On Generalized Inverses of Some Common Covariance Matrices

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Abstract

Considered are the covariance matrices of the multinomial distribution and arbitrary equally correlated random variables. Generalized inverses are derived. These well-known results are succinctly restated.

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1. Introduction. Two common covariance matrices are those of the multinomial random vector $\underline{Y}$ and equally correlated variables $\underline{X}$. The latter arises as the intrablock correlation matrix and also, as the correlation matrix of exchangeable random variables.

To be more specific let n_i, $i = 1, \cdots, k$ be the observed number of observations falling in the i^{th} multinomial class with the probabilities $(p_1, \cdots, p_k)$ $\sum_{i=1}^{k} p_i = 1$ and sample size $\sum_{i=1}^{k} n_i = n$. Then we consider the covariance matrix of $\underline{Y}' = (Y_1, \cdots, Y_k)$ where

$$Y_i = \frac{1}{\sqrt{n}}\left(n_i - np_i\right), \qquad i = 1, \cdots, k \ .$$

In particular

$$\sum_{\underline{Y}} = \mathrm{Var}(\underline{Y}) = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_k \\ -p_1p_2 & p_2(1-p_2) & \cdots & -p_2p_k \\ \vdots & \vdots & \ddots & \vdots \\ -p_2p_k & \cdots & \cdots & p_k(1-p_k) \end{bmatrix}_{k\times k},$$

and is singular with rank $(k-1)$.

Secondly, we consider the covariance matrix

$$\sum_{\underline{X}} = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & & \ddots & \\ b & & \cdots & a \end{bmatrix}_{k\times k} .$$

For $\Sigma_{\underline{Y}}$ and $\Sigma_{\underline{X}}$, we find a generalized inverse and inverse respectively. These results are well known and are stated here for convenience.

2. Results. First consider $\Sigma_{\underline{Y}}$. Since $\Sigma_{\underline{Y}}$ is of rank $k-1$, we will consider the reduced covariance matrix, $\Sigma^*_{\underline{Y}}$,

$$\Sigma^*_{\underline{Y}} = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_{k-1} \\ -p_1p_2 & p_2(1-p_2) & \cdots & -p_2p_{k-1} \\ \vdots & & \ddots & \\ -p_1p_{k-1} & \cdots & & p_{k-1}(1-p_{k-1}) \end{bmatrix}_{(k-1)\times(k-1)}$$

which is of full rank. It follows immediately that

LEMMA 1. If $\sum_{i=1}^{k-1} p_i < 1$,

$$\left(\Sigma^*_{\underline{Y}}\right)^{-1} = p_k^{-1} \begin{bmatrix} (p_1+p_k)/p_1 & 1 & 1 & \cdots & 1 \\ 1 & (p_2+p_k)/p_2 & 1 & \cdots & 1 \\ \vdots & & & \ddots & \vdots \\ 1 & & \cdots & & (p_{k-1}+p_k)/p_{k-1} \end{bmatrix} .$$

COROLLARY 1. If $\sum_{i=1}^{k-1} p_i < 1$,

$$\left(\Sigma_{\underline{Y}}\right)^{-} = \left[\begin{array}{c|c} \left(\Sigma^*_{\underline{Y}}\right)^{-1} & \underline{0}_{(k-1)\times 1} \\ \hline \underline{0}_{1\times(k-1)} & 0 \end{array}\right] .$$

PROOF. The result follows from Lemma 1 and Searle (1971), Section 1.1b.

Now we will consider $\Sigma_{\underline{X}}$.

LEMMA 2. If $a + (k-1)b \neq 0$, then

$$\text{(i)} \quad \left|\Sigma_{\underline{X}}\right| = (a-b)^{k-1}[a+(n-1)b]$$

$$\text{(ii)} \quad \Sigma_{\underline{X}}^{-1} = \begin{bmatrix} c & d & \cdots & d \\ d & c & \cdots & d \\ \vdots & & \ddots & \vdots \\ d & & \cdots & c \end{bmatrix},$$

where

$$c = c(a,b,k) = [a+(k-2)b]/(a-b)[a+(k-1)b]$$

and

$$c = d(a,b,k) = -b/(a-b)[a+(k-1)b] \quad .$$

Now let $\Sigma_{\underline{X}}^{*}$ denote the first $(k-1)$ rows and columns of $\Sigma_{\underline{X}}$.

COROLLARY 2. If $a + (k-1)b = 0$, then

$$\left(\Sigma_{\underline{X}}\right)_{k\times k}^{-} = \left[\begin{array}{c|c} \left(\Sigma_{\underline{X}}^{*}\right)^{-1} & \underline{0}_{(k-1)\times 1} \\ \hline \underline{0}_{1\times(k-1)} & 0 \end{array}\right] \quad .$$

In particular we see that the diagonal elements of $\Sigma_{\underline{X}}^{*}$ are given by $c(a,b,k-1)$ and the off-diagonal elements by $d(a,b,k-1)$.

REFERENCE

Searle, S. R. (1971), Linear Models, John Wiley & Sons.