3-WAY BIB DESIGNS

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Abstract

The concept of 3-way BIB designs is introduced. It is shown that the existence of a certain group difference set implies the existence of such designs. In particular it is proved that if $v \equiv 3 \pmod 4$ and v is a prime power then these designs can be constructed.

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1. Introduction

Let D be a v X v array with cells either empty or filled by the elements of a v-set Ω . Let N_1 be the v X v matrix obtained from D by replacing the non-empty cells by "1" and "0" otherwise. Let also N_2 and N_3 denote the incidence matrices of the symbols-rows and symbols-columns respectively. Then D is said to be a 3-way BIB design if the following conditions hold:

$$N_i N_i' = aI_v + cJ_v \ , \qquad i = 1,2,3 \ ,$$

where a and c are positive scalars, I_v is the identity matrix of order v, and J_v is the v X v matrix with all entries equal to one.

These designs are not only interesting combinatorially, but are useful as statistical designs for eliminating heterogeneity in two directions.

Our main purpose of this note is to investigate the existence and construction of such designs.

2. Main Result

Let $<G,*>$ be an abelian group of odd order v with binary operation $*$. Let $G = \{g_1=e, g_2, \cdots, g_v\}$ and $d = \{d_1, d_2, \cdots, d_k\}$ be a difference set [cf. Bruck (1955)] based on $<G,*>$. Suppose we can find a $b \in Z$, integers, such that $G(b) = \{g_1^{b-1}, g_2^{b-1}, \cdots, g_v^{b-1}\} = G$ and $d(b) = \{d_1^b, d_2^b, \cdots, d_k^b\}$ is also a difference set. b = 2 obviously satisfies these requirements. Develop d into a BIB design with the g_j-th block = $\{d_1*g_j, d_2*g_j, \cdots, d_k*g_j\}$. Let N be the incidence matrix of this BIB design. Note that the rows and columns of N are indexed by $g_1, g_2, \cdots, g_v$. Consider the v X v matrix $B = (b_{ij}) = (g_i^b * g_j^{-1})$. Now superimpose B on N and let D be the matrix obtained from B by keeping the entries of B for which the corresponding entries in N is unity and blank otherwise.

<u>Theorem</u> <u>2.1</u>. D <u>is</u> <u>a</u> <u>3-way</u> <u>BIB</u> <u>design</u>.

<u>Proof</u>. Clearly $N_1N_1' = aI_v + cJ_v$ with $a = k(v-k)/(v-1)$ and $c = k(k-1)/(v-1)$. To show that $N_3N_3' = aI_v + cJ_v$ is equivalent to showing that D is a BIB column-wise. Now to prove this, we note that the entries in the first columns of D = d . In the g_j-th column of D the following k cells are non-empty.

$$(d_i*g_j, g_j) , \quad i = 1,2,\cdots,k .$$

The entries of these cells are

$$g_j^{b-1}*d_i^b , \quad i = 1,2,\cdots,k .$$

Therefore, by assumptions on G and d the matrix D is a BIB column-wise. To prove that D is a BIB row-wise we have to find out how the elements of the g_i-th row of B were chosen. The elements in the g_i-th row were chosen because the elements of d have been transformed to g_i, i.e. there exist a subset $\{f_1, f_2, \cdots, f_k\}$ in G such that

$$d_t*f_t = g_i , \quad t = 1,2,\cdots,k .$$

This implies that

$$f_t = g_i*d_t^{-1} .$$

f,s determine those columns of D in which the g_i-th row has non-zero entries.

Thus the following k cells with the given entries constitute the non-empty cells in the g_i-th row of D .

$$\begin{array}{ll} \text{cells:} & \left(g_i, g_i * d_t^{-1}\right) \\ \text{entries:} & g_i^{b-1} * d_t \end{array} \quad , \qquad t = 1,2,\cdots,k \quad .$$

Thus as g_i runs over the elements of G we obtain a BIB from the rows of D .

Corollary 2.1. *If $v \equiv 3 \pmod 4$ and if v is a prime or prime power then there exists a 3-way BIB design.*

Let d be the set of quadratic residues in the GF(v) and let b = 2 . The remaining argument can be seen from the proof of theorem 2.1.

Example. Let v = 7 and $G = \{0,1,\cdots,b\}$. Then $d = \{1,2,4\}$. The corresponding N, B and D for b = 2 are:

$$N = \begin{bmatrix} 0&0&0&1&0&1&1 \\ 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&0 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \end{bmatrix}, \; B = \begin{bmatrix} 0&6&5&4&3&2&1 \\ 2&1&0&6&5&4&3 \\ 4&3&2&1&0&6&5 \\ 6&5&4&3&2&1&0 \\ 1&0&6&5&4&3&2 \\ 3&2&1&0&6&5&4 \\ 5&4&3&2&1&0&6 \end{bmatrix}, \; D = \begin{bmatrix} -&-&-&4&-&2&1 \\ 2&-&-&-&5&-&3 \\ 4&3&-&-&-&6&- \\ -&5&4&-&-&-&0 \\ 1&-&6&5&-&-&- \\ -&2&-&0&6&-&- \\ -&-&3&-&1&0&- \end{bmatrix}$$

Here $N_i N_i' = 2I_7 + J_7$, $i = 1,2,3$.

Reference

1. R. H. Bruck, Difference sets in a finite group, *Trans. Amer. Math. Soc.* 78 (1955), 464-481.