BRICK VARIETIES AND TORIC MATRIX SCHUBERT VARIETIES

A Dissertation

Presented to the Faculty of the Graduate School

of Cornell University

in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

by

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May 2015

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Cornell University 2015

In the first part of this thesis we study *brick varieties* which are fibers of the Bott-Samelson varieties. Bott-Samelson varieties are a twisted product of $\mathbb{CP}^1$'s with a map into G/B. These varieties are mostly studied in the case in which the map into G/B is birational to the image; however in the first part of this thesis we study a fiber of this map when the map is not birational. We prove that in some cases the general fiber, which we christen a brick variety, is a toric variety. In order to do so we use the moment map of a Bott-Samelson variety to translate this problem into one in terms of the "subword complexes" of Knutson and Miller. Pilaud and Stump realized certain subword complexes as the dual of the boundary of a polytope which generalizes the brick polytope defined by Pilaud and Santos. For a nice family of words, the brick polytope is the generalized associahedron realized by Hohlweg, Lange and Thomas. These stories connect in a nice way: we show that the moment polytope of the brick manifold is the brick polytope. In particular, we give a nice description of the toric variety of the associahedron. We give each brick manifold a stratification dual to the subword complex. In addition, we relate brick manifolds to Brion's resolutions of Richardon varieties.

In the second part of this thesis, which is joint work with Karola Mészáros, we show that a family of subword complexes can be realized geometrically via regular triangulations of root polytopes. This implies that a family of β-Grothendieck polynomials are special cases of reduced forms in the subdivision algebra of root polytopes. We also write the volume and Ehrhart series of root polytopes in terms of β-Grothendieck polynomials. Finally, we explain this geometric realization by characterizing the matrix Schubert varieties which are the product of a toric variety and a vector space.

BIOGRAPHICAL SKETCH

Laura Escobar Vega was born in Medellín, Colombia on the 20th of May, 1986. Her family consists of her parents Beatriz and Fernando, and older siblings Claudia and Felipe. She graduated from Universidad de los Andes with a B.S. in Mathematics and a minor in Japanese Language and Culture. She then completed a M.A. in Mathematics at San Francisco State University under the supervision of Professor Federico Ardila. She began her Ph.D. at Cornell University in 2010 and completed her dissertation under the supervision of Professor Allen Knutson.

This document is dedicated to my mentors, loving family and friends.

ACKNOWLEDGEMENTS

I would like to thank my advisor, Allen Knutson for his helpful guidance and patience. The amount of time he has spent teaching me mathematics as well as what good mathematics is has been very formative. He has constantly motivated me to do mathematics outside of my comfort zone and given me a lot of insight to make this possible. Moreover, he has been a very caring advisor making sure my professional career is a successful one. I would also like to thank Federico Ardila, for teaching me that I am able to do mathematics and continuing to guide me throughout my career. His passion not only for mathematics, but to make a difference for everyone around him is very inspiring. I am also grateful to Karola Mészáros for sharing a beautiful project with me.

I would like to thank the professors at Universidad de los Andes, where I learned how to do correct mathematics; the professors and staff San Francisco State University, for teaching me the beauty of mathematics and being so supportive in a very important stage of my career; and the professors and staff at Cornell University for making graduate school an amazing experience.

I am very grateful to my parents, Beatriz Vega and Fernando Escobar, for teaching me the integrity and curiosity that have led me to numerous accomplishments. I am very grateful to Jason Anema for always believing in me and being such an amazing emotional support. Finally, I would like to thank my family and friends.

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CHAPTER 1

INTRODUCTION

In this thesis we study two objects: in the first part we study general fibers of Bott-Samelson varieties, and in the second one we study matrix Schubert varieties. These two objects are related to subword complexes and part of this thesis relates to polytopal realizations of these simplicial complexes.

The Bott-Samelson varieties were first defined by Raoul Bott and Hans Samelson in [BS55]. Bott-Samelson varieties are a twisted product of $\mathbb{CP}^1$'s with a map into G/B. These varieties have been studied mostly in the case in which the map into G/B is birational. In this thesis we study the general fiber of this map when it is not birational to the image. We show that for some Bott-Samelson varieties this fiber is a toric variety. In order to do so we translate this problem into a purely combinatorial one in terms of subword complexes. These simplicial complexes $\Delta(Q, w)$ depend on a word Q in the generators of the Weyl group W of G and an element $w \in W$. They were defined by Allen Knutson and Ezra Miller in [KM04] to describe the geometry of determinantal ideals and Schubert polynomials. In [CLS14], Cesar Ceballos, Jean-Philippe Labbé and Christian Stump showed that for a nice family of words, subword complexes are isomorphic to the cluster complexes arising from the cluster algebra of the corresponding type. In [PS11], Vincent Pilaud and Christian Stump defined the brick polytope and realized certain subword complexes as the boundary of a polytope dual to the brick polytope. For the family of subword complexes related to the cluster complexes, they obtained that the brick polytopes are generalized associahedra. In Chapter 4 we relate the general fiber of certain Bott-Samelson varieties to brick polytopes. Particularly, in Theorem 4.2.2 we prove that for those words Pilaud and Stump define as "root independent", a fiber of the Bott-Samelson map is the toric variety of the brick polytope. In particular, this provides a description of the toric variety of a generalized associahedron, which in type A we interpret in terms of flags arranged in a poset.

Actually the toric case is just a shadow of a more general situation. We prove in Theorem 4.2.1 that for any word Q and element $w \in W$ the brick polytope is the moment polytope of a fiber of the Bott-Samelson variety. This motivates us to define the brick manifold as the fiber studied here. In Chapter 4 we show a very nice connection between subword complexes, brick polytopes and brick manifolds. In Theorem 4.2.2 we classify the toric brick manifolds. We end this chapter with two results about brick manifolds: we exhibit a stratification of the brick manifolds dual to the subword complex in Theorem 4.3.1 and following [Bri05], show that brick manifolds provide resolutions for Richardson varieties in Theorem 4.4.2.

In the second part of this thesis, Chapter 5, we provide geometric realizations of pipe dream complexes $PD(\pi)$ of permutations $\pi = 1\pi'$, where π' is a dominant permutation on $2, 3, \ldots, n$, as well as the subword complexes which are the cores of the pipe dream complexes $PD(\pi)$. We realize $PD(\pi)$ as (repeated cones of) a regular triangulation of the root polytope $\mathcal{P}(T(\pi))$.

Since the appearance of Knutson's and Miller's work in [KM04, KM05] there has been a flurry of research into the geometric realization of subword complexes with progress in realizing families of spherical subword complexes [Stu11, Ceb12, PP12, PS12, SS12, CLS14, BCL14]. This approach is the first to succeed in realizing a family of subword complexes which are homeomorphic to balls.

Subword complexes were first shown to relate to triangulations of root polytopes by Mészáros in [Mész14c], which work served as the stepping stone for the present project. In the papers [Mész11a, Mész15, Mész14b, Mész14a] Mészáros studied triangulations of root polytopes that we utilize in this work (some of the mentioned papers are in the language of flow polytopes, but in view of [Mész14c, Section 4] some of their content can also be understood in the language of root polytopes).

One of the main results of this chapter is Theorem 5.1.1, in which we give a geometric realization of the pipe dream complex of $1\pi'$ in terms of triangulations of root polytopes. This result has several interesting consequences explored here. In this Chapter, we also describe the geometry behind our polytopal realization of subword complexes. In particular, given a matrix Schubert variety $\overline{X_\pi}$ we use Fulton's essential set to define a variety $Y_\pi \hookrightarrow \overline{X_\pi}$ and in Theorem 5.3.6 we use $D(\pi)$ to characterize the π for which Y_π is a toric variety. Furthermore, we show that when $\pi(1) = 1$ polytope of Y_π can be projected to an acyclic root polytope.

CHAPTER 2

COMBINATORIAL BACKGROUND

We introduce the combinatorial objects that will be relevant to our main theorems. We start by defining *subword complexes*, and a subfamily of these simplicial complexes called *pipe dream complexes*. We then proceed to define *brick polytopes*, which are a geometric realization for a subfamily of subword complexes. In Chapter 5 we will realize geometrically a subfamily of subword complexes using root polytopes; we end this chapter by defining these polytopes.

2.1 Pipe dream complexes and subword complexes

2.1.1 Subword complexes

The main reference for this section is [KM04].

Let W be the Weyl group of a complex Lie group G with respect to a torus T, i.e., W is a crystallographic Coxeter group, and let $S = \{s_i : i \in I\}$ denote its generators. Our examples in this section will be of type A_n which means $W = S_{n+1}$, the symmetric group on $\{1, \ldots, n+1\}$. We will denote the permutation $f \in S_{n+1}$ in one line notation, i.e., $f = [f(1), \ldots, f(n+1)]$. The generators of this group are $s_1, \ldots, s_n$, where s_i transposes $i \leftrightarrow i+1$.

Let $Q = (q_1, \ldots, q_m)$ be a **word** in S, i.e. an ordered sequence of elements of S. A **subword** $J = (r_1, \ldots, r_m)$ of Q is a word obtained from Q by replacing some of its letters by $-$. There are a total of $2^{|Q|}$ subwords of Q. Given a subword J, we denote by $Q \setminus J$ the complementary subword with k-th entry equal to $-$ if $r_k \neq -$

and equal to q_k otherwise for $k = 1, \ldots, m$. For example, $J = (s_1, -, s_3, -, s_2)$ is a subword of $Q = (s_1, s_2, s_3, s_1, s_2)$ and $Q \setminus J = (-, s_2, -, s_1, -)$. Given a subword J we denote by $J_{(k)}$ the product of the leftmost k letters in J with $-$ behaving as the identity, if $k \geq 1$.

Definition 2.1.1. Let $Q = (q_1, \ldots, q_m)$ be a word in S and $w \in W$. The **subword complex** $\Delta(Q, w)$ is the simplicial complex on the vertex set Q whose facets are those subwords F of Q such that $(Q \setminus F)_{(m)}$ is a reduced expression for w.

Example 2.1.2. Consider the Weyl group $W = A_2$. Let $Q = (s_1, s_2, s_1, s_2, s_1)$ and $w = s_1 s_2 s_1$; then the simplicial complex $\Delta(Q, w)$ is shown in Figure 2.1. In order to make the reduced expression more explicit, we are labeling the faces by their complements.

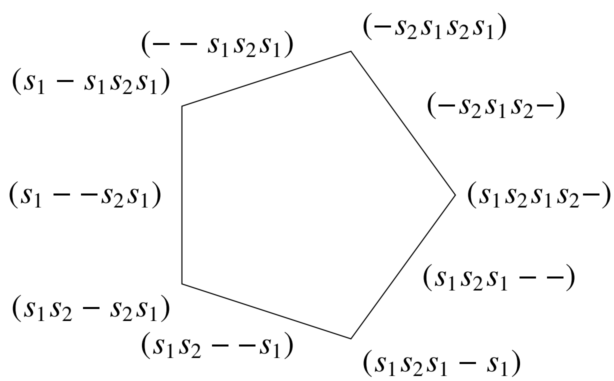


Figure 2.1: Subword complex for $Q = (s_1, s_2, s_1, s_2, s_1)$ and $w = [321]$.

Definition 2.1.3. We define the **Demazure product** of a word Q inductively as follows:

- Dem(empty word) = id
- $\mathrm{Dem}((Q, s)) = \begin{cases} \mathrm{Dem}(Q) \cdot s & \text{if } \ell(\mathrm{Dem}(Q)s) > \ell(\mathrm{Dem}(Q)), \text{ and} \\ \mathrm{Dem}(Q) & \text{if } \ell(\mathrm{Dem}(Q)s) < \ell(\mathrm{Dem}(Q)), \end{cases}$

where (Q, s) is the concatenation of Q and $s \in S$.

In [KM04] the authors prove that $\Delta(Q, w)$ is a sphere if and only if $\mathrm{Dem}(Q) = w$, and a ball for any other w. If we assume Q is reduced and $w = \mathrm{Dem}(Q)$, then $\Delta(Q, w) = \{\emptyset\}$, the (-1)-sphere, so we will not consider this case here.

2.1.2 Pipe dream complexes

Pipe dream complexes are a subfamily of subword complexes of type A, and can be drawn in a combinatorial way inside an $n \times n$ matrix. In [KM05] they were used to describe the geometry of determinantal ideals and their cohomology, i.e. Grothendieck polynomials. The main references for this section are [BB93, KM05].

Let π be a permutation, then associated to this permutation is a **permutation matrix** where its (i, j)-th entry is

$$(\pi)_{(i,j)} = \begin{cases} 1, & \text{if } \pi(j) = i, \\ 0, & \text{else.} \end{cases}$$

We will use π to denote both the permutation and its permutation matrix.

Definition 2.1.4. The **(Rothe) diagram** of a permutation π is the collection of boxes $D(\pi) := \{(\pi_j, i) : i < j, \pi_i > \pi_j\}$. It can be visualized by considering the boxes left in the $n \times n$ grid after we cross out the boxes appearing south or east (but not both) of each 1 in the permutation matrix for π.

Notice that no two permutations can give the same diagram. We will consider permutations of the form $\pi = 1\pi'$ where π' is a dominant permutation of

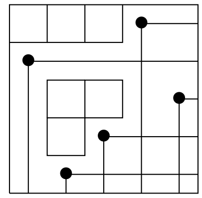


Figure 2.2: The diagram for $\pi = [25413]$.

$\{2, \ldots, n\}$, i.e., the diagram of π is a partition with north-west most box at position $(2, 2)$. Our convention is to encode the partition by the number of boxes in each column.

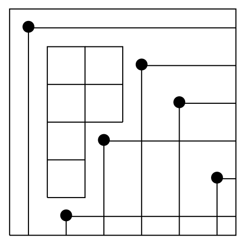


Figure 2.3: The diagram for $\pi = [164235]$ which corresponds to $\lambda = (4, 2)$.

Definition 2.1.5. A **pipe dream** for $\pi \in S_n$ is a tiling of an $n \times n$ matrix with two tiles, crosses ┼ and elbows ╯╭, such that

1. all tiles in the weak south-east triangle are elbows, and
2. if we write $1, 2, \ldots, n$ on the left and follow the strands (ignoring second crossings among the same strands) they come out on the top and read π.

A pipe dream is **reduced** if no two strands cross twice. See Figure 2.4 for an example of a reduced pipe dream.

Definition 2.1.6. The **pipe dream complex** $PD(\pi)$ of a permutation $\pi \in S_n$ is the simplicial complex with vertices given by entries on the northwest triangle of

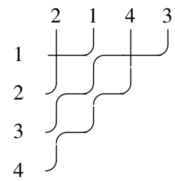


Figure 2.4: A reduced pipe dream for $\pi = [2143]$.

an $n \times n$-matrix and facets given by the positions of the elbows in the reduced pipe dreams for π.

Pipe dream complexes are a special case of subword complexes. We proceed to explain the correspondence. The group S_n is generated by the adjacent transpositions $s_1, \ldots, s_{n-1}$, where s_i transposes $i \leftrightarrow i+1$. Let $Q = (q_1, \ldots, q_m)$ be a word in $\{s_1, \ldots, s_{n-1}\}$, i.e., Q is an ordered sequence. In this language, $PD(\pi)$ is the subword complex $\Delta(Q, \pi)$ corresponding to the triangular word $Q = (s_{n-1}, s_{n-2}, s_{n-1}, \ldots, s_1, s_2, \ldots, s_{n-1})$ and π. The correspondence between pipe dreams and subwords is induced by the labeling of the entries in the northwest triangle of an $n \times n$-matrix by adjacent transpositions, as depicted in Figure 2.5, and by making a $+$ in a pipe dream correspond to a – in a subword and a ⌟⌜ correspond to the s_i in its entry. (This may seem backward, but remember that we take $(Q \setminus F)_{(m)}$ in Definition 2.1.1.) In order to go from a pipe dream to a subword, we read the entries in the northwest triangle from left to right starting at the bottom.

Definition 2.1.7. Let cone(π) be the set of vertices of $PD(\pi)$ that are in all its facets. We define the **core** of π to be the restriction of $PD(\pi)$ to the set of vertices not in cone(π).

Notice that $PD(\pi)$ is obtained from its core by iteratively coning the simplicial complex core(π) over the vertices in cone(π). This is a standard definition for

s_1	s_2	s_3	s_4	$\cdots$
s_2	s_3	s_4		
s_3	s_4			
s_4				
$\vdots$				

Figure 2.5: Labeling of the entries in the northwest triangle by adjacent transpositions.

simplicial complexes. In the language of pipe dream complexes, the core is the restriction to the entries in the $n \times n$ matrix that are a cross in some reduced pipe dream for π. Following the correspondence described in Figure 2.5, this restriction induces a subword Q' of the triangular word and so core(π) is the subword complex $\Delta(Q', \pi)$.

Since we are only considering permutations of the form $1\pi'$ with π' a dominant permutation, core($1\pi'$) is easy to describe. Given a diagram of a permutation there are two natural reduced pipe dreams for π, referred to as the **bottom reduced pipe dream of** π and the **top reduced pipe dream of** π, one obtained by aligning the diagram to the left and replacing the boxes with crosses and the other one by aligning the diagram up. See Figures 2.6 and 2.7.

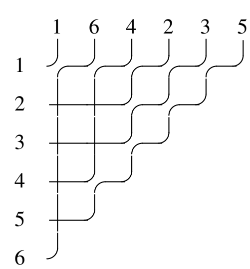


Figure 2.6: The reduced pipe dream for [164235] obtained by aligning the diagram to the bottom.

The core of $1\pi'$ is the simplicial complex obtained by restricting $PD(\pi)$ to the vertices corresponding to the positions of the crosses in the superimposition of

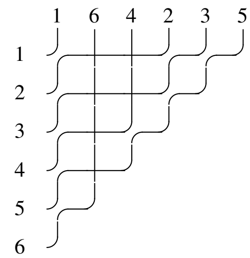


Figure 2.7: The reduced pipe dream for [164235] obtained by aligning the diagram to the top.

these two pipe dreams. We refer to the region itself as the **core region**, and denote it by cr(π). See Figure 2.8 for an example.

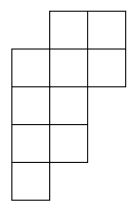


Figure 2.8: The core region of [164235].

In [BB93], Bergeron and Billey introduced an algorithm to construct all reduced pipe dreams for π. Given a reduced pipe dream P for π, a **ladder admitting rectangle** is a connected $k \times 2$ rectangle inside P such that $k \geq 2$ and the only ╯╭ inside this rectangle are in the top row and in the southeast corner, see the diagram on the left in Figure 2.9. A **ladder move** on P moves the ┼ in the southwest corner of a ladder admitting rectangle to the northeast corner. Notice that the resulting pipe dream is a reduced pipe dream for π.

Theorem 2.1.8. *[BB93] All reduced pipe dreams of π can be derived from the bottom reduced pipe dream by a sequence of ladder moves.*

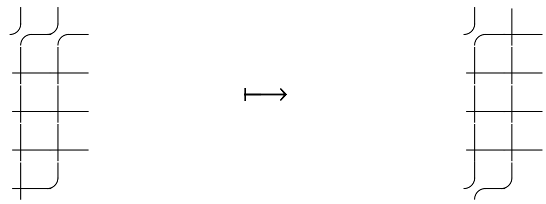


Figure 2.9: A ladder move.

2.2 Brick polytopes

Brick polytopes were first defined in type A by Pilaud and Francisco Santos in [PS12] as geometric realizations of simplicial complexes associated to pseudoline arrangements, a.k.a. type A subword complexes. This was a follow up work to the paper by Pilaud and Michel Pocchiola [PP12] in which they give the structure of simplicial complex to pseudoline arrangements and give a geometric realization of the associahedron in terms of this simplicial complex. Brick polytopes were defined for all types by Pilaud and Stump in [PS11] where they used these polytopes to give geometric realizations of a subfamily of spherical subword complexes for any Coxeter group, and thus giving some insight to a question by Knutson and Miller, see [KM04, Question 6.4].

The main reference for this section is [PS11].

2.2.1 Brick polytopes for $G = SL_n(\mathbb{C})$

The **sorting network** $\mathcal{N}_Q$ of a word $Q = (q_1, \ldots, q_m)$ consists of n horizontal lines (called the **levels**) and m vertical segments (called the **commutators**) drawn from left to right so that each commutator joins consecutive levels, no two commutators share a common endpoint, and if $q_k = s_i$ then the k-th commutator connects levels i and $i+1$. A **brick** of $\mathcal{N}_Q$ is a connected component of its complement, bounded on the left by a commutator.

A **pseudoline** supported by $\mathcal{N}_Q$ is a path on $\mathcal{N}_Q$ traveling monotonically from left to right. A commutator of $\mathcal{N}_Q$ is called a **crossing** between two pseudolines if it is crossed by the two pseudolines and it is called a **contact** otherwise. A **pseudoline arrangement** on $\mathcal{N}_Q$ is a collection of n pseudolines such that each two have at most one crossing and no other intersection.

Example 2.2.1. If $Q = (s_1, s_2, s_1, s_2, s_1)$ then $w_0 = \mathrm{Dem}(Q) = s_1s_2s_1 = s_2s_1s_2$. The sorting network $\mathcal{N}_Q$ and a pseudoline arrangement on $\mathcal{N}_Q$ are depicted on Figures 2.10 and 2.11, respectively.

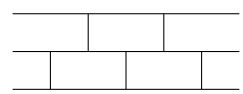


Figure 2.10: The sorting network for $Q = (s_1, s_2, s_1, s_2, s_1)$

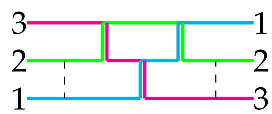


Figure 2.11: A pseudoline arrangement on $\mathcal{N}_Q$.

Given a pseudoline arrangement supported by $\mathcal{N}_Q$, if we let $J = (r_1, \dots, r_m)$ be the subword of Q with $r_i \neq -$ precisely when there is a contact at the i-th commutator, then the product $w = (Q \setminus J)_{(m)}$ is an element of W and the pseudoline ending on the right at level i will start on the left at level $w(i)$. We call such an arrangement a w**-pseudoline arrangement**. There is a one-to-one correspondence between facets J of $\Delta(Q, w)$ and $(Q \setminus J)_{(m)}$-pseudoline arrangements supported by $\mathcal{N}_Q$. The pseudoline arrangement in the previous example corresponds to the subword $J = (s_1, -, -, -, s_1)$. The i-th coordinate of the **brick vector** $B(J)$ is defined to be the number of bricks in $\mathcal{N}_Q$ that lie above the i-th pseudoline with contacts J, and the **brick polytope** $B(Q, w)$ is the following convex hull:

$$B(Q, w) := \text{conv}\{B(J) : J \in \Delta(Q, w) \text{ and } (Q \setminus J)_{(m)} = w\}.$$

Example 2.2.2. Let $Q = (s_1, s_2, s_1, s_2, s_1)$ and $w = s_1 s_2 s_1$. Then the pseudoline arrangement corresponding to the subword $J = (s_1, -, -, -, s_1)$ gives the vector $B(J) = (2, 1, 4)$ obtained by counting bricks above each line. The brick polytope $B(Q, w)$ is pictured in Figure 2.12.

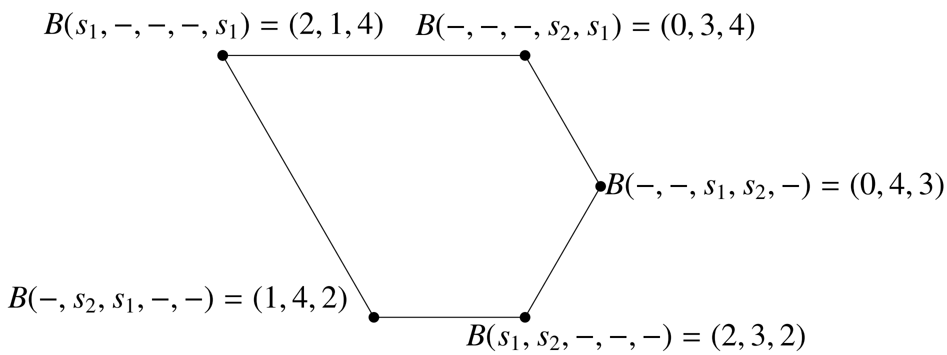


Figure 2.12: The brick polytope $\mathcal{B}(Q, w)$ for $Q = (s_1, s_2, s_1, s_2, s_1)$ and $w = s_1 s_2 s_1$.

See [PS11] for more pictures of brick polytopes of various Q and w.

The associahedron

The n-dimensional associahedron is a simple polytope with vertices corresponding to the triangulations of a convex $(n + 3)$-gon. Moreover, two vertices are adjacent if and only if they differ by one diagonal. It was first noted by Pilaud and Pocchiola in [PP12] that the type A associahedron can be realized as a polytope using pseudoline arrangements, which are a way to describe subword complexes. The works of Stump [Stu11] and Luis Serrano-Stump [SS12] describe the type A associahedron explicitly in terms of subword complexes. See Figures 2.13 and 2.14 for two examples of the associahedron.

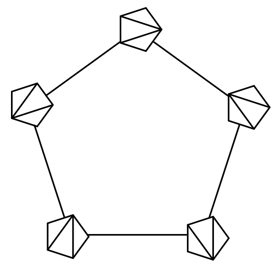


Figure 2.13: The 2-dimensional associahedron.

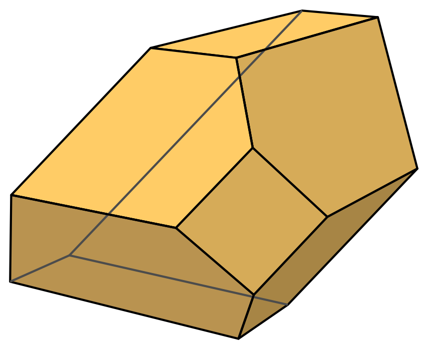


Figure 2.14: A geometric realization of the 3-dimensional associahedron.

Notice that the polytope in Figure 2.12 is a geometric realization of the 2-dimensional associahedron.

2.2.2 General brick polytopes

Let $\Delta(W) := \{\alpha_s : s \in S\}$ be the simple roots of W and let $\nabla(W) := \{\omega_i : s_i \in S\}$ be its fundamental weights. Pilaud and Stump defined brick polytopes and studied their properties in [PS11]. For them, the brick polytope is the convex hull of some conjugates of the fundamental weights of the Weyl group, one per each facet of the subword complex. Our definitions in this section are based on theirs; however we make the brick polytope be the convex hull of the brick vectors corresponding to all the faces in the subword complex such that the product of the complement is w. We prove the equivalence of the definitions in Theorem 4.2.1.

Given a subword complex $\Delta(Q, w)$ with $|Q| = m$ define the **root function**

$$\mathsf{r}(J, \cdot) : \{\text{subwords of } Q\} \to \Delta(W)$$

$$\mathsf{r}(J, k) := (Q \setminus J)_{(k-1)} \cdot (\alpha_{q_k}) \tag{2.1}$$

and the **weight function**

$$\mathsf{w}(J, \cdot) : \{\text{subwords of } Q\} \to \nabla(W)$$

$$\mathsf{w}(J, k) := (Q \setminus J)_{(k-1)} \cdot (\omega_{q_k}). \tag{2.2}$$

Definition 2.2.3. The **brick vector** of a face J of $\Delta(Q, w)$ is defined by

$$B(J) := \sum_{k \in [m]} \mathsf{w}(J, k),$$

and the **brick polytope** is the convex hull of the brick vectors of some faces of $\Delta(Q, w)$

$$B(Q, w) := \operatorname{conv}\{B(J) : J \in \Delta(Q, w) \text{ and } (Q \setminus J)_{(m)} = w\}.$$

Definition 2.2.4. A word Q is **root independent** if for some vertex $B(J)$ of $B(Q, w)$ (or all vertices) we have that the multiset $r(J) := \{\{\mathsf{r}(J, i) : i \in J\}\}$ is linearly independent.

Pilaud and Stump in [PS11] show that if Q is root independent, then the brick polytope is dual to the subword complex. Theorem 4.2.2 in this thesis states that the brick variety of a word Q is toric with respect to a maximal torus of the Lie group exactly when Q is root independent.

2.3 Root polytopes

We follow the exposition of Mészáros in [Mész14c, §4]. A root polytope of type A_n is the convex hull of the origin and some of the points $e_i - e_j$ for $1 \leq i < j \leq n+1$, where e_i denotes the i^{th} coordinate vector in $\mathbb{R}^{n+1}$. The full root polytope is

$$\mathcal{P}(A_n^+) := \mathrm{ConvHull}(0, e_i - e_j : 1 \leq i < j \leq n+1).$$

In this thesis we restrict ourselves to a class of root polytopes including $\mathcal{P}(A_n^+)$, which have subdivision algebras as defined in [Mész11a]. We discuss subdivision algebras in relation to Grothendieck polynomials in Section 5.2.1.

Let Γ be an acyclic graph on the vertex set $[n+1]$. Define

$$\mathcal{V}_\Gamma := \{e_i - e_j : (i, j) \in E(\Gamma), i < j\},$$

a set of positive roots for $SL_n(\mathbb{C})$ associated to Γ;

$$\mathrm{cone}(\Gamma) := \langle \mathcal{V}_\Gamma \rangle := \left\{ \sum_{e_i - e_j \in \mathcal{V}_\Gamma} c_{ij}(e_i - e_j) : c_{ij} \geq 0 \right\},$$

the cone associated to Γ; and

$$\overline{\mathcal{V}}_\Gamma := \Phi^+ \cap \mathrm{cone}(\Gamma),$$

all the positive roots of type A_n contained in cone(Γ), where $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n+1\}$ is the set of positive roots of type A_n.

The **root polytope** $\mathcal{P}(\Gamma)$ associated to the acyclic graph Γ is

$$\mathcal{P}(\Gamma) := \mathrm{ConvHull}(0, e_i - e_j : e_i - e_j \in \overline{\mathcal{V}}_\Gamma). \tag{2.3}$$

The root polytope $\mathcal{P}(\Gamma)$ associated to a graph Γ can also be defined as

$$\mathcal{P}(\Gamma) := \mathcal{P}(A_n^+) \cap \mathrm{cone}(\Gamma). \tag{2.4}$$

Note that $\mathcal{P}(A_n^+) = \mathcal{P}(P_{n+1})$ for the graph P_{n+1} on the vertex set $[n+1]$ where each $i \in [n+1]$ is adjacent to exactly $i-1$ and $i+1$.

The **reduction rule for graphs:** Given a graph Γ_0 on the vertex set $[n+1]$ and $(i,j),(j,k) \in E(\Gamma_0)$ for some $i < j < k$, let $\Gamma_1, \Gamma_2, \Gamma_3$ be graphs on the vertex set $[n+1]$ with edge sets

$$\begin{aligned} E(\Gamma_1) &= E(\Gamma_0)\backslash\{(j,k)\} \cup \{(i,k)\}, \\ E(\Gamma_2) &= E(\Gamma_0)\backslash\{(i,j)\} \cup \{(i,k)\}, \\ E(\Gamma_3) &= E(\Gamma_0)\backslash\{(i,j),(j,k)\} \cup \{(i,k)\}. \end{aligned} \tag{2.5}$$

We say that Γ_0 **reduces** to $\Gamma_1, \Gamma_2, \Gamma_3$ under the reduction rules defined by equations (2.5).

Lemma 2.3.1. *[Mész11a]* ***(Reduction Lemma for Root Polytopes)*** *Given an acyclic graph Γ_0 with d edges let $(i,j),(j,k) \in E(\Gamma_0)$ for some $i < j < k$ and $\Gamma_1, \Gamma_2, \Gamma_3$ as described by equations (2.5). Then*

$$\mathcal{P}(\Gamma_0) = \mathcal{P}(\Gamma_1) \cup \mathcal{P}(\Gamma_2)$$

where all polytopes $\mathcal{P}(\Gamma_0), \mathcal{P}(\Gamma_1), \mathcal{P}(\Gamma_2)$ are d-dimensional and

$$\mathcal{P}(\Gamma_3) = \mathcal{P}(\Gamma_1) \cap \mathcal{P}(\Gamma_2) \text{ is } (d-1)\text{-dimensional.}$$

What the Reduction Lemma really says is that performing a reduction on an acyclic graph Γ_0 is the same as dissecting the d-dimensional polytope $\mathcal{P}(\Gamma_0)$ into two d-dimensional polytopes $\mathcal{P}(\Gamma_1)$ and $\mathcal{P}(\Gamma_2)$, whose vertex sets are subsets of the vertex set of $\mathcal{P}(\Gamma_0)$, whose interiors are disjoint, whose union is $\mathcal{P}(\Gamma_0)$, and whose intersection is a facet of both.

Theorem 2.3.2. *[Mész11a] Let $T_1, \ldots, T_k$ be the noncrossing alternating spanning trees of the directed transitive closure of the acyclic graph Γ. Then $\mathcal{P}(T_1), \ldots, \mathcal{P}(T_k)$ are top dimensional simplices in a regular triangulation of $\mathcal{P}(\Gamma)$.*

We refer to the triangulation from the above theorem as the **canonical triangulation**. Since each simplex $\mathcal{P}(T_i)$, $i \in [k]$, contains the vertex 0, it follows that the canonical triangulation of $\mathcal{P}(\Gamma)$ also induces a triangulation of the vertex figure of $\mathcal{P}(\Gamma)$ at 0, which we also call the canonical triangulation of the vertex figure of $\mathcal{P}(\Gamma)$ at 0. Denote the simplicial complex of the latter triangulation by $C\Gamma)$. The vertices of $C\Gamma)$ are in bijection with edges (i, j) in the directed transitive closure of Γ; the vertex of $C\Gamma)$ corresponding to (i, j) is the intersection of the ray pointing to $e_i - e_j$ and the hyperplane by which we intersect $\mathcal{P}(\Gamma)$ to obtain the considered vertex figure.

CHAPTER 3

GEOMETRIC BACKGROUND

We introduce the geometric objects and tools that will be relevant to our main theorems. We start by defining Bott-Samelson varieties starting with the case $G = SL_n(\mathbb{C})$ as a motivation for the case in which G is a complex semisimple Lie group. We then discuss generic smoothness for algebraic varieties, a tool which will be used in Chapter 4. Then we define matrix Schubert varieties, following [Ful92]. We end the geometric background with moment maps and polytopes for actions of algebraic tori on projective varieties. Moment polytopes will provide a useful tool to study the action of an algebraic torus on brick and matrix Schubert varieties. In particular, in Section 3.4 we will give a criterion to prove that a projective variety is a toric variety.

3.1 Bott-Samelson varieties

Bott-Samelson varieties were first defined by Bott and Samelson in [BS55]. Michel Demazure proved in [Dem74] that these varieties are desingularisations of Schubert varieties. We start with the case $G = SL_n(\mathbb{C})$ as motivation for general complex semisimple Lie groups G. Our main reference for this section is [Mag98].

3.1.1 Bott-Samelson varieties for $SL_n(\mathbb{C})$

Let $G = SL_n(\mathbb{C})$ be the special linear group, that is the set of $n \times n$ matrices over $\mathbb{C}$ with determinant 1. Fix an ordered basis $e_1, \ldots, e_n$ for $\mathbb{C}^n$. Let B be the subgroup of $SL_n(\mathbb{C})$ consisting of upper triangular matrices with respect to this basis. We then get an ascending flag of B-invariant vector spaces

$$\langle e_1 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle,$$

which we refer to as the **base flag**. Let T be the subgroup consisting of all diagonal matrices in G, so T is a maximal algebraic torus contained in B. Let P_i be the minimal parabolic subgroup consisting of all matrices in G that are upper triangular except possibly at the position $(i+1, i)$. The quotient G/B is the **flag variety**, that is, the space of flags $\{0\} \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ where each V_i is an i-dimensional vector space. The Weyl group of G is $W = A_{n-1}$ with generators $S = \{s_1, \ldots, s_{n-1}\}$. The fundamental weights are $\nabla(W) = \{\omega_i : i = 1, \ldots, n-1\}$ where the first i entries of ω_i are 1 and the rest are 0.

We begin the definition of BS^Q with an example.

Example 3.1.1. Let $G = SL_3(\mathbb{C})$ and $Q = (s_1, s_2, s_1, s_2, s_1)$ be a word on the generators S. The Bott-Samelson variety BS^Q is constructed by starting with the base flag and then iteratively reading the word from left to right: if the k-th letter of Q is s_i, we have an i-dimensional vector space V_k such that it contains the preceding $(i-1)$-dimensional space and is contained in the preceding $(i+1)$-dimensional vector space. In this example we have that

$$BS^{(s_1,s_2,s_1,s_2,s_1)} = \{(V_1, V_2, V_3, V_4, V_5) : \text{ the diagram in Figure 3.1 holds}\}.$$

More generally, if $Q = (q_1, \ldots, q_m)$ is a word of length m in the generators S then BS^Q, **the Bott-Samelson variety of** Q, consists of a list $(F_0, \ldots, F_m)$ of $m+1$ flags where the zeroth one is the base flag and such that the k-th one agrees with the previous one except possibly on the k-th subspace V_k. We can give a point in

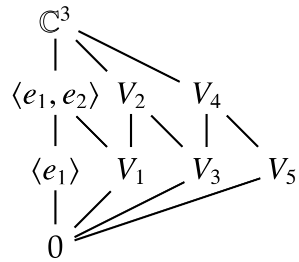


Figure 3.1: The Bott-Samelson variety for $Q = (s_1, s_2, s_1, s_2, s_1)$

BS^Q by listing the subspaces $(V_1, \ldots, V_m)$ such that the incidence relations given by the flags hold. This space carries a B-action given by

$$b \cdot (F_0, \ldots, F_m) := (b \cdot F_0, \ldots, b \cdot F_m),$$

and the map $BS^Q \xrightarrow{\mathrm{m_Q}} G/B$ mapping the list to the last flag is B-equivariant.

Example 3.1.2. Continuing with the previous example, we have that

$$m_Q : BS^{(s_1,s_2,s_1,s_2,s_1)} \to G/B$$

is the map shown in Figure 3.2.

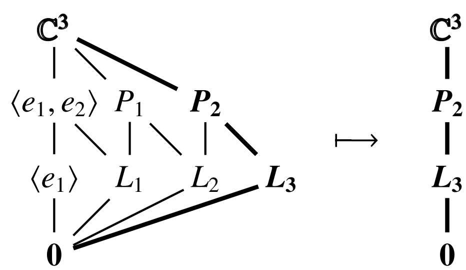


Figure 3.2: Map m_Q for $Q = (s_1, s_s, s_1, s_2, s_1)$.

3.1.2 Bott-Samelson varieties in general

Let G be a complex semisimple Lie group, let B be a Borel subgroup of G, i.e., a maximal solvable subgroup, and T be a maximal torus contained in B. Let W

be the Weyl group of G with generators $S = \{s_1, \ldots, s_n\}$, which correspond to the simple roots $\Delta(W) = \{\alpha_1, \ldots, \alpha_n\}$. Let P be a parabolic subgroup of G, i.e., a subgroup of G for which the quotient B/P is a projective algebraic variety; this condition is equivalent to requiring P to contain B. We denote by P_i the minimal parabolic subgroup corresponding to s_i, we then have that $P_i/B \cong \mathbb{CP}^1$. The torus T acts on this quotient and this action has exactly two T-fixed points: the cosets B and s_iB.

Definition 3.1.3. Let $Q = (s_{i_1}, \ldots, s_{i_m})$ be a word in the generators of W. Then the product $P_{i_1} \times \cdots \times P_{i_m}$ has an action of B^m given by:

$$(b_1, \ldots, b_m) \cdot (p_1, \ldots, p_m) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{m-1}^{-1} p_m b_m).$$

The **Bott-Samelson variety** of Q is the quotient of the product of the P_i's by this action

$$BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m.$$

Bott-Samelson varieties can be constructed as iterated $\mathbb{CP}^1$-bundles and from this construction follows that these varieties are smooth, irreducible and $|Q|$-dimensional; see [Mag98, §1.1] for this construction. These varieties have a B-action given by

$$b \cdot (p_1, p_2, \ldots, p_m) := (b \cdot p_1, p_2, \ldots, p_m), \tag{3.1}$$

and they come equipped with a natural B-equivariant map

$$BS^Q \xrightarrow{\mathrm{m_Q}} G/B$$

$$(p_1, \ldots, p_m) \longmapsto (p_1 \cdots p_m)B.$$

The image of this map is the **opposite Schubert variety**

$$X^w := \overline{BwB}, \tag{3.2}$$

where $w = \mathrm{Dem}(Q)$. In the case in which Q is reduced, this map is a resolution of singularities for X^w; however in this thesis we will study cases in which Q is not reduced.

We can describe points in BS^Q as a list of flags for general G, similarly to the case $G = SL_n(\mathbb{C})$. If $Q = (q_1, \ldots, q_m)$ is a word of size m, then the points in BS^Q consist of lists of $m+1$ flags in G/B

$$(F_0 = B, F_1, \ldots, F_{|Q|}) \in BS^Q$$

such that the k-th flag agrees with the previous one except possibly on the subspace corresponding to q_k, i.e., if $F_{k-1} = gB$ and $F_k = hB$, then $h^{-1}gB \in X^{q_k}$. With this description we have that the B-action is given by

$$b \cdot (F_0, \ldots, F_m) = (b \cdot F_0, \ldots, b \cdot F_m),$$

and m_Q maps a list of flags to the last one.

3.2 Generic smoothness

The reference for this section is [Har77].

In this section we state generic smoothness, the main tool which will allow us to prove that brick varieties are smooth and irreducible and to compute their dimension. Generic smoothness in algebraic geometry is the analogue of Sard's theorem in differential topology, which states that the set of critical values of a smooth function $f : M \to N$ between real manifolds has Lebesgue measure 0. In algebraic geometry, a smooth morphism $f : X \to Y$ is a fiber bundle such that all its fibers are smooth.

Theorem 3.2.1 (Generic smoothness). *[Har77, III, Corollary 10.7] Let $f : X \to Y$ be a morphism of varieties over an algebraically closed field k of characteristic 0, and assume that X is nonsingular. Then there is a nonempty open subset $V \subset Y$ such that $f : f^{-1}(V) \to V$ is smooth. In the case in which $f^{-1}(V) \neq \emptyset$, the fiber $f^{-1}(v)$ is nonsingular and* $\dim(f^{-1}(v)) = \dim(X) - \dim(Y)$ *for all $v \in V$.*

3.3 Matrix Schubert varieties

The main reference for this section is [Ful92].

For this section we consider the case $G = GL_n(\mathbb{C})$. Let B_+ be upper triangular invertible $n \times n$ matrices and B_- be lower triangular invertible $n \times n$ matrices. Let M_n denote the set of $n \times n$ matrices over $\mathbb{C}$. We let $\pi \in S_n$ denote both a permutation and its corresponding permutation matrix, see Section 2.1.2. An $n \times n$-matrix can always be reduced into a partial permutation matrix by multiplying on the left by matrices in B_- and on the right by matrices in B_+. The multiplication on the left corresponds to downward row operations and multiplication on the right corresponds to rightward column operations. This multiplication gives a left action of $B_- \times B_+$ on M_n defined by $(X, Y) \cdot M := XMY^{-1}$.

Given $1 \leq a \leq n$ and $1 \leq b \leq n$, let $M_{(a,b)}$ denote the upper left $a \times b$ submatrix of the matrix M, see Figure 3.3 for an illustration. Define a **rank function** of a matrix M to be $r_M(a, b) := \operatorname{rank}(A_{(a,b)})$. We then have that $M \in B_- \pi B_+$ if and only if $r_M(a, b) = r_\pi(a, b)$ for all $(a, b) \in [n] \times [n]$.

Definition 3.3.1. The **matrix Schubert variety** of π is $\overline{X_\pi} := \overline{B_- \pi B_+}$, i.e. the Zarisksi closure of its $(B_- \times B_+)$-orbit inside $M_n = \mathbb{C}^{n^2}$.

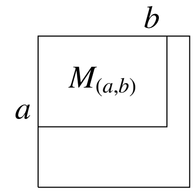


Figure 3.3: The submatrix $M_{(a,b)}$ of M.

William Fulton studied this affine variety in [Ful92]. We summarize some of his results here.

Theorem 3.3.2. *[Ful92, Proposition 3.3] The matrix Schubert variety $\overline{X_\pi}$ is an irreducible variety of dimension $n^2 - \ell(\pi)$ defined as a scheme by the equations $r_M(a,b) \le r_\pi(a,b)$ for all $(a,b) \in [n] \times [n]$.*

Some of these inequalities are implied by others, and Fulton described the minimal set of rank conditions.

Definition 3.3.3. Fulton's essential set, denoted $Ess(\pi)$, is the set consisting of the south-east corners of $D(\pi)$, see Definition 2.1.4.

Theorem 3.3.4. *[Ful92, Lemma 3.10] The ideal defining the variety $\overline{X_\pi}$ is generated by the equations $r_M(a,b) \le r_\pi(a,b)$ for all $(a,b) \in Ess(\pi)$.*

Definition 3.3.5. The **dominant piece**, denoted $dom(\pi)$, of a permutation π is the connected component of the diagram of π containing the box $(1,1)$, or empty if $\pi(1) = 1$.

We have that $r_\pi(a,b) = 0$ if and only if $(a,b) \in dom(\pi)$. Therefore the dominant piece of π consists precisely of the coordinates in M_n that are 0 on $\overline{X_\pi}$. Note that if $\pi(1) = 1$ then its dominant piece is empty.

In [KM05], Knutson and Miller use Gröbner bases to degenerate matrix Schubert varieties into unions of coordinate subspaces corresponding to reduced pipe dreams. This approach gives a geometric origin to reduced pipe dreams. We now give some basic background on Gröbner bases, but see [Eis95, Chapter 15] for further background and geometric interpretations.

A **term order** on $\mathbb{C}[z_1, \ldots, z_n]$ is a well order on the set of monomials such that $1 \leq m$ for all monomials $m \in \mathbb{C}[z_1, \ldots, z_n]$, and it is multiplicative, i.e., if $m < m'$ then for any monomial m'' we have that $m'' \cdot m < m'' \cdot m'$. If we fix a term order $<$, then for any polynomial f there is a unique largest monomial $in_<(f)$ appearing with nonzero coefficient. This monomial is called the **initial term** of f and the **initial ideal** with respect to $<$ of an ideal I is the ideal $in_<(I) = \langle in_<(f) : f \in I \rangle$. A set $\{f_1, \ldots, f_k\}$ is a **Gröbner basis** for I with respect to $<$ if $in_<(I) = \langle in_<(f_1), \ldots, in_<(f_k) \rangle$. Since $in_<(I)$ is a monomial ideal, if we take the variety corresponding to this monomial ideal we get a union of vector spaces.

Let us denote by $I_\pi \subset \mathbb{C}[z_{(1,1)}, \ldots, z_{(n,n)}]$ the ideal defining $\overline{X_\pi}$. Given a minor of a matrix, an **antidiagonal term order** on I_π picks as initial term the product of the entries on the main antidiagonal.

Theorem 3.3.6. *[KM05] The determinants from Fulton's essential set are a Gröbner basis for I_π with respect to any antidiagonal term order. Moreover, the components of the resulting initial variety correspond naturally to the reduced pipe dreams of π.*

Example 3.3.7. Let $\pi = [2143]$ then Fulton's essential set tells us that for $M \in \overline{X_\pi}$ we must have $z_{(1,1)} = 0$ and

$$\begin{vmatrix} z_{(1,1)} & z_{(1,2)} & z_{(1,3)} \\ z_{(2,1)} & z_{(2,2)} & z_{(2,3)} \\ z_{(3,1)} & z_{(3,2)} & z_{(3,3)} \end{vmatrix} = 0.$$

Now $in_<(I_\pi) = \langle z_{(1,1)}, z_{(3,1)}z_{(2,2)}z_{(1,3)}\rangle$. So for a point to be in the initial variety of $\overline{X_\pi}$, we must have $z_{(1,1)}$ and there are three possibilities for $z_{(3,1)}$, $z_{(2,2)}$, or $z_{(1,3)}$ be equal to 0. The variables used in each of the three possibilities correspond precisely with the positions of the elbows in the three reduced pipe dreams for π, see Figure 3.4.

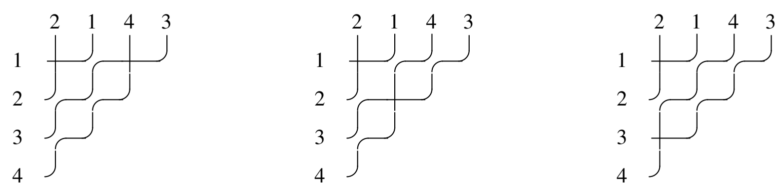


Figure 3.4: The three reduced pipe dreams for $\pi = 2143$.

3.4 Torus actions, moment maps, and toric varieties

In this section we will study the action of an algebraic torus $T^n = (\mathbb{C}^*)^n$ on a projective algebraic variety X with a T^n-equivariant embedding $X \hookrightarrow \mathbb{CP}^k$. We will describe how this setup induces an action of a real compact torus $T^n_{\mathbb{R}} \subset T^n$ on X and a moment map $\Phi : X \to (\mathfrak{t}^*)^n$, where $(\mathfrak{t}^*)^n$ is the dual Lie algebra of $T^n_{\mathbb{R}}$. On the smooth part of X, the $T^n_{\mathbb{R}}$-action is Hamiltonian. We will also define toric varieties and explain how they fit into this story. In the subsections we compute the moment map for Bott-Samelson varieties and matrix Schubert varieties. The main references for this section are [Bri87, KT93, GLS96, Knu00, CdS01].

Let M be a manifold and ω a **pre-symplectic form** on M, which is just a closed 2-form on M. If ω is also nondegenerate at every point of M then it is called a **symplectic form**. Let G be a connected Lie group acting on M preserving the form ω and $\mathfrak{g}$ its Lie algebra. Notice that G acts on itself by conjugation and therefore it acts on $\mathfrak{g}$ and on the dual $\mathfrak{g}^*$.

Definition 3.4.1. Let M be a pre-symplectic manifold with a G-action that preserves ω. A map $\Phi : M \to \mathfrak{g}^*$ is a **moment map** for the action of G on M if the following conditions hold:

1. the map Φ is G-equivariant, i.e., for all $g \in G$ and $m \in M$, we have that $\Phi(g \cdot m) = g\Phi(m)g^{-1}$, and
2. $d\langle \Phi(\cdot), X\rangle = \omega(X|_M, \cdot)$ for each $X \in \mathfrak{g}$.

We say that M is a **pre-Hamiltonian** K-manifold if the action has a moment map. In the case M is a symplectic manifold, M is **Hamiltonian**.

The moment maps we will encounter in this thesis will arise from actions of a real compact torus $T^n_{\mathbb{R}} \simeq (\mathbb{S}^1)^n$. Moreover, we will construct the moment maps for the varieties studied in this thesis using an embedding into $\mathbb{CP}^k$ and the standard moment map for the $T^k_{\mathbb{R}}$ action on $\mathbb{CP}^k$. We proceed to describe this latter moment map. Let $T^k = (\mathbb{C}^*)^k$ be an algebraic torus and define a T^k-action on $\mathbb{CP}^k$ by

$$(t_1, \ldots, t_k) \cdot [z_0, \ldots, z_k] := [z_0 : t_1 z_1 : \ldots : t_k z_k].$$

This space has a natural symplectic form called the Fubini-Study form, which is given by

$$\omega_{FS} := \frac{i}{2\pi}\partial\overline{\partial} \log |z|^2.$$

Let $T^k_{\mathbb{R}}$ be real compact torus obtained by restricting the entries of T^k to those of the form $e^{i\theta}$, where $\theta \in \mathbb{R}$. Notice that $T^k_{\mathbb{R}}$ is isomorphic to $(\mathbb{S}^1)^k$ and the action of T^k on $\mathbb{CP}^k$ induces an action of $T^k_{\mathbb{R}}$ on $\mathbb{CP}^k$. This $T^k_{\mathbb{R}}$-action has moment map $\Phi : \mathbb{CP}^k \to (\mathfrak{t}^*)^k$

$$\Phi([z_0 : \ldots : z_k]) = \frac{1}{2}\left(\frac{|z_0|^2}{\sum_i |z_i|^2}, \frac{|z_1|^2}{\sum_i |z_i|^2}, \ldots, \frac{|z_k|^2}{\sum_i |z_i|^2}\right),$$

where $(\mathfrak{t}^*)^k$ is the dual Lie algebra of $T^n_{\mathbb{R}}$. The image of this map is the simplex with vertices $\frac{1}{2}e_i$ for $i = 0, \ldots, n$.

The construction above together with Propositions 3.4.2 and 3.4.4 below are the main tools we will use to construct moment maps. Proposition 3.4.2 allows us to obtain moment maps for actions of subgroups of G.

Proposition 3.4.2. *Let G act on M with moment map $\Phi_G : M \to \mathfrak{g}^*$ and let $\rho : H \to G$ be a Lie group homomorphism which induces an action of H on M. Then M has H-moment map $\Phi_H : M \to \mathfrak{h}^*$ obtained by composing Φ_G with the map $(d\rho)^* : \mathfrak{g}^* \to \mathfrak{h}^*$.*

Example 3.4.3. Let $G \times G$ act on $M \times N$ with moment map $\Phi_{G\times G}$. Then the inclusion $\iota : G \hookrightarrow G\times G$ gives the diagonal action of G on $M\times N$ and the moment map for this action is the sum of the G-moment maps for M and N, i.e.

$$\Phi_G = (\Phi_{G\times G})|_M + (\Phi_{G\times G})|_N.$$

The following tells us that if we have an embedded G-invariant subvariety X of $\mathbb{CP}^k$, then the restriction of the moment map of $\mathbb{CP}^k$ to X is a moment map for X.

Proposition 3.4.4. *Let N be a G-invariant submanifold of a pre-Hamiltonian manifold M with moment map Φ. Then N is also a pre-Hamiltonian manifold with moment map given by the restriction of Φ to N.*

In particular, if X is a smooth algebraic variety with an action of an algebraic torus $T^n = (\mathbb{C}^*)^n$ and a T^n-equivariant embedding into $\mathbb{CP}^k$, then X is Hamiltonian with symplectic form given by the restriction of ω_{FS} to X and moment map given by the composition $\Phi \circ \iota : X \to (\mathfrak{t}^*)^n$. The embedding $\iota : X \hookrightarrow \mathbb{CP}^k$ is T^n-equivariant if for all $t \in T^n$ and $x \in X$ we have that $\iota(t \cdot x) = t \cdot \iota(x)$.

We have seen that the image of the moment map of $\mathbb{CP}^k$ with respect to the T^k algebraic torus action is a convex polytope (the standard simplex). The following theorem says that this is true in general.

Theorem 3.4.5. *[Bri87, Remark 2 in §4.2] Let X be a closed irreducible subvariety of $\mathbb{CP}^n$ with an action of a complex algebraic torus $T \simeq (\mathbb{C})^k$. Let $\Phi_T : X \to t^*$ be a moment map for this action. Then the image of Φ_T is a convex polytope, the convex hull of the images of the T-fixed points of X.*

Definition 3.4.6. The image $\Phi_T(X)$ described in the previous theorem is called the **moment polytope** of X.

A **toric variety** is an algebraic variety X with an action of an algebraic torus $T = (\mathbb{C}^*)^n$ such that $X = \overline{T \cdot x}$ for some $x \in X$. Unlike Fulton's approach in [Ful93], we do not consider only normal varieties. If X is a normal variety (which is the case is X is smooth) the moment polytope of a toric variety captures the structure of X completely: see [CdS03], and the references within. There is a natural stratification of X given by the faces of $\Phi(X)$: for a face F take $\Phi^{-1}(F^\circ)$, where F° denotes its relative interior.

In this thesis we will consider only toric varieties T^n-equivariantly embedded in $\mathbb{CP}^k$. Let X be a projective variety with a faithful action of an algebraic torus T^n. Let $x \in X$ be a general point, then $\overline{T^n \cdot x} \subset X$ is a projective toric variety with moment polytope $\Phi(X)$. If X is irreducible and it has the same dimension as $\overline{T^n \cdot x}$ then they must be equal. We have that $\dim(\overline{T^n \cdot x}) = \dim(\Phi(X))$ and in turn $\dim(\Phi(X))$ equals the dimension of the vector space spanned by the edge directions out of a vertex of $\Phi(X)$. Thus the following Lemma holds.

Lemma 3.4.7. *X is a toric variety with respect to the $T^n/Stab(T^n)$-action if there is a vertex of $\Phi_T(X)$ such that the span of the edge directions out of this vertex has dimension* $\dim(X)$.

Here $\mathrm{Stab}(T^n)$ denotes the stabilizer of T^n acting on X.

3.4.1 Moment maps of Bott-Samelson varieties

A reference for toric moment maps of quotients G/P for P a parabolic subgroup of G is Chapter 5 of [GLS96]. Let $P_{\hat{i}}$ be the maximal parabolic subgroup of G corresponding to the generators $S_{\hat{i}} := \{s_1, \ldots, \hat{s}_i, \ldots, s_n\}$. Note that for $G = SL_n(\mathbb{C})$ each quotient $G/P_{\hat{i}}$ is a Grassmannian. Let K be the maximal compact subgroup of G. In [Kos70] Kostant observed that we can view $G/P_{\hat{i}}$ as a coadjoint orbit, i.e., a K-orbit through the fundamental weight $\omega_i \in \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K. This interpretation gives us a symplectic structure on $G/P_{\hat{i}}$ with respect to the action of K such that the inclusion

$$G/P_{\hat{i}} \hookrightarrow \mathfrak{k}^*$$

is a moment map for the K-action. Then the composition

$$G/P_{\hat{i}} \hookrightarrow \mathfrak{k}^* \longrightarrow \mathfrak{t}^*$$

is the moment map of $G/P_{\hat{i}}$ with respect to the torus action. Moreover, the moment map for the diagonal T-action on a product $\prod G/P_{\hat{i}}$ is the sum of the moment maps $G/P_{\hat{i}} \longrightarrow \mathfrak{t}^*$.

Let T act on BS^Q by

$$t \cdot (p_1, p_2, \ldots, p_m) = (t \cdot p_1, p_2, \ldots, p_m),$$

this is the restriction of the B-action defined by Equation 3.1. Given $Q = (q_1, \ldots, q_m)$ we have a T-equivariant inclusion

$$BS^Q \overset{\varphi}{\hookrightarrow} \prod_{i: s_i \in Q} G/P_{\hat{i}} \tag{3.3}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)$ and the k-th component is

$$\begin{aligned} BS^Q &\overset{\varphi_k}{\longrightarrow} G/P_{\hat{k}} \\ (p_1, \ldots, p_m) &\longmapsto (\prod_{i<j} p_i)P_{\hat{k}}. \end{aligned}$$

This map makes BS^Q a symplectic submanifold of a product of Grassmannians. The composition

$$BS^Q \overset{\varphi}{\hookrightarrow} \prod_{i: s_i \in Q} G/P_{\hat{i}} \longrightarrow \mathfrak{t}^*$$

gives us a moment map for this Bott-Samelson variety with respect to the T-action. Thus Bott-Samelson varieties are Hamiltonian symplectic manifolds with respect to this torus action. The image of this map is the **moment polytope** and by Michael Atiyah [Ati82], Victor Guillemin - Shlomo Sternberg [GS82], it equals the convex hull of the images of the T-fixed points.

Given a subword $J = (r_1, \ldots, r_m)$ of Q, let $p(J) = (F_0, \ldots, F_m) \in BS^Q$ be the list of flags where $F_0 = B$ and for $i > 0$ we have that

$$F_i = \begin{cases} F_{i-1}, \text{ if } r_i = q_i \\ q_i F_{i-1}, \text{ if } r_i = -. \end{cases}$$

This gives a correspondence between T-fixed points on BS^Q and subwords J of Q: if $p(J)$ is the T-fixed point corresponding to J then

$$m_Q(p(J)) = (Q \setminus J)_{(m)} B \in G/B.$$

This correspondence motivates the relation between fibers of the map m_Q : $BS^Q \longrightarrow G/B$ and subword complexes.

We now describe the image of the T-fixed points under the moment map. For each k we have the moment map

$$\Phi_k : G/P_{\hat{k}} \longrightarrow \mathfrak{t}^*,$$

where $\Phi_k(P_{\hat{k}}) = \omega_k$, the fundamental weight corresponding to s_k, and it maps a general element to a Weyl conjugate of this fundamental weight. Before we finish describing the maps Φ_k, we note that the moment map of BS^Q is then

$$\sum_{k=1}^{m} \varphi_k \circ \Phi_k.$$

Consider the fixed point $(p_1, \ldots, p_m)$ in BS^Q corresponding to the subword J of Q then under the moment map Φ_k each p_j corresponds to either the reflection s_{i_j} if $q_j \in J$ or to the identity in W. In other words, p_j corresponds to s_{i_j} if $p_j \notin B$ and to the identity in W otherwise. In conclusion we have that for J a subword of Q and

$$p_J = \text{ the fixed point corresponding to } J$$

$$BS^Q \overset{\varphi_k \circ \Phi_k}{\longrightarrow} \mathfrak{t}^*$$

$$p_J \longmapsto (J)_{(k-1)} \cdot (\omega_k).$$

It then follows that

$$BS^Q \overset{\Phi}{\longrightarrow} \mathfrak{t}^* \tag{3.4}$$

$$p_J \longmapsto \sum_{k=1}^{m} (J)_{(k-1)} \cdot (\omega_k) \tag{3.5}$$

The moment polytope for BS^Q is the convex hull of the points in (3.5) for all subwords J of Q.

Remark 3.4.8. We now describe the moment map of BS^Q for $G = SL_n(\mathbb{C})$. The moment map is a map

$$\Phi : BS^Q \longrightarrow \mathbb{R}\langle\nabla(W)\rangle,$$

where $\mathbb{R}\langle\nabla(W)\rangle$ is the real span of the fundamental weights of W. Let $\pi_V : \mathbb{C}^n \to V$ denote the orthogonal projection onto V and let P_V be the corresponding matrix with respect to the basis $e_1, \ldots, e_n$. Given $p = (V_1, \ldots, V_m) \in BS^Q$ the moment map of BS^Q is

$$BS^Q \xrightarrow{\Phi} \mathbb{R}^n \tag{3.6}$$

$$(V_1, \ldots, V_m) \overset{\Phi}{\longmapsto} \sum_{i=1}^{m} \operatorname{diag}(P_{V_i}). \tag{3.7}$$

Note that for $G = SL_n(\mathbb{C})$, the B-action on BS^Q restricted to T is just the extension to BS^Q of T acting on vectors in $\mathbb{C}^n$ by multiplication. For this group, the **fixed point $p(J)$ corresponding to the subword $J = (r_1, \ldots, r_m)$** is determined by deciding between = and ≠ in each diamond shown in Figure 3.5 using the rule: for $Q = (q_1, \ldots, q_m)$, we pick "=" if $r_j = q_j$ and "≠" if $r_j = -$. We illustrate this correspondence by an example.

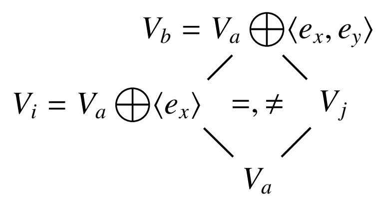


Figure 3.5: Diamond that determines the fixed point corresponding to a subword.

Example 3.4.9. The subword $J = (-, s_2, -, -, s_1)$ of $Q = (s_1, s_2, s_1, s_2, s_1)$ corresponds to the coordinate flags shown in figure 3.6, and

$$m_Q(p(J)) = (Q \setminus J)_{(m)}B = (s_1 s_1 s_2)B = (s_2)B.$$

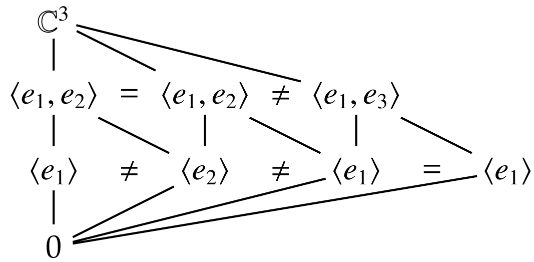


Figure 3.6: The fixed point of BS^Q corresponding to $J = (-, s_2, -, -, s_1)$.

Notice then that the fixed point $p(J)$ consists of a list of vector spaces each spanned by a subset of the basis $\{e_1 \ldots, e_n\}$. The image of the fixed point $p(J) = (V_1, \ldots, V_m)$ under the moment map is the vector

$$\left(\sum_{j=1}^{m} \dim_{e_1}(V_j), \ldots, \sum_{j=1}^{m} \dim_{e_n}(V_j)\right) \in \mathbb{R}^n,$$

where $\dim_{e_i}(V) = 1$ if and only if e_i is part of a basis for V and $\dim_{e_i}(V) = 0$ otherwise.

3.4.2 Moment maps of matrix Schubert varieties

The $(B_- \times B^+)$-action on $\overline{X_\pi}$ described in Section 3.3 restricts to a $(T^n \times T^n)$-action on $\overline{X_\pi}$, where T^n consists of $n \times n$ diagonal matrices. This action is not faithful because scaling acts the same way on both sides, so

$$\mathrm{Stab}(T^{2n}) = \{(a \cdot I, a \cdot I) : a \in \mathbb{C}^*\}.$$

We construct the moment map for $\overline{X_\pi}$ with respect to the torus T^{2n}.

Let $T^{n^2} = (\mathbb{C}^*)^{n^2}$ act on $M_n = \mathbb{C}^{n^2}$ by coordinate-wise multiplication, then $T^{n^2}_{\mathbb{R}} \simeq (\mathbb{S}^1)^{n^2}$. This action has moment map $\Phi_{T^{n^2}} : M_n \to (\mathfrak{t}^{n^2})^* \simeq \mathbb{R}^{n^2}$ given by

$$\Phi_{T^{n^2}}(z_{(1,1)}, \ldots, z_{(n,n)}) = \frac{1}{2}(|z_{(1,1)}|^2, \ldots, |z_{(n,n)}|^2).$$

Let us denote by $\mathbb{1}$ the matrix with all entries equal to 1. The inclusion $\rho : T^{2n-1} \hookrightarrow T^{n^2}$ given by $\rho(A, B) := A\mathbb{1}B^{-1}$ induces the map $(d\rho)^* : (\mathfrak{t}^{n^2})^* \to (\mathfrak{k})^*$ given by $x_{(i,j)} \mapsto x_i - y_j$, where the $x_{(i,j)}$ are a basis for $\mathbb{R}^{n^2}$, the x_i are a basis for $\mathbb{R}^n \times 0$, and the y_j are a basis for $0 \times \mathbb{R}^n$. Since we have the inclusion $\overline{X_\pi} \hookrightarrow M_n$, then by Propositions 3.4.4 and 3.4.4, we have that the T^{2n} action on $\overline{X_\pi}$ has a moment map $\overline{X_\pi} \xrightarrow{\Phi} t^*$ given by the compositon of the maps

$$\overline{X_\pi} \hookrightarrow M_n \to (\mathfrak{t}^{n^2})^* \to (\mathfrak{t}^*)^{2n}.$$

Since the ideal defining $\overline{X_\pi}$ is homogeneous, the variety $\overline{X_\pi}$ is a cone, meaning that for any $z \in \overline{X_\pi}$ and $c \in \mathbb{C}$, we have that $cz \in \overline{X_\pi}$. We can therefore projectivize it, that is we can take

$$\mathbb{P}(\overline{X_\pi}) := \{[z_{(1,1)}, \ldots, z_{(n,n)}] : (z_{(1,1)}, \ldots, z_{(n,n)}) \in \overline{X_\pi}\} \subset \mathbb{CP}^{n^2-1}.$$

This variety inherits a T^{2n-1}-action and it is projective, so we can apply Theorem 3.4.5 to compute its image.

The T^{2n-1}-fixed points of $\mathbb{P}(\overline{X_\pi})$ are the points $\mathbb{PC}E_{i,j} \in \mathbb{P}(\overline{X_\pi})$ where $\mathbb{C}E_{i,j}$ is the pencil of the matrix $E_{i,j}$ with a single nonzero entry at (i, j). Therefore, $\Phi(\mathbb{PC}E_{i,j}) = \frac{1}{2}(x_i - y_j)$ so the moment polytope of $\mathbb{P}(\overline{X_\pi})$ is the convex hull of the points $\frac{1}{2}(x_i - y_j)$ for all (i, j) not in the dominant piece of π. See Definition 3.3.5 for the definition of the dominant piece. The moment cone of $\overline{X_\pi}$ is the cone of this polytope over the origin.

CHAPTER 4

BRICK VARIETIES

In this chapter we define and study brick varieties for complex semisimple Lie groups. Following the exposition of Bott-Samelson varieties in Section 3.1, we motivate brick varieties with the case $G = SL_n(\mathbb{C})$. Brick varieties come equipped with an action of a complex algebraic torus. In Theorem 4.2.2 we characterize the brick varieties that are toric with respect to this torus action. Corollary 4.1.4 is a nice consequence of this theorem, which tells us that the toric variety of the associahedron is a brick variety. In the end of this chapter we give a stratification of the brick variety dual to the corresponding subword complex. We end the chapter noting that brick varieties have appeared as a resolution of singularities of Richardson varieties in Michel Brion's paper [Bri05].

Let G be a complex semisimple Lie group, B be a Borel subgroup of G, and T the maximal torus contained in B. Let W be the Weyl group of G with generators $S = \{s_1, \dots, s_n\}$.

We now define one of the main objects of study in this thesis.

Definition 4.0.10. Let $Q = (q_1, \dots, q_m)$ be a word in the generators of W and $w = \mathrm{Dem}(Q)$, the **brick variety** is the fiber $m_Q^{-1}(wB) \subset BS^Q$.

We will prove in Theorem 4.0.11 that brick varieties for complex semisimple Lie groups are smooth, projective, irreducible and $(|Q| - \ell(w))$-dimensional.

In Section 3.4.1 we described a 1-1 correspondence between T-fixed points in BS^Q and subwords J of Q such that if $p(J)$ is the T-fixed point corresponding to J then

$$m_Q(p(J)) = (Q \setminus J)_{(m)} B \in G/B,$$

where $m = |Q|$. This correspondence motivates the relation between fibers of the map m_Q and subword complexes. The main tool connecting brick polytopes with fibers of Bott-Samelson varieties will be moment maps of symplectic manifolds. See Section 3.4.1 for the construction of the moment map of a Bott-Samelson variety. In this section we will give a precise statement about the relation between brick polytopes and Bott-Samelson varieties.

Theorem 4.0.11. *Brick varieties are smooth (thus normal), projective and irreducible with*

$$\dim(m_Q^{-1}(wB)) = |Q| - \ell(w).$$

Proof. Consider the map $m_Q : BS^Q \to X^w$. Since BS^Q is nonsingular, then we can apply Theorem 3.2.1 and find a nonempty open subset $V \subset X^w$ such that m_Q restricted to $m_Q^{-1}(V)$ is smooth. Since multiplication by any $b \in B$ is a smooth map, then m_Q restricted to $m_Q^{-1}(b \cdot V)$ is smooth for any $b \in B$. Moreover, since the big cell BwB is dense in X^w then there exist $gB \in V \cap BwB$ and $b \in B$ such that $b \cdot gB = wB$. We thus have that $m_Q^{-1} : m_Q^{-1}(BwB) \to BwB$ is smooth so the brick variety $m_Q^{-1}(wB)$ is nonsingular and

$$\begin{aligned}\dim(m_Q^{-1}(wB)) &= \dim(BS^Q) - \dim(X^w) \\ &= |Q| - \ell(w).\end{aligned}$$

Smoothness of $m_Q^{-1}(wB)$ also implies that if $m_Q^{-1}(wB)$ is reducible, then it's disconnected. Since BwB is simply connected, if the brick variety is disconnected, $m^{-1}(BwB)$ is open and disconnected. But then BS^Q is reducible, a contradiction.

The fact that it is projective follows from the following sequence of closed embeddings

$$m_Q^{-1}(wB) \hookrightarrow BS^Q \hookrightarrow \prod G/B,$$

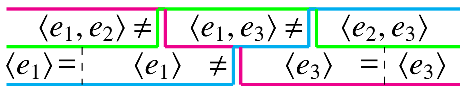
where the second embedding maps a point in BS^Q to its corresponding list of flags.

□

4.1 Brick varieties for $SL_n(\mathbb{C})$ and toric varieties for type A brick polytopes

For $G = SL_n(\mathbb{C})$, the bijective correspondence between subwords J of Q and the T-fixed points of BS^Q tells us that the rightmost flag of the configuration $p(J)$ is the flag corresponding to $(Q \setminus J)_{(m)} \in W$. Therefore, the pseudoline arrangement corresponding to S is an $(Q \setminus S)_{(m)}$-arrangement. We now give an example of the correspondence.

Example 4.1.1. The pseudoline arrangement corresponding to the subword $J = (s_1, -, -, -, s_1)$ gives a T-fixed point of $BS^{(s_1,s_2,s_1,s_2,s_1)}$. The figure below exhibits this correspondence. Each brick of the sorting network corresponds to a coordinate subspace of a point in the Bott-Samelson variety. Given a pseudoline arrangement supported in the sorting network of Q, the j-th subspace corresponding to the j-th brick is the coordinate subspace spanned by the e_i where i ranges over those pseudolines passing below the j-th brick. Note then that two adjacent bricks on the same level do not share a crossing if and only if the corresponding coordinate spaces are equal. This will be proven in the theorem that follows.



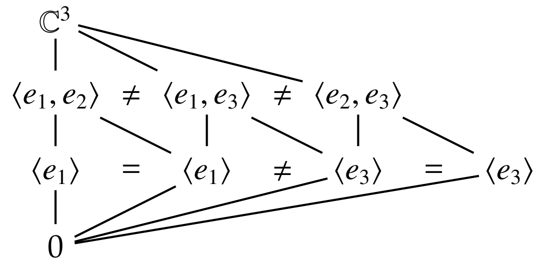


Figure 4.1: Correspondence between pseudoline arrangements and T-fixed points of the brick variety.

Theorem 4.1.2. *Suppose Dem*$(Q) = w$. *There is a bijective correspondence between* w*-pseudoline arrangements supported by* $\mathcal{N}_Q$*, and* T*-fixed points of the brick variety* $m_Q^{-1}(wB)$*. Moreover, this correspondence makes the composite map*

$$m_Q^{-1}(wB)^T \hookrightarrow m_Q^{-1}(wB) \xrightarrow{\Phi} \mathbb{R}^n$$

equivalent to the mapping

$$B : \{w\text{-pseudoline arrangements supported by } \mathcal{N}_Q\} \longrightarrow \mathbb{R}^n$$

given in [PS12].

Proof. The first part of the proposition is proven in the paragraph preceding the example above. We prove the second part of this theorem using induction on $|Q| = m$ to prove that for all subwords J, the brick vector $\mathsf{B}(J)$ equals Φ(the point in BS^Q corresponding to the subword J). Let $Q = (q_1, \ldots, q_{m+1})$. Recall that the rightmost flag of the fixed point $p(J)$ corresponding to the subword J is shown in Figure 4.2, where $w = (Q \setminus J)_{(m+1)}$. Let J be a subword of Q and consider the word $Q' = (q_1, \ldots, q_m)$ and subword $J' = (j_1, \ldots, j_m)$. By induction we have that

$\Phi(p(J')) = \mathsf{B}(J')$. Now notice that

$$\begin{aligned}\Phi(p(J)) &= \Phi(p(J')) + (\dim_{e_1}(V_{k+1}), \ldots, \dim_{e_n}(V_{k+1})) \\ &= \Phi(p(J')) + w \cdot (1, \ldots, 1, 0, \ldots, 0),\end{aligned}$$

where the 0-1 vector has as many ones as $\dim(V_{k+1})$. The vector $w \cdot (1, \ldots, 1, 0, \ldots, 0)$ adds one to the i-th coordinate if and only if the brick corresponding to the commutator q_{k+1} is above the i-th pseudoline. □

$$\begin{array}{c}\langle e_{w(1)}, \ldots, e_{w(n)} \rangle = \mathbb{C}^n \\ | \\ \vdots \\ | \\ \langle e_{w(1)}, e_{w(2)} \rangle \\ | \\ \langle e_{w(1)} \rangle \\ | \\ 0\end{array}$$

Figure 4.2: Rightmost flag corresponding to w.

Theorem 4.1.3. *Let $w = Dem(Q)$. The fiber $m_Q^{-1}(wB)$ is a toric variety with respect to the torus T if and only if Q is root independent and $\ell(w) < |Q|$. Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the polytope $B(Q, w)$.*

We have proved the if part of this theorem; the only if part will follow from Theorem 4.2.2. The following corollary follows from the theorem above and the work of Pilaud and Santos in [PS12]. A **Coxeter word c** in a Coxeter group is defined to be a list of all simple reflections in some order using each reflection only once. Define the **c-sorting word** of w to be the lexicographically first subword of $\mathbf{c}^\infty$ that is a reduced expression for w.

Corollary 4.1.4. *If Q is the concatenation of a word $\mathbf{c}$ representing a Coxeter element c and the $\mathbf{c}$-sorting word for w_0, then $m_Q^{-1}(w_0B)$ is the toric variety of the associahedron as realized in [HLT11] and in [PS12].*

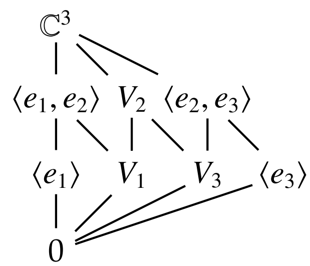


Figure 4.3: The toric variety of the pentagon from Example 2.2.2.

Example 4.1.5. The toric variety of the pentagon from Example 2.2.2, i.e. the associahedron corresponding to the Coxeter element $c = (s_1, s_2)$, is

$$m_Q^{-1}(w_0B) = \{(V_1, V_2, V_3) : \text{ the diagram in Figure 4.3 holds}\},$$

where $Q = (s_1, s_2, s_1, s_2, s_1)$.

4.2 Moment polytopes of brick varieties

We now state and prove one of the main results of this thesis.

Theorem 4.2.1. *Let $w = Dem(Q)$. The image of the brick variety $m_Q^{-1}(wB)$ under the moment map is the brick polytope $B(Q, w)$.*

Proof. T-fixed points of BS^Q are in 1-1 correspondence with subwords J of Q. This induces a 1-1 correspondence between T-fixed points of $m_Q^{-1}(wB)$ and the subwords J of Q with $(Q \setminus J)_{(m)} = w$, where $w = \mathrm{Dem}(Q)$. If the subword J is not a facet of the subword complex $\Delta(Q, w)$ then it gives a non reduced product

$(Q \setminus J)_{(m)}$. This implies that the root configuration $r(J) = \{\{r(J, i) : i \in F\}\}$ has a smaller dimension than the root configuration of a facet and thus it cannot be a vertex. Therefore, the moment polytope is the convex hull of the points corresponding to facets of $\Delta(Q, w)$ and by Equation 3.5 the image of each fixed point is precisely the one defined in Equation 2.1 in Section 2.2 by Pilaud and Stump. □

Note that this theorem does not assume that the fiber is a toric variety, so the relation between brick polytopes and brick varieties is quite strong. The following theorem classifies toric brick varieties.

Theorem 4.2.2. *Let $w = Dem(Q)$. The brick variety $m_Q^{-1}(wB)$ is a toric variety with respect to the torus T if and only if Q is root independent and $\ell(w) < |Q|$. Moreover, $m_Q^{-1}(wB)$ is the toric variety associated to the brick polytope $B(Q, w)$.*

Proof. Note that $\dim(m_Q^{-1}(wB)) \leq \dim(T)$. However, if we have $<$ then we can make the torus smaller and so without loss of generality we can assume the dimensions are equal. It suffices to show that T doesn't have generic stabilizer of positive dimension. This is true if and only if $\Phi(m_Q^{-1}(wB))$ spans $\mathbb{R}^n$ and this happens precisely when Q is root independent. The theorem follows after applying Lemma 3.4.7. □

4.3 Stratification of the brick variety

In this section we give a stratification whose dual, in some sense, is the subword complex. We now introduce and recall some notation. Consider a complex semisimple Lie group G with upper and lower Borel subgroups $B = B^+$ and

B^-, and Weyl group W. For $u \in W$ we have the **Schubert cell** $X_v^\circ := B^- uB$ and the **opposite Schubert cell** $X_\circ^v := B^+ uB$. The Schubert variety X^v and opposite Schubert variety X_u are the closure of X_u° and $X_\circ^u$, respectively. Given $u, v \in W$, the **open Richardson variety** is $\mathring{X}_u^v := X_\circ^v \cap X_u^\circ$. The **Richardson variety** X_u^v is the closure of $\mathring{X}_u^v$. This variety is nonempty if and only if $u \leq v$ in the Bruhat order, and its dimension is $\ell(v) - \ell(u)$. Then $X_u^v = \coprod_{u \leq x < y \leq v} \mathring{X}_x^y$ is a stratification.

Given a Bott-Samelson variety $BS^Q := (P_{i_1} \times \cdots \times P_{i_m})/B^m$ and a subword R of Q, we can realize BS^R inside BS^Q by

$$BS^R = \{(p_1, \ldots, p_m) : p_{i_j} = id \text{ if } s_{i_j} \notin R\};$$

note that $BS^R \cap BS^S = BS^{R \cap S}$. Let $\mathring{BS}_u^R := BS^R \cap m_Q^{-1}(X_u^\circ)$ then these subvarieties yield a stratification of BS^Q, where R ranges over all subwords of Q and $u \in W$. We have that $BS_u^R \neq \emptyset$ if and only if $\mathrm{Dem}(R) \geq u$. Moreover, $BS_u^R \subseteq BS_v^S$ if and only if R is a subword of S and $u \geq v$ in Bruhat order. This induces a stratification of $m_Q^{-1}(wB)$, described in the following theorem, that is dual to the subword complex $\Delta(Q, \mathrm{Dem}(Q))$.

Theorem 4.3.1. *Let $w = Dem(Q)$. Its brick variety has the stratification*

$$m_Q^{-1}(wB) = \coprod_R \mathring{BS}_w^R,$$

where R ranges over all subwords of Q with $Dem(R) = w$. The complement of the open stratum is a simple normal crossings divisor.

Proof. If $p \in \mathring{BS}_w^Q$, then $m_Q(p) \in X_w^w = \{wB\}$ and so $m_Q^{-1}(wB)$ is a stratum of BS^Q. Moreover, if $\mathring{BS}_w^R \subset m_Q^{-1}(wB)$ is nonempty then R is a subword and $\mathrm{Dem}(R) \geq u \geq w$ but then $\mathrm{Dem}(R) = u = w$. Therefore, the stratification of the Bott-Samelson variety restricts to a stratification of the brick variety and $\mathring{BS}_w^R \cap \mathring{BS}_w^S = \mathring{BS}_w^{R \cap S}$. $\square$

4.4 Brick varieties and Richardson varieties

A subfamily of brick varieties were used before by Brion in [Bri05, proof of Theorem 4.2.1] as resolutions of singularities for Richardson varieties. Given a word $Q = (q_1, \ldots, q_m)$, the **opposite Bott-Samelson variety** BS_Q is defined analogously to BS^Q. More precisely,

$$BS_Q := (P^-_{i_1} \times \cdots \times P^-_{i_m})/(B^-)^m,$$

where the P^-_i are the **opposite minimal parabolics**. The natural map to the flag variety is

$$BS_Q \overset{\mathrm{m_Q}}{\longrightarrow} G/B$$

$$(p_1, \ldots, p_m) \longmapsto (p_1 \cdots p_m w_0)B.$$

Given an element $u \in W$ the opposite Bott-Samelson variety BS_Q is a resolution of the Schubert variety X_u, where Q is a reduced word for uw_0. Given $u, v \in W$, let R be a reduced word for v and T be a reduced word word for uw_0, then the fiber product $BS^R \times_{G/B} BS_T$ with the map induced by m_R is Brion's resolution of the Richardson variety X^v_u. We will construct this fiber product as a brick variety.

Given $u, v \in W$, let R be a reduced word for v and S be a reduced word for $u^{-1}w_0$, where w_0 is the longest word in W. Now, if $Q = R + S$, i.e. Q is the concatenation of R and S, and $u \leq v$ then $\mathrm{Dem}(Q) = w_0$. Moreover, the brick variety $m_Q^{-1}(w_0B)$ together with the map to the flag in the middle gives a resolution of the Richardson variety X^v_u.

Example 4.4.1. Let $R = (s_1, s_2, s_3, s_1, s_2)$ and $S = (s_3, s_1, s_2, s_1)$. Then $m_Q^{-1}(w_0B)$ together with the map given by the bold flag in Figure 4.4 is a resolution of singularities for X^v_u with $v = s_1s_2s_3s_1s_2$ and $u = s_1s_2$.

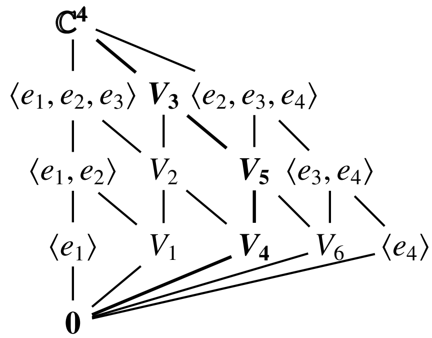


Figure 4.4: Resolution of singularities for $X_{s_1 s_2}^{s_1 s_2 s_3 s_1 s_2}$.

Theorem 4.4.2. *Let $u \leq v$ and $Q = R + S$, where R is a reduced word for v and S is a reduced word for $u^{-1}w_0$. The brick variety $m_Q^{-1}(w_0 B)$ together with the map $m_R : BS_w^R \to G/B$ is a resolution of the singularities of the Richardson variety X_u^v.*

Proof. Let T be the reduced word for uw_0 obtained by taking S^{-1} and conjugating each letter by w_0. The result follows from identifying the fiber product $BS^R \times_{G/B} BS_T$ with the brick variety $m_Q^{-1}(w_0 B)$. If $R = (q_1, \ldots, q_{|R|})$, then then the points in BS^R consist of lists of $m + 1$ flags in G/B

$$(F_0 = B, F_1, \ldots, F_{|R|}) \in BS^R$$

such that the k-th flag agrees with the previous one except possibly on the subspace corresponding to q_k, and if $F_{k-1} = gB$ and $F_k = hB$, then $h^{-1}gB \in X^{q_k}$. Similarly, if $T = (q_1, \ldots, q_{|T|})$, then then the points in $BS_{S^{-1}}$ consist of lists of $m + 1$ flags in G/B

$$(E_0 = w_0 B, E_1, \ldots, E_{|T|}) \in BS_T$$

such that the k-th flag agrees with the previous one except possibly on the subspace corresponding to q_k, and if $E_{k-1} = gB$ and $E_k = hB$, then $h^{-1}gB \in X^{w_0^{-1} q_k w_0}$. Therefore, the fiber product $BS^R \times_{G/B} BS_T$ consists of the lists of flags of the form

$$(F_0 = B, F_1, \ldots, F_{|R|} = E_{|T|}, E_{|T|-1}, \ldots, E_0 = w_0 B)$$

such that consecutive flags agree in the described way, together with the maps $BS^R \xrightarrow{\mathrm{m_R}} G/B$ and $BS_T \xrightarrow{\mathrm{m_T}} G/B$ that map the list of flags to $F_{|R|} = E_{|T|}$. This is precisely the brick variety $m_Q^{-1}(w_0 B)$. □

CHAPTER 5

MATRIX SCHUBERT VARIETIES AND ROOT POLYTOPES

In this chapter we study pipe dream complexes and matrix Schubert varieties and their relation with root polytopes. In Section 5.1 we provide geometric realizations of pipe dream complexes of permutations $\pi = 1\pi'$, where π' is a dominant permutation on $2, 3, \ldots, n$, as well as the subword complexes which are the cores of pipe dream complexes. In Section 5.2 we explore some consequences of the geometric realization of pipe dream complexes via root polytopes. Fulton's essential set gives us a decomposition of a matrix Schubert variety $\overline{X_\pi}$ into the product of an affine variety Y_π and a $\mathbb{C}$-vector space V_π. In Section 5.3 we characterize the matrix Schubert varieties for which Y_π is a toric variety. In Section 5.4 we prove that the moment polytope of Y_π can be specialized into a root polytope $\mathcal{P}(T(\pi))$. This chapter is based on joint work with Karola Mészáros.

5.1 Geometric realizations of pipe dream complexes via root polytopes

In this section we give geometric realizations of pipe dream complexes of permutations $\pi = 1\pi'$, where π' is dominant, in terms of triangulations of root polytopes. Indeed, we construct a geometric realization of the subword complex which is the core of $PD(1\pi')$. The main theorem of this section is the following, which has several interesting consequences explored in this chapter.

Theorem 5.1.1. *Let $\pi = 1\pi' \in S_n$, where π' is dominant. Let $C^2(\pi)$ be the core of $PD(\pi)$ coned over twice. The canonical triangulation of the root polytope $\mathcal{P}(T(\pi))$ (which is a* *regular triangulation) is a geometric realization of $C^2(\pi)$.*

We will give the definitions needed for this theorem throughout this section. To this end we start by defining a tree $T(\pi)$ for each permutation $\pi = 1\pi'$, π' dominant.

Let $\pi = 1\pi'$, where π' is dominant. We now give some notation we will use to define the tree $T(\pi)$. Denote the region which is the union of the box $(1, 1)$ and $\mathrm{cr}(\pi)$ by $\boldsymbol{R(\pi)}$. Denote by $\boldsymbol{\mathcal{S}(\pi)}$ the subword complex which is the restriction of $PD(\pi)$ to the vertices corresponding to the entries inside $R(\pi)$, in other words $\mathcal{S}(\pi)$ is the core(π) coned over the vertex of $PD(\pi)$ corresponding to the entry $(1, 1)$. See Figure 5.1 for an example and compare it with Figure 2.8.

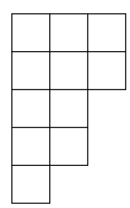


Figure 5.1: The region $\mathcal{S}(\pi)$ for $\pi = [164235]$.

In order to determine the tree $T(\pi)$, we will label the southeast boundary with numbers and we will place some dots in $R(\pi)$, see Figure 5.2. The boundary of the core region starting from the southwest (SW) corner of it to the northeast (NE) corner can be described as a series of east (E) and north (N) steps. Let A be the set consisting of all the N steps together with the E steps picked by the following rule: the step $E \in A$ if the bottom reduced pipe dream is bounded by E but not by the N step directly preceding E. As we traverse this lower boundary from the SW corner we write the numbers $1, \ldots, m$ in increasing fashion below the E steps and to the right of the N steps that belong to A. For the E steps that we did not assign a number, we consider their number to be the number assigned to the N step directly preceding them. The elements in A, labelled by $1,\dots,m$ are the vertices of the tree $T(\pi)$.

Consider the bottom reduced pipe dream drawn inside $R(\pi)$ and with elbows replaced by dots. Drop these dots south. Define $T(\pi)$ to be the tree on m vertices such that there is an edge between vertices $i < j$ if there is a dot in the entry in the column of the E step labeled i and in the row of the N step labeled j. Let $\boldsymbol{t(\pi)}$ denote the number of edges of $T(\pi)$.

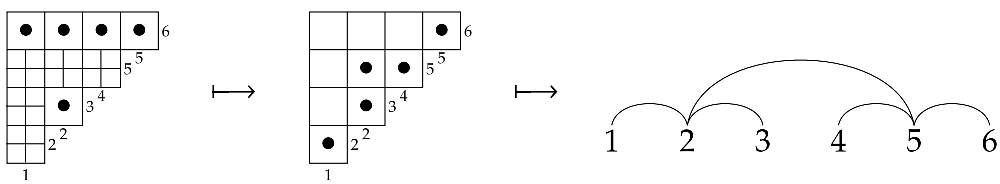


Figure 5.2: The bottom reduced pipe dream for $\pi = [15342]$ drawn inside $R(\pi)$ with dots instead of elbows gives the labeling of the boundary. We then drop the dots to the south to get the edges of $T(\pi)$, which is depicted on the right.

Notice that the first string of E steps are in A. Furthermore, the E steps that can be seen as the boundary of the bottom reduced pipe dream B of π and such that E bounds row r of length l_r of B and the row below row r of length l_{r+1} is at least two boxes shorter than row r and moreover, step E is not $(l_{r+1}+1)^{st}$ from the left side are also in A. Also, we are placing dots in the rightmost boxes of the core region as well as in positions $(l_{r+1}+1)^{st}$ until $(l_r-1)^{th}$ in rows r of the bottom reduced pipe dream B which are longer than row $r+1$ by at least two boxes. In the case in which $\pi = 1\pi'$ where π' is dominant with its diagram having all parts distinct, then the decoration on the core diagram is much simpler. The first string of E steps consist of only one E step and this is also the only E step in A, so the number of vertices of $T(\pi)$ is 2 more than the size of the largest column of the diagram of π. The dots are placed on the rightmost boxes of $\mathrm{cr}(\pi)$, see Figure

5.3.

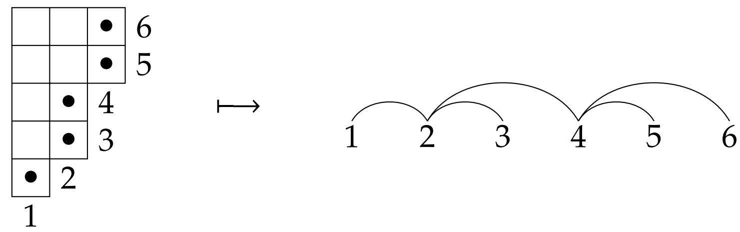


Figure 5.3: Obtaining $T(\pi)$ from $R(\pi)$.

The vertices of $\mathcal{S}(\pi)$ are in bijection with configurations of 1 elbow and $|R(\pi)| - 1$ crosses in $R(\pi)$. Denote these vertices by $v_1, \ldots, v_k$. We define a map M from the vertices of the simplicial complex $\mathcal{S}(\pi)$ to the vertices of $C(\pi) := C(T(\pi))$. Recall that $C(\pi)$ is the canonical triangulation of the vertex figure at 0 of the root polytope $\mathcal{P}(T(\pi))$ and its vertices are in bijection with edges (i, j) in the directed transitive closure of $T(\pi)$. The latter in turn are in bijection with the boxes of $R(\pi)$ by the map that takes a box to the edge (i, j) if the E step below the box and in the boundary of $R(\pi)$ is labeled by i and the N step to the right of the box and in the boundary of $R(\pi)$ is labeled by j. The map M is then defined analogously: consider a vertex of $\mathcal{S}(\pi)$. This can be seen as a sole elbow tile in $R(\pi)$. Map this vertex to the vertex of $C(\pi)$ corresponding to (i, j) if the box containing the elbow tile yields the edge (i, j) in $T(\pi)$ (that is to the intersection of the ray pointing to $e_i - e_j$ and the hyperplane by which we intersect $\mathcal{P}(\Gamma)$ to obtain the considered vertex figure).

Theorem 5.1.2. *The map M described above respects the simplicial complex structure of $\mathcal{S}(\pi)$ and $C(\pi)$. In other words, $C(\pi)$ is a geometric realization of the subword complex $\mathcal{S}(\pi)$.*

Proof. Since both $\mathcal{S}(\pi)$ and $C(\pi)$ are pure simplicial complexes of the same dimension by Lemma 5.1.5, it suffices to show that the map M is a bijection on the facets of $\mathcal{S}(\pi)$ and $C(\pi)$. This is proven in Theorem 5.1.6. □

Proof of Theorem 5.1.1. This follows using Theorem 5.1.2. □

Next we show that from Theorem 5.1.2 it follows that we can also realize core(π) geometrically, which is a subword complex as explained in Section 2.1.2 after Definition 2.1.7. To this end we prove an auxiliary lemma first.

Lemma 5.1.3. *Let $\pi = 1\pi'$, with π' dominant. If $C(\pi)$ has an interior vertex, then it is the unique vertex in $C(\pi)$ on the ray between 0 and $e_1 - e_m$. Moreover, $C(\pi)$ has an interior vertex if and only if $\pi = 1n \ldots 2$.*

Proof. $C(\pi)$ has an interior vertex if and only if the cone generated by $e_i - e_j$ for $(i, j) \in T(\pi)$ has an interior point $e_k - e_l$, where (k, l) is in the directed transitive closure of $T(\pi)$. Since $e_i - e_j$ for $(i, j) \in T(\pi)$ are linearly independent, an interior point can be expressed as $\sum_{(i,j)\in T(\pi)} c_{ij}(e_i - e_j)$ with $c_{ij} > 0$, $(i, j) \in T(\pi)$. If $T(\pi) = ([m], \{(i, i+1) : i \in [m-1]\})$, then $e_1 - e_m = \sum_{(i,j)\in T(\pi)}(e_i - e_j)$ is an interior point; moreover, $e_k - e_l$ for $1 \leq k < l \leq m$ is an interior point if and only if $k = 1$ and $l = m$. We have $T(\pi) = ([m], \{(i, i+1) : i \in [m-1]\})$ exactly for $\pi = 1m(m-1)\ldots 2$. For $T(\pi) \neq ([m], \{(i, i+1) : i \in [m-1]\})$, there is no (k, l) in the directed transitive closure of $T(\pi)$ such that $e_k - e_l = \sum_{(i,j)\in T(\pi)} c_{ij}(e_i - e_j)$ with $c_{ij} > 0$. □

Theorem 5.1.4. *Let $\pi = 1\pi'$, with π' dominant. Let v be the unique vertex in $C(\pi)$ on the ray between 0 and $e_1 - e_m$. For $\pi \neq 1m(m-1)\ldots 2$, v is in the boundary of $C(\pi)$, and* core(π) *is realized by the induced triangulation of the vertex figure of $C(\pi)$ at v. For $\pi = 1m(m-1)\ldots 2$,* core(π) *is realized by the induced triangulation of the boundary of $C(\pi)$.*

Proof. The vertex v, which is the unique vertex in $C(\pi)$ on the ray between 0 and $e_1 - e_m$, is the unique coning point of the geometric realization of $\mathcal{S}(\pi)$. If v is in the boundary of $C(\pi)$, which happens exactly when $\pi \neq 1m(m-1)\dots 2$ by Lemma 5.1.3, then the induced triangulation of the vertex figure at v of $C(\pi)$ is a geometric realization of core(π) (which is homeomorphic to a ball). For $\pi = 1m(m-1)\dots 2$, the coning point v lies in the interior of $C(\pi)$, then since it is the only point in the interior of the canonical triangulation $C(\pi)$ by Lemma 5.1.3, then the induced triangulation of the boundary of $C(\pi)$ is a geometric realization of core(π) (which is homeomorphic to a sphere). □

Lemma 5.1.5. *The core of the pipe dream complex of $\pi = 1\pi'$, where π' is dominant, is of dimension $t(\pi) - 2$. The dimension of the root polytope $\mathcal{P}(T(\pi))$ is $t(\pi)$ and its vertex figure at 0 is of dimension $t(\pi) - 1$. In particular, both $\mathcal{S}(\pi)$ and $C(\pi)$ are of dimension $t(\pi) - 1$.*

Proof. Since subword complexes are pure, then the dimension of the core of the pipe dream complex of π equals the dimension of one of its facets. Consider the facet given by the bottom reduced pipe dream drawn inside the core region. The dimension of this facet equals one less than the number of elbows in the core and from the construction of $T(\pi)$ this equals $t(\pi) - 2$. The dimension of the root polytope $\mathcal{P}(T(\pi))$ is the number of edges in $T(\pi)$, which by definition is $t(\pi)$. □

The map M can be easily extended to a map between pipe dreams P of π drawn inside $R(\pi)$ and forests F on m vertices as follows. For each elbow tile in P add the edge (i, j) corresponding to the box of the elbow to F. Moreover, add the edge $(1, m)$ to F.

Theorem 5.1.6. *The reduced pipe dreams of $\pi = 1\pi'$, where π' is dominant, are in bijection via M with the noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$.*

We prove Theorem 5.1.6 by induction on the number of columns in the diagram. We break it down in several lemmas.

Lemma 5.1.7. *Take the permutation $\pi = 1\pi'$, where the diagram of π' is $\lambda = (k)$. The reduced pipe dreams of π are in bijection with the noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$ via the map M.*

Proof. The edges of $T(\pi)$ for such a π are $(1,2)$ and $(2, j)$ for $j = 3, \ldots, k+2$ and thus for the transitive closure of $T(\pi)$ we add the edges $(1, j)$ with $j = 3, \ldots, k+2$. See Figure 5.4.

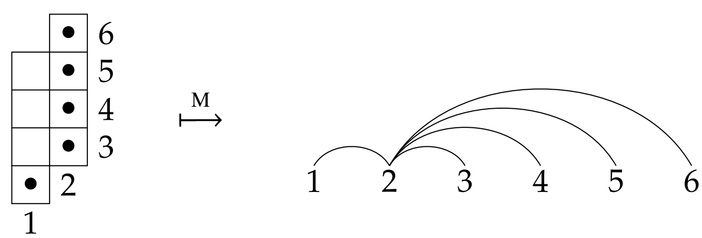


Figure 5.4: The core region and the tree $T(\pi)$ for $\lambda = (4)$

For $l = 2, \ldots, k+2$ let T_l be a tree on the vertex set $[k+2]$ consisting of the edges $(2, i)$ for $2 < i \leq l$ and $(1, j)$ for $j \geq l$. Then T_l, $l = 2, \ldots, k+2$, are all of the noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$. The map M applied to the bottom reduced pipe dream of π yields T_{k+2}. Moreover, after performing $0 \leq i \leq k$ ladder moves (there is only one way to do this) on the bottom reduced pipe dream of π, we obtain a reduced pipe dream whose image under M is T_{k+2-i}. By Theorem 2.1.8 these are indeed all of the reduced pipe dreams of π concluding the proof. □

Lemma 5.1.8. *Given $\pi = 1\pi'$, where π' is dominant and which has more than one nonzero column, let its rightmost (shortest) column be of size k. Then in a reduced pipe dream of π the only configurations of crosses and elbows that can occur in the rightmost column of $\mathrm{cr}(\pi)$ are, as read from above, l crosses and $k-l$ elbows, for $l = 0, ..., k$.*

Proof. This follows immediately from Theorem 2.1.8. □

Lemma 5.1.9. *Let $\pi = 1\pi'$, where the diagram of π' is a partition $\lambda = (\lambda_1, \ldots, \lambda_z)$ which has more than one nonzero column. Consider all reduced pipe dreams of π where the configuration of crosses and elbows in the rightmost column of $\mathrm{cr}(\pi)$ is set to have l crosses and $k-l$ elbows for a fixed $0 \leq l \leq k$. These are in bijection with reduced pipe dreams of the permutation $1w_l$, where w_l has diagram $(\lambda_1-(k-l), \lambda_2-(k-l), \ldots, \lambda_{z-1}-(k-l))$.*

Proof. Since the bottom $k-l$ boxes of the rightmost column of $\mathrm{cr}(\pi)$ are elbows, it can be seen using Theorem 2.1.8 that the $k-l$ rows containing crosses one step to the south and one step to the west of these $k-l$ boxes can never move anywhere.

Moreover, the fixed rows of crosses do not affect the ladder moves we can make on the remaining crosses. This allows us to get exactly the reduced pipe dreams for the permutation $1w$, where the diagram of w is the diagram of π after ignoring the fixed rows and shortest column, i.e., w has diagram $(\lambda_1-(k-l), \lambda_2-(k-l), \ldots, \lambda_{z-1}-(k-l))$. □

The following example illustrates the lemma above.

Example 5.1.10. Let $1\pi = [15423]$ and suppose $l = 1$, i.e., we are fixing one cross in position $(1,4)$ and elbow in position $(2,4)$, see Figure 5.5. The elbow in entry $(2,4)$ causes row 3 to consist of only crosses. The elbow in position $(2,3)$ is also forced because no ladder move of the remaining crosses would yield a cross in this position. Therefore the reduced pipe dreams for [15423] with a cross in entry $(1,4)$ and elbow in $(2,4)$ correspond with the reduced pipe dreams for [1432]. The diagram of this latter permutation has fewer columns.

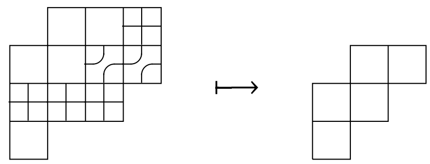


Figure 5.5: New core after applying the reduction of Lemma 5.1.9 for $l = 1$ to the core on the left.

Lemma 5.1.11. *Given $\pi = 1\pi'$, π' dominant, where the length of the shortest column of the diagram of π' is k, the set S of noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$ is a disjoint union $S = S_0 \sqcup \cdots \sqcup S_k$, where*

$$\begin{aligned} S_l = &\{T \in S : (m-k, m-j) \notin E(T) \text{ for } j = 0, \ldots, l-1\} \\ &\cup \{T \in S : (m-k, m-j) \in E(T) \text{ for } j = l, \ldots, k-1\}, \end{aligned} \tag{5.1}$$

for $0 \leq l \leq k$, where m is the number of vertices of $T(\pi)$.

Note that $m-k$ is the label on the bottom of the last column of the core of π. Thus, S_l consists of the noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$ which do not contain the edges corresponding to the top l crosses in last column of the core of π and contain the edges corresponding to the bottom $k-l$ elbows in last column of the core.

Proof of Lemma 5.1.11. This follows immediately from the definition of $T(\pi)$. □

Lemma 5.1.12. *Let $\pi = 1\pi'$, π' dominant of shape $(\lambda_1, \ldots, \lambda_z)$ and let $1w_l$ be the permutation where w_l has diagram $(\lambda_1 - (k-l), \lambda_2 - (k-l), \ldots, \lambda_{z-1} - (k-l))$. Use Lemma*

5.1.9 to draw the core region of $1w_l$ *inside the core region of* π*. Then all the edges corresponding to the entries outside the core region of* $1w_l$ *in a tree* $T \in S_l$ *are forced by the last column.*

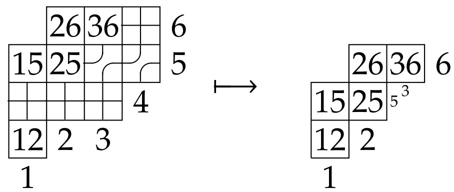


Figure 5.6: Edge labeling for the core of $1w_1$ coming from the core of [154623] on the left.

Proof. If $\lambda_z < \lambda_{z-1}$, then the edges inside $\mathrm{cr}(\pi)$ and outside of $\mathrm{cr}(1w_l)$ are precisely those in the last column and in the $(k-l)$ rows one step to the south of the $k-l$ boxes fixed to be elbows. The boxes in these rows are crosses and thus we conclude that in this case all the edges corresponding to boxes outside $\mathrm{cr}(1w_l)$ are indeed fixed after fixing the last column. If $\lambda_z = \lambda_{z-1} = \cdots = \lambda_{z-j} < \lambda_{z-j-1}$, then aside from the boxes outside of $\mathrm{cr}(1w_l)$ described in the previous sentence, we also have the j boxes to the left of the top most elbow on the last column. We will show that a tree $T \in S_l$ must contain the edge corresponding to these boxes. Let $T \in S_l$, u be the E step below the leftmost of these boxes and v be the N step to the right of these boxes. Since T is an alternating spanning tree, then v must be adjacent either to u or to a vertex before u. Similarly, u must be adjacent either to v or to a vertex after v. The only way noncrossing is preserved is if (u, v) is an edge of T. We continue in this fashion by looking at the second leftmost box and taking the E step below it and again prove that the edge corresponding to that box is in T. □

Lemma 5.1.13. *The set* S_l*,* $0 \le l \le k$*, as in Lemma 5.1.11 is in bijection with reduced pipe dreams of the permutation* $1w_l$*, where* w_l *has diagram* $(\lambda_1 - (k-l), \lambda_2 - (k -$

$l), \ldots, \lambda_{z-1} - (k-l))$, via the map M.

We prove this lemma and Theorem 5.1.6 together by using induction on the number of columns of the diagram of π'.

Proof of Theorem 5.1.6 and Lemma 5.1.13. We use induction on the number of columns of the diagram of π'. The base case for Theorem 5.1.6 where this diagram contains one column is proved in Lemma 5.1.7. Notice that in the proof of this lemma the base case for Lemma 5.1.13 is also proven.

By Lemma 5.1.11 the noncrossing alternating spanning trees of the directed transitive closure of $T(\pi)$ can be broken down into the sets S_l, $l = 0, 1, \ldots, k$. Consider the permutation $1w_l$ where w_l has diagram $(\lambda_1 - (k-l), \lambda_2 - (k-l), \ldots, \lambda_{z-1} - (k-l))$. By the inductive hypothesis, we know that $1w_l$ satisfies Theorem 5.1.6, i.e., its reduced pipe dreams are in bijection with the noncrossing alternating spanning trees of the directed transitive closure of $T(1w_l)$ via the map M. By Lemma 5.1.12 we have that the noncrossing alternating spanning trees of the directed transitive closure of $T(1w_l)$ yield the set S_l. Finally, by Lemma 5.1.9 we know that the reduced pipe dreams of the permutations $1w_l$, as $l = 0, 1, \ldots, k$, are in bijection with the reduced pipe dreams of π, concluding the proof. □

5.2 Consequences of the correspondence between pipe dream complexes and triangulations of root polytopes

5.2.1 Reduced forms in the subdivision algebra and Grothendieck polynomials

In this section we show that Grothendieck polynomials of permutations $\pi = 1\pi'$, π' dominant, are special cases of reduced forms in the subdivision algebra of root polytopes. To this end we start by defining the notions appearing in the previous sentence.

The **subdivision algebra of root polytopes** $\mathcal{S}(\beta)$ is a commutative algebra generated by the variables x_{ij}, $1 \leq i < j \leq n$, over $\mathbb{Q}[\beta]$, subject to the relations $x_{ij}x_{jk} = x_{ik}(x_{ij} + x_{jk} + \beta)$, for $1 \leq i < j < k \leq n$. This algebra is called the subdivision algebra, because its relations can be seen geometrically as subdividing root polytopes via Lemma 2.3.1. The subdivision algebra has been used extensively for subdividing root (and flow) polytopes in [Mész15, MM13, Mész14b, Mész14a, Mész14c, Mész11a, Mész11b].

A **reduced form** of the monomial in the algebra $\mathcal{S}(\beta)$ is a polynomial obtained by successively substituting $x_{ik}(x_{ij} + x_{jk} + \beta)$ in place of an occurrence of $x_{ij}x_{jk}$ for some $i < j < k$ until no further reduction is possible. Note that the reduced forms are not necessarily unique.

A possible sequence of reductions in algebra $\mathcal{S}(\beta)$ yielding a reduced form of $x_{12}x_{23}x_{34}$ is given by

$$
\begin{aligned}
x_{12}\boldsymbol{x_{23}}\boldsymbol{x_{34}} &\to \boldsymbol{x_{12}}x_{24}\boldsymbol{x_{23}} + \boldsymbol{x_{12}}x_{34}\boldsymbol{x_{24}} + \beta\boldsymbol{x_{12}}\boldsymbol{x_{24}} \\
&\to \boldsymbol{x_{24}}x_{13}\boldsymbol{x_{12}} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} + x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} \\
&\quad +\beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} + \beta^2 x_{14} \\
&\to x_{13}x_{14}x_{12} + x_{13}x_{24}x_{14} + \beta x_{13}x_{14} + x_{24}x_{23}x_{13} + \beta x_{24}x_{13} \\
&\quad +x_{34}x_{14}x_{12} + x_{34}x_{24}x_{14} + \beta x_{34}x_{14} + \beta x_{14}x_{12} + \beta x_{24}x_{14} \\
&\quad +\beta^2 x_{14}
\end{aligned} \tag{5.2}
$$

where the pair of variables on which the reductions are performed is in boldface. The reductions are performed on each monomial separately.

Given a noncrossing tree T on the vertex set $[n]$, let $m[T] := \prod_{(i,j)\in T} x_{ij}$. The **canonical reduced form** $\mathrm{Crf}_T(x_{ij} : 1 \leq i < j \leq n)$ of $m[T]$ is the reduced form obtained by performing reductions on the tree T from front to back (or back to front) on the topmost edges always. This can of course be translated into algebra language. For $x_{ij} = t_i$ we denote by $\mathrm{Crf}_T(t_1, \ldots, t_{n-1}) = \mathrm{Crf}_T(x_{ij} : 1 \leq i < j \leq n)$. While the reduced form of a monomial in the subdivision algebra is not necessarily unique, once we set $x_{ij} = t_i$ it becomes unique. This is the statement of the next theorem which we prove in Section 5.2.3.

Theorem 5.2.1. *Given a noncrossing tree T on the vertex set $[n]$, let $R_T(x_{ij} : 1 \leq i < j \leq n)$ be an arbitrary reduced form of $m[T]$. Let $R_T(t_1, \ldots, t_{n-1})$ be the reduced form $R_T(x_{ij} : 1 \leq i < j \leq n)$ when we let $x_{ij} = t_i$. Then,*

$$R_T(t_1, \ldots, t_{n-1}) = \mathrm{Crf}_T(t_1, \ldots, t_{n-1}). \tag{5.3}$$

We will use the notation $\tilde{R}_T(\mathbf{t})$ when instead of setting $x_{ij} = t_i$, we do the following. Let $i_1 < \ldots < i_v$ be the vertices of T which have outgoing edges. Therefore, the only x_{ij}'s appearing in a reduced form must have $i \in \{i_1, \ldots, i_v\}$. The reduced form $\tilde{R}_T(\mathbf{t})$ is then obtained from $R_T(x_{ij} : 1 \leq i < j \leq n)$ by setting $x_{i_k,j} = t_k$ for $k \in [v]$ and all $j \in [n]$.

The following theorem provides a combinatorial way of thinking about double Grothendieck polynomials.

Theorem 5.2.2. *[KM04, FK94] The* ***double Grothendieck polynomial*** $\mathfrak{G}_w(\mathbf{x}, \mathbf{y})$ *for* $w \in S_n$*, where* $\mathbf{x} = (x_1, \ldots, x_{n-1})$ *and* $\mathbf{y} = (y_1, \ldots, y_{n-1})$ *can be written as*

$$\mathfrak{G}_w(\mathbf{x}, \mathbf{y}) = \sum_{P \in \mathrm{Pipes}(w)} (-1)^{codim_{PD(w)} F(P)} wt_{x,y}(P), \tag{5.4}$$

where $\mathrm{Pipes}(w)$ *is the set of all pipe dreams of* w *(both reduced and nonreduced),* $F(P)$ *is the interior face in* $PD(w)$ *labeled by the pipe dream* P*,* $codim_{PD(w)} F(P)$ *denotes the codimension of* $F(P)$ *in* $PD(w)$ *and* $wt_{x,y}(P) = \prod_{(i,j) \in \mathrm{cross}(P)} (x_i - y_j)$*, with* $\mathrm{cross}(P)$ *being the set of positions where* P *has a cross.*

In the spirit of Theorem 5.2.2, we use the following definition for the **double β-Grothendieck polynomial**:

$$\mathfrak{G}^{\beta}_w(\mathbf{x}, \mathbf{y}) = \sum_{P \in \mathrm{Pipes}(w)} \beta^{codim_{PD(w)} F(P)} wt_{x,y}(P). \tag{5.5}$$

Note that if we assume that β has degree -1, while all other variables are of degree 1, then the powers of β's simply make the polynomial $\mathfrak{G}^{\beta}_w(\mathbf{x}, \mathbf{y})$ homogeneous. We chose this definition of β-Grothendieck polynomials, as it will be the most convenient notationwise for our purposes.

Theorem 5.2.3. *Given $\pi = 1\pi'$, π' dominant, we have that for any reduced form of $m[T(\pi)]$*

$$\tilde{R}_{T(\pi)}(\mathbf{t}) = (\prod_{i=1}^{n-1} t_i^{g_i})\mathfrak{G}^{\beta}_{\pi^{-1}}(t_1^{-1}, \ldots, t_{n-1}^{-1}, \mathbf{0}), \tag{5.6}$$

where g_i is the number of boxes in the ith column from the left in $R(\pi)$.

A special case of Theorem 5.2.3 for $\pi = 1n(n-1)\cdots 2$ appears in [Kir14].

Theorem 5.2.3 is a special case of Theorem 5.2.4 below in light of Theorem 5.2.1 and using the fact that the pipe dreams of π and π^{-1} are in bijection when we reflect them across the north-east diagonal. Before we can state and prove Theorem 5.2.4 we need to define a map ϕ from the labels (i, j) that the boxes in the region $R(\pi)$ inherit from the labeling of its boundary (as described in Figure 5.2) to the conventional labeling where rows are labeled increasingly from top to bottom and columns are labeled increasingly from left to right. We call the former labeling the tree labeling and when unclear which labeling we are talking of we put a T index on it: $(i, j)_T$. The map ϕ simply takes the tree label (i, j) to the conventional label $(\phi_{ij}(i), \phi_{ij}(j))$ of the corresponding box. In the example of Figure 5.2 we have that $\phi((1,6)) = (1,1), \phi((2,3)) = (3,2), \phi((5,6)) = (1,4), \phi((4,5)) = (2,3)$, and so forth.

Theorem 5.2.4. *Given $\pi = 1\pi'$, π' dominant, we have that*

$$\mathrm{Crf}_{T(\pi)}\left(x_{ij} = \frac{1}{x_{\phi_{ij}(i)} - y_{\phi_{ij}(j)}} : 1 \leq i < j \leq n\right) = \left(\prod_{(i,j)_T \in R(\pi)} \frac{1}{x_{\phi_{ij}(i)} - y_{\phi_{ij}(j)}}\right)\mathfrak{G}^{\beta}_{\pi}(\mathbf{x}, \mathbf{y}). \tag{5.7}$$

Proof. By definition we have that

$$\mathrm{Crf}_{T(\pi)}(x_{ij} : 1 \leq i < j \leq n) = \sum_{\Gamma \in \mathcal{L}(T(\pi))} \beta^{codim_{\mathcal{P}(T(\pi))}\mathcal{P}(\Gamma)} wt(\Gamma), \tag{5.8}$$

where $wt(\Gamma) = \prod_{(i,j)\in\Gamma} x_{ij}$, $\mathcal{L}(T(\pi))$ denotes the set of graphs corresponding to the terms of the reduced form of $m[T(\pi)]$, and $\mathcal{P}(\Gamma)$ denotes the simplex in the canonical triangulation of $\mathcal{P}(T(\pi))$ corresponding to Γ. Together with Theorem 5.1.2 using the map M and Theorem 5.2.2, we obtain (5.7). □

5.2.2 Volumes and Ehrhart series of root polytopes

In this section we state two immediate corollaries regarding volumes and Ehrhart series of root polytopes following from Theorem 5.1.1.

Theorem 5.2.5. *Let $\pi = 1\pi'$, where π' is a dominant permutation. Then the normalized volume of $\mathcal{P}(T(\pi))$ is equal to the number of reduced pipe dreams of π. This can be written as*

$$\mathrm{vol}(\mathcal{P}(T(\pi))) = \mathfrak{G}_{\pi}^{\beta=0}(\mathbf{1}, \mathbf{0}). \tag{5.9}$$

Recall that for a polytope $\mathcal{P} \subset \mathbb{R}^N$, the t^{th} dilate of $\mathcal{P}$ is $t\mathcal{P} = \{(tx_1, \ldots, tx_N) : (x_1, \ldots, x_N) \in \mathcal{P}\}$. The number of lattice points of $t\mathcal{P}$, where t is a nonnegative integer and $\mathcal{P}$ is a convex polytope, is given by the **Ehrhart function** $i(\mathcal{P}, t)$. If $\mathcal{P}$ has integral vertices then $i(\mathcal{P}, t)$ is a polynomial.

In order to state the Ehrhart series of root polytopes we need the following lemma, which follows from the well-known relationship of f- and h-vectors. We note that we take $h(\mathbb{C}, x) = \sum_{i=0}^{d} h_i x^i$ to be the h-polynomial of a $(d-1)$-dimensional simplicial complex $\mathbb{C}$.

Lemma 5.2.6. *[Sta83] Let $\mathbb{C}$ be a $(d-1)$-dimensional pure simplicial complex homeomorphic to a ball and f_i° be the number of interior faces of $\mathbb{C}$ of dimension i. Then*

$$h(\mathbb{C}, \beta + 1) = \sum_{i=0}^{d-1} f_i^\circ \beta^{d-1-i} \tag{5.10}$$

Theorem 5.2.7. *Let $\pi = 1\pi'$, where π' is a dominant permutation. Then*

$$\mathfrak{G}_\pi^{\beta-1}(\mathbf{1}, \mathbf{0}) = \sum_{m\geq 0} (i(\mathcal{P}(T(\pi)), m)\beta^m)(1-\beta)^{\dim(\mathcal{P}(T(\pi)))+1}. \tag{5.11}$$

Proof. Since the canonical triangulation $\mathbb{C}$ of $\mathcal{P}(T(\pi))$ is unimodular, we have

$$h(\mathbb{C}, \beta) = \sum_{m\geq 0} (i(\mathcal{P}(T(\pi)), m)\beta^m)(1-\beta)^{\dim(\mathcal{P}(T(\pi)))+1}. \tag{5.12}$$

By Theorem 5.1.1 and Lemma 5.2.6 we get that $h(\mathbb{C}, \beta) = h(PD(\pi), \beta)$. On the other hand $h(PD(\pi), \beta) = \mathfrak{G}_\pi^{\beta-1}(\mathbf{1}, \mathbf{0})$ by [Mész14c]. □

5.2.3 Uniqueness of *t*-reduced forms

The aim of this section is to prove Theorem 5.2.1, which states that when we let $x_{ij} = t_i$ for all i, then the reduced form becomes unique. For clarity we call the reduced forms with the substitution $x_{ij} = t_i$, the **t-reduced forms**.

In order to prove Theorem 5.2.1 we recall several definitions and results from [Mész11a].

A reduction on the edges $(i, j), (j, k)$ of a noncrossing graph Γ is **noncrossing** if the graphs resulting from the reduction are also noncrossing. Analogously we can define noncrossing reductions on $m[\Gamma]$.

Theorem 5.2.8. *[Mész11a] Let T be a noncrossing tree on the vertex set $[n]$. Performing noncrossing reductions on $m[T]$, regardless of order, we obtain a unique reduced form $R_T^{noncross}(x_{ij} : 1 \leq i < j \leq n)$ for $m[T]$.*

Consider a noncrossing tree T on $[n]$. We define the **pseudo-components** of T inductively. The unique simple path P from 1 to n is a pseudo-component of T. The graph $T \backslash P$ is an edge-disjoint union of trees $T_1, \ldots, T_k$, such that if v is a vertex of P and $v \in T_l$, $l \in [k]$, then v is either the minimal or maximal vertex of T_l . Furthermore, there are no $k-1$ trees whose edge-disjoint union is $T \backslash P$ and which satisfy all the requirements stated above. The set of pseudo-components of T, denoted by $ps(T)$ is $ps(T) = \{P\} \cup ps(T_1) \cup \cdots \cup ps(T_k)$. A pseudo-component P' is said to be on $[i, j]$, $i < j$ if it is a path with endpoints i and j. A pseudo-component P' on $[i, j]$ is said to be a **left pseudo-component** of T if there are no edges $(s, i) \in E(T)$ with $s < i$ and a **right pseudo-component** if if there are no edges $(j, s) \in E(T)$ with $j < s$. See Figure 5.7 for an example.

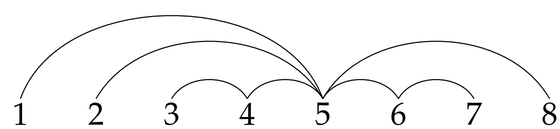


Figure 5.7: The pseudo-components for this tree are described in Example 5.2.9.

Example 5.2.9. Consider the tree in Figure 5.7. The edge sets of the pseudo-components in the graph depicted are $\{(1, 5), (5, 8)\}$, $\{(2, 5)\}$, $\{(3, 4), (4, 5)\}$, $\{(5, 6), (6, 7)\}$. The pseudo-component with edge set $\{(1, 5), (5, 8)\}$ is a both a left and right pseudo-component, while the pseudo-components with edge sets $\{(2, 5)\}$, $\{(3, 4), (4, 5)\}$ are left pseudo-components and the pseudo-component with edge set $\{(5, 6), (6, 7)\}$ is a right pseudo-component.

Theorem 5.2.10. *[Mész11a] Let T be a noncrossing tree. Then $R_T^{noncross}(x_{ij} : 1 \leq i < j \leq n)$ is the sum of the monomials corresponding to the following graphs weighted with powers of β (of degree 1) to obtain a homogeneous polynomial. The graphs are: all noncrossing alternating spanning forests of the directed transitive closure of T on the* *vertex set* $[n]$ *containing edge* $(1, n)$ *and at least one edge of the form* (i_1, j) *with* $i_1 \leq i$ *for each right pseudo-component of* T *on* $[i, j]$ *and at least one edge of the form* (i, j_1) *with* $j \leq j_1$ *for each left pseudo-component of* T *on* $[i, j]$.

We note that in the above we assume that the vertices of our graphs are drawn on a line from left to right in increasing order, $1, 2, \ldots, n$. This condition is of course not an essential condition for the above theorems, and if we rearrange the order of the vertices of our graphs, then we can reinterpret the above results accordingly.

Consider the noncrossing tree T on the vertex set $[n]$ with vertices drawn from left to right in increasing order $1, 2, \ldots, n$. Let $(k, l), (l, m)$ be a pair of non-alternating edges in T. If the reduction performed on $(k, l), (l, m)$ is noncrossing, then we set $T_{klm} = T$ with T_{klm} drawn identically to T. If the reduction performed on $(k, l), (l, m)$ is not noncrossing, then let $C_1 = (V_1, E_1)$ and $C_2 = (V_2, E_2)$ be the connected components containing the vertices k and m, respectively, in the graph $T - \{(k, l), (l, m)\} = ([n], E(T) \backslash \{(k, l), (l, m)\})$. Then we define $T_{klm} = T$ to be drawn with its vertices arranged from left to right in the following order: $v_1^1, \ldots, v_1^p, w_1, \ldots, w_q, v_2^1, \ldots, v_2^r$, where $V_1 = \{v_1^1 < \cdots < v_1^p\}$, $V_2 = \{v_2^1 < \cdots < v_2^r\}$, $[n] \backslash (V_1 \cup V_2) = \{w_1 < \cdots < w_q, \}$. The tree T_{klm} is then a noncrossing tree. See Figure 5.8.

Lemma 5.2.11. *For a noncrossing tree* T *on the vertex set* $[n]$ *and any two edges* $(k, l), (l, m)$ *of* T *which are nonalternating, we have that*

$$R_T^{noncross}(t_i : 1 \leq i \leq n-1) = R_{T_{klm}}^{noncross}(t_i : 1 \leq i \leq n-1). \tag{5.13}$$

Proof. We prove this lemma by induction on the number of increasing paths in T. Suppose there is a vertex $v \neq l$ that is nonalternating. Perform noncrossing

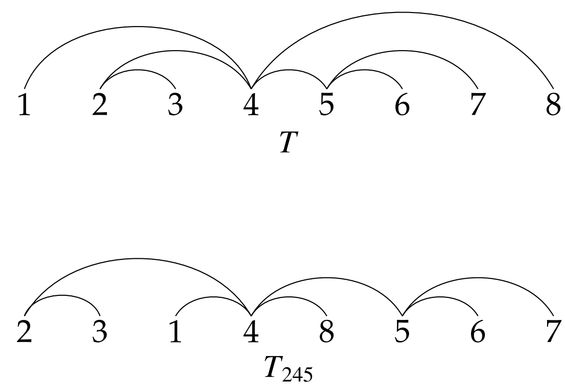


Figure 5.8: Reduction performed on the edges $(2, 4)$, $(4, 5)$ of T is not noncrossing, however when performed on T_{245} it is noncrossing.

reductions at v in both T and T_{klm} obtaining three descendants. Note that the graphs obtained in this fashion from T and T_{klm} are in natural bijection, and they each have fewer number of increasing paths than does T, so by the inductive hypothesis the Lemma is true for them. However, $R_T^{noncross}(t_i : 1 \leq i \leq n-1)$ and $R_{T_{klm}}^{noncross}(t_i : 1 \leq i \leq n-1)$ is the sum of the t-reduced forms corresponding to the mentioned graphs, so we are done.

It remains to prove the case when the only nonalternating vertex of T is l. This is accomplished in Lemma 5.2.12 below. □

Lemma 5.2.12. *For* $T := T^l = ([n], \{(i, l), (l, j) : 1 \leq i < l, l < j \leq n\})$, *for some* $2 \leq l \leq n-1$, *and any two edges* $(k, l), (l, m)$ *of* T *which are nonalternating, we have that*

$$R_T^{noncross}(t_i : 1 \leq i \leq n-1) = R_{T_{klm}}^{noncross}(t_i : 1 \leq i \leq n-1). \tag{5.14}$$

Proof. The only both left and right pseudo-component of T is $\{(1, l), (l, n)\}$, its left pseudo-components are $\{(i, l)\}$, for $2 \leq i \leq l-1$, and its right pseudo-components are $\{(l, i)\}$, for $l+1 \leq i \leq n-1$. Similarly, the only both left and right pseudo-component of T is $\{(k, l), (l, m)\}$, its left pseudo-components are $\{(i, l)\}$, for $i \neq k$,

$1 \le i \le l-1$, and its right pseudo-components are $\{(l,i)\}$, for $i \neq m$, $l+1 \le i \le n$. Using this one can prove by induction on l that there is a bijection between the forests described in Theorem 5.2.10 for T and for those of T_{klm} such that the number of edges of the emanating from any vertex $i \in [n]$ is preserved. While such a proof is not hard, it is technical to describe, and we leave it to the interested reader. □

Proof of Theorem 5.2.1. We proceed by induction on the number of increasing paths in T. If we start by a noncrossing reduction $(k,l),(l,m)$ in T, then no matter how we reduce the three descendants of T which each have fewer number of increasing paths, we obtain that the t-reduced form of $m[T]$ we get is $\mathrm{Crf}_T(\mathbf{t})$.

Suppose we start with a reduction $(k,l),(l,m)$ in T which is a crossing reduction. Redraw the tree T as T_{klm}. Since we can apply the inductive hypothesis to all three descendants of T_{klm}, we get that the t-reduced form of $m[T]$ obtained this way is $\mathrm{Crf}_{T_{klm}}(\mathbf{t})$.

However, by Lemma 5.2.11 $\mathrm{Crf}_T(\mathbf{t}) = \mathrm{Crf}_{T_{klm}}(\mathbf{t})$, thereby proving the theorem. □

5.3 Toric matrix Schubert varieties

On Theorem 5.3.6 we will characterize those π for which $\overline{X_\pi}$ is the product of a vector space and a toric variety (with respect to the torus T^{2n-1}). We introduce some useful notation for the pieces inside the $n \times n$ grid that will play a fundamental role in this section.

Definition 5.3.1. Let $\boldsymbol{NW(\pi)}$ denote the union over $b \in D(\pi)$ of the boxes northwest of b. Let $\boldsymbol{L(\pi)} := NW(\pi) - dom(\pi)$ and let $\boldsymbol{L'(\pi)} := L(\pi) - D(\pi)$.

See Figure 5.9 for an example of these regions.

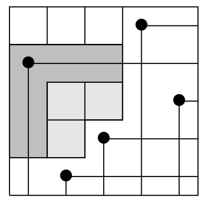


Figure 5.9: Given $\pi = [25413]$, $L(\pi)$ consists of all the gray boxes and $L'(\pi)$ consists of only the darker gray boxes.

Definition 5.3.2. Given a permutation π, let $\boldsymbol{Y_\pi}$ be the projection of $\overline{X_\pi}$ onto the entries inside $L(\pi)$ and let $\boldsymbol{V_\pi}$ be the projection onto the entries not north-west of any box of $D(\pi)$.

Theorem 3.3.4 implies that the entries in V_π are free in $\overline{X_\pi}$. We have that $\overline{X_\pi} = Y_\pi \times V_\pi$. Each of these spaces is a T^{2n}-equivariant subspace of $\overline{X_\pi}$. The dimension of Y_π is

$$\begin{aligned}
\dim(Y_\pi) &= \dim(\overline{X_\pi}) - \dim(V_\pi) \\
&= (n^2 - \ell(\pi)) - (n^2 - |NW(\pi)|) \\
&= (n^2 - |D(\pi)|) - (n^2 - |NW(\pi)|) \\
&= |NW(\pi)| - D(\pi) \\
&= |L'(\pi)|.
\end{aligned}$$

Notice that if π' is a dominant permutation then $L'(1\pi')$ is a hook.

Theorem 5.3.6 characterizes the π for which Y_π is a toric variety with respect to T^{2n-1} in terms of $L'(\pi)$. In order to prove this theorem we will use moment maps; see Section 3.4 for the background on moment maps.

Before we state and prove Theorem 5.3.6, we give an idea of the main tools involved in the proof and how we will use them. The general idea is to use moment maps to find the largest toric variety with respect to T^{2n-1} inside Y_π and then classify the π for which the dimension of this toric variety equals $\dim(Y_\pi)$. The first step in this argument is constructing the T^{2n}-moment map of Y_π. To do this we use Proposition 3.4.4, which tells us we can restrict the moment map of a space into a T^{2n}-equivariantly embedded subspace. We then restrict the moment map of $\overline{X_\pi}$ to obtain the moment map of Y_π.

The second step in our argument is to describe the image of the moment map, using Theorem 3.4.5. This theorem allows us to compute the image of the moment map for a projective variety in terms of its T^{2n-1}-fixed points. Thus, we projectivize Y_π, in order to have a compact space, and compute its T^{2n-1}-fixed points. Theorem 3.4.5 then tells us that the image is the convex hull of the image under the moment map of the T^{2n-1}-fixed points. After constructing the moment polytope, i.e. the image of the moment map, we have all the tools to classify the π for which the toric variety associated to the moment polytope agrees with Y_π.

The following lemma is just a rewording of Lemma 3.4.7 in the language of this chapter.

Lemma 5.3.3. *Y_π is a toric variety if and only if there is a vertex of $\Phi(\mathbb{P}(Y_\pi))$ such that the span of the edge directions out of this vertex has dimension $\dim(Y_\pi) - 1$.*

We now proceed to classify those π for which Y_π is a toric variety with respect to the T^{2n}-action.

Lemma 5.3.4. *If L' is a hook, then Y_π is a toric variety with respect to the T^{2n-1}-action.*

Proof. Consider the moment polytope of $\mathbb{P}(Y_\pi)$. By Lemma 5.3.3 it suffices to find a vertex of this polytope such that the edge directions span $\mathbb{Q}^{\dim(Y_\pi)-1}$. Let (a,b) be the unique box in L' that has no boxes to its north or west. We will show that this vertex has edge directions $x_i - x_a$ for $i = a+1\ldots,w$ where w is the width of $L'(\pi)$ and $y_b - y_j$ for $j = b+1,\ldots,h$ where h is the height of $L'(\pi)$. Note that these vectors are rational and span a vector space with the dimension equal to one less than the boxes inside $L'(\pi)$. Consider the linear functionals $l_i = s_i + s_a - t_b$ with $i = a+1,\ldots,h$. By evaluating l_i on the points we took the convex hull over we see that it is maximized by precisely $x_a - y_b$ and $x_i - y_b$ so this functional tells us that indeed $x_i - x_a$ is an edge direction from the vertex $x_a - y_b$. The result follows from an analogous argument for the vectors $y_b - y_j$. □

Corollary 5.3.5. *If π' is a dominant permutation on $2,3,\ldots,n$ then $Y_{1\pi'}$ is a toric variety.*

Theorem 5.3.6. *Y_π is a toric variety with respect to the T^{2n-1}-action if and only if $L'(\pi)$ consists of disjoint hooks that do not share a row or a column with each other.*

Proof. By Lemma 5.3.4 it follows that if $L'(\pi)$ consists of disjoint hooks then Y_π is the product of as many toric varieties as the number of hooks, where the i-th toric variety corresponding to the entries in $L(\pi)$ south-east of the i-th hook. For the proof of the only if part of the Theorem, we will compute the dimension of $\Phi(Y_\pi)$ in different cases and show that it cannot equal $|L'(\pi)|$ if $L'(\pi)$ does not have the desired shape.

Suppose now that $L'(\pi)$ is connected and it is not a hook. Consider the smallest rectangle that contains $L'(\pi)$, see Figure 5.10. The dimension of $\Phi(Y_\pi)$ is at most $h+w-1$, where h is the height of the rectangle and w is the width. Consider the westernmost box (a,b) on the northernmost row of $L'(\pi)$, then we must have that $\pi(a) = b$. Moreover, this box cannot be on the north-east corner of the rectangle since the rectangle has to contain a box in $D(\pi)$ that is south-east of every box in $L'(\pi)$. We have a similar case for the northernmost box (c, d) on the west column of $L'(\pi)$. If these two boxes are distinct, then $|L'(\pi)| > h + w$ and Y_π is not toric. If the two boxes are the same, since $L'(\pi)$ is not a hook then $|L'(\pi)| \geq h + w$ and again, Y_π is not toric.

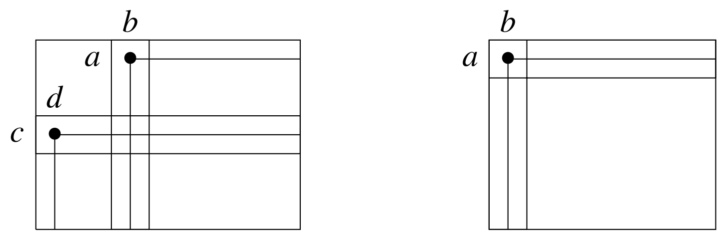


Figure 5.10: The two possibilities for the north-most and west-most boxes in $L'(\pi)$.

Now suppose that $L'(\pi)$ has more than one connected component. If two components C_1 and C_2 have boxes on the same row or on the same column then they must be nested, meaning that if $(i, j) \in C_1$ and $(k, l) \in C_2$, then $i < k$ and $k < l$. We can therefore decompose $L'(\pi) = \sqcup D_i$ such that each D_i is a maximal family of nested components and thus distinct D_i do not have boxes on the same row or column, see Figure 5.11. For each D_i consider the smallest rectangle R_i containing it and the projection of Y_π to the entries in $L(\pi)$ inside R_i, then Y_π is the product of these varieties and the moment polytope of $\mathbb{P}(Y_\pi)$ is the product of the polytopes obtained from these projections.

Consider one of these maximal families of nested components $D_i = C_1 \cup \ldots \cup C_k$, where C_1 is the largest connected component. The dimension of the polytope corresponding to D_i at most $h + w - 1$ where h is the height of R_i and w is the width. Therefore, the variety corresponding to D_i is a toric variety if

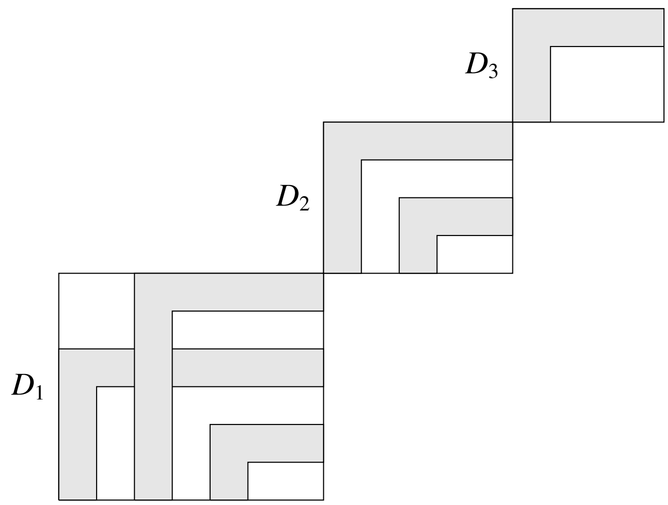


Figure 5.11: The decomposition $L'(\pi) = \bigcup D_i$

and only if $L'(\pi)$ restricted to D_i has $h + w - 1$ boxes which is only possible if D_i is exactly one hook. It then follows that Y_π is a toric variety if and only if $L'(\pi)$ consists of disjoint hooks $H_1, \ldots, H_k$ where for $s \neq t$ we have that

$$\{i : (i, j) \in H_s\} \cap \{i : (i, j) \in H_t\} = \{j : (i, j) \in H_s\} \cap \{j : (i, j) \in H_t\} = \varnothing.$$

□

5.4 Projection of the moment polytope of Y_π onto a root polytope

In this section we give a specialization of the moment polytope of $\mathbb{P}(Y_\pi)$ into a root polytope for $\pi = 1\pi'$, but π' not necessarily dominant. We start by generalizing the map mapping the region $R(\pi)$ to the tree $T(\pi)$ for any $\pi = 1\pi'$, π' not necessarily dominant.

Proposition 5.4.1. *Let $\pi = 1\pi'$ with π' dominant. Then the union of $(1, 1)$ and $cr(\pi)$*

consists of all entries north-west of some box in $D(\pi)$ *that are not inside Fulton's essential set. In equations, this says that* $R(\pi) = NW(\pi) - Ess(\pi)$.

Proof. It is clear that $R(\pi) \subset NW(\pi)$. Given $(i, j) \in Ess(\pi)$, we have that neither the bottom or the top reduced pipe dreams of π have a cross on this position since (i, j) is a south-east corner of $D(\pi)$ and so $(i, j) \notin R(\pi)$. Since $D(\pi)$ is a partition, then $D(\pi)$ restricted to a row or to a column is connected. Therefore, any entry strictly west of an entry in $Ess(\pi)$ must be in the bottom reduced pipe dream of π and thus inside $R(\pi)$. Similarly, any entry strictly north of $Ess(\pi)$ must be in the top reduced pipe dream. □

Recall that the graph associated to $\mathrm{cr}(\pi)$ for $\pi = 1\pi'$ with π' dominant has as vertices the elements of the set A which consists of some E and N steps that bound the SW boundary of $\mathrm{cr}(\pi)$.

Proposition 5.4.2. *Let* $\pi = 1\pi'$ *with* π' *dominant. An* E *step is in* A *if and only if it does not bound a box in* $Ess(\pi)$*.*

Proof. Proposition 5.4.1 tells us that given an E step the N step directly preceding E bounds $R(\pi)$ if and only if E bounds a box in $Ess(\pi)$. □

These propositions allow us to generalize the construction of $T(\pi)$ to any permutation π with empty dominant piece, i.e. $\pi(1) = 1$. Consider the subset of $Ess(\pi)$ consisting of the boxes that are on the boundary of $NW(\pi)$ and denote it by $subEss(\pi)$. In other words, $subEss(\pi)$ consists of the boxes in Fulton's essential set that are needed to determine $NW(\pi)$. Note that there is at most one element in $subEss(\pi)$ per row of $NW(\pi)$.

We construct the graph $T(\pi)$ by looking at the E and N steps bounding the

SE boundary of $NW(\pi) - Ess(\pi)$. Let A be the set consisting of all the N steps together with the E steps that do not bound a box in $Ess(\pi)$. Suppose $|A| = m$, as we transverse this lower boundary from the SW corner, we write $\alpha_1, \ldots, \alpha_m$ in increasing fashion below the E steps and to the right of the N steps that belong to A. For the E steps that we did not assign an α_i, we consider their label to be the α_i assigned to the N step directly preceding them. Let $\overline{T}(\pi)$ be the tree with vertices $V = \{\alpha_1, \ldots, \alpha_m\}$ and edges corresponding to the entries in $NW(\pi) - Ess(\pi)$. More precisely (α_i, α_j) is an edge of $\overline{T}(\pi)$ if the entry $(i, j) \in NW(\pi) - Ess(\pi)$ is north of the E step labeled α_j and east of the N step labeled α_i. We let $T(\pi)$ be the minimal tree such that its transitive closure is $\overline{T}(\pi)$. See Figures 5.12 and 5.13 for an example.

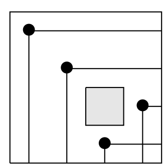


Figure 5.12: The diagram for $\pi = [1243]$.

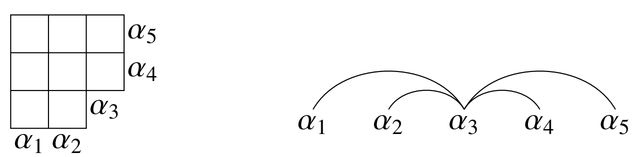


Figure 5.13: On the left we have $NW(\pi) - subEss(\pi)$ for $\pi = [1243]$ with its SW boundary labelled by A. On the right we have the corresponding tree $T(\pi)$.

Note that Propositions 5.4.1 and 5.4.2 tell us that for $\pi = 1\pi'$ with π' dominant this construction coincides with the one we had before.

Theorem 5.4.3. *Given π with empty dominant piece (instead of just $1\pi'$ with π' dominant), the moment polytope $\Phi(\mathbb{P}Y_\pi)$ can be projected onto the root polytope $P(T(\pi))$.*

Proof. We will give a linear map from $\Phi(\mathbb{P}Y_\pi) \twoheadrightarrow P(Y(\pi))$ as a composition of two maps f and g. Let g be the map given by

$$g(x_i) = -e_j, \text{ where } \alpha_j \text{ is the label of step } N \text{ on row } i, \textit{and}$$

$$g(y_i) = \begin{cases} 0 & \text{if } (a,i) \in subEss(\pi) \text{ for some } a, \\ -e_j & \text{where } \alpha_j \text{ is the label of step } E \text{ on column } i. \end{cases}$$

Let f be the map given by

$$f(x_i) = 2x_i$$

$$f(y_j) = \begin{cases} 2x_i & \text{if } (i,j) \in subEss(\pi), \\ 2y_j & \text{if there is no } a \text{ such that } (a,j) \in subEss(\pi). \end{cases}.$$

See Figures 5.14 and 5.15 for an example of these maps. Now notice that if $\frac{1}{2}(x_i - y_j) \in \Phi((\mathbb{P}Y_\pi)^{T^{2n-1}})$, i.e. it is the image of a fixed point, then we have that $g \circ f(\frac{1}{2}(x_i - y_j)) = e_{k_j} - e_{k_i}$, where α_{k_i} is the label of step N in A on row i and α_{k_j} is the label of step E in A on column j. Notice that f corresponds to multiplying by two and subtracting the vector $x_a - y_b$ from each vector $x_i - y_b$ in column b whenever $(a,b) \in subEss(\pi)$. Notice also that g is the map induced by the relabeling of the entries in $NW(\pi) - subEss(\pi)$ into the tree labeling. In the language of ϕ, we then have that $\frac{1}{2}(x_i - y_j) \mapsto e_{\phi^{-1}(j)} - e_{\phi^{-1}(i)}$. The root polytope of $T(\pi)$ is

$$\text{ConvHull}(0, e_{\phi^{-1}(j)} - e_{\phi^{-1}(i)} : (i,j) \in NW(\pi) - subEss(\pi))$$

so the theorem follows. □

$\frac{1}{2}$

$x_1 - y_1$	$x_1 - y_2$	$x_1 - y_3$
$x_2 - y_1$	$x_2 - y_2$	$x_2 - y_3$
$x_3 - y_1$	$x_3 - y_2$	$x_3 - y_3$

$\overset{f}{\longmapsto}$

$x_1 - y_1$	$x_1 - y_2$	$x_1 - x_3$
$x_2 - y_1$	$x_2 - y_2$	$x_2 - x_3$
$x_3 - y_1$	$x_3 - y_2$	0

Figure 5.14: The map f for $Y_{[1243]}$.

$x_1 - y_1$	$x_1 - y_2$	$x_1 - x_3$
$x_2 - y_1$	$x_2 - y_2$	$x_2 - x_3$
$x_3 - y_1$	$x_3 - y_2$	0

$\overset{g}{\longmapsto}$

$e_1 - e_5$	$e_2 - e_5$	$e_3 - e_5$	α_5
$e_1 - e_4$	$e_2 - e_4$	$e_3 - e_4$	α_4
$e_1 - e_3$	$e_2 - e_3$	α_3	
α_1	α_2		

Figure 5.15: The map g for $Y_{[1243]}$.

By Propositions 5.4.1 and 5.4.2 we have that for $\pi = 1\pi'$ with π' dominant the image $g \circ f(\Phi(\mathbb{P}Y_\pi))$ is precisely the root polytope $P(T(\pi))$.

The following Proposition allows us to describe the projection in Theorem 5.4.3 in words.

Proposition 5.4.4. *The vertices of $\Phi(\mathbb{P}Y_\pi)$ corresponding to the entries in $subEss(\pi)$ form a face of $\Phi(\mathbb{P}Y_\pi)$.*

Proof. Let $subEss(\pi) = \{(a_1, b_1), \ldots, (a_k, b_k)\}$ and consider the linear functional $l = (s_{a_1} + \cdots + s_{a_k}) - (t_{b_k} + \cdots + t_{b_k})$. By evaluating l on the points inside $\Phi((\mathbb{P}Y_\pi)^{T^{2n-1}})$ we see that it is maximized precisely by $x_{a_i} - y_{b_i}$ with $(a_i, b_i) \in subEss(\pi)$. It then follows that these points are a face of $\Phi(\mathbb{P}Y_\pi)$. □

The projection $g \circ f : \Phi(\mathbb{P}Y_\pi) \to P(T(\pi))$ consists of contracting the face of $\Phi(\mathbb{P}Y_\pi)$ corresponding to $subEss(\pi)$ to a point and moving this point to the origin while tweaking the vertices of $\Phi(\mathbb{P}Y_\pi)$ that are of the form $\frac{1}{2}(x_i - y_j)$ where (i, j) is north of an entry of $subEss(\pi)$.

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