RELATION BETWEEN POISSON AND MULTINOMIAL DISTRIBUTIONS

Robert G.D. Steel BU-39-M April, 1953

Introduction. It is usual to see the Poisson distribution developed from the binomial distribution by removing the restriction on the exponent. Here it is shown that the imposition of a fixed total on the number of successes observed in several Poisson populations leads to the multinomial distribution. Fisher (1) has obtained the same result under other circumstances.

Theory. Let $x_1, \ldots, x_k$ be the numbers of successes observed in k trials on Poisson populations with means $\mu_1, \ldots, \mu_k$ respectively. The joint distribution of the observations is

$$(1)\quad f(x_1, \ldots, x_k) = \frac{e^{-\Sigma\mu_i}\mu_1^{x_1} \ldots \mu_k^{x_k}}{x_1! \ldots x_k!}, \quad x_i = 0,1,2, \ldots, \; i=1, \ldots, k$$

Set $\Sigma x_i = n$ and consider (1) as the joint distribution of n and some k-1 of the x_i's, say $x_1, \ldots, x_{k-1}$.

For the marginal distribution of n, we require

$$P(\Sigma x_i = n) = \sum_x \frac{e^{-\Sigma\mu_i}\mu_1^{x_1} \ldots \mu_k^{x_k}}{x_1! \ldots x_k!}$$

where $\sum_x$ is the sum for all configurations of the x_i such that $\Sigma x_i = n$.

From

$$(\mu_1 + \ldots + \mu_k)^n = \Sigma \frac{n!}{x_1! \ldots x_k!} \mu_1^{x_1} \ldots \mu_k^{x_k}$$

where $\Sigma x_i = n$, we have

$$\frac{(\mu_1 + \ldots + \mu_k)^n}{n!} = \Sigma \frac{\mu_1^{x_1} \ldots \mu_k^{x_k}}{x_1! \ldots x_k!}.$$

Hence

$$P(\Sigma x_i = n) = \frac{e^{-\Sigma\mu_i}(\Sigma\mu_i)^n}{n!}$$

and the marginal distribution of n is

$$(2) \qquad f(n) = \frac{e^{-\Sigma\mu_i}(\Sigma\mu_i)^n}{n!}, \quad n = 0, 1, 2, \ldots,$$

a Poisson distribution with parameter $\Sigma\mu_i$.

From (1) and (2), the conditional distribution of $x_1, \ldots, x_{k-1}$ is

$$(3) \qquad f(x_1, \ldots, x_{k-1} | n) = \frac{n!}{x_1! \ldots x_k!} \; {}_i \left(\frac{\mu_i}{\Sigma\mu_i}\right), \text{ with } x_k = n - \sum_1^{k-1} x_i.$$

This is clearly a multinomial distribution.

REFERENCE

1. Fisher, R. A. On the distribution of chi-square. Jour. Roy. Stat. Soc. 85, 89(footnote), 1922.