TAIL PROBABILITIES OF SUBADDITIVE FUNCTIONALS ACTING ON LÉVY PROCESSES

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ABSTRACT. We study the tail behavior of the distribution of certain subadditive functionals acting on the sample paths of Lévy processes. The functionals we consider have, roughly speaking, the following property: only the points of the process that lie above a certain curve contribute to the value of the functional. Our assumptions will make sure that the process ends up eventually below the curve. Our results apply to ruin probabilities, distributions of sojourn times over curves, last hitting times and other functionals.

1. INTRODUCTION

Both in the theory and in applications of stochastic processes one is often interested in two types of questions: *When does the process* $\mathbf{X} = \{X(t),\, t \geq 0\}$ *lie above a certain deterministic function (curve)* $\boldsymbol{\mu} = \{\mu(t),\, t \geq 0\}$*, and given the process exceeds this curve, what are its values?* For example, what can be said about the distribution of the biggest excess of the process over the curve and, if both the process and the function are measurable, what is the distribution of the time the process spends above the curve?

In this paper, we outline a general approach to the asymptotic tail behavior of the distributions of these and other subadditive functionals acting on an infinitely divisible process with "not too light" tails. (The latter notion will be made precise soon.) We focus on a particular class of infinitely divisible processes, the well known Lévy processes, and we consider the distributional tails of various subadditive functionals of their paths. These examples will show in detail how successfully this method works and how general it is.

Let $\mathbf{X}$ be an infinitely divisible process without Gaussian component and Lévy measure ν. Following Maruyama (1970), the distribution of $\mathbf{X}$ is characterized as follows:

$$Ee^{i\langle \boldsymbol{\beta}, \mathbf{X}\rangle} = \exp\left\{ \int_{\mathbb{R}^{[0,\infty)}} \left(e^{i\langle \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle} - 1 - i\langle \boldsymbol{\beta}, \boldsymbol{\tau}(\boldsymbol{\alpha})\rangle \right) \nu(d\boldsymbol{\alpha}) \right\}, \quad \boldsymbol{\beta} \in \mathbb{R}^{([0,\infty))}.$$

Here ν is the projective limit of the Lévy measures corresponding to the finite dimensional distributions of $\mathbf{X}$. The symbol $\mathbb{R}^{([0,\infty))}$ denotes the space of real functions $\boldsymbol{\beta}$ defined on $[0,\infty)$ such that $\beta(t) = 0$ for all but finitely many t, and $\langle \boldsymbol{\beta}, \boldsymbol{\alpha}\rangle = \Sigma_{t\in[0,\infty)}\beta(t)\alpha(t)$. Finally, $\tau(\boldsymbol{\alpha})(t) = \alpha(t)\mathbf{1}(|\alpha(t)| \leq 1)$.

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Some examples of the measurable functionals $\phi : \mathbb{R}^{[0,\infty)} \to (-\infty,\infty]$ on $\mathbf{X}$ we consider are

$$(1.1)\quad \phi_{\sup}(\boldsymbol{\alpha}) = \sup_{t\geq 0}\,[\alpha(t)]_+\,, \quad \phi(\boldsymbol{\alpha}) = \sup\{t>0:\,\alpha(t)>0\}\,, \quad \phi(\boldsymbol{\alpha}) = \int_0^\infty [\alpha(s)]_+^p\,ds\,,$$

where $y_+ = \max(0,y)$ and $p \in (0,1]$. The supremum functional $\phi_{\sup}$ has gained particular importance in the context of queuing and insurance, where one is interested in quantitative measures for the excesses of $\mathbf{X}$ over high level thresholds which event is interpreted as buffer overflow or ruin in the different contexts. The above functionals have in common that they are *subadditive*, i.e., for any $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2 \in \mathbb{R}^{[0,\infty)}$,

$$\phi(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2) \leq \phi(\boldsymbol{\alpha}_1)+\phi(\boldsymbol{\alpha}_2)\,.$$

If, with probability 1, $\phi(\mathbf{X}-\boldsymbol{\mu}) < \infty$ is finite, it makes sense to measure the thickness of the distributional tail $P(\phi(\mathbf{X}-\boldsymbol{\mu}) > u)$ for large u. Suppose this tail does not decay “too fast” as $u\to\infty$ and define

$$(1.2)\qquad \psi(u) = \nu(\{\boldsymbol{\alpha}:\ \phi(\boldsymbol{\alpha}-\boldsymbol{\mu}) > u\})\,.$$

The subadditivity of the functional ϕ, the presence of heavy tails and the logic of large deviations saying that unlikely events happen in the most likely way, lead one to the conjecture that $\psi(u)$ and $P(\phi(\mathbf{X}-\boldsymbol{\mu}) > u)$ are *equivalent* in the following sense:

$$(1.3)\qquad \lim_{u\to\infty}\frac{P(\phi(\mathbf{X}-\boldsymbol{\mu})>u)}{\psi(u)} = 1\,.$$

Indeed, relations of type (1.3) were proved in the theory of laws with so-called subexponential tails. For example, Embrechts et al. (1979) considered the overall supremum of Lévy processes, and Rosiński and Samorodnitsky (1993) studied very general subadditive functionals.

The setup in the latter paper is, in fact, close to the present one. However, there is one crucial difference: the functionals in Rosiński and Samorodnitsky (1993) were assumed to be bounded by an almost surely finite pseudonorm of the process. Hence these processes are, in a certain sense, bounded “from above and below”. This assumption is far away from the situation in the present paper. Our functionals are akin to the supremum of a negative drift random walk over the entire infinite horizon. In this sense, they are bounded “only from one side”.

The validity of relation (1.3) has been established for the overall supremum functional $\phi_{\sup}$ and some particular classes of processes with subexponential tails. Those include Lévy processes with a negative linear drift (see Embrechts and Veraverbeke (1982)) and symmetric α-stable processes, $\alpha \in (1,2)$, with stationary ergodic increments and negative linear drift. In general, the precise circumstances under which (1.3) is valid for subadditive functionals are not known, even in the particular case of Lévy processes. The results of the present paper provide a further step in the process of understanding the tail equivalence relation (1.3) for heavy tailed processes. Once again, we will focus on Lévy processes and a large family of subadditive functionals ϕ and deterministic functions $\boldsymbol{\mu}$.

The proof of our main result (Theorem 3.1) shows that we use the “heavy tail large deviations heuristics”. This means that large values of the functional $\phi(\mathbf{X}-\boldsymbol{\mu})$ are essentially due to one very large jump of the Lévy process that occurs early enough, before the negative drift took it ”too far down”. Since Lévy processes are well described by Poisson processes, we extensively make use of the latter tool. In particular, we show that the large deviation idea can be made precise by considering the ”large and occurring early enough” jumps and the ”small or occurring too late” jumps of the underlying Poisson process separately. This leads one to a decomposition of the Lévy process into two independent processes. We show that the process which represents the small jumps is asymptotically negligible, i.e., this process will not contribute to the asymptotic tail behavior of $\phi(\mathbf{X}-\boldsymbol{\mu})$. The crucial part in this decomposition is the process which represents the large jumps of the Lévy process. It has representation as a compound Poisson sum of paths. We show that the tail behavior of $\phi(\mathbf{X}-\boldsymbol{\mu})$ is essentially determined by a single term in that sum.

In related work Hüsler and Piterbarg (1999) considered the tail behavior of the supremum functional $\phi_{\sup}$ of certain Gaussian processes, including fractional Brownian motion, with negative (not necessarily linear) drift. The Gaussian nature of the underlying process causes exponential decay of the tails $P(\phi_{\sup}(\mathbf{X}-\boldsymbol{\mu})>u)$.

This paper is organized as follows. In Section 2 we give the assumptions we impose on the family of the subadditive functionals ϕ, the function (curve) $\boldsymbol{\mu}$ and the distribution of the Lévy processes $\mathbf{X}$. We conclude Section 2 with a discussion on the nature of the assumptions, including some immediate consequences, and we consider situations when they are satisfied. We made these assumptions as general as possible in order to include as large a variety of subadditive functionals as possible. Although some of the conditions may look quite abstract they are easily checked for various well known subadditive functionals, see Section 5. In Section 3 the main theorem on the asymptotic behavior of the tails $P(\phi(\mathbf{X}-\boldsymbol{\mu})>u)$ is formulated. It describes one situation when relation (1.3) is valid. The main steps of the proof are also given in Section 3. However, the proof is quite technical and therefore we postpone various calculations until Section 4. In Section 5 we provide several examples of explicit calculations of the tail asymptotics for a number of important subadditive functionals of Lévy processes. Among those are the functionals in (1.1), but also, for example, the sojourn time of a Lévy process above a curve.

2. Assumptions and notation

Throughout this paper, C stands for a generic positive constant C. Its value will be allowed to change from appearance to appearance, even if we do not mention it explicitly.

Let $\mathbf{X}=\{X(t),\, t\geq 0\}$ be a Lévy process, i.e., a real-valued process with stationary and independent increments, and Lévy measure ρ on $\mathbb{R}$. We refer the reader to Bertoin (1996) and Sato (1999) for encyclopedic treatments of Lévy processes. In particular, one can find detailed proofs of the properties we mention and use below.

Specifically, the marginal distributions of a Lévy process are determined by the characteristic function

$$Ee^{i\theta X(1)}=\exp\left\{\int_{-\infty}^{\infty}\left(e^{i\theta x}-1-i\theta x\mathbf{1}(|x|\leq 1)\right)\rho(dx)\right\},\quad \theta\in\mathbb{R}. \tag{2.1}$$

We always take a version of $\mathbf{X}$ with all sample paths in the Skorokhod space $\mathbb{D}[0,\infty)$, i.e., with paths which are right-continuous at every $t\geq 0$ and have left limits at every $t>0$. This version of $\mathbf{X}$ is automatically measurable; this feature will become useful as we will have many opportunities to integrate the sample paths of $\mathbf{X}$.

The Lévy measure ν of the process $\mathbf{X}$ has the form

$$\nu(A)=\int_0^{\infty}\int_{-\infty}^{\infty}\mathbf{1}\left(x\mathbf{1}_{[s,\infty)}\in A\right)\,\rho(dx)\,ds\,, \tag{2.2}$$

for any measurable set $A \subset \mathbb{R}^{[0,\infty)}$. Therefore the function ψ in (1.2) turns into

$$\psi(u) = \int_0^\infty \int_{-\infty}^\infty \mathbf{1}\left(\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > u\right)\ \rho(dx)\ ds, \quad u > 0\,. \tag{2.3}$$

We denote the right tail of the one dimensional Lévy measure ρ by

$$H(u) = \rho\big([u,\infty)\big)\,, \quad u > 0\,.$$

A few comments on the conditions below. The reader should realize that the number of conditions we had to assume is due to our desire to cover the largest possible number of functionals and processes. The conditions simplify drastically in the special cases of Section 5.

Assumptions on the Lévy measure ρ

Dominance of the right tail of the Lévy measure

We assume that the right tail of the one dimensional Lévy measure ρ dominates its left tail in the sense that there is a constant $A_1 > 0$ such that

$$\rho((-\infty\,, -t]) \le A_1\, \rho([t,\infty)) \ \text{ for all } t \ge 1\,. \tag{2.4}$$

Δ_2 condition

There is $a_1 > 0$ such that

$$H(2u) \ge a_1 H(u) \ \text{ for all } u \ge 1\,. \tag{2.5}$$

Notice that the Δ_2 condition on H yield a bound from below; it excludes exponential decay of $H(u)$.

Bound from above

There is $\beta_1 > 0$ such that

$$H(u) = o(u^{-\beta_1})\,, \quad u \to \infty\,. \tag{2.6}$$

Assumptions on the drift $\boldsymbol{\mu}$

Let $\boldsymbol{\mu} = \{\mu(t),\, t \ge 0\}$ be a nonnegative function satisfying the following assumptions.

Power law bound from below

There are $a_2 > 0$ and $\beta_2 > \max(\beta_1^{-1}, 0.5)$ such that

$$\mu(t) \ge a_2\, t^{\beta_2}\,, \quad t > 0\,. \tag{2.7}$$

Δ_2 condition

There is an $A_2 > 0$ and $t_0 \ge 0$ such that

$$\mu(2t) \le \ A_2\ \mu(t) \quad \text{for all } t \ge t_0. \tag{2.8}$$

The Δ_2 condition on $\boldsymbol{\mu}$ excludes too fast (in particular exponential) growth of $\boldsymbol{\mu}$.

Quasi–monotonicity of $\boldsymbol{\mu}$

There is an $a_3 \in (0,1]$ and $t_0 \ge 0$ such that

$$\inf_{s \ge t} \mu(s) \ge a_3\ \mu(t) \quad \text{for all } t \ge t_0. \tag{2.9}$$

Assumptions on the subadditive functional ϕ

Let $\phi : \mathbb{R}^{[0,\infty)} \to [0,\infty]$ be a measurable subadditive functional satisfying the following conditions.

The functional "lives off only positive values of its argument"

This means that

$$\phi(\mathbf{0}) = 0, \quad \text{and if} \quad \alpha(t) \leq 0 \quad \text{for all } t > t_0, \text{ some } t_0, \text{ then} \quad \phi(\boldsymbol{\alpha}) = \phi\left(\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]}\right). \tag{2.10}$$

Here $\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]} = \{\alpha(t)\mathbf{1}_{[0,t_0]}(t),\, t \geq 0\}$.

The functional is finite on locally bounded functions that are eventually non-positive

This means that

$$\phi(\boldsymbol{\alpha}) = \phi\left(\boldsymbol{\alpha}\mathbf{1}_{[0,t_0]}\right) < \infty \quad \text{if } \alpha(t) \leq 0 \text{ for all } t > t_0, \text{ some } t_0, \text{ and } \sup_{t \leq t_0} \alpha(t) < \infty. \tag{2.11}$$

Monotonicity

This means that

$$\text{if} \quad \alpha(t) \leq \beta(t) \quad \text{for all } t \quad \text{then} \quad \phi(\boldsymbol{\alpha}) \leq \phi(\boldsymbol{\beta}) \tag{2.12}$$

and

$$\phi\left(c\boldsymbol{\alpha}\right) \leq \phi(\boldsymbol{\alpha}) \quad \text{for all } c \in [0,1] \text{ and } \boldsymbol{\alpha} \in \mathbb{R}^{[0,\infty)}. \tag{2.13}$$

Notice that (2.13) is implied by (2.12) if $\alpha(t) \geq 0$ for all $t \geq 0$.

Assumptions involving the triple $(\rho, \phi, \boldsymbol{\mu})$

One can easily give separate sufficient conditions for the assumptions below, i.e., conditions which involve ρ, ϕ and $\boldsymbol{\mu}$ separately. However, when doing so one gets into more restrictive situations. The assumptions we impose are easy to check in applications. Therefore we have chosen to formulate them in the present form.

For $s \geq 0$ and $u > 0$ define

$$T(s,u) = \inf\{x > 0 : \ \phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > u\}, \tag{2.14}$$

and denote

$$T(u) = T(0,u).$$

Relation between $T(s,u)$ and $T(u)$

There is $A_3 > 0$ such that

$$T(s,u) \leq A_3\left[\mu(s) + T(u)\right] \ \text{ for all } s,u > 0. \tag{2.15}$$

A scaling property

There are positive functions $g(\delta)$ and $h(\delta)$, $0 < \delta \leq 1$, satisfying

$$h(\delta) \to 1 \ \text{ as } \delta \uparrow 1, \quad |\log(g(\delta))| \leq O(\delta^{-1}) \ \text{ as } \delta \downarrow 0 \tag{2.16}$$

and such that for every $u > u(\delta)$ and $0 < \delta \leq 1$

$$\int_0^\infty H\left(\delta T(s,\delta u)\right)\, ds \leq h(\delta) \int_0^\infty H\left(T(s,u)\right)\, ds, \tag{2.17}$$

and for every $u \geq u_0$ and $0 < \delta \leq 1$

$$\int_0^\infty H\left(\delta T(s,\delta u)\right)\, ds \leq g(\delta) \int_0^\infty H\left(T(s,u)\right)\, ds\,. \tag{2.18}$$

The latter conditions are easily checked if one assumes appropriate regular variation conditions.

Some implications of the assumptions

We collect some particular consequences of the above assumptions.

Lemma 2.1. *The following statements hold.*

1. *With probability* 1, *for every* $\gamma > 0$, $\phi(|\mathbf{X}| - \gamma\mu) < \infty$ *and therefore* $\phi(\mathbf{X} - \gamma\mu) < \infty$.
2. *For every* $\epsilon > 0$ *and* $u > 0, \gamma > 0$

$$\begin{aligned} \int_0^\infty H\left(\frac{1+\epsilon}{\gamma} T(s,u)\right) ds &\leq \int_0^\infty \int_0^\infty \mathbf{1}\left(\phi\left(\gamma x \mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > u\right) \rho(dx) ds \\ = \psi(u) &\leq \int_0^\infty H\left(\frac{1}{\gamma} T(s,u)\right) ds\,. \end{aligned} \tag{2.19}$$

3. *There is a* $u_1 \geq 0$ *such that for every* $\epsilon > 0$ *and* $u > u_1$

$$\int_0^\infty H(\epsilon T(s,u))\, ds < \infty\,. \tag{2.20}$$

4. *For every* $\gamma > 0$,

$$\phi\left(\gamma x \mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) < \infty \quad \textit{outside a set of measure zero with respect to } \rho \times \mathrm{Leb}\,. \tag{2.21}$$

5. *There exists* $\beta_3 > 0$ *such that, for every* $u \geq 2$,

$$H(u) \geq u^{-\beta_3} \quad \textit{and} \quad \mu(u) \leq u^{\beta_3}\,. \tag{2.22}$$

6. *There is a constant* $A_4 > 0$ *such that for all* $s, t \geq 0$,

$$\mu(s+t) \leq A_4\left[\mu(s) + \mu(t)\right]. \tag{2.23}$$

Proof. 1) Observe first that $\mathbf{X}$ does not have a drift term, by virtue of condition (2.1). Thus, if $\beta_2 > \max(\beta^{-1}, 0.5)$, we conclude from (2.7), (2.6), and standard a.s. limit results (law of the iterated logarithm when $E[X(1)]^2 < \infty$, generalized strong laws of large numbers when $E[X(1)]^2 = \infty$; see Stout (1974)) that $X(t)/t^{\beta_2} \to 0$ a.s. as $t \to \infty$. Therefore

$$|X(t)| - \epsilon\mu(t) \leq |X(t)| - Ct^{\beta_2} \to -\infty\,.$$

So we may conclude from (2.11) that $\phi(|\mathbf{X}| - \epsilon\boldsymbol{\mu}) < \infty$ a.s. (with the usual convention of taking pointwise absolute values of a function).

2) This statement follows from the definition (2.14) of the function T.

3) It follows from (2.10), (2.9), and (2.12) that $T(s,u) \geq \max(a_3\mu(s), T(u))$ for all $s \geq t_0$ and that $T(u) > 0$ for every sufficiently large u (say, $u > u_1$). Hence

$$\int_0^\infty H(\epsilon T(s,u))\, ds \leq (t_0+1) H(\epsilon T(u)) + \int_{t_0+1}^\infty H(a_3 \epsilon \mu(s))\, ds\,.$$

The right hand integral is finite by virtue of (2.6) and (2.7).

4) Relation (2.21) follows from (2.17).

5) The inequalities (2.22) follow from the Δ_2-conditions on H and μ; cf. Bingham et al. (1987).
6) Relation (2.23) is an immediate consequence of (2.8) and (2.9). □

How to verify condition (2.15)?

Here is an easily verifiable sufficient condition for (2.15).

Proposition 2.2. *Assume that the following conditions hold:*

1. *The subadditive functional ϕ satisfies* (2.10) – (2.13)*.*
2. *There exists $\gamma > 0$ such that for all $0 < c < 1$*

$$\phi\left(cx\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \le c^{\gamma}\phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right), \tag{2.24}$$

3. *There exists $a > 0$ such that for every $s, x > 0$*

$$\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s\right) \ge \phi\left(ax\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right), \tag{2.25}$$

   where $\mu_s(t) = \mu\left((t-s)_+\right)$.
4. *$\boldsymbol{\mu}$ is nondecreasing.*

Then (2.15) *holds.*

In fact, condition

$$\phi(c\boldsymbol{\alpha}) \le c^{\gamma}\phi(\boldsymbol{\alpha}) \text{ for every } 0 < c < 1, \tag{2.26}$$

implies, and is more restrictive, than (2.13) and (2.24). Indeed, if (2.26) holds, monotonicity of ϕ implies

$$\phi\left(cx\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) = \phi\left(c\left(x\mathbf{1}_{[0,\infty)} - c^{-1}\boldsymbol{\mu}\right)\right) \le c^{\gamma}\phi\left(x\mathbf{1}_{[0,\infty)} - c^{-1}\boldsymbol{\mu}\right) \le c^{\gamma}\phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right).$$

Moreover, many of the functionals of interest have the property

$$\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s\right) = \phi\left(x\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right). \tag{2.27}$$

which implies (2.25).

The proof of the proposition is based on the following property of the function $T(u)$.

Lemma 2.3. *There is a constant $B > 0$ such that for all $u, v > 0$,*

$$T(u+v) \le B\left[T(u) + T(v)\right]. \tag{2.28}$$

Proof. By monotonicity of ϕ, for every $\epsilon > 0$,

$$\phi\left(\left[T(u) + T(v) + \epsilon\right]\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \ge \max(u, v) \ge \frac{u+v}{2},$$

which implies that

$$T\left(\frac{u+v}{2}\right) \le T(u) + T(v). \tag{2.29}$$

Let $u > 0$ and suppose that $T(2u) > 0$. By the scaling property (2.24) we have for every $\epsilon > 0$

$$u > \frac{1}{2}\phi\left((1-\epsilon)T(2u)\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \ge \phi\left(2^{-1/\gamma}(1-\epsilon)T(2u)\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right),$$

which means that

$$2^{-1/\gamma}\, T(2u) \le T(u).$$

On the other hand, if $T(2u) = 0$ then this relation is trivial. The above relation, together with (2.29), yields the desired relation (2.28). □

Proof of Proposition 2.2. It follows from (2.23) (in which we assume, without loss of generality, that $A_4 \geq 1$), $\mu(t) \leq A_4\,[\mu(t-s)+\mu(s)]$ for $s<t$, and so

$$\boldsymbol{\mu}\mathbf{1}_{[s,\infty)} \leq A_4\,[\boldsymbol{\mu}_s + \mu(s)\mathbf{1}_{[s,\infty)}]\,,$$

implying by monotonicity of ϕ that

$$r(x,s) := \phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\mathbf{1}_{[s,\infty)}\right) \geq \phi\left(A_4\,[(x/A_4-\mu(s))\,\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s]\right)\,.$$

Now (2.13) and (2.25) yield,

$$\begin{aligned} r(x,s) &\geq \phi\left((x/A_4-\mu(s))\;\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}_s\right) \\ &\geq \phi\left(a\,[x/A_4-\mu(s)]\,\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right)\,. \end{aligned} \tag{2.30}$$

Now it follows from the subadditivity that

$$\phi\left(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) \geq \phi\left(a\,[x/A_4-\mu(s)]\,\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) - \phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\,.$$

Let $\epsilon>0$ and choose $x := (1+\epsilon)A_4[\mu(s)+a^{-1}T(u)]$. Then by (2.30),

$$\phi\left((1+\epsilon)A_4\left[\mu(s)+a^{-1}T(u)\right]\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) \geq u - \phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\,,$$

which implies that

$$T\left(s, u-\phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\right) \leq A_4\left[\mu(s)+a^{-1}T(u)\right]\,,$$

and, after a change of variable and with (2.28),

$$\begin{aligned} T(s,u) &\leq A_4\,[\mu(s)+a^{-1}\,T\left(u+\phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\right)] \\ &\leq A_4\,[\mu(s)+a^{-1}\,B\,T(u)+a^{-1}\,B\,T\left(\phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\right)] \\ &=: A_4\,[\mu(s)+a^{-1}\,B\,T(u)]+a^{-1}A_4\,B\,T\left(g(s)\right)\,. \end{aligned} \tag{2.31}$$

It remains to estimate the last term in (2.31). If it is nonzero, according to (2.14) we have

$$\phi\left(\frac{1}{2}T(g(s))\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \leq g(s). \tag{2.32}$$

Assume that

$$T(g(s)) > 4\mu(s)\,.$$

Using (2.24), (2.10) and monotonicity of ϕ we obtain

$$\begin{aligned} \phi\left(\frac{1}{2}T(g(s))\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) &\geq \phi\left(2\mu(s)\mathbf{1}_{[0,\infty)} - \boldsymbol{\mu}\right) \\ &\geq \phi\left(2\mu(s)\mathbf{1}_{[0,s)} - \boldsymbol{\mu}\mathbf{1}_{[0,s)}\right)\,. \end{aligned} \tag{2.33}$$

But $2\mu(s)-\mu(t) \geq \mu(s) \geq \mu(t)$ for $0 \leq t < s$, and so (2.33) is at least

$$\phi\left(\boldsymbol{\mu}\mathbf{1}_{[0,s)}\right) = g(s)\,.$$

However, this contradicts (2.32).

Hence $T(g(s)) \leq 4\mu(s)$ for all $s \geq 0$, which together with (2.31) gives us (2.15). □

3. The main theorem

Here we give our main result which was announced in Section 1 and the main steps of its proof. The latter is quite technical, and therefore we collect some auxiliary results in Section 4.

First recall the definition of the quantity $\psi(u)$ from (2.3).

Theorem 3.1. *Let $\mathbf{X}$ be a Lévy process, $\boldsymbol{\mu}$ a deterministic function and ϕ a subadditive measurable functional satisfying the Assumptions of Section 2. If $\psi(u)$ is regularly varying (at infinity) with exponent $-\alpha < 0$, then $\psi(u)$ and $P(\phi(\mathbf{X}-\boldsymbol{\mu}) > u)$ are equivalent:*

$$\lim_{u\to\infty} \frac{P(\phi(\mathbf{X}-\boldsymbol{\mu}) > u)}{\psi(u)} = 1\,. \tag{3.1}$$

At this point, the large variety of assumptions on $\mathbf{X}$, ϕ, $\boldsymbol{\mu}$ and ψ may look quite restrictive and difficult to verify. We will, however, see in Section 5 that these assumptions hold under very natural conditions for various important subadditive functionals.

Proof of Theorem 3.1

The basic decomposition. For fixed $0 < \tau < 1$ and some $\delta_0 > 0$ we introduce the set

$$B_\tau = \left\{ x\mathbf{1}_{[s,\infty)} : \phi\left(|x|\mathbf{1}_{[s,\infty)} - \tau\boldsymbol{\mu}\right) > \delta_0\,, s \ge 0\,, x \in \mathbb{R} \right\} \subset \mathbb{R}^{[0,\infty)}\,. \tag{3.2}$$

Lemma 3.2. *If $\delta_0 > \max(u_1, 1)$ (see Lemma 2.1) then the set B_τ has finite Lévy measure: $\nu(B_\tau) < \infty$.*

Proof. By definition of the Lévy measure ν (see (2.2)) and since (2.4), (2.20) and (2.13) hold, we have

$$\begin{aligned} \nu(B_\tau) &= \int_0^\infty \int_{-\infty}^\infty \mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in B_\tau\right)\, \rho(dx)\, ds \\ &\le C \int_0^\infty \int_0^\infty \mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in B_\tau\right)\, \rho(dx)\, ds \\ &\le C \int_0^\infty H\left(\frac{\tau}{2} T(s,\delta_0)\right)\, ds\,. \end{aligned}$$

The right hand expression is finite by the choice of δ_0. □

From now on δ_0 is chosen to satisfy the assumptions of Lemma 3.2. Since B_τ and B_τ^c are disjoint there exist two independent infinitely divisible processes $\mathbf{X}_1^{(0)}$ and $\mathbf{X}_2^{(0)}$ such that

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}_1^{(0)} + \mathbf{X}_2^{(0)}$$

with Lévy measures ν_1 and ν_2, respectively, given by

$$\nu_1(A) = \nu(A \cap B_\tau) \quad \text{and} \quad \nu_2(A) = \nu(A \cap B_\tau^c)$$

for any measurable $A \subset \mathbb{R}^{[0,\infty)}$. By virtue of Lemma 3.2, $\nu(B_\tau) < \infty$, and therefore the process $\mathbf{X}_1^{(0)}$ has a representation as compound Poisson sum

$$\mathbf{X}_1^{(0)} \stackrel{\mathrm{d}}{=} \sum_{j=1}^N \mathbf{Y}_j - \boldsymbol{\eta} =: \mathbf{X}_1 - \boldsymbol{\eta}\,, \tag{3.3}$$

where $\mathbf{Y}_1, \mathbf{Y}_2, \ldots$ are iid stochastic processes on $[0,\infty)$ with common law $\nu_1/\nu(B_\tau)$, independent of a Poisson random variable N with mean $\nu(B_\tau)$. Because of (2.1), the drift term $\boldsymbol{\eta}$ has form

$$\eta(t) = t\int_0^\infty \int_{-\infty}^\infty x\mathbf{1}\left(|x|\le 1,\, x\mathbf{1}_{[s,\infty)} \in B_\tau\right)\,\rho(dx)\,ds, \quad t\ge 0\,.$$

Writing $\mathbf{X}_2 = \mathbf{X}_2^{(0)} - \boldsymbol{\eta}$, we have

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{X}_1^{(0)} + \mathbf{X}_2^{(0)} = \mathbf{X}_1 + \mathbf{X}_2\,. \tag{3.4}$$

The following fact will be useful in what follows.

Lemma 3.3. *For every $\gamma > 0$, with probability 1,*

$$\phi(|\mathbf{X}_i| - \gamma\boldsymbol{\mu}) < \infty\,, \quad i = 1,2\,.$$

Proof. Since $P(\mathbf{X}_1 = 0) > 0$ and $\mathbf{X}_1$, $\mathbf{X}_2$ are independent, it follows from part 1 of Lemma 2.1 that $\phi(|\mathbf{X}_2| - \gamma\boldsymbol{\mu}) < \infty$ a.s. for every $\gamma > 0$. In turn, exploiting the subadditivity of ϕ, we conclude that $\phi(|\mathbf{X}_1| - \gamma\boldsymbol{\mu}) < \infty$ a.s. for every $\gamma > 0$. ☐

The upper bound. By (3.4) and subadditivity of ϕ, for every $\epsilon \in (0,1)$,

$$\begin{aligned} P\Big(\phi(\mathbf{X}-\boldsymbol{\mu}) > u\Big) &= P\big(\phi(\mathbf{X}_1+\mathbf{X}_2-\boldsymbol{\mu}) > u\big) \\ &\le P\Big(\phi(\mathbf{X}_1 - (1-\epsilon)\boldsymbol{\mu}) > (1-\epsilon)u\Big) + P\Big(\phi(\mathbf{X}_2 - \epsilon\boldsymbol{\mu}) > \epsilon u\Big) \\ &=: I_1(u) + I_2(u)\,. \end{aligned} \tag{3.5}$$

Lemma 3.4. *Under the assumptions of the theorem for every τ small enough,*

$$\lim_{\epsilon\to 0}\limsup_{u\to\infty} I_1(u)/\psi(u) \le 1\,, \tag{3.6}$$

$$\lim_{\epsilon\to 0}\limsup_{u\to\infty} I_2(u)/\psi(u) = 0\,. \tag{3.7}$$

From Lemma 3.4 and (3.5) we conclude that

$$\limsup_{u\to\infty} \frac{P\Big(\phi(\mathbf{X}-\boldsymbol{\mu}) > u\Big)}{\psi(u)} \le 1\,.$$

This concludes the proof of the upper bound in (3.1).

We proceed with the proof of Lemma 3.4.

Proof of (3.6). The compound Poisson representation (3.3), subadditivity of ϕ and the same argument as in Lemma 2.7 in Mikosch and Samorodnitsky (1999) yield that

$$\begin{aligned} I_1(u) &\le EN\,P\Big(\phi(\mathbf{Y}_1 - (1-\epsilon)^2\boldsymbol{\mu}) > (1-\epsilon)^2 u\Big) \\ &\quad + \sum_{k=2}^\infty k^2 P(N=k)\left[P\left(\phi\left(\mathbf{Y}_1 - \frac{\epsilon(1-\epsilon)}{k}\boldsymbol{\mu}\right) > \frac{\epsilon(1-\epsilon)}{k}u\right)\right]^2 \\ &=: I_{11}(u) + I_{12}(u)\,. \end{aligned}$$

Recalling that $EN = \nu(B_\tau)$ and using the monotonicity properties (2.12), (2.13) of ϕ, we see that

$$\begin{aligned} I_{11}(u) &= \int_0^\infty \int_0^\infty \mathbf{1}\left(\phi\left(x\mathbf{1}_{[s,\infty)} - (1-\epsilon)^2\boldsymbol{\mu}\right) > (1-\epsilon)^2 u\right)\ \rho(dx)\ ds \\ &\le \int_0^\infty \int_0^\infty \mathbf{1}\left(\phi\left((1-\epsilon)^{-2}x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}\right) > (1-\epsilon)^2 u\right)\ \rho(dx)\ ds\,. \end{aligned}$$

Write for $\epsilon \in (0,1)$,

$$\widetilde{h}(\epsilon) = h((1-\epsilon)^2/(1+\epsilon))\,.$$

The function H is decreasing, while $T(s,u)$ is increasing in both arguments. Therefore and in view of (2.18) we can further bound $I_{11}(u)$ as follows for sufficiently large u

$$\begin{aligned} I_{11}(u) &\le \int_0^\infty H\left((1-\epsilon)^2 T(s,(1-\epsilon)^2 u)\right)\ ds \le \widetilde{h}(\epsilon)\int_0^\infty H\left((1+\epsilon)T(s,(1+\epsilon)u)\right)\ ds \\ &\le \widetilde{h}(\epsilon)\int_0^\infty H\left((1+\epsilon)T(s,u)\right)\ ds \le \widetilde{g}(\epsilon)\ \psi(u)\,. \end{aligned}$$

In the last step we used (2.19). Similar arguments and the assumptions on the function g in (2.16) yield for sufficiently large u and sufficiently small ϵ that

$$I_{12}(u) \le [EN]^{-2}\sum_{k=2}^\infty k^2\ P(N=k)\ \left[g\left(\frac{\epsilon(1-\epsilon)}{(1+\epsilon)k}\right)\right]^2 [\psi(u)]^2 \le C\ [\psi(u)]^2\,.$$

Since, by (2.16), $\widetilde{h}(\epsilon) \to 1$ as $\epsilon \to 0$, we finally conclude that

$$\lim_{\epsilon\to 0}\limsup_{u\to\infty} I_1(u)/\psi(u) \le \lim_{\epsilon\to 0}\widetilde{h}(\epsilon) = 1\,.$$

□

Proof of (3.7). Let $\widetilde{\mathbf{X}}_2^{(0)}$ be an independent copy of $\mathbf{X}_2^{(0)}$ and $\widetilde{\mathbf{X}}_2 = \widetilde{\mathbf{X}}_2^{(0)} - \boldsymbol{\eta}$. Consider the symmetrization of $\mathbf{X}_2$, respectively $\mathbf{X}_2^{(0)}$, given by

$$\mathbf{Z} = \mathbf{X}_2 - \widetilde{\mathbf{X}}_2 = \mathbf{X}_2^{(0)} - \widetilde{\mathbf{X}}_2^{(0)}. \tag{3.8}$$

By independence of $\mathbf{X}_2$ and $\widetilde{\mathbf{X}}_2$ and subadditivity of ϕ, we have

$$P\left(\phi\left(\mathbf{Z} - \epsilon\boldsymbol{\mu}/2\right) > \epsilon u/2\right) \ge P\left(\phi(\mathbf{X}_2 - \epsilon\boldsymbol{\mu}) > \epsilon u\right) P\left(\phi\left(\widetilde{\mathbf{X}}_2 - \epsilon\boldsymbol{\mu}/2\right) \le \epsilon u/2\right)\,.$$

By Lemma 3.3, the second factor on the right hand side goes to 1 as $u \to \infty$. Hence (3.7) follows once we proved that for every $\epsilon > 0$ and τ small enough

$$\lim_{u\to\infty}\frac{P\left(\phi(\mathbf{Z} - \epsilon\boldsymbol{\mu}) > \epsilon u\right)}{\psi(u)} = 0\,. \tag{3.9}$$

Observe that $\mathbf{Z}$ is a symmetric infinitely divisible process whose Lévy measure ν_Z is given by

$$\begin{aligned} \nu_Z(A) &= \int_0^\infty \int_{-\infty}^\infty \left[\mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in A \cap B_\tau\right) + \mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in (-A)\cap B_\tau\right)\right]\ \rho(dx)\ ds \\ &= \int_0^\infty \int_{-\infty}^\infty \mathbf{1}\left(x\mathbf{1}_{[s,\infty)} \in A \cap B_\tau\right)\ \rho^*(dx)\ ds \end{aligned} \tag{3.10}$$

for any measurable $A \subset \mathbb{R}^{[0,\infty)}$. Here

$$\rho^*\left((t,\infty)\right) = \rho\left((t,\infty)\right) + \rho\left((-\infty,-t)\right), \quad t > 0,$$

is the symmetrized one dimensional Lévy measure of $\mathbf{X}$. Define two symmetric one dimensional Lévy measures ρ_1 and ρ_2 by

$$\rho_1(\cdot) = \rho^*\left(\cdot \cap \{x : |x| \geq 1\}\right) \quad \text{and} \quad \rho_2(\cdot) = \rho^*\left(\cdot \cap \{x : |x| < 1\}\right). \tag{3.11}$$

Let $\mathbf{Z}_1$, $\mathbf{Z}_2$ be independent infinitely divisible processes with Lévy measures that are obtained by replacing ρ in (3.10) by ρ_1, ρ_2, respectively. Then

$$\mathbf{Z} \stackrel{\mathrm{d}}{=} \mathbf{Z}_1 + \mathbf{Z}_2 \tag{3.12}$$

and therefore subadditivity of ϕ implies

$$P\left(\phi(\mathbf{Z} - \epsilon\boldsymbol{\mu}) > \epsilon u\right) \leq P\left(\phi(\mathbf{Z}_1 - \epsilon\boldsymbol{\mu}/2) > \epsilon u/2\right) + P\left(\phi(\mathbf{Z}_2 - \epsilon/2\boldsymbol{\mu}) > \epsilon u/2\right) =: I_3(u) + I_4(u).$$

It follows from Lemmas 4.1 and 4.2 below that for any $\epsilon > 0$ and $\tau > 0$

$$\lim_{u\to\infty} I_4(u)/\psi(u) = 0, \tag{3.13}$$

and so we proceed to estimate $I_3(u)$.

Let (Γ_j) be the points of a time homogeneous Poisson process on $[0,\infty)$ with rate $\lambda = \rho_1(\mathbb{R})$, and let (V_j) be iid symmetric random variables with common distribution $\rho_1/\rho_1(\mathbb{R})$ and independent of the Poisson process. Define for $j \geq 1$

$$Y_j = V_j \mathbf{1}\left(\phi\left(|V_j|\mathbf{1}_{[\Gamma_j,\infty)} - \tau\boldsymbol{\mu}\right) \leq \delta_0\right). \tag{3.14}$$

Observe that we can represent the process $\mathbf{Z}_1$ in the form

$$Z_1(t) = \sum_{j=1}^{\infty} Y_j \, \mathbf{1}_{[\Gamma_j,\infty)}(t), \quad t \geq 0.$$

(Simply compute the mean measures of the Poisson random measures on both sides of the equation above.) Notice that

$$(\Gamma_1, \pm Y_1, \Gamma_2, \pm Y_2, \dots) \stackrel{\mathrm{d}}{=} (\Gamma_1, Y_1, \Gamma_2, Y_2, \dots) \tag{3.15}$$

for any choice of signs above.

For $u > 0$ and $T(u) = T(0,u)$ let

$$m = m(u) = \inf\{j = 0, 1, 2, \dots : \ \mu(2j) \geq T(u)\} \tag{3.16}$$

and

$$\mathbf{Z}_{1,(m)} = \sum_{j=1}^{m} Y_j \, \mathbf{1}_{[\Gamma_j,\infty)} \quad \text{and} \quad \mathbf{Z}_1^{(m)} = \sum_{j=m+1}^{\infty} Y_j \, \mathbf{1}_{[\Gamma_j,\infty)}. \tag{3.17}$$

Then, again by subadditivity,

$$I_3(u) \leq P\left(\phi(\mathbf{Z}_{1,(m)} - \epsilon\boldsymbol{\mu}/4) > \epsilon u/4\right) + P\left(\phi(\mathbf{Z}_1^{(m)} - \epsilon\boldsymbol{\mu}/4) > \epsilon u/4\right). \tag{3.18}$$

However, the right hand expressions are of the order $o(\psi(u))$ as $u \to \infty$ for every $\epsilon > 0$ and $\tau > 0$ small enough (relatively to ϵ), as follows from Lemmas 4.4 and 4.6. This and (3.13) imply (3.9) and complete the proof of (3.7). □

The lower bound. We again start with the identity

$$P\Big(\phi(\mathbf{X}-\boldsymbol{\mu})>u\Big)=P\big(\phi(\mathbf{X}_1+\mathbf{X}_2-\boldsymbol{\mu})>u\big)\,.$$

Recalling that $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent and ϕ is subadditive, for every $K>0$

$$\begin{aligned} P\Big(\phi(\mathbf{X}-\boldsymbol{\mu})>u\Big) &\geq P\Big(\phi(\mathbf{X}_1-(1+\epsilon)\boldsymbol{\mu})>u+K\Big)P\Big(\phi(-\mathbf{X}_2-\epsilon\boldsymbol{\mu})\leq K\Big) \\ &= I_5(u,K)I_6(K)\,. \end{aligned} \tag{3.19}$$

Lemma 3.5. *Under the assumptions of the theorem,*

$$\lim_{K\to\infty}\liminf_{u\to\infty} I_5(u,K)/\psi(u)\geq 1\,. \tag{3.20}$$

It is immediate from Lemma 3.3 that $\lim_{K\to\infty} I_6(K)=1$. Therefore, from this lemma and (3.19) we conclude that

$$\liminf_{u\to\infty}\frac{P\Big(\phi(\mathbf{X}-\boldsymbol{\mu})>u\Big)}{\psi(u)}\geq 1\,.$$

This finishes the proof of the lower bound in (3.1).

Proof of Lemma 3.5. First recall the compound Poisson structure of $\mathbf{X}_1$ from (3.3). For $k=1,2,\ldots$ and $j\leq k$, consider the disjoint events

$$\begin{aligned} B_{kj} = \Big\{ & N=k,\ \phi\left(\mathbf{Y}_j-(1+\epsilon)^2\boldsymbol{\mu}\right)>u+2K\,, \\ & \phi\left(\mathbf{Y}_i-(1+\epsilon)^2\boldsymbol{\mu}\right)\leq u+2K\,,\quad i=1,\ldots,k\,,\ i\neq j\,, \\ & \phi\Big(-\textstyle\sum_{1\leq i\neq j\leq k}\mathbf{Y}_i-\epsilon(1+\epsilon)\boldsymbol{\mu}\Big)\leq K\Big\}\,. \end{aligned}$$

Subadditivity of ϕ implies

$$\begin{aligned} I_5(u,K) &\geq \sum_{k=1}^{\infty}\sum_{j=1}^{k}P(B_{kj}) \\ &\geq P\left(\phi\left(\mathbf{Y}_1-(1+\epsilon)^2\boldsymbol{\mu}\right)>u+2K\right)\,p_1(K)-p_2(u)\,, \end{aligned}$$

where

$$p_1(K) := \sum_{k=1}^{\infty}\sum_{j=1}^{k}P\left(N=k\,,\phi\left(-\textstyle\sum_{1\leq i\neq j\leq k}\mathbf{Y}_i-\epsilon(1+\epsilon)\boldsymbol{\mu}\right)\leq K\right)\,,$$

$$p_2(u) := \sum_{k=2}^{\infty}P\left(N=k\,,\phi\left(\mathbf{Y}_j-(1+\epsilon)^2\boldsymbol{\mu}\right)>u+2K \text{ for at least 2 different } j\in\{1,\ldots,k\}\right)\,.$$

Using the independence of $\mathbf{Y}_j$ and $\sum_{1\leq i\neq j\leq k}\mathbf{Y}_i$ and again the subadditivity of ϕ, we have

$$p_1(K) \geq EN\ P\left(\phi\left(\mathbf{Y}_1-\epsilon(1+\epsilon)\boldsymbol{\mu}/2\right)\leq K/2\right)\ P\left(\phi\left(-\mathbf{X}_1-\epsilon(1+\epsilon)\boldsymbol{\mu}/2\right)\leq K/2\right)\,.$$

By Lemma 3.3, $\phi(|\mathbf{X}_1|-\gamma\boldsymbol{\mu})<\infty$ a.s. for every $\gamma>0$. Using this fact, (2.12) and (2.21) (recall that the law of $\mathbf{Y}_1$ is $\nu_1/\nu(B_\tau)$, see (3.3)) we see that

$$\liminf_{K\to\infty} p_1(K)/EN\geq 1\,.$$

The argument leading to (3.6) also shows that for every $K > 0$,

$$\liminf_{u\to\infty} \frac{EN\, P\left(\phi\left(\mathbf{Y}_1 - (1+\epsilon)^2\boldsymbol{\mu}\right) > u + 2K\right)}{\psi(u)} \geq \frac{1}{h\left((1+\epsilon)^{-3}\right)},$$

implying that

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\liminf_{u\to\infty} \frac{p_1(K)\; P\left(\phi\left(\mathbf{Y}_1 - (1+\epsilon)^2\boldsymbol{\mu}\right) > u + 2K\right)}{\psi(u)} \geq \lim_{\epsilon\to 0}\frac{1}{h\left((1+\epsilon)^{-3}\right)} = 1\,.$$

Since, as $u \to \infty$,

$$p_2(u) \leq E(N^2)\;\left[P\left(\phi\left(\mathbf{Y}_1 - (1+\epsilon)^2\boldsymbol{\mu}\right) > u + 2K\right)\right]^2,$$

(3.20) follows. □

4. Auxiliary facts and lemmas

In this section we provide some auxiliary results for the proof of Theorem 3.1. In what follows, we always assume that the assumptions of this theorem are satisfied.

We start with a simple lemma connecting the rates of decay of the function $\psi(u)$ in (2.3) and of $1/T(u)$ in (2.14).

Lemma 4.1. *There are constants $q > 0$ and $C > 0$ such that for all $u \geq 1$*

$$\psi(u) \geq C\,[T(u)]^{-q}\,. \tag{4.1}$$

Proof. Recalling the definition of $\psi(u)$ from (2.3), observing that H is decreasing and using (2.15), we obtain

$$\psi(u) \geq \int_0^\infty H(2T(s,u))\,ds \geq \int_0^\infty H\Big(2A_3\left[\mu(s) + T(u)\right]\Big)\,ds\,.$$

Now use that both $\mu(s)$ and $H(u)$ can be bounded by power laws, see (2.22), and change the variable of integration to get the desired result. □

Next we discuss certain properties of the processes occurring in (3.4) and subsequent decompositions. The processes $\mathbf{X}_i$, $\mathbf{X}_i^{(0)}$ have independent (though not necessarily stationary) increments. This property is inherited by the symmetric processes $\mathbf{Z}$ in (3.8) and, subsequently, $\mathbf{Z}_i$ in (3.12).

Let $\mathbf{W}_i$ be Lévy processes on $[0,\infty)$ with one dimensional Lévy measure ρ_i, $i = 1,2$, as defined in (3.11). Then the following identities in law hold:

$$\mathbf{W}_i \stackrel{\mathrm{d}}{=} \mathbf{Z}_i + \mathbf{V}_i\,, \quad i = 1,2\,, \tag{4.2}$$

where, for fixed i, $\mathbf{Z}_i$ and $\mathbf{V}_i$ are independent symmetric infinitely divisible processes with independent increments.

Our next lemma shows that the functional ϕ applied to the process $\mathbf{Z}_2$, representing the "small jumps" of the process $\mathbf{Z}$, has a "light" tailed distribution.

Lemma 4.2. *For every $\gamma > 0$ and $r > 0$, we have*

$$J(u) := P\left(\phi\left(\mathbf{Z}_2 - \gamma\boldsymbol{\mu}\right) > u\right) = o\left([T(u/2)]^{-r}\right) \quad \text{as } u \to \infty\,. \tag{4.3}$$

Proof. Without loss of generality we may assume that $r \geq 2$ and $\gamma \leq 1$. By subadditivity of ϕ,

$$\begin{aligned} J(u) &\leq P\left(\phi(\mathbf{Z}_2 \mathbf{1}_{[0,T(u/2))} - \gamma\boldsymbol{\mu}/2) > u/2\right) + P\left(\phi(\mathbf{Z}_2 \mathbf{1}_{[T(u/2),\infty)} - \gamma\boldsymbol{\mu}/2) > u/2\right) \\ &=: J_1(u) + J_2(u)\,. \end{aligned}$$

The monotonicity properties (2.12), (2.13) of ϕ yield

$$\begin{aligned} J_1(u) &\leq P\left(\phi\left(\sup_{0\leq t\leq T(u/2)} Z_2(t) \mathbf{1}_{[0,T(u/2))} - \gamma\boldsymbol{\mu}/2\right) > u/2\right) \\ &\leq P\left(\sup_{0\leq t\leq T(u/2)} Z_2(t) \geq \gamma T(u/2)/2\right) \\ &\leq 2\,P\left(Z_2(T(u/2)) \geq \gamma T(u/2)/2\right)\,. \end{aligned}$$

In the last step we used Lévy's maximal inequality. Another appeal to this inequality and to the definition of $\mathbf{W}_2$ in (4.2) gives

$$J_1(u) \leq 4\,P\left(W_2(T(u/2)) \geq \gamma T(u/2)/2\right)\,.$$

However, the Lévy process $\mathbf{W}_2$ has a symmetric Lévy measure supported by a compact set. Therefore, it has finite exponential moments. By the Burkholder–Gundy inequality, for every $p \geq 2$ there is a positive constant C such that $E|W_2(t)| \leq Ct^{p/2}$ for all $t > 0$. Applying Markov's inequality, we finally obtain the following bound:

$$J_1(u) = o\left([T(u/2)]^{-r}\right)\,, \quad u \to \infty\,. \tag{4.4}$$

Now we turn to the estimation of $J_2(u)$. We proceed in a similar fashion. First, monotonicity of ϕ together with (2.10) gives

$$\begin{aligned} J_2(u) &\leq P\left(\phi((\mathbf{Z}_2 - \gamma\boldsymbol{\mu}/2)\mathbf{1}_{[T(u/2),\infty)}) > u/2\right) \\ &\leq P\left(Z_2(t) > \gamma\mu(t)/2 \quad \text{for some } t \geq T(u/2)\right) \\ &\leq \sum_{j=1}^{\infty} P\left(Z_2(t) > \gamma\mu(t)/2 \quad \text{for some } \ T(u/2) + (j-1) \leq t < T(u/2) + j\right) =: \sum_{j=1}^{\infty} b_j\,. \end{aligned}$$

Now use again Lévy's maximal inequality, the fact that $\boldsymbol{\mu}$ is quasi-monotone and converges to infinity to obtain the following chain of inequalities:

$$\begin{aligned} b_j &\leq 2\,P\left(Z_2(T(u/2) + j) > a_3\,\gamma\,\mu(T(u/2) + (j-1))/2\right) \\ &\leq 4\,P\left(W_2(T(u/2) + j) > a_3\,\gamma\,\mu(T(u/2) + (j-1))/2\right)\,. \end{aligned}$$

Finally, applying the Burkholder-Gundy and Markov inequalities and choosing $p > 2(\beta_2 + r)/\beta_2$ (see (2.7)), we obtain for any $r \geq 2$,

$$J_2(u) \leq C \sum_{j=1}^{\infty} \frac{[T(u/2) + j]^{p/2}}{[\mu\left(T(u/2) + (j-1)\right)]^p} = o\left([T(u/2)]^{-r}\right)\,.$$

The latter estimate together with (4.4) for $J_1(u)$ establishes the desired bound (4.3) for $J(u)$. □

Now we turn to the processes $\mathbf{Z}_{1,(m)}$ and $\mathbf{Z}_1^{(m)}$ defined in (3.17). In this context, recall that (Y_j) is a sequence of independent symmetric random variables given the points (Γ_k) of a homogeneous Poisson process with rate $\lambda = \rho_1(\mathbb{R})$. Write

$$A_m = \{|Y_j| \le \theta\, T(\Gamma_j, u), j = 1, \ldots, m\}, \quad m \ge 1,$$

$$S_k = Y_1 + \cdots + Y_k, \quad k \ge 1.$$

Lemma 4.3. *Let $m = m(u)$ be defined by (3.16). For every $\gamma > 0$ and $r > 0$, there are positive constants θ and C such that for all $u > 0$,*

$$G(u) := P\left(\phi(\mathbf{Z}_{1,(m)} - \gamma \boldsymbol{\mu}) > u, A_m\right) \le C\, m^{-r}.$$

Proof. Without loss of generality assume that $\gamma \le 1$. By monotonicity of ϕ,

$$\phi\left(\sum_{j=1}^m Y_j \mathbf{1}_{[\Gamma_j,\infty)} - \gamma\boldsymbol{\mu}\right) \le \phi\left(\max_{1\le k\le m} S_k\ \mathbf{1}_{[\Gamma_1,\infty)} - \gamma\boldsymbol{\mu}\right) \le \phi\left(\gamma^{-1} \max_{1\le k\le m} S_k\ \mathbf{1}_{[\Gamma_1,\infty)} - \boldsymbol{\mu}\right).$$

Therefore we have

$$\begin{aligned} G(u) &\le P\left(\max_{1\le k\le m} S_k \ge \gamma T(\Gamma_1, u),\ A_m\right) \\ &\le P\left(\max_{1\le k\le m} S_k \ge \gamma T(\Gamma_1, u),\ A_m, \Gamma_m \le 2\lambda m\right) + P(\Gamma_m > 2\lambda m) =: p_m^{(1)} + p_m^{(2)}. \end{aligned}$$

Obviously, $p_m^{(2)}$ decays to zero at an exponential rate. As to $p_m^{(1)}$, observe that

$$\{A_m, \Gamma_m \le 2\lambda m\} \subset \left\{\max_{j=1,\ldots,m} |Y_j| \le \theta\, T(2\lambda m, u)\right\} =: \widetilde{A}_m.$$

Therefore and by virtue of Lévy's maximal inequality, applied conditonally upon (Γ_k) and $(|Y_k|)$,

$$p_m^{(1)} \le P\left(\max_{1\le k\le m} S_k \ge \gamma T(u), \widetilde{A}_m\right) \le 2\, P\left(S_m \ge \gamma T(u), \widetilde{A}_m\right) \le 4\, P\left(\widetilde{S}_m \ge \gamma T(u)\right).$$

In the last step we applied the contraction principle for sums of independent symmetric random variables. Here

$$\widetilde{S}_m = \sum_{j=1}^m \widetilde{Y}_j, \quad \widetilde{Y}_j = \widetilde{Y}_j^{(m)} = Y_j \mathbf{1}_{|Y_j| \le \theta T(2\lambda m, u)}, \quad j = 1, \ldots, m.$$

Notice that conditionally upon (Γ_k), $\widetilde{S}_m$ is a sum of independent symmetric random variables which are uniformly bounded by $T(2\lambda m, u)$. An application of Prokhorov's exponential inequality (see Prokhorov (1959); cf. Petrov (1995), 2.6.1 on p. 77), conditionally on (Γ_k), yields

$$p_m^{(1)} \le 4\, E \exp\left\{-\frac{\gamma T(u)}{2\theta\, T(2\lambda m, u)} \operatorname{arsinh}\left[\frac{\theta\, T(2\lambda m, u)\ \gamma\ T(u)}{2\mathrm{var}(\widetilde{S}_m \mid (\Gamma_k))}\right]\right\}.$$

Let us consider the case $\beta_1 < 2$ in (2.6); the case $\beta_1 \ge 2$ is analogous. It follows from the representation (3.14) of the random variables Y_j that there is a constant C such that for any realization (Γ_i)

$$\mathrm{var}(\widetilde{S}_m \mid (\Gamma_k)) \le C\, m\, [T(2\lambda m, u)]^{2-\beta_1}.$$

Then use the property (2.15) of $T(2\lambda m,u)$ and the definition (3.16) of $m=m(u)$ to obtain

$$T(2\lambda m,u)\le A_3\,[\mu(2\lambda m)+T(u)]\le C\,[\mu(2\lambda m)+\mu(2m)]\le C\,\mu(m)\,.$$

Similarly,

$$T(2\lambda m,u)T(u)\ge [T(u)]^2\ge[\mu(2(m-1))]^2\ge C\,[\mu(m)]^2\,.$$

Combining the latter estimates and using the growth condition (2.7) on $\boldsymbol{\mu}$, we arrive at the bound

$$\frac{\theta\,T(2\lambda m,u)\;\gamma\,T(u)}{2\mathrm{var}(\widetilde{S}_m\mid(\Gamma_k))}\ge C\,\frac{[\mu(m)]^{\beta_1}}{m}\ge C\,m^{\beta_1\beta_2-1}$$

Since $\beta_2>\max(0.5,\beta_1^{-1})$, the power of m is positive. Similarly,

$$\frac{T(u)}{T(2\lambda m,u)}\ge C\frac{T(u)}{\mu(2\lambda m)+T(u)}\,,$$

which is bounded away from 0 by (3.16). Using the fact that $\mathrm{arsinh}(t)\ge\log(1+t)$, $t>0$, we conclude that

$$p_m^{(1)}\le C\exp\left\{-\frac{\log m}{C\theta}\right\}\,.$$

Collecting all bounds above for $G(u)$, $p_m^{(1)}$ and $p_m^{(2)}$, we obtain our claim by choosing θ small enough. □

Our next lemma shows that the first probability on the right hand side of (3.18) is much smaller than $\psi(u)$.

Lemma 4.4. *Let $m=m(u)$ be defined by 3.16). For every $\epsilon>0$ and $\tau>0$ small enough (relatively to ϵ),*

$$\lim_{u\to\infty}\frac{P\left(\phi(\mathbf{Z}_{1,(m)}-\epsilon\boldsymbol{\mu})>\epsilon u\right)}{\psi(u)}=0\,. \tag{4.5}$$

Proof. Since $\mu(t)$ does not grow faster than a power function, it follows from the definition of $m=m(u)$ that there exist positive constants C, q such that $m\ge C[T(u)]^q$ for large u. By virtue of Lemma 4.3, for every $r>0$ and $\epsilon>0$ there are positive θ and C such that

$$P\left(\phi(\mathbf{Z}_{1,(m)}-\epsilon\boldsymbol{\mu})>\epsilon u\,,A_m\right)\le C\,m^{-r}\le C\,[T(u)]^{-rq}\,.$$

Since $r>0$ can be chosen arbitrarily large, the latter fact in combination with Lemma 4.1 implies that

$$P\left(\phi(\mathbf{Z}_{1,(m)}-\epsilon\boldsymbol{\mu})>\epsilon u,A_m\right)=o(\psi(u))\,.$$

On the other hand, we have

$$P(A_m^c)\le\sum_{j=1}^m P(|Y_j|>\theta T(\Gamma_j,u))=2\,\sum_{j=1}^m P(Y_j>\theta T(\Gamma_j,u))\,. \tag{4.6}$$

By representation (3.14) for the Y_js,

$$Y_j\le\tau T(\Gamma_j,\delta_0)\le\tau T(\Gamma_j,u) \tag{4.7}$$

for u large enough. Thus, for τ small enough, the right hand expression in (4.6) vanishes. This concludes the proof. □

The next lemmas are related to the behavior of the second term on the right hand side of (3.18). Write

$$A_{mn} = \{|Y_j| \le \theta T\left(\Gamma_j, u\right), m < j < n+1\},\ n = m, m+1, \ldots, \infty\,.$$

Lemma 4.5. *For any $\gamma > 0$ and $r > 0$, there are positive constants θ and C such that for all $u > 0$ and $n > m = m(u)$*

$$H_n := P\left(S_n - S_m > \gamma\mu(\Gamma_n)\,, A_{mn}\right) \le C\, n^{-r}\,. \tag{4.8}$$

Proof. For all j and $u > 0$, we have $T(\Gamma_j, u) \le A_3\,[\mu(\Gamma_j) + T(u)] \le C\,[\mu(\Gamma_n) + T(u)]$. Now, the definition of $m = m(u)$ and the Δ_2 condition give for $n > m$, $T(u) \le \mu(2m) \le C\,\mu(2n) \le C\mu(n)$. Therefore

$$T(\Gamma_j, u) \le C\,[\mu(\Gamma_n) + \mu(n)]\,, \quad j \le n\,.$$

Then for $m < n$,

$$H_n \;\le\; P\left(S_n - S_m > \gamma\,\mu(\Gamma_n)\,, \max_{j=m+1,\ldots,n} |Y_j| \le \theta\, C\,[\mu(\Gamma_n) + \mu(n)]\right)\,.$$

Recalling that the Poisson process (Γ_j) has rate λ, write

$$D_n = \{|\Gamma_n - \lambda n| \le 0.5\lambda n\}\,.$$

and notice that $P(D_n^c)$ decays to zero at an exponential rate. Therefore, for any $r > 0$,

$$\begin{aligned}
H_n \;&\le\; P\left(S_n - S_m > \mu(\Gamma_n)\,, \max_{l=m+1,\ldots,n} |Y_j| \le \theta\, C\,[\mu(\Gamma_n) + \mu(n)]\,, D_n\right) + P(D_n^c) \\
&\le\; P\left(S_n - S_m > C^{-1}\,\mu(n)\,, \max_{j=m+1,\ldots,n} |Y_j| \le \theta\, C\,\mu(n)\right) + Cn^{-r} \\
&\le\; 2\,P\left(\widehat{S}_n - \widehat{S}_m \ge C^{-1}\,\mu(n)\right) + C\,n^{-r} \\
&\le\; 4\,P\left(\widehat{S}_n \ge C^{-1}\,\mu(n)\right) + C\,n^{-r}\,.
\end{aligned} \tag{4.9}$$

In the last step we used the contraction principle and Lévy's maximal inequality for the sum of conditionally independent and symmetric random variables Y_j. Here

$$\widehat{S}_n = \sum_{j=1}^n \widehat{Y}_j\,, \quad \widehat{Y}_j = Y_j \mathbf{1}_{|Y_j| \le \theta C\mu(n)}\,, \quad j = 1, \ldots, n\,.$$

Using again Prokhorov's inequality, conditionally on (Γ_k) we can bound the tail probability in (4.9) by

$$E\exp\left\{-\frac{C^{-1}\mu(n)}{2\theta C\mu(n)}\,\mathrm{arsinh}\left[\frac{\theta[\mu(n)]^2}{2\,\mathrm{var}(\widehat{S}_n \mid (\Gamma_k))}\right]\right\}\,.$$

Proceeding as in the proof of Lemma 4.3 and choosing θ small enough, the last expression can be bounded by Cn^{-r} for any $r > 0$. This concludes the proof. ☐

The following statement is now a straightforward conclusion from the previous lemma.

Lemma 4.6. *For every $\gamma > 0$ and $r > 0$, there are positive constants θ and C such that for all $u > 0$ and $m = m(u)$,*

$$R_m = P\left(\phi(\mathbf{Z}_1^{(m)} - \gamma\mu) > 0\,, A_{m\infty}\right) \le C\, m^{-r}\,. \tag{4.10}$$

Moreover,

$$\lim_{u\to\infty} \frac{P\left(\phi(\mathbf{Z}_1^{(m)} - \epsilon\boldsymbol{\mu}) > \epsilon u\right)}{\psi(u)} = 0\,. \tag{4.11}$$

Proof. Using the properties (2.10), (2.9) of ϕ, we obtain the bounds

$$\begin{aligned} R_m &\le P\left(\bigcup_{n=m+1}^{\infty} \{S_n - S_m > a_3\,\gamma\,\mu(\Gamma_n)\} \cap A_{mn}\right) \\ &\le \sum_{n=m+1}^{\infty} P\left(S_n - S_m > a_3\,\gamma\,\mu(\Gamma_n)\,, A_{mn}\right)\,. \end{aligned}$$

Now apply Lemma 4.5 to get (4.10). For u large enough, (4.7) holds for all j. Choosing θ small enough (relatively to ϵ), (4.11) follows from (4.10) and Lemma 4.1. □

5. Some examples of subadditive functionals

In this section we consider several important and common subadditive functionals ϕ acting on Lévy processes. We apply Theorem 3.1 to characterize the tail behavior of the distribution of these functionals. The reader should note that only a few natural and transparent assumptions are needed for the results below to hold.

Throughout this section we assume that the following assumptions hold.

$$H \text{ is regularly varying with exponent } -\alpha \text{ for some } \alpha > 0\,, \tag{5.1}$$

there is a constant $C > 0$ such that

$$\rho((-\infty\,, -t]) \le C\;\rho([t,\infty)) \quad \text{for all } t \ge 1 \tag{5.2}$$

and

$$\mu \text{ is regularly varying with exponent } \beta \text{ for some } \beta > \max(\alpha^{-1}, 0.5)\;\;. \tag{5.3}$$

Of course, the assumption (5.2) is the same as (2.4). Since it is our goal to collect all the relevant assumptions in this section together for easy reference, this assumption is repeated here. The following lemma collects several well known facts on regular varying functions. The reader is referred to Bingham et al. (1987)) for proofs and more information. Let

$$\mu^{\leftarrow}(u) = \sup\{t > 0 :\; \mu(t) \le u\}\,,\; u > 0 \tag{5.4}$$

be the generalized inverse of μ.

Lemma 5.1. *(a) Let $\boldsymbol{\mu}$ be regularly varying at infinity with positive exponent of regular variation. Then there are monotone functions $\boldsymbol{\mu}_*$ and $\boldsymbol{\mu}^*$ such that*

$$\mu_*(t) \le \mu(t) \le \mu^*(t) \quad \textit{for all } t \ge 0 \textit{ and } \lim_{t\to\infty} \frac{\mu^*(t)}{\mu_*(t)} = 1\,. \tag{5.5}$$

(ii) Let $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ be regularly varying with positive exponent α of regular variation and such that $\lim_{t\to\infty}\mu(t)/\eta(t)=1$ (i.e., $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$ are asymptotically equivalent). Then their generalized inverses are regularly varying with exponent $1/\alpha$ and asymptotically equivalent as well. Moreover, let

$$\mu_0(x)=\inf\{y>0:\ \mu^{\leftarrow}(y)>x\}\,,\ x>0.$$

Then

$$\lim_{x\to\infty}\frac{\mu_0(x)}{\mu(x)}=1\,. \tag{5.6}$$

(iii) (**Potter's bounds**) *Let $\boldsymbol{\mu}$ be regularly varying with a positive exponent α of regular variation. For every $C>1$ and $\epsilon>0$ there is $s_0=s_0(C,\epsilon)$ such that for all $s,t\geq s_0$*

$$\frac{\mu(t)}{\mu(s)}\leq C\ \max\left(\left(\frac{t}{s}\right)^{\alpha+\epsilon},\left(\frac{t}{s}\right)^{\alpha-\epsilon}\right)\,. \tag{5.7}$$

5.1. **The overall supremum.** One of the interesting subadditive functionals is the overall supremum

$$\phi_{\sup}(\boldsymbol{\alpha})=\sup_{t\geq 0}\alpha(t)\,.$$

It has numerous applications. among them in insurance mathematics for describing eventual ruin (see Embrechts et al. (1997)) or in queuing for the buffer overflow (see Prabhu (1998)).

Remark 5.2. In this paper we deal with "power–like" tails and, hence, the following theorem that describes the tail behavior of the distribution of the overall supremum of a Lévy process is stated under the assumptions of regular variation. We conjecture, however, that the first asymptotic equivalence in (5.8) below holds in greater generality, perhaps under the assumption of subexponentiality of the tail of H. In fact, if $\alpha>1$ and $\mu(t)=\mu t$ for some $\mu>0$, is a linear function, then the first asymptotic equivalence in (5.8) is just the classical result for the ruin probability as proved by Embrechts and Veraverbeke (1982):

$$P\left(\sup_{t\geq 0}\left(X(t)-\mu(t)\right)>u\right)\sim\frac{1}{\mu}\int_u^\infty H(s)\,ds\,,$$

and the latter result is known to hold when H has a subexponential right tail.

In fact, it is quite possible that the curve $\boldsymbol{\mu}$ may be allowed to belong to a wider class of functions as well.

Theorem 5.3. *Assume* (5.1)–(5.3). *Then*

$$\begin{aligned} P\left(\phi_{\sup}(\mathbf{X}-\boldsymbol{\mu})>u\right) &= P\left(\sup_{t\geq 0}\left(X(t)-\mu(t)\right)>u\right)\\ &\sim \int_0^\infty H(\mu(s)+u))\,ds\sim C(\alpha,\beta)\mu^{\leftarrow}(u)H(u)\end{aligned} \tag{5.8}$$

as $u\to\infty$. Here $C(\alpha,\beta)=\alpha\int_0^\infty z^{1/\beta}(1+z)^{-(1+\alpha)}\,dz$.

Proof. The first step is to note that it is enough to prove the theorem in the case when $\boldsymbol{\mu}$ is a monotone function. Since all the terms in (5.8) are, obviously, monotone in $\boldsymbol{\mu}$, parts (i) and (ii) of Lemma 5.1 show that, knowing that the theorem holds for monotone functions, implies its validity in general.

Assume, therefore, that $\boldsymbol{\mu}$ is a monotone function. The only assumptions in Theorem 3.1 that are not obviously satisfied are (2.17) and (2.18). We will postpone verification of these conditions until the end of the proof. Note that in our case

$$T(s,u) = \mu(s) + u\,,$$

and so it follows from (2.19) and regular variation of H that

$$\psi(u) \sim \int_0^\infty H(\mu(s)+u))\,ds$$

as $u \to \infty$, which together with Theorem 3.1 establishes the first asymptotic equivalence in (5.8). Furthermore, by Potter's bounds (part (iii) of Lemma 5.1), we see, further, that

$$\psi(u) \sim u^\alpha H(u) \int_0^\infty (\mu(s)+u))^{-\alpha}\,ds = \alpha u^\alpha H(u) \int_0^\infty \mu^{\leftarrow}(u)(y+u)^{-(\alpha+1)}\,dy\,.$$

Since $\mu^{\leftarrow}$ is, according to part (ii) of Lemma 5.1, regularly varying, the second asymptotic equivalence in (5.8) is a standard exercise in integration of regularly varying functions.

It remains to check (2.17) and (2.18). For every $0 < \delta < 1$

$$\int_0^\infty H(\delta T(s,\delta u))\,ds = \int_0^\infty H(\delta(\mu(s)+\delta u))\,ds \le \int_0^\infty H(\delta^2(\mu(s)+u))\,ds := I_\delta(u)\,. \tag{5.9}$$

Since we have already proved that $I_1(u)$ is regularly varying, by Potter's bounds, there is a $\theta > 0$ such that $I_1(u) \ge Cu^{-\theta}$ for all $u \ge 1$, while for any $0 < \delta < 1$ and $u \ge \delta^{-2}$,

$$I_\delta(u) \le C\delta^{-\theta} I_1(u)\,.$$

In the case $1 \le u < \delta^{-2}$ we write

$$I_\delta(u) = \int_{\mu(s)\le\delta^{-2}} + \int_{\mu(s)>\delta^{-2}} := I_\delta^{(1)}(u) + I_\delta^{(2)}(u)\,.$$

Using Potter's bounds in the same way as before shows that

$$I_\delta^{(1)}(u) \le C\delta^{-\theta} I_1(u)\,.$$

Moreover, since H is the tail of a Lévy measure, we know that, for some $C > 0$, $H(y) \le Cy^{-2}$ for all $0 < y \le 2$. Therefore, by the regular variation of $\boldsymbol{\mu}$

$$I_\delta^{(1)}(u) \le C\delta^{-4}u^{-2} \int_0^\infty \mathbf{1}(\mu(s) \le \delta^{-2})\,ds \le C\delta^{-\theta_1}$$

for some $\theta_1 > 0$. Putting everything together establishes that for some $C > 0$ and $\theta_2 > 0$

$$\int_0^\infty H(\delta T(s,\delta u))\,ds \le C\delta^{-\theta_2} \int_0^\infty H(T(s,u))\,ds$$

for all $u \ge 1$ and $0 < \delta < 1$. This is, of course, more than enough to prove (2.18). Finally, the assumption (2.17) is an immediate consequence of (5.9) and Potter's bounds. □

5.2. **The time the process spends above zero.** In this section we consider the sojourn time

$$\phi_{\text{sojourn}}(\boldsymbol{\alpha}) = \int_0^\infty \mathbf{1}\left(\alpha(t) > 0\right)\, dt\,,$$

which is easily seen to be a subadditive functional.

Theorem 5.4. *Assume* (5.1)–(5.3). *Then*

$$\begin{aligned} P\left(\phi_{\text{sojourn}}(\mathbf{X}-\boldsymbol{\mu}) > u\right) &= P\left(\int_0^\infty \mathbf{1}\left(X(t)-\mu(t) > 0\right) dt > u\right) \\ &\sim \int_u^\infty H(\mu(s))\, ds \sim C(\alpha,\beta)\, u\, H(\mu(u)) \qquad (5.10) \end{aligned}$$

as $u \to \infty$. *Here* $C(\alpha,\beta) = (\alpha\beta-1)^{-1}$.

Proof. We may and will assume that $\boldsymbol{\mu}$ is monotone. Furthermore, we may assume that $\mu(0) \geq 1$. Indeed, let

$$\tilde{\mu}(t) = \max(\mu(t),\ 1+\log(1+t))\,,\ t \geq 0\,.$$

Then $\tilde{\mu}(0) \geq 1$, $\tilde{\mu}(t) \sim \mu(t)$ as $t \to \infty$, and it is easy to check that monotonicity and subadditivity of the functional ϕ_{sojourn} imply that

$$P\left(\phi_{\text{sojourn}}(\mathbf{X}-\boldsymbol{\mu}) > u\right) \sim P\left(\phi_{\text{sojourn}}(\mathbf{X}-\tilde{\boldsymbol{\mu}}) > u\right)$$

as $u \to \infty$.

Once again, the only assumptions in Theorem 3.1 that are not obviously satisfied are (2.17) and (2.18) (see Proposition 2.2 for (2.15)), and we postpone their verification. Note that in this case

$$T(s,u) = \mu_0(s+u)$$

(see part (ii) of Lemma 5.1), and so it follows from that part of the lemma, (2.19) and regular variation of H that

$$\psi(u) \sim \int_u^\infty H(\mu(s))\, ds$$

as $u \to \infty$.

It remains to verify the conditions (2.17) and (2.18). We have

$$\int_0^\infty H(\delta T(s,\delta u))\, ds = \int_{\delta u}^\infty H(\delta\mu_0(s))\, ds = \delta \int_u^\infty H(\delta\mu_0(\delta s))\, ds := I_\delta(u)\,.$$

Note that the assumption $\mu(0) \geq 1$ implies that $\mu_0(0) \geq 1$. Therefore, if $0 < \delta < 1$ and $u \geq 1/\delta$ we can use Potter's bounds and the fact that $H(y) \leq Cy^{-2}$ for all $0 < y \leq 2$ to see that for some $C > 0$ and $\theta > 0$ we have

$$I_\delta(u) \leq C\delta^{-\theta} I_1(u)\,,$$

whereas if $1 \leq u < 1/\delta$, then

$$I_\delta(u) \leq C\delta^{-\theta}(1 + I_1(u))\,.$$

The already established regular variation at infinity of $I_1(u)$ implies now that

$$I_\delta(u) \leq C\delta^{-\theta} I_1(u)$$

for all $0 < \delta < 1$ and $u \geq 1$, which is, once again, more than enough to prove (2.18). The assumption (2.17) is an immediate consequence of Potter's bounds. $\square$

5.3. **The last hitting time of zero.** In this section we consider the functional

$$\phi_{\text{last}}(\boldsymbol{\alpha}) = \sup\{t > 0 : \alpha(t) \geq 0\}\,.$$

It is not difficult to see that this functional is subadditive.

Theorem 5.5. *Assume* (5.1)–(5.3). *Then*

$$\begin{aligned} P\left(\phi_{\text{last}}(\mathbf{X} - \boldsymbol{\mu}) > u\right) &= P\left(\sup\{t > 0 : X(t) \geq \mu(t)\} > u\right) \\ &\sim uH(\mu(u)) + \int_u^\infty H(\mu(s))\,ds \sim C(\alpha,\beta)\;u\;H(\mu(u)) \end{aligned} \tag{5.11}$$

as $u \to \infty$. *Here* $C(\alpha,\beta) = 1 + (\alpha\beta - 1)^{-1}$.

Proof. The proof is similar to the one for Theorem 5.4. Without loss of generality we may and will assume that $\boldsymbol{\mu}$ is monotone increasing and $\mu(0), \mu_0(0) \geq 1$. The only conditions we have to verify are (2.17) and (2.18) (once again, see Proposition 2.2 for (2.15)). We postpone this calculation until later.

Notice that

$$\phi_{\text{last}}(x\mathbf{1}_{[s,\infty)} - \boldsymbol{\mu}) = \begin{cases} 0 & \text{if } x \leq \mu(s), \\ \mu^{\leftarrow}(x) & \text{if } x > \mu(s). \end{cases}$$

Therefore

$$T(s,u) = \inf\{x : x > \mu(s),\, \mu^{\leftarrow}(x) > u\} = \max(\mu(s),\, \mu_0(u))\,, \tag{5.12}$$

and so we may conclude that

$$\psi(u) \sim uH(\mu_0(u)) + \int_u^\infty H(\mu(s))\,ds\,, \tag{5.13}$$

which together with Karamata's theorem concludes the proof of the theorem.

We now turn to the proof of (2.17) and (2.18). We have by (5.12)

$$\int_0^\infty H(\delta T(s,\delta u))\,ds = \delta u H(\delta\mu_0(\delta u) + \int_{\delta u}^\infty H(\delta\mu(\delta s))\,ds =: H_1(\delta u) + H_2(\delta u)\,.$$

It clearly suffices to show that each term $H_1(u)$ and $H_2(u)$ satisfies (2.17) and (2.18). For $H_2(u)$ this was proved in the proof of Theorem 5.4. Now turn to $H_1(u)$. Let $\delta \in (0,1)$. Then it follows from Potter's bounds and regular variation of $\boldsymbol{\mu}$ and H that for $u > 1/\delta$, say,

$$H_1(\delta u) \leq C\;\delta^\theta H_1(u)\,,$$

where θ is a real constant. For small $u \leq 1/\delta$ one can again proceed as in the proof of Theorem 5.4. Making use of the fact that H is the tail of a one-dimensional Lévy measure and that $\mu(0) \geq 1$, we see that

$$H_1(\delta u) \leq C\;[\delta u]\;H(\delta) \leq C\;[\delta u]\;\delta^{-2} \leq C\;\delta^{-1} u H(\mu_0(u)) = C\;H_1(u)\,.$$

□

5.4. **Integral of a nonnegative subadditive function.** The functional ϕ_{sojourn} of Theorem 5.4 is a particular case of a more general group of subadditive functionals obtained by appropriate space–dependent weighting of the positive values of a process. Consider a nondecreasing nonnegative function f such that $f(x)=0$ for $x\le 0$ and

$$f(x_1+x_2)\le f(x_1)+f(x_2) \quad \text{for } x_1,x_2>0,$$

and let

$$\phi_{I(f)}(\boldsymbol{\alpha})=\int_0^\infty f(\alpha(t))\,dt\,. \tag{5.14}$$

It is clear that $\phi_{I(f)}$ is a subadditive functional. We will not address here the question what functionals $\phi_{I(f)}$ fit in the framework of the theory developed in the present paper. Instead, we will briefly consider the class of functionals corresponding to the power functions

$$f(x)=[x_+]^p,\ 0\le p\le 1\,. \tag{5.15}$$

We will denote the corresponding functional by $\phi_p(\boldsymbol{\alpha})$. The case $p=0$ corresponds to the functional ϕ_{sojourn}.

The tail behavior of the distribution of the functional $\phi_p(\boldsymbol{\alpha})$ is described in the following theorem. Its proof is very similar to that of Theorem 5.4, but quite a bit longer. We omit the argument.

Theorem 5.6. *Assume* (5.1)–(5.3). *Then for every* $0<p\le 1$

$$\begin{aligned} P\left(\phi_p(\mathbf{X}-\boldsymbol{\mu})>u\right) &= P\left(\int_0^\infty [X(t)-\mu(t)]_+^p\,dt>u\right)\\ &\sim C(\alpha,\beta,p)\ u\ \left(F^{\leftarrow}(u)\right)^{-p} H\left(F^{\leftarrow}(u)\right) \end{aligned} \tag{5.16}$$

as $u\to\infty$. *Here*

$$F(x)=x^p\mu^{\leftarrow}(x),\ x>0\,,$$

and $C(\alpha,\beta,p)$ *is a finite positive constant given by*

$$C(\alpha,\beta,p)=\int_0^\infty y(t)^{-\alpha}t^{-\alpha\beta}\,dt\,,$$

where $y(t)=h^{-1}(t^{-(1+p\beta)})$, $t>0$, *and* h *is a strictly increasing continuous function on* $[1,\infty)$ *given by*

$$h(y)=py^p\int_{1/y}^1\left((yz)^\beta-1\right)(z-1)^{p-1}\,dt\,.$$

5.5. **The supremum of the integral of the process.** Here we consider the subadditive functional

$$\phi_{\text{supint}}(\boldsymbol{\alpha})=\sup_{v>0}\int_0^v\alpha(t)\,dt\,.$$

Unlike other functionals considered in this section, this functional is affected by the negative values of the process. The tail behavior of this functional is described in the theorem below. Its proof, once again, is very similar to that of the previous results, but longer. We will omit its argument as well.

Theorem 5.7. *Assume* (5.1)–(5.3). *Then*

$$
\begin{aligned}
P\left(\phi_{\mathrm{supint}}(\mathbf{X}-\boldsymbol{\mu})>u\right) &= P\left(\sup_{v\geq 0}\int_0^v (X(t)-\mu(t))\,dt>u\right)\\
&\sim C(\alpha,\beta)\,\mu_1^{\leftarrow}(u)\,H\left(\frac{u}{\mu_1^{\leftarrow}(u)}\right) \qquad (5.17)
\end{aligned}
$$

as $u\to\infty$. *Here*

$$\mu_1(x)=\int_0^x \mu(y\,dy, \quad x>0\,,$$

and $C(\alpha,\beta)$ *is a finite positive constant given by*

$$C(\alpha,\beta)=\int_0^\infty y(t)^{-\alpha}t^{-\alpha}\,dt\,,$$

where $y(t)=h^{-1}(4^\beta(1+\beta)t^{1+\beta})$, $t>0$, *and* h *is a strictly increasing continuous function on* $[0,\infty)$ *given by*

$$h(y)=\frac{y^{1+\beta}}{(1+y)^\beta}\,.$$

References

J. Bertoin (1996): *Lévy Processes*. Cambridge University Press, Cambridge.

N.H. Bingham, C.M. Goldie, and J.L. Teugels (1987): *Regular Variation*. Cambridge University Press, Cambridge.

P. Embrechts, C. Goldie and N. Veraverbeke (1979): Subexponentiality and infinite divisibility. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete* 49:335–347.

P. Embrechts, C. Klüppelberg and T. Mikosch (1997): *Modelling Extremal Events for Insurance and Finance.* Springer, Heidelberg.

P. Embrechts and N. Veraverbeke (1982): Estimates for the probability of ruin with special emphasis on the possibility of large claims. *Insurance: Mathematics and Economics* 1:55–72.

J. Hüsler and V. Piterbarg (1999): Extremes of a certain class of Gaussian processes. *Stochastic Processes and Their Applications* 83:257–271.

G. Maruyama (1970): Infinitely divisible processes. *Theory Prob. Appl.* 15:1–22.

T. Mikosch and G. Samorodnitsky (1999): Ruin probability with claims modeled by a stationary ergodic stable process. Preprint. Available as rstab.ps at www.orie.cornell.edu/~gennady/techreports.

V. Petrov (1995): *Limit Theorems of Probability Theory*. Oxford University Press, Oxford.

N.U. Prabhu (1998): *Stochastic Storage Processes. Queues, Insurance Risk, Dams, and Data Communication.* 2nd edition. Springer, New York.

Y. Prokhorov (1959): An extremal problem in probability theory. *Theory of Probability and Its Applications* 4:201–204.

S.I. Resnick (1987): *Extreme Values, Regular Variation, and Point Processes*. Springer, New York.

J. Rosiński and G. Samorodnitsky (1993): Distributions of subadditive functionals of sample paths of infinitely divisible processes. *Annals of Probability* 21:996–1014.

K.-i. Sato (1999): *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge.

W.F. Stout (1974): *Almost Sure Convergence.* Academic Press, New York.

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