus E_{v_t}\}$ . So  $s_k(\Gamma) = s_k(\Gamma - v_t)$ .  $\square$

**Corollary 3.3.14.** *We may now bound the difference between  $s_k(\Gamma)$  and  $s_k(\Gamma - v_t)$ ,*

$$0 \leq s_k(\Gamma) - s_k(\Gamma - v_t) \leq \max(\lambda(v_t) - k - 1, 0).$$

*Proof.* It immediately follows from the Deletion Lemma 3.3.12.  $\square$

**Lemma 3.3.15.** *Let  $\Gamma = (V, E)$ , and let  $k \geq \max_i \lambda(v_i) - 1$ . Then  $r_k(\Gamma) = |E|$  and  $s_k(\Gamma) = 0$ . Consequently,  $c_{k+2}(\Gamma) = 0$ .*

*Proof.* The repeated application of the Deletion Lemma will give  $s_k(\Gamma) = 0$ . Hence  $r_k(\Gamma) = |E| - s_k(\Gamma) = |E|$  and it follows from Proposition 3.3.4 that  $c_{k+2} = s_{k+2} - 2s_{k+1} + s_k = 0$ .  $\square$

**Proposition 3.3.16.** *For any  $\Gamma = (V, E)$ , we have*

$$(3.3.17) \quad \sum_{i=0}^{\infty} c_i(\Gamma) = m = |V|,$$

and

$$(3.3.18) \quad \sum_{i=0}^{\infty} i c_i(\Gamma) = |E|.$$

*Proof.* Equation (3.3.17) is already proved in Theorem 3.2.1. Turning to (3.3.18), according to Lemma 3.3.15, we can choose  $k \in \mathbb{N}$  such that  $r_k(\Gamma) = |E|$  and  $c_i = 0$  for all  $i \geq k + 1$ . Then by formula (3.3.9),

$$\sum_{i=0}^k (k+1-i)c_i = (k+1)m - r_k = (k+1)m - |E|.$$

So

$$\sum_{i=0}^{\infty} i c_i = \sum_{i=0}^k i c_i = \sum_{i=0}^k (k+1)c_i - (k+1)m + |E| = |E|.$$

This completes the proof.  $\square$

**Remark 3.3.19.** When  $\Gamma$  is a regular graph of degree  $d$ , then it follows from Proposition 3.3.16 that the average degree of a set of generators of  $H_{\mathbb{T}}^*(\Gamma)$  is  $\frac{d}{2}$ . In this sense, the proposition provides a weak version of Poincare duality.

**Proposition 3.3.20.** *Assume  $\Gamma = (V, E)$  is a connected regular graph of degree  $d$ , then  $c_i = 0$  for  $i > d$  and  $c_d = 0$  or  $1$ .*

*Proof.* The first part of the proposition is just a special case of Lemma 3.3.15.

As for the second part, pick any edge  $e_{ij}$  and let  $\Gamma' = (V, E \setminus \{e_{ij}\})$ . Then we have  $s_{d-2}(\Gamma) \leq s_{d-2}(\Gamma') + 1$ . By repeatedly applying the Deletion Lemma to  $\Gamma'$ , we get  $s_{d-2}(\Gamma') = 0$ . So  $s_{d-2}(\Gamma) \leq 1$ . By Lemma 3.3.15, we have  $s_{d-1}(\Gamma) = s_d(\Gamma) = 0$ . So we conclude

$$c_d(\Gamma) = s_d(\Gamma) + s_{d-2}(\Gamma) - 2s_{d-1}(\Gamma) \leq 1,$$

as desired. □

**Remark 3.3.21.** As the GKM graph of a GKM manifold is a regular graph, Proposition 3.3.20 can be viewed as a weaker version of the fact that the top geometric Betti number of a connected compact oriented manifold is 1. If the connected regular graph  $\Gamma$  of degree  $d$  is also equipped with an axial function in the sense of [GZ99], then one can show by [GZ99, Theorem 2.2] that  $c_d(\Gamma) = 1$  and  $(\prod_{e_{1j} \in E} \alpha(e_{1j}), 0, 0, \dots, 0)$  can be taken as the generator of  $H_{\mathbb{T}}^*(\Gamma)$  in degree  $d$ .

**Example 3.3.22.** Figure 3.3 shows a regular graph of degree 2 whose vertices are in general position. By Proposition 3.3.20,  $c_i = 0$  for  $i \geq 3$ . Also we know  $c_0 = 1$  as the graph is connected. Then the weaker version of Poincare duality, Proposition 3.3.16, forces  $c_2$  to be 1, hence  $c_1 = 3$ .

In general, for regular graph of degree 2 and  $m$  vertices in general position, we have  $c_0 = c_2 = 1$  and  $c_1 = m - 2$ .

Next we study the relations between the characteristic numbers and the combinatorial Betti numbers. Proposition 3.3.23 provides a generalization of [GZ99,



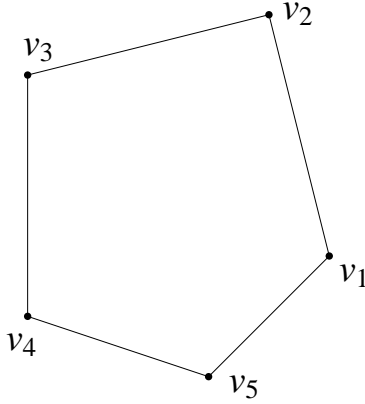


Figure 3.3: A regular graph of degree 2

inequality (2.28)] in dimension two.

**Proposition 3.3.23.** *Given  $\Gamma = (V, E)$ , order the vertices as  $v_1, \dots, v_m$ . Define the index of  $v_i$  as*

$$\mu_i = \left| \{j \in \mathbb{N} \mid j > i \text{ and } e_{ij} \in E\} \right|,$$

and let

$$b_k = \left| \{i \in \mathbb{N} \mid \mu_i = k\} \right|.$$

Then

$$\sum_{i=0}^k (k+1-i)c_i \leq \sum_{i=0}^k (k+1-i)b_i,$$

for any  $k \geq 0$ .

*Proof.* First observe that  $\sum_{i=0}^{\infty} b_i = |V| = m$  because each vertex is counted exactly once. Second, we claim that  $\sum_{i=0}^{\infty} ib_i = |E|$ . To see this, we count the edge set in

the following way. First we look at a vertex  $v_i$  with  $\mu_i = 0$ , and count how many edges connect  $v_i$  to a vertex  $v_j$  whose subscript  $j$  is larger than  $i$ . The answer must be  $\mu_i = 0$  by the definition of  $\mu_i$ . Then we look at a vertex  $v_i$  with  $\mu_i = 1$ . In this case, there should be one edge connecting  $v_i$  to some vertex of larger subscript. We repeat this process until we have made the count for every vertex. Then altogether we will have counted  $\sum_{i=0}^{\infty} ib_i$  many edges. On the other hand, we see that every edge in  $E$  is counted once and only once. So  $\sum_{i=0}^{\infty} ib_i = |E|$ .

Let  $\Gamma' = \Gamma - v_1$ . Applying Corollary 3.3.14 to  $v_1$ , we get the bound

$$s_k(\Gamma) \leq s_k(\Gamma') + \max(\mu_1 - (k + 1), 0).$$

Applying Corollary 3.3.14 to  $v_2, v_3, \dots, v_m$  in order, we get

$$\begin{aligned} s_k(\Gamma) &\leq \sum_{j=1}^m \max(\mu_j - (k + 1), 0) \\ &= \sum_{i=k+2}^{\infty} (i - (k + 1))b_i \\ &= \sum_{i=0}^{\infty} (i - (k + 1))b_i - \sum_{i=0}^k (i - (k + 1))b_i \\ &= \sum_{i=0}^{\infty} ib_i - (k + 1) \sum_{i=0}^{\infty} b_i + \sum_{i=0}^k (k + 1 - i)b_i \\ &= |E| - (k + 1)m + \sum_{i=0}^k (k + 1 - i)b_i. \end{aligned}$$

So

$$\begin{aligned}
\sum_{i=0}^k (k+1-i)c_i &= (k+1)m - r_k \\
&= (k+1)m - (|E| - s_k) \\
&\leq (k+1)m - |E| + |E| - (k+1)m + \sum_{i=0}^k (k+1-i)b_i \\
&= \sum_{i=0}^k (k+1-i)b_i,
\end{aligned}$$

where the first equality is equation (3.3.9). □

We next compute the characteristic numbers of a complete graph. They turn out to be the Betti numbers of complex projective space, as expected.

**Definition 3.3.24.** Define  $\omega \in H_{\mathbb{T}}^1(\Gamma)$  by  $\omega = (x_1x + y_1y, x_2x + y_2y, \dots, x_mx + y_my)$ , where  $(x_i, y_i) = \phi(v_i)$ . We call this the *equivariant symplectic form* of  $\Gamma$ .

Theorem 3.2.1 guarantees that  $H_{\mathbb{T}}^*(\Gamma)$  is a free  $\mathbb{C}[x, y]$ -module. If we denote by  $I$  the ideal of  $H_{\mathbb{T}}^*(\Gamma)$  generated by  $(x, x, \dots, x)$  and  $(y, y, \dots, y)$ , then the quotient ring  $H_{\mathbb{T}}^*(\Gamma)/I$ , which we will denote by  $H^*(\Gamma)$ , is a vector space of dimension  $m$ . A module basis of  $H_{\mathbb{T}}^*(\Gamma)$  descends to a vector space basis of  $H^*(\Gamma)$ . The image of  $\omega$  in  $H^*(\Gamma)$  is denoted by  $\tilde{\omega}$  and is called the *ordinary symplectic form* of  $\Gamma$ . As graded  $\mathbb{C}[x, y]$ -modules, we have  $H_{\mathbb{T}}^*(\Gamma) \cong H^*(\Gamma) \otimes_{\mathbb{C}} \mathbb{C}[x, y]$ .

If  $\Gamma' = (V', E')$  is a subgraph of  $\Gamma = (V, E)$ , then  $\Gamma'$  is equipped with  $\phi'$  and  $\alpha'$  induced by  $\phi$  and  $\alpha$ . If we denote by  $i$  the natural inclusion map  $i : \Gamma' \rightarrow \Gamma$ , then it induces a graded  $\mathbb{C}[x, y]$ -algebra homomorphism  $i^* : H_{\mathbb{T}}^*(\Gamma) \rightarrow H_{\mathbb{T}}^*(\Gamma')$ . This map descends to a graded ring homomorphism  $\tilde{i}^* : H^*(\Gamma) \rightarrow H^*(\Gamma')$ .

**Proposition 3.3.25.** *Given  $\Gamma = (V, E)$  a complete graph on  $m$  vertices, we have  $c_0 = c_1 = \cdots = c_{m-1} = 1$  and  $c_k = 0$  for  $k \geq m$ . Moreover,  $\{\omega^i | 0 \leq i \leq m-1\}$  forms a basis of the free module  $H_{\mathbb{T}}^*(\Gamma)$  over  $\mathbb{C}[x, y]$ .*

*Proof.* Assume  $\phi(v_i) = (x_i, y_i)$  for  $1 \leq i \leq m$ . First we observe that a translation of  $\phi$  to  $\phi' = \phi + (p, q)$ , where  $(p, q) \in \mathbb{R}^2$ , will not affect the graph cohomology at all. The classes  $\{\omega^i | 0 \leq i \leq m-1\}$  form a module basis precisely when the classes  $\{(\omega + \gamma)^i | 0 \leq i \leq m-1\}$  form a module basis, where  $\gamma = (px + qy, px + qy, \dots, px + qy)$ . So without loss of generality, we may assume  $x_i \neq 0$  for all  $i$  and  $\frac{y_i}{x_i} \neq \frac{y_j}{x_j}$  for all  $i \neq j$ .

We prove the statement by induction on  $m$ . When  $m = 1$ , the statement clearly holds.

Now assume we have proved that for  $m = k$  and  $\Gamma$  a complete graph on  $m$  vertices, the classes  $\{\omega^i | 0 \leq i \leq m-1\}$  form a module basis of  $H_{\mathbb{T}}^*(\Gamma)$ . We consider the case  $m = k + 1$ .

Denote by  $\Gamma'$  the complete graph on  $v_1, v_2, \dots, v_k$ . This is a subgraph of  $\Gamma$  and we denote by  $i$  the inclusion map. Now  $\omega$  is the equivariant symplectic form of  $\Gamma$ , and we notice that  $i^*(\omega)$  is the equivariant symplectic form of  $\Gamma'$ . For any  $0 \leq i \leq k-1$ , by our induction hypothesis, we have  $\tilde{i}^*(\tilde{\omega}^i) \neq 0 \in H^i(\Gamma')$ , so  $\tilde{\omega}^i \neq 0 \in H^i(\Gamma)$ , hence  $c_i(\Gamma) \geq 1$ .

Then the relations

$$\sum_{i=0}^k c_i(\Gamma) = k + 1$$

and

$$\sum_{i=0}^k i c_i(\Gamma) = |E| = \frac{k(k+1)}{2}$$

force  $c_0(\Gamma) = c_1(\Gamma) = \dots = c_k(\Gamma) = 1$  and  $c_i(\Gamma) = 0$  for  $i \geq k + 1$ .

Now to complete the induction step, we only need to show  $\tilde{\omega}^k \neq 0 \in H^k(\Gamma)$ .

Assume this is not the case. Then there must exist homogeneous polynomials  $f_i(x, y)$  of degree  $i$  for  $1 \leq i \leq k$ , such that

$$(3.3.26) \quad \omega^k = f_k + f_{k-1}\omega + \dots + f_1\omega^{k-1} \in H_{\mathbb{T}}^k(\Gamma),$$

where we have identified the polynomial  $f_i$  with  $(f_i(x, y), f_i(x, y), \dots, f_i(x, y)) \in H_{\mathbb{T}}^i(\Gamma)$ .

At each vertex  $v_i$  for  $1 \leq i \leq k + 1$ , since  $\phi(v_i) = (x_i, y_i)$ , we have  $\omega|_{v_i} = x_i x + y_i y$ .

It follows from equation (3.3.26) that

$$(3.3.27) \quad (x_i x + y_i y)^k \equiv f_k(x, y) + f_{k-1}(x, y)(x_i x + y_i y) + \dots + f_1(x, y)(x_i x + y_i y)^{k-1},$$

where  $\equiv$  means the two sides are equal as polynomials in  $x$  and  $y$ .

So  $(x_i x + y_i y) | f_k(x, y)$  for  $1 \leq i \leq k + 1$ . Because  $\frac{y_i}{x_i} \neq \frac{y_j}{x_j}$  for all  $i \neq j$ , we have

$\prod_{i=1}^{k+1} (x_i x + y_i y) | f_k(x, y)$ . This is only possible when  $f_k(x, y) \equiv 0$ . Then equation (3.3.27) reduces to

$$(x_i x + y_i y)^{k-1} \equiv f_{k-1}(x, y) + f_{k-2}(x, y)(x_i x + y_i y) + \dots + f_1(x, y)(x_i x + y_i y)^{k-2}$$

for  $1 \leq i \leq k + 1$ . In other words,

$$\omega^{k-1} = f_{k-1} + f_{k-2}\omega + \cdots + f_1\omega^{k-2} \in H_{\mathbb{T}}^{k-1}(\Gamma).$$

This contradicts with that  $\tilde{\omega}^{k-1} \neq 0 \in H^{k-1}(\Gamma)$ .  $\square$

We conclude this section by proving an analogue of the Künneth formula.

**Definition 3.3.28.** Given  $\Gamma_1 = (V_1, E_1)$  and  $\Gamma_2 = (V_2, E_2)$ , assume  $V_1 = \{v_1, v_2, \dots, v_m\}$  and  $V_2 = \{u_1, u_2, \dots, u_n\}$ . The (Cartesian) product graph  $\Gamma_1 \square \Gamma_2 = (V_3, E_3)$  is defined by  $V_3 = V_1 \times V_2$ , and two vertices  $(v_i, u_s)$  and  $(v_j, u_t)$  in  $V_3$  are adjacent if and only if

$$v_i = v_j \in V_1, u_s \text{ and } u_t \text{ are adjacent in } \Gamma_2,$$

or

$$u_s = u_t \in V_2, v_i \text{ and } v_j \text{ are adjacent in } \Gamma_1.$$

**Proposition 3.3.29.** Given  $\Gamma_1 = (V_1, E_1)$  and  $\Gamma_2 = (V_2, E_2)$ , assume  $V_1 = (v_1, \dots, v_m)$ ,  $V_2 = (u_1, \dots, u_n)$  and  $\phi_1 : V_1 \rightarrow \mathbb{R}^2, \phi_2 : V_2 \rightarrow \mathbb{R}^2$  are two moment maps in general position. Let  $\Gamma_3 = \Gamma_1 \square \Gamma_2 = (V_3, E_3)$  and assume that  $\phi_3 : V_3 = V_1 \times V_2 \rightarrow \mathbb{R}^2$  defined by  $\phi_3(v_i, u_s) = a\phi_1(v_i) + b\phi_2(u_s)$  is also in general position, where  $a, b \in \mathbb{R}$  are both non-zero constants. Then we have an isomorphism of  $\mathbb{C}[x, y]$ -algebras:

$$(3.3.30) \quad \begin{aligned} p : H_{\mathbb{T}}^*(\Gamma_1) \otimes_{\mathbb{C}[x, y]} H_{\mathbb{T}}^*(\Gamma_2) &\longrightarrow H_{\mathbb{T}}^*(\Gamma_3) \\ (f_1, f_2, \dots, f_m) \otimes (g_1, g_2, \dots, g_n) &\mapsto (f_1g_1, f_1g_2, \dots, f_1g_n, f_2g_1, \dots, f_mg_n), \end{aligned}$$

where the vertices in  $V_3$  are ordered lexicographically as

$$(v_1, u_1), (v_1, u_2), \dots, (v_1, u_n), (v_2, u_1), \dots, (v_m, u_n).$$

This immediately implies

$$(3.3.31) \quad c_k(\Gamma_3) = \sum_{i=0}^k c_i(\Gamma_1)c_{k-i}(\Gamma_2).$$

*Proof.* It is straightforward to verify that the map  $p$  defined by (3.3.30) is an algebra homomorphism. We are going to prove it is an isomorphism by induction on  $|E_1| + |E_2|$ . If  $|E_1| = 0$  or  $|E_2| = 0$ , the conclusion obviously holds. Now we assume  $|E_1| > 0, |E_2| > 0$  and  $p$  defined by (3.3.30) is an isomorphism for smaller  $|E_1| + |E_2|$ .

Suppose  $v_i$  and  $v_j$  are connected by an edge in  $E_1$ , which we will denote by  $e_{v_i}^{v_j}$ . Suppose  $u_s$  and  $u_t$  are connected by an edge in  $E_2$ , which we will denote by  $e_{u_s}^{u_t}$ . Assume  $\alpha_k : E_k \rightarrow \mathbb{C}[x, y]_1$  is induced by  $\phi_k$  for  $k = 1, 2, 3$ . Denote  $\alpha_1(e_{v_i}^{v_j})$  by  $f$  and  $\alpha_2(e_{u_s}^{u_t})$  by  $g$ . We first observe that  $f$  and  $g$  cannot be multiples of each other, otherwise the three vertices  $(v_i, u_s), (v_j, u_s), (v_j, u_t)$  in  $V_3$  would be on the same line, contradicting the assumption that  $\phi_3$  is in general position. We know both  $H_{\mathbb{T}}^*(\Gamma_1) \otimes_{\mathbb{C}[x, y]} H_{\mathbb{T}}^*(\Gamma_2)$  and  $H_{\mathbb{T}}^*(\Gamma_3)$  are free modules over  $\mathbb{C}[x, y]$  of dimension  $mn$ , we now fix bases for both. Then the map  $p$  can be represented by a matrix  $M(p)$ , and  $p$  is an isomorphism if and only if  $M(p)$  is invertible as a matrix with coefficients in  $\mathbb{C}[x, y]$ . This is equivalent to  $\det(M(p))$  being invertible in  $\mathbb{C}[x, y]$ .

Define  $E'_1 = E_1 \setminus \{e_{v_i}^{v_j}\}$  and  $\Gamma'_1 = (V_1, E'_1)$ . We may define

$$p' : H_{\mathbb{T}}^*(\Gamma'_1) \otimes_{\mathbb{C}[x, y]} H_{\mathbb{T}}^*(\Gamma_2) \rightarrow H_{\mathbb{T}}^*(\Gamma'_1 \square \Gamma_2)$$

just as we have defined  $p$ . It follows from the induction hypothesis that  $p'$  is an isomorphism. For any  $\mathbb{C}[x, y]$ -module  $N$ , we will use  $N_f$  to denote  $N \otimes_{\mathbb{C}[x, y]} \mathbb{C}[x, y]_f$ ,

the localization at the polynomial  $f$ . It follows from the definition of graph cohomology that for  $f = \alpha_1(e_{v_i}^{v_j})$ ,

$$H_{\mathbb{T}}^*(\Gamma_1)_f = H_{\mathbb{T}}^*(\Gamma'_1)_f,$$

and

$$H_{\mathbb{T}}^*(\Gamma_1 \square \Gamma_2)_f = H_{\mathbb{T}}^*(\Gamma'_1 \square \Gamma_2)_f.$$

Then the two maps

$$p_f : H_{\mathbb{T}}^*(\Gamma_1)_f \otimes_{\mathbb{C}[x,y]_f} H_{\mathbb{T}}^*(\Gamma_2)_f \rightarrow H_{\mathbb{T}}^*(\Gamma_1 \square \Gamma_2)_f$$

and

$$p'_f : H_{\mathbb{T}}^*(\Gamma'_1)_f \otimes_{\mathbb{C}[x,y]_f} H_{\mathbb{T}}^*(\Gamma_2)_f \rightarrow H_{\mathbb{T}}^*(\Gamma'_1 \square \Gamma_2)_f$$

can be identified. Because  $p'$  is an isomorphism, so are  $p'_f$  and  $p_f$ . So  $\det(M(p))$  is invertible in  $\mathbb{C}[x, y]_f$ . By the same argument, we know that  $\det(M(p))$  is also invertible in  $\mathbb{C}[x, y]_g$ . Since  $f$  and  $g$  are not multiples of each other, this is only possible when  $\det(M(p)) \in \mathbb{C}^*$ . So  $p$  is an isomorphism. This completes the induction step.

The equation (3.3.31) now follows by counting dimensions. □

### 3.4 Connectivity properties of GKM graphs

In this section we prove two theorems about the connectivity of a regular  $d$ -valent graph with  $c_d = 1$ . As corollaries we deduce connectivity properties of a GKM graph.



We first recall the standard graph theory notations of edge and vertex connectivity.

**Definition 3.4.1.** A graph  $\Gamma = (V, E)$  is  $k$ -edge-connected for some  $k \in \mathbb{N}$ , if for any subset  $F = \{e_{i_1j_1}, e_{i_2j_2}, \dots, e_{i_tj_t}\} \subseteq E$  with  $t < k$ , the subgraph  $\Gamma' = (V, E \setminus F)$  is connected.

**Definition 3.4.2.** A graph  $\Gamma = (V, E)$  is  $k$ -vertex-connected for some  $k \in \mathbb{N}$ , if for any subset  $U = \{v_{i_1}, v_{i_2}, \dots, v_{i_t}\} \subseteq V$  with  $t < k$ , the subgraph  $\Gamma - U = (V \setminus U, E \setminus E_U)$  is connected.

Now we are ready to state the main theorems in this section.

**Theorem 3.4.3.** *Given  $\Gamma = (V, E)$  a connected regular graph of degree  $d$ , if  $c_d(\Gamma) = 1$ , then  $\Gamma$  is  $d$ -edge-connected.*

**Corollary 3.4.4.** *If a  $2d$ -dimensional compact connected Hamiltonian GKM manifold has a moment map in general position, then its GKM graph is  $d$ -edge-connected.*

Let  $\lceil \frac{d}{2} \rceil$  denote the least integer greater than or equal to  $\frac{d}{2}$ .

**Theorem 3.4.5.** *Given  $\Gamma = (V, E)$  a connected regular graph of degree  $d$ , if  $c_d(\Gamma) = 1$ , then  $\Gamma$  is  $(\lceil \frac{d}{2} \rceil + 1)$ -vertex-connected.*

**Corollary 3.4.6.** *If a  $2d$ -dimensional compact connected Hamiltonian GKM manifold has a moment map in general position, then its GKM graph is  $(\lceil \frac{d}{2} \rceil + 1)$ -vertex-connected.*

**Remark 3.4.7.** As we remarked earlier in Remark 3.3.21,  $c_d = 1$  is a weaker assumption than the existence of an axial function. So our theorems also apply to the graphs studied by Guillemin and Zara [GZ99].

**Notation 3.4.8.** Given  $\Gamma = (V, E)$ , we say a vector  $\mathbf{u} = (u_1, u_2, \dots, u_{(k+1)m}) \in \mathbb{C}^{(k+1)m}$  vanishes on  $v_i$ , or  $\mathbf{u}|_{v_i} = 0$ , if  $u_i = u_{m+i} = \dots = u_{km+i} = 0$ . We say  $\mathbf{u}$  vanishes on  $U \subseteq V$ , or  $\mathbf{u}|_U = 0$ , if  $\mathbf{u}$  vanishes on every point in  $U$ . We denote the set of vectors in  $\mathbb{C}^{(k+1)m}$  that vanishes on  $V \setminus U$  by  $W_U^k$ . We will use  $W_{v_i}^k$  for  $W_{\{v_i\}}^k$  when the set has only one point. It is easy to see that  $\mathbf{v}_{ij}^k \in W_{\{v_i, v_j\}}^k$ .

We denote by  $P_U^k : \mathbb{C}^{(k+1)m} \rightarrow W_U^k$  the natural projection which sets all coordinates corresponding to  $V \setminus U$  to 0.

For any  $U \subseteq V$ , we use  $K(U)$  to denote the edge set of the complete graph on  $U$  and use  $K_U = (U, K(U))$  to denote the complete graph itself. Note that when  $U$  has only one element,  $K(U) = \emptyset$  and  $K_U$  is just a single vertex.

**Lemma 3.4.9.** *Assume  $\Gamma = (V, E)$  and  $U \subseteq V$  is a non-empty subset. Then*

$$\langle \mathbf{v}_{ij}^k \mid e_{ij} \in E \rangle \cap W_U^k \subseteq \langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(U) \rangle,$$

where  $\langle S \rangle$  means the subspace of  $\mathbb{C}^{(k+1)m}$  spanned by  $S$ .

*Proof.* Our vertex set is  $V = \{v_1, v_2, \dots, v_m\}$ , and W.L.O.G. we assume  $U = \{v_1, v_2, \dots, v_n\}$ . It is enough to show

$$(3.4.10) \quad \langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(V) \rangle \cap W_U^k = \langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(U) \rangle.$$

It is clear that in the above equation, the RHS is contained in the LHS. It remains to show the other containment. We divide the proof into three cases.

**Case 1:**  $n \geq k + 1$ .

$$\begin{aligned}
(3.4.11) \quad & \dim \langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(V) \rangle = r_k(K_V) \\
& = (k+1)m - c_k(K_V) - 2c_{k-1}(K_V) - \cdots - (k+1)c_0(K_V) \\
& = (k+1)m - \frac{(k+1)(k+2)}{2}.
\end{aligned}$$

The first equality is the definition of  $r_k$ , the second equality is by formula (3.3.9), the third equality is a consequence of Proposition 3.3.25.

It is immediate that

$$(3.4.12) \quad \dim W_U^k = (k+1)n.$$

Now for any  $t \in \mathbb{N}$  with  $n+1 \leq t \leq m$ , and any  $i \in N$  with  $1 \leq i \leq k+1 \leq n$ , we have  $\mathbf{v}_{it}^k \in W_{\{v_i, v_t\}}^k$  and  $P_{\{v_t\}}^k(\mathbf{v}_{it}^k) \in \langle \mathbf{v}_{it}^k \rangle + W_{v_i}^k$ . More explicitly, we have

$$P_{\{v_t\}}^k(\mathbf{v}_{it}^k) = (0, \dots, -1, 0, \dots, -a_{it}, 0, \dots, -a_{it}^k, 0, \dots, 0),$$

where the only nonzero entries of the vector are the  $t, (m+t), \dots, (km+t)$ -th entries and they are  $-1, -a_{it}, \dots, -a_{it}^k$  respectively. Then the elementary facts about Vandermonde matrices tell us that  $\{P_{\{v_t\}}^k(\mathbf{v}_{it}^k) \mid 1 \leq i \leq k+1\}$  forms a basis for  $W_{v_t}^k$ .

So

$$W_{v_t}^k \subseteq \sum_{i=1}^{k+1} (\langle \mathbf{v}_{it}^k \rangle + W_{v_i}^k) \subseteq \langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(V) \rangle + W_U^k,$$

for  $n+1 \leq t \leq m$ . So  $\langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(V) \rangle + W_U^k = W_V^k$  and

$$(3.4.13) \quad \dim (\langle \mathbf{v}_{ij}^k \mid e_{ij} \in K(V) \rangle + W_U^k) = m(k+1).$$

Now we put (3.4.11), (3.4.12), (3.4.13) together to get

$$\begin{aligned}
& \dim (\langle \mathbf{v}_{ij}^k | e_{ij} \in K(V) \rangle \cap W_U^k) \\
&= \dim \langle \mathbf{v}_{ij}^k | e_{ij} \in K(V) \rangle + \dim W_U^k - \dim (\langle \mathbf{v}_{ij}^k | e_{ij} \in K(V) \rangle + W_U^k) \\
&= (k+1)m - \frac{(k+1)(k+2)}{2} + (k+1)n - m(k+1) \\
&= (k+1)n - \frac{(k+1)(k+2)}{2} \\
&= \dim \langle \mathbf{v}_{ij}^k | e_{ij} \in K(U) \rangle.
\end{aligned}$$

The last equality is obtained in exactly the same way as we obtained (3.4.11). Thus we conclude (3.4.10) holds in Case 1.

**Case 2:**  $k = 1, n = 1$ . We observe that each vector

$$\mathbf{u} = (u_1, u_2, \dots, u_{2m}) \in \langle \mathbf{v}_{ij}^1 | e_{ij} \in K(V) \rangle$$

must satisfy the linear equations

$$\sum_{i=1}^m u_i = 0 \text{ and } \sum_{i=m+1}^{2m} u_i = 0.$$

This forces the intersection of  $\langle \mathbf{v}_{ij}^1 | e_{ij} \in K(V) \rangle$  with  $W_{v_1}^1$  to be  $\mathbf{0}$ . So (3.4.10) holds since  $K(\{v_1\}) = \emptyset$ .

**Case 3:**  $k \geq 2, n < k + 1$ .

For a fixed  $n \geq 1$ , we prove (3.4.10) using induction on  $k$ . If  $n \geq 2$ , we use  $k = n - 1$  as the base case. If  $n = 1$ , we use  $k = 1$  as the base case. Thus these base cases have been verified in Case 1 and Case 2 respectively. So now we have  $k \geq 2, k > n - 1$  and we assume (3.4.10) holds for smaller values of  $k$ .

Assume  $\mathbf{u} \in (\langle \mathbf{v}_{ij}^k | e_{ij} \in K(V) \rangle \cap W_U^k)$ , so we may write it as

$$(3.4.14) \quad \mathbf{u} = \sum_{e_{ij} \in K(V)} u_{ij} \mathbf{v}_{ij}^k \in W_U^k$$

for some  $u_{ij} \in \mathbb{C}$ .

For any vector  $\mathbf{w} = (w_1, w_2, \dots, w_{(k+1)m}) \in \mathbb{C}^{(k+1)m}$ , let

$$\mathbf{w}^{ini} = (w_1, w_2, \dots, w_{km}) \in \mathbb{C}^{km},$$

$$\mathbf{w}^{end} = (w_{m+1}, w_{m+2}, \dots, w_{(k+1)m}) \in \mathbb{C}^{km}, \text{ and}$$

$$\mathbf{w}^{mid} = (w_{m+1}, w_{m+2}, \dots, w_{km}) \in \mathbb{C}^{(k-1)m}.$$

Then in particular we have

$$(\mathbf{v}_{ij}^k)^{ini} = \mathbf{v}_{ij}^{k-1}, (\mathbf{v}_{ij}^k)^{end} = a_{ij} \mathbf{v}_{ij}^{k-1}, \text{ and } (\mathbf{v}_{ij}^k)^{mid} = a_{ij} \mathbf{v}_{ij}^{k-2}.$$

Then it follows from (3.4.14) that

$$\begin{aligned} \mathbf{u}^{ini} &= \sum_{e_{ij} \in K(V)} u_{ij} \mathbf{v}_{ij}^{k-1} \in W_U^{k-1} \text{ and} \\ \mathbf{u}^{end} &= \sum_{e_{ij} \in K(V)} u_{ij} a_{ij} \mathbf{v}_{ij}^{k-1} \in W_U^{k-1}. \end{aligned}$$

Then by the induction hypothesis, we have

$$(3.4.15) \quad \begin{aligned} \mathbf{u}^{ini} &= \sum_{e_{ij} \in K(U)} x_{ij} \mathbf{v}_{ij}^{k-1} \text{ and} \\ \mathbf{u}^{end} &= \sum_{e_{ij} \in K(U)} y_{ij} \mathbf{v}_{ij}^{k-1}, \end{aligned}$$

for some  $x_{ij}, y_{ij} \in \mathbb{C}$ . Hence

$$(3.4.16) \quad \mathbf{u}^{mid} = \sum_{e_{ij} \in K(U)} x_{ij} a_{ij} \mathbf{v}_{ij}^{k-2} = \sum_{e_{ij} \in K(U)} y_{ij} \mathbf{v}_{ij}^{k-2}.$$

Since  $k > n - 1$ , we may apply Lemma 3.3.15 to  $K_U$  to obtain  $s_{k-2}(K_U) = 0$ , which means the vectors  $\{\mathbf{v}_{ij}^{k-2} | e_{ij} \in K(U)\}$  are linearly independent. Thus (3.4.16) implies that  $x_{ij}a_{ij} = y_{ij}$ . So the coefficients of the linear combinations in (3.4.15) are “compatible” and we may write

$$\mathbf{u} = \sum_{e_{ij} \in K(U)} x_{ij} \mathbf{v}_{ij}^k.$$

This exactly implies (3.4.10), completing the induction.  $\square$

We need the following technical lemma in the proof of Lemma 3.4.18.

**Lemma 3.4.17.** *If  $1 \leq p \leq n, 1 \leq q \leq n$ , then*

$$\frac{p(p-1)}{2} + \frac{q(q-1)}{2} + n < \frac{(p+q)(n+1)}{2}.$$

*Proof.*

$$\begin{aligned} & \frac{p(p-1)}{2} + \frac{q(q-1)}{2} + n - \frac{(p+q)(n+1)}{2} \\ &= \frac{1}{2} (p^2 - (n+2)p + q^2 - (n+2)q) + n \\ &= \frac{1}{2} \left( \left(p - \frac{n+2}{2}\right)^2 + \left(q - \frac{n+2}{2}\right)^2 \right) - \left(\frac{n+2}{2}\right)^2 + n \\ &\leq \left(\frac{n}{2}\right)^2 - \left(\frac{n+2}{2}\right)^2 + n \\ &= -1 < 0. \end{aligned}$$

$\square$

We now study the effect that disconnecting the graph by deleting edges has on the statistic  $s_k$ . We call the following lemma the *Cut Lemma*. It will be used to prove Theorem 3.4.3, and will also be used repeatedly in Section 3.5.

**Lemma 3.4.18.** *Given a connected graph  $\Gamma = (V, E)$  and a cut-set  $F = \{e_{i_1 j_1}, e_{i_2 j_2}, \dots, e_{i_n j_n}\}$  of size  $n$ , assume  $(V, E \setminus F) = \Gamma_1 \sqcup \Gamma_2$ , then*

$$s_k(\Gamma) = s_k(\Gamma_1) + s_k(\Gamma_2)$$

for any  $k \geq n - 1$ .

*Proof.* Assume  $\Gamma_1 = (V_1, E_1)$  and  $\Gamma_2 = (V_2, E_2)$ . For any  $1 \leq t \leq n$ , either  $v_{i_t} \in V_1$  and  $v_{j_t} \in V_2$ , or  $v_{i_t} \in V_2$  and  $v_{j_t} \in V_1$ . Without loss of generality, we may assume  $v_{i_t} \in V_1$  for all  $1 \leq t \leq n$ . Let  $V_3 = \{v_{i_t} | 1 \leq t \leq n\}$ ,  $V_4 = \{v_{j_t} | 1 \leq t \leq n\}$ ,  $E_5 = \{e_{i_1 j_1}, e_{i_2 j_2}, \dots, e_{i_n j_n}\}$ . Note that even though  $e_{i_s j_s}$  and  $e_{i_t j_t}$  are assumed to be distinct for  $s \neq t$ , it could still happen that  $v_{i_s} = v_{i_t}$  or  $v_{j_s} = v_{j_t}$ . So we have  $|V_3| \leq n$ ,  $|V_4| \leq n$  and  $|E_5| = n$ . Define  $\Gamma' = (V', E')$  by  $V' = V_3 \cup V_4$ ,  $E' = K(V_3) \cup K(V_4) \cup E_5$ .

We claim the graph  $\Gamma'$  has at least one vertex of degree less than  $n + 1$ . Otherwise,  $\Gamma'$  should have at least  $\frac{(|V_3| + |V_4|)(n + 1)}{2}$  edges. By Lemma 3.4.17, this is greater than  $\frac{|V_3|(|V_3| + 1)}{2} + \frac{|V_4|(|V_4| + 1)}{2} + n$ , which is the actual number of edges of  $\Gamma'$ , a contradiction.

Now assume without loss of generality that  $v_{i_1}$  is of degree less than  $n + 1$ . Define  $\Gamma'' = (V' \setminus \{v_{i_1}\}, E' \setminus E_{v_{i_1}})$ . Then we may apply the Deletion Lemma 3.3.12 to  $v_{i_1}$  and  $\Gamma'$  to get  $s_k(\Gamma') = s_k(\Gamma'')$ .

By the same argument we can show  $\Gamma''$  also has at least one vertex of degree less than  $n + 1$ . Repeatedly applying the Deletion Lemma, we finally conclude

$$s_k(\Gamma') = 0.$$

Now assume there is a linear relation among  $\{\mathbf{v}_{ij}^k | e_{ij} \in E\}$ :

$$(3.4.19) \quad \sum_{e_{ij} \in E} u_{ij} \mathbf{v}_{ij}^k = \mathbf{0}.$$

We may split the left hand side and rewrite it as

$$(3.4.20) \quad \sum_{e_{ij} \in E_1} u_{ij} \mathbf{v}_{ij}^k + \sum_{e_{ij} \in E_2} u_{ij} \mathbf{v}_{ij}^k + \sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^k = \mathbf{0}.$$

As both  $\sum_{e_{ij} \in E_2} u_{ij} \mathbf{v}_{ij}^k$  and  $\sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^k$  vanish on  $V_1 \setminus V_3$ , so does  $\sum_{e_{ij} \in E_1} u_{ij} \mathbf{v}_{ij}^k$ . Then we may apply Lemma 3.4.9 to  $\Gamma_1$  and  $\sum_{e_{ij} \in E_1} u_{ij} \mathbf{v}_{ij}^k$  to get

$$\sum_{e_{ij} \in E_1} u_{ij} \mathbf{v}_{ij}^k = \sum_{e_{ij} \in K(V_3)} m_{ij} \mathbf{v}_{ij}^k,$$

for some  $m_{ij} \in \mathbb{C}$ . Similarly, we have

$$\sum_{e_{ij} \in E_2} u_{ij} \mathbf{v}_{ij}^k = \sum_{e_{ij} \in K(V_4)} n_{ij} \mathbf{v}_{ij}^k,$$

for some  $n_{ij} \in \mathbb{C}$ . So (3.4.20) becomes

$$\sum_{e_{ij} \in K(V_3)} m_{ij} \mathbf{v}_{ij}^k + \sum_{e_{ij} \in K(V_4)} n_{ij} \mathbf{v}_{ij}^k + \sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^k = \mathbf{0}.$$

Because  $s_k(\Gamma') = 0$ , the above equation forces the coefficients  $m_{ij} = 0$  for each  $e_{ij} \in K(V_3)$ ,  $n_{ij} = 0$  for each  $e_{ij} \in K(V_4)$  and  $u_{ij} = 0$  for each  $e_{ij} \in E_5$ . So (3.4.20), hence (3.4.19), is in fact the sum of two linear relations:

$$\begin{aligned} \sum_{e_{ij} \in E_1} u_{ij} \mathbf{v}_{ij}^k &= \mathbf{0}, \text{ and} \\ \sum_{e_{ij} \in E_2} u_{ij} \mathbf{v}_{ij}^k &= \mathbf{0}. \end{aligned}$$

So  $s_k(\Gamma) = s_k(\Gamma_1) + s_k(\Gamma_2)$ . □



Now we are ready to prove Theorem 3.4.3.

*Proof of Theorem 3.4.3.* We prove this by contradiction. Assume to the contrary that  $\Gamma$  is not  $d$ -edge-connected, then there exists a cut-set  $F = \{e_{i_1j_1}, e_{i_2j_2}, \dots, e_{i_nj_n}\}$  of size  $n < d$ .

Assume  $(V, E \setminus F) = \Gamma_1 \sqcup \Gamma_2$ . Then by Lemma 3.4.18,  $s_{d-2}(\Gamma) = s_{d-2}(\Gamma_1) + s_{d-2}(\Gamma_2)$ . Repeated application of the Deletion Lemma to  $\Gamma_1$  will yield  $s_{d-2}(\Gamma_1) = 0$  and similarly  $s_{d-2}(\Gamma_2) = 0$ . Therefore  $s_{d-2}(\Gamma) = 0$ .

We always have  $s_{d-1}(\Gamma) = s_d(\Gamma) = 0$  by Lemma 3.3.15. So  $c_d = s_d + s_{d-2} - 2s_{d-1} = 0$ . This contradicts our assumption that  $c_d = 1$ .  $\square$

We prove Theorem 3.4.5 in a similar fashion.

*Proof of Theorem 3.4.5.* We prove this by contradiction. Assume to the contrary that  $\Gamma$  is not  $(\lceil \frac{d}{2} \rceil + 1)$ -vertex connected, so that there exists a smallest  $n \leq \lceil \frac{d}{2} \rceil$ , such that there are  $n$  vertices, which we may assume without loss of generality to be  $U = \{v_1, v_2, \dots, v_n\}$ , such that the graph  $\Gamma - U = (V \setminus U, E \setminus E_U)$  is disconnected.

By the minimality of  $n$ , we know  $\Gamma - U$  consists of exactly two connected components, denoted by  $\Gamma_1 = (V_1, E_1)$  and  $\Gamma_2 = (V_2, E_2)$ . For each vertex  $v_i$  in  $U$ , the edge set  $E_{v_i}$  decomposes into three parts:

$$E_{v_i} = E_{v_i}^1 \sqcup E_{v_i}^2 \sqcup E_{v_i}^U,$$

where  $E_{v_i}^1, E_{v_i}^2$ , and  $E_{v_i}^U$  are the set of edges connecting  $v_i$  to  $V_1, V_2$  and  $U$  respectively. If  $E_{v_i}^1 = \emptyset$ , then removing  $U \setminus \{v_i\}$  already disconnects  $\Gamma$ , contradicting the minimality of  $n$ . So  $E_{v_i}^1 \neq \emptyset$ . For same reason,  $E_{v_i}^2 \neq \emptyset$ .

Let  $E_3 = \bigcup_{v_i \in U} E_{v_i}^1, E_4 = \bigcup_{v_i \in U} E_{v_i}^2$  and  $E_5 = \bigcup_{v_i \in U} E_{v_i}^U = E \cap K(U)$ . A linear relation among  $\{\mathbf{v}_{ij}^{d-2} \mid e_{ij} \in E\}$

$$(3.4.21) \quad \sum_{e_{ij} \in E} u_{ij} \mathbf{v}_{ij}^{d-2} = \mathbf{0}$$

may be rewritten as

$$(3.4.22) \quad \sum_{e_{ij} \in E_1 \cup E_3} u_{ij} \mathbf{v}_{ij}^{d-2} + \sum_{e_{ij} \in E_2 \cup E_4} u_{ij} \mathbf{v}_{ij}^{d-2} + \sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^{d-2} = \mathbf{0}.$$

Since  $|E_{v_1}^1| + |E_{v_1}^2| + |E_{v_1}^U| = d$ , either  $|E_{v_1}^1| \leq \frac{d}{2}$  or  $|E_{v_1}^2| \leq \frac{d}{2}$ . We may assume without loss of generality that  $|E_{v_1}^1| \leq \frac{d}{2}$ . Since both  $\sum_{e_{ij} \in E_2 \cup E_4} u_{ij} \mathbf{v}_{ij}^{d-2}$  and  $\sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^{d-2}$  vanish on  $V_1$ , it follows from (3.4.22) that  $\sum_{e_{ij} \in E_1 \cup E_3} u_{ij} \mathbf{v}_{ij}^{d-2}$  also vanishes on  $V_1$ . Define

$\Gamma_3 = (V_1 \cup U, E_1 \cup E_3 \cup K(U))$ , then apply Lemma 3.4.9 to  $\Gamma_3$  and  $\sum_{e_{ij} \in E_1 \cup E_3} u_{ij} \mathbf{v}_{ij}^{d-2}$ ,

we have

$$(3.4.23) \quad \sum_{e_{ij} \in E_1 \cup E_3} u_{ij} \mathbf{v}_{ij}^{d-2} = \sum_{e_{ij} \in K(U)} x_{ij} \mathbf{v}_{ij}^{d-2},$$

for some  $x_{ij} \in \mathbb{C}$ .

**Claim:**  $s_{d-2}(\Gamma_3) = 0$ .

**Proof of the claim:** The degree of  $v_1$  inside  $\Gamma_3$  is  $\leq \frac{d}{2} + \lceil \frac{d}{2} \rceil - 1 \leq \frac{d}{2} + \frac{d}{2} + \frac{1}{2} - 1 < d$ . So we may apply the Deletion Lemma to  $\Gamma_3$  and  $v_1$  to remove  $v_1$  from  $\Gamma_3$  without

changing  $s_{d-2}$ . If there is a vertex in the new graph of degree less than  $d$ , we may repeat this to form another new graph. We continue this process as long as there is a vertex in the new graph of degree less than  $d$ . This process will only stop when we are left with a subgraph of  $\Gamma_3$  whose vertices are all of degree greater than or equal to  $d$ . We will show the only such subgraph is the empty graph. Assume to the contrary that there is a non-empty subgraph  $\Gamma_6 = (V_6, E_6)$  with  $V_6 \subseteq V_1 \cup U$ ,  $E_6 \subseteq E_1 \cup E_3 \cup K(U)$ , such that all vertices of  $\Gamma_6$  are of degree  $\geq d$ . First we show  $V_6 \cap V_1 = \emptyset$ . If not, assume  $v_t \in V_6 \cap V_1$ . Since  $E_{v_1}^1 \neq \emptyset$ , we may assume  $e_{1s} \in E_{v_1}^1$ , then  $v_s \in V_1$ . As  $\Gamma_1$  is connected, there is a path from  $v_t$  to  $v_s$  in  $\Gamma_1$ , which we denote by  $e_{ti_1} = e_{i_0i_1}, e_{i_1i_2}, \dots, e_{i_{p-1}i_p}, e_{i_pi_{p+1}} = e_{i_ps}$ . Let  $u = \max\{j \mid 0 \leq j \leq p+1, v_j \in V_6\}$ . Then  $\lambda_{\Gamma_6}(v_u) < \lambda_{\Gamma}(v_u) = d$ , contradicting our assumption on  $\Gamma_6$ . So  $V_6 \subseteq U$ . But  $|U| \leq \lfloor \frac{d}{2} \rfloor$ , contradicting the assumption that every vertex in  $\Gamma_6$  has degree greater than or equal to  $d$ .

Therefore  $s_{d-2}(\Gamma_3) = 0$  and we have proved the claim.

The above claim and equation (3.4.23) now force  $u_{ij} = 0$  for  $e_{ij} \in E_1 \cup E_3$ . Then it follows from (3.4.22) that

$$(3.4.24) \quad \sum_{e_{ij} \in E_2 \cup E_4} u_{ij} \mathbf{v}_{ij}^{d-2} + \sum_{e_{ij} \in E_5} u_{ij} \mathbf{v}_{ij}^{d-2} = \mathbf{0}.$$

Define  $\Gamma_4 = (V_2 \cup U, E_2 \cup E_4 \cup E_5)$ . Applying the Deletion Lemma to  $\Gamma_4$  gives  $s_{d-2}(\Gamma_4) = 0$ . So (3.4.24) forces  $u_{ij} = 0$  for all  $e_{ij} \in E_2 \cup E_4 \cup E_5$ .

So (3.4.22), hence (3.4.21) is a trivial linear relation, which means  $s_{d-2}(\Gamma) = 0$ . So  $c_d(\Gamma) = s_d(\Gamma) + s_{d-2}(\Gamma) - 2s_{d-1}(\Gamma) = 0$ , a contradiction.  $\square$

### 3.5 An upper bound for the second Betti number of compact Hamiltonian GKM manifolds

We now turn to the geometric consequences.

**Theorem 3.5.1.** *Given  $\Gamma = (V, E)$  a connected regular graph of degree  $d \geq 2$ , if  $c_d(\Gamma) = 1$ , then  $c_{d-1}(\Gamma) \leq \frac{m-2}{d-1}$ . As a consequence, if  $M$  is a  $2d$ -dimensional compact connected Hamiltonian GKM manifold whose moment map is in general position, we have  $\beta_2(M) = \beta_{2d-2}(M) \leq \frac{m-2}{d-1}$ , where  $\beta_i(M)$  is the  $i$ -th geometric Betti number of  $M$ ,  $m$  is the sum of all the Betti numbers, which is equal to the number of vertices in the GKM graph of the manifold.*

**Corollary 3.5.2.** *If  $M$  is a 8 or 10-dimensional compact connected Hamiltonian GKM manifold whose moment map is in general position, then  $M$  has nondecreasing even Betti numbers up to half dimension:  $\beta_0(M) \leq \beta_2(M) \leq \beta_4(M)$ .*

*Proof of Corollary.* This is a straightforward calculation using Theorem 3.5.1 and Poincare duality.

In the case of 8-dimensional manifold, it follows from  $\beta_2(M) \leq \frac{m-2}{3}$  that  $\beta_4(M) = m - 2 - 2\beta_2(M) \geq m - 2 - \frac{2(m-2)}{3} = \frac{m-2}{3} \geq \beta_2(M)$ .

In the case of 10-dimensional manifold, it follows from  $\beta_2(M) \leq \frac{m-2}{4}$  that  $\beta_4(M) = \frac{1}{2}(m - 2 - 2\beta_2(M)) \geq \frac{m-2}{4} \geq \beta_2(M)$ .

In both cases, we have  $\beta_0(M) = 1 \leq \beta_2(M)$  since the symplectic form represents a nontrivial cohomology class in  $H^2(M)$ .  $\square$

**Remark 3.5.3.** Corollary 3.5.2 gives an affirmative answer to Question 3.1.7 in the case of 8 and 10 dimensional Hamiltonian GKM manifolds whose moment map is in general position. These are the first two non-trivial cases, as the inequalities are automatic for 2-, 4- and 6-dimensional manifolds.

**Definition 3.5.4.** A graph  $\Gamma = (V, E)$  is called *k-trimmed* for some positive integer  $k$ , if

- each vertex of  $\Gamma$  is of degree at least  $k + 1$ ,
- each connected component of  $\Gamma$  is  $(k + 1)$ -edge-connected.

**Lemma 3.5.5.** *Every graph  $\Gamma = (V, E)$  has a unique maximal k-trimmed subgraph, which we will denote by  $\tilde{\Gamma}^k$ . It can be obtained by the following algorithm:*

1. *If  $\Gamma$  is k-trimmed, stop.*
2. *If  $\Gamma$  contains a vertex  $v_i$  of degree less than or equal to  $k$ , we define  $\Gamma_1 = (V_1, E_1)$  by  $V_1 = V \setminus \{v_i\}$  and  $E_1 = E \setminus E_{v_i}$ , where  $E_{v_i}$  is the set of edges incident to  $v_i$ .*
3. *If every vertex of  $\Gamma$  is of degree  $k + 1$  or higher, and there exists  $E' = \{e_{i_1j_1}, \dots, e_{i_tj_t}\} \subseteq E$ , such that  $t \leq k$  and removing these edges would increase the number of connected components of  $\Gamma$ , but removing any  $t - 1$  among them would not, then we define  $\Gamma_1 = (V_1, E_1)$  by  $V_1 = V$ , and  $E_1 = E \setminus E'$ .*
4. *Repeat the algorithm on  $\Gamma_1$ .*

This process will finally stop. The resulting graph, which could be empty, is  $\tilde{\Gamma}^k$ .

By the Deletion Lemma 3.3.12 and the Cut Lemma 3.4.18, we see that

$$s_{k-1}(\tilde{\Gamma}^k) = s_{k-1}(\Gamma).$$

*Proof.* The union of two  $k$ -trimmed subgraphs of  $\Gamma$  is also  $k$ -trimmed, so there is a unique maximal  $k$ -trimmed subgraph.

If  $\Gamma$  is itself  $k$ -trimmed, the lemma is trivial. Otherwise, following the algorithm, we get a sequence of graphs  $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ , where  $\Gamma_{i+1}$  is a subgraph of  $\Gamma_i$  obtained from  $\Gamma_i$  either as in Case (2) or as in Case (3) of the algorithm, and  $\Gamma_p$  is  $k$ -trimmed. This sequence is not necessarily unique, but we will show in any case  $\Gamma_p = \tilde{\Gamma}^k$ .

Assume  $\tilde{\Gamma}^k = (\tilde{V}^k, \tilde{E}^k)$ . First,  $\Gamma_p$  is a subgraph of  $\tilde{\Gamma}^k$  since  $\tilde{\Gamma}^k$  is the unique maximal  $k$ -trimmed subgraph of  $\Gamma$ . Secondly, if  $\Gamma$  is as in Case (2), then  $v_i \notin \tilde{V}^k$ , so  $\tilde{\Gamma}^k$  is a subgraph of  $\Gamma_1$ . If  $\Gamma$  is as in Case (3), then  $E' \cap \tilde{E}^k = \emptyset$ , so  $\tilde{\Gamma}^k$  is also a subgraph of  $\Gamma_1$ . Then we can show inductively that  $\tilde{\Gamma}^k$  is a subgraph of  $\Gamma_p$ . So we must have  $\Gamma_p = \tilde{\Gamma}^k$ .  $\square$

We make the following definition so we can make the statements and proofs in the rest of the section more concise.

**Definition 3.5.6.** For any  $d \geq 2$ , we call a graph  $\Gamma = (V, E)$  is of type  $T_d$  if

- each vertex of  $\Gamma$  is of degree  $d$  or  $(d - 1)$ ;

- each connected component of  $\Gamma$  has at least one vertex of degree  $(d - 1)$ ;
- each connected component of  $\Gamma$  is  $(d - 1)$ -edge-connected.

**Proposition 3.5.7.** *Assume  $\Gamma = (V, E)$  is a graph of type  $T_d$ , then*

$$(3.5.8) \quad s_{d-3}(\Gamma) \leq \frac{n_d(\Gamma)}{d-1} + \pi_0(\Gamma),$$

where  $n_d(\Gamma) = \{v_i \in V \mid \lambda(v_i) = d\}$  is the number of vertices of degree  $d$ ,  $\pi_0(\Gamma)$  is the number of connected components of  $\Gamma$ .

*Proof.* We use induction on the size of  $|V|$ . The graph of type  $T_d$  with the fewest vertices is the empty graph, and (3.5.8) holds as  $0 \leq 0$  in this case.

Now we consider  $\Gamma = (V, E)$  with  $|V| = m > 0$  and assume (3.5.8) holds for any graph of type  $T_d$  and  $|V| < m$ . If  $\pi_0(\Gamma) > 1$ , then each connected component of  $\Gamma$  is still of type  $T_d$ , and has fewer vertices. So the induction hypothesis implies (3.5.8) holds for each connected component. We may add them up to show (3.5.8) holds for  $\Gamma$  as well. Now we assume  $\Gamma$  is connected.

Since  $\Gamma$  is of type  $T_d$ , we may pick a vertex  $v_t$  of degree  $d-1$ . Define  $\Gamma' = (V', E')$  by  $V' = V \setminus \{v_t\}$  and  $E' = E \setminus E_{v_t}$ , where  $E_{v_t}$  is the set of edges incident to  $v_t$ . By Corollary 3.3.14, we have  $s_{d-3}(\Gamma) \leq s_{d-3}(\Gamma') + 1$ . Let  $\tilde{\Gamma}'^{d-2}$  be the maximal  $(d-2)$ -trimmed subgraph of  $\Gamma'$ , then we have  $s_{d-3}(\tilde{\Gamma}'^{d-2}) = s_{d-3}(\Gamma')$ . Assume  $\pi_0(\tilde{\Gamma}'^{d-2}) = p$ ,  $\tilde{\Gamma}'^{d-2} = \bigsqcup_{i=1}^p \Gamma_i$ , and  $\Gamma_i = (V_i, E_i)$ .

Since  $\Gamma$  is  $(d-1)$ -edge-connected, for each  $\Gamma_i$  there must be at least  $d-1$  vertices

in  $V_i$  whose degrees in  $\Gamma$  were  $d$  but now have degree  $d - 1$  in  $\Gamma_i$ . So

$$\sum_{i=1}^p n_d(\Gamma_i) \leq n_d(\Gamma) - (d - 1)p.$$

Since  $\tilde{\Gamma}^{d-2}$  is also of type  $T_d$  and with fewer vertices than  $\Gamma$ , by the induction hypothesis we have

$$s_{d-3}(\tilde{\Gamma}^{d-2}) \leq \frac{n_d(\tilde{\Gamma}^{d-2})}{d - 1} + \pi_0(\tilde{\Gamma}^{d-2}).$$

So

$$\begin{aligned} s_{d-3}(\Gamma) &\leq s_{d-3}(\tilde{\Gamma}^{d-2}) + 1 \\ &\leq \frac{n_d(\tilde{\Gamma}^{d-2})}{d - 1} + \pi_0(\tilde{\Gamma}^{d-2}) + 1 \\ &= \frac{1}{d - 1} \sum_{i=1}^p n_d(\Gamma_i) + p + 1 \\ &\leq \frac{1}{d - 1} (n_d(\Gamma) - (d - 1)p) + p + 1 \\ &= \frac{n_d(\Gamma)}{d - 1} + 1 \\ &= \frac{n_d(\Gamma)}{d - 1} + \pi_0(\Gamma). \end{aligned}$$

This completes the induction step. □

Now Theorem 3.5.1 follows easily.

*Proof of Theorem 3.5.1.* Since  $c_d(\Gamma) = 1$ , by Theorem 3.4.3 we know that  $\Gamma$  is  $d$ -edge-connected. Pick any edge  $e_{ij} \in E$  and form a new graph  $\Gamma' = (V, E \setminus \{e_{ij}\})$ . Then  $\Gamma'$  is of type  $T_d$ . By Proposition 3.5.7, we have

$$s_{d-3}(\Gamma') \leq \frac{m - 2}{d - 1} + 1.$$



So  $s_{d-3}(\Gamma) \leq s_{d-3}(\Gamma') + 1 \leq \frac{m-2}{d-1} + 2$ . Hence

$$c_{d-1}(\Gamma) = s_{d-1}(\Gamma) + s_{d-3}(\Gamma) - 2s_{d-2}(\Gamma) \leq 0 + \frac{m-2}{d-1} + 2 - 2 = \frac{m-2}{d-1},$$

where we have used the facts  $s_{d-1}(\Gamma) = 0$  by the Deletion Lemma, and  $s_{d-2}(\Gamma) = c_d(\Gamma) - (s_d(\Gamma) - 2s_{d-1}(\Gamma)) = 1$ . □

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