ITERATIVE ESTIMATION OF VARIANCE COMPONENTS IN THE 2-WAY CROSSED CLASSIFICATION, MIXED MODEL, WITH INTERACTION, USING UNBALANCED DATA

by

R. R. Corbeil and S. R. Searle

Biometrics Unit, Cornell University, Ithaca, N. Y.

BU-460-M April 1973

Abstract

Thompson's (1969) iterative estimation method for unbalanced data in a 2-way classification without interaction is extended to the case with interaction. Specific computing formulae are developed.

1. Introduction

Iterative methods for estimating variance components in the 2-way crossed classification mixed model from data with unequal subclass numbers are developed by Cunningham and Henderson (1968) and corrected by Thompson (1969). This paper extends Thompson's method to include an interaction component between the random and fixed effects of the model.

We follow the notation of Searle (1971) and re-write the usual 2-way interaction model as

$$y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + e_{ijk}$$

in the more useful matrix form

$$\underset{\sim}{y} = \underset{\sim}{X}\underset{\sim}{b} + \underset{\sim}{Z}\underset{\sim}{u} + \underset{\sim}{W}\underset{\sim}{v} + \underset{\sim}{e} \ . \qquad (1)$$

The vector of observations, $\underline{y}$, has order $n_{..} = N$, its elements y_{ijk} for $i = 1,\ldots,a$, $j = 1,\ldots,b$ and $k = 1,\ldots,n_{ij}$ are random variables with means given by $E(y) = \underline{X}\underline{b}$ and variance-covariance matrix

$$\underline{V} = \underline{Z}\underline{Z}'\sigma_u^2 + \underline{W}\underline{W}'\sigma_v^2 + \underline{I}\sigma_e^2 .$$

The vector $\underline{b}$ of order b has elements $\mu + \beta_j$, the fixed effects, and the corresponding incidence matrix $\underline{X}$ has full column rank. The vector $\underline{u}$ contains the random effects α_i and $\underline{v}$ contains the random interaction effects γ_{ij} for s filled cells of the data. Thus in (1) $\underline{X}$ is $N \times b$, $\underline{Z}$ is $N \times a$ and $\underline{W}$ is $N \times s$.

2. Sums of squares

One problem in estimating fixed effects in a mixed model for known variance components is the practical one of having to invert $\text{var}(\underline{y}) = \underline{V}$, which is of order $N \times N$ and consequently, often large. Henderson, et al. (1959) show that the solution to the general least squares equations

$$\underline{X}'\underline{V}^{-1}\underline{X}\hat{\underline{b}} = \underline{X}'\underline{V}^{-1}\underline{y}$$

is identical to the solution of a substitute set of equations constructed from the normal equations of a fixed effects model, after adding the inverse of the variance-covariance matrix of each random effects factor to appropriate submatrices of the normal equations [see Searle (1971), p. 460]. This modification to the normal equations for our model (1) is

$$\begin{bmatrix} \underline{X}'\underline{X} & \underline{X}'\underline{Z} & \underline{X}'\underline{W} \\ \underline{Z}'\underline{X} & \underline{Z}'\underline{Z} + \frac{\sigma_e^2}{\sigma_u^2}\underline{I} & \underline{Z}'\underline{W} \\ \underline{W}'\underline{X} & \underline{W}'\underline{Z} & \underline{W}'\underline{W} + \frac{\sigma_e^2}{\sigma_v^2}\underline{I} \end{bmatrix} \begin{bmatrix} \hat{\underline{b}} \\ \hat{\underline{u}} \\ \hat{\underline{v}} \end{bmatrix} = \begin{bmatrix} \underline{X}'\underline{y} \\ \underline{Z}'\underline{y} \\ \underline{W}'\underline{y} \end{bmatrix} . \qquad (2)$$

When σ_u^2, σ_v^2 and σ_e^2 are known, the estimator of $\underline{b}$ is $\hat{\underline{b}}$, the same as the solution to $\underline{X}'\underline{V}^{-1}\underline{X}\hat{\underline{b}} = \underline{X}'\underline{V}^{-1}\underline{y}$. This estimator is also the maximum likelihood estimator under normality assumptions. Usually the variance components in (2) are unknown and a solution is obtained by replacing them by estimates, often the fitting-constants-method estimates.

Although σ_u^2, σ_v^2 and σ_e^2 <u>can</u> be estimated by the fitting constants method of estimating variance components (Henderson's Method 3), equation (2) provides an alternative method, iterative in nature, in the manner of Thompson [1969]. This is based upon the reduction in sum of squares for fitting a model for which equations (2) would be the normal equations. By this we mean in general that for normal equations expressed as $\underline{X}'\underline{X}\beta^o = \underline{X}'\underline{y}$ for any $\underline{X}$, the corresponding reduction in sum of squares is $\underline{y}'\underline{X}(\underline{X}'\underline{X})^-\underline{X}'\underline{y}$ where $(\underline{X}'\underline{X})^-$ is the generalized inverse of $\underline{X}'\underline{X}$. Consequently on writing

$$\lambda_u = \sigma_e^2/\sigma_u^2 \quad \text{and} \quad \lambda_v = \sigma_e^2/\sigma_v^2$$

the sum of squares we consider here, based on (2), is

$$R^*(b,u,v) = (\underline{y}'\underline{X} \quad \underline{y}'\underline{Z} \quad \underline{y}'\underline{W}) \begin{bmatrix} \underline{X}'\underline{X} & \underline{X}'\underline{Z} & \underline{X}'\underline{W} \\ \underline{Z}'\underline{X} & \underline{Z}'\underline{Z} + \lambda_u \underline{I} & \underline{Z}'\underline{W} \\ \underline{W}'\underline{X} & \underline{W}'\underline{Z} & \underline{W}'\underline{W} + \lambda_v \underline{I} \end{bmatrix}^{-1} \begin{bmatrix} \underline{X}'\underline{y} \\ \underline{Z}'\underline{y} \\ \underline{W}'\underline{y} \end{bmatrix} . \qquad (3)$$

The matrix inverse in (3) exists because the effect of the linear dependencies among columns of $\underline{X}$, $\underline{W}$, and $\underline{Z}$ is nullified when λ_u and λ_v are non-zero.

We define

$$\underline{P} = \underline{Z}'\underline{Z} + \lambda_u \underline{I}, \qquad \underline{T} = \underline{I} - \underline{Z}\underline{P}^{-1}\underline{Z}' = \underline{T}'$$

$$\underline{S}_u = \underline{X}'\underline{T}\underline{X} \quad \text{and} \quad \underline{S}_v = \underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{W} + \lambda_v \underline{I} \qquad (4)$$

where $\underline{P}$, $\underline{S}_u$ and $\underline{S}_v$ are non-singular. Then it can be shown (see appendix) that from (3)

$$R^*(\underline{b},\underline{u},\underline{v}) = \underline{y}'[\underline{Z}\underline{P}^{-1}\underline{Z}' + \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T} + (\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{W}\underline{S}_v^{-1}\underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})]\underline{y} . \qquad (5)$$

The no-interaction model considered by Cunningham and Henderson [1968] and Thompson [1969] is, in terms of (1)

$$\underline{y} = \underline{X}\underline{b} + \underline{Z}\underline{u} + \underline{e} , \qquad (6)$$

simply a sub-model of (1) with $\underline{W}\underline{v}$ omitted. The reduction in sum of squares analogous to (5) is then

$$R^*(\underset{\sim}{b},\underset{\sim}{u}) = (\underset{\sim}{y}'\underset{\sim}{X} \quad \underset{\sim}{y}'\underset{\sim}{Z}) \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} \\ \underset{\sim}{Z}'\underset{\sim}{X} & \underset{\sim}{Z}'\underset{\sim}{Z} + \lambda_u \underset{\sim}{I} \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{y} \\ \underset{\sim}{Z}'\underset{\sim}{y} \end{bmatrix} . \tag{7}$$

$$= \underset{\sim}{y}'(\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}' + \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\,\underset{\sim}{y} . \tag{8}$$

The difference between the reductions (5) and (8), which might be called a reduction due to interaction, is

$$R^*(\underset{\sim}{v}|\underset{\sim}{b},\underset{\sim}{u}) = R^*(\underset{\sim}{b},\underset{\sim}{u},\underset{\sim}{v}) - R^*(\underset{\sim}{b},\underset{\sim}{u})$$

$$= \underset{\sim}{y}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{W}\underset{\sim}{S}_v^{-1}\underset{\sim}{W}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{y} . \tag{9}$$

In order to estimate σ_u^2, σ_v^2 and σ_e^2 we need three quadratic forms, of which one is (9). A second is

$$R^*(\underset{\sim}{u},\underset{\sim}{v}|\underset{\sim}{b}) = R^*(\underset{\sim}{b},\underset{\sim}{u},\underset{\sim}{v}) - R^*(\underset{\sim}{b}) \tag{10}$$

where

$$R^*(\underset{\sim}{b}) = \underset{\sim}{y}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{y} \tag{11}$$

which is also the familiar reduction $R(\underset{\sim}{b})$ due to fitting $E(\underset{\sim}{y}) = \underset{\sim}{X}\underset{\sim}{b}$. And finally we use

$$SSE^* = \underset{\sim}{y}'\underset{\sim}{y} - R^*(\underset{\sim}{b},\underset{\sim}{u},\underset{\sim}{v}). \tag{12}$$

3. Expected values

Equations for estimating σ_u^2, σ_v^2 and σ_e^2 are derived by equating observed values of (9), (10) and (12) to their expected values. We assume normality and use the general result for $\underline{y} \sim N(\underline{\mu}, \underline{V})$ that

$$E(\underline{y}'\underline{Q}\underline{y}) = \underline{\mu}'\underline{Q}\underline{\mu} + tr(\underline{Q}\underline{V}).$$

In our case, the model (1), $\underline{\mu} = \underline{X}\underline{b}$ and $\underline{V} = \underline{Z}\underline{Z}'\sigma_u^2 + \underline{W}\underline{W}'\sigma_v^2 + \underline{I}\sigma_e^2$ so that

$$E(\underline{y}'\underline{Q}\underline{y}) = \underline{b}'\underline{X}'\underline{Q}\underline{X}\underline{b} + \sigma_u^2 tr(\underline{Z}'\underline{Q}\underline{Z}) + \sigma_v^2 tr(\underline{W}'\underline{Q}\underline{W}) + \sigma_e^2 tr(\underline{Q})$$

$$= \underline{b}'\underline{X}'\underline{Q}\underline{X}\underline{b} + \sigma_v^2 tr(\underline{W}'\underline{Q}\underline{W}) + tr[(\sigma_u^2\underline{Z}\underline{Z}' + \sigma_e^2\underline{I})\underline{Q}]$$

$$= \underline{b}'\underline{X}'\underline{Q}\underline{X}\underline{b} + \sigma_v^2 tr(\underline{W}'\underline{Q}\underline{W}) + \sigma_u^2 tr[(\underline{Z}\underline{Z}' + \lambda_u\underline{I})\underline{Q}] \ . \qquad (13)$$

Deriving the expected values of the quadratic forms (9), (10) and (12) now involves using for $\underline{Q}$ in (13), the matrices of those quadratic forms. For example, writing (9) as $R^*(\underline{v}|\underline{b},\underline{u}) = \underline{y}'\underline{Q}_1\underline{y}$ we have

$$\underline{Q}_1 = (\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{W}\underline{S}_v^{-1}\underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T}),$$

and replacing $\underline{Q}$ of (13) by $\underline{Q}_1$ leads, as shown in the appendix, to

$$E\ R^*(\underline{v}|\underline{b},\underline{u}) = \sigma_v^2[tr(\underline{S}_v) - s\lambda_v] \ . \qquad (14)$$

Similarly

$$E\ R^*(\underset{-}{u},\underset{-}{v}|\underset{-}{b}) = (n_{..} - k_4)(\sigma_u^2 + \sigma_v^2) \tag{15}$$

and

$$E\ (SSE^*) = (n_{..} - b)\sigma_e^2, \tag{16}$$

where

$$k_4 = \sum_j \frac{1}{n_{.j}} \sum_i n_{ij}^2 .$$

4. Estimation

Equating the expected values on the right-hand sides of (14), (15) and (16) to observed values yields equations in σ_u^2, σ_v^2 and σ_e^2 that cannot be solved explicitly. This is because the quadratic forms $R^*(\underset{-}{v}|\underset{-}{b},\underset{-}{u})$, $R^*(\underset{-}{u},\underset{-}{v}|\underset{-}{b})$ and SSE^* involve $\lambda_u = \sigma_e^2/\sigma_u^2$ and $\lambda_v = \sigma_e^2/\sigma_v^2$, through $\underset{-}{P}$, $\underset{-}{T}$ $\underset{-}{S}_u$ and $\underset{-}{S}_v$ of (4). However, the equations can be solved iteratively. Writing them as

$$\tilde{\sigma}_e^2 = \frac{SSE^*}{n_{..} - b} = \frac{\underset{-}{y}'\underset{-}{y} - R^*(\underset{-}{b},\underset{-}{u},\underset{-}{v})}{n_{..} - b} \tag{17}$$

$$\tilde{\sigma}_v^2 = \frac{R^*(\underset{-}{v}|\underset{-}{b},\underset{-}{u})}{tr(\underset{-}{S}_v) - s\lambda_v} \tag{18}$$

and

$$\tilde{\sigma}_u^2 = \frac{R^*(\underset{-}{u},\underset{-}{v}|\underset{-}{b})}{n_{..} - k_4} - \tilde{\sigma}_v^2 , \tag{19}$$

we first take initial values of λ_u and λ_v, calculate (17), (18) and (19), use the results to get second values of λ_u and λ_v and continue the process until some satisfactory degree of convergence is (hopefully) achieved.

Note in passing that when $\sigma_v^2 \equiv 0$, (17) and (19) reduce to Thompson's (1969) results. [See also Searle (1971), p. 469.] In both cases, for balanced data the results are identical to that of the ANOVA method.

5. Fixed effects

Estimates of the fixed effects represented in the model by $\underline{b}$ are the solutions $\hat{\underline{b}}$ to equation (2). As shown in the appendix this solution reduces to

$$\hat{\underline{b}} = \underline{S}_u^{-1}\underline{X}'\underline{T}[\underline{I} - \underline{W}\underline{S}_v^{-1}\underline{W}'(\underline{I} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})]\underline{y}, \qquad (20)$$

which is calculated after $\tilde{\sigma}_u^2$, $\tilde{\sigma}_v^2$ and $\tilde{\sigma}_e^2$ have been obtained from (17), (18) and (19).

6. Computing formulae

Easy terms to compute are $\underline{y}'\underline{y}$ and k_4. Harder ones are $R^*(\underline{v}|\underline{b},\underline{u})$, $R^*(\underline{u},\underline{v}|\underline{b})$ and $tr(\underline{S}_v)$. However, from the no-interaction case, Searle [1973], we have expressions for $R^*(\underline{b})$, $R^*(\underline{u})$ and $R^*(\underline{b}|\underline{u})$. These and the other terms needed are now listed.

$\underline{n}_i' = [n_{i1} \cdots n_{ib}]$, including zero n_{ij}'s

$\underline{\Delta}(\underline{n}_i')$ = diagonal matrix of the elements of $\underline{n}_i'$

$\underline{\Delta}_1(\underline{n}_i') = \underline{\Delta}(\underline{n}_i')$ omitting the null rows (but not the null columns)

$(\underline{M}_i)_{b\times b} = \dfrac{1}{n_{i\cdot} + \lambda_u}\{n_{ij}\, n_{ij'}\}$ for $j, j' = 1, \ldots, b$

$\underline{M}_{1,i} = \underline{M}_i$ omitting the null rows (but not the null columns)

$\underset{\sim}{M}_{2,i} = \underset{\sim}{M}_i$ omitting the null rows and null columns

$$\underset{\sim}{y}'\underset{\sim}{y} = \Sigma\,\Sigma\,\Sigma\, y^2_{ijk}$$

$$R^*(\underset{\sim}{b}) = \sum_j \frac{y^2_{\cdot j\cdot}}{n_{\cdot j}}$$

$$k_4 = \sum_j \frac{1}{n_{\cdot j}} \sum_j n^2_{ij}$$

$$R^*(\underset{\sim}{u}) = \sum_i \frac{y^2_{i\cdot\cdot}}{n_{i\cdot} + \lambda_u}$$

$$(\underset{\sim}{r}_*)_{b\times 1} = \underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{y} = \left\{y_{\cdot j\cdot} - \sum_i \frac{n_{ij}y_{i\cdot\cdot}}{n_{i\cdot} + \lambda_u}\right\} \quad \text{for } j = 1,\cdots,b$$

and

$$(\underset{\sim}{C}_*)_{b\times b} = \underset{\sim}{S}_u = \underset{\sim}{D}\{n_{\cdot j}\} - \left\{\sum_i \frac{n_{ij}n_{ij'}}{n_{i\cdot} + \lambda_u}\right\} \quad \text{for } j,j' = 1,\cdots,b$$

$$R^*(\underset{\sim}{b}|\underset{\sim}{u}) = \underset{\sim}{r}_*'\underset{\sim}{C}_*^{-1}\underset{\sim}{r}_*$$

$$\underset{\sim}{q}_{s\times 1} = \underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{y} = \left\{y_{ij\cdot} - \frac{n_{ij}y_{i\cdot\cdot}}{n_{i\cdot} + \lambda_u}\right\} \quad \text{for } i = 1,\cdots,a \text{ and } j = 1,\cdots,b \text{ in lexicon order}$$

$$\underset{\sim}{H}_{s\times b} = \underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{X} = \left\{\underset{\sim}{\Delta}_1(\underset{\sim}{n}_i') - \underset{\sim}{M}_{1,i}\right\} \quad \text{for } i = 1,\cdots,a$$

$$= \begin{bmatrix} \underset{\sim}{\Delta}_1(\underset{\sim}{n}_1') & \cdots & \underset{\sim}{M}_{1,1} \\ \underset{\sim}{\Delta}_1(\underset{\sim}{n}_2') & \cdots & \underset{\sim}{M}_{1,2} \\ \vdots & & \vdots \\ \underset{\sim}{\Delta}_1(\underset{\sim}{n}_a') & \cdots & \underset{\sim}{M}_{1,a} \end{bmatrix}$$

$$G_{s \times s} = W'TW$$

$$= D\{n_{ij}\} - \Sigma_i^+ M_{2,i}$$

$$S_v = G - HC_*^{-1}H' + \lambda_v I_s$$

$$R^*(v|b,u) = y'(T - TXS_u^{-1}X'T)WS_v^{-1}W'(T - TXS_u^{-1}X'T)y$$

$$= (W'Ty - W'TXS_u^{-1}X'Ty)'S_v^{-1}(W'Ty - W'TXS_u^{-1}X'Ty).$$

$$= (q - HC_*^{-1}r_*)'S_v^{-1}(q - HC_*^{-1}r_*)$$

$$R^*(b,u,v) = R^*(v|b,u) + R^*(b|u) + R^*(u)$$

$$SSE^* = y'y - R^*(b,u,v)$$

$$R^*(u,v|b) = R^*(b,u,v) - R^*(b)$$

$$k_* = tr(S_v) - s\lambda_v$$

$$\tilde{\sigma}_e^2 = \frac{SSE^*}{n_{..} - b}$$

$$\tilde{\sigma}_v^2 = \frac{R^*(v|b,u)}{k_*}$$

$$\tilde{\sigma}_u^2 = \frac{R^*(u,v|b)}{n_{..} - k_4} - \tilde{\sigma}_v^2$$

Iteration commences by using pre-assigned values for $\lambda_u = \sigma_e^2/\sigma_u^2$ and $\lambda_v = \sigma_e^2/\sigma_v^2$; with these, first estimates $\tilde{\sigma}_u^2$, $\tilde{\sigma}_v^2$ and $\tilde{\sigma}_e^2$ are obtained from which new values $\tilde{\lambda}_u$ and $\tilde{\lambda}_v$ are calculated and the process repeated. On convergence (hopefully), the fixed effects estimates are calculated from (20) as

$$\hat{\underline{b}} = \underline{C}_*^{-1}\underline{r}_* - \underline{C}_*^{-1}\underline{H}'\underline{S}_v^{-1}(\underline{q} - \underline{H}\underline{C}_*^{-1}\underline{r}_*)$$

Acknowledgement

This work was partially supported by Grant No. GJ31746 from the National Science Foundation.

References

Cunningham, E. P. and Henderson, C. R. [1968]. An iterative procedure for estimating fixed effects and variance components in mixed model situations. Biometrics, 24, 13-25. Corrigenda, Biometrics, 25, 777-778, 1969.

Searle, S. R. [1971]. Linear Models, Wiley, New York.

Searle, S. R. [1973]. Computing procedures for estimating variance components from unbalanced data in the 2-way crossed classification, no interaction, mixed model. Paper No. BU-450-M in the Biometrics Unit, Cornell University.

Thompson, R. [1969]. Iterative estimation of variance components for non-orthogonal data. Biometrics, 25, 767-773.

Henderson, C. R., Kempthorne, O., Searle, S. R., and von Krosigk, C. N. [1959]. Estimation of environmental and genetic trends from records subject to culling. Biometrics, 15, 192-218.

Appendix

1A. Introduction

By the nature of the model (1)

$$\underline{X} = \left\{ \sum_{j=1}^{b}{}^{*} \underline{1}_{n_{ij}} \right\}_{s\times b} \quad \text{for } i = 1,\ldots,a \tag{A1}$$

$$\underline{Z} = \sum_{i=1}^{a}{}^{+} \underline{1}_{n_{i\cdot}} \tag{A2}$$

and

$$\underline{W} = \sum_{i=1}^{a}{}^{+} \sum_{j=1}^{b}{}^{+} \underline{1}_{n_{ij}} \quad \text{for } i = 1,\ldots,a \text{ and } j = 1,\ldots,b \text{ in lexicon order and for only those } n_{ij} \text{ for which } n_{ij} > 0, \tag{A3}$$

where Σ^{+} represents the operation of direct sum and $\sum_{j=1}^{b}{}^{*} \underline{1}_{n_{ij}}$ is a matrix of b columns, a direct sum of vectors $\underline{1}_{n_{ij}}$ modified by the convention that the t'th column contains no $\underline{1}$-vector when $n_{it} = 0$ and so is null. For example with,

$$n_{rj} = 2,0,4,3,0,6$$

$$\sum_{j=1}^{b}{}^{*} \underline{1}_{n_{rj}} = \begin{bmatrix} \underline{1}_2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \underline{1}_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \underline{1}_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \underline{1}_6 \end{bmatrix}.$$

These forms imply

$$\underline{X}'\underline{X} = \underline{D}\{n_{.j}\}_{b\times b} \quad \underline{Z}'\underline{Z} = \underline{D}\{n_{i.}\}_{a\times a} \quad \text{and} \quad \underline{W}'\underline{W} = \underline{D}\{n_{ij}\}_{s\times s} \tag{A4}$$

together with

$$\underline{X}'\underline{Z} = \{n_{ij}\}_{a\times b} \tag{A5}$$

$$\underline{W}'\underline{Z} = \Sigma_i^+ \underline{\nu}_i \tag{A6}$$

for $\underline{\nu}_i$ being the vector $\{n_{ij}\}$ for $j = 1,\ldots,b$ and $n_{ij} \neq 0$, i.e. with the zero n_{ij}'s omitted; and

$$\underline{W}'\underline{X} = \left\{\lambda_{(i,j),k}\right\}_{s\times b} = \left\{n_{ij}\,\delta_{jk}\right\}_{s\times b} \tag{A7}$$

for $i = 1,\ldots,a$ and $j = 1,\ldots,b$, in lexicon order, for those (i,j) for which $n_{ij} \neq 0$, and for $k = 1,\ldots,b$,

where $\delta_{jk} = 0$ when $j \neq k$ and $\delta_{jk} = 1$ when $j = k$. With

$$\lambda_u = \sigma_e^2/\sigma_u^2 \quad \text{and} \quad \lambda_v = \sigma_e^2/\sigma_v^2, \tag{A8}$$

and

$$\underline{P} = \underline{Z}'\underline{Z} + \lambda_u\underline{I} = \underline{D}\{n_{i.} + \lambda_u\}, \tag{A9}$$

and

$$\underline{T} = \underline{I} - \underline{Z}\underline{P}^{-1}\underline{Z}' = \underline{T}' \tag{A10}$$

we then have

$$\underline{Z}\underline{Z}'\underline{T} = \underline{Z}\underline{Z}'(\underline{I} - \underline{Z}\underline{P}^{-1}\underline{Z}') = \underline{Z}(\underline{I} - \underline{Z}'\underline{Z}\underline{P}^{-1})\underline{Z}' = \underline{Z}[\underline{I} - (\underline{P} - \lambda_u\underline{I})\underline{P}^{-1}]\underline{Z}' = \lambda_u\underline{Z}\underline{P}^{-1}\underline{Z}'$$

so that

$$(\underline{Z}\underline{Z}' + \lambda_u\underline{I})\underline{T} = \lambda_u(\underline{Z}\underline{P}^{-1}\underline{Z}' + \underline{T}) = \lambda_u\underline{I} \quad \text{by (A10)} . \tag{A11}$$

2A. Sums of squares

On defining

$$\underset{\sim}{K} = \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} \\ \underset{\sim}{Z}'\underset{\sim}{X} & \underset{\sim}{P} \end{bmatrix}, \quad \underset{\sim}{R}' = \begin{bmatrix} \underset{\sim}{X}' \\ \underset{\sim}{Z}' \end{bmatrix} \text{ and } \underset{\sim}{L} = \underset{\sim}{R}'\underset{\sim}{W} = \begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{W} \\ \underset{\sim}{Z}'\underset{\sim}{W} \end{bmatrix} \tag{A12}$$

we have for (3)

$$\begin{bmatrix} \underset{\sim}{X}'\underset{\sim}{X} & \underset{\sim}{X}'\underset{\sim}{Z} & \underset{\sim}{X}'\underset{\sim}{W} \\ \underset{\sim}{Z}'\underset{\sim}{X} & \underset{\sim}{Z}'\underset{\sim}{Z} + \lambda_u \underset{\sim}{I} & \underset{\sim}{Z}'\underset{\sim}{W} \\ \underset{\sim}{W}'\underset{\sim}{X} & \underset{\sim}{W}'\underset{\sim}{Z} & \underset{\sim}{W}'\underset{\sim}{W} + \lambda_v \underset{\sim}{I} \end{bmatrix}^{-1} = \begin{bmatrix} \underset{\sim}{K} & \underset{\sim}{L} \\ \underset{\sim}{L}' & \underset{\sim}{W}'\underset{\sim}{W} + \lambda_v \underset{\sim}{I} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \underset{\sim}{K}^{-1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} + \begin{bmatrix} -\underset{\sim}{K}^{-1}\underset{\sim}{L} \\ \underset{\sim}{I} \end{bmatrix} \underset{\sim}{S}_v^{-1} (-\underset{\sim}{L}'\underset{\sim}{K}^{-1} \quad \underset{\sim}{I}) \tag{A13}$$

where

$$\underset{\sim}{S}_v = \underset{\sim}{W}'\underset{\sim}{W} + \lambda_v \underset{\sim}{I} - \underset{\sim}{L}'\underset{\sim}{K}^{-1}\underset{\sim}{L} = \underset{\sim}{W}'\underset{\sim}{W} + \lambda_v \underset{\sim}{I} - \underset{\sim}{W}'\underset{\sim}{R}\underset{\sim}{K}^{-1}\underset{\sim}{R}'\underset{\sim}{W} \; . \tag{A14}$$

Also, from (A12)

$$\underset{\sim}{K}^{-1} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{P}^{-1} \end{bmatrix} + \begin{bmatrix} \underset{\sim}{I} \\ -\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X} \end{bmatrix} \underset{\sim}{S}_u^{-1} (\underset{\sim}{I} \quad -\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}) \tag{A15}$$

where

$$\underset{\sim}{S}_u = \underset{\sim}{X}'\underset{\sim}{X} - \underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X} = \underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{X} \; . \tag{4a}$$

The inverses in (A13) and (A15) are standard results for the inverse of a partitioned matrix. To simplify (A14) note from (A12) and (A15) that

$$\underline{RK}^{-1}\underline{R}' = \underline{ZP}^{-1}\underline{Z}' + (\underline{I} - \underline{ZP}^{-1}\underline{Z}')\underline{XS}_u^{-1}\underline{X}'(\underline{I} - \underline{ZP}^{-1}\underline{Z}')$$

$$= \underline{I} - \underline{T} + \underline{TXS}_u^{-1}\underline{X}'\underline{T} \tag{A16}$$

so that (A14) becomes

$$\underline{S}_v = \underline{W}'(\underline{T} - \underline{TXS}_u^{-1}\underline{X}'\underline{T})\underline{W} + \lambda_v\underline{I} \ . \tag{4b}$$

Using (A12) and (A13) we then have (3) as

$$R\ (\underline{b},\underline{u},\underline{v}) = \underline{y}'(\underline{R}\ \ \underline{W})\left[\begin{pmatrix}\underline{K}^{-1} & \underline{0}\\ \underline{0} & \underline{0}\end{pmatrix} + \begin{pmatrix}-\underline{K}^{-1}\underline{0}\\ \underline{I}\end{pmatrix}\underline{S}_v^{-1}(-\underline{L}'\underline{K}^{-1}\ \ \underline{I})\right]\begin{pmatrix}\underline{R}'\\ \underline{W}'\end{pmatrix}\underline{y}$$

$$= \underline{y}'[\underline{RK}^{-1}\underline{R}' + (\underline{W} - \underline{RK}^{-1}\underline{L})\underline{S}_v^{-1}(\underline{W}' - \underline{L}'\underline{K}^{-1}\underline{R}')]\underline{y}$$

$$= \underline{y}'[\underline{RK}^{-1}\underline{R}' + (\underline{I} - \underline{RK}^{-1}\underline{R}')\underline{WS}_v^{-1}\underline{W}'(\underline{I} - \underline{RK}^{-1}\underline{R}')]\underline{y}$$

$$= \underline{y}'[\underline{ZP}^{-1}\underline{Z}' + \underline{TXS}_u^{-1}\underline{X}'\underline{T} + (\underline{T} - \underline{TXS}_u^{-1}\underline{X}'\underline{T})\underline{WS}_v^{-1}\underline{W}'(\underline{T} - \underline{TXS}_u^{-1}\underline{X}'\underline{T})]\underline{y}\ . \tag{5}$$

Also, using (A15), $R^*(\underline{b},\underline{u})$ of (7) is

$$R^*(\underline{b},\underline{u}) = \underline{y}'[\underline{ZP}^{-1}\underline{Z}' + (\underline{I} - \underline{ZP}^{-1}\underline{Z}')\underline{XS}_u^{-1}\underline{X}'(\underline{I} - \underline{ZP}^{-1}\underline{Z}')]\underline{y}$$

$$= \underline{y}'(\underline{ZP}^{-1}\underline{Z}' + \underline{TXS}_u^{-1}\underline{X}'\underline{T})\underline{y}. \tag{8}$$

3A. Expected values

We need to evaluate

$$E(\underline{y}'\underline{Q}\underline{y}) = \underline{b}'\underline{X}'\underline{Q}\underline{X}\underline{b} + \sigma_v^2 tr(\underline{W}'\underline{Q}\underline{W}) + \sigma_u^2 tr[(\underline{Z}\underline{Z}' + \lambda_u \underline{I})\underline{Q}] \tag{13}$$

for the values of $\underline{Q}$ implicit in (9), (10) and (12).

First note that, using (4a),

$$(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{X} = \underline{T}\underline{X} - \underline{T}\underline{X}(\underline{X}'\underline{T}\underline{X})^{-1}\underline{X}'\underline{T}\underline{X} = 0. \tag{A17}$$

For $R^*(\underline{v}|\underline{b},\underline{u}) = \underline{y}'\underline{Q}_1\underline{y}$ we see from (9) that

$$\underline{Q}_1 = (\underline{T} - \underline{T}\underline{X}\underline{S}^{-1}\underline{X}'\underline{T})\underline{W}\underline{S}_v^{-1}\underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T}),$$

so that in using $\underline{Q}_1$ for $\underline{Q}$ in (13) we have from (A17)

$$\underline{b}'\underline{X}'\underline{Q}_1\underline{X}\underline{b} = 0, \tag{A18}$$

and also

$$\begin{aligned}\sigma_v^2 tr(\underline{W}'\underline{Q}_1\underline{W}) &= \sigma_v^2 tr[\underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{W}\underline{S}_v^{-1}\underline{W}'(\underline{T} - \underline{T}\underline{X}\underline{S}_u^{-1}\underline{X}'\underline{T})\underline{W}] \\ &= \sigma_v^2 tr[\underline{S}_v - \lambda_v\underline{I})\underline{S}_v^{-1}(\underline{S}_v - \lambda_v\underline{I})] \text{ from (4b)} \\ &= \sigma_v^2 tr(\underline{S}_v - \lambda_v\underline{I}) - \sigma_e^2 tr(\underline{I} - \lambda_v\underline{S}_v^{-1}),\end{aligned} \tag{A19}$$

and

$$\sigma_u^2 \mathrm{tr}[(\underset{\sim}{Z}\underset{\sim}{Z}' + \lambda_u \underset{\sim}{I})\underset{\sim}{Q}_1] = \sigma_u^2\lambda_u \mathrm{tr}[(\underset{\sim}{I} - \underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{W}\underset{\sim}{S}_v^{-1}\underset{\sim}{W}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T}) \quad \text{from (A11)},$$

$$= \sigma_e^2 \mathrm{tr}[\underset{\sim}{S}_v^{-1}\underset{\sim}{W}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{W}] - \sigma_e^2 \mathrm{tr}[\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{W}\underset{\sim}{S}_v^{-1}\underset{\sim}{W}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{X}]$$

$$= \sigma_e^2 \mathrm{tr}[\underset{\sim}{S}_v^{-1}(\underset{\sim}{S}_v - \lambda_v \underset{\sim}{I})] - \sigma_e^2 \mathrm{tr}(\underset{\sim}{0}), \quad \text{from (4b) and (A17)}$$

$$= \sigma_e^2 \mathrm{tr}(\underset{\sim}{I} - \lambda_v \underset{\sim}{S}_v^{-1}). \tag{A20}$$

Adding (A18) through (A20) gives, for using $\underset{\sim}{Q}_1$ in (13),

$$\mathrm{E}\, R^*(\underset{\sim}{v}|\underset{\sim}{b},\underset{\sim}{u}) = \sigma_v^2 \mathrm{tr}(\underset{\sim}{S}_v - \lambda_v \underset{\sim}{I})$$

$$= \sigma_v^2[\mathrm{tr}(\underset{\sim}{S}_v) - s\lambda_v]. \tag{14}$$

For $R^*(\underset{\sim}{b},\underset{\sim}{u},\underset{\sim}{v}) = \underset{\sim}{y}'\underset{\sim}{Q}_2\underset{\sim}{y}$ we have from (5) and (9)

$$\underset{\sim}{Q}_2 = \underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}' + \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T} + \underset{\sim}{Q}_1$$

so that

$$\underset{\sim}{b}'\underset{\sim}{X}'\underset{\sim}{Q}_2\underset{\sim}{X}\underset{\sim}{b} = \underset{\sim}{b}'(\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X} + \underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{X} + \underset{\sim}{X}'\underset{\sim}{Q}_1\underset{\sim}{X})\underset{\sim}{b}$$

$$= \underset{\sim}{b}'(\underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X} + \underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{X})\underset{\sim}{b}, \quad \text{from (4a) and (A18)}$$

$$= \underset{\sim}{b}'\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b}, \quad \text{from (A9)}. \tag{A21}$$

$$\sigma_v^2 \mathrm{tr}(\underset{\sim}{W}'\underset{\sim}{Q}_2\underset{\sim}{W}) = \sigma_v^2 \mathrm{tr}[\underset{\sim}{W}'(\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}' + \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T})\underset{\sim}{W}] + \sigma_v^2 \mathrm{tr}(\underset{\sim}{W}'\underset{\sim}{Q}_1\underset{\sim}{W})$$

$$= \sigma_v^2 \mathrm{tr}[\underset{\sim}{W}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{W} + \underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{W} + \lambda_v \underset{\sim}{I} - \underset{\sim}{S}_v]$$

$$+ \sigma_v^2 \mathrm{tr}(\underset{\sim}{S}_v - \lambda_v \underset{\sim}{I}) - \sigma_e^2 \mathrm{tr}(\underset{\sim}{I} - \lambda_v \underset{\sim}{S}_v^{-1}), \quad \text{from (4b) and (A19)}$$

$$= \sigma_v^2 tr[\underline{W}'(\underline{ZP}^{-1}\underline{Z}' + \underline{T})\underline{W}] - \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1})$$

$$= \sigma_v^2 tr(\underline{W}'\underline{W}) - \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}), \text{ from (A9)}$$

$$= n_{..}\sigma_v^2 - \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}), \text{ from (A4).} \quad \text{(A22)}$$

And

$$\sigma_u^2 tr[\underline{ZZ}' + \lambda_u \underline{I})\underline{Q}_2$$

$$= \sigma_u^2 tr[(\underline{ZZ}' + \lambda_u \underline{I})(\underline{ZP}^{-1}\underline{Z}' + \underline{TXS}_u^{-1}\underline{X}'\underline{T} + \underline{Q}_1)]$$

$$= \sigma_u^2 tr[(\underline{ZZ}' + \lambda_u \underline{I})(\underline{I} - \underline{T} + \underline{TXS}_u^{-1}\underline{X}'\underline{T})] + \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}), \text{ from (A10)\&(A20)}$$

$$= \sigma_u^2 tr(\underline{ZZ}' + \lambda_u \underline{I} - \lambda_u \underline{I} + \lambda_u \underline{XS}_u^{-1}\underline{X}'\underline{T}) + \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}), \text{ from (A11)}$$

$$= \sigma_u^2 tr(\underline{Z}'\underline{Z}) + \lambda_u \sigma_u^2 tr(\underline{X}'\underline{TXS}_u^{-1}) + \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1})$$

$$= n_{..}\sigma_u^2 + \sigma_e^2 tr(\underline{I}_b) + \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}) \text{ from (4a)}$$

$$= n_{..}\sigma_u^2 + b\sigma_e^2 + \sigma_e^2 tr(\underline{I} - \lambda_v \underline{S}_v^{-1}). \quad \text{(A23)}$$

Substituting (A21) - (A23) in (13) gives

$$E\, R^*(\underline{b},\underline{u},\underline{v}) = \underline{b}'\underline{X}'\underline{Xb} + n_{..}\sigma_u^2 + n_{..}\sigma_v^2 + b\sigma_e^2\,. \quad \text{(A24)}$$

Writing $R^*(\underline{b}) = \underline{y}'\underline{Q}_3\underline{y}$ gives $\underline{Q}_3 = \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'$ for which

$$\underline{b}'\underline{X}'\underline{Q}_3\underline{Xb} = \underline{b}'\underline{X}'\underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{Xb} = \underline{b}'\underline{X}'\underline{Xb}, \quad \text{(A25)}$$

$$\sigma_v^2 tr(\underline{W}'\underline{Q}_3\underline{W}) = \sigma_v^2 tr[\underline{W}'\underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'\underline{W}] \quad \text{(26)}$$

and

$$\sigma_u^2 \mathrm{tr}[(\underset{\sim}{Z}\underset{\sim}{Z}' + \lambda_u \underset{\sim}{I})\underset{\sim}{Q}_3] = \sigma_u^2 \mathrm{tr}[(\underset{\sim}{Z}\underset{\sim}{Z}' + \lambda_u \underset{\sim}{I})\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}']$$

$$= \sigma_u^2 \mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Z}] + \lambda_u \sigma_u^2 \mathrm{tr}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{X}$$

$$= \sigma_u^2 \mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Z}] + b\sigma_e^2. \tag{A27}$$

Hence, on adding (A25) through (A27)

$$\mathrm{E}\, R^*(\underset{\sim}{b}) = \underset{\sim}{b}'\underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{b} + \sigma_u^2 \mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Z}] + \sigma_v^2 \mathrm{tr}[\underset{\sim}{W}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W}] + b\sigma_e^2$$

and so

$$\mathrm{E}\, R^*(\underset{\sim}{u},\underset{\sim}{v}|\underset{\sim}{b}) = \mathrm{E}\, R^*(\underset{\sim}{b},\underset{\sim}{u},\underset{\sim}{v}) - \mathrm{E}\, R^*(\underset{\sim}{b})$$

$$= \left\{n_{..} - \mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Z}]\right\}\sigma_u^2 + \left\{n_{..} - \mathrm{tr}[\underset{\sim}{W}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W}]\right\}\sigma_v^2 . \tag{A28}$$

The trace operations needed here are derived by first noting from (1) that, as in Searle [1973],

$$\mathrm{tr}[\underset{\sim}{Z}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{Z}] = k_4 = \sum_j \frac{1}{n_{.j}} \sum_i n_{ij}^2. \tag{A29}$$

Also, from (A4) and (A7)

$$\underset{\sim}{W}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W} = \left\{n_{ij}\delta_{jk}\right\} \underset{\sim}{D}\left\{1/n_{.j}\right\} \left\{n_{ij}\delta_{jk}\right\}'$$

$$= \left\{\frac{n_{ij}\delta_{jk}}{n_{.j}}\right\} \left\{n_{ij}\delta_{jk}\right\}'$$

so that

$$\mathrm{tr}[\underset{\sim}{W}'\underset{\sim}{X}(\underset{\sim}{X}'\underset{\sim}{X})^{-1}\underset{\sim}{X}'\underset{\sim}{W}] = \sum_i \sum_j \sum_k \frac{n_{ij}^2 \delta_{jk}^2}{n_{.j}} = \sum_i \sum_j \frac{n_{ij}^2}{n_j} = \sum_j \frac{1}{n_{.j}} \sum_i n_{ij}^2 = k_4.$$

Thus (A28) is

$$E\, R^*(\underline{u},\underline{v}|\underline{b}) = (n_{..} - k_4)(\sigma_u^2 + \sigma_v^2). \tag{15}$$

Finally, for $E(SSE^*)$ it is well known that

$$E(\underline{y}\underline{y}') = \underline{b}'\underline{X}'\underline{X}\underline{b} + n_{..}\sigma_u^2 + n_{..}\sigma_v^2 + n_{..}\sigma_e^2$$

so that on subtracting (A24)

$$E(SSE^*) = (n_{..} - b)\sigma_e^2 \tag{16}$$

5A. Fixed effects estimators

The solution to (2) for $\hat{\underline{b}}$ is

$$\hat{\underline{b}} = [\text{first row of (A13)}] \begin{bmatrix} \underline{X}' \\ \underline{Z}' \\ \underline{W}' \end{bmatrix} \underline{y} \tag{A30}$$

and in using (A15) in (A13) only the first row of (A15) is to be involved. Re-define $\underline{K}^{-1}$ of (A15) as

$$\underline{K}^{-1} = \begin{bmatrix} \underline{B}_{11} & \underline{B}_{12} \\ \underline{B}_{21} & \underline{B}_{22} \end{bmatrix} \text{ for } \underline{B}_{21} = \underline{B}_{12}' . \tag{A31}$$

Then from (A12)

$$\underset{\sim}{K}^{-1}\underset{\sim}{L} = \begin{bmatrix} (\underset{\sim}{B}_{11}\underset{\sim}{X}' + \underset{\sim}{B}_{12}\underset{\sim}{Z}')\underset{\sim}{W} \\ (\underset{\sim}{B}_{21}\underset{\sim}{X}' + \underset{\sim}{B}_{22}\underset{\sim}{Z}')\underset{\sim}{W} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{F}_1 \\ \underset{\sim}{F}_2 \end{bmatrix} \text{ say,} \tag{A32}$$

So that from (A15) and (A31)

$$\underset{\sim}{F}_1 = \underset{\sim}{S}_u^{-1}(\underset{\sim}{X}' - \underset{\sim}{X}'\underset{\sim}{X}\underset{\sim}{P}^{-1}\underset{\sim}{Z}')\underset{\sim}{W} = \underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T}\underset{\sim}{W} \tag{A33}$$

and

$$\underset{\sim}{F}_2 = \underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{W} - \underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X}\underset{\sim}{S}_u^{-1}(\underset{\sim}{X}' - \underset{\sim}{X}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}')\underset{\sim}{W} = \underset{\sim}{P}^{-1}\underset{\sim}{Z}'(\underset{\sim}{W} - \underset{\sim}{X}\underset{\sim}{F}_1) \ . \tag{A34}$$

With these values (A13) is

$$\begin{bmatrix} \underset{\sim}{B}_{11} & \underset{\sim}{B}_{12} & \underset{\sim}{0} \\ \underset{\sim}{B}_{21} & \underset{\sim}{B}_{22} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \end{bmatrix} + \begin{bmatrix} -\underset{\sim}{F}_1 \\ -\underset{\sim}{F}_2 \\ \underset{\sim}{I} \end{bmatrix} \underset{\sim}{S}_v^{-1}[\underset{\sim}{F}_1' \quad -\underset{\sim}{F}_2' \quad \underset{\sim}{I}]$$

so that (A30) becomes

$$\hat{\underset{\sim}{b}} = (\underset{\sim}{B}_{11}\underset{\sim}{X}' + \underset{\sim}{B}_{12}\underset{\sim}{Z}')\underset{\sim}{y} - \underset{\sim}{F}_1\underset{\sim}{S}_v^{-1}(-\underset{\sim}{F}_1'\underset{\sim}{X}' - \underset{\sim}{F}_2'\underset{\sim}{Z}' + \underset{\sim}{W}')\underset{\sim}{y} \ .$$

But

$$\begin{aligned} -\underset{\sim}{F}_1'\underset{\sim}{X}' - \underset{\sim}{F}_2'\underset{\sim}{Z}' + \underset{\sim}{W}' &= -\underset{\sim}{F}_1'\underset{\sim}{X}' - (\underset{\sim}{W}' - \underset{\sim}{F}_1'\underset{\sim}{X}')\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}' + \underset{\sim}{W}', \text{ from (A34)} \\ &= \underset{\sim}{W}'\underset{\sim}{T} - \underset{\sim}{F}_1'\underset{\sim}{X}'\underset{\sim}{T}, \text{ using (A10)} \\ &= \underset{\sim}{W}'(\underset{\sim}{T} - \underset{\sim}{T}\underset{\sim}{X}\underset{\sim}{S}_u^{-1}\underset{\sim}{X}'\underset{\sim}{T}), \text{ from (A33)} \ . \end{aligned}$$

Hence, using (A32) and (A33)

$$\hat{b} = S_u^{-1}X'Ty - S_u^{-1}X'TWS_v^{-1}W'(T - TXS_u^{-1}X'T)y . \tag{20}$$

6A. Computing formulae

Development

To simplify (21) and (22) we use the following results taken from Searle [1973].

$$Ty = \left\{y_{ijk} - \frac{y_{i..}}{n_{i.}+\lambda_u}\right\}_{s\times 1} \text{ for } i = 1,\ldots,a,\ j = 1,\ldots,b \text{ in lexicon order} \tag{A35}$$

$$X'Ty = r_* = \left\{y_{.j.} - \sum_i \frac{n_{ij}y_{i..}}{n_{i.}+\lambda_u}\right\} \text{ for } j = 1,\ldots,b \tag{A36}$$

and, on also using (4a),

$$S_u = X'TX = C_* = D\left\{n_{.j}\right\} - \left\{\sum_i \frac{n_{ij}n_{ij'}}{n_{i.}+\lambda_u}\right\}_{b\times b} \text{ for } j,j' = 1,\ldots,b. \tag{A37}$$

Thus (21) and (22) are

$$R^*(v|u,b) = (W'Ty - W'TXC_*^{-1}r_*)'S_v^{-1}(W'Ty - W'TXC_*^{-1}r_*) \tag{A38}$$

and

$$k_* = tr(W'TW - W'TXC_*^{-1}X'TW) \tag{A39}$$

with, from (4b),

$$S_v = W'TW - W'TXC_*^{-1}X'TW + \lambda_v I. \tag{A40}$$

To calculate (A38) - (A40) we need

$$\underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{y} = \left\{ y_{ij\cdot} - \frac{n_{ij}y_{i\cdot\cdot}}{n_{i\cdot}+\lambda_u} \right\}_{s\times 1} \text{ for } i = 1,\ldots,a \text{ and } j = 1,\ldots,b \text{ in lexicon order} \tag{A41}$$

from (A3) and (A35);

$$\underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{W} = \underset{\sim}{W}'\underset{\sim}{W} - \underset{\sim}{W}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{W} \quad \text{from (A10)}$$

$$= \underset{\sim}{D}\left\{n_{ij}\right\} - \left(\sum_i^+ \frac{1}{n_{i\cdot}+\lambda_u} \underset{\sim}{\nu}_i \right)\left(\sum_i^+ \underset{\sim}{\nu}_i' \right), \text{ from (A4), (A6) and (A9)}$$

$$= \underset{\sim}{D}\left\{n_{ij}\right\} - \sum_i^+ \left[\left\{ \frac{n_{ij}n_{ij'}}{n_{i\cdot}+\lambda_u} \right\} \text{ for } j,j' = 1,\ldots,b \right] \text{ and } n_{ij},\ n_{ij'} \neq 0; \tag{A42}$$

and finally

$$\underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{X} = \underset{\sim}{W}'\underset{\sim}{X} - \underset{\sim}{W}'\underset{\sim}{Z}\underset{\sim}{P}^{-1}\underset{\sim}{Z}'\underset{\sim}{X}$$

$$= \left\{ n_{ij}\delta_{jk} \right\}_{s\times b} - \left(\sum_{i=1}^{a}{}^{+} \underset{\sim}{\nu}_i \right)_{s\times a} \left\{ \frac{n_{ij}}{n_{i\cdot}+\lambda_u} \right\}_{a\times b} \tag{A43}$$

from (A5), (A6), (A7) and (A9) .

Define

$$\underset{\sim}{n}_i' = [n_{i1} \;\; n_{i2} \; \cdots \; n_{ib}] \text{ including zero } n_{ij}\text{'s}$$

$$= i^{th} \text{ row of } \underset{\sim}{Z}'\underset{\sim}{X} = \left\{ n_{ij} \right\} \text{ for } i = 1,\ldots,a \text{ and } j = 1,\ldots,b \text{ .}$$

$\underset{\sim}{\Delta}(\underset{\sim}{n}_i')$ = diagonal matrix of the elements of $\underset{\sim}{n}_i'$

$\underset{\sim}{\Delta}_1(\underset{\sim}{n}_i') = \underset{\sim}{\Delta}(\underset{\sim}{n}_i')$ omitting null rows (but not the null columns)

$(\underset{\sim}{M}_i)_{b\times b} = \frac{1}{n_{i\cdot}+\lambda_u}\underset{\sim}{n}_i\underset{\sim}{n}_i' = \frac{1}{n_{i\cdot}+\lambda_u}\left\{n_{ij}n_{ij'}\right\}$ for $j,j' = 1,\ldots,b$

$\underset{\sim}{M}_{1,i} = \underset{\sim}{M}_i$ omitting the null rows (but not the null columns)

$\underset{\sim}{M}_{2,i} = \underset{\sim}{M}_i$ " " " " and the null columns.

Then in (A42) and (A43)

$$(\underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{W})_{s\times s} = \underset{\sim}{D}\left\{n_{ij}\right\} - \sum_i^+ \underset{\sim}{M}_{2,i} \tag{A44}$$

$$(\underset{\sim}{W}'\underset{\sim}{T}\underset{\sim}{X})_{s\times b} = \left\{\Delta_1(\underset{\sim}{n}_i') - \underset{\sim}{M}_{1,i}\right\} \text{ for } i = 1,\ldots,a \tag{A45}$$

$$= \begin{bmatrix} \underset{\sim}{\Delta}_1(\underset{\sim}{n}_1') - M_{1,1} \\ \underset{\sim}{\Delta}_1(\underset{\sim}{n}_2') - M_{1,2} \\ \underset{\sim}{\Delta}_1(\underset{\sim}{n}_3') - M_{1,3} \\ \vdots \qquad \vdots \\ \underset{\sim}{\Delta}_1(\underset{\sim}{n}_a) - M_{1,a} \end{bmatrix} .$$