Condition Numbers for Polyhedra With Real Number Data

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Abstract: We develop a condition-based complexity analysis for homogeneous polyhedra with real number data. We analyze the dependency of primal-dual interior point algorithm efficiency on this condition number for finding a point in a polyhedron.

Key words: polyhedron, interior point algorithms, condition-based complexity.

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1 Introduction

Consider the following polyhedron:

$$\mathcal{P} = \{\mathbf{x} : \ A\mathbf{x} = \mathbf{0}, \ \mathbf{e}^T\mathbf{x} = 1, \ \mathbf{x} \geq \mathbf{0}\},$$

where $A \in \mathbf{R}^{m \times n}$ with rank m is given, $\mathbf{e}$ is the vector of all ones, and ^{T} denotes transpose. This is the homogeneous form proposed by Karmarkar [1]. $\mathcal{P}$ is said to be feasible if and only if $\mathcal{P} \neq \emptyset$. Given an A, there is unique partition of the columns of A, $A = (B, N)$, such that the set

$$\mathcal{P}_p = \{\mathbf{x}_B : \ B\mathbf{x}_B = \mathbf{0}, \ \mathbf{e}^T\mathbf{x}_B = 1, \ \mathbf{x}_B \geq \mathbf{0}\},$$

has a strictly feasible point or an interior point in the positive orthant, and the dual set

$$\mathcal{P}_d = \{(\mathbf{y}, \mathbf{s}) : \ B^T\mathbf{y} = \mathbf{0}, \ N^T\mathbf{y} \leq \mathbf{0}, \ \mathbf{e}^T(-N^T\mathbf{y}) = 1, \ \mathbf{s} = -A^T\mathbf{y}\},$$

has a strictly feasible point, i.e., a feasible $(\mathbf{y}, \mathbf{s})$ with $\mathbf{s}_N = -N^T\mathbf{y} < \mathbf{0}$. (We use $\mathbf{s}$, the slack vector, to simplify notations.) It is also known that $\mathbf{x}_N = \mathbf{0}$ for any $\mathbf{x} \in \mathcal{P}$. Thus, $\mathcal{P}$ is infeasible if and only if $A = N$.

The assumption that $\mathcal{P}$ is in homogeneous form is without loss of generality (although see the concluding remark). Consider the standard nonhomogeneous linear feasibility system:

$$\mathcal{K} = \{\mathbf{x} : \ \bar{A}\mathbf{x} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}\}.$$

We can construct a related homogeneous system $\mathcal{P}$ using

$$A = (\bar{A}, -\mathbf{b}).$$

Then, $\mathcal{K}$ is feasible if and only if column $\mathbf{b}$ is in B, the unique partition identified for $\mathcal{P}$.

Suppose we want to answer the following question using an interior point method:

$$(P) \quad \text{Is} \quad \mathcal{P} \quad \text{feasible?}$$

In this note, we develop a condition-based complexity analysis of problem (P), and show how a condition number of A may affect algorithm efficiency in solving problem (P). Then, we show how this condition number relates to another condition number of A defined by [6] and [7]. Other condition numbers of A were used in error bound analysis [3][2] and convergence analysis [5].

2 An interior-point algorithm for solving (P)

First, we propose to apply an interior-point algorithm to solve a related linear programming (LP) problem:

$$\begin{array}{lll} (LP) & \text{minimize} & x_0 \\ & \text{subject to} & -(A\mathbf{e})x_0 + A\mathbf{x} = \mathbf{0}, \ \mathbf{e}^T\mathbf{x} \leq 1, \ (x_0, \mathbf{x}) \geq \mathbf{0}, \end{array}$$

The dual of (LP) is

$$\begin{array}{lll}(LD) & \text{maximize} & y_0 \\ & \text{subject to} & \mathbf{e}y_0 + A^T\mathbf{y} \le \mathbf{0},\ -\mathbf{e}^T A^T \mathbf{y} \le 1,\ y_0 \le 0.\end{array}$$

We use $s_0 = 1+\mathbf{e}^T A^T \mathbf{y}$ and $\mathbf{s} = -\mathbf{e}y_0 - A^T\mathbf{y}$ to denote the slack variables of (LD). Obviously, $x_0^0 = 1/(n+1)$ and $\mathbf{x}^0 = \mathbf{e}/(n+1)$, $\mathbf{y}^0 = \mathbf{0}$ and $y_0^0 = -1$ with slack $s_0 = 1$ and $\mathbf{s}^0 = \mathbf{e}$ are feasible points for the (LP) and (LD), respectively. Moreover, they are on the central path with initial duality gap

$$\mu^0 = \frac{x_0^0 s_0^0 + (\mathbf{x}^0)^T \mathbf{s}^0 + (1-\mathbf{e}^T\mathbf{x}^0)(-y_0^0)}{n+2} = \frac{1}{n+1}.$$

Starting from this point, an $O(\sqrt{n}L)$ (primal-dual) interior-point algorithm will generate a sequence of $\{(\mathbf{x}^k, \mathbf{y}^k)\}$, starting from $(\mathbf{x}^0, \mathbf{y}^0)$, such that

$$\min_{0\le j\le n}(x_j^k s_j^k) \ge \alpha\mu^k \quad \text{and} \quad \mu^k \le (1-\beta/\sqrt{n+2})\mu^{k-1} \tag{1}$$

for some constants $0 < \alpha,\ \beta < 1$.

Consider a condition number of A, defined as

$$\begin{array}{rll}\sigma_p & = \min_{j\in B}\{\max & x_j : \quad \mathbf{x}_B \in \mathcal{P}_p\} \\ \sigma_d & = \min_{j\in N}\{\max & s_j : \quad (\mathbf{y},\mathbf{s}) \in \mathcal{P}_d\} \\ \sigma(A) & = \min(\sigma_p, \sigma_d). & \end{array} \tag{2}$$

(we assign σ_p (σ_d) to 1 if B (N) is null.) It has been shown that the interior-point algorithm generates a sequence of partitions $(B^k, N^k) = A$ such that, after $O(\sqrt{n}(|\log\sigma(A)| + \log n))$ iterations, we have convergence to $B^k = B$ and $N^k = N$ (see Ye [11]). It seems that this result could be used to answer question (P) to determine whether $A = N$ or not. However, this requires $\sigma(A)$ as *a priori* information. Without knowledge of $\sigma(A)$, we have to provide a witness whether $A = N$ or not.

Here is an example of the analysis in [11]. It has been shown that if $A = N$, then for any k, the sequence $(\mathbf{x}^k, \mathbf{y}^k)$ satisfies

$$s_j^k \ge \alpha s_j^*/(n+2), \quad 1 \le j \le n$$

for any $(\mathbf{y}^*, \mathbf{s}^*)$ in $\mathcal{P}_d$ (for example, see Ye [11]). Thus,

$$s_j^k \ge \alpha\sigma(A)/(n+2), \quad 1 \le j \le n. \tag{3}$$

Consider a least-squares projection at the kth iteration of the algorithm

$$\begin{array}{ll}\text{minimize} & \|\mathbf{d}_s\| \\ \text{subject to} & A^T\mathbf{d}_y + S^k\mathbf{d}_s = \mathbf{e}y_0^k,\end{array}$$

where S^k is diag$(\mathbf{s}^k)$. We have a closed form for $\mathbf{d}_s$ that is

$$\mathbf{d}_s = y_0^k P_{A(S^k)^{-1}}(S^k)^{-1}\mathbf{e} \tag{4}$$

where P_A is the projection matrix to the null space of A. Let

$$\mathbf{y}^+ = \mathbf{y}^k + \mathbf{d}_y \quad \text{and} \quad \mathbf{s}^+ = \mathbf{s}^k + S^k \mathbf{d}_s = -A^T(\mathbf{y} + \mathbf{d}_y) = -A^T \mathbf{y}^+.$$

Note that from (3) and (4)

$$\|\mathbf{d}_s\| \leq |y_0^k| \cdot \|(S^k)^{-1}\| \cdot \|\mathbf{e}\| \leq \frac{(n+2)^{1.5}}{\alpha\sigma(A)} \cdot |y_0^k|.$$

Thus, if

$$|y_0^k| = -y_0^k \leq x_0^k - y_0^k = (n+2)\mu^k \leq \frac{\alpha\sigma(A)}{(n+2)^{1.5}},$$

then $\|\mathbf{d}_s\| < 1$ and

$$\mathbf{s}^+ = S^k(\mathbf{e} + \mathbf{d}_s) > 0.$$

That is, after a constant-factor scaling, $\mathbf{y}^+$ is an interior point in $\mathcal{P}_d$ with $A = N$, therefore, proving $A = N$. Thus, a witness that (P) is infeasible can be also found in $O(\sqrt{n}(|\log \sigma(A)| + \log n))$ iterations. Similarly, the case $A = B$ can be proved in the same number of iterations. In fact, all other cases can be completed in about the same number of iterations to obtain (B, N), and to generate feasible points in $\mathcal{P}_p$ and $\mathcal{P}_d$, respectively. Thus, $\sigma(A)$ represents a measure of difficulty in solving (P): the smaller $\sigma(A)$, the harder the problem.

3 Relation to another condition number

Let A be an $m \times n$ matrix, and let $\|\cdot\|$ be some p-norm. Let $\mathcal{D}$ be the set of all positive definite $n \times n$ diagonal matrices. Let

$$S = \{\mathbf{s} \in \mathbf{R}^n : \|\mathbf{s}\| = 1 \text{ and } \mathbf{s} = A^T\mathbf{y} \text{ for some } \mathbf{y} \in \mathbf{R}^m\}.$$

Let

$$X = \{\mathbf{x} \in \mathbf{R}^n : AD\mathbf{x} = \mathbf{0} \text{ for some } D \in \mathcal{D}\}.$$

Define

$$\rho_0(A) = \inf\{\|\mathbf{s} - \mathbf{x}\| : \mathbf{x} \in X, \mathbf{s} \in S\}. \tag{5}$$

Theorem 1 *(Stewart [6]) For any nonzero matrix A, $\rho_0(A) > 0$.*

We now define $\bar{\chi}(A) = 1/\rho_0(A)$. In the case that A is full rank, there is an alternative definition:

Theorem 2 *(Stewart [6], O'Leary [4]) Let A be an $m \times n$ matrix of rank n. Then*

$$\bar{\chi}(A) = \sup\{\|A^T(ADA^T)^{-1}AD\| : D \in \mathcal{D}\}. \tag{6}$$

This quantity has been independently analyzed by Todd [7].

Suppose Z_A is a basis for the nullspace of A, that, is $AZ_A = 0$ and any $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ can be written as $\mathbf{x} = Z_A\mathbf{q}$. Then one checks that $1/\bar{\chi}(Z_A^T)$ is precisely equal to the infimum of the distance between

$$S' = \{\mathbf{s} \in \mathbf{R}^n : \mathbf{s} = DA^T\mathbf{w} \text{ for some } \mathbf{w} \in \mathbf{R}^m, D \in \mathcal{D}\},$$

and

$$X' = \{\mathbf{x} \in \mathbf{R}^n : A\mathbf{x} = \mathbf{0} \text{ and } \|\mathbf{x}\| = 1\}.$$

Notice that this fact means that $\bar{\chi}(Z_A^T)$ does not depend on which nullspace basis is chosen.

In this section, we explore the relation between $\sigma(A)$ and $\bar{\chi}(A)$. More specifically, we show

$$\sigma(A) \geq \frac{1}{\bar{\chi}(A) + 1}. \tag{7}$$

Therefore, to solve problem (P) we need at most $O(\sqrt{n}(\log \bar{\chi}(A) + \log n))$ interior-point algorithm iterations.

We first have the following lemma.

Lemma 1 *Let A be an $m \times n$ nonzero matrix, and suppose the columns of A are partitioned arbitrarily as $[B, N]$. Then*

1. $\bar{\chi}(A) \geq 1$*;*
2. *Assuming A has rank m,* $|\bar{\chi}(A) - \bar{\chi}(Z_A^T)| \leq 1$*;*
3. $\bar{\chi}(Z_B^T) \leq \bar{\chi}(Z_A^T)$*, where Z_B, Z_A are nullspace bases for the nullspaces of B, A respectively.*

PROOF. The first two inequalities are proved in Vavasis [9]. We now prove the third one.

Let the size of B be $m \times p$. For simplicity, assume B is composed of the initial p columns of A. Suppose $\mathbf{u}, \mathbf{v} \in \mathbf{R}^p$ are chosen so that $\mathbf{u} = DB^T\mathbf{w}$ for some $D \in \mathcal{D}$ and $\mathbf{w} \in \mathbf{R}^m$, and so that $\|\mathbf{v}\| = 1$ and $B\mathbf{v} = \mathbf{0}$. We must prove that $\|\mathbf{u} - \mathbf{v}\| \geq 1/\bar{\chi}(Z_A^T)$. Let $\mathbf{u}' \in \mathbf{R}^n$ be defined by

$$\mathbf{u}' = D'A^T\mathbf{w}$$

where $D' = \mathrm{diag}(D, \epsilon I)$. Here, I denotes the $(n-p) \times (n-p)$ identity and $\epsilon > 0$ is a small parameter. Observe that

$$\left\| \begin{pmatrix} \mathbf{u} \\ \mathbf{0} \end{pmatrix} - \mathbf{u}' \right\|$$

can be made arbitrarily small as we let ϵ tend to zero. Let $\mathbf{v}'$ be the extension of $\mathbf{v}$ to an n-vector obtained by filling in zeros. Observe that $\|\mathbf{v}'\| = \|\mathbf{v}\| = 1$. Observe also that $A\mathbf{v} = \mathbf{0}$. Thus, by definition,

$$\|\mathbf{u}' - \mathbf{v}'\| \geq 1/\bar{\chi}(Z_A^T).$$

But this implies that the same inequality must hold for $\mathbf{u}$ and $\mathbf{v}$ because the last $n-p$ components of $\mathbf{u}' - \mathbf{v}'$ are arbitrarily small. ∎

We now prove several relations between $\sigma(.)$ and $\bar{\chi}(.)$.

Theorem 3 *Assume $\mathcal{P}_p$ has an interior feasible point. Then*

$$\sigma_p \geq 1/\bar{\chi}(Z_B^T).$$

PROOF. Assume B has n_1 columns. For each $i \in B$ and any $\mu > 0$, consider the optimization problem

$$\max \quad x_i + \mu \sum_{j \in B} \log(x_j), \quad \text{subject to} \quad \mathbf{x}_B \in \mathcal{P}_p.$$

This problem has a unique solution satisfying

$$X_B(-\mathbf{e}_i - B^T\mathbf{y} - \mathbf{e}\lambda) = \mu\mathbf{e},$$

where X_B is diag$(\mathbf{x}_B)$, $\mathbf{e}_i$ is the ith unit vector. Thus, upon taking the inner product with $\mathbf{e}$,

$$-\lambda = n_1\mu + x_i$$

and

$$\mathbf{x}_B - X_B B^T \bar{\mathbf{y}} = \frac{\mu}{n_1\mu + x_i}\mathbf{e} + \frac{1}{n_1\mu + x_i} \cdot X_B\mathbf{e}_i,$$

where $\bar{\mathbf{y}} = \mathbf{y}/(n_1\mu + x_i)$. Note that as $\mu \to 0$, x_i approaches the maximizer x_i^* of

$$\max \quad x_i, \quad \text{subject to} \quad \mathbf{x}_B \in \mathcal{P}_p,$$

which is positive since $\mathcal{P}_p$ has nonempty interior. Thus, $\mathbf{x}_B - X_B B^T\mathbf{y}$ tends to zero as $\mu \to 0$ except for the ith entry. Choose a diagonal matrix D such that D has μ in the ith diagonal position and 1's elsewhere. Then $\|\mathbf{x}_B - DX_B B^T\bar{\mathbf{y}}\| \to x_i$ as $\mu \to 0$. Hence,

$$x_i \geq \frac{1}{\bar{\chi}(Z_B^T)}$$

since $\mathbf{x}_B$ in the range of Z_B^T, $\|\mathbf{x}_B\|_1 = 1$, and $(DX_B)^{-1}\mathbf{w}$, $\mathbf{w} = DX_B B^T\bar{\mathbf{y}}$, is in the null space of Z_B. Thus,

$$x_i^* \geq x_i \geq \frac{1}{\bar{\chi}(Z_B^T)}, \quad \text{for each } i \in B,$$

that is

$$\sigma_p = \min_i(x_1^*, \ldots, x_n^*) \geq \frac{1}{\bar{\chi}(Z_B^T)}.$$

∎

We have similar result for the dual

Theorem 4 *Assume $\mathcal{P}_d$ has an interior feasible point. Then*

$$\sigma_d \geq 1/\bar{\chi}(A).$$

PROOF. Assume N has n_1 columns. For each $i \in N$ and any $\mu > 0$, consider the optimization problem

$$\max \quad s_i + \mu \sum_{j \in N} \log(s_j), \quad \text{subject to} \quad (\mathbf{y}, \mathbf{s}) \in \mathcal{P}_d.$$

This problem has a unique solution satisfying

$$S_N(-\mathbf{e}_i - \mathbf{x}_N - \lambda \mathbf{e}) = \mu \mathbf{e}, \quad \mathbf{s}_N + N^T \mathbf{y} = \mathbf{0}, \quad \mathbf{s}_B = \mathbf{0} \quad \text{and} \quad B\mathbf{x}_B + N\mathbf{x}_N = \mathbf{0}$$

where $\mathbf{x}_N, \mathbf{x}_B$ are Lagrange multipliers, S_N is $\text{diag}(\mathbf{s}_N)$, and $\mathbf{e}_i$ is the ith unit vector. Since $\mathbf{s}_N^T \mathbf{x}_N = \mathbf{s}^T \mathbf{x} = 0$, we again have

$$-\lambda = n_1 \mu + s_i$$

and

$$S_N \bar{\mathbf{x}}_N + \mathbf{s}_N = S_N \bar{\mathbf{x}}_N - N^T \mathbf{y} = \frac{\mu}{n_1 \mu + s_i} \mathbf{e} + \frac{1}{n_1 \mu + s_i} \cdot S_N \mathbf{e}_i,$$

where $\bar{\mathbf{x}} = -\mathbf{x}/(n_1 \mu + s_i)$. Note also that as $\mu \to 0$, s_i approaches the maximizer s_i^* of

$$\max \quad s_i, \quad \text{subject to} \quad (\mathbf{y}, \mathbf{s}) \in \mathcal{P}_d.$$

Choose a diagonal matrix D such that $D_B = \mu I$ and D_N has μ in the ith diagonal position and 1's elsewhere. Then

$$\left\| \begin{pmatrix} D_B & 0 \\ 0 & D_N S_N \end{pmatrix} \bar{\mathbf{x}} - A^T \mathbf{y} \right\| \to s_i$$

as $\mu \to 0$. Hence,

$$s_i^* \geq s_i \geq \frac{1}{\bar{\chi}(A)},$$

since $A\bar{\mathbf{x}} = 0$ and $\mathbf{s} = -A^T y$ with $\|\mathbf{s}\|_1 = 1$. This implies

$$\sigma_d \geq \frac{1}{\bar{\chi}(A)}.$$

∎

Finally, for any A, we have

Theorem 5

$$\sigma(A) \geq \frac{1}{\bar{\chi}(A) + 1}$$

PROOF. For any partition $A = (B, N)$, we have from Lemma 1 and Theorem 3

$$\sigma_p \geq \frac{1}{\bar{\chi}(Z_B^T)} \geq \frac{1}{\bar{\chi}(Z_A^T)} \geq \frac{1}{1 + \bar{\chi}(A)}$$

where Z_B is the null space basis for B. Moreover, from Theorem 4 we have

$$\sigma_d \geq \frac{1}{\bar{\chi}(A)}.$$

Therefore, we have desired result for $\sigma(A) = \min(\sigma_p, \sigma_d)$. ∎

4 A bound on $\bar{\chi}(A)$ for polyhedra with rational data

Finally, if A is rational, we provide a bound for $\bar{\chi}(A)$ in terms of size of A. A similar result is due to Tuncel [8].

Theorem 6 *Let A be rational and L be its bit size. Then*

$$\bar{\chi}(A) \le 2^{O(L)}.$$

PROOF. Consider the least squares problem

$$\min_{\mathbf{y}} \quad \|D(A^T\mathbf{y} - \mathbf{c})\|.$$

For all $D \in \mathcal{D}$, its minimizer

$$\bar{\mathbf{y}} = (AD^2A^T)^{-1}AD^2\mathbf{c},$$

and $\bar{\mathbf{y}}$ is in a bounded polyhedron of the form

$$P(D) = \{\mathbf{y} : A^T\mathbf{y} \ \le, \ =, \ or \ \ge \ \mathbf{c}\}$$

where the actual relation for each inequality depends on D (Todd [7]). Thus, $\bar{\mathbf{y}}$ can be written as a convex combination from $m+1$ vertices of $P(D)$, i.e,

$$\bar{\mathbf{y}} = \sum_{i=1}^{m+1} \alpha_i (B_i^T)^{-1}\mathbf{c}_i, \quad \sum \alpha_i = 1, \quad \alpha_i \ge 0, \ i = 1, \dots, m+1.$$

where B_i is a basis of A and $\mathbf{c}_i$ is the subvector of $\mathbf{c}$ corresponding B_i. Thus, $\|(B_i^T)^{-1}\| \le 2^{O(L)}$ for $i = 1, \dots, m+1$, and

$$\begin{aligned}
\|\bar{\mathbf{y}}\| &= \|\sum_{i=1}^{m+1} \alpha_i (B_i^T)^{-1}\mathbf{c}_i\| \\
&\le \sum_{i=1}^{m+1} \alpha_i \|(B_i^T)^{-1}\mathbf{c}_i\| \\
&\le \sum_{i=1}^{m+1} \alpha_i \|(B_i^T)^{-1}\| \cdot \|\mathbf{c}_i\| \\
&\le \sum_{i=1}^{m+1} \alpha_i \|(B_i^T)^{-1}\| \cdot \|\mathbf{c}\| \\
&\le \sum_{i=1}^{m+1} \alpha_i 2^{O(L)} \|\mathbf{c}\| \\
&= 2^{O(L)}\|\mathbf{c}\|.
\end{aligned}$$

Therefore,

$$\|A^T(AD^2A^T)^{-1}AD^2\mathbf{c}\| = \|A^T\bar{\mathbf{y}}\| = \|A^T\| \cdot \|\bar{\mathbf{y}}\| \le 2^{O(L)}\|\mathbf{c}\|,$$

which implies the theorem. ∎

5 Remarks

We have shown that the complexity of finding an interior point for a homogeneous polyhedron is bounded by $\bar{\chi}(A)$. As mentioned in the introduction, this result can be generalized to nonhomogenous polyhedra. The difficulty with this generalization is that appending $\mathbf{b}$ as a column of A could increase the value of $\bar{\chi}(A)$. This increase is particularly undesirable for problems like flow problems, in which the constraints have the form $A\mathbf{x} = \mathbf{b}$ with small integers entries for A but arbitrary real numbers in the right-hand side vector. It is not hard to construct nonhomogeneous problems with near-degeneracies that make σ_p and σ_d arbitrarily small, independent of A.

A different approach to nonhomogeneous problems is to apply a new kind of interior point method insensitive to such near-degeneracies. This is the subject of a longer paper [10] by the authors.

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