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OPTIMAL SATURATED MAIN EFFECT PLANS FOR THE 2^n FACTORIAL AND v, k, λ CONFIGURATIONS

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ABSTRACT

This paper shows that the saturated main effect plan for the 2^n factorial consisting of the n+1 treatment combinations 00...0, 011...1, 1011...1, ..., 11...10 is optimal (in the sense of maximum absolute value of the determinant of the design matrix,) in the class of all saturated main effect plans where the factors occur n times at the low level.

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1. Introduction and Summary. Consider the estimation of the n main effects and the overall mean in the 2^n factorial with n+1 observations taken at n+1 treatment combinations. Such a plan is known in the literature as a saturated main effect plan (e.g. see Addelman [1963]). Following Ryser [1963] we define a v, k, λ configuration to be an arrangement of v elements into v sets such that each set contains exactly k distinct elements and such that each pair of sets has exactly λ elements in common, where $0 < \lambda < k < v$. In BIB terminology a v, k, λ configuration is a balanced incomplete block design with parameters v, b=v, k=r, r and λ. The v × v,(0,1)-incidence matrix A of a v, k, λ configuration satisfies the properties:

(1.1) $$A'A = AA'$$

(1.2) $$|\det A| = k(k-\lambda)^{\frac{1}{2}(v-1)}$$

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Now, let Q be a (0,1)-matrix of order v, containing exactly t 1's. Let $k = {}^{t}/v$ and set $\lambda = k(k-1)/(v-1)$, with $0 < \lambda < k < v$, then Ryser [1956] has proved that:

$$|\det Q| \leq k(k-\lambda)^{\frac{1}{2}(v-1)} \tag{1.3}$$

with equality holding if and only if Q is the incidence matrix of a v, k, λ configuration.

In this paper, we show that the saturated main effect plan for the 2^n factorial consisting of the treatment combinations 00...0, 011...1, 1011...1, ..., 11...10 is optimal (in the sense of maximum absolute value of the determinant of the design matrix) in the class of all saturated main effect plans where the factors occur exactly 2n times at the low level. We prove this by utilizing Williamson's [1946] and Ryser's [1956] results.

2. Saturated main effect plans, (0,1)-matrices and v, k, λ configurations.

Let X_{n+1} be the (-1,1)-design matrix of order (n+1) of any saturated main effect plan of the 2^n factorial. The first column of X_{n+1} consists of 1's, i.e. X is in semi-normalized form (see Raktoe and Federer [1969]). In searching for an optimal plan in the sense of maximum absolute value of the determinant of X_{n+1}, Williamson [1946] has shown that one may always take the first row of X to consist of -1's, except the element in the first column. Relating this result to saturated main effect plans of the 2^n factorial, we have the following theorem:

THEOREM 2.1

The study of optimal saturated main effect plans for the 2^n factorial in the sense of maximum absolute value of the determinant of X_{n+1} is equivalent to the study of the class of seminormalized square (-1,1)-matrices with -1's as entries in the first row except the element in the first column.

Now, following Williamson [1946] or Raktoe and Federer [1969] we obtain the following relation by adding the first column of X to all other columns:

(2.1) $$|\det X_{n+1}| = 2^n |\det X_n^*|$$

where X^* is an nxn matrix of 0's and 1's. This result leads us to a slightly different theorem than the one obtained by Raktoe and Federer [1969], viz.:

THEOREM 2.2

The study of optimal saturated main effect plans for the 2^n factorial in the sense of maximum absolute value of the determinant of the (-1,1)-matrix X_{n+1} is equivalent to the study of the class of (0,1)-matrices in the sense of maximum absolute value of the determinant of X_n^*.

Historically, the study of optimal saturated main effect plans has been divided into two categories, namely the case where n+1 = 4t (we assume n+1 ≥ 2), so that Hadamard matrices are conjectured to exist and the non-orthogonal case with n+1 ≢ 4t. Plackett and Burman [1946] have given tables of orthogonal main effect plans, while Raghavarao [1959] has studied optimal saturated main effect plans assuming that $X'_{n+1}X_{n+1}$ is of the form a I_{n+1} + b J_{n+1}. The plans corresponding to these two classes has been recently characterized by Raktoe and Federer [1969].

The fundamental paper by Williamson [1946] provides us optimal matrices $\mathring{X}_n^*$ for n = 2, 3, 4, 5 and 6. We exhibit here examples of these for the benefit of the reader:

$$\mathring{X}_2^* = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad \mathring{X}_3^* = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathring{X}_4^* = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} , \quad \mathring{X}_5^* = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathring{X}_6^* = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

The absolute values of the determinants of these five matrices are 1, 2, 3, 5 and 9 respectively. The optimal saturated main effect plans are simply obtained by adjoining the treatment combination 00...0 as a row to each of the matrices above. The design matrices are obtained by identifying 0's by -1's and by bordering the resulting matrix with a column of 1's. The determinant of X_{n+1} is then simply obtained from (2.1).

Now, let t denote the number of 1's in the matrix X_n^*, then it follows that for the 2^n factorial the range of t is:

(2.2) $$n \leq t \leq n^2 - n + 1$$

Set $k = {}^t/n$ and $\lambda = k(k-1)/n^2(n-1)$ then from Ryser [1956] we know that for given t and $0 < \lambda < k < n$ the result (1.3) is valid. When t is not fixed, then Williamson [1946], Ryser [1956] and later on Raktoe and Federer [1969] have shown that:

(2.3) $$|\det X_n^*| \leq 2^{-n} (n+1)^{\frac{n+1}{2}}$$

with equality holding if and only if X_n^* is obtained from a Hadamard matrix, i.e. in the case of equality with n+1 = 4t, we have:

(2.4) $$|\det X_n^*| = 2^{-(4t-1)}(4t)^{2t}$$

The result (2.3) of Williamson [1946] and Ryser [1956] is in reality stronger than the same result obtained by Raktoe and Federer [1969], since the class of (0,1)-matrices considered by the last two authors were of the semi-normalized type and of order (n+1). Also, in the case where (2.4) holds Ryser [1956] has proved that $v = 4\lambda-1$ (=n), $k = 2\lambda$ and $\lambda = \lambda$ for the incidence matrix of the v, k, λ configuration. In using (1.3) we will presently be concerned with the attainment of this upperbound, i.e. with the existence and construction of a v=n, k, λ configuration for every value of t in the range given by (2.2). Hence, we have another equivalence set forth in the following theorem:

THEOREM 2.3

The study of optimal saturated main effect plans for the 2^n factorial in the sense maximum absolute value of the (-1,1)-matrix X_{n+1} is equivalent to the study of each class of (0,1)-matrices corresponding to each value of t as given by (2.2) in the sense of maximum absolute value of the determinant of X_n^*.

3. Optimal plans and v, k, λ configurations of a certain type. In utilizing (1.3) we have $k = {}^t/n$ and $\lambda = \dfrac{t(t-n)}{n^2(n-1)}$; both of these quantities must equal positive integers in order to make sense as a v, k, λ configuration. This implies that t must equal a multiple of n. In other words, we have to consider the values n, 2n, 3n,...,(n-1)n for t. But from $\lambda = t(t-n)/n^2(n-1)$ it then follows that λ will be a positive integer if and only if $t = n(n-1)$.

Hence, for a given 2^n factorial, there will be a v=n, k=n-1, λ=n-2 configuration if and only if t = (n-1)n. This implies that the upperbound (1.3) will be achieved if and only if t = (n-1)n. Hence we have proved the following theorem:

THEOREM 3.1

The upperbound (1.3) is achievable for the class of saturated main effect plans for the 2^n factorial if and only if the number of 1's in X_n^* is equal to (n-1)n.

That there actually exist X_n^*'s with t = (n-1)n, such that these will be incidence matrices of v=n, k=n-1, λ=n-2 configurations can be readily seen by considering the saturated main effect plan consisting of the origin 00...0 and the treatment combinations having exactly one factor at the low level, i.e., 011...1, 1011...1,...., 11...10 . Here $X_n^* = -I_n + J_n$, where I_n is an nXn identity matrix and J_n is an nXn matrix of 1's. It is easily verified that for this plan the following is true:

$$(3.1) \qquad |\det X_n^*| = (n-1) \text{ and } |\det X_{n+1}| = 2^n(n-1)$$

Hence, (1.2) is satisfied and since we can also verify that (1.1) is true, we have the following theorem:

THEOREM 3.2

For the 2^n factorial with t = (n-1)n there exists an optimal saturated main effect plan corresponding to the v=n, k=n-1, λ=n-2 configuration.

4. Discussion. In this paper we have looked at a subset of the possible $\binom{2^n}{n+1}$ saturated main effect plans of the 2^n factorial, namely those having

exactly t = (n-1)n 1's in the matrix X_n^* . For this class we have exhibited an optimal plan corresponding to a v=n, k=n-1, λ=n-2 configuration. Observe that $\mathring{X}_2^*$, $\mathring{X}_3^*$ and $\mathring{X}_4^*$ of Williamson [1956] are of the type given by theorem 3.2; $\mathring{X}_5^*$ and $\mathring{X}_6^*$ are on the other hand not of the type, as obviously they do not belong to the class having t equal to (n-1)n . As can be observed, the major problem still remains because we have in this paper not touched upon classes corresponding to other values of t (i.e. t $\neq$ (n-1)n). This problem is currently under research.

5. Literature.

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